On the Law of the Iterated Logarithm

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CHAPTER I

A SUFFICIENT CONDITION FOR THE LAW OF THE ITERATED LOGARITHM

1. Introduction

Let (Ω, F, P) be a probability space, and assume that $(S_n, F_n, n \geq 1)$ is a stochastic sequence (i.e. $F_1 \subset F_2 \subset \ldots \subset F$ are sigma-fields and, for each n, the random variable S_n is F_n-measurable). We say that S_n obeys the Law of the Iterated Logarithm if there is a sequence $b_1, b_2, \ldots$ of positive real numbers such that $\limsup_{n \to \infty} S_n/b_n = 1$ a.e. The name of this law is derived from the fact that each number b_n involves the function "log log" in the results proved to date. (Note: throughout this thesis we will refer only to natural logarithms.)

Perhaps the best known result in this field is the celebrated Kolmogorov Law of the Iterated Logarithm (see [11] or p. 260 of [12]), which appeared in 1929. The result states: Let $X_1, X_2 \ldots$ be a sequence of independent random variables with mean 0 and finite variance for each n. Define $S_n = X_1 + X_2 + \ldots + X_n$, $s_n^2 = ES_n^2$, and $t_n = (2 \log \log s_n^2)^{\frac{1}{2}}$. If $s_n \to \infty$ and $|X_n| \leq m_n = o(s_n/t_n)$ as $n \to \infty$, then $\limsup_{n \to \infty} S_n/(s_n t_n) = 1$ a.e.

This result generalizes Khintchine's result [10], which proved the same result for the coin-tossing case (i.e. X_n takes value 1 or -1 with equal probability, for each n). It should be noted that the condition $m_n=o(s_n/t_n)$ cannot, in general, be replaced by the weaker condition $m_n=O(s_n/t_n)$. This fact was demonstrated by Marcinkiewicz and Zygmund in [13].

The proof of Kolmogorov's result involved the ingenious use of Kolmogorov's exponential bounds (see [11] or p. 254 of [12]; indeed, most of the iterated logarithm results on record since 1929 are based, to some extent at least, on Kolmogorov's method.

Let S_1, S_2, . . . be random variables (r.v.). A sequence b_1, b_2, . . . of positive real numbers is said to belong to the upper class or lower class of S_1, S_2, . . . according as $P[S_n>b_n$ i.o.] is 0 or 1 respectively. So, for example, to prove Kolmogorov's result it is evident that one must show that, for every $\varepsilon>0$, the sequence $(1+\varepsilon)s_nt_n$ is in the upper class of S_n, while the sequence $(1-\varepsilon)s_nt_n$ is in the lower class.

In [3], Feller completes the Kolmogorov result, in the sense that he provides some necessary and sufficient conditions for sequences $\{b_n\}$ to belong to the upper or lower classes of S_n, given the situation that X_1, X_2, . . . are independent with mean 0 and finite variance, $S_n=X_1+ \cdots +X_n$, $s_n^2 = ES_n^2\to\infty$, and for all n, $\max_{1\leq k\leq n}|X_k|\leq\lambda_n s_n$, where $\lambda_n\downarrow 0$.

The results outlined heretofore have related only to certain sequences of bounded random variables. However, Hartman and Wintner [8] presented the following result in

1941: If $X_1, X_2, \ldots$ are independent, identically distributed (i.i.d.) r.v.'s with mean 0 and variance 1, then $\limsup_{n\to\infty} (X_1 + \ldots + X_n)/(2n \log\log n)^{\frac{1}{2}} = 1$ a.e.

Recently, Strassen [18] has proved a converse to their result: Let $X_1, X_2, \ldots$ be i.i.d. and $S_n = X_1 + \ldots + X_n$. Then the condition $\limsup_{n\to\infty} |S_n|/(2n \log\log n)^{\frac{1}{2}} < \infty$ implies $EX_1 = 0$ and $EX_1^2 < \infty$. In fact, X_1 has mean 0 and variance 1 if and only if $\limsup_{n\to\infty} S_n/(2n \log\log n)^{\frac{1}{2}} \doteq 1$ and $\liminf_{n\to\infty} S_n/(2n \log\log n)^{\frac{1}{2}} = -1$ a.e.

Some other results concerning the Law of the Iterated Logarithm will be outlined in following chapters.

As noted above, the proof of Kolmogorov's theorem depends on the exponential bounds. However, an analysis of the proof of these exponential inequalities shows that they are actually a consequence of certain properties of the moment-generating functions Ee^{tX_n}, properties that are implied by the hypotheses of Kolmogorov's theorem. In this chapter we will prove theorems which follow from these as yet unspecified conditions on the moment-generating functions, Our theorems will be shown to imply some known results, including that of Kolmogorov, as well as some new results.

2. A Generalization of Kolmogorov's Theorem

LEMMA 1.1 Let S be a r.v. such that $Ee^{tS} < \exp((t^2/2)(1+tc/2))$ for some $c>0$ and all $0<tc\leq 1$. Let $\varepsilon>0$.

(i) if $\varepsilon c \leq 1$ then $P[S>\varepsilon] < \exp(-(\varepsilon^2/2)(1-\varepsilon c/2))$;

(ii) if $\varepsilon c \geq 1$ then $P[S>\varepsilon]<\exp(-\varepsilon/(4c))$.

PROOF. By hypothesis, if $tc\leq 1$ we have

$$P[S>\varepsilon]=P[e^{tS}>e^{t\varepsilon}]<e^{-t\varepsilon}\cdot Ee^{tS} < \exp(-t\varepsilon+(t^2/2)(1+tc/2)).$$

Then (i) follows by setting $t=\varepsilon$ in this inequality, while (ii) will follow if we set $t=c^{-1}$.

THEOREM 1.1. Let $X_1, X_2, \ldots$ be independent r.v. with $EX_n=0$, $EX_n^2=\sigma_n^2<\infty$ for each n. Define $S_n=X_1+\ldots+X_n$, $s_n^2 = ES_n^2$, $t_n^2 = 2 \log\log s_n^2$. Suppose there exists a sequence $c_1, c_2, \ldots$ of positive numbers such that $c_n = o(t_n^{-1})$ and, for each sufficiently large n, $Ee^{tS_n/s_n} < \exp((t^2/2)(1+tc_n/2))$ provided $0<tc_n\leq 1$. If, furthermore, (i) $s_n\to\infty$ and (ii) $\sigma_n/s_n\to 0$ as $n\to\infty$, then $\limsup_{n\to\infty} S_n/(s_nt_n)\leq 1$ a.e.

PROOF. Our proof will closely follow that of Kolmogorov in [11] or on page 260 of [12].

First, note that $s_n\sim s_{n+1}$,[1] since, by (ii),

$$1 \leq \frac{s_{n+1}^2}{s_n^2} = \frac{1}{1-\sigma_{n+1}^2/s_{n+1}^2} \to 1 \quad \text{as } n\to\infty.$$

Let $\delta>\delta'>\delta''>0$. For $c>1$, let n_k be the first integer m such that $s_m>c^k$; such an integer exists by (i). Then $s_{n_k}>c^k\geq s_{n_k-1}\sim s_{n_k}$, so $s_{n_k}\sim c^k$ as $k\to\infty$. Note also

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[1] Throughout this thesis the statement "$a_n\sim b_n$" will be equivalent to the statement "$\lim_{n\to\infty} a_n/b_n=1$."

that $t_{n_k} \sim t_{n_{k-1}}$.

For each k, define $S^*_{n_k} = \max_{n_{k-1} < n \leq n_k} S_n$. Choose $c > 1$ so close to 1 that $(1+\delta)/c > 1+\delta'$. Then, since $(1-\delta)s_{n_{k-1}} t_{n_{k-1}} \sim s_{n_k} t_{n_k}(1-\delta)/c$,

$$A_\delta \equiv [S_n \geq (1+\delta)s_n t_n \text{ i.o.}] \subset [S^*_{n_k} \geq (1+\delta)s_{n_{k-1}} t_{n_{k-1}} \text{ i.o.}]$$

$$\subset [S^*_{n_k} \geq (1+\delta')s_{n_k} t_{n_k} \text{ i.o.}]$$

Choose k so large that $(1+\delta'') < (1+\delta') - \sqrt{2}/t_{n_k}$. Then, by a variation of Levy's inequality due to Kolmogorov (see [12], p. 248),

$$P[S^*_{n_k} \geq (1+\delta')s_{n_k} t_{n_k}] \leq 2P[S_{n_k} \geq (1+\delta')s_{n_k} t_{n_k} - \sqrt{2}\, s_{n_k}]$$

$$\leq 2P[S_{n_k} \geq (1+\delta'')s_{n_k} t_{n_k}].$$

Let $\varepsilon_k = (1+\delta'')t_{n_k}$; clearly $\varepsilon_k \to \infty$. By hypothesis, $t_{n_k} c_{n_k} \to 0$ as $k \to \infty$, so $t_{n_k} c_{n_k} \leq 1$ for all large k. Furthermore, for all sufficiently large k, $(1 - \varepsilon_k c_{n_k}/2) > (1+\delta'')^{-1}$. Now, since we are given that, for all large k, $E\exp(tS_{n_k}/s_{n_k}) < \exp((t^2/2)(1+tc_{n_k}/2))$ if $0 < tc_{n_k} \leq 1$, we may apply lemma 1.1 (i) to find that, for all sufficiently large k,

$$P[S_{n_k} \geq (1+\delta'')s_{n_k} t_{n_k}] < \exp(-(1+\delta'')^2 (t^2_{n_k}/2)(1-\varepsilon_k c_{n_k}/2))$$

$$< (\log s^2_{n_k})^{-(1+\delta'')} \sim (2k \log c)^{-1-\delta''}.$$

$$\therefore \Sigma_{k=1}^{\infty} P[S_{n_k} \geq (1+\delta'')s_{n_k} t_{n_k}] < \infty.$$

$$\therefore \Sigma_{k=1}^{\infty} P[S^*_{n_k} > (1+\delta')s_{n_k} t_{n_k}] < \infty.$$

Hence, by the Borel-Cantelli lemma, $P[S^*_{n_k} > (1+\delta')s_{n_k}t_{n_k}$ i.o.$]$ $= 0$. Therefore, $PA_\delta = 0$ for all $\delta>0$, so $\limsup_{n\to\infty} S_n/(s_n t_n) \leq 1$ a.e. Q.E.D.

LEMMA 1.2. Let S be a r.v. such that

(1) $\exp((t^2/2)(1-tc)) < Ee^{tS} < \exp((t^2/2)(1+tc/2))$

for some $c>0$ and all $0<tc\leq 1$. Assume $\varepsilon>0$. For any given $\gamma>0$, there exist numbers $\varepsilon_o>0$ and $\eta_o>0$ (depending on γ) such that, if $\varepsilon>\varepsilon_o$ and $\varepsilon c<\eta_o$, then $P[S>\varepsilon]>\exp(-(\varepsilon^2/2)(1+\gamma))$.

PROOF. The following proof virtually duplicates that of Kolmogorov (see pp. 255-257 of [12]).

Choose $0<\beta<1$ such that $(1+2\beta+\beta^2/2)/(1-\beta)^2 \leq 1+\gamma$. Let t_o be the smallest value of t for which

(2) $9t^2 e^{-\beta^2 t^2/8} \leq 1/4$, $e^{\beta^2 t^2/8} \geq 4\beta t^2$ and $e^{t^2/4} \geq 8$,

and define $\varepsilon_o = t_o(1-\beta)$, $\eta_o = (1-\beta)\beta^2/(8(1+\beta)^2)$. Let $t=\varepsilon/(1-\beta)$. If $\varepsilon>\varepsilon_o$, then $t>t_o$. If $\varepsilon c<\eta_o$, then clearly $8tc \lesssim \beta^2/(1+\beta)^2$; in particular, $4tc \lesssim \beta^2/(1+\beta)^2 < \beta/(1+\beta)$, so $1<(1-4tc)^{-1}<1+\beta$. Hence we have

(3) $8tc \lesssim \beta^2/(1+\beta)^2$ and $1-\beta<(1-4tc)^{-1}<1+\beta$.

