

THE LIMITING DISTRIBUTION OF THE MAXIMUM TERM
IN A SEQUENCE OF RANDOM VARIABLES DEFINED
ON A MARKOV CHAIN*

by

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0. SUMMARY

Gnedenko's classical work [1] on the limit of the distribution of the maximum of a sequence of independent random variables is extended to the distribution of the maximum of a sequence of random variables defined on a finite Markov chain.

1. INTRODUCTION

Let $\{J_n, n \geq 0\}$ be a finite, irreducible Markov chain on the state space $\{1, \dots, m\}$ with transition probability matrix $P = \{P_{ij}\}$ and let X_1, X_2, \dots be a sequence of random variables which are conditionally independent, given the Markov chain $\{J_n\}$. Furthermore we assume that:

$$P\{X_n \leq x | J_{n-1} = i, J_n = j\} = F_{ij}(x), \quad n \geq 1$$

independently of n . For a pair (i, j) such that $P_{ij} = 0$, $F_{ij}(x)$ is arbitrary. We set:

$$Q_{ij}(x) = p_{ij} F_{ij}(x) = P\{X_n \leq x, J_n = j | J_{n-1} = i\}, \quad n \geq 1. \quad (1)$$

If the random variables X_n are nonnegative and nontrivial, they may be considered as the sojourn times of a semi-Markov process whose transition matrix is $Q(\cdot) = \{Q_{ij}(\cdot)\}$.

The semi-Markov process defined by Q is thought of as a process which remains a time X_n in each state J_{n-1} before jumping to state J_n , with a transition into state J_0 assumed to have occurred at time zero.

Given a matrix R of real constants, the probability that, given the initial state i , the Markov chain $\{J_n\}$ is in state j after the n th transition and that for all n transitions the corresponding X -variable did not exceed the constant $R_{k\ell}$ whenever the transition was from state k to state ℓ is given by:

$$P(J_n = j, X_v \leq R_{J_{v-1}, J_v} (v = 1, \dots, n) | J_0 = i) = [(Q(R))^n]_{ij}, \quad (2)$$

i.e., the (i,j) th element of the n th matrix power of the matrix whose (k,ℓ) th element is $Q_{k\ell}(R_{k\ell})$. For $n = 1$, this formula is just (1). Induction on n gives the rest quite simply.

The matrix power here is the analogue of the ordinary real-number power of the distribution function in the formula for the distribution of the maximum of a sequence of independent, identically distributed random variables. It is our purpose to extend Gnedenko's classical work on the limits of such distributions [1] to the present case.

2. THE APERIODIC CASE

Our principal result is the following:

Theorem 1. Suppose we are given an irreducible, $m \times m$ matrix of transition distributions, as defined above, in which the Markov chain J_n is aperiodic, so that

$$\lim_{n \rightarrow \infty} P^n = \Pi$$

where Π is a stochastic matrix with every row the same vector $\pi = (\pi_1, \dots, \pi_m)$. If a_{ijn} and b_{ijn} ($i, j = 1, \dots, m; n = 1, 2, \dots$) are real constants such that

$$\lim_{n \rightarrow \infty} [F_{ij}(a_{ijn}x + b_{ijn})]^n = \phi_{ij}(x), \quad (3)$$

let T_n ($n = 1, 2, \dots$) be $m \times m$ matrices such that

$$[T_n]_{ij} = Q_{ij}(a_{ijn}R_{ij} + b_{ijn})$$

Then

$$\lim_{n \rightarrow \infty} T_n^n = \chi^\Pi$$

where χ is the scalar $\prod_{i,j=1}^m [\phi_{ij}(R_{ij})]^{\pi_i P_{ij}}$.

The $\phi_{ij}(\cdot)$ may be some of the possible extreme value distributions given by Gnedenko or may be improper. The result, in intuitive terms, is that certain events which are dependent for finite n , become asymptotically independent as $n \rightarrow \infty$. The limiting probability sought is merely the limiting probability of being in state j multiplied by a factor representing the limiting probability of never exceeding any of the

$a_{ijn} R_{ij} + b_{ijn}$. Since each state is visited infinitely often, the factor χ is a product of the ϕ_{ij} 's, but with exponents to correct for the fact that $a_{ijn} R_{ij} + b_{ijn}$ is the argument associated in (3) with n visits to phase state (i,j) , whereas in n steps of the semi-Markov process in equilibrium only an expected number $\pi_i P_{ij}^n$ of visits are made.

