LIMIT LAWS FOR MAXIMA OF A SEQUENCE OF

RANDOM VARIABLES DEFINED ON A MARKOV CHAIN*

by

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Limit Laws for Maxima of a Sequence of Random Variables Defined on a Markov Chain*

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Abstract

Consider the bivariate sequence of r.v.'s $\{(J_n, X_n), n \geq 0\}$ with $X_0 = -\infty$ a.s. The marginal sequence $\{J_n\}$ is an irreducible, aperiodic, m-state M.C., $m < \infty$, and the r.v.'s X_n are conditionally independent given $\{J_n\}$. Furthermore $P\{J_n=j, X_n \leq x | J_{n-1} = i\} = p_{ij}H_i(x) = Q_{ij}(x)$, where $H_1(\cdot), \ldots, H_m(\cdot)$ are c.d.f.'s. Setting $M_n = \max\{X_1, \ldots, X_n\}$, we obtain $P\{J_n=j, M_n \leq x | J_0=i\} = [\underset{\sim}{Q}^n(x)]_{i,j}$, where $\underset{\sim}{Q}(x) = \{Q_{ij}(x)\}$. The limiting behavior of this probability and the possible limit laws for $\{M_n\}$ are characterized:

Theorem: Let $\rho(x)$ be the Perron-Frobenius eigenvalue of $\underset{\sim}{Q}(x)$ for real x; then: a) $\rho(x)$ is a c.d.f. b) if for a suitable normalization $\{Q^n_{ij}(a_{ijn}x+b_{ijn})\}$ converges completely to a matrix $\{U_{ij}(x)\}$ whose entries are nondegenerate distributions, then $U_{ij}(x) = \pi_j \rho_U(x)$, where

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$\pi_j = \lim_{n\to\infty} p_{ij}^n$ and $\rho_U(x)$ is an extreme value distribution. c) the normalizing constants need not depend on i,j . d) $\rho^n(a_n x+b_n)$ converges completely to $\rho_U(x)$. e) The maximum M_n has a nontrivial limit law $\rho_U(x)$ iff $Q^n(x)$ has a nontrivial limit matrix $U(x) = \{U_{ij}(x)\} = \{\pi_j \rho_U(x)\}$ or equivalently iff $\rho(x)$ or the c.d.f. $\prod_{i=1}^{m} H_i^{\pi_i}(x)$ is in the domain of attraction of one of the extreme value distributions. Hence the only possible limit laws for $\{M_n\}$ are the extreme value distributions which generalizes the results of Gnedenko for the i.i.d. case.

I. Introduction

The limit laws for the maxima of a sequence of independent, identically distributed (i.i.d.) random variables were fully characterized by B.V. Gnedenko [3]. They are the so-called extreme value distributions. Precisely, if $\{X_n, n \geq 1\}$ is a sequence of i.i.d. random variables with distribution function $F(\cdot)$, let $M_n = \max(X_1, X_2, \ldots, X_n)$. Then if there exist normalizing constants $a_n > 0$ and b_n such that

$$P[a_n^{-1}(M_n - b_n) \leq x] = F^n(a_n x + b_n) \xrightarrow{c} \Phi(x)$$ where $\Phi(x)$ is a nondegenerate limiting distribution, then $\Phi(x)$ belongs to the type of one of the following three distributions:

$$\Lambda(x) = \exp\{-e^{-x}\} \qquad -\infty < x < \infty$$

$$\Phi_\alpha(x) = \begin{cases} 0 & x \leq 0 \\ \exp\{-x^{-\alpha}\} & x > 0 \end{cases}$$

$$\Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^{\alpha}\} & x \leq 0 \\ 1 & x > 0 \end{cases}$$

where α is a positive constant.

Consider the analogous problem for random variables defined on a finite Markov chain (M.C.) which are conditionally independent given the chain. Let $\{J_n, n \geq 0\}$ be an m-state M.C. whose transition matrix $P = \{p_{ij}\}$ is irreducible and aperiodic. The random variables $X_n, n \geq 1$, are conditionally independent given the M.C. $\{J_n\}$ and $P\{X_n \leq x | J_{n-1} = i\} = H_i(x)$.

The distributions $H_i(x)$, $i = 1,\ldots,m$ are assumed to be nondegenerate and honest $(H_i(+\infty) = +1)$. Let $M_n = \max\{X_1,\ldots,X_n\}$ and $\underset{\sim}{Q}(x) = \{p_{ij}H_i(x)\}$ $i,j = 1,2,\ldots,m$. The Q-matrix governs the system. (There is no loss of generality in allowing the distribution of X_n to depend only on J_{n-1} -Pyke [5, p 1751]. The case where the distribution of X_n depends on the pair (J_{n-1},J_n) can be reduced to this case.)

By induction we establish that:

$$Q^n_{ij}(x) = H_i(x)\sum_{k_1=1}^{m}\sum_{k_2=1}^{m}\cdots\sum_{k_{n-1}=1}^{m} p_{ik_1}H_{k_1}(x)p_{k_1k_2}\cdots p_{k_{n-2}k_{n-1}}H_{k_{n-1}}(x)p_{k_{n-1}j}$$

where $\underset{\sim}{Q}^n(x) = \{Q^n_{ij}(x)\}$ is the n-th power of the Q-matrix (In this paper, we are not concerned with matrix-convolution powers.) Using this formula and the conditional independence of the X_n , we get

$$(1.1)\quad P\{J_n = j, M_n \leq x \mid J_0 = i\} = Q^n_{ij}(x) .$$

We concern ourselves with the existence of normalizing constants $a_{ijn} > 0$ and b_{ijn} $i,j = 1,\ldots,m,$ $n \geq 1$ such that the expressions $P\{J_n = j,\ a^{-1}_{ijn}(M_n - b_{ijn}) \leq x \mid J_0 = i\} = Q^n_{ij}(a_{ijn}x + b_{ijn})$ converge to nondegenerate mass-functions $U_{ij}(x)$ at all continuity points of the latter and such that $\sum_{j=1}^{m} U_{ij}(x)$, $i = 1,\ldots,m$, is an honest distribution function. If such normalizing constants exist, what are the possible limit matrices $\{U_{ij}(\cdot)\}$?

Finally we establish basic properties of the normalizing constants a_{ijn} and b_{ijn} and discuss the limiting behavior of the marginal distribution of M_n.

2. Preliminaries

A semi-Markov matrix (S.M.M.) $\underset{\sim}{Q}(x) = \{Q_{ij}(x)\}$ is a matrix whose entries $Q_{ij}(x)$, i,j, = 1,...,m are mass functions such that $\sum_{j=1}^{m} Q_{ij}(+\infty) \leq 1$.

A S.M.M. is honest if for all i = 1,...,m, equality holds, otherwise it is dishonest. Unless otherwise specified all distribution functions and S.M.M.'s are honest.

Let $\{_n Q(x)\}$ be a sequence of S.M.M.'s. The sequence of S.M.M.'s converges completely to a limit matrix $\underset{\sim}{Q}(x)$ iff $\underset{\sim}{Q}(x)$ is honest and for each i,j $_nQ_{ij}(\cdot) \xrightarrow{w} Q_{ij}(\cdot)$. We write $_n\underset{\sim}{Q}(\cdot) \xrightarrow[=]{c} \underset{\sim}{Q}(\cdot)$.