Note that, from (3), it easily follows that $8tc \lesssim 1$, $8tc \lesssim \beta^2/(1-\beta)^2$ and $\beta^2/4 > tc$.

Now

$$Ee^{tS} = -\int_{-\infty}^{\infty} e^{tx} dP[S>x] = t\int_{-\infty}^{\infty} e^{tx} P[S>x]dx$$

$$= \left(\int_{-\infty}^{0} + \int_{0}^{t(1-\beta)} + \int_{t(1-\beta)}^{t(1+\beta)} + \int_{t(1+\beta)}^{8t} + \int_{8t}^{\infty}\right) te^{tx} P[S>x]dx$$

$\equiv J_1+J_2+J_3+J_4+J_5$. We shall estimate these integrals separately.

Clearly $J_1 \lesssim t \int_{-\infty}^{0} e^{tx}dx = 1$.

If $x \geqslant 8t$ and $xc \gtrsim 1$, then $P[S>x] < e^{-x/(4c)} \leqslant e^{-2tx}$ by lemma 1.1 (ii) and since $1/c \geqslant 8t$ by (3). On the other hand, if $x \gtrsim 8t$ and $xc \leqslant 1$, then, by lemma 1.1 (i), $P[S>x] < \exp(-(x^2/2)(1-xc/2)) \lesssim e^{-x^2/4} \lesssim e^{-2tx}$. Hence $P[S>x] < e^{-2tx}$ if $x \geqslant 8t$, so $J_5 \lesssim t\int_{8t}^{\infty} e^{-tx}dx \leqslant 1$. Therefore

(4) $J_1+J_5 \lesssim 2$.

Now let $0<x<8t$. Note that, by (3), $xc<8tc\lesssim 1$. Hence,

(5) $e^{tx}P[S>x] < \exp(tx - (x^2/2)(1-xc/2))$

$\lesssim \exp(tx-(x^2/2)(1-4tc))$ since $x<8t$.

Let $g(x) = tx - (x^2/2)(1-4tc)$. Then $\frac{dg}{dx} = t-(1-4tc)x$ and $\frac{d^2g}{dx^2} = -(1-4tc)<0$ by (3), so $g(x)$ assumes its maximum value at $x=t/(1-4tc)$. Note that $t(1-\beta) < t/(1-4tc) < t(1+\beta)$ by (3). If $0<x<t(1-\beta)$, then, since g is increasing on $(0, t(1-\beta))$, we have $g(x) \lesssim g(t(1-\beta)) = t^2(1-\beta)-(1-\beta)^2(t^2/2)(1-4tc)$. But $8tc \lesssim \beta^2/(1-\beta)^2$, so $1-4tc>(1-2\beta+\beta^2/2)/(1-\beta)^2$. Therefore, $g(x) \lesssim (t^2/2)(1-\beta^2/2)$. So, by (5),

$$J_2 \le t\int_0^{t(1-\beta)} e^{g(x)}dx \le t^2(1-\beta)e^{t^2(1-\beta^2/2)/2}.$$

Similarly, since $g(x)$ is decreasing on $(t(1+\beta),\ 8t)$, we have, for $t(1+\beta)<x<8t$,

$$g(x)\le g(t(1+\beta))=t^2(1+\beta)-(1+\beta)^2(t^2/2)(1-4tc)$$

$$\le t^2(1+\beta)-(1+2\beta+\beta^2/2)t^2/2=(t^2/2)(1-\beta^2/2).$$

Hence, by (5), $J_4\le t\int_{t(1+\beta)}^{8t} e^{g(x)}dx\le 8t^2e^{t^2(1-\beta^2/2)/2}$.

Thus $J_2+J_4\le 9t^2\ \exp((t^2/2)(1-\beta^2/2))=9t^2\exp(-\beta^2t^2/8)\circ$

$$\exp((t^2/2)(1-\beta^2/4))\le\ 1/4\cdot\exp((t^2/2)(1-tc))<Ee^{tS}/4$$

by (1), (2) and (3).

Furthermore, by (1), (2) and (4), and since $1-tc>1/2$ by (3), $J_1+J_5\le 2\le 1/4\cdot e^{t^2/4}<1/4\cdot e^{t^2(1-tc)/2}<Ee^{tS}/4$.

$\therefore$ by definition, $J_3\ge Ee^{tS}/2>1/2\cdot e^{t^2(1-tc)/2}$. But

$$J_3=t\int_{t(1-\beta)}^{t(1+\beta)} e^{tx}P[S>x]dx\le te^{t^2(1+\beta)}(t(1+\beta)-t(1-\beta))P[S>t(1-\beta)]$$

$$=2\beta t^2e^{t^2(1+\beta)}P[S>\varepsilon].$$

$$\therefore\ P[S>\varepsilon]>1/2\cdot\exp(t^2(1-\beta^2/4)/2-t^2(1+\beta))\cdot(2\beta t^2)^{-1}$$

$$=(4\beta t^2)^{-1}\cdot\exp(\beta^2t^2/8)\cdot\exp(-(1+2\beta+\beta^2/2)t^2/2)$$

$\ge \exp(-(\varepsilon^2/2)(1+2\beta+\beta^2/2)/(1-\beta)^2)$ by (2) and definition of t,

$\ge \exp(-(\varepsilon^2/2)(1+\gamma))$ by the definition of β. Q.E.D.

DEFINITION 1.1. Let $X_1, X_2, \ldots$ be independent r.v.'s with $EX_n=0$ and $EX_n^2=\sigma_n^2<\infty$ for each n. Define, for each $n\geq 1$, $S_n=X_1+\ldots+X_n$, and $s_n^2=ES_n^2$. Then the sequence X_n satisfies HYPOTHESIS A if there exists a sequence $c_1, c_2, \ldots$ of positive real numbers such that $c_n=o((\log\log s_n^2)^{-\frac{1}{2}})$ and, for all sufficiently large n, if t is any non-zero real number such that $|t|c_n\leq 1$, then, for each $k\leq n$,

$$\exp\left(\frac{\sigma_k^2 t^2}{2s_n^2}(1-|t|c_n)\right)<Ee^{tX_k/s_n} < \exp\left(\frac{\sigma_k^2 t^2}{2s_n^2}(1+|t|c_n/2)\right).$$

REMARK. Hypothesis A sets forth the conditions on the moment-generating functions Ee^{tX_n} mentioned in the introduction. As stated, this definition appears rather complicated. However, it is easily seen that if $X_1, X_2, \ldots$ have normal distribution with mean 0 and variance σ_n^2 (we will denote such a distribution by $N(0,\sigma_n^2)$ throughout this thesis), then Hypothesis A holds. From the arguments on page 255 of [12], it is clear that if $|X_n|\leq m_n = o(s_n/(\log\log s_n^2)^{\frac{1}{2}})$ where $s_n^2=\sigma_1^2+\ldots+\sigma_n^2$, then Hypothesis A is valid. The latter condition holds, in particular, if the r.v.'s are uniformly bounded.

The following theorem completes Theorem 1.1:

THEOREM 1.2. Let $X_1, X_2, \ldots$ be independent r.v.'s with $EX_n=0$ and $EX_n^2=\sigma_n^2<\infty$ for each $n\geq 1$. Define, for each n, $S_n=X_1+\ldots+X_n$, $s_n^2=ES_n^2$ and $t_n^2=2\log\log s_n^2$. Then, if

(i) $s_n \to \infty$ (ii) $\sigma_n / s_n \to 0$, and (iii) Hypothesis A holds, we have

$$\limsup_{n \to \infty} S_n/(s_n t_n) = 1 \text{ a.e.}$$

PROOF. The method below is similar to the proof of Kolmogorov's theorem. In view of theorem 1.1. we need only prove that $\limsup_{n\to\infty} S_n/(s_n t_n) \geq 1$ a.e. Note that $s_n \sim s_{n+1}$, by (ii). Define, for $c>1$, n_k to be the smallest integer such that $s_{n_k} > c^k$. As in the proof of theorem 1.1, $s_{n_k} \sim c^k$.

Let $u_k^2 = s_{n_k}^2 - s_{n_{k-1}}^2$ and $v_k^2 = 2 \log \log u_k^2$ for each $k=1,2,\ldots$. Note that $u_k^2 \sim s_{n_k}^2 - s_{n_k}^2/c^2 = s_{n_k}^2 \cdot (c^2-1)/c^2$, and $v_k \sim t_{n_k}$. Define $c_k' = s_{n_k} c_{n_k}/u_k$, and suppose $0<tc_k' \leq 1$. Then $(ts_{n_k}/u_k)c_{n_k} \leq 1$, so for all sufficiently large k, it follows from Hypothesis A that

$$\exp((\sigma_j^2\ t^2/2u_k^2)(1-tc_k')) < Ee^{tX_j/u_k} < \exp((\sigma_j^2\ t^2/2u_k^2)(1+tc_k'/2))$$

for all $j \leq n_k$ and all t such that $0<tc_k' \leq 1$.

$\therefore$ (1) holds for the r.v. $(S_{n_k} - S_{n_{k-1}})/u_k$ by independence. Let $0<\delta<\delta'<1$. Define $\gamma = (1-\delta)^{-2} - 1$ and $\varepsilon_k = (1-\delta)v_k$. Clearly $\varepsilon_k \to \infty$. Note that $c_k' = o(t_{n_k}^{-1}) = o(v_k^{-1})$, so that $\varepsilon_k c_k' \to 0$ as $k \to \infty$. Hence, for all sufficiently large k, we may apply lemma 1.2 to find

$$P_k' \equiv P[S_{n_k} - S_{n_{k-1}} > u_k \varepsilon_k] > \exp(-(1-\delta)^2 (1+\gamma) v_k^2/2))$$

$$= (\log u_k^2)^{-1} \sim (\log s_{n_k}^2)^{-1} \sim (2k \cdot \log c)^{-1}.$$

$\therefore \sum_{k=1}^{\infty} P_k' = \infty$, and it follows from the Borel Zero-One Law that $P[S_{n_k} - S_{n_{k-1}} > u_k \varepsilon_k \text{ i.o.}] = 1$.

Therefore $\limsup_{k \to \infty} (S_{n_k} - S_{n_{k-1}})/(u_k v_k) \geq 1-\delta$ a.e., in fact, we have $\limsup_{k \to \infty} (S_{n_k} - S_{n_{k-1}})/(s_{n_k} t_{n_k}) \geq (1-\delta)(c^2-1)^{\frac{1}{2}}/c$ a.e.

Hypothesis A holds for the sequence $-X_1, -X_2, \ldots$, so, by theorem 1.1, $\limsup_{n \to \infty} -S_n/(s_n t_n) \leq 1$ a.e.; from here it follows easily that $\liminf_{k \to \infty} S_{n_{k-1}}/(s_{n_k} t_{n_k}) \geq -1/c$ a.e.

But $\limsup_{k \to \infty} S_{n_k}/(s_{n_k} t_{n_k}) \geq \limsup_{k \to \infty} (S_{n_k} - S_{n_{k-1}})/(s_{n_k} t_{n_k})$

$$+ \liminf_{k \to \infty} S_{n_{k-1}}/(s_{n_k} t_{n_k})$$

$$\geq (1-\delta)(c^2-1)^{\frac{1}{2}}/c - 1/c \text{ a.e.}$$

$$> 1+\delta' \text{ a.e. if } c>1 \text{ is chosen}$$

appropriately large. But δ' is arbitrary, hence,

$$\limsup_{k \to \infty} S_{n_k}/(s_{n_k} t_{n_k}) \geq 1 \text{ a.e.}$$

Therefore $\limsup_{n \to \infty} S_n/(s_n t_n) \geq 1$ a.e. Q.E.D.

