ON SOME PROBLEMS IN THE THEORY OF OPTIMAL STOPPING RULES AND LOG LOG LAW*

by

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CHAPTER I

EXISTENCE OF OPTIMAL STOPPING RULES FOR REWARD SEQUENCE S_n/n

1.1 Introduction and Summary

Let (Ω,F,P) be a probability space. A nondecreasing sub σ-fields of F is called a stochastic basis. A stochastic sequence $(X_n, F_n, n \geq 1)$ consists of a stochastic basis (F_n) and a sequence (X_n) of random variables (r.v.) such that X_n is F_n-measurable. A given sequence $\{X_n\}$ of r.v. is a stochastic sequence if we put $F_n = B(X_1, X_2,\ldots, X_n)$, the σ-field generated by $X_1,X_2,\ldots X_n$. For a given stochastic basis a r.v. t with values $n=1,2\ldots$. ∞ such that $(w|\ t(w) = n) \epsilon F_n$ for each $n \geq 1$ is called a Stopping time (s.t.). A s.t. is called a stopping rule(s.r.) if $P(t < \infty) = 1$. We observe the X's sequentially and must decide when to stop sampling. If we stop at time n we receive a reward $Z_n = f(X_1,X_2,\ldots X_n)$, which depends upon past observation only.

Unless otherwise stated we shall always assume $E(Z_n^-) < \infty$ where $\bar{Z} = \max(-Z,0)$. Let

$$C = (t \mid E(Z_t^-) < \infty,\ t \text{ is a s.r.})$$

$$C_n = (t \mid t \in C,\ P(t \geq n) = 1)$$

$$V_n = \sup_{t\in C_n} E(Z_t),\ V = \sup_{t\in C} E(Z_t)$$

Let t be a s.r. such that $E(Z_t)$ exists.

The fundamental problem in the theory of optimal stopping rule is how can we find the value of V and what s. r. will achieve V or come close to it? We shall recall that the ess sup of a family of r.v.'s q_t, $t \epsilon T$ is a r.v. Q such that (1) $Q \geq q_t$, $t \epsilon T$, and (2) if Q' is a r.v. such that $Q' \geq q_t$, $t \epsilon T$, then $Q' \geq Q$. It is known that the ess sup of a family of r.v. 's always exists, and can be assumed to be the sup of some countable sub-family.

Let $\gamma_n = \underset{t \epsilon c_n}{\text{ess sup}} E(Z_t | F_n)$. Then the Fundamental theorem says:

(a) $\gamma_n = \max (Z_n, E(\gamma_{n+1} | F_n))$a.e.

(b) $E(\gamma_n) = V_n$

Then the Functional equation rule (FER) is defined as

$$\sigma = \text{first } n \geq 1 \text{ such that } Z_n = \gamma_n$$
$$= \infty \text{ if } Z_n < \gamma_n \text{ for all n.}$$

In general $P(\sigma < \infty) < 1$ and σ is not a s.r. Seigmund (see [21]) has shown, however, that if we enlarge our class of procedures by enlarging to s.t., with the convention that $Z_t = Z_\infty = \limsup Z_n$ when $t = \infty$, then V is not increased and under the condition $E(\sup Z_n^+) < \infty$, σ is optimal in the extended class.

Let $(X_n, F_n, n \geq 1)$ be a stochastic sequence. Let $S_n = \sum_1^n X_i$. Let the reward sequence be of the form $Z_n = h_n(s_n)$. We are concerned here with finding stopping rule t which maximizes our expected reward, $E(h_n(S_n))$. As history of the problem, we shall state and compare a few results known in the literature:

(1) Chow and Robbins [4]

Let $X_1, X_2, \ldots$ be a sequence of independent identical (i.i.d.) r.v. with $P(X=1)=P(X=-1) = \frac{1}{2}$. For the reward sequence (R.S.) $\frac{i+S_1}{j+1}$, $\frac{i+S_2}{j+2}$, ... $i=0, \pm 1, \pm 2, \ldots$ and $j=0,1,w,\ldots$ there exists a minimal optimal stopping rule τ (i) defined by τ (i)=first $n \geq 1$ such that $a_{j+n}(i+S_n)=0$

$= \infty$ if no such n exists

where $a_N^N(i)=o$, $a_n(i) = \lim_{N\to\infty} a_n^N(i)$

$$a_n^N(i) = \max(i^+/n, \sup_{t\epsilon T_{N-n}} E(i+S_t)^+ /n+t)) - i^+/n$$

where T_{N-n} = class of all s.r. $\leq$ N-n. n=1,2,... N.