A matrix analogue of the classical weak compactness theorem for distribution functions holds for S.M.M.s: Given a sequence of S.M.M.'s $_n\underset{\sim}{Q}(x)$, there exists a subsequence n_k and a limit S.M.M. $\underset{\sim}{Q}(x)$ (not necessarily honest) such that $_{n_k}\underset{\sim}{Q}(\cdot) \xrightarrow{w} \underset{\sim}{Q}(\cdot)$; that is, for i,j = 1,...,m,

$$_{n_k}Q_{ij}(\cdot) \xrightarrow{w} Q_{ij}(\cdot) .$$

Two S.M.M.'s $\underset{\sim}{U}(x)$, $\underset{\sim}{V}(x)$ are of the same type if there exist constants $A > 0$ and B such that for each i,j $V_{ij}(x) = U_{ij}(Ax+B)$. The following lemma of Khintchin is useful [2,p. 246]:

Lemma (2.1) Let $U(\cdot)$ and $V(\cdot)$ be two non-degenerate distribution functions. If for a sequence $\{F_n(\cdot)\}$ of distribution functions and constants $a_n > 0$, b_n and $\alpha_n > 0$, β_n :

(2.2) $$F_n(a_n x + b_n) \xrightarrow{w} U(x) \ , \ F_n(\alpha_n x + \beta_n) \xrightarrow{w} V(x)$$

Then:

(2.3) $$\frac{\alpha_n}{a_n} \longrightarrow A \neq 0 \ , \ \frac{\beta_n - b_n}{a_n} \longrightarrow B$$

and then

(2.4) $$V(x) = U(Ax + B)$$

Conversely if (2.3) holds, then each of the two relations (2.2) implies the other and (2.4) .

The set of normalizing constants $a_n > 0$, b_n ,$n \geq 1$ is <u>asymptotically equivalent</u> to the set of normalizing constants $\alpha_n > 0$,β_n , $n \geq 1$ iff

$$\frac{\alpha_n}{a_n} \longrightarrow 1 \ , \ \frac{\beta_n - b_n}{a_n} \longrightarrow 0$$

A S.M.M. $\underset{\sim}{Q}(x)$ is a non-negative matrix for every x ; hence the Perron-Frobenius theory is applicable. For a matrix $\underset{\sim}{A}$ with real entries, we write $\underset{\sim}{A} \geq \underset{\sim}{0}$ ($> \underset{\sim}{0}$) if $a_{ij} \geq 0$ ($a_{ij} > 0$) for each i,j . For a complex matrix $\underset{\sim}{B} = \{b_{ij}\}$, $|\underset{\sim}{B}|$ denotes the matrix $\{|b_{ij}|\}$. We use the following theorem [6, p.30]:

<u>Theorem 2.5</u> Let $\underset{\sim}{A} \geq \underset{\sim}{0}$ be an irreducible $m \times m$ matrix. Then:

1. $\underset{\sim}{A}$ has a simple, positive eigenvalue equal to its spectral radius $\rho_{\underset{\sim}{A}}$.
2. To the eigenvalue $\rho_{\underset{\sim}{A}}$ corresponds a positive eigenvector $\underset{\sim}{X} > 0$.
3. $\rho_{\underset{\sim}{A}}$ increases when any entry of $\underset{\sim}{A}$ increases. (If $\underset{\sim}{A}$ is reducible, then $\rho_{\underset{\sim}{A}}$ does not decrease when any entry of $\underset{\sim}{A}$ increases.)

<u>Theorem (2.6)</u> [6, pp. 28,47]: Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be two $m \times m$ matrices with $\underset{\sim}{0} \leq |\underset{\sim}{B}| \leq \underset{\sim}{A}$. Then $\rho_{\underset{\sim}{B}} \leq \rho_{\underset{\sim}{A}}$. If $\underset{\sim}{A}$ is irreducible then $\rho_{\underset{\sim}{B}} = \rho_{\underset{\sim}{A}}$ implies that $|\underset{\sim}{B}| = \underset{\sim}{A}$.

<u>Theorem (2.7)</u> [6, p. 13]: If $\underset{\sim}{A}$ is an $m \times m$ complex matrix, then $\underset{\sim}{A}^n \to \underset{\sim}{0}$ entrywise iff $\rho_{\underset{\sim}{A}} < 1$.

For fixed x, $\underset{\sim}{Q}(x)$ is a positive matrix whose spectral radius we denote by $\rho(x)$. <u>$\rho(x)$ is a distribution function.</u> $\underset{\sim}{Q}(+\infty)$ is stochastic; hence $\rho(+\infty) = 1$. $\underset{\sim}{Q}(-\infty) = \underset{\sim}{0}$; hence $\rho(-\infty) = 0$. $\rho(x)$ is nondecreasing by Theorem (2.5-3).

Furthermore:

<u>Lemma (2.8)</u> (1) If $\underset{\sim}{Q}(x)$ is (right, left) continuous at x_0, then $\rho(x)$ is (right,left) continuous at x_0.

(2) If $\rho(x)$ is right continuous at x_0 and $\underset{\sim}{Q}(x_0)$ is irreducible, then $\underset{\sim}{Q}(x)$ is right continuous at x_0. If $\rho(x)$ is left continuous at x_0 and $\underset{\sim}{Q}(x)$ is irreducible for $x > x_0 - \varepsilon$ for some $\varepsilon > 0$, then $\underset{\sim}{Q}(x)$ is left continuous at x_0.

<u>Proof:</u> (1) If $Q(x)$ is left continuous at x_0, select a sequence $x_n \uparrow x_0$. Then $\underset{\sim}{Q}(x_n) \longrightarrow \underset{\sim}{Q}(x)$ and hence $\rho(x_n) \uparrow \rho(x)$. Hence $\rho(x)$ is left continuous at x_0. Similarly for right continuity.

(2) Suppose $\rho(x)$ is left continuous at x_0. Choose a sequence $\{x_n\}$ such that $x_0 - \varepsilon < x_n \uparrow x_0$. Then $\underset{\sim}{Q}(x_n) \longrightarrow \underset{\sim}{Q}(x_0-) \leq \underset{\sim}{Q}(x_0)$. If there exists (i,j) such that $Q_{ij}(x_0-) < Q_{ij}(x_0)$ then $\rho(x_0-) < \rho(x_0)$ by Theorem (2.5-3), contradicting the left continuity of $\rho(x)$ at x_0. Similarly for right continuity.

Lemma (2.9): Let $\{_n\mathbf{Q}(\cdot)\}$ be a sequence of S.M.M.'s and $_n\mathbf{Q}(\cdot) \xrightarrow{c} \mathbf{Q}(\cdot)$. Then $\rho_n(\cdot) \xrightarrow{c} \rho(\cdot)$ where $\rho(x)$ and $\rho_n(x)$ are the spectral radii of $\mathbf{Q}(x)$ and $_n\mathbf{Q}(x)$ respectively.

Proof: Weak convergence of distribution functions is equivalent to pointwise convergence on a set everywhere dense on the real line, so $_n\mathbf{Q}(\cdot) \xrightarrow{c} \mathbf{Q}(\cdot)$ implies that for $x \varepsilon D$, $_nQ(x) \to Q(x)$; D is an everywhere dense subset of R . Hence for $x \varepsilon D$, $\rho_n(x) \to \rho(x)$ and hence $\rho_n(\cdot) \xrightarrow{w} (\cdot)$. But $\mathbf{Q}(\cdot)$ is honest, so $\mathbf{Q}(+\infty)$ is stochastic. Thus $\rho(+\infty) = 1$ and $\rho_n(\cdot) \xrightarrow{c} \rho(\cdot)$.