COROLLARY 1.1 (Kolmogorov [11]). Let $X_1, X_2, \ldots$ be independent r.v.'s with $EX_n = 0$ and $EX_n^2 = \sigma_n^2 < \infty$ for each n.

Define $S_n=X_1+\ldots+X_n$, $s_n^2=ES_n^2$ and $t_n^2=2\log\log s_n^2$. If $s_n\to\infty$ as $n\to\infty$ and if, for each $n\geq 1$, $\frac{|X_n|}{s_n}\leq\frac{a_n}{t_n}$ where $a_n\to 0$ as $n\to\infty$, then $\limsup_{n\to\infty} S_n/(s_nt_n)=1$ a.e.

PROOF. Hypothesis A holds as in the proof of the Kolmogorov exponential bounds ([12], p. 255) with $c_n=a_n/t_n$. But $\sigma_n^2/s_n^2\leq a_n^2/t_n^2\to 0$ as $n\to\infty$, so the corollary follows from Theorem 1.1.

COROLLARY 1.2. (see [9]). Let $X_1,X_2,\ldots$ be independent such that, for each n, X_n is normally distributed with mean 0 and variance σ_n^2 (i.e. X_n is $N(0,\sigma_n^2)$). Let $S_n=X_1+\ldots+X_n$, $s_n^2=ES_n^2$ and $t_n^2=2\log\log s_n^2$. If $s_n\to\infty$ and $\sigma_n/s_n\to 0$ as $n\to\infty$, then $\limsup_{n\to\infty} S_n/(s_nt_n)=1$ a.e.

In particular, suppose $Y_1,Y_2,\ldots$ are i.i.d. with $N(0,1)$ distribution and that $a_1,a_2,\ldots$ are positive reals such that $\Sigma_{k=1}^{\infty} a_k^2=\infty$ but $a_n/B_n\to 0$ as $n\to\infty$, where $B_n^2=a_1^2+\ldots+a_n^2$. Then $\limsup_{n\to\infty}\dfrac{a_1Y_1+\ldots+a_nY_n}{B_n(2\log\log B_n^2)^{1/2}}=1$ a.e.

PROOF. For all t real, $Ee^{tX_k/s_n}=\exp(\sigma_k^2t^2/(2s_n^2))$, so Hypothesis A holds for any sequence $c_n=o(t_n^{-1})$ as $n\to\infty$. So let $c_n=s_n^{-1}$ and apply the theorem.

REMARK. Hartman's result [9] is slightly stronger than corollary 1.2. He proves that the result of corollary

1.2 is valid if $s_n \to \infty$ as $n \to \infty$ and $\limsup_{n\to\infty} \sigma_n/s_n < 1$. Furthermore, his proof is much more direct because use of the exponential bounds is avoided; the relation $1-\Phi(x) \sim \frac{e^{-x^2/2}}{\sqrt{2\pi}x}$ as $x \to \infty$ (see [4], p. 166 for example), where $\Phi(x)$ represents the normal distribution function, is used instead.

THEOREM 1.3. Let $X_1, X_2, \ldots$ be independent, each with mean 0 and variance 1. For any $\alpha \geq 0$, define $Y_n = n^\alpha X_n$ and $Z_n = Y_1 + \ldots + Y_n$. If $X_1, X_2, \ldots$ satisfy Hypothesis A, then $Y_1, Y_2, \ldots$ also satisfy Hypothesis A and $\limsup_{n\to\infty} Z_n/(n^\alpha(2n \log\log n/(2\alpha+1))^{\frac{1}{2}}) = 1$ a.e.

PROOF. By assumption, there exist numbers $c_1, c_2, \ldots$ such that $0 < c_n = o((\log\log n)^{-\frac{1}{2}})$ and, for all large n;

$$(7) \quad \exp((t^2/(2n))\,(1-|t|c_n)) < Ee^{tX_k/n} < \exp((t^2/(2n))\,(1+|t|c_n/2))$$

for all $k \leq n$ provided $0 < |t|c_n \leq 1$.

Let $z_n^2 = EZ_n^2 = \Sigma_{k=1}^n k^{2\alpha}$. Note that $z_n^2 \sim n^{2\alpha+1}/(2\alpha+1)$; in fact,

(8) $z_n^2 \geq n^{2\alpha+1}/(2\alpha+1)$, as is easily seen by geometric considerations. Let $c_n' = (2\alpha+1)^{\frac{1}{2}} c_n$ and $v_n^2 = 2 \log\log z_n^2 \sim 2 \log\log n$.

Hence $c_n' = o(v_n^{-1})$. Suppose $0 < |t|c_n' \leq 1$, for n so large that (7) holds. Then, defining $t_k' = k^\alpha \sqrt{n}\, t/z_n$ for each $k \leq n$, we have, by (8),

$0<|t_k'|c_n \leq n^{\alpha+1/2}|t|c_n/z_n \leqslant |t|c_n' \leq 1$. Hence, replacing t by t_k' in (7), it follows that

$$\exp\left(\frac{k^{2\alpha}t^2}{2z_n^2}(1-|t_k'|c_n)\right)<Ee^{tY_k/z_n}<\exp\left(\frac{k^{2\alpha}t^2}{2z_n^2}(1+|t_k'|c_n/2)\right)$$

for all $k \leq n$ if $0<|t|c_n' \leq 1$. That the sequence $Y_1, Y_2, \ldots$ satisfies Hypothesis A follows immediately, since

$|t_k'|c_n \leqslant |t|c_n'$ for all $k \leq n$.

It remains only to note that $z_n \to \infty$ while $n^{\alpha}/z_n = O(n^{-\frac{1}{2}})$, so that theorem 1.2 implies that $\limsup_{n \to \infty} Z_n/(n^{\alpha}(2n \log\log n/(2\alpha+1))^{\frac{1}{2}})=1$ a.e.

An immediate corollary is the following:

COROLLARY 1.3. Let $X_1, X_2, \ldots$ be independent with $EX_n=0$, $EX_n^2=1$. If $|X_n| \leq m_n = o((n/\log\log n)^{\frac{1}{2}})$, then, for any $\alpha \geqslant 0$,

$$\limsup_{n \to \infty} \frac{\Sigma_{k=1}^{n} k^{\alpha} X_k}{n^{\alpha}\sqrt{\frac{2}{2\alpha+1}\, n \log\log n}} = 1 \text{ a.e.}$$

PROOF. As remarked earlier (and established in the proof of corollary 1.1), the condition concerning the bounds on the X_n sequence implies that Hypothesis A holds for $X_1, X_2, \ldots$. The result then follows by theorem 1.3.

REMARK. A special case of Corollary 1.3 occurs when the $X_1, X_2, \ldots$ sequence is uniformly bounded. Such a result is used by Gaposhkin in [7].

The theorems of this chapter may not be very strong results. However, the following corollary will provide a result which is implied by the results of this chapter, but which certainly doesn't follow from the Kolmogorov theorem and does not seem to be a consequence of any other known result.

COROLLARY 1.4. Let $Y_1, Y_2, \ldots$ be i.i.d. with density function $f(x)=e^{-|x|}/2$, $-\infty<x<\infty$, i.e. Laplace distribution. Define, for all $n \geq 1$, $X_n=\sqrt{n}Y_n$, and $S_n=X_1+\ldots+X_n$. Then

$\limsup_{n\to\infty} S_n/(2n \log\log n)^{\frac{1}{2}} = 1$ a.e.

PROOF. It is easily checked that $EY_1=0$, $EY_1^2=2$, and $Ee^{tY}=(1-t^2)^{-1}$ if $|t|<1$. If $Y_1, Y_2, \ldots$ satisfy Hypothesis A, then, applying theorem 1.3 with $\alpha=1/2$, the desired result follows. (Note that the $\sqrt{2}$ factor in the result follows since $EY^2=2$.) Hence, we need only establish that Hypothesis A holds for $Y_1, Y_2 \ldots$.

We will utilize the following inequality (see [4], p. 50):

(9) if $0<t<1$, then $\exp(-t/(1-t))<1-t<\exp(-t)$.

Define $c_n=2n^{-1/4}$ for each $n \geq 4$. Note that if $n \geq 4$ and $0<tc_n \leq 1$, then $0<t<n^{1/4}$ and $(2t/n)(1-t^2/n)^{-1}=2t/(n-t^2)$

$$<2n^{1/4}/(n-\sqrt{n}) = \frac{2n^{1/4}}{\sqrt{n}(\sqrt{n}-1)} \leq 2n^{-1/4} = c_n.$$

So $0<t^2/n<1$ and $(1-t^2/n)^{-1} = 1+(t^2/n)(1-t^2/n)^{-1}$

$$= 1+(t/2)(2t/n(1-t^2/n)^{-1}<1+tc_n/2.$$

Hence, by (9), $Ee^{tY/\sqrt{n}}=(1-t^2/n)^{-1}<\exp((t^2/n)(1-t^2/n)^{-1})$

$$<\exp((t^2/n)(1+tc_n/2)).$$

And, again by (9), $Ee^{tY/\sqrt{n}} >\exp(t^2/n)>\exp((t^2/n)(1-tc_n))$ if $n\geq 4$ and $0<tc_n<1$. Since the distribution of Y is symmetric, it follows that Hypothesis A is indeed valid for the $Y_1,Y_2,$ . . . sequence, as required.

CHAPTER II

SOME RELATIONS BETWEEN THE CENTRAL LIMIT THEOREM AND THE LAW OF THE ITERATED LOGARITHM

1. Introduction

In this chapter we will consider a sequence $X_1, X_2, \ldots$ of independent random variables, each with mean 0 and finite variance. Let $S_n = X_1 + \ldots + X_n$, $s_n^2 = ES_n^2$, $t_n^2 = 2 \log \log s_n^2$, and for all x real, $F_n(x) = P[S_n/s_n \leq x]$. We will denote by $\Phi(x)$ the distribution function of a N(0,1) r.v., i.e.

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt.$$ Then the sequence

$X_1, X_2, \ldots$ is said to obey the Central Limit Theorem if, for all real x, $\lim_{n\to\infty} F_n(x) = \Phi(x)$.

If Hypothesis A of Chapter I is valid for $X_1, X_2, \ldots$, then it is clear that, for any real t, $\lim_{n\to\infty} Ee^{tS_n/s_n} = e^{t^2/2}$, which is the moment-generating function of the N(0,1) distribution; so the Central Limit Property would imply that the Law of the Iterated Logarithm is valid.

It is not true in general that the Central Limit property implies the Law of the Iterated Logarithm. Both Marcinkiewicz and Zygmund [13] and Weiss [19] have constructed counter-examples of bounded random variables $X_1, X_2, \ldots$ such that $s_n \to \infty$ and $|X_n| \leq M_n = O(s_n/t_n)$ as $n \to \infty$ for which the Law of the

Iterated Logarithm is not valid (in fact, $\limsup_{n\to\infty} S_n/(s_n t_n) < 1$ a.e. in [13] and $\limsup_{n\to\infty} S_n/(s_n t_n) > 1$ a.e. in [19]), but the Central Limit Theorem is valid.

Petrov [14] proved a result giving conditions which, along with the Central Limit Property, imply the Law of the Iterated Logarithm; these conditions include assumptions on the rate of the convergence $F_n \to \Phi$. Assuming the notation at the beginning of the chapter and defining

$M_n = \sup_{-\infty<x<\infty} |F_n(x) - \Phi(x)|$, Petrov's result states that $\limsup_{n\to\infty} S_n/(s_n t_n) = 1$ a.e. if $s_n \to \infty$, $s_n \sim s_{n+1}$, and $\exists \delta > 0$ such that $M_n = O((\log s_n^2)^{-1-\delta})$ as $n \to \infty$.