The main points in Chow and Robbins' [4] method are by usual backward induction (see, e.g. [28]) they have shown that there exists a minimal optimal s.r. for their R.S. and then passing to the limit $N \to \infty$ it is shown that there exists an optimal element in C iff $\tau_j^*(i) = \lim_{N\to\infty} \tau_j^N(i)$ is in C i.e. iff $P(\overset{*}{\tau}_j(i) < \infty) = 1$. To prove $P(\tau_j^*(i) < \infty) = 1$ their lemma 1 states; $a_n(0)= \sup_{t\epsilon C} E(S_t^+/(n+t)) \leq 1/n^{\frac{1}{2}}$

which was proved by solving some difference equations and applying Stirling's approximation, suited only for coin tossing r.v. Their lemma 4 showed that for $n \geq n_o$ and $i > 13n^{\frac{1}{2}}$ implies there is no favorable continuation. The Law of Iterated Logarithm implies that the latter probability is one.

(2) Dvoretzky [10]

Let $X_1, X_2, \ldots$. be i.i.d. r.v. with mean zero and positive finite variance σ^2. Then there is a s.r. $\tau \epsilon C$ such that $E(S_\tau/\tau^\alpha) = \sup_{t \epsilon C} E(S_t/t^\alpha)$ for $\alpha > \frac{1}{2}$ and $0 < E(S_\tau/_\tau) < \pi\sigma/6^{\frac{1}{2}}$. Dvoretzky's method consists of proving lemma 1 of Chow and Robbins [4] by taking into consideration of second moment. Then by series of lemmas and repeated applications of Kolmogorov's inequality he proved his lemma 8 which is the generalization of lemma 4 of Chow and Robbins [4]. But instead of considering truncated optimal rules he proved $E(\text{Sup } S^+_{n/n}) < \infty$ (which is lemma 9 of [10]) and then appealing to theorem 1 of Chow and Robbins [28] and the Law of Iterated Logarithm he proved the existence of optimal s.r.

(3) Teicher and Wolfowitz [27]

Let $X_1, X_2, \ldots$ be i.i.d. r.v. with $E(X) = 0$, $E(X^2) < \infty$. Let the R.S. be $C_n S_n^j$ $j=1,2$ and $C_n > 0$, $C_{n+1} \leq C_{n+2} C_n$, $(n+1)^j C_{n+1} \leq n^j C_n$ then there exists a S.R. t^* such that $E(C_{t^*} S^j_{t^*}) = \sup_{t \epsilon C} E(C_t S^j_t)$ for $j=1,2$.

Teicher and Wolfowitz used the classical sequential analysis method of Wald and Wolfowitz. Lemma 5 of them follows from Dubins and Freedman [9] and is comparable to lemma 1 of [4] & lemma 3 of [10]. their lemma 6 showed that for large K and n sufficiently large $S_n > Kn^{\frac{1}{2}}$ implies there is no favorable continuation. This lemma follows from an invariance theorem of Kac and Erdos and is comparable to lemma 4 of [4] and lemma 8 of [10]. Lemma 4 follows from Weiner's Dominated Ergodic theorem and is comparable to lemma 3 of [4].

(4) Siegmund, Simmons, and Feder [21]

Let X_1, X_2,...... be i.i.d. r.v. with $E(X) = 0$. Let the reward sequence be $(Z_n) = (n^{-\alpha} |S_n|^{\beta})$ where $2\alpha > \beta > 0$ and $E(|X|^{\max(2,\beta)}) < \infty$ then the FER is optimal and also there exists a $K > 0$ for which the FER stops at (n,y) whenever $y > K n^{\frac{1}{2}}$.

Using this basic R.S. they examined the problem of optimality of R.S. of the form $C_n S_n$, $C_n |S_n|^{\beta}$, $n^{-\alpha} \log^{+}|S_n|$, etc.

Departing from the traditional approach of requiring that $E(\sup_n h_n(S_n)) < \infty$, they consider the class of procedure by dropping the requirement that $P(t < \infty) = 1$, and introduce the extended s.r. Then by modification of Teicher and Wolfowitz [27] and Dvoretsky's [10] methods they are able to prove existence of certain reward sequences h_n of more complicated form by relating to them to a particularly simple form.

(5) Siegmund [24] in an unpublished work proved that there exists an extended optimal s.r. for the R.S. S_n/n when X_1, X_2 .... be independent r.v. (not necessarily identical) with $E(X_n) = 0$, $E(X_n^2) = 1$ for all n and if moreover $P(S_n \geq Kn^{\frac{1}{2}}$ i.o.$) = 1$ then FER is optimal.