By identifying the c_{ijn} with the $F_{ij}(a_{ijn} R_{ij} + b_{ijn})$ and the ϕ_{ij} with the $\phi_{ij}(R_{ij})$, the theorem may be stated more generally:

Theorem 1'. If P is an $m \times m$ stochastic matrix such that $\lim_{n \rightarrow \infty} P^n = \Pi$ where $\Pi_{ij} = \pi_j$, and T_n is the matrix defined by $[T_n]_{ij} = c_{ijn} P_{ij}^n$ where the $c_{ijn} \in [0,1]$ and $\lim_{n \rightarrow \infty} (c_{ijn})^n = \phi_{ij}$, then

$$\lim_{n \rightarrow \infty} T_n^n = \chi \Pi$$

where $\chi = \prod_{i,j=1}^m \phi_{ij}^{\pi_i P_{ij}}$.

Proof: We first note that if for any index pair (α, β) , $\phi_{\alpha\beta} = 0$ and $P_{\alpha\beta} > 0$, then $\lim T_n^n = 0$. For, consider the matrix A : $A_{ij} = P_{ij}$ if $(i,j) \neq (\alpha, \beta)$ and $A_{\alpha\beta} = P_{\alpha\beta}/2$. Since P is finite and irreducible, A is also, and so by the Frobenius theory of positive matrices (see for example [2], p. 475ff) the spectral radius of A , $\lambda(A) > 0$, is an eigenvalue associated with a left eigenvector with all positive components. From this, since $\sum_j A_{\alpha j} < 1$, it is easy to see that $\lambda(A) < 1$. Since for sufficiently large n , $[T_n]_{ij} \leq A_{ij}$ for all (i,j) , we have $\lambda(T_n) \leq \lambda(A) < 1$ for all sufficiently large n , so $T_n^n \rightarrow 0$.

If $P_{\alpha\beta} = 0$, then $F_{\alpha\beta}$ is irrelevant and we can arbitrarily define $\phi_{\alpha\beta} = 1$. Thus we only have to prove the theorem for the case $\phi_{ij} > 0$ for all (i,j) . In this case, since $c_{ijn}^n \rightarrow \phi_{ij}$ we can say $c_{ijn} = 1 + \lambda_{ij}/n + o(1/n)$, where, in fact, $\lambda_{ij} = \log \phi_{ij}$. Thus we can express

$$T_n = P + \frac{\Gamma}{n} + o\left(\frac{1}{n}\right)$$

where Γ is the constant matrix with elements $\Gamma_{ij} = \lambda_{ij} P_{ij}$ and $o(1/n)$ is a matrix whose norm is $o(1/n)$. The norm of a matrix is here the maximum of the absolute row sums.

Next we point out that in the limit the term $o(1/n)$ in T_n can be omitted. Writing $A_n = P + \Gamma/n$, we have $\|A_n\| \leq 1$, for sufficiently large n , since Γ has all non-positive elements. Then

$$\begin{aligned} \|T_n^n - A_n^n\| &\leq \sum_{j=1}^n \binom{n}{j} \|A_n\|^{n-j} \left(\frac{o(1)}{n}\right)^j \\ &\leq \sum_{j=1}^n \frac{[o(1)]^j}{j!} \leq e^{o(1)} - 1 \rightarrow 0. \end{aligned}$$

To determine the limit of $(P + \frac{\Gamma}{n})^n$, consider the expansion

$$\left(P + \frac{\Gamma}{n}\right)^n = \sum_{j=0}^n u_j$$

where $u_0 = P^n$, $u_1 = P^{n-1} \frac{\Gamma}{n} + P^{n-2} \frac{\Gamma}{n} P + \dots + \frac{\Gamma}{n} P^{n-1}, \dots$, and u_j

consists of $\binom{n}{j}$ terms of the form $p^{m_0} \frac{\Gamma}{n} p^{m_1} \frac{\Gamma}{n} \dots \frac{\Gamma}{n} p^{m_j}$ ($m_0 + m_1 + \dots + m_j = n-j$).

