ASYMPTOTIC EXPANSIONS FOR DISTRIBUTIONS OF THE CHARACTERISTIC ROOTS OF TWO MATRICES FROM CLASSICAL AND COMPLEX GAUSSIAN POPULATIONS*

by

Hung-chiang Li

Department of Statistics

Division of Mathematical Sciences

Mimeograph Series No. 196

August 1969

*This research was partly supported by the National Science Foundation Contract No. GP-7663.

Asymptotic expansions for distributions of the characteristic roots of two matrices from classical and complex gaussian populations

by

Hung-chiang Li

ERRATA

Page	Line	Change	To
vii	2	Chapter	Chapters
5	8	$-4\underset{\sim}{ASLS}^3 + 6\underset{\sim}{AS}^2\underset{\sim}{LS}^2$	$-4\underset{\sim}{RSLS}^3 + 6\underset{\sim}{RS}^2\underset{\sim}{LS}^2 -$
		$-4\underset{\sim}{AS}^3\underset{\sim}{LS} + \underset{\sim}{AS}^4\underset{\sim}{L}$	$-4\underset{\sim}{RS}^3\underset{\sim}{LS} + \underset{\sim}{RS}^4\underset{\sim}{L}$
8	6	$= \frac{1}{3} C_{ij} - \frac{1}{2} C_{ij}^2$	$= - \frac{1}{3} C_{ij} - \frac{1}{2} C_{ij}^2$
16	5'	o (squares of S_{ij}'s) }	O (squares of S_{ij}'s) }
18	4	o ( ) }	O ( ) }
18	3'	$o\left(\frac{1}{n}\right)$	$O\left(\frac{1}{n}\right)$
19	6	$o\left(\frac{1}{n}\right)$	$O\left(\frac{1}{n}\right)$
38	3	$n^{\frac{1}{2}p(2n-p+3)}$	$n^{\frac{1}{2}p(2n-p+1)}$
40	3	$o\left(\frac{1}{n^2}\right)$	$O\left(\frac{1}{n^2}\right)$
73	13	$\rightarrow (\underset{\sim}{B}\ \underset{\sim}{X}_1\ \underset{\sim}{F}_\circ, ---)$	$\rightarrow (\underset{\sim}{B}\ \underset{\sim}{X}_1\ \underset{\sim}{F}_1, ----)$
75	4	$j = 1, \ldots, P$.	$j = 1, \ldots, p$.
77	7	$-\ tr\ w\ \bar{w}\ *$	$-\ tr\ w\ w\ *$
78	-6	$\alpha\underset{\sim}{\delta}_1 + (1-\alpha)\ \underset{\sim}{\delta}\quad \epsilon\omega$	$\alpha\underset{\sim}{\delta} + (1-\alpha)\ \underset{\sim}{\delta}_2\ \epsilon\omega$
78	10	$\ell = \min\ (p-s)$	$\ell = \min\ (p, s)$
78	16	comples	complex
79	-6	$\int_D p_j(\underset{\sim}{\eta}_j + \ell_j\underset{\sim}{\delta}_j)d\underset{\sim}{\eta}_j \geq$	$\int_D p_j(\underset{\sim}{\eta}_j + \ell_j\underset{\sim}{\delta}_j)d\underset{\sim}{\eta}_j \geq$
79	-6	$\int_D p_j(\underset{\sim}{\eta}_j + \ell_j^o\underset{\sim}{\delta}_j)d\underset{\sim}{\eta}_j$	$\int_D p_j(\underset{\sim}{\eta}_j + \ell_j^o\underset{\sim}{\zeta}_j)d\underset{\sim}{\eta}_j$

Page	Line	Change	To
79	-5	$\nu_j' = (\nu_{1j},\ldots,\nu_{pj})$,	$\underset{\sim}{\nu}_j' = (\nu_{1j},\ldots,\nu_{pj})$,
79	-5	$\underset{\sim}{\S}_j' = (\alpha_{1j}, \beta_{1j},\ldots,$	$\underset{\sim}{\zeta}_j' = (\alpha_{1j}, \beta_{1j},\ldots,$
79	-3	$=\int_D p_j(\underset{\sim}{\eta}_j + \ell_j^o \underset{\sim}{\S}_j) d\underset{\sim}{\eta},$	$\int_D p_j(\underset{\sim}{\eta}_j + \ell_j^o \underset{\sim}{\zeta}_j) d\underset{\sim}{\eta}_j,$
80	2'	$(\underset{\sim}{w}\ \underset{\sim}{\bar{w}}^*)^{-1}$	$(\underset{\sim}{w}\ \underset{\sim}{w}^*)$
81	3	$(VV^*)\ (W\bar{W}^*)^{-1}$,	$(VV^*\ (WW^*)^{-1}$,
82	3	$\underset{\sim}{Z}_j$'c	$\underset{\sim}{Z}_j$'s
92	3	two - sample sizes	two sample sizes
27	1	$n^{(n-1)p}\{\Gamma_p \ldots$	$n^{np}\{\Gamma_p \ldots$

CHAPTER I

AN ASYMPTOTIC EXPANSION FOR THE DISTRIBUTION OF THE CHARACTERISTIC ROOTS OF $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$

1. Introduction and Summary

The distribution of the characteristic (ch.) roots of $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ depends on a definite integral over the group of orthogonal matrices. This integral involves the ch. roots of both the population covariance matrix and the sample covariance matrix. Usually the integral is expressed as a hypergeometric series involving zonal polynomials [8], [16]. Unfortunately, this series converges slowly unless the ch. roots of the argument matrices are small. Furthermore the computation of this series is not so easy. In the one sample case, Anderson [1] has obtained an asymptotic expansion for the distribution of the ch. roots of the sample covariance matrix. His expansion is given in increasing powers of n^{-1}, where n is the sample size less one, up to the first three terms. In the two sample case, however, the situation is more complicated. Chang [6] has obtained an asymptotic expansion for the first term. In this chapter, we extend his result to the second term. We also compare the asymptotic expansion for the two sample case with that of one sample case [1].

Let $\underset{\sim}{S}_i(p \times p)$ $(i = 1, 2)$ be independently distributed as Wishart $(n_i, p, \underset{\sim}{\Sigma}_i)$, and let the ch. roots of $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ and $(\underset{\sim}{\Sigma}_1\underset{\sim}{\Sigma}_2^{-1})^{-1}$ be ℓ_j and a_j $(j = 1, \ldots, p)$ respectively such that

$\ell_1 > \ell_2 > \ldots > \ell_p > 0$ and $0 < a_1 < a_2 < \ldots < a_p$. Further, let us denote

$$\underset{\sim}{A} = \text{diag}\,(a_1, a_2, \ldots, a_p) \quad ,$$

$$\underset{\sim}{L} = \text{diag}\,(\ell_1, \ell_2, \ldots, \ell_p) \quad ,$$

$n = n_1 + n_2$, and $\underset{\sim}{I}$ is a $p \times p$ identity matrix throughout this chapter. Then the joint distribution of $\ell_1, \ell_2, \ldots, \ell_p$ is given by [16], [19],

$$(1.1) \qquad C \prod_{i=1}^{p} a_i^{\frac{1}{2}n_1}\, \ell_i^{\frac{1}{2}(n_1-p-1)} \prod_{i<j}^{p} (\ell_i - \ell_j) \prod_{i=1}^{p} d\ell_i$$

$$\cdot \int_{O(p)} |\underset{\sim}{I} + \underset{\sim}{AHLH'}|^{-\frac{n}{2}} \; (\underset{\sim}{H'}\; d\underset{\sim}{H}) \quad ,$$

where

$$(1.2) \qquad C = \Gamma_p\left(\frac{n_1 + n_2}{2}\right)\left\{ 2^p\, \Gamma_p(\tfrac{1}{2}n_1)\, \Gamma_p(\tfrac{1}{2}n_2) \right\}^{-1} ,$$

$$\Gamma_b(x) = \pi^{\frac{1}{4}b(b-1)} \prod_{j=1}^{b} \Gamma\,(x - \tfrac{1}{2}j + \tfrac{1}{2}) \quad ,$$

and $(\underset{\sim}{H'}d\underset{\sim}{H})$ is the invariant measure on the group $O(p)$.

From (1.1) we know that the distribution of the ch. roots of $\underset{\sim}{S_1}\underset{\sim}{S_2}^{-1}$ depends on the definite integral

$$(1.3) \qquad \mathcal{J} = \int_{O(p)} |\underset{\sim}{I} + \underset{\sim}{AHLH'}|^{-\frac{n}{2}} \; (\underset{\sim}{H'}\; d\underset{\sim}{H})$$

2. The Asymptotic Expansion of $\mathcal{J}$

Transform first

(2.1) $$\underset{\sim}{H} = e^{\underset{\sim}{S}}$$

where $\underset{\sim}{S}$ (p x p) is a skew symmetric matrix. The Jacobian of this transformation has been computed by Anderson (c.f. (2.3) of [1]), and is given by

(2.2) $$J = 1 + \frac{p-2}{4!} \operatorname{tr} \underset{\sim}{S}^2 + \frac{8-p}{4(6!)} \operatorname{tr} \underset{\sim}{S}^4 + \frac{5p^2 - 20p + 14}{8(6!)} (\operatorname{tr} \underset{\sim}{S}^2)^2 + \ldots .$$

Lemma 2.1. Let $\underset{\sim}{A}$ and $\underset{\sim}{L}$ be defined as before, then $f(\underset{\sim}{H}) = |\underset{\sim}{I} + \underset{\sim}{AHLH'}|$, $\underset{\sim}{H} \in O(p)$, attains its identical minimum value $|\underset{\sim}{I} + \underset{\sim}{AL}|$ when $\underset{\sim}{H}$ is of the form

(2.3) $$\begin{pmatrix} \pm 1 & & & 0 \\ & \pm 1 & & \\ & & \ddots & \\ 0 & & & \pm 1 \end{pmatrix}$$

Proof: See [1] and [6].

Lemma 2.1 allows us to claim that, for large n, the integrand in (1.3) is negligible except for small neighborhoods about each of these matrices of (2.3) and $\underset{\sim}{I}$ consists of identical contributions from each of these neighborhoods, so that

$$(2.4) \qquad \mathcal{J} = 2^p \int_{N(\underset{\sim}{I})} |\underset{\sim}{I} + \underset{\sim}{AHLH'}|^{-\frac{n}{2}} (\underset{\sim}{H'}\ d\underset{\sim}{H}) \quad ,$$

where $N(\underset{\sim}{I})$ is a neighborhood of the identity matrix on the orthogonal manifold.

Lemma 2.2. Let b_j $(j = 1, \ldots, p)$ be the ch. roots of $\underset{\sim}{B}$ $(p \times p)$ such that

$$\max_{1 \leq j \leq p} |b_j| < 1$$

then

$$|\underset{\sim}{I} + \underset{\sim}{B}|^m = \exp \{m \operatorname{tr}(\underset{\sim}{B} - \tfrac{1}{2}\underset{\sim}{B}^2 + \tfrac{1}{3}\underset{\sim}{B}^3 - \ldots)\} .$$

Proof: See [6].

Since we want to compute up to the second term in the asymptotic expansion of $\mathcal{J}$, we need to investigate the groups of terms up to the fourth order of $\underset{\sim}{S}$. Under transformation (2.1), we have

$$\begin{aligned}\underset{\sim}{AHLH'} = {} & \underset{\sim}{AL} + (\underset{\sim}{ASL} - \underset{\sim}{ALS}) + \tfrac{1}{2}(\underset{\sim}{ALS}^2 + \underset{\sim}{AS}^2\underset{\sim}{L} - 2\underset{\sim}{ASLS}) \\ & + \tfrac{1}{6}(\underset{\sim}{AS}^3\underset{\sim}{L} - 3\underset{\sim}{AS}^2\underset{\sim}{LS} + 3\,\underset{\sim}{ASLS}^2 - \underset{\sim}{ALS}^3) \\ & + \tfrac{1}{24}(\underset{\sim}{ALS}^4 - 4\underset{\sim}{ASLS}^3 + 6\underset{\sim}{AS}^2\underset{\sim}{LS}^2 - 4\underset{\sim}{AS}^3\underset{\sim}{LS} + \underset{\sim}{AS}^4\underset{\sim}{L}) + \ldots\end{aligned}$$

Hence

$$|I + AHLH'|^{-\frac{n}{2}} = |I + AL|^{-\frac{n}{2}} |I + \{S\} + \{S^2\} + \{S^3\} + \{S^4\} + \dots|^{-\frac{n}{2}},$$

where

$$R = (I + AL)^{-1}A = \begin{pmatrix} r_1 & & & 0 \\ & r_2 & & \\ & & \ddots & \\ 0 & & & r_p \end{pmatrix}, \quad r_j = \frac{a_j}{1+a_j\ell_j} \quad (j = 1,\dots, p)$$

$$\{S\} = RSL - RLS \quad ,$$

$$\{S^2\} = \frac{1}{2}(RLS^2 + RS^2L - 2\,RSLS) \quad ,$$

$$\{S^3\} = \frac{1}{6}(RS^3L - 3\,RS^2LS + 3RSLS^2 - RLS^3)$$

and

$$\{S^4\} = \frac{1}{24}(RLS^4 - 4ASLS^3 + 6AS^2LS^2 - 4\,AS^3LS + AS^4L) \quad .$$

Under transformation (2.1), $N(I) \to N(S = 0)$. If we put $G = \{S\} + \{S^2\} + \{S^3\} + \{S^4\} + \dots$, then in the neighborhoods of $S = 0$, the elements of S are very small, and hence the maximum ch. roots of G can be assumed to be less than unity. Therefore Lemma 2.2 is applicable. By Lemma 2.2, we obtain

$$|I + AHLH'|^{-\frac{n}{2}} = |I + AL|^{-\frac{n}{2}} |I + G|^{-\frac{n}{2}}$$

$$= |I + AL|^{-\frac{n}{2}} \exp\{-\frac{n}{2}\,\mathrm{tr}(\,[S] + [S^2] + [S^3] + [S^4] + \dots)\}$$

where

$$[\underset{\sim}{S}] = \{\underset{\sim}{S}\} \quad ,$$

$$[\underset{\sim}{S}^2] = \{\underset{\sim}{S}^2\} - \frac{1}{2}\{\underset{\sim}{S}\}^2 \quad ,$$

$$[\underset{\sim}{S}^3] = \{\underset{\sim}{S}^3\} - \frac{1}{2}\{\underset{\sim}{S}\}\{\underset{\sim}{S}^2\} - \frac{1}{2}\{\underset{\sim}{S}^2\}\{\underset{\sim}{S}\} + \frac{1}{3}\{\underset{\sim}{S}\}^3 \quad ,$$

and

$$[\underset{\sim}{S}^4] = \{\underset{\sim}{S}^4\} - \frac{1}{2}\{\underset{\sim}{S}\}\{\underset{\sim}{S}^3\} - \frac{1}{2}\{\underset{\sim}{S}^3\}\{\underset{\sim}{S}\} - \frac{1}{2}\{\underset{\sim}{S}^2\}^2 + \frac{1}{3}\{\underset{\sim}{S}\}^2\{\underset{\sim}{S}^2\}$$

$$+ \frac{1}{3}\{\underset{\sim}{S}\}\{\underset{\sim}{S}^2\}\{\underset{\sim}{S}\} + \frac{1}{3}\{\underset{\sim}{S}^2\}\{\underset{\sim}{S}\}^2 - \frac{1}{4}\{\underset{\sim}{S}\}^4 \quad .$$

Since $\underset{\sim}{S} = (s_{ij})$ $s_{ji} = -s_{ij}$ for all $i,j = 1, \ldots, p$, now we have

$$\mathrm{tr}[\underset{\sim}{S}] = \mathrm{tr}(\underset{\sim}{RSL} - \underset{\sim}{RLS}) = 0$$

$$\mathrm{tr}[S^2] = \mathrm{tr}(\{S^2\} - \tfrac{1}{2}\{S\}^2)$$

$$= \mathrm{tr}(\tfrac{1}{2}\underset{\sim}{RLS}^2 + \tfrac{1}{2}\underset{\sim}{RS}^2\underset{\sim}{L} - \underset{\sim}{RSLS} - \tfrac{1}{2}(\underset{\sim}{RLSRLS} + \underset{\sim}{RSLRSL} - \underset{\sim}{RLSRSL} - \underset{\sim}{RSLRLS}))$$

$$= \mathrm{tr}(\underset{\sim}{LS} - \underset{\sim}{SL})(\underset{\sim}{I} - \underset{\sim}{RL})\underset{\sim}{SR}$$

$$= \sum_{i<j}^{p} C_{ij}s_{ij}^2$$

where

$$C_{ij} = (r_{ji} - r_i r_j \ell_{ij})\, \ell_{ij} = -C_{ji}$$

(2.5)

$$r_{ij} = r_i - r_j \quad \text{and} \quad \ell_{ij} = \ell_i - \ell_j \quad .$$

Let us note that

$$\mathrm{tr}\{\underset{\sim}{S}\}\{\underset{\sim}{S}^2\} = \mathrm{tr}\{\underset{\sim}{S}^2\}\{\underset{\sim}{S}\} \quad ,$$

$$\mathrm{tr}\{\underset{\sim}{S}\}\{\underset{\sim}{S}^3\} = \mathrm{tr}\{\underset{\sim}{S}^3\}\{\underset{\sim}{S}\} \quad ,$$

and

$$\mathrm{tr}\{\underset{\sim}{S}\}^2\{\underset{\sim}{S}^2\} = \mathrm{tr}\{\underset{\sim}{S}\}\{\underset{\sim}{S}^2\}\{\underset{\sim}{S}\} = \mathrm{tr}\{\underset{\sim}{S}^2\}\{\underset{\sim}{S}\}^2 \quad .$$

Similarly, after simplification, we find

$$\mathrm{tr}[\underset{\sim}{S}^3] = \mathrm{tr}\{\underset{\sim}{S}^3\} - \mathrm{tr}\{\underset{\sim}{S}\}\{\underset{\sim}{S}^2\} + \tfrac{1}{3}\{\underset{\sim}{S}\}^3$$

$$= \sum_{i<j<k}^{p} f\, s_{ij} s_{jk} s_{ki} \quad ,$$

where

$$f = f(i,j,k)$$

$$= r_{ij}\ell_{jk} - r_{jk}\ell_{ij} + r_i r_{jk}\ell_{ij}\ell_{ik} + r_j r_{ik}\ell_{ij}\ell_{jk}$$

$$+ r_k r_{ij}\ell_{ik}\ell_{jk} - 2r_i r_j r_k \ell_{ij}\ell_{jk}\ell_{ki} \quad ,$$

and

$$\mathrm{tr}[\underset{\sim}{S}^4] = \mathrm{tr}\{\underset{\sim}{S}^4\} - \mathrm{tr}\{\underset{\sim}{S}\}\{\underset{\sim}{S}^3\} - \tfrac{1}{2}\mathrm{tr}\{\underset{\sim}{S}^2\}^2 + \mathrm{tr}\{\underset{\sim}{S}\}^2\{\underset{\sim}{S}^2\} - \tfrac{1}{4}\{\underset{\sim}{S}\}^4$$

$$= \sum_{i<j}^{p} \varphi s_{ij}^4 + \sum_{i<j<k}^{p} \psi_1 s_{ij}^2 s_{ik}^2 + \sum_{i<j<k}^{p} \psi_2 s_{ij}^2 s_{jk}^2$$

$$+ \sum_{i<j<k}^{p} \psi_3 s_{ik}^2 s_{jk}^2 + \sum_{i<j}^{p} g s_{ij} s_{jk} s_{kt} s_{ti} \quad ,$$

where

(2.7) $$\varphi = \varphi(i,j) = (r_i r_j \ell_{ij}^2 - \frac{1}{3}) r_{ji} \ell_{ij} + (\frac{1}{3} r_i r_j - \frac{1}{2} r_{ji}^2)\ell_{ij}^2 - \frac{1}{2} r_i^2 r_j^2 \ell_{ij}^4$$

$$= \frac{1}{3} c_{ij} - \frac{1}{2} c_{ij}^2 \quad ,$$

(2.8) $$\psi_1 = \psi_1(i,j,k)$$

$$= -\frac{1}{3} r_{ji}\ell_{ij} - \frac{1}{3} r_{ki}\ell_{ik} + \frac{1}{4} r_{kj}\ell_{jk} + \frac{1}{3} r_i (r_j \ell_{ij}^2 + r_k \ell_{ik}^2)$$

$$+ r_i (r_j + r_k)\ell_{ij}\ell_{ik} - r_i^2 \ell_{ij}\ell_{ik} - \frac{1}{4} r_j r_k (\ell_{ij} + \ell_{ik})^2$$

$$- r_i (r_j \ell_{ij} r_{ik} + r_k r_{ij} \ell_{ik} + r_i r_j r_k \ell_{ij}\ell_{ik})\ell_{ij}\ell_{ik}$$

$$= -\frac{1}{3}(c_{ij} + c_{ik}) + \frac{1}{4} c_{jk} - c_{ij} c_{ik}$$

and

$$g = g(i,j,k,t)$$

$$= \frac{1}{2}(r_{ki}\ell_{ik}+r_{tj}\ell_{jt}) - \frac{1}{3}(r_{ji}\ell_{ij}+r_{kj}\ell_{jk}+r_{tk}\ell_{kt}+r_{it}\ell_{ti})$$

$$+ \frac{1}{3}r_i r_j \ell_{ij}(\ell_{ij}+3\ell_{kt}) + \frac{1}{3}r_j r_k \ell_{jk}(\ell_{jk}+3\ell_{ti}) + \frac{1}{3}r_k r_t \ell_{kt}(\ell_{kt}+3\ell_{ij})$$

$$+ \frac{1}{3}r_t r_i \ell_{ti}(\ell_{ti}+3\ell_{jk}) - \frac{1}{2}r_i r_k(\ell_{ij}-\ell_{jk})(\ell_{kt}-\ell_{ti})$$

$$+ \frac{1}{2}r_j r_t(\ell_{ij}-\ell_{ti})(\ell_{jk}-\ell_{kt}) - r_i r_j \ell_{ij}\ell_{jk} r_{kt}\ell_{ti} - r_j r_k \ell_{ij}\ell_{jk}\ell_{kt} r_{ti}$$

$$- r_k r_t r_{ij}\ell_{jk}\ell_{kt}\ell_{ti} - r_t r_i \ell_{ij} r_{jk}\ell_{kt}\ell_{ti} - 2r_i r_j r_k r_t \ell_{ij}\ell_{jk}\ell_{kt}\ell_{ti} \quad .$$

Note that ψ_2 and ψ_3 can be obtained from ψ_1 cyclically, i.e., changing i to j, j to k, k to i, then ψ_1 becoming ψ_2, ψ_2 becoming ψ_3 and ψ_3 becoming ψ_1. Moreover, we need not know the value of g, because any term containing an odd power of a factor s_{ij} when integrated with respect to s_{ij} reduces to zero (see below). From (2.6), it is not difficult to show that $f^2 = C_{ij}^2+C_{ik}^2+C_{jk}^2-2(C_{ij}C_{ik}+C_{ij}C_{jk}+C_{ik}C_{jk})-4C_{ij}C_{jk}C_{ik}$.

Finally, we can write (2.4) to be

$$(2.9) \qquad \mathcal{J} = 2^p \prod_{i=1}^{p} (1 + a_i\ell_i)^{-\frac{n}{2}} \int_{N(\underset{\sim}{S}=\underset{\sim}{0})} \exp\left(-\frac{n}{2}\sum_{i<j}^{p} C_{ij}s_{ij}^2\right)$$

$$\cdot \exp\left(-\frac{n}{2}\,\mathrm{tr}[\underset{\sim}{S}^3] - \frac{n}{2}\,\mathrm{tr}[\underset{\sim}{S}^4] - \ldots\right) J \prod_{i<j}^{p} ds_{ij} \quad .$$

If this integration is to be performed term by term on the expansion of $\exp\,(-\frac{n}{2}\,\mathrm{tr}[\underset{\sim}{S}^3]-\ldots)J$ then for large n the limits for each s_{ij} can be put to $\pm\infty$, since each integration is of the form

$$\int_{N(\underset{\sim}{S}=\underset{\sim}{0})} \exp\,(-\frac{n}{2}\sum_{i<j}^{p} C_{ij}s_{ij}^2)\prod_{i<j}^{p} s_{ij}^{m_{ij}}\,ds_{ij}$$

and most of this integral is given in a small neighborhood of $\underset{\sim}{S}=\underset{\sim}{0}$. The m_{ij}'s are positive even integers or zero since any term containing an odd power of an s_{ij} as a factor will integrate to zero. We expand $\exp\,(-\frac{n}{2}\,\mathrm{tr}[\underset{\sim}{S}^3]-\ldots)J$, writing the terms in groups, each group corresponding to a certain value of m. We have

$$(2.10)\qquad \exp\,(-\frac{n}{2}\,\mathrm{tr}[\underset{\sim}{S}^3]-\frac{n}{2}\,\mathrm{tr}[\underset{\sim}{S}^4]-\ldots)J$$

$$=1-\frac{n}{2}\,\mathrm{tr}[\underset{\sim}{S}^4]+\frac{n^2}{8}(\mathrm{tr}[\underset{\sim}{S}^3])^2+\frac{p-2}{4!}\,\mathrm{tr}\underset{\sim}{S}^2$$

$$-\frac{n}{2}\,\mathrm{tr}[\underset{\sim}{S}^6]+\frac{n^2}{8}(\mathrm{tr}[\underset{\sim}{S}^4])^2+\ldots\quad.$$

Using (2.6a) and (2.6b) of [1] we obtain the following theorem.