We can say more about the spectral properties of a S.M.M. $\mathbf{Q}(x)$. Suppose there exists $x_0 < \infty$ such that for $x > x_0$ $\mathbf{Q}(x)$ is irreducible. Now let $\mathbf{r}(x) = (r_1(x),\ldots,r_m(x))$, $\boldsymbol{\ell}(x) = (\ell_,(x),\ldots,\ell_m(x))$ be right and left eigenvectors of $\mathbf{Q}(x)$ corresponding to $\rho(x)$. The components of $\mathbf{r}(x)$ and $\boldsymbol{\ell}(x)$ can be chosen to be non-negative and for $x > x_0$ all components are then strictly positive (2.5-2) . As functions of x, $\mathbf{r}(x)$ and $\boldsymbol{\ell}(x)$ are only determined up to arbitrary factors, since for any scalar functions $k_1(x)$ and $k_2(x)$, $k_1(x)\mathbf{r}(x)$ and $k_2(x)\boldsymbol{\ell}(x)$ are also eigenvectors. In order to discuss continuity properties and limiting behavior of $\mathbf{r}(x)$ and $\boldsymbol{\ell}(x)$ we must specify a version of the eigenvectors.

Lemma (2.10): Let $\mathbf{Q}(x)$, $\mathbf{r}(x)$, $\boldsymbol{\ell}(x)$ be as above. Restrict attention to the domain $x > x_0$ where $\mathbf{Q}(x)$ is irreducible. We normalize $\mathbf{r}(x)$ and $\boldsymbol{\ell}(x)$ by: $\sum_{i=1}^{m} r_i(x) = \sum_{i=1}^{m} \ell_i(x) = 1$. Suppose $\mathbf{P} = \mathbf{Q}(+\infty)$ is primitive.

We have

(1) $\lim_{x\to\infty} \underset{\sim}{r}(x) = (m^{-1},\ldots,m^{-1})$

$\lim_{x\to\infty} \underset{\sim}{\ell}(x) = (\pi_1,\ldots,\pi_m)$ where $(\pi_1,\ldots,\pi_m)$ are the stationary probabilities associated with $\underset{\sim}{P}$. Also $\underset{\sim}{P}^n \longrightarrow \underset{\sim}{\Pi}$ where $\Pi_{ij} = \pi_j$

(2) If $\underset{\sim}{Q}(x)$ is (right, left) continuous at $x_1 > x_0$, then $\underset{\sim}{r}(x)$ and $\underset{\sim}{\ell}(x)$ are (right, left) continuous at x_1.

Proof: (1) $\underset{\sim}{r}(x)$ is in a compact set. For any sequence $x_n \uparrow +\infty$, $\{\underset{\sim}{r}(x_n)\}$ must have a convergent subsequence, say $\{\underset{\sim}{r}(x_{n_k})\}$. Suppose $\lim_{k\to\infty} \underset{\sim}{r}(x_{n_k}) = r = (r_1,\ldots,r_m)$. Since $\sum_{i=1}^{m} r_i = 1$, not all components of $\underset{\sim}{r}$ can vanish. Then $\lim_{k\to\infty} \underset{\sim}{Q}(x_{n_k})\underset{\sim}{r}(x_{n_k}) = \lim_{k\to\infty} \rho(x_{n_k})\underset{\sim}{r}(x_{n_k})$, so $\underset{\sim}{P}\underset{\sim}{r} = \underset{\sim}{r}$. Since P is stochastic and irreducible, its right eigenvector corresponding to Perron-Frobenius eigenvalue 1 is uniquely determined up to a factor and hence $r_i = m^{-1}$, $i = 1,\ldots,m$. Since every convergent subsequence of $\{\underset{\sim}{r}(x_n)\}$ converges to the same limit, $\lim_{n\to\infty} \underset{\sim}{r}(x_n) = (m^{-1},\ldots,m^{-1})$. Similarly for $\underset{\sim}{\ell}(x)$.

(2) Suppose $\underset{\sim}{Q}(x)$ is left continuous at x_1. Pick any sequence $\{x_n\}$ such that $x_0 < x_n \uparrow x_1$. Then $\underset{\sim}{Q}(x_n) \longrightarrow \underset{\sim}{Q}(x_1)$ and $\rho(x_n) \to \rho(x_1)$. By compactness, these exists a subsequence n_k and $\underset{\sim}{s} = (s_1,\ldots,s_m)$ such that $\sum_{i=1}^{m} s_i = 1$ and $\lim_{k\to\infty} \underset{\sim}{r}(x_{n_k}) = \underset{\sim}{s}$.

Hence $\lim_{k\to\infty} \underset{\sim}{Q}(x_{n_k})\underset{\sim}{r}(x_{n_k}) = \lim_{k\to\infty} \rho(x_{n_k})\underset{\sim}{r}(x_{n_k})$, i.e. $\underset{\sim}{Q}(x_1)\underset{\sim}{s} = \rho(x_1)\,\underset{\sim}{s}$. But since $\underset{\sim}{Q}(x_1)$ is irreducible $\underset{\sim}{s} = \underset{\sim}{r}(x_1)$. All convergent subsequences have the same limit; hence $\lim_{n\to\infty} \underset{\sim}{r}(x_n) = \underset{\sim}{r}(x_1)$. Similarly for $\underset{\sim}{\ell}(x)$ and for right continuity.

Now let $\underset{\sim}{Q}(x) = \{p_{ij}H_i(x)\}$ i,j, = 1,...,m where $\underset{\sim}{P} = \{p_{ij}\}$ is an irreducible, aperiodic, stochastic matrix and $\underset{\sim}{P}^n \longrightarrow \underset{\sim}{\Pi}$ and $H_1(\cdot),\ldots,H_m(\cdot)$ are nondegenerate distribution functions. There exist an integer k' such that $\underset{\sim}{P}^k > \underset{\sim}{0}$ for $k > k'$ and a real number x_0 , such that for $x > x_0$ $\min\{H_1(x),\ldots,H_m(x)\} > 0$. We may limit ourselves to the domain $x > x_0$ where $\underset{\sim}{Q}^k(x) > \underset{\sim}{0}$.

The conditions $\sum_{i=1}^m \ell_i(x)r_i(x) = 1$ and $\sum_{i=1}^m r_i(x) = 1$ determine a version of the right and left eigenvectors possessing the continuity properties and limiting behavior discussed in Lemma (2.10). This version can be obtained from the one satisfying $\sum_{i=1}^m r_i(x) = \sum_{i=1}^m \ell_i(x) = 1$ through the transformations $r_i(x) \longrightarrow \dfrac{r_i(x)}{\sum_{i=1}^m r_i(x)\ell_i(x)}$, i = 1,...,m . We assume henceforth that $\underset{\sim}{r}(x)$ and $\underset{\sim}{\ell}(x)$ are so normalized.

Form the matrix $\underset{\sim}{M}(x) = \{r_i(x)\ \ell_j(x)\}$, i,j = 1,...,m . It is known [4, p.248]:

(2.11) $\lim_{x\to\infty} \underset{\sim}{M}(x) = \underset{\sim}{\Pi}$

(2.12) $\underset{\sim}{M}^2(x) = \underset{\sim}{M}(x)$

(2.13) For any vector $\underset{\sim}{V} = (V_1,\ldots,V_m)$ we have:

$\underset{\sim}{M}(x)\underset{\sim}{V} = (\underset{\sim}{V}, \underset{\sim}{\ell}(x))\underset{\sim}{r}(x)$ and $\underset{\sim}{V}\underset{\sim}{M}(x) = (\underset{\sim}{V},\underset{\sim}{r}(x))\underset{\sim}{\ell}(x)$.

(2.14) $\underset{\sim}{Q}(x)\underset{\sim}{M}(x) = \underset{\sim}{M}(x)\underset{\sim}{Q}(x) = \rho(x)\underset{\sim}{M}(x)$

(2.15) $\lim_{n\to\infty} \rho(x)^{-n}\underset{\sim}{Q}^n(x) = \underset{\sim}{M}(x)$.