Let U_j^* be $\binom{n}{j} \frac{\Gamma}{n} \frac{\Gamma}{n} \dots \frac{\Gamma}{n}$, where there are j factors of Γ/n . Since $\Pi^i = \Pi$ for all $i > 0$, U_j^* resembles, though is not exactly the same as, U_j with Π substituted for p .

Since Π has all rows alike, straightforward matrix multiplication yields $\Pi\Pi = \kappa\Pi$, $\Pi\Pi\Pi = \kappa^2\Pi$, etc., where κ is the scalar $\sum_{ij} \pi_i \Gamma_{ij}$. Thus $U_j^* = \binom{n}{j} \left(\frac{\kappa}{n}\right)^j \Pi$, and $\lim_{n \rightarrow \infty} \sum_{j=0}^n U_j^* = e^{\kappa\Pi}$. Since $\Gamma_{ij} = P_{ij} \log \phi_{ij}$, $e^{\kappa} = \prod_{i,j=1}^m \phi_{ij}^{\pi_i P_{ij}}$, which gives for $\lim \Sigma U_j^*$ the result predicted for $\lim T_n^n$.

The remainder of the proof consists of showing that for sufficiently large n , the substitution of ΣU_j^* for ΣU_j introduces an error $\Delta = \left| \left| \Sigma U_j - \Sigma U_j^* \right| \right|$ which is less than ϵ . To this end, for any $\eta > 0$, choose K so that $\left| p^i - \Pi \right| < \eta$ if $i \geq K$. Now let us classify the terms of U_j into "good terms", in which all the exponents of the P 's, m_0, \dots, m_j , are at least K , and "bad terms", in which $m_i < K$ for at least one i .

Of the $\binom{n}{j}$ terms in U_j , $\binom{n - (j+1)K}{j}$ are good terms, and the essence of our argument is that the good terms in the expansion ΣU_j correspond closely to terms in ΣU_j^* and contribute small errors to Δ , but that for sufficiently large n , they constitute most of the terms. (In this discussion, to eliminate special cases and to simplify the limits on many sums, we use the binomial coefficient and factorial power symbols $\binom{m}{i}$ and $m^{(i)}$ with their customary meanings if $m \geq 0$

since $n^j > n^{(j)} > n^{(j)} - [n - (j+1)K]^{(j)}$. The second sum is the tail of a convergent series, so J can be chosen to make the second sum less than $\epsilon/3$ uniformly in n . With J fixed, we can take n sufficiently large to guarantee that in the first sum $n - (j+1)K > 0$ so that $n^{(j)} - [n - (j+1)K]^{(j)}$ is a polynomial of degree $j-1$. The first sum is then $O(1/n)$ and can be made less than $\epsilon/3$ by taking n sufficiently large. Thus we have shown Δ to consist of three terms, each less than $\epsilon/3$, which completes the proof.

3. THE PERIODIC CASE

In the case of a process with Markov chain J_n periodic with period d we do not have convergence of P^n to a single matrix Π , but rather $\lim_{n \rightarrow \infty} P^{nd} = \Pi_0$ and $\lim_{n \rightarrow \infty} P^{nd+i} = \Pi_i = \Pi_0 P^i = P^i \Pi_0$ ($i = 1, \dots, d-1$).