Theorem 2.1. Let $\underset{\sim}{A}$ and $\underset{\sim}{L}$ be diagonal matrices with $0<a_1<a_2<\ldots<a_p$ and $\ell_1>\ell_2>\ldots>\ell_p>0$. Then for large n, the first two terms in the expansion for $\mathcal{J}$ are given by

$$(2.11)\qquad \mathcal{J}=2^p\prod_{i=1}^{p}(1+a_i\ell_i)^{-\frac{n}{2}}\prod_{i<j}^{p}\left(\frac{2\pi}{nC_{ij}}\right)^{\frac{1}{2}}\left\{1+\frac{1}{2n}[\Sigma C_{ij}^{-1}+\alpha(p)]+\ldots\right\},$$

where

(2.12) $$\alpha(p) = p(p-1)(2p+5) / 12 \quad .$$

Proof: In the proof, we include only terms without an odd power of an s_{ij}. First note that only the second, third and fourth terms on the right hand side of (2.10) contribute the factor n^{-1}. After integration, the first term unity has been shown [1] to give

(2.13) $$K = \prod_{i<j}^{p} \left(\frac{2\pi}{nC_{ij}} \right)^{\frac{1}{2}} .$$

The second term $-n \operatorname{tr}[\underset{\sim}{S}^4] / 2$ contributes

(2.14) $$K\left\{\frac{1}{2n} \sum_{i<j}^{p} C_{ij}^{-1} + \frac{3}{4n}\binom{p}{2} + \frac{p-2}{3n} \sum_{i<j}^{p} C_{ij}^{-1} - \frac{1}{8n} \sum_{i<j<k}^{p} \left(\frac{C_{jk}}{C_{ij}C_{ik}} + \frac{C_{ik}}{C_{ij}C_{jk}} + \frac{C_{ij}}{C_{ik}C_{jk}} \right) + \frac{3}{2n}\binom{p}{3}\right\} ,$$

and the third term $n^2(\operatorname{tr}[\underset{\sim}{S}^3])^2/ 8$ gives

(2.15) $$K\left\{\frac{1}{8n} \sum_{i<j<k}^{p} \left(\frac{C_{jk}}{C_{ij}C_{ik}} + \frac{C_{ik}}{C_{ij}C_{jk}} + \frac{C_{ij}}{C_{ik}C_{jk}} \right) - \frac{p-2}{4n} \sum_{i<j}^{p} C_{ij}^{-1} - \frac{1}{2n}\binom{p}{3}\right\} .$$

Finally, since $\operatorname{tr} \underset{\sim}{S}^2 = -2 \sum_{i<j}^{p} s_{ij}^2$, it is easy to see that $(p-2) \operatorname{tr} \underset{\sim}{S}^2/4!$ contributes

$$- \frac{p-2}{12n} K \sum_{i<j}^{p} C_{ij}^{-1} \quad . \tag{2.16}$$

Add (2.13) - (2.16) and factoring K out, we obtain (2.11).

<u>Theorem 2.2</u>. The asymptotic distribution of the ch. roots, $\ell_1>\ell_2>\ldots>\ell_p>0$, of $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ for large degrees of freedom $n = n_1 + n_2$ when the roots of $\underset{\sim}{\Sigma}_1\underset{\sim}{\Sigma}_2^{-1}$ are $\lambda_1>\lambda_2>\ldots>\lambda_p>0$ where $\lambda_j=a_j^{-1}$ $(j=1,\ldots,p)$, is given by

$$C\, 2^p \prod_{i=1}^{p} a_i^{\frac{1}{2}n_1} \ell_i^{\frac{1}{2}(n_1-p-1)} (1 + a_i\ell_i)^{-\frac{n}{2}} \prod_{i<j}^{p} (\ell_i - \ell_j) \tag{2.17}$$

$$\cdot \prod_{i=1}^{p} d\ell_i \prod_{i<j}^{p} \left(\frac{2\pi}{nC_{ij}}\right)^{\frac{1}{2}} \left\{1 + \frac{1}{2n}[\Sigma\, C_{ij}^{-1} + \alpha(p)] + \ldots\right\} \quad ,$$

where C, C_{ij} and $\alpha(p)$ are defined by (1.2), (2.5) and (2.12) respectively.

3. <u>Remarks</u>

In (2.11), if we write

$$\omega = \omega(a,\ell) = 2^p \prod_{i=1}^{p} (1 + a_i\ell_i)^{-\frac{n}{2}} \prod_{i<j}^{p} \left(\frac{2\pi}{nC_{ij}}\right)^{\frac{1}{2}} \quad ,$$

then the first and second approximation for ϑ are ω and $\omega\left\{1 + [\Sigma C_{ij}^{-1} + \alpha(p)] / 2n\right\}$, and hence we know that $\omega[\Sigma C_{ij}^{-1}+\alpha(p)] / 2n$ is omitted when the first approximation is used.

It is interesting to compare (2.11) with the corresponding formula in the one sample case (c.f. (2.8) of [1]). We find that there is an extra term $\alpha(p)/2n$ (in the second term of the asymptotic expansion for ϑ in the two sample case) which is a function of n and p only. Moreover, if we replace ℓ_j by $n_1\ell_j/n_2$ in (2.17) and let n_2 tend to infinity, then the asymptotic distribution of the ch. roots of $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ reduces that of $\underset{\sim}{S}_1$, which was given by Anderson (c.f. (2.8) of [1]).

CHAPTER II

AN ASYMPTOTIC EXPANSION FOR THE DISTRIBUTION OF THE CHARACTERISTIC ROOTS OF $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ WHEN ROOTS ARE NOT ALL DISTINCT

1. Introduction and Summary

James [17] has studied the Bartlett-Lawley tests of equality of the smaller characteristic (ch.) roots of the covariance matrix using a conditional distribution of the smaller sample roots given the larger roots, obtained from a gamma type asymptotic approximation to the roots distribution with linkage factors between sample roots corresponding to larger and smaller population roots. In the two sample case, we obtain a beta type asymptotic approximation to the roots distribution. If n_2, the sample size of the second sample less one, tends to infinity, then the problem reduces to the one sample case discussed by James [17]. We also derive a general formula which includes the formulae of Anderson [1], James [17], and Chang [6] as limiting or special cases.

2. The Asymptotic Expansion of $\mathcal{J}$ When Roots Are Not All Distinct

Let $\underset{\sim}{S}_i(p \times p)$ $(i = 1, 2)$ be distributed as in the previous chapter and $0 < a_1 < \ldots < a_k < a_{k+1} = \ldots = a_p = a$, $(1 \leq k \leq p-1)$. Then

$$\underset{\sim}{A} = \text{diag}(a_1, \ldots, a_k, a, \ldots, a) \quad ,$$

and the joint distribution of $\ell_1, \ell_2, \ldots, \ell_p$ of (1.1) in Chapter I becomes

$$\text{(2.1)} \qquad C a^{\frac{1}{2}qn_1} \prod_{i=1}^{k} a_i^{\frac{1}{2}n_1} \int_{O(p)} |\underset{\sim}{I} + \underset{\sim}{AHLH'}|^{-\frac{n}{2}} (\underset{\sim}{H}'d\underset{\sim}{H})$$

$$\cdot \prod_{i=1}^{p} \ell_i^{\frac{1}{2}(n_1-p-1)} \prod_{i<j}^{p} (\ell_i - \ell_j) \prod_{i=1}^{p} d\ell_i \quad ,$$

where $q = p - k$. As in Chapter I consider the integral

$$\text{(2.2)} \qquad \mathcal{J} = \int_{O(p)} |\underset{\sim}{I} + \underset{\sim}{AHLH'}|^{-\frac{n}{2}} (\underset{\sim}{H}'d\underset{\sim}{H}) \quad .$$

Now we partition the matrix $\underset{\sim}{H}$ into the submatrices $\underset{\sim}{H_1}$ and $\underset{\sim}{H_2}$ consisting of its first k and the remaining q rows of $\underset{\sim}{H}$ respectively. If the integrand of (2.2) does not depend on $\underset{\sim}{H_2}$, then we can integrate over $\underset{\sim}{H_2}$ for fixed $\underset{\sim}{H_1}$ by the formula

$$\text{(2.3)} \qquad \int_{\underset{\sim}{H_2}} C_1 \, (d\underset{\sim}{H}) = C_2(d\underset{\sim}{H_1})$$

where

$$C_1 = \pi^{\frac{1}{2}p^2}\left\{ \Gamma_p\left(\frac{p}{2}\right)\right\}^{-1} \quad , \qquad C_2 = \pi^{\frac{1}{2}kp}\left\{ \Gamma_k\left(\frac{p}{2}\right)\right\}^{-1}$$

and the symbol $(d\underset{\sim}{H}_1)$ denotes the invariant volume element on the Stiefel manifold of orthonormal k-frames in p-space normalized to make its integral unity.

Make the transformation

$$\underset{\sim}{H} = e^{\underset{\sim}{S}} \tag{2.4}$$

where $\underset{\sim}{S}$ is defined as in Chapter I, and the Jacobian is given by (2.2) of Chapter I.

A parameterization of $\underset{\sim}{H}_1$ may be obtained by writing

$$\underset{\sim}{H} = \begin{pmatrix} \underset{\sim}{H}_1 \\ \underset{\sim}{H}_2 \end{pmatrix} = \exp\left\{\begin{pmatrix} \underset{\sim}{S}_{11} & \underset{\sim}{S}_{12} \\ -\underset{\sim}{S}_{12}' & \underset{\sim}{0} \end{pmatrix}\right\} \tag{2.5}$$

where $\underset{\sim}{S}_{11}$ is a $k \times k$ skew symmetric matrix and $\underset{\sim}{S}_{12}$ is a $k \times q$ rectangular matrix. From (2.2) of Chapter I, it is not difficult to show that

$$C_2(d\underset{\sim}{H}_1) = (d\underset{\sim}{S}_{11})(d\underset{\sim}{S}_{12})\{1 + o(\text{squares of } s_{ij}\text{'s})\}$$

where the symbols $(d\underset{\sim}{S}_{11})$ and $(d\underset{\sim}{S}_{12})$ stand for $\prod_{i<j}^{k} ds_{ij}$ and $\prod_{i=1}^{k} \prod_{j=k+1}^{p} ds_{ij}$ respectively.

Before we find the asymptotic expansion of (2.2) we prove the following lemma.

Lemma 2.1. Let $\underset{\sim}{A}$ and $\underset{\sim}{L}$ be defined as before with $\ell_1 > \ell_2 > \ldots > \ell_p > 0$
$0 < a_1 < \ldots < a_k < a_{k+1} = \ldots = a_p = a$, $(1 \leq k \leq p-1)$.

If we partition $\underset{\sim}{H} = \begin{pmatrix} \underset{\sim}{H}_1 \\ \underset{\sim}{H}_2 \end{pmatrix}$ such that $\underset{\sim}{H}_2$ consists of the last q rows of $\underset{\sim}{H}$, then $|\underset{\sim}{I} + \underset{\sim}{AHLH'}|$ does not depend on $\underset{\sim}{H}_2$.

Proof: See [7].

Since we are only interested in the first term, all we need to investigate is the groups of terms up to the second order of $\underset{\sim}{S}$ which is denoted by $[\underset{\sim}{S}^2]$. As we did in Chapter I, but remembering the last q ch. roots of $\underset{\sim}{A}$ are equal, it is easy to show that

$$\mathrm{tr}[\underset{\sim}{S}^2] = \sum_{i<j}^{k} C_{ij} s_{ij}^2 + \sum_{i=1}^{k} \sum_{j=k+1}^{p} C_{ij}^{o} s_{ij}^2 \quad ,$$

where

$$(2.6) \quad \begin{aligned} C_{ij} &= r_{ji}\ell_{ij} - r_i r_j \ell_{ij}^2 = -C_{ji} \qquad i, j = 1, \ldots, k, \quad i < j \\ C_{ij}^{o} &= r_{ji}\ell_{ij} - r_i r_j \ell_{ij}^2 = -C_{ji}^{o} \qquad i = 1, \ldots, k, \quad j = k+1, \ldots, p. \end{aligned}$$

$$r_i = \begin{cases} \dfrac{a_i}{1 + a_i \ell_i} & \text{if } i = 1, \ldots, k \\ \dfrac{a}{1 + a\ell_i} & \text{if } i = k+1, \ldots, p \end{cases} ,$$

$$r_{ij} = r_i - r_j \qquad \text{and} \qquad \ell_{ij} = \ell_i - \ell_j \quad .$$

Therefore

$$(2.7) \qquad |\underset{\sim}{I} + \underset{\sim}{AHLH'}|^{-\frac{n}{2}} = \prod_{i=1}^{k} (1 + a_i \ell_i)^{-\frac{n}{2}} \prod_{i=k+1}^{p} (1 + a\ell_i)^{-\frac{n}{2}}$$

$$\cdot \prod_{i<j}^{k} \exp(-\tfrac{1}{2}nC_{ij}s_{ij}^2) \prod_{i=1}^{k} \prod_{j=k+1}^{p} \exp(-\tfrac{1}{2}nC_{ij}^{o}s_{ij}^2)$$

$$\cdot \{1 + o(\text{squares of } s_{ij}\text{'s})\} \ .$$

Substituting (2.7) into (2.4) of Chapter I, and using

$$\int_{O(p)} |\underset{\sim}{I} + \underset{\sim}{AHLH'}|^{-\frac{n}{2}} (\underset{\sim}{H}'d\underset{\sim}{H}) = 2^p C_1 \int_{O(p)} |\underset{\sim}{I} + \underset{\sim}{AHLH'}|^{-\frac{n}{2}} (d\underset{\sim}{H})$$

yields

$$(2.8) \qquad \mathcal{J} = \frac{\pi^{\frac{1}{2}q^2}}{\Gamma_q\left(\frac{q}{2}\right)} \prod_{i=1}^{k} (1 + a_i \ell_i)^{-\frac{n}{2}} \prod_{i=k+1}^{p} (1 + a\ell_i)^{-\frac{n}{2}}$$

$$\cdot \int_{\underset{\sim}{S}_{11}} \int_{\underset{\sim}{S}_{12}} \prod_{i<j}^{k} \exp(-\tfrac{n}{2}C_{ij}s_{ij}^2)\, ds_{ij}$$

$$\cdot \prod_{i=1}^{k} \prod_{j=k+1}^{p} \exp(-\tfrac{n}{2}C_{ij}^{o}s_{ij}^2)\, ds_{ij} \{1 + o(\tfrac{1}{n})\} \ .$$

For large n and a_i's and ℓ_i's $(i = 1, \ldots, k)$ well spaced, most of the integral in (2.8) will be obtained from small values of the elements of $\underset{\sim}{S}_{11}$ and $\underset{\sim}{S}_{12}$. Hence, to obtain an asymptotic series, we can replace the finite range of s_{ij} by the range of all real values of s_{ij}. Thus

$$\mathcal{I} = \frac{\pi^{\frac{1}{2}q^2}}{\Gamma_q\left(\frac{q}{2}\right)} \prod_{i=1}^{k} (1 + a_i\ell_i)^{-\frac{n}{2}} \prod_{i=k+1}^{p} (1 + a\ell_i)^{-\frac{n}{2}}$$

$$\cdot \prod_{i<j}^{k} \int_{-\infty}^{\infty} \exp\left(-\frac{n}{2}C_{ij}s_{ij}^2\right) ds_{ij}$$

$$\cdot \prod_{i=1}^{k} \prod_{j=k+1}^{p} \int_{-\infty}^{\infty} \exp\left(-\frac{n}{2}C_{ij}^{o}s_{ij}^2\right) ds_{ij}\{1 + o(\tfrac{1}{n})\} .$$

Hence we have the following theorem:

<u>Theorem</u>. The asymptotic distribution of the ch. roots, $\ell_1>\ell_2>\ldots>\ell_p>0$ of $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$, for large degrees of freedom $n = n_1 + n_2$, when ch. roots of $(\underset{\sim}{\Sigma}_1\underset{\sim}{\Sigma}_2^{-1})^{-1}$ are $0<a_1<\ldots<a_k<a_{k+1} = \ldots = a_p = a$, $(1 \leq k \leq p-1)$ is given by

$$(2.9) \quad C_3 a^{\frac{1}{2}qn_1} \prod_{i=1}^{k} a_i^{\frac{1}{2}n_1} \prod_{i=1}^{p} \ell_i^{\frac{1}{2}(n_1-p-1)} \prod_{i=1}^{k} (1 + a_i\ell_i)^{-\frac{n}{2}} \prod_{i=k+1}^{p} (1 + a\ell_i)^{-\frac{n}{2}}$$

$$\cdot \prod_{i<j}^{p} (\ell_i - \ell_j) \prod_{i<j}^{k} \left(\frac{2\pi}{nC_{ij}}\right)^{\frac{1}{2}} \prod_{i=1}^{k} \prod_{j=k+1}^{p} \left(\frac{2\pi}{nC_{ij}^{o}}\right)^{\frac{1}{2}} \prod_{i=1}^{p} d\ell_i ,$$

where

$$C_3 = \pi^{\frac{1}{2}q^2} \Gamma_p\left(\frac{n_1 + n_2}{2}\right)\left\{\Gamma_q\left(\frac{q}{2}\right)\Gamma_p(\tfrac{1}{2}n_1)\Gamma_p(\tfrac{1}{2}n_2)\right\}^{-1} ,$$

and C_{ij} and C^o_{ij} defined by (2.6).

The result (2.9) was given by Chang [7], but he had an error in the constant ; he had

$$\frac{\pi^{\frac{1}{4}p(p-1)-\frac{1}{2}kp}}{\left[\Gamma_k\left(\frac{p}{2}\right)\right]^{-1}} \prod_{i=1}^{p} \Gamma\left(\frac{i}{2}\right)\Gamma_p\left(\frac{n_1+n_2}{2}\right)\left\{\Gamma_p\left(\frac{p}{2}\right)\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\right\}^{-1} \prod_{i=1}^{p} a_i^{\frac{1}{2}n_1}$$

instead of $C_3\, a^{\frac{1}{2}qn_1} \prod_{i=1}^{k} a_i^{\frac{1}{2}n_1}$. He had also another error in the factors, he had $\prod_{i=k+1}^{p}(1+a_i\ell_i)^{-\frac{n_1+n_2}{2}} \prod_{i=1}^{k}(1+a_i\ell_i)^{\frac{n_1+n_2}{2}} \prod_{i=1}^{k}\prod_{j=k+1}^{p}\left\{\frac{2\pi}{(n_1+n_2)C_{ij}}\right\}^{\frac{1}{2}}$

instead of $\prod_{i=1}^{k}(1+a_i\ell_i)^{-\frac{n}{2}} \prod_{i=k+1}^{p}(1+a\ell_i)^{-\frac{n}{2}} \prod_{i=1}^{k}\prod_{j=k+1}^{p}\left(\frac{2\pi}{nC^o_{ij}}\right)^{\frac{1}{2}}$.

3. Special and Limiting Cases

For $k = 0$, $\prod_{i=1}^{k}(1+a_i\ell_i)^{-\frac{n}{2}}$, $\prod_{i=1}^{k}\prod_{j=k+1}^{p}\left(\frac{2\pi}{nC^o_{ij}}\right)^{\frac{1}{2}}$ products should be assumed to be unity. Similarly for $k=p$, $\prod_{i=k+1}^{p}(1 + a\ell_i)^{-\frac{n}{2}}$ etc. are unity, and define $\Gamma_o(x) = 1$, then $1 \leq k \leq p-1$ can be written $0 \le k \le p$.

(i) If $k = 0$, i.e. $q = p$, then $a_1 = \dots = a_p = a$. In other words, all ch. roots of population covariance matrix $\Sigma_1 \Sigma_2^{-1}$ are equal, i.e., $a_1^{-1} = \dots = a_p^{-1}$ and (2.9) reduces to

$$(3.1) \qquad \Gamma_p\left(\frac{n_1+n_2}{2}\right) \pi^{\frac{1}{2}p^2} \left\{\Gamma_p\left(\frac{n_1}{2}\right) \Gamma_p\left(\frac{n_2}{2}\right) \Gamma_p\left(\frac{p}{2}\right)\right\}^{-1} a^{\frac{1}{2}pn_1}$$

$$\cdot \prod_{i=1}^{p} \ell_i^{\frac{1}{2}(n_1-p-1)} \prod_{i=1}^{p} (1 + a\ell_i)^{-\frac{n}{2}} \prod_{i<j}^{p} (\ell_i - \ell_j) \prod_{i=1}^{p} d\ell_i \ .$$

(3.1) is the joint distribution of $\ell_1, \ell_2, \dots, \ell_p$ under null hypothesis $\Sigma_1 = a\Sigma_2$ [28], and is an exact form where we assume no asymptotic condition. Moreover, in this case, the integrand of (2.2) is independent of H .

(ii) If $k = p$, i.e. $q = 0$, then $0 < a_1 < a_2 < \dots < a_p$. In other words, all ch. roots of $\Sigma_1 \Sigma_2^{-1}$ are distinct, and (2.9) reduces to

$$(3.2) \qquad \Gamma_p\left(\frac{n_1+n_2}{2}\right) \left\{\Gamma_p\left(\frac{n_1}{2}\right) \Gamma_p\left(\frac{n_2}{2}\right)\right\}^{-1} \prod_{i=1}^{p} a_i^{\frac{1}{2}n_1} \ell_i^{\frac{1}{2}(n_1-p-1)} (1+a_i\ell_i)^{-\frac{n}{2}}$$

$$\cdot \prod_{i<j}^{p} (\ell_i - \ell_j) \prod_{i<j}^{p} \left(\frac{2\pi}{nC_{ij}}\right)^{\frac{1}{2}} \prod_{i=1}^{p} d\ell_i \ .$$

This is Chang's result under condition $0 < a_1 < a_2 < \dots < a_p$ (c.f. [6]).

Now let $\ell_i = n_1 v_i / n_2 \quad (i = 1, \ldots, p)$ and let n_2 tend to infinity, then (2.9), (3.1) and (3.2) reduce to the limiting forms

$$(3.3) \qquad C_4 a^{\frac{1}{2}qn_1} \prod_{i=1}^{k} a_i^{\frac{1}{2}n_1} \prod_{i=1}^{p} v_i^{\frac{1}{2}(n_1-p-1)} \exp(-\tfrac{1}{2}n_1 \sum_{i=1}^{k} a_i v_i) \exp(-\tfrac{1}{2}n_1 a \sum_{i=k+1}^{p} v_i)$$

$$\cdot \prod_{i<j}^{p} (v_i - v_j) \prod_{i<j}^{k} t_{ij}^{-\frac{1}{2}} \prod_{i=1}^{k} \prod_{j=k+1}^{p} t_{ij}^{o\ -\frac{1}{2}} \prod_{i=1}^{p} dv_i \quad ,$$

$$(3.4) \qquad C_5 a^{\frac{1}{2}pn_1} \prod_{i=1}^{p} v_i^{\frac{1}{2}(n_1-p-1)} \exp(-\tfrac{1}{2}n_1 a \sum_{i=1}^{p} v_i) \prod_{i<j}^{p} (v_i - v_j) \prod_{i=1}^{p} dv_i \quad ,$$

and

$$(3.5) \qquad C_6 \prod_{i=1}^{p} a_i^{\frac{1}{2}n_1} v_i^{\frac{1}{2}(n_1-p-1)} \exp(-\tfrac{1}{2}n_1 \sum_{i=1}^{p} a_i v_i) \prod_{i<j}^{p} \left(\frac{v_i - v_j}{a_j - a_i} \right)^{\frac{1}{2}} \prod_{i=1}^{p} dv_i$$

respectively, where

$$C_4 = \pi^{\frac{1}{4}k(k-1)+\frac{1}{2}pq} \left(\frac{n_1}{2} \right)^{\frac{1}{2}pn_1 - \frac{1}{4}k(k-1) - \frac{1}{2}kq} \left\{ \Gamma_q\left(\frac{q}{2}\right) \Gamma_p\left(\frac{n_1}{2}\right) \right\}^{-1} ,$$

$$C_5 = \left(\frac{n_1}{2} \right)^{\frac{1}{2}pn_1} \pi^{\frac{1}{2}p^2} \left\{ \Gamma_p\left(\frac{n_1}{2}\right) \Gamma_p\left(\frac{p}{2}\right) \right\}^{-1} ,$$

$$C_6 = \left(\frac{n_1}{2}\right)^{\frac{1}{2}pn_1 - \frac{1}{4}p(p-1)} \pi^{\frac{1}{4}p(p-1)} \left\{\Gamma_p\left(\frac{n_1}{2}\right)\right\}^{-1} ,$$

$$t_{ij} = (a_j - a_i)(v_i - v_j) \qquad i, j = 1, \ldots, k \text{ and } i < j ,$$

and

$$t^o_{ij} = (a - a_i)(v_i - v_j) \qquad i = 1, \ldots, k , \quad j = k+1, \ldots, p .$$

Note that (3.4) is the joint distribution of $\ell_1, \ell_2, \ldots, \ell_p$ under null hypothesis $\underset{\sim}{\Sigma} = a\underset{\sim}{I}$ [28] and (3.5) is the first approximation of (1.8) in [1]. This is when F to be taken as 1 . Furthermore, (3.3) can be rewritten as

$$(3.6) \quad C_4 \prod_{i=1}^{k} a_i^{\frac{1}{2}n_1} \exp(-\tfrac{1}{2}n_1 \sum_{i=1}^{k} a_i v_i) \prod_{i=1}^{k} v_i^{\frac{1}{2}(n_1-p-1)} \prod_{i<j}^{k} \left(\frac{v_i - v_j}{a_j - a_i}\right)^{\frac{1}{2}} \prod_{i=1}^{k} dv_i$$

$$\cdot \prod_{i=1}^{k} \prod_{j=k+1}^{p} \left(\frac{v_i - v_j}{a - a_i}\right)^{\frac{1}{2}} a^{\frac{1}{2}qn_1} \exp(-\tfrac{1}{2}n_1 a \sum_{i=k+1}^{p} v_i) \prod_{i=k+1}^{p} v_i^{\frac{1}{2}(n_1-p-1)}$$

$$\cdot \prod_{k+1 \le i < j}^{p} (v_i - v_j) \prod_{i=k+1}^{p} dv_i ,$$

which is exactly the same as (3.12) of James [17].