We examine (2.15) in detail. Set $\underset{\sim}{B}(x) = \underset{\sim}{Q}(x) - \rho(x)\underset{\sim}{M}(x)$. Then by (2.12) and (2.14) , we have $\underset{\sim}{B}^n(x) = \underset{\sim}{Q}^n(x) - \rho^n(x)\underset{\sim}{M}(x)$.

Theorem (2.15): Let $\underset{\sim}{Q}(x) = \{p_{ij}H_i(x)\}$, $\underset{\sim}{M}(x)$, $\underset{\sim}{B}(x)$ be as above. There exists a real number M such that $\lim_{n\to\infty} \underset{\sim}{B}^n(x) = \lim_{n\to\infty} [\underset{\sim}{Q}^n(x) - \rho^n(x)\underset{\sim}{M}(x)] = \underset{\sim}{Q}$ uniformly in $x > M$. Equivalently:

(2.17) $\underset{\sim}{Q}^n(x) = \rho^n(x)\underset{\sim}{M}(x) + \underset{\sim}{o}(1)$ where $\lim_{n\to\infty} \underset{\sim}{o}(1) = \underset{\sim}{0}$ uniformly in $x > M$.

Proof: We can show by induction that $|\underset{\sim}{B}^n| \leq |\underset{\sim}{B}|^n$ for integral n . Let $\underset{\sim}{E}$ be the $m \times m$ matrix $E_{ij} = 1$ and $\underset{\sim}{B}(x) = \{B_{ij}(x)\}$. Fix N , a positive integer such that $\max_{i,j}|p_{ij}^N - \pi_j| < m^{-1}$. Set $\alpha = \max_{i,j} |p_{i,j}^N - \pi_j|$. Pick $\varepsilon > 0$ such that $\alpha + \varepsilon < m^{-1}$. Since $\lim_{x\to\infty} \underset{\sim}{B}^n(x) = \underset{\sim}{P}^n - \underset{\sim}{\Pi}$, there exists M_N such that for $x > M_N$, $|B_{ij}^N(x)| \leq \alpha + \varepsilon, i,j=1,\dots,m$. Then $|\underset{\sim}{B}^N(x)| = \{|B_{ij}^N(x)|\} \leq (\alpha + \varepsilon)\underset{\sim}{E} < m^{-1}E$. The spectral radius of $\underset{\sim}{E}$ is m so the spectral radius of $(\alpha + \varepsilon)\underset{\sim}{E}$ is strictly less than 1 ; hence $((\alpha+\varepsilon)\underset{\sim}{E})^n \longrightarrow \underset{\sim}{0}$ as $n \to \infty$ by Theorems (2.6), (2.7) . So for $x > M_N$, $|\underset{\sim}{B}^N(x)|^n \longrightarrow \underset{\sim}{0}$ uniformly in x and since $|\underset{\sim}{B}^N(x)|^n \geq |\underset{\sim}{B}^{nN}(x)|$ we have that $|\underset{\sim}{B}^{nN}(x)| \xrightarrow{n\to\infty} \underset{\sim}{0}$ uniformly in $x > M_N$.

Now for any n , write

$$|\underset{\sim}{B}^n(x)| = |\underset{\sim}{B}^{[\frac{n}{N}]N}(x)\underset{\sim}{B}^{n-[\frac{n}{N}]N}(x)| \leq |\underset{\sim}{B}^{[\frac{n}{N}]N}(x)|\ |B^{n-[\frac{n}{N}]N}(x)| .$$

For any n , the second factor is one of the following: $|\underset{\sim}{B}^0(x)|, |\underset{\sim}{B}^1(x)|, \ldots, |\underset{\sim}{B}^{N-1}(x)|$. For $k = 1,2,\ldots,N-1$ there exist real numbers $M_1,\ldots,M_{N-1}$ such that $x > M_k$ implies $|\underset{\sim}{B}^k(x)| \leq \underset{\sim}{E}$. So for $X > M = \max\{M_1,\ldots,M_{N-1},M_N\}$ the second factor is bounded by $\underset{\sim}{E}$; the first factor approaches $\underset{\sim}{0}$ uniformly in $x > M$. This completes the proof.

We use the following lemma [1]:

<u>Lemma (2.18)</u>: Let $\underset{\sim}{P} = \{p_{ij}\}$ be an $m \times m$, irreducible, aperiodic, stochastic matrix such that $\lim_{n\to\infty} \underset{\sim}{P}^n = \underset{\sim}{\Pi}$. Suppose there are constants c_{ijn} with $0 \leq c_{ijn} \leq 1$, $n \geq 1$, $i,j = 1,2,\ldots,m$, such that $\lim_{n\to\infty} (c_{ijn})^n = \phi_{ij}$. Then:

$$\lim_{n\to\infty} \{c_{ijn}p_{ij}\}^n = [\prod_{i,j=1}^{m} \phi_{ij}^{\pi_i p_{ij}}]\underset{\sim}{\Pi}$$

3. Limit Laws

<u>Theorem (3.1): Limit Laws for the Q-Matrix:</u> Let $\underset{\sim}{Q}(x) = \{p_{ij}H_i(x)\}$ where $\underset{\sim}{P} = \{p_{ij}\}$ is irreducible, aperiodic, stochastic, $\lim_{n\to\infty} \underset{\sim}{P}^n = \underset{\sim}{\Pi}$ and $H_1(\cdot),\ldots,H_m(\cdot)$ are nondegenerate, honest distribution functions. If there exist $a_{ijn} > 0$ and b_{ijn} , $i,j = 1,2,\ldots,m$ and $n \geq 1$, such that

$$\{P[J_n = j,\ a_{ijn}^{-1}(M_n - b_{ijn}) \leq x | J_0 = i]\} = \{Q_{ij}^n(a_{ijn}x + b_{ijn})\} \xrightarrow{c} \{U_{ij}(x)\} .$$

where $U_{ij}(x)$ is nondegenerate, then

(1) $U_{ij}(x)$ is independent of i and is given by $\rho_U(x)\ \pi_j$; $\rho_U(x)$ is an honest, nondegenerate distribution function, the Perron-Frobenius eigenvalue of $\{U_{ij}(x)\}$.

(2) $\rho_U(x)$ is an extreme value distribution. In fact for all i,j

$$\rho^n(a_{ijn}x + b_{ijn}) \xrightarrow{c} \rho_U(x)$$

(3) a_{ijn} and b_{ijn} may be chosen independently of i,j . $\rho_U(x)$ is of the form $\prod_{i=1}^{m} \phi_i^{\pi_i}(x)$ where $\phi_i^{\pi_i}(x)$ is an honest distribution function such that $H_i^{n_k}(a_{n_k} x + b_{n_k}) \xrightarrow{c} \phi_i(x)$ for some subsequence n_k .

(4) The domain of attraction of $\rho_U(x)$ includes also $\prod_{i=1}^{m} H_1^{\pi_i}(x)$.

The proof of part (2) requires a lemma. We state it now but defer its proof until after the proof of Theorem 3.1 . Recall the representation $\underset{\sim}{Q}^n(x) = \rho^n(x)\underset{\sim}{M}(x) + \underset{\sim}{o}(1)$ where $\lim_{n\to\infty} \underset{\sim}{o}(1) = \underset{\sim}{0}$ uniformly in $x \in [K,\infty]$ for a suitably chosen K .

<u>Lemma 3.2</u>: If $\rho_U(x) > 0$ then: $\lim_{n\to\infty} M_{ij}(a_{ijn}x + b_{ijn}) = \pi_j$ for all i,j .