He first showed that $E(\sup_{n \geq 1} S_n/n) < \infty$ which implies that FER is optimal in the extended class.

His lemma 2 states that if $K > 4$ then for any extended s.r. t $E(S_t/t) < S_n/n$ on the set $(P(t=\infty | F_n) > 4/K,\ S_n > k\, n^{\frac{1}{2}})$ which follows from Haje k-Renyi inequality. Then under the condition $P(S_n \geq Kn^{\frac{1}{2}}$ i.o.$) = 1$, he proved by a contradiction argument that

FER stops with probability 1.

(6) Recently Chow [7] has proved that if $(S_n, F_n\ n\geq 1)$ be Martingale with $E(y_n^2|F_{n-1}) = \sigma_n^2$, where $y_n = S_n - S_{n-1}, F_o \subset F_1 \subset F_2 \ldots$ σ-fields, $X_n = S_n/s_n$, $o < s_n = \sum_1^n \sigma_k^2 \to \infty$ & (X_n) uniformly intégrable, $E(\frac{1}{s_1^{1/2}}) < \infty$, then FER is optimal in the extended class.

Moreover if, for some $K > 1$, $P(S_n \geq K\, n^{\frac{1}{2}}$ i.o.$) = 1$, then FER stops with probability 1. He appealed to his general theory, i.e. usual γ_n , V_n, & σ and proved that FER σ is optimal in the extended sense by showing that $X_n \to 0$ a.e. Then as in the proof of Siegmund's result [24], replacing Hajek-Renyl inequality by his Martingale extension of the last inequality he proved that FER stops with probability 1.

Motivated by Siegmund [2] and Chow's work we extended Siegmund, Simmons and Feder's work [21] in the Martingale difference sequences. Our method is a modification of Chow's [7] and Siegmund's [24] method and an application of Burkholder's inequality [2].

It is worth to compare some unpublished work of A. Dvoretzky [11] to the last mentioned Chow's work and our work. Dvoretzky proved that if $(X_n, F_n, n\geq 1)$ be a martingale difference sequence with (a) $E(X_n^2 \mid F_{n-1}) = \sigma^2$, constant $< \infty$ (b) $1/n \sum_1^n (X_1^2|F_{i-1}) > 0$ (i.e. all r.v. are not degenerate)

(c) $1/n \sum_1^n \int_{|X_i| > \sigma n^{1/2}} X_i^2 \to 0$ as $n \to \infty$ (Lindeberg's cond.)

then FER is optimal for the R.S. (S_n/n) where $S_n = X_1 + \ldots + X_n$.

This time his proof consists of proving all lemmas in conditional expection form. Then he proved the deep central limit theorem and law of iterated logarithm for martingale difference sequences (which is a slight modification of Levy's results) to get his result.

In i.i.d. case C.L.T. and Hartman & Wintner's law of iterated log implies $P(S_n > K n^{\frac{1}{2}} \text{ i.o.}) = 1$ for $0 < k < \infty$

Also in i.i.d. case if second moment exists and the r.v. are not degenerate) Lindeberg's condition holds.

Therefore in i.i.d. Case Dvoretzky's conditions and Chow & Siegmund's conditions are equivalent.

It is to be noted that if $X_1, X_2, \ldots$ be independent r.v. with $E(X_n) = 0$, $E(X_n^2) = 1$ and $S_n/n^{\frac{1}{2}} \xrightarrow{\mathscr{L}} N(0,1)$ then

$$
\begin{aligned}
P(S_n > K n^{\frac{1}{2}} \text{ i.o.}) &= P(\limsup S_n/n^{\frac{1}{2}} > K) \\
&= \lim_{i \to \infty} P(\sup_{n \geq i} S_n/n^{\frac{1}{2}} > K) \\
&\geq \lim_{i \to \infty} P(S_i/i^{\frac{1}{2}} > K) = 1 - \Phi(K) > 0
\end{aligned}
$$

$$\text{if } -\infty < K < \infty$$

where $\Phi(x)$ is standard normal d.f.

Hence Siegmund's result [24] in the independent case covers Dvoretzky's [11] result.

It is worth mentioning that whenever law of iterated logarithm holds (not necessarily independent case) Siegmund and Chow's condition $P(S_n > K n^{\frac{1}{2}} \text{ i.o.}) = 1$ holds. We do not know any simple sufficient condition (besides law of iterated log) for $P(S_n > K n^{\frac{1}{2}} \text{i.o.}) = 1$

when (S_n) is a martingale.

So far in the literature the finiteness of variance of r.v. is an essential condition.