Theorem 2.1 will state a result which contains Petrov's result, namely, if $s_n \sim s_{n+1}$, $s_n \to \infty$, $1 - F_n(x_n) \sim 1 - \Phi(x_n)$, and $F_n(-x_n) \sim \Phi(-x_n)$ as $n \to \infty$ for certain sequences $\{x_n\}$, as $n \to \infty$, then $\limsup_{n\to\infty} S_n/(s_n t_n) = 1$ a.e. This result will be shown to imply some new results, including a useful corollary relating to the Berry-Esséen bounds.

2. Results and Corollaries

LEMMA 2.1. Let $\Phi(x)$ represent the distribution function of a $N(0,1)$ r.v. and suppose $\varepsilon > 1$. Then (i) $1 - \Phi(\varepsilon) < e^{-\varepsilon^2/2}$, and, (ii) for any given $\gamma > 0$, if ε is sufficiently large (depending on γ), then $1 - \Phi(\varepsilon) > e^{-(1+\gamma)\varepsilon^2/2}$.

PROOF. By lemma 2 on p. 166 of [3],

$1 - \Phi(\varepsilon) < \frac{1}{\sqrt{2\pi}\varepsilon} e^{-\varepsilon^2/2} < e^{-\varepsilon^2/2}$, while

$$1-\Phi(\varepsilon)>\frac{1}{\sqrt{2\pi}}\left(\frac{1}{\varepsilon}-\frac{1}{\varepsilon^3}\right) e^{\gamma\varepsilon^2/2}\circ e^{-(1+\gamma)\varepsilon^2/2}>e^{-(1+\gamma)\varepsilon^2/2} \quad \text{if } \varepsilon$$

is large.

THEOREM 2.1. Let $X_1, X_2, \ldots$ be independent with $EX_n=0$, $EX_n^2=\sigma_n^2<\infty$; Define, for $n\geq 1$, $S_n=X_1+\ldots+X_n$, $s_n^2=ES_n^2$, $t_n^2=2\log\log s_n^2$, $F_n(x)=P[S_n\leq x\circ s_n]$, and let $\Phi(x)$ be the $N(0,1)$ distribution function. For any number $a>0$, define the sequence $a_n=\sqrt{a}\circ t_n$. If (i) $s_n\to\infty$, (ii) $\sigma_n/s_n\to 0$ and (iii) $1-F_n(a_n)\sim 1-\Phi(a_n)$ as $n\to\infty$, for all $0<a<A$ for some $A>1$, then $\limsup_{n\to\infty} S_n/(s_n t_n)\leq 1$ a.e. If, furthermore, (iv) $F_n(-a_n)\sim\Phi(-a_n)$, then

(1) $\limsup_{n\to\infty} S_n/(s_n t_n)=1$ a.e.

REMARK. In the proof of [14], Petrov uses lemma 2.1 and the restriction on M_n to derive exponential inequalities to replace the Kolmogorov exponential bounds. In the following proof, we shall use the limit comparison test, i.e. "if $\lim_{n\to\infty} a_n/b_n=1$ for two sequences $\{a_n\}$ and $\{b_n\}$ of positive real numbers, then Σa_n converges if and only if Σb_n converges, where both summations are over $n=1,2,\ldots$" (See p. 360 of [1]).

Our proof below, like that of Petrov, will follow the general method of the demonstration of the Kolmogorov Law of the Iterated Logarithm, (see [11] or p. 260 of [12]).

PROOF. From (ii) it immediately follows that $s_n \sim s_{n+1}$ as $n\to\infty$. Let $0<\delta''<\delta'<\delta<A-1$ be arbitrary, and select $c>1$ so close to 1 that $(1+\delta)/c>1+\delta'$.

By (i) we can define, for each $k\geq 1$, n_k to be the least integer satisfying $s_{n_k}>c^k\geq s_{n_k-1}$. We shall consider only k so large that

(2) $n_k<n_{k+1}$ . . . and $\sqrt{2}<(\delta'-\delta'')t_{n_k}$.

Clearly $s_{n_k}\sim c^k$ and $t_{n_k}\sim t_{n_{k-1}}$ as $k\to\infty$. Define $S^*_{n_k}=\max_{n_{k-1}\leq n<n_k} S_n$. By a variation of Levy's inequality (see p. 248 of [12]),

(3) $P[S^*_{n_k}\geq(1+\delta')s_{n_k}t_{n_k}]\leq 2P[S_{n_k}\geq(1+\delta')s_{n_k}t_{n_k}-\sqrt{2}s_{n_k}]$

$\leq 2P[S_{n_k}\geq(1+\delta'')s_{n_k}t_{n_k}]$ by (2).

Let $\varepsilon_k=(1+\delta'')t_{n_k}$. Then, by lemma 2.1 (i),

$$1-\Phi(\varepsilon_k)<e^{-\varepsilon_k^2/2} = (\log s_n^2)^{-(1+\delta'')^2}\sim(2\log c\cdot k)^{-(1+\delta'')^2}.$$

Hence $\sum_{k=1}^{\infty}[1-\Phi(\varepsilon_k)]<\infty$.

But $1-F_{n_k}(\varepsilon_k)\sim 1-\Phi(\varepsilon_k)$ as $k\to\infty$ by (iii), so we can apply the limit comparison test ([1], p. 360) to yield

$$\sum_{k=1}^{\infty} P[S_{n_k}>s_{n_k}\varepsilon_k] = \sum_{k=1}^{\infty}[1-F_{n_k}(\varepsilon_k)]<\infty.$$

Hence, by (3), $\sum_{k=1}^{\infty} P[S^*_{n_k}>(1+\delta')s_{n_k}t_{n_k}]<\infty$.

By the Borel-Cantelli lemma, then,

$$P[S^*_{n_k} > (1+\delta')s_{n_k}t_{n_k} \text{ i.o.}] = 0.$$

Therefore, since $s_{n_{k-1}}t_{n_{k-1}} \sim s_{n_k}t_{n_k}/c$ and by choice of c,

$$P[S_n > (1+\delta)s_n t_n \text{ i.o.}] \leq P[S^*_{n_k} > (1+\delta)s_{n_{k-1}}t_{n_{k-1}} \text{ i.o.}]$$

$$\leq P[S^*_{n_k} > (1+\delta')s_{n_k}t_{n_k} \text{ i.o.}] = 0.$$

So, since δ is arbitrary, we have $\limsup_{n\to\infty} S_n/(s_n t_n) \leq 1$ a.e.

Now we will assume (iv) and will establish the second part of the theorem. Let $G_n(x) = P[-S_n \leq x \circ s_n] = 1 - F_n(-x)$. Then, by (iv), we find that $1-G_n(a_n) = F_n(-a_n) \sim \Phi(-a_n) = 1-\Phi(a_n)$. So the sequence $-X_1, -X_2, \ldots$ obeys the conditions (i), (ii) and (iii). Hence it follows from the first part of the theorem that

$$(4) \quad \limsup_{n\to\infty} -S_n/(s_n t_n) \leq 1 \text{ a.e.}$$

Now let $0<\delta<\delta'<1$, $u_k^2 = s_{n_k}^2 - s_{n_{k-1}}^2$, $v_k^2 = 2 \log\log u_k^2$. Note that $u_k^2 \sim \frac{c^2-1}{c^2} \circ s_{n_k}^2$ and $v_k \sim t_{n_k}$. If A and B are any two events, then $PAB = PA - PAB^c \geq PA - PB^c$. (This relation is also used by Petrov in [15].

Define $A_k = [S_{n_k} - S_{n_{k-1}} > (1-\delta)u_k v_k]$, $k=1,2,\ldots\ldots$

Then

$$PA_k \geq P\{[S_{n_k} > (1-\delta/2)u_k v_k] \cap [S_{n_{k-1}} \leq u_k v_k \delta/2]\}$$

$$\geq P[S_{n_k} > (1-\delta/2)u_k v_k] - P[S_{n_{k-1}} > u_k v_k \delta/2]$$

$$\geq P[S_{n_k} > (1-\delta/2)u_k v_k] - P[S_{n_{k-1}} > (c^2-1)s_{n_{k-1}} t_{n_k} \delta/3]$$

for all sufficiently large k,

$\sim 1-\Phi(\varepsilon_k') - (1-\Phi(\varepsilon_k''))$, where $\varepsilon_k' = (1-\delta/2)t_{n_k}$

and $\varepsilon_k'' \doteq (c^2-1)t_{n_k}\delta/3$, by (iii).

Let $\gamma = (1-\delta/2)^{-2} - 1$. Then, by lemma 2.1,

$$[1-\Phi(\varepsilon_k')] - [1-\Phi(\varepsilon_k'')] > \exp\{-t_{n_k}^2/2\} - \exp\{-(c^2-1)^2\delta^2 t_{n_k}^2/18\}$$

$$= (\log s_{n_k}^2)^{-1}[1-\exp\{(1-\frac{(c^2-1)^2\delta^2}{9})t_{n_k}^2/2\}]$$

If we choose c>1 so large that $(c^2-1)\delta/3>1$. Then, for all sufficiently large k, then,

$$PA_k > \tfrac{1}{2}(\log s_{n_k}^2)^{-1} \sim \tfrac{1}{2}(2 \log c \cdot k)^{-1}. \quad \text{Then } \sum_{k=1}^{\infty} PA_k = \infty.$$

But the events $A_1, A_2, \ldots$ are independent, so the Borel Zero-One Law assures us that $P[A_k \text{ i.o.}]=1$. This means that $\limsup_{k\to\infty}(S_{n_k} - S_{n_{k-1}})/(u_k v_k) > 1-\delta$ a.e., and,

therefore, $\limsup_{k\to\infty} (S_{n_k} - S_{n_{k-1}})/(s_{n_k} t_{n_k}) > (1-\delta)(c^2-1)^{\frac{1}{2}}/c$ a.e.

Choose c>1 so large that $(1-\delta)(c^2-1)^{\frac{1}{2}}/c \doteq 1/c > 1-\delta'$, and note that it follows easily from (4) that

$$\liminf_{k\to\infty} S_{n_{k-1}}/(s_{n_k} t_{n_k}) > -1/c \text{ a.e.}$$

Therefore,

$$\limsup_{k\to\infty} S_{n_k}/(s_{n_k}t_{n_k}) \gtrsim \limsup_{k\to\infty} (S_{n_k}-S_{n_{k-1}})/(s_{n_k}t_{n_k})$$
$$+ \liminf_{k\to\infty} S_{n_{k-1}}/(s_{n_k}t_{n_k}) > 1-\delta' \text{ a.e.}$$

But δ' is arbitrary, so $\limsup_{k\to\infty} S_{n_k}/(s_{n_k}t_{n_k}) \gtrsim 1$ a.e. Hence $\limsup_{n\to\infty} S_n/(s_nt_n) \gtrsim 1$ a.e., which, with the first part of the theorem, establishes (1).

REMARK. An obvious consequence of Theorem 2.1 is corollary 1.2: if $X_1, X_2, \ldots$ are independent such that X_n is $N(0,\sigma_n^2)$ for each n, $s_n^2=\sigma_1^2+\ldots+\sigma_n^2$ and $\sigma_n/s_n\to 0$ as $n\to\infty$, then (1) holds. In this case, the asymptotic relations in the proof are replaced by equality, and our method reduces to that of Hartman [9].

We shall now show that Petrov's result follows from our result.