We can show more. If $\rho_U(x) > 0$ then:

(a) $\lim_{n\to\infty} H_i(a_{ijn}x + b_{ijn}) = 1$

(b,1) If there exists some i_0 such that $H_{i_0}(x) < 1$ for all x , then

$$\lim_{n\to\infty} a_{ijn}x + b_{ijn} = +\infty \text{ for all } i,j .$$

(b,2) If $H_i(x_i) = 1$ and $H_i(x_i - \varepsilon) < 1$ for all $\varepsilon > 0$, $i = 1,2,\dots,m$ and $x_0 = \max\ \{x_1,\dots,x_m\} < \infty$, then for x fixed

either (b,2,i) $a_{ijn}x + b_{ijn} > x_0$ for finitely many n and

$$\lim_{n\to\infty} a_{ijn}x + b_{ijn} = x_0$$

or (b,2,ii) $a_{ijn}x + b_{ijn} > x_0$ infinitely often and

$$\underset{\sim}{Q}^n(a_{ijn}x + b_{ijn}) \longrightarrow \underset{\sim}{\Pi} \text{ and } \rho_U(x) = 1 .$$

(Note in $\underset{\sim}{Q}^n(a_{ijn}x + b_{ijn})$ we evaluate each component $Q^n_{k\ell}(\cdot)$ at $a_{ijn}x + b_{ijn}$ for $k,\ell = 1,2,\dots,m$.)

<u>Proof of Theorem 3.1</u>: (1) We have:

$$\{Q_{ij}(a_{ijn}x + b_{ijn})\}^n = \{p_{ij}\ H_i(a_{ijn}x + b_{ijn})\}^n .$$

There exists a subsequence n_k such that for all i,j

$$H_i(a_{ijn_k}x + b_{ijn_k})^{n_k} \xrightarrow{w} \phi_{ij}(x) \quad \text{for distributions } \phi_{ij}(x) \text{ by the}$$

weak compactness theorem. For a given x, if there exists an index pair (i,j) such that $\phi_{ij}(x) = 0$, then $\{p_{ij}H_i(a_{ijn_k}x + b_{ijn_k})^{n_k} \longrightarrow \underset{\sim}{0}$ [1]. Since $\{p_{ij}H_i(a_{ijn_k}x + b_{ijn_k})^{n_k} \xrightarrow{c} \{U_{ij}(x)\}$ we have that, if for any (i,j) $U_{ij}(x) > 0$, then for all i,j $\phi_{ij}(x) > 0$. For x such that $\phi_{ij}(x) > 0$ for all i,j we have

$$\{p_{ij}H_i(a_{ijn_k}x + b_{ijn_k})\}^{n_k} \longrightarrow [\prod_{i,j=1}^{m} \phi_{ij}^{\pi_i p_{ij}}(x)]\ \underset{\sim}{\Pi}$$

by (2.18) and also

$$\{p_{ij}H_i(a_{ijn_k}x + b_{ijn_k})\}^{n_k} \longrightarrow \{U_{ij}(x)\}$$

so that

$$U_{ij}(x) = [{}_{i,}\prod_{j=1}^{m}\phi_{ij}^{\pi_i \cdot p_{ij}}]_{\pi_j} \quad \text{and } U_{ij}(x) \text{ is independent of } i \text{ .}$$

Set $\rho_U(x) = \prod_{i,j=1}^{m} \phi_{ij}^{\pi_i p_{ij}}(x)$. Then $\rho_U(x)$ is independent of the choice of subsequence and $\rho_U(x) = \sum_{j=1}^{m} U_{ij}(x)$ for all i and

(3.3) $\rho_U(x) > 0$ implies that $\phi_{ij} > 0$ for all i,j

Since $\sum_{j=1}^{m} U_{ij}(x) = \rho_U(x)$, we have $\rho_U(x)$ is honest and nondegenerate (by definition of complete convergence of S.M.M.'s). If $p_{ij} > 0$ for all i,j , we see that none of the $\phi_{ij}(\cdot)$ can be dishonest. This will be seen to hold true even if some of the p_{ij}'s vanish. Hence

$$H_i(a_{ijn_k}x + b_{ijn_k})^{n_k} \xrightarrow{c} \phi_{ij}(x) \text{ .}$$

At least one of the $\phi_{ij}(\cdot)$ is nondegenerate since if this were not the case $\rho_U(x)$ would be degenerate.

(3.4) Furthermore: $[\prod_{i,j=1}^{m} H_i^{\pi_i p_{ij}}(a_{ijn}x + b_{ijn})]^n \xrightarrow{c} \rho_U(x)$

since every convergent subsequence will converge to $\rho_U(x)$.

(2) For x such that $\rho_U(x) > 0$, we have

$$\lim_{n\to\infty} Q_{ij}^n(a_{ijn}x + b_{ijn}) = \lim_{n\to\infty} [\rho^n(a_{ijn}x + b_{ijn}) M_{ij}(a_{ijn}x + b_{ijn}) + o(1)] .$$

Therefore

$$\rho_U(x)\pi_j = \lim_{n\to\infty} \rho^n(a_{ijn}x + b_{ijn})\pi_j$$

by Lemma (3.2) and

(3.5) $$\rho_U(x) = \lim_{n\to\infty} \rho^n(a_{ijn}x + b_{ijn}) \quad \text{for all } i,j .$$

Therefore $\rho_U(x)$ is an extreme value distribution [3] .

(3) Since (3.5) holds for all i,j a_{ijn} and b_{ijn} may be chosen independently of i and j (Lemma 2.1)

For a suitably chosen subsequence n_k we have that

$$H_i(a_{ijn_k}x + b_{ijn_k})^{n_k} \xrightarrow{w} \phi_{ij}(x) .$$

Since a_{ijn} and b_{ijn} need not depend on i,j ,

$$H_i(a_{n_k}x + b_{n_k})^{n_k} \xrightarrow{w} \phi_{ij}(x) ;$$

Therefore $\phi_{ij}(\cdot)$ is independent of j . This implies that

$$\rho_U(x) = \prod_{i,j=1}^{m} \phi_{ij}^{\pi_i p_{ij}}(x) = \prod_{i=1}^{m} \phi_i^{\pi_i}(x) .$$

So each $\phi_i(\cdot)$ is honest and $H_i(a_{n_k}x + b_{n_k})^{n_k} \xrightarrow{c} \phi_i(x)$.

(4) Since $[\prod_{i,j=1}^{m} H_i^{\pi_i p_{ij}}(a_{ijn}x + b_{ijn})]^n \xrightarrow{c} \rho_U(x)$

by (3.4), we have

$$[\prod_{i,j=1}^{m} H_i^{\pi_i p_{ij}}(a_{ijn}x + b_{ijn})]^n = [\prod_{i=1}^{m} H_i^{\pi_i}(a_n x + b_n]^n \xrightarrow{c} \rho_U(x) .$$

So $\prod_{i=1}^{m} H_i^{\pi_i}(\cdot)$ is in the domain of attraction of $\rho_U(x)$.