Dvoretzky [10] conjectured about the existence of an Optimal Stopping rule for the R.S. $\{S_n/n\}$ when $\{X_n\}$ are i.i.d., $E(X) = 0$, $E(X^\alpha) < \infty$ for $1 < \alpha < 2$.

In section 1.3 we partially proved his conjecture when $\{X_n\}$ are i.i.d. with common symmetrical Stable distribution with characteristic exponent $1 < \alpha < 2$.

1.2 Reward Sequence $\{|S_n|^\beta/n^\alpha\}$.

Lemma 1.1

Let $(S_n = X_1 + X_2 + \dots + X_n \,, F_n \,, n \geq 1)$ be a martingale with $E(|X_n|^{\max(2,\beta)}) \leq C < \infty$ Then there exists a constant $A_\beta > 0$ such that $E(Z_n) \leq A_\beta n^{\frac{\beta}{2} - \alpha}$ where $Z_n = \left(\frac{|S_n|^\beta}{n^\alpha}\right)$ $\beta > 0$ and hence $\lim_{n \to \infty} E(Z_n) = 0$ where $\alpha > \beta/2$.

(This lemma is the martingale extension of lemma 2 of Siegmund, Simmons and Feder [21])

Proof. Without loss of generality let $C = 1$

If $0 < \beta < 2$,

$$E(Z_n) \leq E^{\beta/2}(Z_n^{2/\beta}) = E^{\beta/2}(n^{-2\alpha/\beta}|S_n|^2) \leq A_\beta n^{\beta/2-\alpha}$$

If $\beta \geq 2$,

then by Burkholder's inequality [2]

$$E(|S_n|^\beta/n^\alpha) \leq M_\beta n^{-\alpha} E\left(\sum_{i=1}^{n} X_i^2\right)^{\beta/2}$$

by Holder's inequality $\leq M_\beta n^{-\alpha} n^{\beta/2-1} \; E\left(\sum_{i=1}^{n} |X_i|^\beta\right)$

$$\leq A_\beta \; n^{\beta/2-\alpha}$$

Lemma 1.2 Let $(S_n = X_1 + X_2 + \ldots\ldots + X_n,\ F_n,\ n \geq 1)$ be a martingale and $E(|X_n|^{\max(2,\beta)} \;|F_{n-1}) \leq 1$ a.e. for $n=1,2,\ldots$ $0<\beta<2\alpha$ let t be a stopping time. Then

$$E\left(\frac{|X_{n+1}+\ldots\ldots+X_t|^\beta}{t^\alpha} I_{(t<\infty)} \mid F_n\right) \leq B_\beta n^{\beta/2-\alpha} \text{ a.e. on } (t\geq n+1)$$

(This lemma is due to Chow [7] if $\alpha = \beta = 2$ and $B_2 = 1$)

Proof. Let $\beta > 2$,

Define $S_N^{(n)} = X_{n+1} + \ldots\ldots + X_N$

Let $A \in F_n$

Then $(S_k^{(n)} I_{(n<t)A}, F_k, k > n)$ is a martingale with difference sequence $X_k I_{(n < t)A}$.

Since $|S_k^{(n)}|^\beta$ is a submartingale with respect to F_k and $(n < t \leq k-1) \in F_{k-1}$,

$$\int_{(t>n)A} \frac{|S_t^{(n)}|^\beta}{t^\alpha} = \sum_{k=n+1}^{\infty} \int_{A(k\leq t<k+1)} \frac{|S_t^{(n)}|^\beta}{t^\alpha} = \sum_{k=n+1}^{\infty} k^{-\alpha} \int_{(t=k)A} |S_k^{(n)}|^\beta$$

$$= \sum_{k=n+1}^{\infty} k^{-\alpha} \left[\int_{A(n<t\leq k)} |S_k^{(n)}|^{\beta} - \int_{A(n<t\leq k-1)} |S_k^{(n)}|^{\beta} \right]$$

$$= \sum_{k=n+1}^{\infty} k^{-\alpha} \int_{A(n<t\leq k)} |S_k^{(n)}|^{\beta} - \int_{A(n<t\leq k-1)} |S_{k-1}^{(n)}|^{\beta} \Big]$$

$$= \sum_{k=n+1}^{\infty} k^{-\alpha} C_k \quad \text{Where } C_k = \int_{A(n<t\leq k)} |S_k^{(n)}|^{\beta} - \int_{A(n<t\leq k-1} |S_{k-1}^{(n)}|^{\beta}$$

By Burkholder's inequality [see [2]]

for $N=n+1$ $n+2$ ...