COROLLARY 2.1. (Petrov [14]). Let $X_1, X_2, \ldots$ be independent with $EX_n=0$ and $EX_n^2=\sigma_n^2$. Define $S_n=X_1+\ldots+X_n$, $s_n^2=ES_n^2$, $t_n^2=2\log\log s_n^2$; for all real x let $F_n(x)=P[S_n\leq x\cdot s_n]$, $\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt$, and let $M_n=\sup_{-\infty<x<\infty}|F_n(x)-\Phi(x)|$. If $s_n\to\infty$, $\sigma_n/s_n\to 0$ and there exists $\delta>0$ such that $M_n=0((\log s_n^2)^{-1-\delta})$ as $n\to\infty$, then

$$\limsup_{n\to\infty} S_n/(s_n t_n)=1 \text{ a.e.}$$

PROOF. We need only verify (iii) and (iv) of theorem 2.1.

For any $0<a<1+\delta$, define $a_n=\sqrt{a}\cdot t_n$. Suppose $K>0$ is such that $M_n \leq K(\log s_n^2)^{-1-\delta}$ for all n. Then, using the relation (p. 166 of [4]) $1-\Phi(x)\sim\frac{e^{-x^2/2}}{\sqrt{2\pi}\ x}$ as $x\to\infty$, it follows that

$$M_n(1-\Phi(a_n))^{-1} \leq K\cdot(\log s_n^2)^{-1-\delta}\cdot(1-\Phi(a_n))^{-1}$$

$$\sim 2K(a\pi \log\log s_n^2)^{\frac{1}{2}}(\log s_n^2)^{a-1-\delta} \to 0 \text{ as } n\to\infty.$$

But, by definition of M_n, $\left|\frac{1-F_n(a_n)}{1-\Phi(a_n)}-1\right| \leq M_n(1-\Phi(a_n))^{-1}$

and $\left|\frac{F_n(-a_n)}{\Phi(-a_n)}-1\right| = \left|\frac{F_n(-a_n)-\Phi(-a_n)}{1-\Phi(a_n)}\right| \leq M_n(1-\Phi(a_n))^{-1}$.

It then follows that $1-F_n(a_n)\sim 1-\Phi(a_n)$ and $F_n(-a_n)\sim\Phi(-a_n)$

as $n\to\infty$, so the desired result follows from theorem 2.1.

Q.E.D.

COROLLARY 2.2. Let $X_1, X_2, \ldots$ be independent with mean zero and $EX_n^2=\sigma_n^2$. Define $S_n=X_1+\ldots+X_n$, $s_n^2=ES_n^2$, $t_n^2=2\log\log s_n^2$, and, for $\varepsilon>0$, $g_n(\varepsilon)=s_n^{-2}\sum_{k=1}^{\infty}\int_{|X_k|\geq\varepsilon s_n} X_k^2$.

For $a>0$, define the sequence $a_n=\sqrt{a}\cdot t_n$. Suppose there exist a number $A>1$ and a sequence of positive numbers $p_1, p_2, \ldots$ such that (i) $g_n(p_n)\leq p_n^3$ for each n, and (ii) for all $0<a<A$, $p_n(1-\Phi(a_n))^{-1}\to 0$ as $n\to\infty$. If $s_n\to\infty$

and $\sigma_n/s_n \to 0$ as $n\to\infty$, then $\limsup_{n\to\infty} S_n/(s_n t_n) = 1$ a.e.

PROOF. Again we need only show that (iii) and (iv) of theorem 2.1 are true. Let us note first that, as in the proof of corollary 2.1, both of the quantities

$$\left|\frac{1-F_n(a_n)}{1-\Phi(a_n)}-1\right| \text{ and } |F_n(-a_n)/\Phi(-a_n)-1| \text{ are less than}$$

or equal to $M_n(1-\Phi(a_n))^{-1}$, where M_n is defined as in corollary 2.1.

By Theorem 4 of Berry's paper [2], an absolute constant $B>0$ exists such that if $g_n(\varepsilon)<\varepsilon^3$ for some $n\geq 1$ and $\varepsilon>0$, then $M_n\leq B\varepsilon$. Hence $M_n(1-\Phi(a_n))^{-1} \leq Bp_n(1-\Phi(a_n))^{-1}\to 0$ as $n\to\infty$ if $a<A$, by (ii). So $1-F_n(a_n)\sim 1-\Phi(a_n)$ and $F_n(-a_n)\sim\Phi(-a_n)$ as required. Q.E.D.

COROLLARY 2.3 Let $X_1,X_2,\ldots$ be independent, $EX_n=0$, $EX_n^2=\sigma_n^2<\infty$. Define $S_n=X_1+\ldots+X_n$, $s_n^2=ES_n^2$, and $t_n^2=2\log\log s_n^2$. If (i) $s_n\to\infty$ as $n\to\infty$, and (ii) $\sup_{n\geq 1} E|X_n|^3/\sigma_n^2 \equiv \lambda<\infty$, then $\limsup_{n\to\infty} S_n/(s_n t_n) = 1$ a.e.

PROOF. First, note that $\sigma_n^2=E^3(X_n^2)/\sigma_n^4 \leq E^2|X_n|^3/\sigma_n^4 \leq \lambda^2$;

by (i) it is clear that $\sigma_n/s_n\to 0$ as $n\to\infty$.

Since $\lambda<\infty$, by the well-known Berry-Esséen theorem (see, for example, [5], p. 521) there exists an absolute constant $D>0$ such that, for any real number x,

$$|F_n(x)-\Phi(x)|\leq Ds_n^{-1}, \text{ where } F_n(x) = P[S_n\leq x\cdot s_n].$$

Hence, as in the proof of corollary 2.1, both

$|(1-F_n(a_n)(1-\Phi(a_n))^{-1} -1|$ and $|F_n(-a_n)/\Phi(-a_n) -1|$ are bounded above by $Ds_n^{-1}(1-\Phi(a_n))^{-1}$, where $a_n = \sqrt{a}\cdot t_n$ for $a>0$. But $s_n(1-\Phi(a_n)) \sim s_n(4\pi a \log\log s_n^2)^{-\frac{1}{2}}(\log s_n^2)^{-a} \to\infty$ as $n\to\infty$.

Clearly, then, conditions (iii) and (iv) of theorem 2.1 are satisfied, so the required result is a consequence of theorem 2.1. Q.E.D.

REMARK. Theorem 1.1, Theorem 2.1 and Corollaries 2.1, 2.2, and 2.3 are all aimed at obtaining some results on the Law of the Iterated Logarithm for unbounded random variables. However, of all the above-mentioned results, the most useful seems to be Corollary 2.3.; verifying the conditions of the other results will be a very difficult task in general, it would seem, so that those results would not appear to be very useful in many cases.

On the other hand, Corollary 2.3 depends on conditions which can generally be quickly checked in some particular cases not covered by the Kolmogorov [11] or Hartman-Wintner [8] results. It seems that corollary 2.3 will be most useful in the case of unbounded random variables with bounded variances. The following two examples present two such cases; these results appear to be new, as they are not obtainable from other results listed in Chapters

I and II.

EXAMPLE 2.1. Let $Y_1, Y_2, \ldots$ be i.i.d. with density function $f(x)=e^{-|x|}/2$, $-\infty<x<\infty$ (i.e. Laplace distribution). Let $a_1, a_2, \ldots$ be any sequence of positive real numbers such that (i) $\sup_{n\geq 1} a_n \equiv a<\infty$ and (ii) $s_n^2 \equiv \sum_{k=1}^{n} a_k^2 \to \infty$. Define $X_n=a_nY_n$, $S_n=X_1+\ldots+X_n$, and $t_n^2 = 2 \log \log s_n^2$. Then

$\limsup_{n\to\infty} S_n/(s_nt_n) = 1$ a.e.

PROOF. It is easily verified that $EY_1=0$, $EY_1^2 = 2$, $E|Y_1|^3 = 6$.

Hence $\sup_{n\geq 1} \frac{E|X_n|^3}{EX_n^2} \leq 3a<\infty$. Therefore, corollary 2.3 applies.

EXAMPLE 2.2. Suppose $X_1, X_2, \ldots$ are independent, $X_n=\frac{\sqrt{a_n}W_n}{\sqrt{V_n}}$ where $a_n \geq 4$ are integers, W_n is $N(0,1)$ independent of V_n, which has chi-square distribution with a_n degrees of freedom (i.e. each X_n has Student's t-distribution). Define $S_n=X_1+\ldots+X_n$, $\sigma_n^2 = EX_n^2$, $s_n^2=ES_n^2$, and $t_n^2= 2 \log \log s_n^2$. If $\sup_{n\geq 1} a_n \equiv a<\infty$, then

$$\limsup_{n\to\infty} S_n/(s_nt_n) = 1 \text{ a.e.}$$

PROOF. $E|W_n|^3 = 2\sqrt{2/\pi}$ and $E|V_n|^{-3/2} = \frac{\Gamma((a_n-3)/2)}{2^{3/2}\Gamma(a_n/2)}$.

$\therefore$ By independence, $E|X_n|^3 = \frac{\Gamma((a_n-3)/2)\cdot a_n^{3/2}}{\sqrt{\pi}\cdot\Gamma(a_n/2)}$

But $\sigma_n^2 = \frac{a_n\Gamma((a_n-2)/2)}{2\Gamma(a_n/2)}$

$$\therefore \frac{E|X_n|^3}{\sigma_n^2} = \frac{2\circ\sqrt{a}_n\circ\Gamma((a_n-3)/2)}{\sqrt{\pi}\circ\Gamma((a_n-2)/2)} = 0(\sqrt{a_n}) = 0(\sqrt{a})$$

But $\sigma_n^2 = a_n/(a_n-2)>1$, so $s_n\to\infty$. Therefore, corollary 2.3 applies.

CHAPTER III

ON THE LAW OF THE ITERATED LOGARITHM FOR SOME WEIGHTED AVERAGES OF INDEPENDENT RANDOM VARIABLES

1. Introduction

Let $X_1, X_2, \ldots$ be independent random variables, each with mean 0 and variance 1. In this chapter we will be interested in establishing some Law of the Iterated Logarithm results for sequences of r.v.'s of the form

$S_n = \sum_{m=1}^{n} f(m/n) X_m$, where f is a real-valued function, continuous on the interval [0,1].

The first result of this type appeared in 1951 and was due to Gal [6]: Let r_k represent the k^{th} Rademacher function (i.e. for $0 \leq x \leq 1$, $r_k(x) = \text{sign}(\sin 2^{k+1}\pi x)$; it is known that $r_1, r_2, \ldots$ are independent with mean 0 and variance 1 with respect to Lebesgue measure) and let $S_n = r_1 + \ldots + r_n$. Then

$$\limsup_{n \to \infty} \frac{\sum_{k=1}^{n} (1 - \frac{k-1}{n}) r_k}{\sqrt{\frac{2}{3} n \log \log n}} = \limsup_{n \to \infty} \frac{\sum_{k=1}^{n} S_k}{n\sqrt{\frac{2}{3} n \log \log n}} \leq 1 \text{ a.e.}$$

Some thirteen years elapsed before Stackelberg [16] completed Gal's result by proving that, in fact, equality holds in the above relation.

In the same vein, Strassen [17] has shown that if $X_1, X_2, \ldots$ are i.i.d., $EX_1 = 0$, $EX_1^2 = 1$, and $S_n = X_1 + \ldots + X_n$, then, defining $F(t) = \int_t^1 f(x)dx$, where f is any integrable real function on [0,1],

$$\limsup_{n \to \infty} (2n^3 \log \log n)^{-\frac{1}{2}} \sum_{m=1}^{n} f(m/n) S_m = \left(\int_0^1 F^2(t)dt\right)^{\frac{1}{2}} \text{ a.e.}$$

Furthermore, under the same conditions, he proved that, for any $a \geq 1$,

$$\limsup_{n \to \infty} \frac{\sum_{m=1}^{n} |S_m|^a}{n(2n \log \log n)^{a/2}} = \frac{2(a+2)^{a/2-1} \cdot a^{-a/2}}{\left(\int_0^1 (1-t^a)^{-\frac{1}{2}}\right)^a} \text{ a.e.}$$

In particular, if a=1, we get

$$\limsup_{n \to \infty} \frac{\sum_{k=1}^{n} |S_k|}{n\sqrt{\frac{2}{3} n \log \log n}} = 1 \text{ a.e.}$$

Further work on the Law of the Iterated Logarithm for Cesaro's method of summation was done by V.F. Gaposhkin in [7]. He considered $X_1, X_2, \ldots$ independent, $EX_n = 0$, $EX_n^2 = 1$, $|X_n| \leq M < \infty$ a.e. for all n. Then for any $\alpha > 0$, he proved that

$$\limsup_{n \to \infty} \frac{\sum_{k=1}^{n} (1-k/n)^{\alpha} X_k}{\sqrt{\frac{2}{2\alpha+1} n \log \log n}} = 1 \text{ a.e.}$$

In particular, the Gal-Stackelberg result follows from Gaposhkin's theorem by setting $\alpha=1$.