It only remains to prove Lemma (3.2):

<u>Proof of Lemma (3.2)</u>: (a) We fix x such that $\rho_U(x) > 0$ and pick a subsequence n_k such that $H_i(a_{ijn_k}x + b_{ijn_k})$ converges. Suppose that $\lim_{k\to\infty} H_i(a_{ijn_k}x + b_{ijn_k}) = \ell$. There exists a further subsequence n_k' such that

$$H_i(a_{ijn_k'}x + b_{ijn_k'})^{n_k'} \longrightarrow \psi_{ij}(x)$$

and because of (3.3) and the assumption that $\rho_U(x) > 0$ we have $\psi_{ij}(x) > 0$. So taking logarithms:

$$n_k' \log \; H_i(a_{ijn_k'}x + b_{ijn_k'}) \longrightarrow \log \psi_i(x)$$

and therefore

$$\log \; H_i(a_{ijn_k'}x + b_{ijn_k'}) \longrightarrow 0$$

and

$$H_i(a_{ijn_k'}x + b_{ijn_k'}) \longrightarrow 1 .$$

This identifies $\ell = 1$ and since any convergent subsequence must converge to 1 we have the desired result.

(b,1) If $H_{i_0}(x) < 1$ for all x then $\rho(x) < 1$ for all x by (2.6)

and for all x $\lim_{n\to\infty} Q^n(x) = Q$ by (2.7) . Suppose $a_{ijn}x + b_{ijn}$ does

not converge to $+\infty$. Then there is a subsequence n_k and a real number K^0

such that $a_{ijn_k}x + b_{ijn_k} \leq K^0 < +\infty$ for all k .

Then

$$Q^{n_k}(a_{ijn_k}x + b_{ijn_k}) \leq Q^{n_k}(K^0) \longrightarrow 0 \text{ as } k \longrightarrow \infty.$$ In particular

$$Q_{ij}^{n_k}(a_{ijn_k}x + b_{ijn_k}) \longrightarrow 0 .$$ Since

$$Q_{ij}^{n_k}(a_{ijn_k}x + b_{ijn_k}) \longrightarrow \rho_U(x)\pi_j > 0$$ we have a contradiction.

For this case, since $\lim_{n\to\infty} a_{ijn}x + b_{ijn} = +\infty$, we have immediately

from (2.10) and the fact that $M_{ij}(x) = r_i(x)\, \ell_j(x)$ that

$$\lim_{n\to\infty} M_{ij}(a_{ijn}x + b_{ijn}) = \pi_j .$$

(b,2,i) If $a_{ijn}x + b_{ijn} > x_0$ for only finitely many n then

there exists a positive integer N_x such that if $n > N_x$ then

$a_{ijn}x + b_{ijn} \leq x_0$. Pick a convergent subsequence n_k and suppose

$a_{ijn_k}x + b_{ijn_k} \longrightarrow x' \leq x_0$ as $k \to \infty$. If $x' < x_0$ then there

is an $\varepsilon > 0$ such that $x' < x_0 - \varepsilon$. Then for all n_k sufficiently large

$$\underset{\sim}{Q}^{n_k}(a_{ijn_k}x + b_{ijn_k}) \le \underset{\sim}{Q}^{n_k}(x_0 - \varepsilon) \longrightarrow \underset{\sim}{Q} \quad \text{as} \quad k \to \infty \quad \text{but also}$$

$$Q_{ij}^{n_k}(a_{ijn_k}x + b_{ijn_k}) \longrightarrow \rho_U(x)\pi_j > 0 \quad \text{which gives a contradiction.}$$

Hence $x' = x_0$. Since any convergent subsequence converges to x_0 , the sequence converges to x_0 .

Hence for $n > N_x$, $x_0 \ge a_{ijn}x + b_{ijn} \to x_0$, $n \to \infty$; we have

$H_i(a_{ijn}x + b_{ijn}) \longrightarrow H_i(x_0-)$. However from

(3.9) $H_i(a_{ijn}x + b_{ijn}) \longrightarrow 1$, whence $H_i(x_0 -) = 1 = H_i(x_0)$. So

$H_i(\cdot)$, $i = 1,\ldots,m$. are continuous at x_0 and hence so is $\underset{\sim}{Q}(\cdot)$.

By Lemma (2.10) $\rho(\cdot)$, $\underset{\sim}{r}(\cdot)$, $\underset{\sim}{\ell}(\cdot)$ and hence $\underset{\sim}{M}(\cdot)$ are all continuous at x_0 . Therefore $\lim_{n\to\infty} M_{ij}(a_{ijn}x + b_{ijn}) = M_{ij}(x_0) = \pi_j$.

(b,2,ii) If $a_{ijn}x + b_{ijn} > x_0$, for infinitely many n , then for infinitely many n $\underset{\sim}{Q}^n(a_{ijn}x + b_{ijn}) = \underset{\sim}{P}^n$. Hence $Q_{ij}^n(a_{ijn}x + b_{ijn}) \longrightarrow \pi_j$ and by Theorem (3.8) this suffices for $\underset{\sim}{Q}^n(a_{ijn}x + b_{ijn}) \longrightarrow \underset{\sim}{\Pi}$ and

$\rho^n(a_{ijn}x + b_{ijn}) \longrightarrow 1$ as $n \to \infty$. So $\lim_{n\to\infty} Q_{ij}^n(a_{ijn}x + b_{ijn}) =$

$= \lim_n\{\rho^n(a_{ijn}x + b_{ijn})M_{ij}(a_{ijn}x + b_{ijn}) + o(1)\}$, whence $\pi_j = \lim_{n\to\infty} M_{ij}(a_{ijn}x+b_{ijn})$.

The lemma is completely proved.

If there are constants $a_{ijn} > 0$, b_{ijn}, $n \ge 1$, $i,j = 1,\ldots,m$ for which $\{Q_{ij}^n(a_{ijn}x + b_{ijn})\} \xrightarrow{c} \underset{\sim}{U}(x)$ with $U_{ij}(x)$ nondegenerate, then by Theorem (3.1), part (2), for fixed (i_0,j_0) the set of constants $a_{i_0 j_0 n} > 0$, $b_{i_0 j_0 n}$, $n \geq 1$ is asymptotically equivalent to each of the sets $a_{k\ell n} > 0$, $b_{k\ell n}$ for $k,\ell = 1,\ldots,m$. Without loss of generality we henceforth assume that normalizing constants are chosen independently of i and j.

Corollary (3.6) Convergence to Types: If for given constants $\alpha_n > 0$, β_n and $a_n > 0$, b_n:

$$\{Q_{ij}^n(\alpha_n x + \beta_n)\} \xrightarrow{c} \underset{\sim}{V}(x) = \{V_{ij}(x)\} \text{ and}$$

$$\{Q_{ij}^n(a_n x + b_n) \xrightarrow{c} \underset{\sim}{U}(x) = \{U_{ij}(x)\}$$

where $U_{ij}(x)$, $V_{ij}(x)$ are nondegenerate for each (i,j), then $\underset{\sim}{U}(x)$ and $\underset{\sim}{V}(x)$ are of the same type. There exist $A > 0$ and B such that

$$A = \lim_{n\to\infty} \alpha_n^{-1} a_n \text{ and } B = \lim_{n\to\infty} a_n^{-1}(\beta_n - b_n) \text{ and}$$

$$\{V_{ij}(x)\} = \underset{\sim}{V}(x) = \underset{\sim}{U}(Ax + B) = \{U_{ij}(Ax + B)\}. \text{ Furthermore } \underset{\sim}{U}(x) = \rho_U(x)\underset{\sim}{\Pi},$$

where $\rho_U(x)$ is an extreme value distribution and $\underset{\sim}{V}(x) = \rho_U(Ax + B)\underset{\sim}{\Pi}$.