$$\text{Let } d_N = \sum_{n+1}^{N} C_k \leq \int_{A(n<t)} |S_N^{(n)}|^{\beta} = E(|S_N^{(n)}|^{\beta} I_{(n<t)A})$$

$$\leq A_{\beta} E((I_{A(n\ t)}(X_{n+1}^2 + \ldots\ldots + X_N^2)\ )^{\beta/2})$$

$$= A_{\beta} \int_{A(n<t)} (X_{n+1}^2 + \ldots\ldots + X_N^2)^{\beta/2})$$

By Holder's inequality,

$$d_N \leq A_{\beta} \int_{A(n<t)} (\sum_{j=n+1}^{N} |X_j|^{\beta})) \ (\sum_{j=n+1}^{N} 1\ )^{\beta/2-1}$$

$$\leq A_{\beta} (N-n)^{\beta/2} . P(A(n< t))$$

Now by Abel's summation method

$$\sum_{k=n+1}^{N} k^{-\alpha} c_k = \sum_{k=n+1}^{N} k^{-\alpha}(d_k - d_{k-1}) \quad \sum_{k=n+1}^{N} (k^{-\alpha}-(k+1)^{-\alpha})d_k + (N+1)^{-\alpha}d_N$$

$$\leq \sum_{k=n+1}^{N} (k^{-\alpha}-(k+1)^{-\alpha}) A_\beta (k-n)^{\beta/2} P(A(n<t))$$

$$+ (N+1)^{-\alpha}d_N (N-n)^{\frac{\beta-\alpha}{2}} A_\beta P(A(n<t))$$

$$= A_\beta P(A(n<t)) \sum_{k=n+1}^{N} k^{-\alpha} \left((k-n)^{\beta/2} -(k-1-n)^{\beta/2} \right)$$

$$\therefore \sum_{k=n+1}^{\infty} k^{-\alpha} c_k \leq A_\beta P(A(n<t)) \sum_{k=n+1}^{\infty} k^{-\alpha}\left((k-n)^{\beta/2}-(k-1-n)^{\beta/2}\right)$$

Therefore $E\left(\left|\dfrac{S_t^{(n)}}{t^{\alpha}}\right|^{\beta} I_{(t<\infty)} \Big| F_n\right) \leq A_\beta \displaystyle\sum_{k=1}^{\infty} \frac{k^{\beta/2} - (k-1)^{\beta/2}}{(n+k)^{\alpha}}$ on $(t>n)$

$$\leq A_\beta \text{ const} \sum_{1}^{\infty} k^{\frac{\beta}{2}-1}/(n+k)^{\alpha} \leq \text{const} \int_0^{\infty} u^{\frac{\beta}{2}-1}(n+u)^{-\alpha} du$$

$$= \text{const} \int_0^{\infty} (n+u)^{-\alpha+\frac{\beta}{2}+1} du = B_\beta n^{\beta/2-\alpha} \quad \text{a.e on } (t\geq n+1)$$

If $\beta<2$ $E(t^{-\alpha}|X_{n+1}+\ldots\ldots + X_t|^{\beta} I_{(t<\infty)}| F_n)$

By Holder's inequality

$$\leq E^{\beta/2}(t^{-2\alpha/\beta}|X_{n+1}+ \ldots\ldots+X_t|^2 I_{(t<\infty)}|F_n)$$

$$\leq (B_\beta n^{1-2/\beta})^{\beta/2} = \text{const. } n^{\beta/2-\alpha}$$

Lemma 1.3 Let $(Z_n, G_n, n\geq 1)$ be an integrable stochastic sequence and $T_n = Z_1 + \ldots\ldots + Z_n$. Suppose that for positive integer n and some s.t.t (relative to G_n) for which

$E(|T_t|^\beta / t^\alpha)$ exists and for $0<\beta<2\alpha$

$$E\left(\frac{|Z_{n+1} + \ldots\ldots + Z_t|^\beta}{t^\alpha} \Big| G_n\right) \leq B_\beta n^{\beta/2-\alpha} \text{ a.e. on } (t > n)$$

then for any $K > \dfrac{2^\alpha B_\beta}{2^\alpha - 1}$

$$E\left(|T_t|^\beta / t^\alpha \Big| G_n\right) < |T_n|^\beta / n^\alpha \quad \text{on}$$

$$A = \left(t > n > m_0 \ , \ P(t=\infty | G_n) > \frac{2^\alpha B_\beta / K}{2^\alpha - 1} \ , \ |T_n|^\beta \geq K n^{\beta/2}\right)$$

m_o is sufficiently large

(This lemma is due to Chow [7] in case $\beta = \alpha = 2$, $B_2=1$)