In this chapter, we will consider a sequence of independent r.v.'s, each with mean 0 and variance 1, which satisfy Hypothesis A (definition 1.1). Let f be a continuous function on [0,1] and define $S_n = \Sigma_{m=1}^{n} f(m/n)X_m$. We will provide some iterated logarithm results for this S_n sequence; these results contain Gaposhkin's theorem as a special case.

2. Preliminary Results

LEMMA 3.1. Let (a_{nm}), n=1,2, . . . , m=1,2, . . . , n, be a double sequence of non-negative numbers: define $s_n^2 = \Sigma_{m=1}^{n} a_{nm}^2$, $t_n^2 = 2 \log \log s_n^2$. Let $X_1, X_2, \ldots$ be independent, each with mean 0 and variance 1, and define $S_n = \Sigma_{m=1}^{n} a_{nm}X_m$. Suppose there exist positive numbers $c_n = o(t_n^{-1})$ and an increasing sequence (n_k), $k \geq 1$, of positive integers such that, for any $k \geq 1$, if $0 < tc_{n_k} \leq 1$, then $e^{(t^2/2)(1-tc_{n_k})} < Ee^{tS_{n_k}/s_{n_k}} < e^{(t^2/2)(1+tc_{n_k}/2)}$.

Define $P_k = P[S_{n_k} > as_{n_k}t_{n_k}]$, for a>0.

(i) If a>1 and $\Sigma_{k=1}^{\infty}(\log s_{n_k}^2)^{-a} < \infty$, then $\Sigma_{k=1}^{\infty}P_k < \infty$.

(ii) If 0<a<1 and $\Sigma_{k=1}^{\infty}(\log s_{n_k}^2)^{-a} = \infty$, then $\Sigma_{k=1}^{\infty}P_k = \infty$.

PROOF. (i). Let a>1 and $\Sigma_{k=1}^{\infty}(\log s_{n_k}^2)^{-a} < \infty$. Choose k_o so large that $a \cdot t_{n_k}c_{n_k} \leq 1$ and $a(1-a \cdot c_{n_k}t_{n_k}/2) > 1$ for all $k \geq k_o$.

We may apply lemma 1.1 (i) to the random variables S_{n_k}/s_{n_k} to find that, if $k \geq k_o$,

$$P_k < \exp(-a^2(1-a \cdot c_{n_k} t_{n_k}/2) \log \log s_{n_k}^2) < (\log s_{n_k}^2)^{-a}.$$

It is clear that (i) follows.

(ii). Let $0<a<1$, and define $\gamma = a^{-1} - 1$. Applying lemma 1.2 to the random variables S_{n_k}/s_{n_k}, we have, for all large k,

$$P_k > \exp(-a^2(1+\gamma) \log \log s_{n_k}^2) = (\log s_{n_k}^2)^{-a}. \quad \text{(ii)}$$

follows immediately.

LEMMA 3.2. Let $X_1, X_2, \ldots$ be independent random variables, each with mean 0 and variance 1, which satisfy Hypothesis A; i.e. there exist positive numbers $c_n = o((\log \log n)^{-\frac{1}{2}})$ such that, for all sufficiently large n,

$$(1) \quad e^{(t^2/2)(1-|t|c_n)} < Ee^{tX_k/\sqrt{n}} < e^{(t^2/2)(1+|t|c_n/2)}$$

for all $k \leq n$, provided $0<|t|c_n \leq 1$. Let (a_{nm}), $n \geq 1$, $m \leq n$, be a double sequence of non-negative reals; define $S_n = \Sigma_{m=1}^n a_{nm} X_m$, $s_n^2 = ES_n^2$, $t_n^2 = 2 \log \log s_n^2$, and

$A_n = \max_{1 \leq m \leq n} (a_{nm})$. Let $a>0$ and let $\{n_k\}$ be any increasing sequence of positive integers.

(i) If $a>1$, $\Sigma_{k=1}^{\infty} (\log s_{n_k}^2)^{-a} < \infty$ and $A_n \sqrt{n} \cdot c_n = o(s_n/t_n)$, then $\Sigma_{k=1}^{\infty} P[|S_{n_k}|>a\cdot s_{n_k}t_{n_k}]<\infty$ and, hence,

$$\limsup_{k\to\infty} |S_{n_k}|/(s_{n_k}t_{n_k})\leq a \quad \text{a.e.}$$

(ii) If $a<1$, $\Sigma_{k=1}^{\infty} (\log s_{n_k}^2)^{-a}=\infty$ and $A_n\sqrt{n}\cdot c_n=o(s_n/t_n)$, then $\Sigma_{k=1}^{\infty} P[S_{n_k}>a\cdot s_{n_k}t_{n_k}]=\infty$.

(iii) In particular, if $s_n^2\sim B^2\cdot n$ for some $B>0$, if there exists A such that $A_n\leq A$ for all n, and if $s_{n_k}\sim Dc^k$ for some $c>1$ and $D>0$, then $\Sigma_{k=1}^{\infty} P[|S_{n_k}|>a\cdot s_{n_k}t_{n_k}]<\infty$ for all $a>1$ (so $\limsup_{k\to\infty}|S_{n_k}|/(s_{n_k}t_{n_k})\leq 1$ a.e.), and

$\Sigma_{k=1}^{\infty} P[S_{n_k}>a\cdot s_{n_k}t_{n_k}] =\infty$ for all $0<a<1$.

PROOF. Choose k_o so large that (1) holds for all n_k such that $k\geq k_o$. Define $c_n'=A_n\sqrt{n}\cdot c_n/s_n = o(t_n^{-1})$. If

$0<tc_{n_k}'\leq 1$, then $(t\cdot a_{nm}\sqrt{n}/s_n)c_n\leq 1$ for any $m\leq n_k$; hence, it follows directly from (1) that

$$\exp((a_{nm}^2 t^2/(2s_n^2))(1-tc_{n_k}'))<Ee^{ta_{nm}X_m/s_n}$$

$$<\exp((a_{nm}^2 t^2/(2s_n^2))(1+tc_{n_k}'/2))$$

for all $m\leq n_k$. By independence, the hypotheses of lemma 3.1 are fulfilled. (ii) is immediate from lemma 3.1. Note that $-X_1,-X_2,\ldots$ satisfy Hypothesis A, so that this sequence also satisfies the conditions of lemma 3.1. Hence, if $a>1$ and $\Sigma_{k=1}^{\infty}(\log s_{n_k}^2)^{-a}<\infty$, then both of the series $\Sigma_{k=1}^{\infty} P[S_{n_k} > a \circ s_{n_k} t_{n_k}]$ and

$\Sigma_{k=1}^{\infty} P[-S_{n_k} > a \circ s_{n_k} t_{n_k}]$ converge. Then (i) follows from lemma 3.1 (i), since $P[|S_{n_k}| > a \circ s_{n_k} t_{n_k}] = P[S_{n_k} > a \circ s_{n_k} t_{n_k}]$

$+ P[-S_{n_k} > a \circ s_{n_k} t_{n_k}]$.

If $s_n^2 \sim B^2 \circ n$, $A_n \leq A$; and $s_{n_k} \sim D \circ c^k$ for some $c>1$, then

$$A_n \sqrt{n} \circ c_n / s_n \leq A \sqrt{n} \circ c_n / s_n \sim ABc_n = o((2 \log \log n)^{-\frac{1}{2}})$$
$$= o((2 \log \log s_n^2)^{-\frac{1}{2}}).$$

Furthermore, $\log s_{n_k}^2 \sim 2k \log c$, so, for $a>0$,

$\Sigma_{k=1}^{\infty} (\log s_{n_k}^2)^{-a}$ converges if and only if $a>1$. Hence, (iii) follows from (i) and (ii). Q.E.D.

DEFINITION 3.1. Let f be a real-valued function which is continuous on [0,1]. Define $f^* = \max_{0 \leq x \leq 1} |f(x)|$ and

$||f|| = (\int_0^1 f^2(t)dt)^{\frac{1}{2}}$, (i.e. the L^2-norm of f on [0,1].)

LEMMA 3.3. Let $X_1, X_2, \ldots$ be independent, $EX_n = 0$, $EX_n^2 = 1$, and assume that Hypothesis A holds. Let f be real-valued continuous function on [0,1], with $||f|| = 1$, and define $S_n = \Sigma_{m=1}^{n} f(m/n) X_m$, $s_n^2 = ES_n^2$, and $t_n^2 = 2 \log \log s_n^2$. For $1 < c < \sqrt{2}$, and each $k \geq 1$, define n_k to be the integral part of c^{2k}. Then

$$\limsup_{k\to\infty} \max_{n_{k-1}<n\leq n_k} \frac{|X_{n_{k-1}+1}+\cdots+X_n|}{s_{n_{k-1}} t_{n_{k-1}}} \leq 1 \text{ a.e.}$$

PROOF. Let us first note that

$$(2) \quad s_n^2 = \Sigma_{m=1}^n f^2(m/n) \sim n||f||^2 = n.$$

So n_k is a strictly increasing sequence if k is sufficiently large. Furthermore, $n_k \sim c^{2k}$ and $t_{n_k} \sim t_{n_{k-1}}$ as $k\to\infty$.

Let $\varepsilon>0$ and define, for each $k\geq 1$, and each $n_{k-1}<n\leq n_k$, $S_n^{(k)} = X_{n_{k-1}+1}+ \cdots +X_n$. Note that $ES_n^{(k)} = 0$ and $ES_n^{(k)^2} = n-n_{k-1}$. Now

$$(1+2\varepsilon)s_{n_{k-1}} t_{n_{k-1}} - \sqrt{2(n_k-n_{k-1})} = \sqrt{(n_k-n_{k-1})}\, t_{n_{k-1}}((1+2\varepsilon)\cdot s_{n_{k-1}}(n_k-n_{k-1})^{-\frac{1}{2}} - \sqrt{2}\, t_{k-1}^{-1})$$

$$\sim \sqrt{(n_k-n_{k-1})}\, t_{n_k}((1+2\varepsilon)(c^2-1)^{-\frac{1}{2}} - \sqrt{2}\, t_{n_{k-1}}^{-1}).$$

But $c^2-1<1$, and $\sqrt{2}\, t_{n_{k-1}}^{-1} < \varepsilon$ if k is sufficiently large, so $(1\pm 2\varepsilon)s_{n_{k-1}} t_{n_{k-1}} - \sqrt{2(n_k-n_{k-1})} > (1+\varepsilon)\sqrt{(n_k-n_{k-1})}\, t_{n_k}$ for all large k.

$$\therefore P'_k \equiv P[\max_{n_{k-1}<n\leq n_k} |S_n^{(k)}| > (1+2\varepsilon)s_{n_{k-1}} t_{n_{k-1}}]$$

$$\leq 2P[|S_{n_k}^{(k)}| > (1+2\varepsilon)s_{n_{k-1}} t_{n_{k-1}} - \sqrt{2(n_k-n_{k-1})}] \text{ by}$$

Levy's inequality

$$\leq 2P[|S_{n_k}^{(k)}| > (1+\varepsilon)\sqrt{(n_k-n_{k-1})}\, t_{n_k}] \text{ for large k.}$$

For each n such that $n_{k-1}<n\leq n_k$, and each $m\leq n$, define $a_{nm}=0$ or 1 accordingly as $m\leq n_{k-1}$ or $m>n_{k-1}$. Applying lemma 3.2 (iii) for this double sequence, we find

$$\Sigma_{k=1}^{\infty} P[|S_{n_k}^{(k)}|>(1+\varepsilon)\sqrt{(n_k-n_{k-1})}\,t_{n_k}]<\infty.$$

Therefore $\Sigma_{k=1}^{\infty} P_k' <\infty$ for each $\varepsilon>0$, so the result follows by the Borel-Cantelli lemma. Q.E.D.