Corollary (3.7): Asymptotic Independence: Given

$$\{P[J_n = j, a_n^{-1}(M_n - b_n) \leq x | J_0 = i]\} \to \{U_{ij}(x)\} = \rho_U(x)\underset{\sim}{\Pi} \quad \text{then}$$

$$P[a_n^{-1}(M_n - b_n) \leq x] \xrightarrow{c} \rho_U(x) \text{ and } \lim_{n\to\infty} P[J_n = j, a_n^{-1}(M_n - b_n) \leq x] =$$

$$= \lim_{n\to\infty} P[J_n = j] \lim_{n\to\infty} P[a_n^{-1}(M_n - b_n) \leq x].$$

Proof: We have that

$$\lim_{n\to\infty} P[J_n = j,\ a_n^{-1}(M_n - b_n) \le x | J_0 = i] = \rho_U(x)\pi_j \quad \text{so}$$

$$\lim_{n\to\infty} P[a_n^{-1}(M_n - b_n) \le x | J_0 = 1] = \rho_U(x) \quad \text{and}$$

$$\lim_{n\to\infty} P[a_n^{-1}(M_n - b_n) \le x] = \rho_U(x) \ .$$ Therefore M_n has a limiting distribution which is an extreme value distribution. Next we have that

$$\lim_{n\to\infty} P[J_n = j,\ a_n^{-1}(M_n - b_n) \le x] =$$

$$= \lim_{n\to\infty} P[J_n = j, a_n^{-1}(M_n - b_n) \le x | J_0 = 1] = \pi_j \rho_U(x) =$$

$$= \lim_{n\to\infty} P[J_n = j] \lim_{n\to\infty} P[a_n^{-1}(M_n - b_n) \le x]$$ which completes the proof.

That the norming constants can be chosen to be independent of i,j is not surprising. When we take the nth power of the Q-matrix we sum over all paths of length n starting at i and ending at j . This entails sufficient mixing of the distributions involved so that the effects of the endpoints i and j become negligible for large n .

A further reflection of this thorough mixing when taking powers of the Q-matrix is given in:

Theorem (3.8): There exist norming constants $a_n > 0$, b_n , $n \ge 1$ and an index pair (i_0, j_0) , $1 \le i_0, j_0 \le m$, such that

(3.9) $$Q^n_{i_0 j_0}(a_n x + b_n) \xrightarrow{c} U_{i_0 j_0}(x) \cdot$$

with $U_{i_0j_0}(x)$ nondegenerate iff:

$$Q^n(a_nx + b_n) \xrightarrow{c} \{U_{ij}(x)\}$$

where $U_{ij}(x) = \rho_U(x)\pi_j$ and $\rho_U(x)$ is an extreme value distribution and

$$\pi_{j_0}^{-1} U_{i_0j_0}(x) = \rho_U(x) .$$

Proof: We need only show that (3.9) implies convergence of the Q-matrix. Focus attention on any $(i,j) \neq (i_0,j_0)$. Pick a convergent subsequence n_k and suppose $Q_{ij}^{n_k}(a_{n_k}x + b_{n_k}) \xrightarrow{w} U_{ij}(x)$. We wish to identify $U_{ij}(x)$ and so we select a further subsequence n_k' such that

$$H_i^{n_k'}(a_{n_k'}x + b_{n_k'}) \xrightarrow{w} \phi_i(x) , \; 1 \leq i \leq m ; \quad \phi_i(x) \text{ is a mass function.}$$

Hence $Q^{n_k'}(a_{n_k'}x + b_{n_k'}) \xrightarrow{w} [\prod_{i=1}^{m} \phi_i^{\pi_i}(x)]\pi$ by Lemma (2.18), which identifies

$$U_{ij}(x) = [\prod_{i=1}^{m} \phi_i^{\pi_i}(x)]\pi_j . \quad \text{But } [\prod_{i=1}^{m}\phi_i^{\pi_i}(x)]\pi_{j_0} = U_{i_0j_0}(x) \text{ and}$$

therefore $[\prod_{i=1}^{m} \phi_i^{\pi_i}(x)] = \pi_{j_0}^{-1}U_{i_0j_0}(x)$; this is a nondegenerate, honest probability distribution function, since the convergence in (3.9) is complete. So $\lim_{k\to\infty} Q_{ij}^{n_k}(a_{n_k}x + b_{n_k}) = U_{ij}(x) = [\pi_{j_0}^{-1} U_{i_0j_0}(x)]\pi_j$. Since this holds for any convergent subsequence

$$\lim_{n\to\infty} Q_{ij}^n(a_nx + b_n) = [\pi_{j_0}^{-1} U_{i_0j_0}(x)]\pi_j .$$

The pair (i,j) is arbitrary, which completes the proof.

Our results are related to those of Gnedenko by the following theorem.

Theorem (3.10): There exist norming constants $a_n > 0$, b_n, $n \geq 1$ such that $P[a_n^{-1}(M_n - b_n) \leq x] \xrightarrow{c} \rho_U(x)$ where $\rho_U(x)$ is a nondegenerate distribution function iff $\underset{\sim}{Q}^n(a_n x + b_n) \xrightarrow{c} \rho_U(x)\underset{\sim}{\Pi}$. Hence $\rho_U(x)$ is an extreme value distribution and the only possible limiting distributions for the sequence $\{M_n\}$ are the extreme value types.

Proof: Given the convergence of the Q-matrix, the desired result follows from (3.1) and (3.6).

Now we suppose that $\lim\limits_{n} P[a_n^{-1}(M_n - b_n) \leq x] = \rho_U(x)$.

For some initial distribution (p_i), $i = 1,\ldots,m$ we have from (1.1) that

$$(3.11) \qquad \lim_{n\to\infty} P[a_n^{-1}(M_n - b_n) \leq x] = \lim_{n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{m} Q_{ij}^{n}(a_n x + b_n)\, p_i = \rho_U(x) .$$

By the weak compactness theorem for S.M.M.'s we can select a subsequence n_k such that, for some limit: $\underset{\sim}{U}(x) = \{U_{ij}(x)\}$, $\lim\limits_{k\to\infty} \{Q_{ij}^{n_k}(a_{n_k} x + b_{n_k})\} = \{U_{ij}(x)\}$. We will identify $\{U_{ij}(x)\}$. From (3.11) we have:

$$(3.12) \qquad \sum_{k=1}^{m} \sum_{\ell=1}^{m} U_{k\ell}(x) p_k = \rho_U(x) .$$

There exists a further subsequence n_k' such that $H^{n_k'}(a_{n_k'} x + b_{n_k'}) \xrightarrow{w} \phi_i(x)$ with the $\phi_i(x)$ mass functions. We have $\underset{\sim}{Q}^{n_k'}(a_{n_k'} x + b_{n_k'}) \longrightarrow \underset{\sim}{U}(x)$ and also $\underset{\sim}{Q}^{n'_k}(a_{n'_k}x + b_{n'_k}) \longrightarrow [\prod_{i=1}^{m} \phi_i^{\pi_i}(x)]\underset{\sim}{\Pi}$ by (2.18) .

So $U_{ij}(x) = [\prod_{i=1}^{m} \phi_i^{\pi_i}(x)]\pi_j$ and from (3.12)

$$\rho_U(x) = \sum_{k=1}^{m} \sum_{\ell=1}^{m} [\prod_{i=1}^{m} \phi_i^{\pi_i}(x)]\pi_\ell p_k = \prod_{i=1}^{m} \phi_i^{\pi_i}(x) .$$

Therefore $U_{ij}(x) = \rho_U(x)\pi_j$ and $\{Q_{ij}^{n_k}(a_{n_k}x + b_{n_k})\} \longrightarrow \rho_U(x)\underset{\sim}{\Pi}$. Since this holds for any convergent subsequence we have $\underset{\sim}{Q}^n(a_n x + b_n) \xrightarrow{c} \rho_U(x)\underset{\sim}{\Pi}$.

By (3.1) $\rho_U(x)$ is an extreme value distribution.