Also the Cesaro limit exists:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n P^j = \Pi' = \frac{1}{d}(\Pi_0 + \dots + \Pi_{d-1})$$

where Π' has for every row the m -tuple (π_1', \dots, π_m') . An intuitive argument similar to the one after the statement of theorem 1 leads us to expect the following:

Theorem 2. Given a finite, irreducible matrix of transition distributions with Markov chain J_n periodic with period d , for T_n defined as in theorem 1 and the Π -matrices defined above, we have

$$\lim_{n \rightarrow \infty} (T_{nd+i})^{nd+i} = \chi \Pi_i \quad (i = 0, 1, \dots, d-1) ,$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n T_j^j = \chi \Pi'$$

where χ is now defined in terms of the Cesaro limit probabilities:

$$\chi = \prod_{i,j=1}^m [\phi_{ij}(R_{ij})]^{\pi_i' P_{ij}} .$$

Proof: Notice that with appropriate renumbering of states, P falls into blocks ${}^i P$ as

$$P = \begin{bmatrix} 0 & {}^1 P & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & {}^2 P & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & {}^3 P & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & {}^{d-1} P \\ {}^d P & 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix},$$

and $U = P^d$ falls into diagonal blocks ${}^i U$ ($i = 1, \dots, d$). Similarly,

T_{nd}^d , which we shall call V_n for short, falls into diagonal blocks

${}^i V_n$ and Π_0 into diagonal blocks ${}^i \Pi_0$. For each i , ${}^i U^n \rightarrow {}^i \Pi_0$ and

$[{}^i V_n]_{jk} = c_{jkn} [{}^i U]_{jk}$ where each c_{jkn} is some polynomial in the

$F_{\mu\nu} (a_{\mu\nu} R_{\mu\nu} + b_{\mu\nu})$ ($\mu, \nu = 1, \dots, m$) with coefficients in $[0, 1]$, and

so $\lim_{n \rightarrow \infty} (c_{jkn})^n$ exists. Thus ${}^i V_n$ satisfies the hypotheses of

Theorem 1, and for some $\chi_i \in [0, 1]$ we have ${}^i V_n^n \rightarrow \chi_i {}^i \Pi_0$.

We call Λ_0 the matrix of the diagonal blocks ${}^i \Lambda_0 = \chi_i {}^i \Pi_0$ so

we may say $V_n^n \rightarrow \Lambda_0$. Since $T_n = P + \frac{\Gamma}{n} + o(\frac{1}{n})$ and thus $\|T_n - P\| \rightarrow 0$,

we have $\lim_{n \rightarrow \infty} T_{nd}^{nd} P = \lim_{n \rightarrow \infty} T_{nd}^{nd+1} = \lim_{n \rightarrow \infty} P T_{nd}^{nd}$ so that $P \Lambda_0 = \Lambda_0 P$.

This matrix and Π_1 have the same block structure as P , but the top

block of $P \Lambda_0$ is ${}^1 (P \Lambda_0) = {}^1 P {}^2 \Lambda_0 = {}^1 P \chi_2 {}^2 \Pi_0 = \chi_2 {}^1 \Pi_1$ and the same

block of $\Lambda_0 P$ is ${}^1 (\Lambda_0 P) = {}^1 \Lambda_0 {}^1 P = \chi_1 {}^1 \Pi_0 {}^1 P = \chi_1 {}^1 \Pi_1$. Thus $\chi_1 = \chi_2$.

Similar examination of the other blocks gives $\chi_i = \chi_{i+1}$ ($i = 2, \dots, d-1$) so that $\Lambda_0 = \chi \Pi_0$ for some χ .

To determine χ , write $V_n = U + \frac{\Delta}{n} + o(\frac{1}{n})$. For each diagonal block we have established ${}^i V_n \rightarrow \chi {}^i \Pi_0$. But from the proof of the theorem 1'

we know $\chi = e^\kappa$ where κ is defined by ${}^i \Pi_0 \Delta {}^i \Pi_0 = \kappa {}^i \Pi_0$, and since this is true for each i , we have $\Pi_0 \Delta \Pi_0 = \kappa \Pi_0$ for the whole matrix.

Multiplying on the right by P and using the relationships $\Pi_{i+1} =$

$P \Pi_i = \Pi_i P$ and $\Pi' = \frac{1}{d}(\Pi_0 + \dots + \Pi_{d-1})$, we get $\Pi_0 \Delta \Pi_i = \kappa \Pi_i$ and so

$\Pi_0 \Delta \Pi' = \kappa \Pi'$, whence left-multiplication by P yields $\Pi' \Delta \Pi' = \kappa \Pi'$.