(3.6) can be written as $dF_1 \cdot dF_2$, where

$$dF_1 = dF_1(v_1, \ldots, v_k)$$

$$= \text{const.} \prod_{i=1}^{k} a_i^{\frac{1}{2}n_1} \prod_{i=1}^{k} \prod_{j=k+1}^{p} (a-a_i)^{-\frac{1}{2}} \exp\left(-\tfrac{1}{2}n_1 \sum_{i=1}^{k} a_i v_i\right)$$

$$\cdot \prod_{i=1}^{k} v_i^{\frac{1}{2}(n_1-p-1)} \prod_{i<j}^{k} \left(\frac{v_i-v_j}{a_j-a_i}\right)^{\frac{1}{2}} \prod_{i=1}^{k} dv_i \quad ,$$

and

$$(3.7) \qquad dF_2 = dF_2(v_{k+1}, \ldots, v_p \mid v_1, \ldots, v_k)$$

$$= \text{const.} \quad a^{\frac{1}{2}qn_1} \prod_{i=1}^{k} \prod_{j=k+1}^{p} (v_i-v_j)^{\frac{1}{2}} \exp\left(-\tfrac{1}{2}n_1 a \sum_{i=k+1}^{p} v_i\right)$$

$$\cdot \prod_{i=k+1}^{p} v_i^{\frac{1}{2}(n_1-p-1)} \prod_{k+1\leq i<j}^{p} (v_i-v_j) \prod_{i=k+1}^{p} dv_i \quad .$$

From dF_1 we know that the first k sample roots $v_1, \ldots, v_k$ are asymptotic sufficient for the population roots $a_1^{-1}, \ldots, a_k^{-1}$. dF_2 is the conditional distribution of the last roots, $v_{k+1}, \ldots, v_p$ given the first $v_1, \ldots, v_k$, which does not depend on the population parameters $a_1, \ldots, a_k$.

CHAPTER III

AN ASYMPTOTIC EXPANSION FOR THE DISTRIBUTION OF THE CHARACTERISTIC ROOTS OF A COVARIANCE MATRIX IN THE COMPLEX CASE

1. Introduction and Summary

Let $\underset{\sim}{\xi}(p \times 1)$ be distributed multivariate complex normal $N(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$ where $E[\underset{\sim}{\xi}] = \underset{\sim}{\mu}$ and $\underset{\sim}{\Sigma}$ is positive definite Hermitian. There is a unitary matrix $\underset{\sim}{U}_1$ such that

$$(1.1) \qquad \underset{\sim}{U}_1^* \underset{\sim}{\Sigma} \underset{\sim}{U}_1 = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_p \end{pmatrix} = \underset{\sim}{\Lambda} ,$$

where $\underset{\sim}{U}_1^*$ is the conjugate transpose of $\underset{\sim}{U}_1$ and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p > 0$ are characteristic (ch.) roots of $\underset{\sim}{\Sigma}$. Let $\underset{\sim}{\eta} = \underset{\sim}{U}_1^*(\underset{\sim}{\xi} - \underset{\sim}{\mu})$. Then $\underset{\sim}{\eta} \sim N(\underset{\sim}{0}, \underset{\sim}{\Lambda})$.

Clearly, $\underset{\sim}{\xi} = \underset{\sim}{\mu} + \eta_1 \underset{\sim}{u}_1 + \ldots + \eta_p \underset{\sim}{u}_p$, where $\underset{\sim}{u}_j$ is the jth column of $\underset{\sim}{U}_1$. If $j > r$, then λ_j are very small the corresponding $|\eta_j|$, the absolute value of η_j are nearly zero and with small error, we may write

$$\underset{\sim}{\xi} = \underset{\sim}{\mu} + \eta_1 \underset{\sim}{u}_1 + \ldots + \eta_r \underset{\sim}{u}_r .$$

Thus we are interested in those principal components η_j which have large variance.

Let $\underset{\sim}{Z}(p \times N)$ be the sample matrix of N observations from $N(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$, then the sample covariance matrix $\underset{\sim}{S}$ is given by

$$\underset{\sim}{S} = (s_{jk}) = n^{-1} \sum_{t=1}^{N} (z_{jt} - \nu_j)(\bar{z}_{kt} - \bar{\nu}_k)$$

where $n = N - 1$ and $\nu_j = \frac{1}{N} \sum_{t=1}^{N} z_{jt}$.

It is known that $\underset{\sim}{W} = n\underset{\sim}{S}$ has complex Wishart distribution on n degrees of freedom. Since $\underset{\sim}{S}$ is positive definite Hermitian, we can write

$$\underset{\sim}{S} = \underset{\sim}{U}\underset{\sim}{L}\underset{\sim}{U}^{*} ,$$

where $\underset{\sim}{U}$ is the group $U(p)$ of $p \times p$ unitary matrices with real diagonal elements, and

$$\underset{\sim}{L} = \begin{pmatrix} \ell_1 & & & 0 \\ & \ell_2 & & \\ & & \ddots & \\ 0 & & & \ell_p \end{pmatrix}$$

where $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_p > 0$ are ch. roots of $\underset{\sim}{S}$. Then from James [16] the distribution of the ch. roots $\ell_1, \ell_2, \ldots, \ell_p$ can be expressed in the form

$$(1.2) \qquad n^{(n+1)p} \left\{ \tilde{\Gamma}_p(n) \; |\underset{\sim}{\Sigma}|^n \right\}^{-1} |\underset{\sim}{L}|^{n-p} \prod_{j<k}^{p} (\ell_j - \ell_k)^2 \prod_{j=1}^{p} d\ell_j$$

$$\cdot \int_{U(p)} \exp [-n \operatorname{tr} \underset{\sim}{\Sigma}^{-1} \underset{\sim}{U}\underset{\sim}{L}\underset{\sim}{U}^*] \, (\underset{\sim}{U}^* d\underset{\sim}{U})$$

where $(\underset{\sim}{U}^* d\underset{\sim}{U})$ is the invariant measure on the group $U(p)$. The group $U(p)$ has volume

$$V(p) = \int_{U(p)} (\underset{\sim}{U}^* d\underset{\sim}{U}) = \frac{\pi^{p(p-1)}}{\tilde{\Gamma}_p(p)}$$

where $\tilde{\Gamma}_p(p)$ as defined in [16], i.e.

$$\tilde{\Gamma}_p(p) = \pi^{\frac{1}{2}p(p-1)} \prod_{j=1}^{p} \Gamma(p - j + 1) \quad .$$

Replace $\underset{\sim}{U}$ by $\underset{\sim}{U}_1\underset{\sim}{U}$ with $\underset{\sim}{U}_1$ defined in (1.1) then the distribution (1.2) depends on the integral

$$(1.3) \qquad \mathscr{I}_1 = \int_{U(p)} \exp [-n \operatorname{tr} \underset{\sim}{A}\underset{\sim}{U}\underset{\sim}{L}\underset{\sim}{U}^*] \, (\underset{\sim}{U}^* d\underset{\sim}{U})$$

where $\underset{\sim}{A} = \underset{\sim}{\Lambda}^{-1}$, so that $a_j = \lambda_j^{-1}$ and $0 < a_1 \leq a_2 \leq \cdots \leq a_p$.

Since $\mathscr{I}_1$ can be written

$$(1.4) \qquad \mathscr{I}_1 = V(p) \sum_{k=0}^{\infty} \sum_{\kappa} \tilde{C}_\kappa(-\underset{\sim}{A}) \, \tilde{C}_\kappa(n\underset{\sim}{L})[k! \; \tilde{C}_\kappa(\underset{\sim}{I}_p)]^{-1}$$

where $\tilde{C}_\kappa(\underset{\sim}{B})$ is the zonal polynomial of a Hermitian matrix $\underset{\sim}{B}$ as defined in [16]. The use of (1.4) in (1.2) gives a power series expansion, but the convergence of this series is very slow, unless the ch. roots of the argument matrices are small. For the real case Anderson [1] has obtained an asymptotic expansion for the integral and his expansion is given in increasing powers of n^{-1}, where n is the sample size less one. In his paper, for $p = 2$, he defines

$$O^{\pm}(2) = \{\underset{\sim}{H} \in O(2) \mid |\underset{\sim}{H}| = \pm 1\}$$

where $O(2)$ is the group of 2×2 orthogonal matrices, then

$$\int_{O(2)} \exp\left[-\frac{n}{2} \operatorname{tr} \underset{\sim}{AHLH'}\right] (\underset{\sim}{H}'d\underset{\sim}{H}) = 2\int_{O^+(2)} \exp\left[-\frac{n}{2} \operatorname{tr} \underset{\sim}{AHLH'}\right] (\underset{\sim}{H}'d\underset{\sim}{H})$$

Unfortunately, for the unitary group we do not have the similar property. In order to overcome this difficulty, we need to impose conditions on $U(p)$, the number of conditions imposed is equal to p, the order of $\underset{\sim}{U} \in U(p)$ (for the reason see Section 2).

2. The Asymptotic Expansion of $\mathcal{I}_1$

Since the procedure used to find the asymptotic expansion for $\mathcal{I}_1$ requires that $\lambda_1 > \lambda_2 > \ldots > \lambda_p$ and $\ell_1 > \ell_2 > \ldots > \ell_p$, hence in this chapter, we consider only the case of distinct ch. roots of the population covariance matrix.

Lemma 2.1. Let $\underset{\sim}{A}$, $\underset{\sim}{U} = (u_{jk})$ and $\underset{\sim}{L}$ be defined as before. Then $f(\underset{\sim}{U}) = \exp(-n \operatorname{tr} \underset{\sim}{AULU}^*) = \exp(-n \sum_{j=1}^{p} \sum_{k=1}^{p} a_j \ell_k u_{jk} \bar{u}_{jk})$ has identical maximum values of $\exp(-n \operatorname{tr} \underset{\sim}{AL})$ at each of the matrices of the form

$$(2.1) \qquad \begin{pmatrix} e^{i\varphi_1} & & & 0 \\ & e^{i\varphi_2} & & \\ & & \ddots & \\ 0 & & & e^{i\varphi_p} \end{pmatrix} ,$$

where $\bar{u}_{jk}$ is the conjugate of u_{jk} and $0 \leq \varphi_j < 2\pi$ $(j = 1, \ldots, p)$.

Proof: Since $\underset{\sim}{U}^* \underset{\sim}{U} = \underset{\sim}{I}_p$ hence

$$d\underset{\sim}{U}^* = - \underset{\sim}{U}^* \cdot d\underset{\sim}{U} \cdot \underset{\sim}{U}^* ,$$

$$df = -n \exp(-n \operatorname{tr} \underset{\sim}{AULU}^*) \operatorname{tr} (\underset{\sim}{A} \cdot d\underset{\sim}{U} \cdot \underset{\sim}{LU}^* + \underset{\sim}{AUL}\, d\underset{\sim}{U}^*)$$

$$= -n \exp(-n \operatorname{tr} \underset{\sim}{AULU}^*) \operatorname{tr} (\underset{\sim}{LU}^* \underset{\sim}{A} - \underset{\sim}{U}^* \underset{\sim}{AULU}^*)\, d\underset{\sim}{U}$$

for every $d\underset{\sim}{U}$. Therefore $df = 0$ implies $\underset{\sim}{LU}^* \underset{\sim}{A} = \underset{\sim}{U}^* \underset{\sim}{AULU}^*$ i.e. $\underset{\sim}{LU}^* \underset{\sim}{AU} = \underset{\sim}{U}^* \underset{\sim}{AUL}$ which means $\underset{\sim}{L}$ and $\underset{\sim}{U}^* \underset{\sim}{AU}$ commute. But $\underset{\sim}{L}$ is a diagonal matrix with real distinct elements, implies $\underset{\sim}{U}^* \underset{\sim}{AU}$ is a diagonal matrix. This can happen if and only if $\underset{\sim}{U}$ is of the form with $e^{i\varphi_j}$ in one position in the jth row and certain column and zero in other positions. After substituting those stationary values into $f(\underset{\sim}{U})$ we obtain a general form

(2.2) $$\exp\left(-n \sum_{j=1}^{p} a_j \ell_{\tau_j} \right)$$

where ℓ_{τ_j} is any permutation of ℓ_j $(j = 1, \ldots, p)$ or $f(\underset{\sim}{U})$ attains its identical maximum value $\exp(-n \operatorname{tr} \underset{\sim}{A}\underset{\sim}{L})$ when $\underset{\sim}{U}$ is of the form (2.1).

The matrices of the form (2.1) are unitary and with ch. roots $e^{i\varphi_j}$ $(j = 1, \ldots, p)$. Now we impose p conditions on $\underset{\sim}{U}$ (reason see later), namely all of the ch. roots are positive real. Then (2.1) reduces to $\underset{\sim}{I}_p$.

Under these restrictions, for large n , the integrand is negligible except for small neighborhood about identity matrix, so that

(2.3) $$\mathcal{J}_1 = \int_{N(\underset{\sim}{I})} \exp\left[-n \operatorname{tr} \underset{\sim}{A}\underset{\sim}{U}\underset{\sim}{L}\underset{\sim}{U}^* \right] (\underset{\sim}{U}^* d\underset{\sim}{U})$$

where $N(\underset{\sim}{I})$ is a neighborhood of the identity matrix on the unitary manifold.

Lemma 2.2. Let $\underset{\sim}{U}(p \times p)$ be a unitary matrix, and make the transformation

(2.4) $$\underset{\sim}{U} = e^{i\underset{\sim}{H}}$$

where $\underset{\sim}{H}$ is Hermitian matrix. Then the Jacobian of this transformation is

(2.5) $$J = 1 - \frac{p}{12}\,\mathrm{tr}\,\underset{\sim}{H}^2 + \frac{1}{12}(\mathrm{tr}\,\underset{\sim}{H})^2 + \frac{1}{2(6!)}\{5(\mathrm{tr}\,\underset{\sim}{H})^4 - p\,\mathrm{tr}\,\underset{\sim}{H}^4$$

$$- 11\,\mathrm{tr}\,\underset{\sim}{H}^3\mathrm{tr}\,\underset{\sim}{H} - 10p\,\mathrm{tr}\,\underset{\sim}{H}^2(\mathrm{tr}\,\underset{\sim}{H})^2 + (5p^2 - 3)(\mathrm{tr}\,\underset{\sim}{H}^2)^2\} + \ldots$$

Proof: Let

$$\underset{\sim}{\Theta} = \begin{pmatrix} \theta_1 & & & 0 \\ & \theta_2 & & \\ & & \ddots & \\ 0 & & & \theta_p \end{pmatrix}$$

where θ_j $(j = 1, \ldots, p)$ are distinct real numbers. Since $\underset{\sim}{U}$ is unitary, there exists a unitary matrix $\underset{\sim}{U}_2$ with real diagonal elements, such that

$$\underset{\sim}{U} = e^{i\underset{\sim}{U}_2^*\underset{\sim}{\Theta}\underset{\sim}{U}_2}$$

Put $\underset{\sim}{H} = (h_{jk}) = \underset{\sim}{U}_2^*\underset{\sim}{\Theta}\underset{\sim}{U}_2$, then from Murnaghan [25], we have

(2.6) $$(\underset{\sim}{U}^*d\underset{\sim}{U}) = \prod_{j<k}^{p} 4\sin^2 \tfrac{1}{2}(\theta_j - \theta_k) \prod_{j=1}^{p} d\theta_j(\underset{\sim}{U}_2^*d\underset{\sim}{U}_2)$$

Since $\underset{\sim}{H}$ is Hermitian, from Khatri [18], we have

(2.7) $$\prod_{j=1}^{p} dh_{jj} \prod_{j<k}^{p} dh_{jkR}dh_{jkI} = \prod_{j<k}^{p}(\theta_j - \theta_k)^2 \prod_{j=1}^{p} d\theta_j(\underset{\sim}{U}_2^*d\underset{\sim}{U}_2)$$

where h_{jj} $(j = 1, \ldots, p)$ are real diagonal elements of $\underset{\sim}{H}$ and h_{jkR} and h_{jkI} are real and imaginary parts of h_{jk} . Note that

$$\text{tr } \underset{\sim}{H}^m = \sum_{j=1}^{p} \theta_j^m .$$

Then using (2.6) and (2.7) we obtain (2.5).

Substitution of (2.4) into tr $\underset{\sim}{AULU}^*$ yields

(2.8) $$\text{tr } \underset{\sim}{AULU}^* = \text{tr } \underset{\sim}{AL} + \text{tr } (\underset{\sim}{AHLH} - \underset{\sim}{ALH}^2) + \text{tr } (\mathcal{I}m \ \underset{\sim}{AHLH}^2)$$
$$+ \text{tr } (\tfrac{1}{12} \underset{\sim}{ALH}^4 - \tfrac{1}{3} \text{Re } \underset{\sim}{AH}^3\underset{\sim}{LH} + \tfrac{1}{4} \underset{\sim}{AH}^2\underset{\sim}{LH}^2) + \ldots .$$

This is rewritten using brackets to define the expressions in parentheses so that

$$\text{tr } \underset{\sim}{AULU}^* = \text{tr } \underset{\sim}{AL} + \text{tr } \{\underset{\sim}{H}^2\} + \text{tr } \{\underset{\sim}{H}^3\} + \text{tr } \{\underset{\sim}{H}^4\} + \ldots$$

where Re $\underset{\sim}{B}$ and $\mathcal{I}m$ $\underset{\sim}{B}$ denote real and imaginary parts of $\underset{\sim}{B}$. Since

(2.9) $$\text{tr } \{\underset{\sim}{H}^2\} = \sum_{j<k}^{p} \tilde{C}_{jk} h_{jk} \bar{h}_{jk}$$

where $\tilde{C}_{jk} = (a_k - a_j)(\ell_j - \ell_k) > 0$, for $j,k = 1, \ldots, p$ and $j < k$.

Under transformation (2.4), it has $N(\underset{\sim}{I}) \rightarrow N(\underset{\sim}{H} = \underset{\sim}{0})$. Then (2.3) can be written

(2.10) $$\mathcal{J}_1 = \exp(-n \text{ tr } \underset{\sim}{AL}) \int_{N(\underset{\sim}{H}=\underset{\sim}{0})} \exp(-n \sum_{j<k}^{p} \tilde{C}_{jk} h_{jk} \bar{h}_{jk})$$
$$\cdot \exp[-n \text{ tr } \{\underset{\sim}{H}^3\} - n \text{ tr } \{\underset{\sim}{H}^4\} + \ldots] J \prod_{j<k}^{p} dh_{jj} \prod_{j<k}^{p} dh_{jkR} dh_{jkI} .$$

Since h_{jj} $(j = 1, \ldots, p)$ are real, each one may range in a certain interval, and since they do not occur in the right hand side of (2.9) and may lead to the divergence of the integral [22]. So we need to impose conditions on $\underset{\sim}{H}$. We may put h_{jj} $(j = 1, \ldots, p)$ to be constants, but the result is quite complicated (see Remark). For simplicity, we set $h_{jj} = 0$ $(j = 1, \ldots, p)$. In view of (2.4), this is equivalent to imposing p conditions on $\underset{\sim}{U}$. Thus each side of (2.4) contains $p^2 - p$ parameters. Under these conditions, (2.5) and (2.10) reduce to

$$(2.11) \qquad J = 1 - \frac{p}{12} \operatorname{tr} \underset{\sim}{H}^2 + \frac{1}{2(6!)} \left[(5p^2 - 3)(\operatorname{tr} \underset{\sim}{H}^2)^2 - p \operatorname{tr} \underset{\sim}{H}^4 \right] + \ldots$$

and

$$(2.12) \qquad \mathcal{J}_1 = \exp(-n \operatorname{tr} \underset{\approx}{A}\underset{\approx}{L}) \int_{N(\underset{\sim}{H}=\underset{\sim}{0})} \exp\Big(-n \sum_{j<k}^{p} \tilde{C}_{jk} h_{jk} \bar{h}_{jk}\Big)$$

$$\cdot \exp[-n \operatorname{tr} \{\underset{\sim}{H}^3\} - n \operatorname{tr} \{\underset{\sim}{H}^4\} - \ldots] \, J \prod_{j<k}^{p} dh_{jkR} dh_{jkI}$$

respectively.

Expand $\exp[-n \operatorname{tr} \{\underset{\sim}{H}^3\} - n \operatorname{tr} \{\underset{\sim}{H}^4\} - \ldots]J$ and write the terms in groups. We have

$$(2.13) \qquad \exp[-n \operatorname{tr} \{\underset{\sim}{H}^3\} - n \operatorname{tr} \{\underset{\sim}{H}^4\} - \ldots]J =$$

$$1 - \frac{p}{12} \operatorname{tr} \underset{\sim}{H}^2 - n \operatorname{tr} \{\underset{\sim}{H}^3\} - n \operatorname{tr} \{\underset{\sim}{H}^4\} + \frac{n^2}{2} (\operatorname{tr} \{\underset{\sim}{H}^3\})^2$$

$$+ \frac{1}{2(6!)} \left[(5p^2 - 3)(\operatorname{tr} \underset{\sim}{H}^2)^2 - p \operatorname{tr} \underset{\sim}{H}^4 \right] + \ldots \quad .$$

If the integration of (2.12) is to be performed term by term on the expansion of (2.13) then for large n, the limits for each h_{jkR} and h_{jkI} can be put to $\pm\infty$, since each integration is of the form

$$\int_{N(\underset{\sim}{H}=\underset{\sim}{0})} \exp\left(-n \sum_{j<k}^{p} \tilde{C}_{jk} h_{jk} \bar{h}_{jk}\right) \prod_{j<k}^{p} h_{jkc}^{m_{jk}} \prod_{j<k}^{p} dh_{jkR} dh_{jkI}$$

where h_{jkc} denotes h_{jkR} or h_{jkI}, and most of this integral is concentrated in a small neighborhood of $\underset{\sim}{H} = \underset{\sim}{0}$. The m_{jk}'s are positive even integers or zero, since any term containing an odd power of an h_{jkc} will integrate to zero. Since

$$(2.14) \quad \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left(-n \sum_{j<k}^{p} \tilde{C}_{jk} h_{jk} \bar{h}_{jk}\right) \prod_{j<k}^{p} dh_{jkR} dh_{jkI}$$

$$= \prod_{j<k}^{p} \frac{\pi}{n\tilde{C}_{jk}} = \left(\frac{\pi}{n}\right)^{\frac{1}{2}p(p-1)} \prod_{j<k}^{p} \tilde{C}_{jk}^{-1} = C ,$$

$$(2.15) \quad \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left(-n \sum_{j<k}^{p} \tilde{C}_{jk} h_{jk} \bar{h}_{jk}\right) h_{stc}^{2m} \prod_{j<k}^{p} dh_{jkR} dh_{jkI}$$

$$= C \cdot 1 \cdot 3 \cdot 5 \cdots (2m-1)(2n\tilde{C}_{st})^{-m}$$

and

$$(2.16) \quad \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left(-n \sum_{j<k}^{p} \tilde{C}_{jk} h_{jk} \bar{h}_{jk}\right) (h_{st}\bar{h}_{st})^{m} \prod_{j<k}^{p} dh_{jkR} dh_{jkI}$$

$$= \frac{C(m!)}{(n\tilde{C}_{st})^{m}} .$$

<u>Theorem 2.1</u>. Let $\underset{\sim}{A}$ and $\underset{\sim}{L}$ be diagonal matrices with $0 < a_1 < a_2 < \ldots < a_p$ and $\ell_1 > \ell_2 > \ldots > \ell_p > 0$. Then for large n the first two terms in the expansion for $\mathcal{J}_1$ are given by

$$(2.17) \qquad \mathcal{J}_1 = \exp\,(-n \operatorname{tr} \underset{\sim\sim}{AL}) \prod_{j<k}^{p} \frac{\pi}{n\tilde{C}_{jk}} \left\{ 1 + \frac{1}{3n} \sum_{j<k}^{p} \tilde{C}_{jk}^{-1} + \ldots \right\} .$$

<u>Proof</u>: For simplicity, we include only terms without an odd power of an h_{jkc} and do not write C which appears with each term after integration, and denote

$$S' = \sum_{j<k}^{p} \tilde{C}_{jk}^{-1}$$

and
$$S'' = \sum_{j<k<s}^{p} (\tilde{C}_{ks}/\tilde{C}_{jk}\tilde{C}_{js} + \tilde{C}_{js}/\tilde{C}_{jk}\tilde{C}_{ks} + \tilde{C}_{jk}/\tilde{C}_{js}\tilde{C}_{ks}) .$$

Since $\operatorname{tr} \underset{\sim}{H}^2 = 2 \sum_{j<k}^{p} h_{jk}\bar{h}_{jk}$, it is easy to see that $-p \operatorname{tr} \underset{\sim}{H}^2/12$ gives

$$(2.18) \qquad - \frac{p}{6n} S' .$$

From (2.8)

$$12 \operatorname{tr} \{\underset{\sim}{H}^4\} = \sum_{j,k,s,t}^{p} f(j,k,s) \operatorname{Re} h_{jk}h_{ks}h_{st}h_{tj} ,$$

where
$$f(j,k,s) = a_j(\ell_j - 4\ell_k + 3\ell_s) .$$

In detail we have

$$12 \text{ tr } \{\underset{\sim}{H}^4\} = \sum_{j,k,s,s\neq j} [f(j,k,s) + f(k,j,k)]\, h_{jk}\bar{h}_{jk}h_{ks}\bar{h}_{ks} + \sum_{j<k} f(j,k,j)(h_{jk}\bar{h}_{jk})^2$$

$$= \sum_{j<k<s} \Big\{[g(j,k,s) + g(s,k,j)]\, h_{jk}\bar{h}_{jk}h_{ks}\bar{h}_{ks} + [g(s,j,k) + g(k,j,s)]\, h_{jk}\bar{h}_{jk}h_{js}\bar{h}_{js} + [g(k,s,j) + g(j,s,k)]\, h_{js}\bar{h}_{js}h_{ks}\bar{h}_{ks}\Big\}$$

$$+ \sum_{j<k} (-4\tilde{C}_{jk})(h_{jk}\bar{h}_{jk})^2$$

where $g(j,k,s) = f(j,k,s) + f(k,j,k)$. But $g(j,k,s) + g(s,k,j) = -4\tilde{C}_{jk} - 4\tilde{C}_{ks} + 3\tilde{C}_{js}$ so that after term by term integration, $12 \text{ tr } \{\underset{\sim}{H}^4\}$ contributes

$$(-8/n^2) \sum_{j<k<s} (\tilde{C}^{-1}_{jk} + \tilde{C}^{-1}_{js} + \tilde{C}^{-1}_{ks}) + (3/n^2)S'' - (8/n^2)S' .$$

Since

$$\sum_{s<j<k} \tilde{C}^{-1}_{jk} + \sum_{j<s<k} \tilde{C}^{-1}_{jk} + \sum_{j<k<s} \tilde{C}^{-1}_{jk} = (p-2)S' .$$

Therefore $-n \text{ tr } \{\underset{\sim}{H}^4\}$ contributes

$$(2.19) \qquad [2(p-2)/3n]S' - (1/4n)S'' + (2/3n)S' .$$

Again from (2.8)

$$\text{tr}\ \{\underset{\sim}{H}^3\} = \text{tr}\ \mathcal{I}m\ \underset{\sim}{AHLH}^2$$

$$= \sum_{j<k<s} -\frac{i}{2}\,\psi(j,k,s)(h_{jk}h_{ks}h_{sj} - \overline{h_{jk}h_{ks}h_{sj}})$$

where $\psi(j,k,s) = a_j(\ell_k - \ell_s) + a_k(\ell_s - \ell_j) + a_s(\ell_j - \ell_k)$.