Criteria for the existence of a limiting distribution for $\{M_n\}$ are given in

Theorem (3.13): There exist constants $a_n > 0$, b_n, $n \geq 1$ such that:

(3.14) $\underset{\sim}{Q}^n(a_n x + b_n) \xrightarrow{c} \rho_U(x)\underset{\sim}{\Pi}$ where $\rho_U(x)$ is a nondegenerate (extreme value) distribution function

(3.15) iff $\rho^n(a_n x + b_n) \xrightarrow{c} \rho_U(x)$,

or:

(3.16) iff $[\prod_{i=1}^{m} H_i^{\pi_i}(a_n x + b_n)]^n \xrightarrow{c} \rho_U(x)$. It follows that M_n has a limiting extreme value distribution $\rho_U(x)$ iff $\rho(x)$ or equivalently $\prod_{i=1}^{m} H_i^{\pi_i}(x)$ are in the domain of attraction of $\rho_U(x)$.

Proof: Given (3.14), the latter two statements follow from theorem (3.1) .

Assuming (3.15), there are two cases:

Case I: If $\rho(x) < 1$, $x < \infty$, (3.15) implies $\rho(a_n x + b_n) \longrightarrow 1$, $n \to \infty$, for all x such that $\rho_U(x) > 0$ and $a_n x + b_n \longrightarrow \infty$. Hence

$$\lim_{n\to\infty} M_{ij}(a_n x + b_n) = \pi_j \text{ . Therefore}$$

$$\lim_{n\to\infty} \underset{\sim}{Q}^n(a_n x + b_n) = \lim_{n\to\infty} [\rho^n(a_n x + b_n)\underset{\sim}{M}(a_n x + b_n) + \underset{\sim}{o}(1)] \text{ and}$$

$$\lim_{n\to\infty} \underset{\sim}{Q}^n(a_n x + b_n) = \rho_U(x)\underset{\sim}{\Pi} .$$

Case II: There exists $x_0 < \infty$ such that $\rho(x_0) = 1$ and $\rho(x_0 - \varepsilon) < 1$ for all $\varepsilon > 0$. For a fixed x such that $\rho_U(x) > 0$, suppose $a_n x + b_n > x_0$ for only finitely many n, then for n sufficiently large $a_n x + b_n \leq x_0$. In fact $a_n x + b_n \longrightarrow x_0$ as $n \to \infty$. To show this, suppose there is a subsequence n_k with $a_{n_k} x + b_{n_k} \longrightarrow x' < x_0$ as $k \to \infty$,

Then for some $\varepsilon > 0$, $x' < x_0 - \varepsilon$. Now $\lim_{n\to\infty} \rho(a_n x + b_n) = 1$ [3. p. 439] and

$$\lim_{k\to\infty} \rho(a_{n_k} x + b_{n_k}) = 1 \text{ . But}$$

$$\lim_{k\to\infty} \rho(a_{n_k} x + b_{n_k}) \leq \rho(x') \leq \rho(x_0 - \varepsilon) < 1$$

yielding a contradiction. There are no subsequential limits less than x_0 and hence $a_n x + b_n \to x_0$. Thus $\rho(a_n x + b_n) \to \rho(x_0 -)$ and since also $\rho(a_n x + b_n) \longrightarrow 1$, $\rho(x_0 -) = 1 = \rho(x_0)$ and $\rho(\cdot)$ is continuous at x_0. So $Q(\cdot)$, $r(\cdot)$, $\ell(\cdot)$, $M(\cdot)$ are all continuous at x_0 (2.8-2), 2.10-2) and

$$\lim_{n\to\infty} M_{ij}(a_n x + b_n) = \pi_j \text{ . Therefore}$$

$$\lim_{n\to\infty} \underset{\sim}{Q}^n(a_n x + b_n) = \lim_{n\to\infty} [\rho^n(a_n x + b_n)\underset{\sim}{M}(a_n x + b_n) + \underset{\sim}{o}(1)] \quad \text{and}$$

$$\lim_{n\to\infty} \underset{\sim}{Q}(a_n x + b_n) = \rho_U(x) \underset{\sim}{\Pi} .$$

Suppose $a_n x + b_n > x_0$ for infinitely many n, then $\rho_U(x) = 1$ and $\underset{\sim}{Q}^n(a_n x + b_n) = \underset{\sim}{P}^n$ for such n. If $a_n x + b_n \le x_0$ for only finitely many n, then $\lim_{n\to\infty} Q^n(a_n x + b_n) = \underset{\sim}{\Pi}$, as was to be proved. If $a_n x + b_n \le x_0$ for infinitely many n then we partition the set of positive integers into sets $\{n_1\}$ and $\{n_2\}$ such that $a_{n_1} x + b_{n_1} \le x_0$ for all n_1 and $a_{n_2} x + b_{n_2} > x_0$ for all n_2. As above $a_{n_1} x + b_{n_1} \longrightarrow x_0$ as $n_1 \uparrow +\infty$ and $M(\cdot)$ is continuous at x_0, so

$$\lim_{n_1\to\infty} \underset{\sim}{Q}^{n_1}(a_{n_1} x + b_{n_1}) = \lim_{n_1\to\infty} [\rho^{n_1}(a_{n_1} x + b_{n_1})\underset{\sim}{M}(a_{n_1} x + b_{n_1}) + \underset{\sim}{o}(1)] \quad \text{and}$$

$\lim_{n_1\to\infty} \underset{\sim}{Q}^{n_1}(a_{n_1} x + b_{n_1}) = \underset{\sim}{\Pi}$. Since $\underset{\sim}{Q}^{n_2}(a_{n_2} x + b_{n_2}) = \underset{\sim}{\Pi}$ for all n_2 we have

$\lim_{n\to\infty} \underset{\sim}{Q}^n(a_n x + b_n) = \underset{\sim}{\Pi}$ as was to be shown.

Now assume (3.16). By the weak compactness theorem for S.M.M.'s we can select a convergent subsequence n_k such that

$\{Q_{ij}^{n_k}(a_{n_k} x + b_{n_k})\} \to \{U_{ij}(x)\}$. To identify $U_{ij}(x)$ as $\rho_U(x)\pi_j$, we select a further subsequence n_k' such that for

$1 \le i \le m$, $H_i^{n_k'}(a_{n_k'} x + b_{n_k'}) \xrightarrow{w} \phi_i(x)$ with the $\phi_i(x)$ mass functions, and therefore

$$Q^{n_k'}(a_{n_k'}x + b_{n_k'}) \longrightarrow [\prod_{i=1}^{m} \phi_i^{\pi_i}(x)]\underset{\sim}{\Pi} \text{ by (2.18) . But}$$

$$[\prod_{i=1}^{m} H_i^{\pi_i}(a_{n_k'}x + b_{n_k'})]^{n_k'} \longrightarrow \prod_{i=1}^{m} \phi_i^{\pi_i}(x) \text{ and also}$$

$$[\prod_{i=1}^{m} H_i^{\pi_i}(a_{n_k'}x + b_{n_k'})]^{n_k'} \longrightarrow \rho_U(x) \quad \text{so} \quad \prod_{i=1}^{m} \phi_i^{\pi_i}(x) = \rho_U(x) \quad \text{and}$$

$$\{Q_{ij}^{n_k}(a_{n_k}x + b_{n_k})\} \longrightarrow \{U_{ij}(x)\} = \rho_U(x)\underset{\sim}{\Pi} \; .$$

This holds for all convergent subsequences, and hence for the full sequence.

<u>Remark</u>: Minor difficulties of a technical nature arise when $\underset{\sim}{P}$ may be reducible and/or periodic. The details are forthcoming.