Proof. Case 1 $\beta \leq 1$

On A we have by C_r-inequality

$$E\left(|T_t|^\beta / t^\alpha \Big| G_n\right) \leq E\left(\frac{|T_n|^\beta + |Z_{n+1} + \ldots + Z_t|^\beta}{t^\alpha} \Big| G_n\right)$$

$$= |T_n|^\beta E(1/t^\alpha | G_n) + E\left(\frac{|Z_{n+1} + \ldots\ldots + Z_t|^\beta}{t^\alpha} \Big| G_n\right)$$

$$\leq |T_n|^\beta \left(\frac{P(t\leq 2n | G_n)}{n^\alpha} + \frac{P(t>2n | G_n)}{(2n)^\alpha}\right) + B_\beta n^{\beta/2-\alpha}$$

$$= |T_n|^\beta (1/n^\alpha - \frac{(2^\alpha-1)P(t\geq 2n)}{(2n)^\alpha} G_n)) + B_\beta n^{\beta/2-\alpha}$$

$$\leq |T_n|^\beta (1/n^\alpha - \frac{(2^\alpha-1)P(t=\infty|G_n)}{(2n)^\alpha}) + B_\beta n^{\beta/2-\alpha}$$

$$\leq |T_n|^\beta/n^\alpha - (2^\alpha-1)\ (Kn^{\beta/2})/(2n)^\alpha\ P(t=\infty|G_n) + B_\beta n^{\beta/2-\alpha}$$

$$< |T_n|^\beta/n^\alpha - B_\beta n^{\beta/2-\alpha} + B_\beta n^{\beta/2-\alpha} = |T_n|^\beta/n^\alpha$$

Case $\beta > 1$. On A, by Minkowsky's inequality

$$E^{1/\beta}\left(\frac{|T_t|^\beta}{t^\alpha}\Big| G_n\right) \leq E^{1/\beta}\left(|T_n|^\beta/t^\alpha | G_n\right) + E^{1/\beta}\left(\frac{|Z_{n+1}+\dots\dots+Z_t|^\beta}{t^\alpha}\Big| G_n\right)$$

$$\leq E^{1/\beta}(|T_n|^\beta/t^\alpha | G_n) + (B_\beta n^{\beta/2-\alpha})^{1/\beta}$$

As in the proof of case 1,

$$E^{1/\beta}(|T_t| \quad |G_n) < (|T_n|^\beta/n^\alpha - B_\beta n^{\beta/2-\alpha})^{1/\beta} + (B_\beta n^{\beta/2-\alpha})^{1/\beta}$$

Therefore, $E(|T_t|^\beta/t^\alpha\ |G_n) < |T_n|^\beta/n^\alpha$ for $n \geq m_o$

and $E\left(\frac{|T_n|^\beta}{n^\alpha}\Big| G_n\right) < |T_n|^\beta/n^\alpha$ on A

Remark;

Chow's lemma and this lemma are the martingale generalization of corresponding lemma of Siegmund [24]. Incidently lemma 2 of Ruiz-Monacayo[20] is a trivial special case of the above mentioned lemmas.

We state a lemma due to Chow [7] without proof

<u>lemma</u> 1.4 If $E(|\gamma_n|) \to 0$ and $\gamma_n \to 0$ a.e.

then $\int_{\sigma<\infty} |\gamma_\sigma| < \infty$ and $V = \int_{\sigma<\infty} X_\sigma$

<u>Theorem</u> 1.1 Let $(S_n = X_1 + \ldots\ldots + X_n\ ,\ F_n,\ n \geq 1)$ be a martingale and $E(|X_n|^{\max(2,\beta)}\ |F_{n-1}) \leq C < \infty$ a.e.

let $Z_n = |S_n|^\beta / n^\alpha$ for $0 < \beta < 2\alpha$

then $E(Z_\sigma) = V$. If for some $K > (2^\alpha B_\beta) / 2^\alpha - 1$

$$P(|S_n|^\beta \geq K\, n^{\beta/2} \text{ i.o.}) = 1$$

then $P(\sigma < \infty) = 1$

(This theorem is due to Chow [7] in case $\beta = \alpha = 2$, $B_\beta = 1$)

Proof. $E\left(\sum_{1}^{n} X_k / k^{\alpha/\beta}\right)^2 \leq C \sum_{1}^{n} 1/k^{2\alpha/\beta} < \infty$

By Kronecker's lemma $|S_n| / n^{\alpha/\beta} \to 0$ a.e.

Therefore $Z_n \to 0$ a.e.