It is easy to check that

$$\psi^2(j,k,s) = \tilde{C}^2_{jk} + \tilde{C}^2_{js} + \tilde{C}^2_{ks} - 2(\tilde{C}_{jk}\tilde{C}_{js} + \tilde{C}_{jk}\tilde{C}_{ks} + \tilde{C}_{js}\tilde{C}_{ks})$$

so that after integration, $(\text{tr}\ \{\underset{\sim}{H}^3\})^2$ contributes

$$(1/2n^3)\sum_{j<k<s}(\tilde{C}_{jk}/\tilde{C}_{js}\tilde{C}_{ks} + \tilde{C}_{js}/\tilde{C}_{jk}\tilde{C}_{ks} + \tilde{C}_{ks}/\tilde{C}_{jk}\tilde{C}_{js})$$

$$- (1/n^3)\sum_{j<k<s}(\tilde{C}^{-1}_{jk} + \tilde{C}^{-1}_{js} + \tilde{C}^{-1}_{ks}) \quad ,$$

i.e., $(1/2n^3)S'' - [(p-2)/n^3]S'$, hence $(n^2/2)(\text{tr}\ \{\underset{\sim}{H}^3\})^2$ gives

(2.20) $$(1/4n)S'' - [(p-2)/2n]S' \quad .$$

Add (2.18) - (2.20), we obtain (2.17).

By Theorem 2.1, we have the following theorem:

<u>Theorem 2.2</u>. The asymptotic distribution of the ch. roots, $\ell_1 > \ell_2 > \ldots > \ell_p > 0$ of $\underset{\sim}{S}$ for large degrees of freedom n, when the ch. roots of $\underset{\sim}{\Sigma}$ are $\lambda_1 > \lambda_2 > \ldots > \lambda_p > 0$ and $a_j = \lambda_j^{-1}$ $(j = 1, \ldots, p)$ is given by

$$(2.21) \qquad K \exp(-n \sum_{j=1}^{p} a_j \ell_j) \prod_{j<k}^{p} \tilde{C}_{jk}^{-1} (\ell_j - \ell_k)^2 \prod_{j=1}^{p} a_j^n \ell_j^{n-p} d\ell_j \{1 + \frac{1}{3n} \sum_{j<k}^{p} \tilde{C}_{jk}^{-1} + \ldots\} ,$$

where

$$K = n^{\frac{1}{2}p(2n-p+3)} \pi^{\frac{1}{2}p(p-1)} \{ \tilde{\Gamma}_p(n) \}^{-1} .$$

3. The Limiting Case

Since $\tilde{C}_{jk} = (a_k - a_j)(\ell_j - \ell_k)$ $j,k = 1, \ldots, p$ and $a_j = \lambda_j^{-1}$ $(j = 1, \ldots, p)$. Hence (2.20) can be rewritten

$$G(\underset{\sim}{\Sigma}) \prod_{j<k}^{p} (\ell_j - \ell_k)/(\lambda_j - \lambda_k) \prod_{j=1}^{p} \ell_j^{n-p} e^{-n\ell_j/\lambda_j} d\ell_j$$

where $G(\underset{\sim}{\Sigma})$ is a function of the ch. roots of $\underset{\sim}{\Sigma}$. It depends on λ_j but not on ℓ_j . For n large enough, by a method used analogous to Anderson [1], we can show

$$\prod_{j<k}^{p} (\ell_j - \ell_k)/(\lambda_j - \lambda_k)$$

to tend to unity with probability 1 , and the chi-square distributions tend to normals which corresponds to the real case for the asymptotic normality proved by Girshick [12].

<u>Remark</u>: for $p = 2$, set $h_{11} = \alpha$, $h_{22} = \beta$ where α and β are constants, then we have

$$\mathcal{J}_1 = \exp(-n \operatorname{tr} \underset{\sim}{A}\underset{\sim}{L}) \frac{\pi}{n\widetilde{C}_{12}} \left\{ F(\alpha,\beta) + G(\alpha,\beta)\frac{1}{n\widetilde{C}_{12}} + B(\alpha,\beta)\frac{1}{n^2\widetilde{C}^2_{12}} + \cdots \right\} ,$$

where

$$F(\alpha,\beta) = \left\{1 + \frac{(\alpha-\beta)^2}{12} + \frac{(\alpha-\beta)^4}{240}\right\} f ,$$

$$G(\alpha,\beta) = g + \frac{2}{3}f + \frac{11}{90}(\alpha-\beta)^2 f + \frac{1}{6}(\alpha-\beta)^2 g + \frac{11}{720}(\alpha-\beta)^4 g ,$$

$$B(\alpha,\beta) = \frac{16}{15}f + 2g + 2b + \frac{8}{15}(\alpha-\beta)^2 g + \frac{1}{2}(\alpha-\beta)^2 b + \frac{1}{12}(\alpha-\beta)^4 b ,$$

$$f = f(\alpha,\beta) = 1 - \frac{1}{12}(\alpha-\beta)^2 - \frac{11}{2(6!)}\left\{\alpha^4 + \alpha^3\beta - \frac{24}{11}\alpha^2\beta^2 + \alpha\beta^3 + \beta^4\right\} + \cdots$$

$$g = g(\alpha,\beta) = -\frac{1}{3} - \frac{1}{2(6!)}\left\{13\alpha^2 + 154\alpha\beta + 13\beta^2\right\} + \cdots$$

and

$$b = b(\alpha,\beta) = \frac{2}{45} + \cdots$$

If $\alpha = \beta$, then $\mathcal{J}_1$ reduces to

$$\mathcal{J}_1 = \exp(-n \operatorname{tr} \underset{\sim}{A}\underset{\sim}{L}) \frac{\pi}{n\widetilde{C}_{12}} \left\{ 1 - \frac{1}{72}\alpha^4 + \left(\frac{1}{3} - \frac{\alpha^2}{8} - \frac{\alpha^4}{108}\right)\frac{1}{n\widetilde{C}_{12}} + \left(\frac{22}{45} - \frac{\alpha^2}{4} - \frac{\alpha^4}{135}\right)\frac{1}{n^2\widetilde{C}^2_{12}} + \cdots \right\}$$

If $\alpha = \beta = 0$, then $\mathcal{J}_1$ becomes

$$\mathcal{J}_1 = \exp\,(-n\ \mathrm{tr}\ \underset{\sim}{AL})\ \frac{\pi}{n\tilde{C}_{12}} \left\{1 + \frac{1}{3n\tilde{C}_{12}} + \frac{22}{45n^2\tilde{C}_{12}^2} + \cdots \right\}$$

or approximately (see Erdélyi [11]) write

$$\mathcal{J}_1 \sim \exp\,[-n(a_1\ell_1 + a_2\ell_2)]\ \frac{\pi}{n\tilde{C}_{12}} \left\{ 1 + \frac{1}{3n\tilde{C}_{12}} + o\left(\frac{1}{n^2}\right)\right\} .$$

CHAPTER IV

AN ASYMPTOTIC EXPANSION FOR THE DISTRIBUTION OF THE CHARACTERISTIC ROOTS OF $S_1S_2^{-1}$ IN THE COMPLEX CASE

1. Introduction and Summary

Let $S_j(p \times p)$ $(j = 1, 2)$ be independently distributed as complex Wishart (n_j, p, Σ_j), and let $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_p > 0$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p > 0$ be the ch. roots of $S_1S_2^{-1}$ and $\Sigma_1\Sigma_2^{-1}$ respectively. To save notations, let $L = \text{diag}(\ell_1, \ell_2, \ldots, \ell_p)$, $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p)$ $A = \Lambda^{-1}$ so that $a_j = \lambda_j^{-1}$ $(j = 1, \ldots, p)$ $0 < a_1 \leq a_2 \leq \cdots \leq a_p$. Furthermore, let U^* be the conjugate transpose of U, $n = n_1 + n_2$ and I denote I_p throughout this chapter, unless otherwise stated. Then the distribution of $\ell_1, \ell_2, \ldots, \ell_p$ can be expressed in the form [16],

$$(1.1) \qquad C_1|A|^{n_1}|L|^{n_1-p} \prod_{j<k}^{p} (\ell_j - \ell_k)^2 \int_{U(p)} |I + AULU^*|^{-n}(U^*dU)$$

where

$$(1.2) \qquad C_1 = \frac{\tilde{\Gamma}_p(n_1 + n_2)}{\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2)} ,$$

$U(p)$ is the group of all $p \times p$ unitary matrices and $(\underset{\sim}{U}^* d\underset{\sim}{U})$ is the invariant measure on the unitary group $U(p)$.

However, this form is not convenient for further development. Since

$$(1.3) \qquad \mathscr{I}_2 = \int_{U(p)} |\underset{\sim}{I} + \underset{\sim}{AULU}^*|^{-n} (\underset{\sim}{U}^* d\underset{\sim}{U})$$

$$= C_2 \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[n]_\kappa \widetilde{C}_\kappa(-\underset{\sim}{A}) \widetilde{C}_\kappa(\underset{\sim}{L})}{k! \; \widetilde{C}_\kappa(\underset{\sim}{I})} ,$$

where $C_2 = \pi^{p(p-1)} \{\widetilde{\Gamma}_p(p)\}^{-1}$,

and $[b]_\kappa$ and the zonal polynomial of a Hermitian matrix $\underset{\sim}{B}$, $\widetilde{C}_\kappa(\underset{\sim}{B})$ are defined in James [16]. The use of (1.3) in (1.1) gives a power series expansion, but the convergence of this series is very slow, unless the ch. roots of the argument matrices are small. In the one sample case, we have obtained a gamma type asymptotic expansion for the distribution of the ch. roots of the sample covariance matrix. In this chapter, we obtain a beta type asymptotic expansion of the roots distribution of $\underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$ involving linkage factors between sample roots and corresponding population roots. If the roots are distinct, the limiting distribution as n_2 tends to infinity has the same form as that of (2.17) in Chapter III. If, moreover, n_1 is assumed also large, then it corresponds to Girshick's result [12] in the real case.

2. The Asymptotic Expansion of $\mathcal{J}_2$

Same as in Chapter III, we here still require that $\lambda_1 > \lambda_2 > \dots > \lambda_p > 0$ and $\ell_1 > \ell_2 > \dots > \ell_p > 0$. It is easy to see that $|I + AULU^*|$ is positive real for all L and every $U \in U(p)$.

Lemma 2.1. Let A and L be defined as before, then $f(U) = |I + AULU^*|$, $U \in U(p)$, attains its identical minimum value $|I + AL|$ when U is of the form

$$U = \begin{pmatrix} e^{i\varphi_1} & & & 0 \\ & e^{i\varphi_2} & & \\ & & \ddots & \\ 0 & & & e^{i\varphi_p} \end{pmatrix} \tag{2.1}$$

where $0 \le \varphi_j < 2\pi \quad j = 1, \dots, p$.

Proof: Since A is positive definite

$$|I + AULU^*| = |I + A^{\frac{1}{2}}ULU^*A^{\frac{1}{2}}|$$

$$\begin{aligned} df(U) &= d|I + A^{\frac{1}{2}}ULU^*A^{\frac{1}{2}}| \\ &= |I + A^{\frac{1}{2}}ULU^*A^{\frac{1}{2}}| \; \mathrm{tr}\; (I + A^{\frac{1}{2}}ULU^*A^{\frac{1}{2}})^{-1}(A^{\frac{1}{2}}dU\cdot LU^*A^{\frac{1}{2}} + A^{\frac{1}{2}}ULdU^*\cdot A^{\frac{1}{2}}) \\ &= |I + AULU^*| \; \mathrm{tr}\; (A^{-1} + ULU^*)^{-1}(dU\cdot LU^* - ULU^*dU\cdot U^*) \\ &= |I + AULU^*| \; \mathrm{tr}\; (LU^*(A^{-1} + ULU^*)^{-1} - U^*(A^{-1} + ULU^*)^{-1}ULU^*)dU \;. \end{aligned}$$

for every dU. Therefore $df(U) = 0$ implies $\mathrm{tr}\;(LU^*(A^{-1} + ULU^*)^{-1} - U^*(A^{-1} + ULU^*)^{-1}ULU^*) = 0$, for every L and U, implies

$\underset{\sim}{L}\underset{\sim}{U}^*(\underset{\sim}{A}^{-1}+\underset{\sim}{U}\underset{\sim}{L}\underset{\sim}{U}^*)^{-1} = \underset{\sim}{U}^*(\underset{\sim}{A}^{-1}+\underset{\sim}{U}\underset{\sim}{L}\underset{\sim}{U}^*)^{-1}\underset{\sim}{U}\underset{\sim}{L}\underset{\sim}{U}^*$,

i.e. $\underset{\sim}{L}\underset{\sim}{U}^*(\underset{\sim}{A}^{-1}+\underset{\sim}{U}\underset{\sim}{L}\underset{\sim}{U}^*)^{-1}\underset{\sim}{U} = \underset{\sim}{U}^*(\underset{\sim}{A}^{-1}+\underset{\sim}{U}\underset{\sim}{L}\underset{\sim}{U}^*)^{-1}\underset{\sim}{U}\underset{\sim}{L}$ which means $\underset{\sim}{L}$ and $\underset{\sim}{U}^*(\underset{\sim}{A}^{-1}+\underset{\sim}{U}\underset{\sim}{L}\underset{\sim}{U}^*)^{-1}\underset{\sim}{U}$ commute. But $\underset{\sim}{L}$ is a diagonal matrix with positive distinct elements. This implies that $\underset{\sim}{U}^*(\underset{\sim}{A}^{-1}+\underset{\sim}{U}\underset{\sim}{L}\underset{\sim}{U}^*)^{-1}\underset{\sim}{U}$ is a diagonal matrix, say $\underset{\sim}{\Delta}$. Thus $\underset{\sim}{A}^{-1} = \underset{\sim}{U}(\underset{\sim}{\Delta}^{-1}-\underset{\sim}{L})\underset{\sim}{U}^*$. This can happen only if $\underset{\sim}{U}$ is of the form with $e^{i\varphi_j}$ in one position in the jth row and zero in other positions. After substituting those stationary values in $f(\underset{\sim}{U})$, we get

$$(2.2) \qquad \prod_{j=1}^{p} (1 + a_j \ell_{\tau_j})$$

where ℓ_{τ_j} is any permutation of ℓ_j $(j = 1, \ldots, p)$. It is easy to see that (2.2) attains its identical minimum value $|\underset{\sim}{I} + \underset{\sim}{A}\underset{\sim}{L}|$ when $\underset{\sim}{U}$ is of the form (2.1).

Now we impose conditions on $\underset{\sim}{U}$, all $e^{i\varphi_j}$ $(j = 1, \ldots, p)$ are positive real say. Then $e^{i\varphi_j} = 1$ for all j , and (2.1) reduces to $\underset{\sim}{I}$.

The above lemma allows us to claim that, for large n , the integrand of $\mathcal{J}_2$ is negligible except for small neighborhood of $\underset{\sim}{I}$. Therefore

$$(2.3) \qquad \mathcal{J}_2 = \int_{N(\underset{\sim}{I})} |\underset{\sim}{I} + \underset{\sim}{A}\underset{\sim}{U}\underset{\sim}{L}\underset{\sim}{U}^*|^{-n} \, (\underset{\sim}{U}^* d\underset{\sim}{U})$$

where $N(\underset{\sim}{I})$ is a neighborhood of the identity matrix on the unitary manifold.

Lemma 2.2. Let q_j $(j = 1, \ldots, p)$ be the real ch. roots of $Q(p \times p)$ if $\max_{1 \le j \le p} |q_j| < 1$. Then

$$|I + Q|^{-m} = \exp\{-m \operatorname{tr}(Q - \tfrac{1}{2}Q^2 + \tfrac{1}{3}Q^3 - \ldots)\} .$$

Proof:

$$|I + Q|^{-m} = e^{-m \log \prod_{j=1}^{p} (1+q_j)}$$

$$= e^{-m \sum_{j=1}^{p} \log (1+q_j)}$$

$$= e^{-m \operatorname{tr} (Q - \frac{1}{2}Q^2 + \frac{1}{3}Q^3 - \ldots)} .$$

Since we want to compute up to the second term in the asymptotic expansion of $\mathcal{S}_2$, we need to investigate the groups of terms up to the fourth order of S. Under transformation (2.4) of Chapter III, we have

$$AULU^* = AL + i(AHL - ALH) + (AHLH - \tfrac{1}{2}ALH^2 - \tfrac{1}{2}AH^2L)$$

$$+ \tfrac{i}{6}(ALH^3 - 3AHLH^2 + 3\,AH^2LH - AH^3L)$$

$$+ \tfrac{1}{24}(ALH^4 - 4AHLH^3 + 6AH^2LH^2 - 4AH^3LH + AH^4L) + \ldots .$$

Hence

$$|I + AULU^*|^{-n} = |I + AL|^{-n}\, |I + \{H\} + \{H^2\} + \{H^3\} + \{H^4\} + \ldots|^{-n} ,$$

where

$$\underset{\sim}{R} = (\underset{\sim}{I}+\underset{\sim}{AL})^{-1}\underset{\sim}{A} = \begin{pmatrix} r_1 & & & 0 \\ & r_2 & & \\ & & \ddots & \\ 0 & & & r_p \end{pmatrix} \quad r_j = \frac{a_j}{1 + a_j \ell_j} \quad (j = 1, \ldots, p) \quad ,$$

$$\{\underset{\sim}{H}\} = i(\underset{\sim}{RHL} - \underset{\sim}{RLH}) \quad ,$$

$$\{\underset{\sim}{H}^2\} = \underset{\sim}{RHLH} - \tfrac{1}{2}\underset{\sim}{RLH}^2 - \tfrac{1}{2}\underset{\sim}{RH}^2\underset{\sim}{L} \quad ,$$

$$\{\underset{\sim}{H}^3\} = \tfrac{i}{6}(\underset{\sim}{RLH}^3 - 3\underset{\sim}{RHLH}^2 + 3\underset{\sim}{RH}^2\underset{\sim}{LH} - \underset{\sim}{RH}^3\underset{\sim}{L})$$

and

$$\{\underset{\sim}{H}^4\} = \tfrac{1}{24}(\underset{\sim}{RLH}^4 - 4\underset{\sim}{RHLH}^3 + 6\underset{\sim}{RH}^2\underset{\sim}{LH}^2 - 4\underset{\sim}{RH}^3\underset{\sim}{LH} + \underset{\sim}{RH}^4\underset{\sim}{L}) \quad .$$

Under transformation (2.4) of Chapter III, it has $N(\underset{\sim}{I}) \to N(\underset{\sim}{H}=\underset{\sim}{0})$. If we put $\underset{\sim}{Q} = \{\underset{\sim}{H}\} + \{\underset{\sim}{H}^2\} + \{\underset{\sim}{H}^3\} + \{\underset{\sim}{H}^4\} + \ldots$, then in the neighborhood of $\underset{\sim}{H} = \underset{\sim}{0}$, the absolute values of the elements of $\underset{\sim}{H}$ are very small, and hence the maximum ch. roots of $\underset{\sim}{Q}$ can be assumed to be less than unity. Therefore Lemma 2.2 is applicable. Thus we have

$$|\underset{\sim}{I} + \underset{\sim}{AULU}^*|^{-n} = |\underset{\sim}{I} + \underset{\sim}{AL}|^{-n} \cdot |\underset{\sim}{I} + \underset{\sim}{Q}|^{-n}$$

$$= |\underset{\sim}{I} + \underset{\sim}{AL}|^{-n} \exp\left\{-n \operatorname{tr}\left([\underset{\sim}{H}] + [\underset{\sim}{H}^2] + [\underset{\sim}{H}^3] + [\underset{\sim}{H}^4] + \ldots\right)\right\} \quad ,$$

where

$$[\underset{\sim}{H}] = \{\underset{\sim}{H}\} \quad ,$$

$$[\underset{\sim}{H}^2] = \{\underset{\sim}{H}^2\} - \tfrac{1}{2}\{\underset{\sim}{H}\}^2 \quad ,$$

$$[\underset{\sim}{H}^3] = \{\underset{\sim}{H}^3\} - \tfrac{1}{2}\{\underset{\sim}{H}\}\{\underset{\sim}{H}^2\} - \tfrac{1}{2}\{\underset{\sim}{H}^2\}\{\underset{\sim}{H}\} + \tfrac{1}{3}\{\underset{\sim}{H}\}^3$$

and

$$[\underset{\sim}{H}^4] = \{\underset{\sim}{H}^4\} - \tfrac{1}{2}\{\underset{\sim}{H}\}\{\underset{\sim}{H}^3\} - \tfrac{1}{2}\{\underset{\sim}{H}^3\}\{\underset{\sim}{H}\} - \tfrac{1}{2}\{\underset{\sim}{H}^2\}^2 + \tfrac{1}{3}\{\underset{\sim}{H}\}^2\{\underset{\sim}{H}^2\}$$
$$+ \tfrac{1}{3}\{\underset{\sim}{H}\}\{\underset{\sim}{H}^2\}\{\underset{\sim}{H}\} + \tfrac{1}{3}\{\underset{\sim}{H}^2\}\{\underset{\sim}{H}\}^2 - \tfrac{1}{4}\{\underset{\sim}{H}\}^4 .$$

Since $\underset{\sim}{H} = (h_{jk})$ $h_{jk} = \bar{h}_{kj}$ for all $j,k = 1, \ldots, p$ under conditions $h_{jj} = 0$ $(j = 1, \ldots, p)$ we have

$$\text{tr } [\underset{\sim}{H}] = i \text{ tr } (\underset{\sim}{RHL} - \underset{\sim}{RLH}) = 0 \quad ,$$

$$\text{tr } [\underset{\sim}{H}^2] = \text{tr } (\{\underset{\sim}{H}^2\} - \tfrac{1}{2}\{\underset{\sim}{H}\}^2)$$

$$= \text{tr } (\underset{\sim}{RHLH} - \underset{\sim}{RLH}^2 + \tfrac{1}{2}(\underset{\sim}{RHLRHL} + \underset{\sim}{RLHRLH} - \underset{\sim}{RHLRLH} - \underset{\sim}{RLHRHL}))$$

$$= \text{tr } (\underset{\sim}{HL} - \underset{\sim}{LH})(\underset{\sim}{I} - \underset{\sim}{RL})\underset{\sim}{HR}$$

$$= \sum_{j<k}^{p} C_{jk} h_{jk} \bar{h}_{jk} \quad ,$$

where

(2.4) $$C_{jk} = (r_{kj} - r_j r_k \ell_{jk})\ell_{jk} = -C_{kj}$$

$$r_{jk} = r_j - r_k \quad \text{and} \quad \ell_{jk} = \ell_j - \ell_k \quad .$$

Let us note that

$$\text{tr } \{\underset{\sim}{H}\}\{\underset{\sim}{H}^2\} = \text{tr } \{\underset{\sim}{H}^2\}\{\underset{\sim}{H}\} \quad ,$$

$$\text{tr } \{\underset{\sim}{H}\}\{\underset{\sim}{H}^3\} = \text{tr } \{\underset{\sim}{H}^3\}\{\underset{\sim}{H}\} \quad ,$$

and

$$\operatorname{tr}\{\underset{\sim}{H}\}^2\{\underset{\sim}{H}^2\} = \operatorname{tr}\{\underset{\sim}{H}\}\{\underset{\sim}{H}^2\}\{\underset{\sim}{H}\} = \operatorname{tr}\{\underset{\sim}{H}^2\}\{\underset{\sim}{H}\}^2 .$$

Similarly, after simplification, we find

$$\operatorname{tr}[\underset{\sim}{H}^3] = \sum_{j<k<s}^{p} F(h_{jk}h_{ks}h_{sj} - \overline{h_{jk}h_{ks}h_{sj}}) ,$$

where

$$(2.5)\quad F = F(j,k,s) = \tfrac{1}{2}(r_{kj}\ell_{ks} - r_{sk}\ell_{jk} + r_j r_{sk}\ell_{jk}\ell_{js}$$
$$+ r_k r_{sj}\ell_{jk}\ell_{ks} + r_s r_{sk}\ell_{js}\ell_{ks} - 2r_j r_k r_s \ell_{jk}\ell_{ks}\ell_{js}) ,$$

and

$$\operatorname{tr}[\underset{\sim}{H}^4] = \sum_{j<k}^{p} \Phi\cdot(h_{jk}\overline{h}_{jk})^2 + \sum_{j<k<s}^{p} \Psi_1\cdot h_{jk}\overline{h}_{jk}h_{js}\overline{h}_{js} + \sum_{j<k<s}^{p} \Psi_2\cdot h_{jk}\overline{h}_{jk}h_{ks}\overline{h}_{ks}$$
$$+ \sum_{j<k<s}^{p} \Psi_3\cdot h_{js}\overline{h}_{js}h_{ks}\overline{h}_{ks} + \sum_{j<k\neq s\neq t}^{p} G\cdot(h_{jk}h_{ks}h_{st}h_{tj} + \overline{h_{jk}h_{ks}h_{st}h_{tj}}) ,$$

where

$$(2.6)\quad \Phi = \Phi(j,k) = (r_j r_k \ell_{jk}^2 - \tfrac{1}{3})r_{kj}\ell_{jk} + (\tfrac{1}{3}r_j r_k - \tfrac{1}{2}r_{kj}^2)\ell_{jk}^2 - \tfrac{1}{2}r_j^2 r_k^2 \ell_{jk}^4$$
$$= -\tfrac{1}{3}C_{jk} - \tfrac{1}{2}C_{jk}^2 ,$$

$$(2.7)\quad \Psi_1 = \Psi_1(j,k,s)$$
$$= -\tfrac{1}{3}r_{kj}\ell_{jk} - \tfrac{1}{3}r_{sj}\ell_{js} + \tfrac{1}{4}r_{sk}\ell_{ks} + \tfrac{1}{3}r_j(r_k\ell_{jk}^2 + r_s\ell_{js}^2) + r_j(r_k + r_s)\ell_{jk}\ell_{js}$$

$$-r_j^2\ell_{jk}\ell_{js}-\frac{1}{4}r_kr_s(\ell_{jk}+\ell_{js})^2$$

$$-r_j(r_kr_{js}\ell_{jk}+r_sr_{jk}\ell_{js}+r_jr_kr_s\ell_{jk}\ell_{js})\ell_{jk}\ell_{js}$$

$$=-\frac{1}{3}(C_{jk}+C_{js})+\frac{1}{4}C_{ks}-C_{jk}C_{js} \quad ,$$

and

$$G = G(j,k,s,t)$$

$$= \frac{1}{4}(r_{sj}\ell_{js}+r_{tk}\ell_{kt})-\frac{1}{6}(r_{kj}\ell_{jk}+r_{sk}\ell_{ks}+r_{ts}\ell_{st}+r_{jt}\ell_{tj})$$

$$+\frac{1}{6}[r_jr_k\ell_{jk}(\ell_{jk}+3\ell_{st})+r_kr_s\ell_{ks}(\ell_{ks}+3\ell_{tj})+r_sr_t\ell_{st}(\ell_{st}+3\ell_{jk})$$

$$+r_tr_j\ell_{tj}(\ell_{tj}+3\ell_{ks})]-\frac{1}{4}[r_jr_s(\ell_{jk}+\ell_{sk})(\ell_{jt}+\ell_{st})$$

$$+r_kr_t(\ell_{jk}+\ell_{jt})(\ell_{sk}+\ell_{st})]-\frac{1}{2}[r_jr_k\ell_{jk}\ell_{ks}r_{st}\ell_{tj}+r_kr_s\ell_{jk}\ell_{ks}\ell_{st}r_{tj}$$

$$+r_sr_tr_{jk}\ell_{ks}\ell_{st}\ell_{tj}+r_tr_j\ell_{jk}r_{ks}\ell_{st}\ell_{tj}]-r_jr_kr_sr_t\ell_{jk}\ell_{ks}\ell_{st}\ell_{tj} \quad .$$

From (2.5), it is not difficult to show that $F^2 = -\frac{1}{4}\{C_{jk}^2+C_{js}^2+C_{ks}^2-2(C_{jk}C_{js}+C_{jk}C_{ks}+C_{js}C_{ks})-4\,C_{jk}C_{ks}C_{js}\}$. Also note that Ψ_2 and Ψ_3 can be obtained from Ψ_1 cyclically, i.e., changing j to k, k to s, and s to j, then Ψ_1 becoming Ψ_2, Ψ_2 becoming Ψ_3 and Ψ_3 becoming Ψ_1. Moreover, we need not know the value of G, because any term containing an odd power of a factor h_{jkR} or h_{jkI} will integrate to zero, where h_{jkR} and h_{jkI} are the real and imaginary parts of h_{jk}.