For completeness' sake, the following useful result is stated.

LEMMA 3.4. Let $X_1, X_2, \ldots$ be independent, each with mean 0 and variance 1, and assume Hypothesis A holds. Then, for any integer $j\geq 0$,

$$\limsup_{n\to\infty} \frac{\Sigma_{m=1}^{n} m^j X_m}{n^j\sqrt{\frac{2}{2j+1}\, n \log\log n}} = 1 \text{ a.e.}$$

PROOF. Immediate from theorem 1.3.

3. Main Results

THEOREM 3.1. Let $X_1, X_2, \ldots$ be independent, each with mean 0 and variance 1, which satisfy Hypothesis A. Let f be a real-valued, continuous function on [0,1] with the additional property that the set $\{0<x<\beta \mid f(x) = 0\}$ has Lebesgue measure zero, for some $\beta>0$. Define

$S_n = \Sigma_{m=1}^{n} f(m/n)X_m$. Then $\limsup_{n\to\infty} S_n/(2n \log\log n)^{\frac{1}{2}} \geq ||f||$ a.e.

REMARK. The proof of theorem 3.1 given below follows a pattern similar to that of the proof of the corresponding half of Gaposhkin's result (see [7]). However, appropriate modifications to his proof have been made to accomodate the more general sequence of random variables. Lemma 3.2 is used to reduce the computation.

PROOF. Define $s_n^2 = ES_n^2$, $t_n^2 = 2 \log \log s_n^2$. Without losing generality, we may assume $||f|| = 1$. Then $s_n^2 \sim n$, by (2), and $t_n^2 \sim 2 \log \log n$.

For $c>1$, to be appropriately chosen later, define n_k to be the integral part of c^{2k}, $k \geq 1$. As will become apparent as the proof progresses, we will only be concerned with large values of k, so we note here that there exists a number $k_o > 0$ such that $n_k < n_{k+1}$ for all $k \geq k_o$. We will restrict ourselves to values of k in excess of k_o.

For $k>k_o$, define $T_k = \Sigma_{m=n_{k-1}+1}^{n_k} f(m/n_k)X_m$ and $\sigma_k^2 = ET_k^2$. It is easily shown that

(3) $\sigma_k^2 \sim n_k(1-I(c^{-2}))$, where $I(x)=\int_0^x f^2(t)dt$. By hypothesis, the function I is strictly increasing on $(0,\beta)$, so that $\sigma_k^2 \to \infty$ as $k \to \infty$ if we choose c so large that $c^{-2} < \beta$.

Let $0<\varepsilon<1$. For each $n_{k-1} < n \leq n_k$, and $m \leq n$, define $a_{nm}=0$ or $f(m/n_k)$ accordingly as $m \leq n_{k-1}$ or $m > n_{k-1}$. Then, by lemma 3.2 (iii), $\Sigma_{k=1}^{\infty} P[T_k>(1-\varepsilon)\sigma_k t_{n_k}] = \infty$. But the r.v.'s $T_{k_o}, T_{k_o+1}, \ldots$ are independent, so the Borel Zero-One Law implies

(4) $\limsup_{k\to\infty} T_k/(\sigma_k t_{n_k}) > (1-\varepsilon)$ a.e.

Define $N_c = \sqrt{I(c^{-1})}$ and $\bar{\sigma}_k^2 = s_{n_k}^2-\sigma_k^2$. Note that $N_c \to 0$ as $c\to\infty$. Choose $c>1$ so large that $N_c<\varepsilon/2$ and $1<\beta c$.

From (3) and the fact that $s_{n_k}^2 \sim n_k$, we have $\bar{\sigma}_k^2/s_{n_k}^2 \sim I(c^{-2})$

$<I(c^{-1})$, since $c^{-1}<\beta$.

(5) $\therefore$ $\bar{\sigma}_k/s_{n_k}<N_c<\varepsilon/2$ for all large k.

Now let, for $n_{k-1}<n\leq n_k$, and all $m\leq n$, $a_{nm}=f(m/n)$ or 0 accordingly as $m\leq n_{k-1}$ or $m>n_{k-1}$. Applying lemma 3.2 (iii) for this double sequence it follows that

$\limsup_{k\to\infty} |S_{n_k}-T_k|/(\bar{\sigma}_k t_{n_k}) \leq 1$ a.e. Hence, by (5),

$$\limsup_{k\to\infty} |S_{n_k}-T_k|/(s_{n_k} t_{n_k}) = \limsup_{k\to\infty} (\bar{\sigma}_k/s_{n_k})\cdot$$

$$(|S_{n_k}-T_k|)/(\bar{\sigma}_k t_{n_k}) \leq \varepsilon/2 \text{ a.e.}$$

But, note that for all large k, it follows (5) that

$$\sigma_k^2/s_{n_k}^2=1-\bar{\sigma}_k^2/s_{n_k}^2>1-N_c^2>1-(\varepsilon/2)^2>(1-\varepsilon/2)^2.$$

(6) i.e. $\sigma_k/s_{n_k} > 1-\varepsilon/2$ for all large k.

From (4), (5) and (6), it is easily seen that

$\limsup_{k\to\infty} S_{n_k}/(s_{n_k}t_{n_k}) \geq 1$ a.e. But, since $s_n^2 \sim n$ and $t_n^2 \sim 2\log\log n$, it follows that

$$\limsup_{n\to\infty} S_n/(2n\log\log n)^{\frac{1}{2}} \geq 1 \text{ a.e. as required.} \quad \text{Q.E.D.}$$

In the following two theorems, we will furnish conditions on f which are sufficient for equality to hold in the result of theorem 3.1.

THEOREM 3.2. Let $X_1, X_2, \ldots$ be independent, each with mean 0 and variance 1. Assume that Hypothesis A holds. Let f be a polynomial defined on [0,1], say,

$f(x) = a_0 + a_1x + \ldots + a_px^p$. Define $S_n = \Sigma_{m=1}^n f(m/n)X_m$.

Then $\limsup_{n\to\infty} S_n/(2n\log\log n)^{\frac{1}{2}} = ||f||$ a.e.

REMARK. Gaposhkin's method breaks down for part of the proof of this theorem. We have to use a new routine to prove that $R_k^{(2)}$ (defined below) becomes small as $k\to\infty$.

PROOF. The result is obvious if f is identically zero a.e. So we may assume $||f||>0$; in fact, there will be no loss of generality if we assume that $||f||=1$. In view of theorem 3.1, we need only prove the "$\leq$" part of the result.

For $c>1$, to be chosen later, and each $k\geq 1$, define n_k to be the integral part of c^{2k}. For each $n\geq 1$, define

$s_n^2 = ES_n^2$ and $t_n^2 = 2 \log \log s_n^2$. For all $k \geq 1$, let

$R_k = \max_{n_{k-1}<n\leq n_k} (s_n t_n)^{-1} |S_n - S_{n_{k-1}}|$, $R_k^{(1)} = \max_{n_{k-1}<n\leq n_k} (s_n t_n)^{-1} \cdot$
$|\Sigma_{m=1}^{n_{k-1}} \{f(m/n) - f(m/n_{k-1})\} X_m|$,

$R_k^{(2)} = \max_{n_{k-1}<n\leq n_k} (s_n t_n)^{-1} |\Sigma_{m=n_{k-1}+1}^{n} f(m/n) X_m|$ and let

$A = \Sigma_{m=0}^{p} |a_m|$. Note that $A>0$ and $R_k \leq R_k^{(1)} + R_k^{(2)}$.

Now $|\Sigma_{m=1}^{n_{k-1}} \{f(m/n) - f(m/n_{k-1})\} X_m| = |\Sigma_{j=0}^{p} a_j (n^{-j} - n_{k-1}^{-j}) \Sigma_{m=1}^{n_{k-1}} m^j X_m|$

$$\leq \Sigma_{j=0}^{p} |a_j| \left(\frac{n_k^j - n_{k-1}^j}{n_k^j}\right) n_{k-1}^{-j} \; |\Sigma_{m=1}^{n_{k-1}} m^j X_m|.$$

Since $n_k \sim c^{2k}$ as $k\to\infty$, $\lim_{k\to\infty} \frac{n_k^j - n_{k-1}^j}{n_k^j} = \lim_{k\to\infty} \frac{c^{2jk} - c^{2j(k-1)}}{c^{2jk}}$

$$= \frac{c^{2j} - 1}{c^{2j}} .$$

Therefore, by lemma 3.4, $\limsup_{k\to\infty} R_k^{(1)} \leq \Sigma_{j=0}^{p} |a_j| (c^{2j} - 1) c^{-2j}$.

Choose $c>1$ so close to 1 that

(7) $\Sigma_{j=0}^{p} |a_j| (c^{2j} - 1) c^{-2j} < \varepsilon/2$ and $c - 1 < \varepsilon/4$.

Hence $\limsup_{k\to\infty} R_k^{(1)} < \varepsilon/2$.

Now consider $R_k^{(2)}$: First,

(8) $|\Sigma_{m=n_{k-1}+1}^{n} f(m/n) X_m| \leq \Sigma_{j=0}^{p} |a_j| n_{k-1}^{-j} \; |\Sigma_{m=n_{k-1}+1}^{n} m^j X_m|$.

For $0<\varepsilon'<\varepsilon/A$, choose $c>1$ so close to 1 that (7) holds and

(9) $\varepsilon'^2/(c^{4p+2} - 1) > 4$.

For $0\leq j\leq p$, let $W_k=W_k(j) = \Sigma_{m=n_{k-1}+1}^{n_k} m^j X_m$ and $\tilde{\sigma}_{j\,k}^2 = \tilde{\sigma}_k^2 = EW_k^2(j)$.

Note that $\Sigma_{m=1}^{n} m^{2j} \sim n^{2j+1} \cdot \int_0^1 x^{2j}dx = n^{2j+1}/(2j+1)$, so

$$(10)\quad \tilde{\sigma}_k^2 \sim (n_k^{2j+1}-n_{k-1}^{2j+1})(2j+1)^{-1} \sim n_{k-1}^{2j} s_{n_{k-1}}^2 (c^{4j+2}-1)(2j+1)^{-1}$$

$$< n_{k-1}^{2j} s_{n_{k-1}}^2 t_{n_{k-1}}^2 (\varepsilon/A-\varepsilon')^2/2 \text{ for all large } k.$$

Furthermore, in view of (9) and (10), we have

$$(11)\quad \varepsilon'^2 n_{k-1}^{2j} s_{n_{k-1}}^2/\tilde{\sigma}_k^2 \xrightarrow{k\to\infty} \varepsilon'^2(2j+1)(c^{4j+2}-1)^{-1}$$

$$> \varepsilon'^2/(c^{4p+2}-1) > 4.$$

Hence, $\overline{P}_k=P[\max_{n_{k-1}<n\leq n_k} |\Sigma_{m=n_{k-1}+1}^{n} m^j X_m| > (\varepsilon/A)n_{k-1}^j s_{n_{k-1}} t_{n_{k-1}}]$

$\leq 2P[|W_k|>(\varepsilon/A)n_{k-1}^j s_{n_{k-1}} t_{n_{k-1}} -\sqrt{2}\,\tilde{\sigma}_k]$

$\leq 2P[|W_k|/\tilde{\sigma}_k > \varepsilon' n_{k-1}^j s_{n_{k-1}} t_{n_{k-1}}/\tilde{\sigma}_k]$ by (10)

$\leq 2P[|W_k|/\tilde{\sigma}_k > 2t_{n_{k-1}}]$ by (11).