By lemma 1.1 $E(Z_n) \to 0$

$$|\gamma_n| = \left|\operatorname*{ess\ sup}_{t \geq n} E(|Z_t|\ |F_n)\right|$$

By C_r-inequality

$$|\gamma_n| \leq |\underset{t\geq n}{\text{ess sup}}\ E(C_\beta \frac{|S_n|^\beta + |X_{n+1} + \cdots + X_t|^\beta}{t^\alpha} | F_n)$$

by lemma 1.2

$$\leq C_\beta |S_n|^\beta / n^\alpha + C_\beta \underset{t\geq n}{\text{ess sup}}\ E(|(X_{n+1} + \cdots\cdots + X_t)|^\beta / t^\alpha | G_n)$$

$$< C_\beta(|S_n|^\beta / n^\alpha + B_\beta n^{\beta/2-\alpha})$$

$$= C_\beta(Z_n + B_\beta n^{\beta/2-\alpha}) \to 0$$

$$E(|\gamma_n|) \leq C_\beta\ E(Z_n) + C_\beta \cdot B_\beta n^{\beta/2-\alpha}) \to 0 \quad \text{(by lemma 1.1)}$$

By lemma 1.4, $E(Z_\sigma)=V$

If $\sigma \leq m_o$, then theorem is obviously true. So we can assume σ m_o.

Now assume $P(|S_n|^\beta \geq K\, n^{\beta/2}$ i.o.$) = 1$ for some $K > \dfrac{2^\alpha B_\beta}{2^\alpha - 1}$

$P(\sigma=\infty) > 0$.

Define $t=\inf(n > m_o \ ||S_n|^\beta > K^{\beta/2}$, $P(\sigma = \infty\ |F_n\ ) > \dfrac{2^\alpha B_\beta / K)}{2^\alpha - 1}$

Then $t < \infty$ a.e. on $(\sigma = \infty)$

Since $P(\sigma = \infty | F_n) \to I_{(\sigma = \infty)}$ a.e., (Doob [8] pp 331)

Pub $t' = \min(t\ ,\ \sigma)$, then $P(t' < \infty\ ) = 1$ on $(\sigma = \infty)$

By lemma 1.3

$$\int_{t<\sigma} |S_\sigma|^\beta/\sigma^\alpha = \sum_{n=m_0+1}^{\infty} \int_{t=n<\sigma} E(|S_\sigma|^\beta/\sigma^\alpha | F_n)$$

$$< \sum_{m_0+1}^{\infty} \int_{t=n<\sigma} |S_n|^\beta/n^\alpha = \int_{t<\sigma} |S_t|^\beta/t^\alpha = \int_{t'<\sigma} |S_{t'}|^\beta/t'^\alpha$$

Again,
$$\int_{t\geq\sigma} |S_\sigma|^\beta/\sigma^\alpha = \int_{t'=\sigma} |S_{t'}|^\beta/t'^\alpha$$

Therefore $V=E(Z_\sigma) < E(|S_{t'}|^\beta/t'^\alpha)$ is a contradiction.

1.3. Existence of Optimal Stopping Rule When Second Moment is Infinite

In this section we shall assume that $X_1,\ldots, X_n$ be i.i.d. r.v. with symmetric Stable distribution with characteristic exponent $1<\alpha < 2$. Let $S_n=X_1 +\ldots+ X_n$

Lemma 1.5

$$E(\sup_{k>n} |S_k-S_n|/k) < B_\alpha\, n^{1/\alpha-1}$$

for $n\geq 1$ and $1 < \alpha < 2$

and $E\left(\sup_{n\geq 1} \frac{|S_n|}{n}\right) < \infty$

Proof.

Since X's are stable with exponent $1<\alpha< 2$, $E(X^{\alpha'})< \infty$ for $1 \leq \alpha' \leq \alpha$ and $E(|S_k|/k^{1/\alpha}) \leq C(\alpha,\alpha')$, a constant independent of k.

By Chow's generalization of Hajek-Renyi inequality

$$P(\max_{N \geq\ > n} (|S_k - S_n|)/k \geq u)$$

$$\leq 1/u^{\alpha'} \left(\sum_{n+1}^{N-1} (k^{-\alpha'} - (k+1)^{-\alpha'}) E(|S_k|^{\alpha'}) + 1/N^{\alpha'} E(|S_N|^{\alpha'}) \right), \quad 1 < \alpha' < \alpha$$

$$\leq 1/u^{\alpha'} \left(\sum_{n+1}^{N-1} \alpha' (k^{-\alpha' -1 + \frac{\alpha'}{\alpha}} C(\alpha,\alpha') + C(\alpha,\alpha') N^{\frac{\alpha'}{\alpha} - \alpha'} \right)$$