Finally, we can write (2.3) to be

$$(2.8) \qquad \mathcal{J}_2 = \prod_{j=1}^{p} (1 + a_j \ell_j)^{-n} \int_{N(\underset{\sim}{H}=\underset{\sim}{0})} \exp\left(-n \sum_{j<k}^{p} C_{jk} h_{jk} \bar{h}_{jk}\right)$$

$$\cdot \exp\left(-n \operatorname{tr} [\underset{\sim}{H}^3] - n \operatorname{tr} [\underset{\sim}{H}^4] - \ldots\right) J \prod_{j<k}^{p} dh_{jkR} dh_{jkI} \quad .$$

where J is found in (2.5) of Chapter III.

If this integration is to be performed term by term on the expansion of $\exp(-n \operatorname{tr} [H^3] - \ldots) J$ then for large n the limits for each h_{jkc} can be put to $\pm \infty$, where h_{jkc} denotes either h_{jkR} or h_{jkI}. Since each integration is of the form

$$\int_{N(\underset{\sim}{H}=\underset{\sim}{0})} \exp\left(-n \sum_{j<k}^{p} C_{jk} h_{jk} \bar{h}_{jk}\right) \prod_{j<k}^{p} h_{jkc}^{m_{jk}} \prod_{j<k}^{p} dh_{jkR} dh_{jkI} \quad ,$$

and most of this integral is concentrated in a small neighborhood of $\underset{\sim}{H}=\underset{\sim}{0}$. The m_{jk}'s are positive even integers or zero, since any term containing an odd power of an h_{jkc} will integrate to zero. Now we expand $\exp(-n \operatorname{tr} [\underset{\sim}{H}^3] - \ldots) J$, writing the terms in groups, each group corresponding to a certain value of m. We have

$$(2.9) \qquad \exp\left(-n \operatorname{tr} [\underset{\sim}{H}^3] - n \operatorname{tr} [\underset{\sim}{H}^4] - \ldots\right) J$$

$$= 1 - n \operatorname{tr} [\underset{\sim}{H}^4] + \frac{n^2}{2} (\operatorname{tr} [\underset{\sim}{H}^3])^2 - \frac{p}{12} \operatorname{tr} \underset{\sim}{H}^2$$

$$+ \frac{1}{2(6!)} \left\{ (5p^2 - 3)(\operatorname{tr} \underset{\sim}{H}^2)^2 - p \operatorname{tr} \underset{\sim}{H}^4 \right\} + \ldots \quad .$$

Using formulas (2.14), (2.15) and (2.16) in Chapter III, we obtain the following theorem:

Theorem 2.1. Let $\underset{\sim}{A}$ and $\underset{\sim}{L}$ be diagonal matrices with $0 < a_1 < a_2 < \ldots < a_p$ and $\ell_1 > \ell_2 > \ldots > \ell_p > 0$. Then for large n, the first two terms in the expansion for $\mathcal{J}_2$ are given by

$$(2.10) \qquad \mathcal{J}_2 = \prod_{j=1}^{p} (1 + a_j\ell_j)^{-n} \prod_{j<k}^{p} \frac{\pi}{nC_{jk}} \left\{1 + \frac{1}{3n}\left[\sum_{j<k}^{p} C_{jk}^{-1} + \beta(p)\right] + \ldots\right\} ,$$

where

$$(2.11) \qquad \beta(p) = p(p - 1)(2p - 1)/2 .$$

Proof: In the proof, we include only terms without an odd power of an h_{jkc}, and do not write C (where C is defined in (2.14) of Chapter III) which appears with each term after integration, and denote

$$S' = \sum_{j<k}^{p} C_{jk}^{-1}$$

and

$$S'' = \sum_{j<k<s}^{p} (C_{ks}/C_{jk}C_{js} + C_{js}/C_{jk}C_{ks} + C_{jk}/C_{js}C_{ks}) .$$

Note that only the second, third and fourth terms on the right hand side of (2.9) contribute the factor n^{-1}, using formulas (2.14) - (2.16) in Chapter III. After integration, the second term $-n \operatorname{tr} [\underset{\sim}{H}^4]$ contributes

$$(2.12) \qquad \frac{2}{3n}S' + \frac{1}{n}\binom{p}{2} + \frac{2(p - 2)}{3n} S' - \frac{1}{4n}S'' + \frac{3}{n}\binom{p}{3} ,$$

and the third term $n^2(\mathrm{tr}\,[\underset{\sim}{H}^3])^2/2$ gives

$$(2.13) \qquad \frac{1}{4n} S'' - \frac{p-2}{2n} S' - \frac{1}{n}\binom{p}{3} \quad .$$

Since $\mathrm{tr}\,\underset{\sim}{H}^2 = 2 \sum_{j<k}^{p} h_{jk}\bar{h}_{jk}$,

it is not difficult to see that $-p\,\mathrm{tr}\,\underset{\sim}{H}^2/12$ gives

$$(2.14) \qquad -\frac{p}{6n} S' \quad .$$

Adding (2.12) - (2.14) we obtain (2.10).

Theorem 2.2. The asymptotic distribution of the ch. roots, $\ell_1 > \ell_2 > \ldots > \ell_p > 0$, of $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ for large degrees of freedom $n = n_1 + n_2$ when the roots of $\underset{\sim}{\Sigma}_1\underset{\sim}{\Sigma}_2^{-1}$ are $\lambda_1 > \lambda_2 > \ldots > \lambda_p > 0$, where $\lambda_j = a_j^{-1}$ $(j = 1, \ldots, p)$ is given by

$$(2.15) \qquad C_1 \prod_{j=1}^{p} a_j^{n_1} \ell_j^{n_1-p} (1 + a_j\ell_j)^{-n} \prod_{j<k}^{p} (\ell_j - \ell_k)^2 \prod_{j=1}^{p} d\ell_j$$

$$\cdot \prod_{j<k}^{p} \frac{\pi}{nC_{jk}} \left\{1 + \frac{1}{3n}\left[\sum_{j<k}^{p} C_{jk}^{-1} + \beta(p)\right] + \ldots\right\} \quad ,$$

where C_1 , C_{jk} and $\beta(p)$ are defined by (1.2), (2.4) and (2.11) respectively.

3. Comparisons

It is interesting to compare (2.10) with the corresponding formula in the one-sample case, i.e., (2.17) of Chapter III. We find that there is an extra term $\beta(p)/3n$ (in the second term of the asymptotic expansion) which is a function of n and p only. It is also interesting to compare (2.10) with the corresponding formula in the real case (c.f. (2.11) of Chapter I). The term corresponding to $\beta(p)/3n$ is $\alpha(p)/2n$ there.

Finally, let us note that if ℓ_j in (2.15) replaced by $n_1\ell_j/n_2$ $(j = 1, \ldots, p)$ and let n_2 tend to infinity, then (2.15) reduces to (2.21) in Chapter III.

CHAPTER V

THE DISTRIBUTION OF CHARACTERISTIC VECTORS CORRESPONDING TO THE TWO LARGEST ROOTS OF A MATRIX

1. Summary

The distribution of the characteristic (ch.) vectors of a sample covariance matrix was found by Anderson [2], when the population covariance matrix is a scalar matrix $\Sigma = \sigma^2 I$. The asymptotic distribution for arbitrary Σ also was obtained by Anderson [3]. For unknown Σ , the distribution of the ch. vector corresponding to the largest root of a covariance matrix was found by Sugiyama [30] and Khatri and Pillai [20]. In this chapter, for arbitrary Σ , we obtain the joint distribution of the ch. vectors corresponding to the two largest roots for the non-central linear case, i.e. when the rank of the mean matrix is one.

2. Notations and Some Useful Results

Matrices will be denoted by bold face capital letters, and their dimensions will be indicated parenthetically. The m x m identity matrix will be denoted by I_m , and in particular, I denotes I_{p-2} throughout this chapter. $|\alpha|$ denotes the absolute value of α , and $|X|$ denotes the determinant of X . $O(n)$ denotes the group of all orthogonal n x n matrices.

Let $S(m \times m)$ be any symmetric positive definite matrix. The zonal polynomials $Z_\kappa(\underset{\sim}{S})$ are defined for each partition $\kappa=(k_1,k_2,\ldots k_m)$, $k_1 \geq k_2 \geq \ldots \geq k_m \geq 0$ of k into not more than m parts, as certain symmetric polynomials in the ch. roots of $\underset{\sim}{S}$, (see James [14], [15], [16] and Constantine [8]). Further (see Constantine [8])

$$(2.1) \qquad \int_{O(m)} C_\kappa(\underset{\sim}{H}'\underset{\sim}{S}\underset{\sim}{H}\underset{\sim}{T})d\underset{\sim}{H} = C_\kappa(\underset{\sim}{S})C_\kappa(\underset{\sim}{T})/C_\kappa(\underset{\sim}{I}_m)$$

where $d\underset{\sim}{H}$ is the invariant Haar measure on the orthogonal group $O(m)$, normalized to make the volume of the group manifold unity. Also note that (see [8])

$$(2.2) \qquad \int_{\underset{\sim}{0}}^{\underset{\sim}{I}_m} |\underset{\sim}{S}|^{t-\frac{1}{2}(m+1)} |\underset{\sim}{I}_m - \underset{\sim}{S}|^{u-\frac{1}{2}(m+1)} C_\kappa(\underset{\sim}{T}\underset{\sim}{S})d\underset{\sim}{S} = \frac{\Gamma_m(t,\kappa)\Gamma_m(u)}{\Gamma_m(t+u,\kappa)} C_\kappa(\underset{\sim}{T})$$

where $\underset{\sim}{T}$ is a positive definite matrix,

$$\Gamma_m(u) = \pi^{\frac{1}{4}m(m-1)} \prod_{i=0}^{m-1} \Gamma(u-\tfrac{1}{2}i)$$

and

$$\Gamma_m(t,\kappa) = \pi^{\frac{1}{4}m(m-1)} \prod_{i=0}^{m-1} \Gamma(t+k_i-\tfrac{1}{2}i) .$$

Let $\underset{\sim}{R}(n \times n)$ be an orthogonal matrix such that the first $r(\leq n)$ columns have random elements and the remaining $(n-r)$ columns depend on these random elements. We will denote $d\underset{\sim}{R}^{(n,r)}$ a normalized measure over this space, i.e.

$$\int_{O(n)} d\underset{\sim}{R}^{(n,r)} = 1 .$$

In terms of Roy's notation [28], let $J(\underset{\sim}{R}) = 2^n / \left| \frac{\partial(\underset{\sim}{R}\underset{\sim}{R}')}{\partial(\underset{\sim}{R}_D)} \right|_{\underset{\sim}{R}_I}$

Thus $J(\underset{\sim}{R})$ is a function of random elements of $\underset{\sim}{R}$. We will write

(2.3) $$d\underset{\sim}{R}^{(n,r)} = \pi^{-\frac{1}{2}rn} \Gamma_r(\tfrac{n}{2}) J(\underset{\sim}{R}) .$$

Lemma 2.1. Let $\underset{\sim}{U}(p \times n)$ and $\underset{\sim}{V}((p-r) \times (n-r))$ be random matrices, $(p \leq n)$ and let

(2.4) $$\underset{\sim}{U} = \underset{\sim}{H} \begin{pmatrix} \alpha_1 & & & 0 \\ & \ddots & & \\ & & \alpha_r & \\ 0 & & & \underset{\sim}{V} \end{pmatrix} \underset{\sim}{G}'$$

be a transformation such that the first $r(\leq p)$ column vectors of orthogonal matrices $\underset{\sim}{H}(p \times p)$ and $\underset{\sim}{G}(n \times n)$ contain random elements, and $\alpha_i \neq 0$ $(i = 1, \ldots, r)$ and $\alpha_1^2 > \alpha_2^2 > \ldots > \alpha_r^2 > 0$ be the first r non-zero largest ordered ch. roots of $\underset{\sim}{U}\underset{\sim}{U}'$. Then the Jacobian of the transformation is given by

(2.5) $$J(\underset{\sim}{U} : \underset{\sim}{H}\, \alpha_1, \alpha_2, \ldots, \alpha_r, \underset{\sim}{V}, \underset{\sim}{G}) =$$

$$C \prod_{i=1}^{r} |\alpha_i|^{n-p} |\alpha_i^2 \underset{\sim}{I} - \underset{\sim}{V}\underset{\sim}{V}'| \prod_{i<j} (\alpha_i^2 - \alpha_j^2) d\underset{\sim}{H}^{(p,r)} d\underset{\sim}{G}^{(n,r)}$$

where

$$C = \pi^{\frac{1}{2}r(p+n)} \left\{\Gamma_r(\tfrac{p}{2})\Gamma_r(\tfrac{n}{2})\right\}^{-1} .$$

Proof: Taking differentials of (2.4), and both sides, pre- and post-multiply $\underset{\sim}{H}'$ and $\underset{\sim}{G}$, we obtain

$$\underset{\sim}{H}' (d\underset{\sim}{U}) \underset{\sim}{G} =$$

$$\underset{\sim}{H}' d\underset{\sim}{H} \begin{pmatrix} \alpha_1 & & & 0 \\ & \ddots & & \\ & & \alpha_r & \\ 0 & & & \underset{\sim}{V} \end{pmatrix} + \begin{pmatrix} d\alpha_1 & & & 0 \\ & \ddots & & \\ & & d\alpha_r & \\ 0 & & & d\underset{\sim}{V} \end{pmatrix} + \begin{pmatrix} \alpha_1 & & & 0 \\ & \ddots & & \\ & & \alpha_r & \\ 0 & & & \underset{\sim}{V} \end{pmatrix} (d\underset{\sim}{G}')\underset{\sim}{G}$$

Let

$$\underset{\sim}{H}' (d\underset{\sim}{U})\underset{\sim}{G} = \underset{\sim}{W} = \begin{pmatrix} w_{11} & \cdots & w_{1n} \\ \cdots & \cdots & \cdots \\ w_{p1} & \cdots & w_{pn} \end{pmatrix}$$

Since $\underset{\sim}{H}' d\underset{\sim}{H}$ and $(d\underset{\sim}{G}')\underset{\sim}{G}$ are $p \times p$ and $n \times n$ skew symmetric matrices, hence we can put

$$\underset{\sim}{H}' d\underset{\sim}{H} = \underset{\sim}{A} = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1p} \\ -a_{12} & 0 & \cdots & a_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ -a_{1p} & -a_{2p} & \cdots & 0 \end{pmatrix}$$

and

$$(d\underset{\sim}{G}')\underset{\sim}{G} = \underset{\sim}{B} = \begin{pmatrix} 0 & b_{12} & \cdots & b_{1n} \\ -b_{12} & 0 & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ -b_{1n} & -b_{2n} & \cdots & 0 \end{pmatrix}$$

Denote $\underset{\sim}{W}_{(r,r)}$ to be the matrix from $\underset{\sim}{W}$ by deleting the first r rows and the first r columns of $\underset{\sim}{W}$. Similarly for $\underset{\sim}{A}_{(r,r)}$ and $\underset{\sim}{B}_{(r,r)}$. Then

$$(w_{ij}, w_{ji}) = (a_{ij}, b_{ij}) \begin{pmatrix} \alpha_j & -\alpha_i \\ \alpha_i & -\alpha_j \end{pmatrix} ,$$

and

$$(w_{i,r+1}, \ldots, w_{in}, w_{r+1,i}, \ldots, w_{pi}) =$$

$$(a_{i,r+1}, \ldots, a_{ip}, b_{i,r+1}, \ldots, b_{in}) \begin{pmatrix} \underset{\sim}{V} & -\alpha_i \underset{\sim}{I} \\ \alpha_i \underset{\sim}{I}_{n-r} & -\underset{\sim}{V}' \end{pmatrix}$$

imply

$$J_{ij} = J(w_{ij}, w_{ji} : a_{ij}, b_{ij}) = \alpha_i^2 - \alpha_j^2 \quad i,j = 1, \ldots, r \text{ and } i < j ,$$

and

$$J_i = J(w_{ik}, W_{\ell i} : a_{i\ell}, b_{ik}; \quad k = r+1, \ldots, n; \quad \ell = r+1, \ldots, p)$$

$$= |\alpha_i|^{n-p} |\alpha_i^2 \underset{\sim}{I} - \underset{\sim}{V}\underset{\sim}{V}'| \qquad i = 1, \ldots, r .$$

Moreover $J(d\underset{\sim}{U} : \underset{\sim}{W}) = 1 \quad J_i^* = J(w_{ii} : d\alpha_i) = 1 \quad , \quad i = 1, \ldots, r$

and
$$\underset{\sim}{W}_{(r,r)} = \underset{\sim}{A}_{(r,r)}\underset{\sim}{V} + d\underset{\sim}{V} + \underset{\sim}{V}\underset{\sim}{B}_{(r,r)}$$

implies
$$J(\underset{\sim}{W}_{(r,r)} : d\underset{\sim}{V}) = 1 \quad .$$

Finally,

$$J_i^{**} = J(a_{i\ell}, b_{ik} : dh_{i\ell}, dg_{ik}; k = r + 1, \ldots, n \, , \ell = r + 1, \ldots, p)$$
$$= 1 \qquad i = 1, \ldots, r \quad .$$

where h_{ij} and g_{ij} are the ith row and jth column elements of $\underset{\sim}{H}$ and $\underset{\sim}{G}$ respectively. Therefore

$$J(\underset{\sim}{U} : \underset{\sim}{H}, \alpha_1, \ldots, \alpha_r, \underset{\sim}{V}, \underset{\sim}{G})$$
$$= J(d\underset{\sim}{U} : d\underset{\sim}{H}, d\alpha_1, \ldots, d\alpha_r, d\underset{\sim}{V}, d\underset{\sim}{G})$$
$$= J(d\underset{\sim}{U} : \underset{\sim}{W})J(\underset{\sim}{W} : dh_{1\ell}, \ldots, dh_{r\ell}, d\alpha_1, \ldots, d\alpha_r, d\underset{\sim}{V}, dg_{1k},$$
$$\ldots, dg_{rk}; \; k = r + 1, \ldots, n; \; \ell = r + 1, \ldots, p)J(\underset{\sim}{H})J(\underset{\sim}{G})$$
$$= \prod_{i=1}^{r} J_i J_i^* J_i^{**} \prod_{i<j} J_{ij} J(d\underset{\sim}{U} : \underset{\sim}{W}) J(\underset{\sim}{W}_{(r,r)} : d\underset{\sim}{V}) J(\underset{\sim}{H}) J(\underset{\sim}{G}) \quad .$$

Using (2.3), we obtain (2.5).

In (2.5) if we put $\lambda_i = \alpha_i^2 \quad i = 1, \ldots, r$ and notice that each λ_i corresponding α_i and $-\alpha_i$, then (2.5) can be written

(2.6) $$J(\underset{\sim}{U} : \underset{\sim}{H}, \lambda_1, \lambda_2, \ldots, \lambda_r \underset{\sim}{V}, \underset{\sim}{G})$$

$$= C \prod_{i=1}^{r} \lambda_i^{\frac{1}{2}(n-p-1)} |\lambda_i \underset{\sim}{I} - \underset{\sim}{VV}'| \prod_{i<j} (\lambda_i - \lambda_j) d\underset{\sim}{H}^{(p,r)} d\underset{\sim}{G}^{(n,r)} .$$

Lemma 2.2. Let $\underset{\sim}{V}(m \times t)$ be a random matrix and $\underset{\sim}{A}(m \times m)$ be a symmetric matrix. For definiteness, assume $m \leq t$. Then

(2.7) $$\int_{\mathscr{D}} |a\underset{\sim}{I}_m - \underset{\sim}{VV}'|^{\alpha} |b\underset{\sim}{I}_m - \underset{\sim}{VV}'|^{\beta} C_{\kappa}(\underset{\sim}{AVV}') d\underset{\sim}{V}$$

$$= \pi^{\frac{1}{2}mt} \Gamma_m(\beta + \tfrac{m+1}{2}) C_{\kappa}(\underset{\sim}{A}) \left\{\Gamma_m(\beta + \tfrac{t+m+1}{2}) C_{\kappa}(\underset{\sim}{I}_m)\right\}^{-1}$$

$$\cdot \sum_{q=0}^{\infty} \sum_{\eta} \sum_{\delta} \frac{(-\alpha)_{\eta}}{q!} g^{\delta}_{\kappa,\eta} \, C_{\delta}(\underset{\sim}{I}_m) (\tfrac{t}{2})_{\delta} \left\{(\beta + \tfrac{t+m+1}{2})_{\delta}\right\}^{-1} a^{m\alpha - qbm(\beta + \frac{1}{2}t) + d} ,$$

where $\alpha > \frac{1}{2}(m - 1)$, $\beta > \frac{1}{2}(m - 1)$, $a > b > 0$,

$\mathscr{D} = \mathscr{D}\,\{\underset{\sim}{V}$ such that $b\underset{\sim}{I}_m - \underset{\sim}{VV}'$ is positive definite$\}$

$\kappa = \{k_1, k_2, \ldots, k_m\}$, $\sum_{i=1}^{m} k_i = k$,

$(x)_{\kappa} = \Gamma_m(x,\kappa)/\Gamma_m(x)$ if x is such that the gamma functions are defined and

$$(2.8)\quad \begin{cases} \delta = (\delta_1, \delta_2, \ldots, \delta_m), \quad \delta_1 \geq \delta_2 \geq \ldots \geq \delta_m \geq 0 , \\ \sum_{i=1}^{m} \delta_i = q + k = d \\ g_{\kappa,\eta}^{\delta} \text{ is the coefficient of } C_\delta(\underset{\sim}{B}) \text{ in the product} \\ \qquad \text{of } C_\kappa(\underset{\sim}{B})C_\eta(\underset{\sim}{B}) \quad . \end{cases}$$

Proof: Let us write

$$h = \int_{\mathcal{D}} |a\underset{\sim}{I}_m - \underset{\sim}{V}\underset{\sim}{V}'|^{\alpha} |b\underset{\sim}{I}_m - \underset{\sim}{V}\underset{\sim}{V}'|^{\beta}\, C_\kappa(\underset{\sim}{A}\underset{\sim}{V}\underset{\sim}{V}')\, d\underset{\sim}{V}$$

Then

$$h = \int_{\mathcal{D}} |a\underset{\sim}{I}_m - \underset{\sim}{V}\underset{\sim}{V}'|^{\alpha} |b\underset{\sim}{I}_m - \underset{\sim}{V}\underset{\sim}{V}'|^{\beta}\, C_\kappa(\underset{\sim}{A}\underset{\sim}{V}\underset{\sim}{V}')d\underset{\sim}{V} \int_{O(m)} d\underset{\sim}{H} \quad .$$

Making transformation $\underset{\sim}{V} \to \underset{\sim}{H}\underset{\sim}{V}$ and notice $C_\kappa(\underset{\sim}{H}'\underset{\sim}{A}\underset{\sim}{H}\underset{\sim}{V}\underset{\sim}{V}') = C_\kappa(\underset{\sim}{A}\underset{\sim}{H}\underset{\sim}{V}\underset{\sim}{V}'\underset{\sim}{H}')$ then

$$h = \frac{C_\kappa(\underset{\sim}{A})}{C_\kappa(\underset{\sim}{I}_m)} \int_{\mathcal{D}} |a\underset{\sim}{I}_m - \underset{\sim}{V}\underset{\sim}{V}'|^{\alpha} |b\underset{\sim}{I}_m - \underset{\sim}{V}\underset{\sim}{V}'|^{\beta}\, C_\kappa(\underset{\sim}{V}\underset{\sim}{V}')d\underset{\sim}{V}$$

by (2.1). Let $\underset{\sim}{V}\underset{\sim}{V}' = \underset{\sim}{S}$, then

$$h = C_o \int_{\underset{\sim}{0}}^{b\underset{\sim}{I}_m} |\underset{\sim}{S}|^{\frac{1}{2}t - \frac{1}{2}(m+1)}\, |a\underset{\sim}{I}_m - \underset{\sim}{S}|^{\alpha} |b\underset{\sim}{I}_m - \underset{\sim}{S}|^{\beta}\, C_\kappa(\underset{\sim}{S})d\underset{\sim}{S}$$

where

$$C_o = \pi^{\frac{1}{2}mt}\, C_\kappa(\underset{\sim}{A})\, \{\Gamma_m(\tfrac{t}{2})C_\kappa(\underset{\sim}{I}_m)\}^{-1} \quad .$$

Next, put $\underset{\sim}{S} = a\underset{\sim}{S}_1$, we get

$$h = C_o a^{m(\alpha+\beta+\frac{1}{2}t)+k} \int_0^{\frac{b}{a}\underset{\sim}{I}_m} |\underset{\sim}{S}_1|^{\frac{1}{2}t-\frac{1}{2}(m+1)} |\underset{\sim}{I}_m - \underset{\sim}{S}_1|^{\alpha} |\frac{b}{a}\underset{\sim}{I}_m - \underset{\sim}{S}_1|^{\beta} C_\kappa(\underset{\sim}{S}_1) d\underset{\sim}{S}_1 \quad .$$

By Constantine [9], notice that

$$\sum_{q=0}^{\infty} \sum_{\eta} \frac{(z)_\eta}{q!} C_\eta(\underset{\sim}{S}) = |\underset{\sim}{I}_m - \underset{\sim}{S}|^{-z}$$

h can be written as

$$h = C_o A^{m(\alpha+\beta+\frac{1}{2}t)+k} \sum_{q=0}^{\infty} \sum_{\eta} \sum_{\delta} \frac{(-\alpha)_\eta}{q!} g^{\delta}_{\kappa,\eta}$$

$$\cdot \int_{\underset{\sim}{0}}^{\frac{b}{a}\underset{\sim}{I}_m} |\underset{\sim}{S}_1|^{\frac{1}{2}t-\frac{1}{2}(m+1)} |\frac{b}{a}\underset{\sim}{I}_m - \underset{\sim}{S}_1|^{\beta} C_\delta(\underset{\sim}{S}_1) d\underset{\sim}{S}_1 \quad ,$$

where δ, δ_i and $g^{\delta}_{\kappa,\eta}$ are defined by (2.8).