But, letting $a_{nm} = 0$ or m^j accordingly as $m\leq n_{k-1}$ or $n_{k-1}<m\leq n\leq n_k$, we can apply lemma 3.2 (iii) to find that

$\Sigma_{k=1}^{\infty} P[|W_k|>2\tilde{\sigma}_k t_{n_{k-1}}] <\infty.$

$\therefore\ \Sigma_{k=1}^{\infty} \overline{P}_k <\infty.$

So, for all $0\leq j\leq p$, $\limsup_{k\to\infty} \max_{n_{k-1}<n\leq n_k} |W_k(j)|/(n_{k-1}^j s_{n_{k-1}} t_{n_{k-1}})$

$< \varepsilon/A$ a.e.

From (8), then, it follows that

$$\limsup_{k \to \infty} R_k^{(2)} \leq (\varepsilon/A) \Sigma_{j=0}^{p} |a_j| = \varepsilon \text{ a.e.}$$

(12) $\therefore \limsup_{k \to \infty} R_k \leq 3\varepsilon/2.$

For brevity, let $v_n = (s_n t_n)^{-1}$. Then

(13) $v_{n_k}/v_{n_{k-1}} \sim \sqrt{n_{k-1}/n_k} \sim 1/c$ and

(14) $$\max_{n_{k-1}<n\leq n_k} |v_n S_n - v_{n_{k-1}} S_{n_{k-1}}| \leq R_k + (1 - v_{n_k}/v_{n_{k-1}}) \circ v_{n_{k-1}} |S_{n_{k-1}}|.$$

If we define $a_{nm} = f(m/n)$, for each $n \geq 1$ and $m \leq n$, in lemma 3.2 (iii), then

(15) $\limsup_{k \to \infty} v_{n_k} S_{n_k} \leq 1$ a.e.

Hence, if k is sufficiently large,

$$\max_{n_{k-1}<n\leq n_k} |v_n S_n| \leq \max_{n_{k-1}<n\leq n_k} |v_n S_n - v_{n_{k-1}} S_{n_{k-1}}| + v_{n_{k-1}} |S_{n_{k-1}}|$$

$\leq R_k + (1 - v_{n_k}/v_{n_{k-1}}) v_{n_{k-1}} |S_{n_{k-1}}| + v_{n_{k-1}} |S_{n_{k-1}}|$

by (14)

$< 2\varepsilon + (c-1)(1+\varepsilon) + 1 + \varepsilon$, by (12), (13) & (15)

$\leq 1 + 3\varepsilon + \varepsilon \circ (1+\varepsilon)/4$ by (7)

$\leq 1 + 5\varepsilon.$

$\therefore 0 = P[\max_{n_{k-1}<n\leq n_k} |S_n|/(s_n t_n) > 1+5\varepsilon \text{ i.o.}]$

$\geq P[S_n/(s_n t_n) > 1+5\varepsilon \text{ i.o.}]$

Hence, $\limsup_{n \to \infty} S_n/(s_n t_n) \leq 1+5\varepsilon$ a.e. for all $\varepsilon > 0$.

Therefore, $\limsup_{n \to \infty} S_n/(s_n t_n) \lesssim 1$ a.e. as required. Q.E.D.

REMARK. It follows from the First Weierstrass Theorem that if f is a continuous function on [0,1], then there exists a sequence $p_1, p_2, \ldots$ of polynomials on [0,1] such that $p_n \to f$ uniformly as $n \to \infty$. So it is plausible that we should be able to replace the hypothesis "f is a polynomial" in Theorem 3.2 by the more general condition "f is a continuous function." The proof of this conjecture, however, would require the interchange of a limit and a limit superior; we have not yet been able to establish such a result.

Nevertheless, theorem 3.2 can be extended to include functions which are power series; this is done in theorem 3.3.

THEOREM 3.3. Let $X_1, X_2, \ldots$ be independent, each with mean 0 and variance 1. Assume that Hypothesis A is satisfied by the $\{X_n\}$ sequence. Let f(x) be a power series on [0,1], say, $f(x) = \Sigma_{j=0}^{\infty} c_j x^j$. Define $S_n = \Sigma_{m=1}^{n} f(m/n) X_m$, and $s_n^2 = ES_n^2$. Then $\limsup_{n \to \infty} S_n/(2n \log \log n)^{\frac{1}{2}} = ||f||$ a.e.

REMARK. While the following proof will resemble that of Gaposhkin [7], some stronger arguments are required. Gaposhkin makes use of the fact that the function $(1-x)^{\alpha}$ is zero when x=1; but we have made no such assumption about f in our theorem.

PROOF. If $f(x)=0$ a.e., then the theorem is obvious. So we may assume that $||f||>0$; in fact, we will again assume without loss of generality that $||f||=1$.

In view of theorem 3.1, it is clear that we need only prove the "$\leq$" part of the result.

For $c>1$, to be chosen later, and each $k\geq 1$, let n_k be the integral part of c^{2k}. Note that since f is a power series, it is continuous (in fact, it is absolutely continuous), it is uniformly convergent and absolutely convergent, in particular, $\Sigma_{j=0}^{\infty} |c_j| <\infty$.
Define $t_n=(2 \log \log n)^{\frac{1}{2}}$, and $R_k, R_k^{(1)}, R_k^{(2)}$ as in the proof of theorem 3.2. Define, for $k\geq 1$, $n_{k-1}<n\leq n_k$, $S_n^{(k)}=X_{n_{k-1}+1}+ \cdots +X_n$. Then

$$(16) \quad \Sigma_{m=n_{k-1}+1}^{n} f(m/n)X_m = \Sigma_{m=n_{k-1}+1}^{n} [f(j/n)-f(j+1/n)]S_j^{(k)},$$

where we define $f(x)=0$ if $x>1$.

Let $\varepsilon>0$. By the definition of absolute continuity, $\exists\ \delta>0$ such that for any finite number of essentially disjoint closed intervals contained in $[0,1]$, say, $[a_n,b_n]$, $n=1,2, \ldots, N$, if $\Sigma_{n=1}^{N}|b_n-a_n|<\delta$, then $\Sigma_{n=1}^{N}|f(b_n)-f(a_n)|<\varepsilon/2$.

Choose $c>1$ so close to 1 that $c^2<2$, $c-1<\varepsilon/4$, and $(c^2-1)/c^2<\delta/2$. Then, for all k large, $1-n_{k-1}/n_k<\delta$, so that if $n_{k-1}<n\leq n_k$, then $1-(n_{k-1}+1)/n<\delta$.

$$\therefore \Sigma_{j=n_{k-1}+1}^{n} |f(j/n)-f(j+1/n)| < \varepsilon/2 + |f(1)|.$$

So, by (16), $R_k^{(2)} < (\varepsilon/2 + |f(1)|) \max_{n_{k-1}<n\leq n_k} |S_n^{(k)}|/(s_n t_n)$.

Applying lemma 3.3, we have $\limsup_{k\to\infty} R_k^{(2)} \leq |f(1)|+\varepsilon/2$.

Now we will determine $\limsup_{k\to\infty} R_k^{(1)}$: Proceeding exactly in accord with Gaposhkin's method, we find:

$$\Sigma_{m=1}^{n_{k-1}} [f(m/n)-f(m/n_{k-1})]X_m \leq \Sigma_{j=0}^{\infty} |c_j| \frac{n_k^j - n_{k-1}^j}{n_k^j n_{k-1}^j} |\Sigma_{m=1}^{n_{k-1}} m^j X_m|.$$

So, by the definition of $R_k^{(1)}$, Fatou's lemma, and lemma 3.4,

$$\limsup_{k\to\infty} R_k^{(1)} \leq \Sigma_{j=0}^{\infty} |c_j| \limsup_{k\to\infty} \left\{ \frac{n_k^j - n_{k-1}^j}{n_k^j} \cdot \limsup_{k\to\infty} \frac{|\Sigma_{m=1}^{n_{k-1}} m^j X_m|}{n_{k-1}^j s_{n_{k-1}} t_{n_{k-1}}} \right\}$$

$$= \Sigma_{j=0}^{\infty} |c_j| \frac{c^{2j}-1}{c^{2j}} < \varepsilon/2 \text{ if } c \text{ is chosen}$$

close enough to 1. Hence $\limsup_{k\to\infty} R_k < |f(1)|+\varepsilon$.

Let $v_n = (s_n t_n)^{-1}$. Then, as in the proof of theorem 3.2, for all large k

s

$$\sup_{n_{k-1}<n\leq n_k} |v_n S_n| \leq R_k + (1 - v_{n_k}/v_{n_{k-1}}) v_{n_{k-1}} |S_{n_{k-1}}| + v_{n_{k-1}} |S_{n_{k-1}}|$$

by (14)

$$\leq |f(1)| + 2\varepsilon + (c-1)(1+\varepsilon) + 1 + \varepsilon \quad \text{by (15)}$$

$$\leq 1 + |f(1)| + 3\varepsilon + \varepsilon\circ(1+\varepsilon)/4 \text{ by choice of}$$

$c<1+\varepsilon/4$.

$$\lesssim 1+|f(1)|+5\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, it follows that

$$\limsup_{n\to\infty} S_n/(s_n t_n) \lesssim 1+|f(1)| \text{ a.e.}$$

Hence it follows that, in general, because $s_n \sim n$,

$$(17)\quad \limsup_{n\to\infty} S_n/(2n\log\log n)^{\frac{1}{2}} \lesssim ||f||\cdot(1+|f(1)|) \text{ a.e.}$$

Up to this point, the proof has virtually duplicated Gaposhkin's method; indeed, if $f(1)=0$, as in the case Gaposhkin considered, then the proof would be complete. However, since we have not made such an assumption, we shall now provide arguments to show that the $f(1)$ term of (17) may be deleted.

For $m\gtrsim 0$, define $g_m(x) = \Sigma_{j=0}^m c_j x^j$ and $h_m(x)=f(x)-g_m(x)$. Then, $g_m(x)$ is a polynomial and $h_m(x)$ is a power series for each $m\gtrsim 0$, and, since $g_m(x) \to f(x)$ uniformly in x as $m\to\infty$, we have $||g_m||\to||f||$, and $||h_m||\to 0$.

Hence, for any $m\gtrsim 0$,

$$\limsup_{n\to\infty} S_n/(2n\log\log n)^{\frac{1}{2}} \lesssim \limsup_{n\to\infty} \frac{\Sigma_{k=1}^n g_m(k/n)X_k}{(2n\log\log n)^{\frac{1}{2}}}$$

$$+ \limsup_{n\to\infty} \frac{\Sigma_{k=1}^n h_m(k/n)X_k}{(2n\log\log n)^{\frac{1}{2}}}$$

$$\lesssim ||g_m||+||h_m||\cdot(1+|h_m(1)|) \text{ a.e.}$$

by Theorem 3.2 and (17).

Letting $m\to\infty$, we get $\limsup_{n\to\infty} S_n/(2n\log\log n)^{\frac{1}{2}} \lesssim ||f||$ a.e.

REMARK. Since the function $f(x)=(1-x)^{\alpha}$, for any $\alpha>0$, has a power series representation (using the Binomial Theorem), Gaposhkin's result in [7] follows from Theorem 3.3.

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