But $V_n = (P + \frac{\Gamma}{nd} + o(\frac{1}{n}))^d$ so $\Pi' V_n \Pi' =$

$$= \Pi' (P^d + \sum_{j=0}^{d-1} P^j \frac{\Gamma}{nd} P^{d-j} + o(\frac{1}{n})) \Pi'$$

$$= \Pi' + \Pi' \frac{\Gamma}{n} \Pi' + o(\frac{1}{n}). \quad \text{But also, } \Pi' V_n \Pi' =$$

$$= \Pi' (U + \frac{\Delta}{n} + o(\frac{1}{n})) \Pi' = \Pi' + \Pi' \frac{\Delta}{n} \Pi' + o(\frac{1}{n})$$

so that $\Pi' \Delta \Pi' = \Pi' \Gamma \Pi'$ and κ is also the solution of $\Pi' \Gamma \Pi' = \kappa \Pi'$,

which is $\kappa = \sum_{ij} \pi_i' \Gamma_{ij}$, which establishes the limit of T_{nd} .

Noting as before that $\|T_n - P\| \rightarrow 0$, the other limits follow trivially.

The results do not depend on particular state labeling, so the original numbering may be restored and the proof is complete.

but define them both to be zero for any $m < 0$.)

For a good term, writing $||\Gamma|| = \gamma$ and noting that $||\Pi|| = ||P^i|| = 1$, we have

$$\begin{aligned} & \left| \left| P^{m_0} \frac{\Gamma}{n} \dots \frac{\Gamma}{n} P^{m_j} - \Pi \frac{\Gamma}{n} \dots \frac{\Gamma}{n} \Pi \right| \right| \\ & \leq \left| \left| P^{m_0} \frac{\Gamma}{n} P^{m_1} \frac{\Gamma}{n} \dots \frac{\Gamma}{n} P^{m_j} - \Pi \frac{\Gamma}{n} P^{m_1} \frac{\Gamma}{n} \dots \frac{\Gamma}{n} P^{m_j} \right| \right| \\ & + \dots + \left| \left| \Pi \frac{\Gamma}{n} \dots \frac{\Gamma}{n} \Pi \frac{\Gamma}{n} P^{m_j} - \Pi \frac{\Gamma}{n} \dots \frac{\Gamma}{n} \Pi \frac{\Gamma}{n} \Pi \right| \right| < (j+1) n \left(\frac{\gamma}{n}\right)^j . \end{aligned}$$

Thus the total contribution to Δ of errors associated with good terms is

$$\begin{aligned} \Delta_g & < \sum_{j=0}^{\infty} \binom{n - (j+1)K}{j} (j+1) n \left(\frac{\gamma}{n}\right)^j \\ & \leq n \sum_{j=0}^{\infty} (j+1) \frac{\gamma^j}{j!} = n(1+\gamma)e^\gamma . \end{aligned}$$

So, given $\epsilon > 0$, choose K so that $||P^i - \Pi|| < \epsilon / (3(1+\gamma)e^\gamma)$ for all $i \geq K$, then $\Delta_g < \epsilon/3$, independent of n .

The norm of any term from U_j or U_j^* is less than $\left(\frac{\gamma}{n}\right)^j$, so the contribution to Δ from bad terms of the U_j 's and from terms of the U_j^* 's not accounted for in Δ_g at worst is

$$\begin{aligned} \Delta_b & < 2 \sum_{j=0}^{\infty} \left[\binom{n}{j} - \binom{n - (j+1)K}{j} \right] \left(\frac{\gamma}{n}\right)^j \\ & < 2 \sum_{j=0}^J \frac{n \binom{j}{j} - [n - (j+1)K] \binom{j}{j}}{j!} \left(\frac{\gamma}{n}\right)^j + 2 \sum_{j=J+1}^{\infty} \frac{\gamma^j}{j!} , \end{aligned}$$

4. REFERENCES

- [1] B. Gnedenko, "Sur la Distribution Limite du Terme Maximum d'une Série Aléatoire." Annals of Math. (second series) 44 (1943), 423-453.
- [2] S. Karlin, A First Course in Stochastic Processes. New York, Academic Press, 1966. xi + 502 pp.

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