Finally, put $\underset{\sim}{S}_1 = \frac{b}{a}\underset{\sim}{T}$ then by (2.2) and $\Gamma_m(\frac{t}{2}, \delta) = \Gamma_m(\frac{t}{2})(\frac{t}{2})_\delta$ we have

$$h = C_o \sum_{q=0}^{\infty} \sum_{\eta} \sum_{\delta} a^{m\alpha - qbm(\beta+\frac{1}{2}t)+q+k} \frac{(-\alpha)_\eta}{q\,!} g^{\delta}_{\kappa,\eta}$$

$$\cdot \int_{\underset{\sim}{0}}^{\underset{\sim}{I}_m} |\underset{\sim}{T}|^{\frac{1}{2}t-\frac{1}{2}(m+1)} |\underset{\sim}{I}_m - \underset{\sim}{T}|^{\beta} C_\delta(\underset{\sim}{T}) d\underset{\sim}{T}$$

$$= \pi^{\frac{1}{2}mt} C_\kappa(\underset{\sim}{A}) \left\{C_\kappa(\underset{\sim}{I}_m)\right\}^{-1} \sum_{q=0}^{\infty} \sum_{\eta} \sum_{\delta} \frac{(-\alpha)_\eta}{q!} g^{\delta}_{\kappa,\eta} C_\delta(\underset{\sim}{I}_m) (\frac{t}{2})_\delta$$

$$\cdot \Gamma_m(\beta + \frac{m+1}{2}) \left\{\Gamma_m(\beta + \frac{t+m+1}{2}, \delta)\right\}^{-1} a^{m\alpha - qbm(\beta+\frac{1}{2}t)+q+k} \quad .$$

After rearranging, we obtain (2.7).

3. Distribution of the Characteristic Vectors Corresponding to the Two Largest Roots of a Matrix in the Non-Central Case

Let the matrix $\underset{\sim}{X}(p \times n)$ be distributed as

$$(2\pi)^{-\frac{1}{2}pn} \, |\underset{\sim}{\Sigma}|^{-\frac{1}{2}n} \exp\left\{-\tfrac{1}{2} \operatorname{tr} \underset{\sim}{\Sigma}^{-1} (\underset{\sim}{X} - \underset{\sim}{M})(\underset{\sim}{X} - \underset{\sim}{M})'\right\}$$

i.e.

$$(3.1) \qquad (2\pi)^{-\frac{1}{2}pn} \, |\underset{\sim}{\Sigma}|^{-\frac{1}{2}n} \exp\left\{-\tfrac{1}{2} \operatorname{tr} \underset{\sim}{\Sigma}^{-1}\underset{\sim}{M}\underset{\sim}{M}' + \operatorname{tr} \underset{\sim}{\Sigma}^{-1}\underset{\sim}{M}\underset{\sim}{X}' - \tfrac{1}{2} \operatorname{tr} \underset{\sim}{\Sigma}^{-1}\underset{\sim}{X}\underset{\sim}{X}'\right\}$$

where $E[\underset{\sim}{X}] = \underset{\sim}{M}$.

Making transformation

$$\underset{\sim}{X} = \underset{\sim}{L} \begin{pmatrix} \alpha_1 & & & 0 \\ & \ddots & & \\ & & \alpha_r & \\ 0 & & & \underset{\sim}{Y} \end{pmatrix} \underset{\sim}{Q}'$$

where $r \leq p$, $\underset{\sim}{L}(p \times p)$ and $\underset{\sim}{Q}(n \times n)$ are orthogonal matrices, $\underset{\sim}{Y}$ is an $(p-r) \times (n-r)$ matrix and $\alpha_1^2 > \alpha_2^2 > \ldots > \alpha_r^2 > 0$ are the first r largest ordered ch. roots of $\underset{\sim}{X}\underset{\sim}{X}'$. Using Lemma 2.1, the joint density function of $\underset{\sim}{L}$, $\alpha_1, \alpha_2, \ldots, \alpha_r$, $\underset{\sim}{Y}$ and $\underset{\sim}{Q}$ is given by

$$(3.2) \qquad C_1 |\Sigma|^{-\frac{1}{2}n} \prod_{i=1}^{r} |\alpha_i|^{n-p} |\alpha_i^2 I - YY'| \prod_{i<j} (\alpha_i^2 - \alpha_j^2)$$

$$\cdot \exp\left\{-\tfrac{1}{2} \operatorname{tr} \Sigma^{-1} MM' + \operatorname{tr} \Sigma^{-1} MQ \begin{pmatrix} \alpha_1 & & & 0 \\ & \ddots & & \\ & & \alpha_r & \\ 0 & & & Y' \end{pmatrix} L'\right.$$

$$\left. -\tfrac{1}{2} \operatorname{tr} \Sigma^{-1} L \begin{pmatrix} \alpha_1^2 & & & 0 \\ & \ddots & & \\ & & \alpha_r^2 & \\ 0 & & & YY' \end{pmatrix} L' \right\} dL^{(p,r)} dQ^{(n,r)} \quad ,$$

where

$$(3.3) \qquad C_1 = C(2\pi)^{-\frac{1}{2}pn} = \pi^{\frac{1}{2}r(p+n)} \left\{(2\pi)^{\frac{1}{2}pn} \Gamma_r(\tfrac{p}{2}) \Gamma_r(\tfrac{n}{2})\right\}^{-1} \quad .$$

In integrating α_i $(i = 1, \ldots, r)$, Y, Q, or L , we only consider the non-central linear case, i.e. when the rank of the mean matrix M is one, because the general problem is extremely difficult.

For $r = 1$, we get the same result as given by Khatri and Pillai [20].

For $r = 2$, let

$$L = (\ell_1, \ell_2, L_2) \quad ,$$

where ℓ_1 and ℓ_2 are the first two columns of L , corresponding to the two largest ordered ch. roots λ_1 and λ_2 of XX', having random elements and the others L_2 depend on these random elements.

Then (3.2) becomes

$$(3.4) \qquad C_2|\alpha_1\alpha_2|^{n-p}(\alpha_1^2 - \alpha_2^2)|\alpha_1^2 \underset{\sim}{I} - \underset{\sim}{Y}\underset{\sim}{Y}'|\,|\alpha_2^2 \underset{\sim}{I} - \underset{\sim}{Y}\underset{\sim}{Y}'|$$

$$\cdot \exp\left\{\operatorname{tr} \underset{\sim}{\Sigma}^{-1}\underset{\sim}{M}\underset{\sim}{Q} \begin{pmatrix} \alpha_1 & 0 & \\ 0 & \alpha_2 & \underset{\sim}{0} \\ \underset{\sim}{0} & & \underset{\sim}{Y}' \end{pmatrix} \underset{\sim}{L}' - \tfrac{1}{2} \operatorname{tr} \underset{\sim}{\Sigma}^{-1}\underset{\sim}{L} \begin{pmatrix} \alpha_1^2 & 0 & \\ 0 & \alpha_2^2 & \underset{\sim}{0} \\ \underset{\sim}{0} & & \underset{\sim}{Y}\underset{\sim}{Y}' \end{pmatrix} \underset{\sim}{L}'\right\}$$

$$\cdot\ d\underset{\sim}{L}^{(p,2)} d\underset{\sim}{Q}^{(n,2)} \quad ,$$

where

$$C_2 = C_1|\underset{\sim}{\Sigma}|^{-\frac{1}{2}n}\, e^{-\frac{1}{2} \operatorname{tr} \underset{\sim}{\Sigma}^{-1}\underset{\sim}{M}\underset{\sim}{M}'} \quad .$$

Integrating (3.4) with respect to $\underset{\sim}{Q}$, we obtain

$$(3.5) \qquad C_2|\alpha_1\alpha_2|^{n-p}(\alpha_1^2 - \alpha_2^2) \sum_{k=0}^{\infty}\left\{k!\ \left(\tfrac{n}{2}\right)_k\right\}^{-1} d\underset{\sim}{L}^{(p,2)} f_k(\alpha_1, \alpha_2, \underset{\sim}{L})$$

or

$$(3.5') \qquad C_2(\lambda_1\lambda_2)^{\frac{1}{2}(n-p-1)}(\lambda_1 - \lambda_2) \sum_{k=0}^{\infty}\left\{k!\ \left(\tfrac{n}{2}\right)_k\right\}^{-1} d\underset{\sim}{L}^{(p,2)} f_k(\lambda_1, \lambda_2, \underset{\sim}{L}) \ ,$$

where

(3.6) $$f_k(\lambda_1, \lambda_2, \underset{\sim}{L}) =$$

$$\int_{\mathcal{D}} |\lambda_1 \underset{\sim}{I} - \underset{\sim}{YY}'| \cdot |\lambda_2 \underset{\sim}{I} - \underset{\sim}{YY}'| \cdot \left\{ \mathrm{tr} \left[\tfrac{1}{4} \underset{\sim}{\Sigma}^{-1} \underset{\sim}{L} \begin{pmatrix} \lambda_1 & 0 & \\ 0 & \lambda_2 & \underset{\sim}{0} \\ & \underset{\sim}{0} & \underset{\sim}{YY}' \end{pmatrix} \underset{\sim}{L}' \underset{\sim}{\Sigma}^{-1} \underset{\sim}{MM}' \right] \right\}^k$$

$$\cdot \exp \left\{ -\tfrac{1}{2} \,\mathrm{tr}\, \underset{\sim}{\Sigma}^{-1} \underset{\sim}{L} \begin{pmatrix} \lambda_1 & 0 & \\ 0 & \lambda_2 & \underset{\sim}{0} \\ & \underset{\sim}{0} & \underset{\sim}{YY}' \end{pmatrix} \underset{\sim}{L}' \right\} d\underset{\sim}{Y} \quad ,$$

where

$$\mathcal{D} = \mathcal{D}\, \{\underset{\sim}{Y} \text{ such that } \lambda_2 \underset{\sim}{I} - \underset{\sim}{YY}' \text{ is a positive definite}\}$$

or

(3.7) $$f(\theta, \lambda_1, \lambda_2, \underset{\sim}{L}) = \sum_{k=0}^{\infty} \frac{\theta^k}{k!} f_k\, (\lambda_1, \lambda_2, \underset{\sim}{L})$$

$$= \int_{\mathcal{D}} |\lambda_1 \underset{\sim}{I} - \underset{\sim}{YY}'| \cdot |\lambda_2 \underset{\sim}{I} - \underset{\sim}{YY}'|$$

$$\cdot \exp \left\{ -\tfrac{1}{2} \,\mathrm{tr}\, \underset{\sim}{\Sigma}^{-1} \underset{\sim}{L} \begin{pmatrix} \lambda_1 & 0 & \\ 0 & \lambda_2 & \underset{\sim}{0} \\ & \underset{\sim}{0} & \underset{\sim}{YY}' \end{pmatrix} \underset{\sim}{L}' (\underset{\sim}{I} - \tfrac{1}{2}\, \theta\, \underset{\sim}{\Sigma}^{-1} \underset{\sim}{MM}') \right\} d\underset{\sim}{Y}$$

$$= \exp \{ -\tfrac{1}{2} \lambda_1\, \underset{\sim}{\ell}_1'\, \underset{\sim}{\Delta}\, \underset{\sim}{\ell}_1 - \tfrac{1}{2} \lambda_2\, \underset{\sim}{\ell}_2'\, \underset{\sim}{\Delta}\, \underset{\sim}{\ell}_2 \}$$

$$\cdot \int_{\mathcal{D}} |\lambda_1 \underset{\sim}{I} - \underset{\sim}{YY}'| \cdot |\lambda_2 \underset{\sim}{I} - \underset{\sim}{YY}'| \, \exp \{ -\tfrac{1}{2} \,\mathrm{tr}\, \underset{\sim}{L}_2'\, \underset{\sim}{\Delta}\, \underset{\sim}{L}_2 \underset{\sim}{YY}' \}\, d\underset{\sim}{Y} \quad ,$$

where

(3.8) $$\underset{\sim}{\Delta} = \underset{\sim}{\Sigma}^{-1} - \tfrac{1}{2}\, \theta\, \underset{\sim}{\Sigma}^{-1}\, \underset{\sim}{MM}'\, \underset{\sim}{\Sigma}^{-1} \quad .$$

Let $\underset{\sim}{H} \in O(p-2)$ such that $\int_{O(p-2)} d\underset{\sim}{H} = 1$.

Making transformation $\underset{\sim}{Y} \to \underset{\sim}{H}\underset{\sim}{Y}$, and notice that

$$\int_{O(p-2)} \exp\{-\tfrac{1}{2} \underset{\sim}{L}_2' \underset{\sim}{\Delta} \underset{\sim}{L}_2 \underset{\sim}{H}\underset{\sim}{Y}\underset{\sim}{Y}'\underset{\sim}{H}'\} \, d\underset{\sim}{H}$$

$$= {}_0F_0\,(-\tfrac{1}{2} \underset{\sim}{L}_2' \underset{\sim}{\Delta} \underset{\sim}{L}_2, \underset{\sim}{Y}\underset{\sim}{Y}')$$

$$= \sum_{k=0}^{\infty} \sum_{\kappa} (-1)^k C_\kappa(\tfrac{1}{2} \underset{\sim}{L}_2' \underset{\sim}{\Delta} \underset{\sim}{L}_2) C_\kappa\,(\underset{\sim}{Y}\underset{\sim}{Y}')\,\{k!\; C_\kappa(\underset{\sim}{I})\}^{-1}$$

and put

(3.9) $$\omega_i = \tfrac{1}{2} \underset{\sim}{\ell}_i' \underset{\sim}{\Delta} \underset{\sim}{\ell}_i \qquad i = 1, 2 \;.$$

Then (3.7) can be written

(3.7') $$f(\theta, \lambda_1, \lambda_2, \underset{\sim}{L}) =$$

$$e^{-\omega_1\lambda_1 - \omega_2\lambda_2} \sum_{k=0}^{\infty} \sum_{\kappa} (-1)^k C_\kappa(\tfrac{1}{2} \underset{\sim}{L}_2' \underset{\sim}{\Delta} \underset{\sim}{L}_2)\,\{k!\; C_\kappa(\underset{\sim}{I})\}^{-1}\, h_k(\lambda_1,\lambda_2) \;,$$

where

$$h_k(\lambda_1, \lambda_2) = \int_{\mathfrak{D}} |\lambda_1\underset{\sim}{I} - \underset{\sim}{Y}\underset{\sim}{Y}'| \cdot |\lambda_2\underset{\sim}{I} - \underset{\sim}{Y}\underset{\sim}{Y}'|\, C_\kappa(\underset{\sim}{Y}\underset{\sim}{Y}')\, d\underset{\sim}{Y} \;.$$

Using Lemma 2.2, and since $\alpha = 1$, then for $q \geq (p-2) + 1$, all coefficients in (2.7) vanish, so that the function reduces to a polynomial of degree $p - 2$. Hence

$$h_k(\lambda_1, \lambda_2) = \pi^{\frac{1}{2}(p-2)(n-2)} \Gamma_{p-2}(\tfrac{p+1}{2}) \left\{\Gamma_{p-2}(\tfrac{p+n-1}{2})\right\}^{-1} \tag{3.10}$$

$$\cdot \sum_{q=0}^{p-2} \sum_{\eta} \sum_{\delta} \frac{(-1)_\eta}{q!} g^{\delta}_{\kappa,\eta} \, C_\delta(\underset{\sim}{I}) (\tfrac{n-2}{2})_\delta$$

$$\cdot \left\{(\tfrac{p+n-1}{2})_\delta\right\}^{-1} \lambda_1^{p-q-2} \lambda_2^{\frac{1}{2}n(p-2)+q+k} \quad .$$

At this stage, we integrate

$$(\lambda_1\lambda_2)^{\frac{1}{2}(n-p-1)}(\lambda_1 - \lambda_2)\, e^{-\omega_1\lambda_1 - \omega_2\lambda_2} \lambda_1^{p-q-2} \lambda_2^{\frac{1}{2}n(p-2)+q+k}$$

i.e.
$$\lambda_1^{\xi-q} \lambda_2^{\zeta+q+k} (\lambda_1-\lambda_2)\, e^{-\omega_1\lambda_1-\omega_2\lambda_2}$$

with respect to λ_1 and λ_2 , where

$$\xi = \tfrac{1}{2}(p+n-5) \quad , \qquad \zeta = \tfrac{1}{2}(pn-n-p-1) \; .$$

First, integrating

$$\lambda_1^{\xi-q} \lambda_2^{\zeta+q+k} (\lambda_1 - \lambda_2)\, e^{-\omega_2\lambda_2}$$

with respect to λ_2 from 0 to λ_1 and using formula

$$\int_0^a x^{b-1} e^{-cx} dx = e^{-ca} \sum_{i=0}^{\infty} \frac{c^i a^{b+i} \Gamma(b)}{\Gamma(b+1+i)}$$

we obtain

$$\int_0^{\lambda_1} \lambda_1^{\xi-q} \lambda_2^{\zeta+q+k} (\lambda_1 - \lambda_2) e^{-\omega_2\lambda_2} d\lambda_2$$

$$= \sum_{i=0}^{\infty} \left\{\frac{\Gamma(v)}{\Gamma(v+1+i)} - \frac{\Gamma(v+1)}{\Gamma(v+2+i)}\right\} \omega_2^i \lambda_1^{s+i-1} e^{-\omega_2\lambda_1}$$

where

(3.11) $$v = \zeta + q + k + 1 , \quad s = \xi + \zeta + k + 3 .$$

Next, integrating $\lambda_1^{s+i-1} e^{-(\omega_1+\omega_2)\lambda_1}$ with respect to λ_1 from 0 to ∞, and notice that

$$\sum_{i=1}^{\infty} \frac{(1+i)\Gamma(v)}{\Gamma(v+2+i)} \Gamma(s+i) \left(\frac{\omega_2}{\omega_1+\omega_2}\right)^i$$

$$= \frac{\Gamma(v)\Gamma(s)}{\Gamma(v+2)} F(2, s, v+2; \frac{\omega_2}{\omega_1+\omega_2}) ,$$

where $F(\alpha, \beta, \gamma; x)$ is hypergeometric function defined as in [16]. Therefore,

$$\int_0^{\infty}\int_0^{\lambda_1} \lambda_1^{\xi-q} \lambda_2^{\zeta+q+k} (\lambda_1 - \lambda_2) e^{-\omega_1\lambda_1-\omega_2\lambda_2} d\lambda_2 d\lambda_1$$

$$= \Gamma(v)\, \Gamma(s)\, \{(\omega_1+\omega_2)^s\, \Gamma(v+2)\}^{-1} F(2, s, v+2; \frac{\omega_2}{\omega_1+\omega_2}) .$$

Substituting (3.10) into (3.7') we get $f(\theta, \lambda_1, \lambda_2, \underset{\sim}{L})$ and the coefficient of $\theta^k/k!$ from $f(\theta, \lambda_1, \lambda_2, L)$ gives $f_k(\lambda_1, \lambda_2, \underset{\sim}{L})$. Then using this value in (3.5'), we get the joint density function of $\underset{\sim}{\ell}_1$, $\underset{\sim}{\ell}_2$, λ_1, and λ_2. Integrating λ_1 and λ_2 we obtain the joint density function of $\underset{\sim}{\ell}_1$ and $\underset{\sim}{\ell}_2$. Hence we have the following theorem:

Theorem. Let the matrix $\underset{\sim}{X}(p \times n)$ be distributed as (3.1), and $\lambda_1 > \lambda_2 > 0$ be the two largest ordered ch. roots of $\underset{\sim}{X}\underset{\sim}{X}'$ and let $\underset{\sim}{L} = (\underset{\sim}{\ell}_1, \underset{\sim}{\ell}_2, \underset{\sim}{L}_2)$ where $\underset{\sim}{\ell}_1$ and $\underset{\sim}{\ell}_2$ are the two columns of $\underset{\sim}{L}$, corresponding to the two largest ordered ch. roots λ_1 and λ_2 of $\underset{\sim}{X}\underset{\sim}{X}'$, having random elements and the others $\underset{\sim}{L}_2$ depend on these random elements. Let the rank of $\underset{\sim}{M}$ be one. Then the joint density function of ℓ_1 and ℓ_2 is given by

$$|\underset{\sim}{\Sigma}|^{-\frac{1}{2}n}\, e^{-\frac{1}{2}\operatorname{tr}\underset{\sim}{\Sigma}^{-1}\underset{\sim}{M}\underset{\sim}{M}'} \sum_{k=0}^{\infty} \left\{k!\ (\tfrac{n}{2})_k\right\}^{-1} d\underset{\sim}{L}^{(p,2)}\, f_k(\underset{\sim}{L})$$

where $f_k(\underset{\sim}{L})$ satisfying

$$\begin{aligned} f(\theta,\underset{\sim}{L}) &= \sum_{k=0}^{\infty} \frac{\theta^k}{k!} f_k(\underset{\sim}{L}) \\ &= C_3 \sum_{k=0}^{\infty} \sum_{\kappa} (-1)^k\, C_\kappa(\tfrac{1}{2}\underset{\sim}{L}_2' \underset{\sim}{\Delta}\, \underset{\sim}{L}_2)\, \{k!\ C_\kappa(\underset{\sim}{I})\}^{-1} \\ &\cdot \sum_{q=0}^{p-2} \sum_{\eta} \sum_{\delta} (-1)_\eta\, g^{\delta}_{\kappa,\eta}\ C_\delta(\underset{\sim}{I})(\tfrac{n-2}{2})_\delta \Gamma(v)\Gamma(s) \\ &\cdot \left\{q!\ (\tfrac{p+n-1}{2})_\delta (\omega_1+\omega_2)^s \Gamma(v+2)\right\}^{-1} F(2,s,v+2;\ \frac{\omega_2}{\omega_1+\omega_2}) \,, \end{aligned}$$

where

$$(3.12) \qquad C_3 = \pi^2 \Gamma_{p-2}(\tfrac{p+1}{2}) \left\{ 2^{\frac{1}{2}pn} \Gamma_{p-2}(\tfrac{p+n-1}{2}) \Gamma_2(\tfrac{p}{2}) \Gamma_2(\tfrac{n}{2}) \right\}^{-1}$$

and v and s , $\underset{\sim}{\Delta}$ and ω_i are defined by (3.11), (3.8) and (3.9).

Note that the explicit expression will be obtained by evaluating the coefficient of $\theta^k/k!$ from $f(\theta, \underset{\sim}{L})$.

4. Remarks

(I). If $\underset{\sim}{M} = \underset{\sim}{0}$, then $\underset{\sim}{\Delta} = \underset{\sim}{\Sigma}^{-1}$ and $\underset{\sim}{Q}$ and $(\lambda_i = \alpha_i^2, \underset{\sim}{Y}, \underset{\sim}{L})$ are independently distributed and their respective density functions are given by

$$d\underset{\sim}{Q}^{(n,r)}$$

and

$$(4.1) \qquad C_1 |\underset{\sim}{\Sigma}|^{-\frac{1}{2}n} \prod_{i=1}^{r} \lambda_i^{\frac{1}{2}(n-p-1)} |\lambda_i \underset{\sim}{I} - \underset{\sim}{Y}\underset{\sim}{Y}'| \prod_{i<j} (\lambda_i - \lambda_j)$$

$$\exp\left\{-\tfrac{1}{2} \operatorname{tr} \underset{\sim}{\Sigma}^{-1} \underset{\sim}{L} \begin{pmatrix} \lambda_1 & & & 0 \\ & \ddots & & \\ & & \lambda_r & \\ 0 & & & \underset{\sim}{Y}\underset{\sim}{Y}' \end{pmatrix} \underset{\sim}{L}' \right\} d\underset{\sim}{L}^{(p,r)}$$

where C_1 is defined by (3.3).

For $r = 1$, integrating (4.1) with respect to $\lambda\ (=\lambda_1)$ and $\underset{\sim}{Y}$, we get the same density function of $\underset{\sim}{L}$ as given by Sugiyama [30] and Khatri and Pillai [20].

For $r = 2$, we obtain the joint density function of $\underset{\sim}{\ell}_1$ and $\underset{\sim}{\ell}_2$ is given by

$$(4.2) \qquad C_3|\underset{\sim}{\Sigma}|^{-\frac{1}{2}n} \sum_{k=0}^{\infty} \sum_{\kappa} (-1)^k\, 2^{-k}\, C_\kappa(\underset{\sim}{L}_2' \underset{\sim}{\Sigma}^{-1} \underset{\sim}{L}_2)\{k!\ C_\kappa(\underset{\sim}{I})\}^{-1}$$

$$\cdot \sum_{q=0}^{p-2} \sum_{\eta} \sum_{\delta} (-1)_\eta\, g_{\kappa,\eta}^{\delta}\, C_\delta(\underset{\sim}{I}) (\tfrac{n-2}{2})_\delta\, \Gamma(v)\Gamma(s)$$

$$\cdot \left\{q!\ (\tfrac{p+n-1}{2})_\delta (\omega_1 + \omega_2)^s \Gamma(v+2)\right\}^{-1} F(2,\ s,\ v+2;\ \frac{\omega_2}{\omega_1 + \omega_2})\ d\underset{\sim}{L}^{(p,2)}$$

where C_3 is defined by (3.12). (4.2) is a special case of the Theorem.

(II). Put $r = p$, integrating (4.1) with respect to $\underset{\sim}{L}$, we get the same distribution of ch. roots $\lambda_1, \ldots, \lambda_p$ of $\underset{\sim}{X}\underset{\sim}{X}'$ as given by James [16].

(III). If $n \leq p$, then in the all adequate formulas change the roles of p and n.

Chapter VI

MONOTONICITY OF THE POWER FUNCTIONS OF SOME TESTS OF HYPOTHESES CONCERNING MULTIVARIATE COMPLEX NORMAL DISTRIBUTIONS

1. Summary

Consider the test procedures invariant under certain groups of transformations [21]; (i) for testing the hypothesis $\Sigma_1 = \Sigma_2$ against one-sided alternatives, [5], [21], which is invariant under the transformation $X_j \to BX_j + b_j$, $j=1,2$, where X_j are distributed as multivariate normal, and B is any nonsingular matrix and b_1 and b_2 are any vectors; (ii) for testing the general multivariate linear hypothesis, [10], [21], which is invariant under the transformation $(X_1(p\times s),\ X_2(p\times(n-r)),\ X_3(p\times(r-s))) \to (BX_1F_1,\ BX_2F_2,\ BX_3F_3 + G)$ where B is nonsingular and F_1, F_2 and F_3 are orthogonal matrices; and (iii) for testing independence between two sets of normally distributed variates, [4], [21], which is invariant under the transformation $\begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} F$, where B_1, B_2 are nonsingular matrices of order p and q respectively, and F is orthogonal. In the real case, sufficient conditions on the procedure for the power function to be a monotonically increasing function of each of the parameters, for (i) are obtained by Anderson and Das Gupta [5]; for (ii), by Das Gupta, Anderson and Mudholkar [10]; and for (iii) by Anderson and Das Gupta [4]. Furthermore, for (ii) and (iii) Mudholkar [24] has shown that the power functions of the members of a class of invariant tests based on statistics, which are symmetric guage functions of increasing convex functions of the

maximal invariants, are monotone increasing functions of the relevant noncentrality parameters. The monotonicity of the power function of Roy's test has been shown by Roy and Mikhail [23], [29]. Further, Pillai and Jayachandran, [26], [27], have carried out exact power function comparisons for these tests based on four criteria for the two-roots case.

In Section 2, we derive some distributions in the complex case and in Section 3, prove a lemma, which helps to extend to the complex case, some results on convex sets in the real case. In Sections 4, 5 and 6 are briefly stated the theorems which can be proved from the real case with necessary changes, and finally, in Section 7 follows a discussion of special cases of tests: the likelihood-ratio test; Roy's maximum root test; and Hotelling's trace test for (i), (ii) and (iii).

2. Introduction and Notations

Matrices will be denoted by bold face capital letters and their dimensions will be indicated parenthetically. The pxp identity matrix will be denoted by I_p and zero matrix by 0. The complex conjugate of a matrix A will be denoted by $\bar{A}$, the transpose of A by A', and the conjugate transpose by A^*. The notation dA denotes the volume element associated with A. $U(pxn)$ will denote a semi-unitary matrix, where $UU^* = I_p$ for $p < n$ or $U^*U = I_n$ for $n < p$, and $U(nxn)$ is unitary matrix if $UU^* = U^*U = I_n$. The characteristic (ch.) roots of A will be denoted by $ch[A]$ and $ch_j[A]$ denotes the jth ordered characteristic root of A if A has real roots.

Let $\underset{\sim}{\xi}' = (Z_1,\ldots,Z_p)$ be a p-variate complex normal random variable such that the vector of real and imaginary parts $\underset{\sim}{\eta}' = (X_1,Y_1,\ldots,X_p,Y_p)$ is 2p-variate normal distributed, where $Z_j = X_j + i\,Y_j \quad j = 1,\ldots,P$. Then the distribution of $\underset{\sim}{\xi}$ was found by Wooding [31] and Goodman [13] and is given by

$$p(\underset{\sim}{\eta}) = p(\underset{\sim}{\xi}) = \pi^{-p}\,|\underset{\sim}{\Sigma}|^{-1}\, e^{-(\underset{\sim}{\xi} - \underset{\sim}{\nu})^{*}\underset{\sim}{\Sigma}^{-1}(\underset{\sim}{\xi} - \underset{\sim}{\nu})} \tag{2.1}$$

where $\underset{\sim}{\nu} = E[\underset{\sim}{\xi}]$ and $\underset{\sim}{\Sigma} = \Sigma_{\underset{\sim}{\xi}}$ (pxp) is a positive definite Hermitian matrix.

Now let $\underset{\sim}{Z}$(pxn) be a complex random matrix whose columns are independently distributed, each distributed as (2.1). Then the distribution of $\underset{\sim}{Z}$, is given by, [13], [16],

$$p(\underset{\sim}{Z};\underset{\sim}{\Sigma},n) = \pi^{-pn}\,|\underset{\sim}{\Sigma}|^{-n}\, e^{-\mathrm{tr}\,\underset{\sim}{\Sigma}^{-1}\,(\underset{\sim}{Z}-\underset{\sim}{\mu})(\underset{\sim}{Z}-\underset{\sim}{\mu})^{*}} \tag{2.2}$$

where $\underset{\sim}{\mu} = E\,[\underset{\sim}{Z}]$ is a matrix of pn complex parameters. In the more general case, $\underset{\sim}{Z}$(pxn) can be assumed to be distributed as

$$p(\underset{\sim}{Z};\underset{\sim}{\Sigma},n) = \pi^{-pn}\,|\underset{\sim}{\Sigma}|^{-n}\, e^{-\mathrm{tr}\,\underset{\sim}{\Sigma}^{-1}\,(\underset{\sim}{Z}-\underset{\sim}{\mu}\underset{\sim}{A})(\underset{\sim}{Z}-\underset{\sim}{\mu}\underset{\sim}{A})^{*}} \tag{2.3}$$

where $\underset{\sim}{A}$ is a known mxn matrix of rank r [assume $r \le \min(m,n-p)$] and $\underset{\sim}{\mu}$ is a pxm matrix of unknown parameters. If $\underset{\sim}{\mu} = \underset{\sim}{0}$, (2.2) and (2.3) reduce to

$$p(\underset{\sim}{Z};\underset{\sim}{\Sigma},n) = \pi^{-pn}\,|\underset{\sim}{\Sigma}|^{-n}\, e^{-\mathrm{tr}\,\underset{\sim}{\Sigma}^{-1}\,\underset{\sim}{Z}\underset{\sim}{Z}^{*}} \tag{2.4}$$

For later use, we use the same techniques as those in Roy's [28] to derive some distributions. Transform

$$A(m\times n) = \begin{pmatrix} \tilde{T}_1 \\ T_2 \end{pmatrix} U_1(r\times n)$$

where $\tilde{T}_1(r\times r)$ is nonsingular, T_2 is $(m-r)\times r$ matrix, and U_1 is a semi-unitary, i.e. $U_1U_1^* = I_r$. Let $U_2((n-r)\times n)$ be the completion of U_1. Then make the unitary transformation $\Delta = Z(U_1^*U_2^*) = (\zeta\ Z_2)$ say i.e. $Z = \zeta\ U_1 + Z_2U_2$ where ζ is $p\times r$ and Z_2 is $p\times(n-r)$ matrix.

Making unitary transormation again $\Delta_1 = \zeta(\nu_1'\nu_2') = (Z_1Z_3)$ say, where ν_1 is $s\times r$, ν_3 is $(r-s)\times r$, Z_1 is $p\times s$ and Z_3 is $p\times(r-s)$ matrix respectively, and $\begin{pmatrix} \nu_1 \\ \nu_3 \end{pmatrix}$ is unitary, then

$$\zeta = Z_1\nu_1 + Z_3\nu_3$$

Similarly put $\mu_1\ (p\times s) = \mu\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}\nu_1^*$, $\mu_3(p\times(r-s)) = \mu\begin{pmatrix} \tilde{T}_1 \\ T_2 \end{pmatrix}\nu_3^*$, then we have

$$(2.5)\quad p(Z_1,Z_2,Z_3) = \pi^{-pn}|\Sigma|^{-n} \exp[-\mathrm{tr}\ \Sigma^{-1}\{(Z_1-\mu_1)(Z_1-\mu_1)^* + Z_2Z_2^* + (Z_3-\mu_3)(Z_3-\mu_3)^*\}].$$

Put $\mu_1\mu_1^* = \begin{pmatrix} \S_1 \\ \S_2 \end{pmatrix} D_\theta(t\times t)\ (\S_1^*\S_2^*)$, and $\Sigma(p\times p) = \begin{pmatrix} \S_1 & \tilde{\S}_3 \\ \S_2 & \S_4 \end{pmatrix}\begin{pmatrix} \S_1^* & \S_2^* \\ \S_3^* & \S_4^* \end{pmatrix} = \S\S^*$

where $\S_1((p-t)\times t)$, $\S_2(t\times t)$, $\tilde{\S}_3((p-t)\times(p-t))$, and $\S_4(t\times(p-t))$; and $\S$ and $\tilde{\S}_3$ are nonsingular; and D_θ denotes the diagonal matrix with ch. roots $\theta_1 \geq \dots \geq \theta_t$ of $\mu_1\mu_1^*\Sigma^{-1}$ as its diagonal elements, and $t = \min(p,s)$.

Put $\mu_1 = \begin{pmatrix} \mu_1^{(1)} \\ \mu_1^{(2)} \end{pmatrix} = \begin{pmatrix} \S_1 \\ \S_2 \end{pmatrix} D_{\sqrt{\theta}}\,\phi\ (t\times s)$ where ϕ is determined by $\phi = D_{\sqrt{\theta}}^{-1}\ \S_2^{-1}\mu_1^{(2)}$ and $\phi\phi^* = I_t$ and complete $\phi^*(s\times t)$ into a unitary matrix $\Psi^*(s\times s)$. Finally, let

$$\underset{\sim}{\S}^{-1}\underset{\sim}{Z}_1\underset{\sim}{\Psi}^* = \underset{\sim}{V}, \quad \underset{\sim}{\S}^{-1}\underset{\sim}{Z}_2 = \underset{\sim}{W}$$

From (2.5) we obtain

$$(2.6) \quad p(\underset{\sim}{V},\underset{\sim}{W}) = \pi^{-p(n-r+s)} \exp\{-\mathrm{tr}(\underset{\sim}{W}\underset{\sim}{W}^*+\underset{\sim}{V}\underset{\sim}{V}^* - 2\mathrm{Re}\, \overline{\underset{\sim}{V}}\underset{\sim}{\Psi}_{J_\theta} + \underset{\sim}{\Psi}_\theta)\}$$

where $\underset{\sim}{\Psi}_\theta(p\mathrm{x}p) = \begin{pmatrix} \underset{\sim}{D}_\theta & \underset{\sim}{O} \\ \underset{\sim}{O} & \underset{\sim}{O} \end{pmatrix}$ and $\underset{\sim}{\Psi}_{J_\theta} = \begin{pmatrix} \underset{\sim}{D}_{J_\theta} & \underset{\sim}{O} \\ \underset{\sim}{O} & \underset{\sim}{O}_1 \end{pmatrix}$ and $\underset{\sim}{O}_1$ is $(s-t)\mathrm{x}(p-t)$ zero matrix.

If $\underset{\sim}{V} = (v_{jk})$ $j=1,\ldots,p$; $k=1,\ldots,s$, then (2.6) can be rewritten

$$(2.7) \quad p(\underset{\sim}{V},\underset{\sim}{W}) = \pi^{-p(n-r+s)} \exp\{-\mathrm{tr}\underset{\sim}{W}\overline{\underset{\sim}{W}}^* - \sum_{j=1}^{t} (v_{jj} - \theta_j^{\frac{1}{2}})(\overline{v}_{jj} - \theta_j^{\frac{1}{2}}) - \sum_{j=t+1}^{p} v_{jj}\overline{v}_{jj} - \sum_{\substack{j=1 \\ j\neq k}}^{p} \sum_{k=1}^{s} v_{jk}\overline{v}_{jk}\} .$$

3. Tests of Multivariate Linear Hypothesis.

Let random complex matrix $\underset{\sim}{Z}(p\mathrm{x}n)$ have density (2.3) and we wish to test the hypothesis H_o: $\underset{\sim}{\mu}\,\underset{\sim}{C} = \underset{\sim}{O}(p\mathrm{x}s)$ where $\underset{\sim}{C}$ is a known mxs matrix of rank $s(\leq r)$ such that $\underset{\sim}{\mu}\underset{\sim}{C}$ is estimable, against all alternatives. By Section 2, this problem can be transformed into the canonical form

$$\underset{\sim}{Z} \rightarrow (\underset{\sim}{Z}_1(p\mathrm{x}s),\ \underset{\sim}{Z}_2(p\mathrm{x}(n-r)),\ \underset{\sim}{Z}_3(p\mathrm{x}(r-s))),\ s \leq r \leq n-p$$

with expectations.

$$E[\underset{\sim}{Z}_1] = \underset{\sim}{\mu}_1(p\mathrm{x}s),\ E[\underset{\sim}{Z}_2] = \underset{\sim}{O}(p\mathrm{x}(n-r)),\ E[\underset{\sim}{Z}_3] = \underset{\sim}{\mu}_3(p\mathrm{x}(r-s)).$$

The hypothesis H_o is equivalent to the hypothesis $\underset{\sim}{\mu}_1 = \underset{\sim}{0}(pxs)$. The matrices of sums of products due to hypotehsis and due to error are given by $\underset{\sim}{S}_h = \underset{\sim}{Z}_1\underset{\sim}{Z}_1^*$ and $\underset{\sim}{S}_e = \underset{\sim}{Z}_2\underset{\sim}{Z}_2^*$ respectively. The problem is invariant under the transformation

$$(\underset{\sim}{Z}_1,\underset{\sim}{Z}_2,\underset{\sim}{Z}_3) \rightarrow (\underset{\sim}{B}\underset{\sim}{Z}_1\underset{\sim}{F}_1,\underset{\sim}{B}\underset{\sim}{Z}_2\underset{\sim}{F}_2,\underset{\sim}{B}\underset{\sim}{Z}_3\underset{\sim}{F}_3 + \underset{\sim}{G})$$

where $\underset{\sim}{B}$ is nonsingular and $\underset{\sim}{F}_1,\underset{\sim}{F}_2$ and $\underset{\sim}{F}_3$ are unitary matrices. These invariant test procedures depend on $C_1 \geq \ldots \geq C_p$, the ch. roots of $\underset{\sim}{S}_h\underset{\sim}{S}_e^{-1}$, and it is known [16] that the power function of any such test depends on the parameters $\theta_1,\ldots,\theta_t$ where $\theta_1 \geq \ldots \geq \theta_t$ are the possible nonzero ch. roots of $\underset{\sim}{\mu}_1\underset{\sim}{\mu}_1^* \underset{\sim}{\Sigma}^{-1}$ and t=min(p-s).

<u>Lemma 3.1</u>. Let $\underset{\sim}{\xi}' = (Z_1,\ldots,Z_p)$ and $\underset{\sim}{\eta}' = (X_1,Y_1,\ldots,X_p,Y_p)$ where $Z_j = X_j + iY_j$ j=1,...,p, and let T be a one-one transformation between $\underset{\sim}{\xi}$ and $\underset{\sim}{\eta}$ such that $T[\underset{\sim}{\xi}] = \underset{\sim}{\eta}$ with the following properties:

(1) $T[\underset{\sim}{\xi}_1 + \underset{\sim}{\xi}_2] = T[\underset{\sim}{\xi}_1] + T[\underset{\sim}{\xi}_2]$ and

(2) $T[a\underset{\sim}{\xi}] = aT[\underset{\sim}{\xi}]$ where a is a real number.

Let ω be a subset of $\underset{\sim}{\xi}$'s in p-dimensional comples sample space C^p; and Ω be its corresponding subset of $\underset{\sim}{\eta}$'s in the 2p-dimensional real sample space R^{2p}. If ω is convex in C^p and symmetric in $\underset{\sim}{\xi}$. Then Ω is convex in R^{2p} and symmetric in $\underset{\sim}{\eta}$ and conversely.

<u>Proof:</u> Let $\underset{\sim}{\eta}_1, \underset{\sim}{\eta}_2 \varepsilon \Omega$ then $T^{-1}[\underset{\sim}{\eta}_k] = \underset{\sim}{\xi}_k$ for some $\underset{\sim}{\xi}_k \varepsilon \omega$, k=1,2. Since ω is convex in C^p, hence $\alpha\underset{\sim}{\xi}_1 + (1-\alpha)\underset{\sim}{\xi}\varepsilon\omega$

$0 \leq \alpha \leq 1$, and $T[\alpha\underset{\sim}{\xi}_1 + (1-\alpha)\underset{\sim}{\xi}_2]\varepsilon\Omega$ i.e. $\alpha\underset{\sim}{\eta}_1 + (1-\alpha)\underset{\sim}{\eta}_2$ $\varepsilon\Omega$. This shows Ω is convex in R^{2p}.

Let Ω^- be the set of all $-\underset{\sim}{\eta}$ such that $\underset{\sim}{\eta}\varepsilon\Omega$. If any $-\underset{\sim}{\eta}\varepsilon\Omega^-$ then $\underset{\sim}{\eta}\varepsilon\Omega$ and $T^{-1}[\underset{\sim}{\eta}] = \underset{\sim}{\xi}$ for some $\underset{\sim}{\xi}\varepsilon\omega$. Since ω is symmetric in $\underset{\sim}{\xi}$, hence $\omega = \omega^-$, where ω^- is a set of all $-\underset{\sim}{\xi}$ for which $\underset{\sim}{\xi}\varepsilon\omega$, implies

$-\underset{\sim}{\xi}\,\epsilon\,\omega$, and then $T\,[-\underset{\sim}{\xi}]\,\epsilon\,\Omega$, i.e. $-\underset{\sim}{\eta}\,\epsilon\,\Omega$. Therefore $\Omega^{-} \subset \Omega$. Using the same argument, we can show $\Omega^{-} \supset \Omega$ and hence $\Omega = \Omega^{-}$.

Similarly for the converse.

<u>Theorem 3.1</u>. Let the random complex vectors $\underset{\sim}{\xi}_j$ $(j=1,\ldots,s)$ and the complex matrix $\underset{\sim}{\phi}$ be mutually independent, the distribution of $\underset{\sim}{\xi}_j$ being $N(\ell_j \underset{\sim}{\nu}_j, \underset{\sim}{\Sigma}_j)$ $j=1,\ldots,s$. If a set ω in the sample space is convex and symmetric in each $\underset{\sim}{\xi}_j$ given the other $\underset{\sim}{\xi}_h$'s and $\underset{\sim}{\phi}$. Then $Pr(\omega)$ decreases with respect to each $\ell_j (\geq 0)$.

<u>Proof</u>: Let $\underset{\sim}{\xi}_j' = (Z_{1j},\ldots,Z_{pj})$ and $\underset{\sim}{\eta}_j' = (X_{1j},Y_{1j},\ldots,X_{pj},Y_{pj})$ where $Z_{kj}=X_{kj} + i\,Y_{kj}$ $k=1,\ldots,p$; $j=1,\ldots,s$ and let Ω be the corresponding set of ω in the sample space R^{2p}. Then by Lemma 3.1 we know that Ω is convex and symmetric in each $\underset{\sim}{\eta}_j$. Denote

$$\mathcal{D} = \omega\{\underset{\sim}{\xi}_j \mid \underset{\sim}{\xi}_h,\ h\neq j,\ h=1,\ldots,s;\ \underset{\sim}{\phi}\} \text{ and}$$

$$D = \Omega\{\underset{\sim}{\eta}_j \mid \underset{\sim}{\eta}_h\ \ h\neq j,\ h=1,\ldots,s;\ \underset{\sim}{X},\underset{\sim}{Y}\}$$

where $\underset{\sim}{\phi} = \underset{\sim}{X} + i\,\underset{\sim}{Y}$. Since the $\underset{\sim}{\xi}_j$'s and $\underset{\sim}{\phi}$ are mutually independent, hence the $\underset{\sim}{\eta}_j$'s and $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ are mutually independent (but $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ are not independent). Define $p_j(\underset{\sim}{\eta}_j)$ to be the density of $N(\underset{\sim}{0},\underset{\sim}{\Sigma}_j)$ at $\underset{\sim}{\eta}_j$. Then by Theorem 1 of [10] , we have

$\int_D p_j(\underset{\sim}{\eta}_j + \ell_j \underset{\sim}{\xi}_j)\,d\underset{\sim}{\eta}_j \geq \int_D p_j(\underset{\sim}{\eta}_j + \ell_j^{0} \underset{\sim}{\xi}_j)\,d\underset{\sim}{\eta}_j$ where $0 \leq \ell_j \leq \ell_j^{0}$, $\nu_j' = (\nu_{1j},\ldots,\nu_{pj})$, $\nu_{kj} = \alpha_{kj} + i\ \beta_{kj}$ and $\underset{\sim}{\xi}_j' = (\alpha_{1j},\beta_{1j},\ldots,\alpha_{pj},\beta_{pj})$ $k=1,\ldots,p$; $j=1,\ldots,s$. But $\int_{\mathcal{D}} p_j(\underset{\sim}{\xi}_j + \ell_j \underset{\sim}{\nu}_j)\,d\underset{\sim}{\xi}_j = \int_D p_j(\underset{\sim}{\eta}_j + \ell_j \underset{\sim}{\zeta}_j)\,d\underset{\sim}{\eta}_j$ and $\int_{\mathcal{D}} p_j(\underset{\sim}{\xi}_j + \ell_j^{0} \underset{\sim}{\nu}_j)\,d\underset{\sim}{\xi}_j = \int_D p_j(\underset{\sim}{\eta}_j + \ell_j^{0} \underset{\sim}{\xi}_j)\,d\underset{\sim}{\eta}_j$,

hence

$$\int_{\mathcal{D}} p_j(\underset{\sim}{\xi}_j + \ell_j \underset{\sim}{\nu}_j)\,d\underset{\sim}{\xi}_j \geq \int_{\mathcal{D}} p_j(\underset{\sim}{\xi}_j + \ell_j^{0} \underset{\sim}{\nu}_j)\,d\underset{\sim}{\xi}_j. \tag{3.1}$$

Multiplying both sides of inequality (3.1) by the joint density of the temporarily fixed variables and integrating with respect to them we obtain $\Pr\{\omega \mid \ell_1, \ldots, \ell_j, \ldots, \ell_s\} \geq \Pr\{\omega \mid \ell_1, \ldots, \ell_j^o, \ldots, \ell_s\}$ for $0 \leq \ell_j \leq \ell_j^o$ and any ℓ_h's $(h \neq j)$.

Theorem 3.2. If the acceptance region of an invariant test is convex in the space of each column vector of $\underset{\sim}{V}$ for each set of fixed values of $\underset{\sim}{W}$ (see equation (2.6)) and of the other column vectors of $\underset{\sim}{V}$, then the power of the test increases monotonically in each θ_j.

The proof of the above theorem is as straight forward as [10].

Corollary 3.1. If the acceptance region of an invariant test is convex in $\underset{\sim}{V}$ for each fixed $\underset{\sim}{W}$, then the power of the test increases monotonically in each θ_j.

Lemma 3.2. For any Hermitian matrix $\underset{\sim}{H}(n\times n)$ the region

$$\mathcal{D} = \{\underset{\sim}{A}(n\times s) \mid ch_1[\underset{\sim}{A}\underset{\sim}{A}^*\underset{\sim}{H}] \leq \lambda\}$$

is convex in $\underset{\sim}{A}$.

Proof: Since the Cauchy-Schwarz inequality is also valid for complex vectors, hence the proof is as straight forward as Lemma 1 of [10].

Corollary 3.2. The maximum root test of Roy, the acceptance region of which is given by

$$ch_1[(\underset{\sim}{V}\underset{\sim}{V}^*)(\underset{\sim}{W}\overline{\underset{\sim}{W}}^*)^{-1}] \leq \lambda \quad ,$$

has a power function which is monotonically increasing in each θ_j.

The proof of the above corollary follows from Corollary 3.1 and Lemma 3.2.

Let $c_1 \geq \dots \geq c_p$ be the ch. roots of $(\underset{\sim}{V}\underset{\sim}{V}^*)(\underset{\sim}{W}\underset{\sim}{\bar{W}}^*)^{-1}$, and $d_j = 1+c_j$ $(j=1,\dots,p)$. Let Q_k be the sum of all different products of $d_1,\dots,d_p$ taken k $(k=1,\dots,p)$ at a time. Consider a complex matrix $\underset{\sim}{M}(p\times n)=(\underset{\sim}{M}_1,\dots,\underset{\sim}{M}_n)$ where $\underset{\sim}{M}_\ell$'s are the column vectors of $\underset{\sim}{M}$. Define $Q_k(\underset{\sim}{M})$ as the sum of all k-rowed principal minors of $\underset{\sim}{M}\underset{\sim}{M}^* + \underset{\sim}{I}_p$, or equivalently as the sum of all different products of ch. roots of $\underset{\sim}{M}\underset{\sim}{M}^* + \underset{\sim}{I}_p$ taken k at a time.

<u>Theorem 3.3</u>. An invariant test having acceptance region $\sum_{k=1}^{p} a_k Q_k \leq \lambda$ (a_k's ≥ 0) has a power function which is monotonically increasing in each θ_j.

The proof of Theorem 3.3 is analogous to that of Theorem 4 in [10].

In the real case, Das Gupta, Anderson and Mudholkar [10] have given another sufficient condition on the acceptance region. The same is true for the complex case, we only state the corresponding theorem, because the proof is quite similar in [10] with minor changes.

<u>Theorem 3.4</u>. For each j $(j=1,\dots,s)$ and for each set of fixed values of $\underset{\sim}{V}_k$'s $(k\neq j)$ and $\underset{\sim}{W}$, suppose there exists a unitary transformation: $\underset{\sim}{V}_j \to \underset{\sim}{U}\underset{\sim}{V}_j = \underset{\sim}{V}_j^{o} = (V_{1j}^{o},\dots,V_{pj}^{o})'$ such that the region $\omega_j(\underset{\sim}{V}_j)$ is transformed into the region $\omega_j^{o}(\underset{\sim}{V}_j^{o})$ which has the following property: Any section of $\omega_j^{o}(\underset{\sim}{V}_j^{o})$ for fixed values of $V_{\ell j}^{o}$ $(\ell\neq k)$ is a region symmetric about $V_{kj}^{o} = 0$. Then the power function of the test, having the acceptance region ω, monotonically increases in each θ_j.

4. Tests of Independence Between Two Sets of Variates.

Consider a $(p+q)x(n+1)$ complex random matrix $\underset{\sim}{Z}$, $(p \leq q,\ p+q < n+1)$ whose column vectors $\underset{\sim}{Z}_j$'c $(j=1,\ldots,n+1)$ are independently distributed as a $(p+q)$-variate complex normal distribution $N(\underset{\sim}{\nu},\underset{\sim}{\Sigma})$ where $\underset{\sim}{\Sigma}((p+q)x(p+q))$ is positive definite Hermitian and be partitioned as follows:

$$\underset{\sim}{\Sigma} = \begin{pmatrix} \underset{\sim}{\Sigma}_{11} & \underset{\sim}{\Sigma}_{12} \\ \underset{\sim}{\Sigma}^{*}_{12} & \underset{\sim}{\Sigma}_{22} \end{pmatrix},$$

where $\underset{\sim}{\Sigma}_{11}$, $\underset{\sim}{\Sigma}_{12}$ and $\underset{\sim}{\Sigma}_{22}$ are pxp, pxq and qxq matrices.

Consider the problem of testing the hypothesis

$$H_o: \quad \underset{\sim}{\Sigma}_{12} = \underset{\sim}{0}\ (pxq)$$

against all alternatives. Let the sample covariance matrix be $\underset{\sim}{S}$ which is similarly partitioned as

$$\underset{\sim}{S} = \begin{pmatrix} \underset{\sim}{S}_{11} & \underset{\sim}{S}_{12} \\ \underset{\sim}{S}^{*}_{12} & \underset{\sim}{S}_{22} \end{pmatrix}$$

where $n\,\underset{\sim}{S} = \underset{\sim}{Z}\underset{\sim}{Z}^{*} - (n+1)\ \underset{\sim}{Z}^{o}\underset{\sim}{Z}^{o*}$ and $\underset{\sim}{Z}^{o} = \sum_{j=1}^{n+1} \underset{\sim}{Z}_j \,/\, (n+1)$. This problem is invariant under transformations

$$\underset{\sim}{Z} \rightarrow \begin{pmatrix} \underset{\sim}{B}_1 & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{B}_2 \end{pmatrix} \underset{\sim}{Z}\,\underset{\sim}{U} \qquad \underset{\sim}{Z}_j \rightarrow \underset{\sim}{Z}_j + \underset{\sim}{b} \qquad j=1,\ldots,n+1,$$

where $\underset{\sim}{B}_1$ and $\underset{\sim}{B}_2$ are nonsingular matrices of order p and q respectively, and $\underset{\sim}{U}$ is unitary. A test procedure which is invariant under these transformations depends only on the ch. roots $r_1^2 \geq \ldots \geq r_p^2$ of

$S_{11}^{-1} S_{12} S_{22}^{-1} S_{12}^{*}$. For convenience let us denote $e_j = r_j^2$ $(j=1,\ldots,p)$. The power function of such a test depends only on the ch. roots $\rho_1^2 \geq \cdots \geq \rho_p^2$ of $\Sigma_{11}^{-1} \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{12}^{*}$ which are the squares of the possible nonzero population canonical correlation coefficients [16]. The distribution of the ch $[S_{11}^{-1} S_{12} S_{22}^{-1} S_{12}^{*}]$ is the same as the distribution of the ch $[(\xi\xi^{*})^{-1}(\xi\zeta^{*})(\zeta\zeta^{*})^{-1}(\zeta\xi^{*})]$ where the density of the matrices ξ (pxn) $= (\xi_{jk})$ and ζ (qxn) $= (\zeta_{jk})$ can be given in the form

$$\pi^{-(p+q)n} \prod_{j=1}^{p} (1-\rho_j^2)^{-n}$$

$$\cdot\exp\{-\sum_{j=1}^{p} (1-\rho_j^2)^{-1} \sum_{k=1}^{n} [\xi_{jk}\bar{\xi}_{jk} + \zeta_{jk}\bar{\zeta}_{jk} - 2\rho_j \mathrm{Re}(\xi_{jk}\bar{\zeta}_{jk})]$$

$$- \sum_{j=p+1}^{q} \sum_{k=1}^{n} \zeta_{jk}\bar{\zeta}_{jk}\} \quad ,$$

or

$$(4.1) \qquad \pi^{-(p+q)n} \prod_{j=1}^{p} (1-\rho_j^2)^{-n}$$

$$\exp\{-\sum_{j=1}^{p} (1-\rho_j^2)^{-1} \sum_{k=1}^{n} (\xi_{jk}-\rho_j\zeta_{jk})(\bar{\xi}_{jk}-\rho_j\bar{\zeta}_{jk}) - \sum_{j=1}^{q} \sum_{k=1}^{n} \zeta_{jk}\bar{\zeta}_{jk}\} ,$$

and H_o holds if and only if $\rho_1 = \ldots = \rho_p = 0$.

From (4.1) we find that given ζ, the column vectors ξ_j's of ξ are independently distributed each according to a p-variate complex normal distribution with covariance matrix D which is a diagonal matrix with diagonal elements $1-\rho_1^2,\ldots,1-\rho_p^2$. The marginal distribution of ζ does not depend on ρ_j's. Moreover, the conditional expectation of $\S$ given ζ is $E[\S|\zeta] = A_1\zeta$ where $A_1(p\times q) = (\Delta\ 0)$ and Δ is the diagonal matrix with diagonal elements $\rho_1,\ldots,\rho_p$.

Define
$$S_h = (\S\zeta^*)(\zeta\zeta^*)^{-1}(\zeta\S^*)$$
$$S_e = (\S\S^*) - (\S\zeta^*)(\zeta\zeta^*)^{-1}(\zeta\S^*).$$

If e_j is the jth largest root of $(\S\S^*)^{-1}(\S\zeta^*)(\zeta\zeta^*)^{-1}(\zeta\S^*)$, then $e_j(1-e_j)^{-1}$ is the jth largest root of $S_hS_e^{-1}$. Thus the class of test procedures based on the ch$[(\S\S^*)^{-1}(\S\zeta^*)(\zeta\zeta^*)^{-1}(\zeta\S^*)]$ is the same as the class of test procedures based on the ch $[S_hS_e^{-1}]$. Let

$$V(p\times q) = B\S F, \qquad W(p\times(n-q)) = B\S G$$

where $B(p\times p)$ is nonsingular, and $F(n\times q)$ and $G(n\times(n-q))$ are such that

$$FF^* = \zeta^*(\zeta\zeta^*)^{-1}\zeta, \qquad GG^* = I_n - \zeta^*(\zeta\zeta^*)^{-1}\zeta.$$

Then the roots of $S_hS_e^{-1}$ are the same as the roots of $(VV^*)(WW^*)^{-1}$. The matrices B, F and G can be found to use the methods in Section 2, such that the conditional density of $V=(v_{jk})$ and $W=(w_{jk})$ given ζ is

$$(4.2)\quad \pi^{-pn}\exp\left\{-\mathrm{tr}(WW^*) - \sum_{j=1}^{p}(v_{jj}-\tau_j)(\bar{v}_{jj}-\tau_j) - \sum_{j=1}^{p}\sum_{\substack{k=1\\k\neq j}}^{q} v_{jk}\bar{v}_{jk}\right\},$$

where $\tau_1^2 \geq \ldots \geq \tau_p^2$ are the ch. roots of $A\zeta\ (A\zeta)^*\ D^{-1}$.

Theorem 4.1. An invariant test for which the acceptance region is convex in each column vector of $\underset{\sim}{V}$ for each fixed $\underset{\sim}{W}$ and fixed values of the other column vectors of $\underset{\sim}{V}$ has a power function which is monotonically increasing in each ρ_j.

The proof of the above theorem is similar to that of Anderson and Das Gupta [4] with necessary changes.

Let $c_1 \geq \ldots \geq c_p$ be the roots of $(\underset{\sim}{V}\underset{\sim}{V}^*)(\underset{\sim}{W}\underset{\sim}{W}^*)^{-1}$. Then $c_j = e_j(1-e_j)^{-1}$. Thus the relation $e_j \leq \lambda$ is equivalent to the relation $c_1 \leq \lambda(1-\lambda)^{-1} = \lambda^o$ (say). Let $d_j = 1+c_j$ $(j=1,\ldots,p)$ and let Q_k be the sum of all different products of $d_1,\ldots,d_p$ taken k at a time $(k=1,\ldots,p)$. In particular,

$$Q_p = \prod_{j=1}^{p} d_j = \prod_{j=1}^{p} (1-e_j)^{-1}.$$

The following theorem is obtained from Section 3 and Theorem 4.1.

Theorem 4.2. A test having the acceptance region $\sum_{j=1}^{p} a_j Q_j \leq \lambda$ $(a_j\text{'s} \geq 0)$ has a power function which is monotonically increasing in each ρ_j.

5. Symmetric Gauge Functions and Convex Functions of Matrices

A real valued function

$$\psi(a) = \psi(a_1,\ldots,a_p)$$

on the p-dimensional space of p-tuples of real numbers is said to be a gauge function if

(1) $\Psi(a_1,\ldots,a_p) \geq 0$ with equality if and only if $a_1 = \ldots = a_p = 0$.

(2) $\Psi(ca_1,\ldots,ca_p) = |c|\Psi(a_1,\ldots,a_p)$ for any real number c.

(3) $\Psi(a_1 + b_1,\ldots,a_p + b_p) \leq \Psi(a_1,\ldots,a_p) + \Psi(b_1,\ldots,b_p)$.

$\Psi(G)$ is said to be a symmetric gauge function if, in addition to (1), (2) and (3), it also satisfies

(4) $\Psi(\varepsilon_1 a_{j1},\ldots,\varepsilon_p a_{jp}) = \Psi(a_1,\ldots,a_p)$ where $\varepsilon_j = \pm 1$ for all j and $j_1,\ldots,j_p$ is a permutation of $1,\ldots,p$.

Let $\underset{\sim}{A}(p\times n)$, $p \leq n$ be a complex matrix, then $\underset{\sim}{A}\underset{\sim}{A}^*$ is Hermitian and all its ch. roots are non-negative. Let $\alpha_1 \geq \ldots \geq \alpha_p$ be its ordered roots. For any increasing convex function f on the positive half of the real line and any symmetric gauge function Ψ of p variables, define

$$||\underset{\sim}{A}||_{\Psi,f} = \Psi(f(\alpha_1^{\frac{1}{2}}),\ldots,f(\alpha_p^{\frac{1}{2}})).$$

<u>Theorem 5.1</u>. $||\underset{\sim}{A}||_{\Psi,f}$ is a convex function of $\underset{\sim}{A}$.

The proof is analogous to Theorem 4 of [24] with minor changes.

Let $c_1 \geq \ldots \geq c_p$ be the ch. roots of $\underset{\sim}{S}_h\underset{\sim}{S}_e^{-1}$ in Section 3 and let $\mathfrak{D} = \mathfrak{D}(c_1,\ldots,c_p)$ be a region in the space of $c_1,\ldots,c_p$.

<u>Theorem 5.2</u>. The power function of an invariant test, which accepts the general multivariate linear hypothesis over $\mathfrak{D}$: $\psi(f(c_1^{\frac{1}{2}}),\ldots,f(c_p^{\frac{1}{2}})) \leq \lambda$, where ψ, f and λ are, respectively a symmetric gauge function of p variables, an increasing convex function on the positive half of the real line and a constant determined by the significance level of the test, is a monotonically increasing function in each θ_j.

The proof follows that of Theorem 5 of [24] with necessary changes.

Now let $e_1 \geq \cdots \geq e_p$ be the ch. roots of $(\underset{\sim}{\S}\underset{\sim}{\S}^*)^{-1}(\underset{\sim}{\S}\underset{\sim}{\zeta}^*)$ $\cdot(\underset{\sim}{\zeta}\underset{\sim}{\zeta}^*)^{-1}(\underset{\sim}{\zeta}\underset{\sim}{\S}^*)$ in Section 5, and let $c_j = e_j(1-e_j)^{-1}$ $j=1,\ldots,p$. Then we have, in view of Theorem 4.1, the following:

Theorem 5.3. The power of an invariant test which accepts the independence hypothesis over $\mathfrak{D}$, increases monotonically in each population canonical correlation coefficient ρ_j $(j=1,\ldots,p)$.

6. Tests of the Equality of Two Covariance Matrices

Samples of size N_1 and N_2 are drawn from $N(\underset{\sim}{\nu}_1,\underset{\sim}{\Sigma}_1)$ and $N(\underset{\sim}{\nu}_2,\underset{\sim}{\Sigma}_2)$ respectively, where $N(\underset{\sim}{\nu}_j,\underset{\sim}{\Sigma}_j)$ $j=1,2$ are (2.1). On the basis of these data we wish to test the null hypothesis:

$$H_o: \ \underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$$

Since the null hypothesis is invariant under the transformations

$$\underset{\sim}{\S}_j \to \underset{\sim}{B}\underset{\sim}{\xi}_j + \underset{\sim}{b}_j \qquad j=1,2$$

where $\underset{\sim}{\xi}_j$ are distributed as (2.1) and $\underset{\sim}{B}$ is any non-singular matrix and $\underset{\sim}{b}_1$ and $\underset{\sim}{b}_2$ are any vectors. As in the real case, it is known [16] that the power of any invariant test depends on the parameters only through the ch. roots $\gamma_1 \geq \cdots \geq \gamma_p$ of $\underset{\sim}{\Sigma}_1\underset{\sim}{\Sigma}_2^{-1}$. The null hypothesis can then be restated as

$$H_o: \gamma_1 = \cdots = \gamma_p = 1$$

In this chapter we consider the following alternatives

$$H_1: \ \gamma_j \geq 1 \quad j=1,\ldots,p \qquad \sum_{j=1}^{p} \gamma_j > p$$

or

$$H_2:\ \gamma_j \leq 1 \quad j=1,\ldots,p \qquad \sum_{j=1}^{p} \gamma_j < p \ .$$

Consider only the problem of testing H_o against H_1 (for H_o against H_2, we consider the test procedures having the above acceptance regions as rejection regions, then the power of such a test will decrease as each ordered root of $\Sigma_1\Sigma_2^{-1}$ increase.)

Theorem 6.1. Let $Z(p\times n)$, $p \leq n$, be a complex random matrix having density (2.4) and let $c_1 \geq \ldots \geq c_p$ be the ch. roots of ZZ^* and ω be a set in the space of $c_1,\ldots,c_p$ such that when a point $(c_1,\ldots,c_p)$ is in ω so is every point $(c_1^o,\ldots,c_p^o)$ for $c_j^o \leq c_j$ $(j=1,\ldots,p)$. Then the probability of the set ω depends on Σ only through ch $[\Sigma]$ and is a monotonically decreasing function of each of the ch. roots of Σ.

Theorem 6.2. Let Z_1 and Z_2 are independently distributed as (2.4) i.e. $p(Z_1,\Sigma_1, n_1)$ and $p(Z_2;\Sigma_2,n_2)$ respectively, and let ω be a set in the space of ch. roots of $(Z_1Z_1^*)\ (Z_2Z_2^*)^{-1}$ [here also called the c_j's] satisfying the condition stated in Theorem 6.1. Then the probability of ω depends on Σ_1 and Σ_2 only through ch $[\Sigma_1\Sigma_2^{-1}]$ and is a monotonically decreasing function of each of the ch. roots of $\Sigma_1\Sigma_2^{-1}$.

The proof of the above two theorems are analogous to those of Theorem 1 and 2 in [5] with necessary changes.

Corollary 6.1. If an invariant test has an acceptance region such that if $(c_1,\ldots,c_p)$ is in the region, so is $(c_1^o,\ldots,c_p^o)$ for

$c_j^o \leq c_j$ $(j=1,\ldots,p)$, then the power of the test is a monotonically increasing function of each γ_j.

Corollary 6.2. If $g(c_1,\ldots,c_p)$ is monotonically increasing in each of the arguments, a test with acceptance region $g(c_1,\ldots,c_p) \leq \lambda$ has a monotonically increasing power function in each γ_j.

7. Remarks

The following discussion of special cases of tests generalizes to the complex case, the results of previous authors in the real case.

(I) The likelihood-ratio test for (ii) and (iii) has the acceptance regions of the form

$$\prod_{j=1}^{p} (1+c_j) \leq \lambda_1.$$

The power function of such test is monotonically increasing in each of the parameters, for (ii) guaranteed by Theorem 3.3, and for (iii) by Theorem 4.2. However, for test (i), it is very difficult to investigate tests with reasonable power against all alternatives, because the acceptance region of such a test is

$$g(c_1,\ldots,c_p) = \prod_{j=1}^{p} \frac{(1+c_j)^{n_1+n_2}}{c_j^{n_1}} \leq \lambda_2$$

and $g(c_1,\ldots,c_p)$ is an increasing function of $c_1,\ldots,c_p$ or not, depending on the values of degrees of freedom n_1 and n_2.

(II)For Roy's maximum root test, the acceptance regions for (i) to (iii) are of the form

$$c_1 \leq \lambda_3.$$

The power function of such test is monotonically increasing in each of the parameters, for (i) guaranteed by Corollary 6.2; for (ii) and (iii) by Corollary 3.2.

(III)For Hotelling's trace test, the acceptance regions for (i) to (iii) are of the form

$$\sum_{j=1}^{p} c_j \leq \lambda_4.$$

The power function of such test is monotonically increasing in each of the parameters, for (i) guaranteed by Corollary 6.2; for (ii) by Theorem 3.3 and for (iii) by Theorem 4.2.

CHAPTER VII

SUMMARY AND CONCLUSION

1. Summary

In the first four chapters, the distribution problems considered are generally those of (1) characteristic roots of a single sample covariance matrix, (2) a matrix from two-sample case, and (3) both (1) and (2) in the complex situation. The primary objective has been to give an asymptotic expansion of the distribution of the characteristic roots in the two-sample case, from which the one-sample expansion is obtained as a limiting case. The distribution of characteristic roots of one or two-sample case each depends on a definite integral over the group of orthogonal (or unitary) matrices. This integral defines a function of the characteristic roots of both the population covariance matrix and the sample covariance matrix. To approximate this integral, two different cases are considered. In Chapter I, all population roots are assumed to be distinct and in Chapter II, not all population roots are distinct. Chapters III and IV deal with the same problem in the complex situation, but we omit the case of not all population roots to be distinct, since it is easy to derive it from the real case. The main idea used here is to localize a whole integral in the neighborhood of the identity elements of the orthogonal (or unitary) group and then map them into the Euclidean space. The mappings $\underset{\sim}{H} = \exp \underset{\sim}{S}$ and $\underset{\sim}{U} = \exp (i\underset{\sim}{H})$ are well known and their properties allow us to develop the integral as a power series in increasing powers of n^{-1}, where n for one-sample case is sample size less one, for two-sample case is the sum of two-sample sizes less two, and then evaluate it asymptotically.

In Chapter V, we have considered some Jacobian problems, and the distribution of the characteristic vectors corresponding to the two largest roots of a matrix for the non-central linear case.

In Chapter VI is discussed the monotonicity property of the power functions of three tests based on some criteria, and also some special cases.

2. Suggestions for Further Research

Several problems which are closely related to this dissertation which still remain to be solved are listed below.

(i) The third or succeeding terms (for all population roots are distinct) and the second or succeeding terms (for not all population roots are distinct) in the asymptotic expansion need to be investigated so that the effect of neglecting terms in the asymptotic approximation can be measured.

(ii) Any subset of adjacent roots of p population roots in the two-sample case discussed in Chapter II can be equal, for instance: $a_1 = \cdots = a_k < a_{k+1} < \cdots < a_{k+t} = a_{k+t+1} = \cdots = a_p$. In this case the theorems proved in Chapter II do not apply and a new method should be formulated.

(iii) The approach developed in the first four chapters could be used to find an asymptotic expansion for other distribution problems for example, MANOVA and canonical correlation, which involve a definite integral.

(iv) Estimation problems based on the asymptotic expansions obtained in the two-sample case need to be investigated.

(v) Extend the distribution of the characteristic vectors corresponding to the first k largest roots of a matrix for the non-central case.

LIST OF REFERENCES

[1] Anderson, G. A. (1965). An asymptotic expansion for the distribution of the latent roots of the estimated covariance matrix. Ann. Math. Statist. 36, 1153-1173.

[2] Anderson, T. W. (1951). The asymptotic distribution of certain characteristic roots and vectors. Proc. Second Berkeley Symp. Math. Statist. Prob. 103-130. Univ. of California Press.

[3] Anderson, T. W. (1963). Asymptotic theory for principal component analysis. Ann. Math. Statist. 34, 122-148.

[4] Anderson, T. W. and Das Gupta, S. (1964). Monotonicity of the power functions of some tests of independence between two sets of variates. Ann. Math. Statist. 35, 206-208.

[5] Anderson, T. W. and Das Gupta, S. (1964). A monotonicity property of the power functions of some tests of the equality of two covariance matrices. Ann. Math. Statist. 35, 1059-1063.

[6] Chang, T. C. (1968). On an asymptotic representation of the distribution of the characteristic roots of $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$. Mimeo. Series No. 165. Department of Statistics, Purdue University.

[7] Chang, T. C. (1969). On an asymptotic representation of the distribution of the characteristic roots of $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ when roots are not all distinct. Mimeo. Series No. 180. Department of Statistics, Purdue University.

[8] Constantine, A. G. (1963). Some non-central distribution problems in multivariate analysis. Ann. Math. Statist. 34, 1270-1285.

[9] Constantine, A. G. (1966). The distribution of Hotelling's generalized T_0^2 . Ann. Math. Statist. 37, 215-225.

[10] Das Gupta, S. and Anderson, T. W. and Mudholkar, G. S. (1964). Monotonicity of the power functions of some tests of the multivariate linear hypothesis. Ann. Math. Statist. 35, 200-205.

[11] Erdélyi, A. (1956). Asymptotic Expansions. Dover, New York.

[12] Girshick, M. A. (1939). On the sampling theory of roots of determinantal equations. Ann. Math. Statist. 10, 203-224.

[13] Goodman, N. R. (1963). Statistical analysis based on a certain multivariate complex Gaussian distribution (An introduction). Ann. Math. Statist. 34, 152-177.

[14] James, A. T. (1961). The distribution of non-central means with known covariance. Ann. Math. Statist. 32, 874-882.

[15] James, A. T. (1961). Zonal polynomial of the real positive definite symmetric matrices. Ann. of Math. 74, 456-469.

[16] James, A. T. (1964). Distribution of matrix variates and latent roots derived from normal samples. Ann. Math. Statist. 35, 475-501.

[17] James, A. T. (1968). Tests of equality of latent roots of the covariance matrix. University of Adelaide.

[18] Khatri, C. G. (1965). Classical statistical analysis based on a certain multivariate complex Gaussian distribution. Ann. Math. Statist. 36, 98-114.

[19] Khatri, C. G. (1967). Some distribution problems connected with the characteristic roots of $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$. Ann. Math. Statist. 38, 944-948.

[20] Khatri, C. G. and Pillai, K. C. S. (1968). Distributions of vectors corresponding to the largest roots of three matrices. Mimeo. Series No. 160, Department of Statistics, Purdue University.

[21] Lehmann, E. L. (1959). Testing Statistical Hypotheses. Wiley New York.

[22] Littlewood, D. E. (1940). The Theory of Group Characters. 2nd Ed. Oxford, 1958.

[23] Mikhail, W. F. (1962). On a property of a test for the equality of two normal dispersion matrices against one-sided alternatives. Ann. Math. Statist. 33, 1463-1465.

[24] Mudholkar, G. S. (1965). A class of tests with monotone power functions for two problems in multivariate statistical analysis. Ann. Math. Statist. 36, 1794-1801.

[25] Murnaghan, F. D. (1938). The Theory of Group Representations. The Johns Hopkins Press, Baltimore.

[26] Pillai, K. C. S. and Jayachandran, K. (1967). Power comparisons of tests of two multivariate hypotheses based on four criteria. Biometrika 54, 195-210.

[27] Pillai, K. C. S. and Jayachandran, K. (1968). Power comparisons of tests of equality of two covariance matrices based on four criteria. Biometrika 55, 335-342.

[28] Roy, S. N. (1958). Some Aspects of Multivariate Analysis. Wiley and Sons, New York.

[29] Roy, S. N. and Mikhail, W. F. (1961). On the monotonic character of the power functions of two multivariate tests. Ann. Math. Statist. 32, 1145-1151.

[30] Sugiyama, T. (1966). On the distribution of the largest latent root and the corresponding latent vector for principal component analysis. Ann. Math. Statist. 37, 995-1001.

[31] Wooding, R. A. (1956). The multivariate distribution of complex normal variates. Biometrika 43, 212-215.