Let $N \to \infty$ then

$$P(\sup_{k > n} (|S_k - S_n|)/k \geq u) = 1/u^{\alpha'} (\alpha' \sum_{n+1}^{\infty} k^{-\alpha' - 1 + \alpha'/\alpha} C(\alpha\ \alpha'))$$

$$E(\sup_{k > n} (|S_k - S_n|)/k) = \int_0^{\infty} P(|S_k - S_n| \geq ku \text{ for some } k > n) du$$

$$\leq \int_0^{n^{1/\alpha - 1}} 1.du + \int_{n^{1/\alpha - 1}}^{\infty} A(\alpha, \alpha') \sum_{n+1}^{\infty} k^{-\alpha' - 1 + \alpha'/\alpha} 1/u^{\alpha'} du$$

$$\leq n^{\frac{1}{\alpha} - 1} + B'_{\alpha} n^{-\alpha' + \alpha'/\alpha} . n^{-(\alpha' - 1)(1/\alpha - 1)} = n^{1/\alpha^{-1}} + B'_{\alpha} n^{1/\alpha^{-1}}$$

$$= B_{\alpha} n^{1/\alpha^{-1}}$$

<u>Lemma</u> 1.6.

$E(S_t/t | F_n < S_n/n$ on ($P(t = \infty | F_n) > 2/K\ B_{\alpha}, S_n > K\ n^{1/\alpha}$) $= A$

for every stopping time t.

Proof.

$$E(S_t/t | F_n) = S_n E(1/t | F_n) + E\ ((S_t - S_n)/t | F_n)$$

by lemma 1.5.

$\leq S_n(1/n\ P(t\leq 2n|F_n)+1/2nP\ (t>2n|F_n))+B_\alpha n^{1/\alpha^{-1}}$

$\leq S_n/n - 1/2n\ P(t=\infty|F_n)\cdot S_n + B_\alpha\ n^{1/\alpha^{-1}}$

$< S_n/n$ on A

<u>Theorem</u> 1.2 Let $X_1, \ldots, X_n, \ldots,$ be i.i.d. r.v. with common Symmetric Stable distribution with characteristic exponent $1<\alpha<2$. Then the functional equation rule σ is optimal for the reward sequence $\{S_n/n\}$ where $S_n = X_1+\ldots X_n$

Proof.

Since by lemma 1.5 $E(\sup S_n/n) < \infty$ σ is optimal in the extended class (see Siegmund et.al [21])

Suppose $P(\sigma = \infty) > 0$

Let $\tau = \min(\sigma, \inf(n \geq 1 : P(\sigma=\infty|F_n) > 2B_\alpha/K,\ S_n > Kn^{1/\alpha})), K > 2B_\alpha$

then $\tau < \infty$ on $(\sigma = \infty)$

$$\int_{\tau<\sigma} S_\sigma/\sigma = \sum_{n=1}^{\infty} \int_{\tau=n<\sigma} E(S_\sigma/\sigma\ |F_n)$$

$$< \sum_{1}^{\infty} \int_{\tau=n<\sigma} S_n/n \qquad \text{(by lemma 1.6)}$$

$$= \int_{\tau<\sigma} S_\tau/\tau$$

Therefore $E(S_\tau/_\tau) > E(S_\sigma/_\sigma)$ is a contradiction .

Therefore $P(\sigma < \infty)=1$, if $P(S_n > Kn^{1/\alpha}$ i.o.$) = 1$

J. Chover [3] proved that $P(\lim_{n\to\infty} (n^{-\alpha^{-1}} |S_n|)^{(\log_2 n)^{-1}} = e^{\alpha^{-1}}) = 1$

if $\{X_n\}$ satisfies conditions of our theorem which implies

$P(S_n > Kn^{1/\alpha}$ i.o.$) = 1$ for $0 < K < \infty$.

1.4 Remarks

1. Following Siegmund, Simmons, and Feder [21] we define $Z_n = h_n(S_n)$. We will say that FER σ_Z stops at (n,y) if σ_Z says to stop at n whenever $S_n = y$. Let $A_m = (y: \sigma_Z$ stops at $(m,y))$ let $Y_n = g_n(S_n)$ and define

$B_m = (y$: there exists $a > 0$, b such that $g_n(z) \leq ah_n(z) + b$ for all z and all $n \geq m$ and $g_m = a\, h_m(y) + b)$

Then by the application of Principles I and II of [21] we find that FER σ_y (for the reward sequence Y_n) stops at (n,y) whenever $y \in A_n \cap B_n$. Clearly $\sigma_y < \infty$ a.e. if $P(S_n \in A_n \cap B_n$ for some $n) = 1$ i.e. $\sigma_y < \infty$ a.e. if $P(S_n \in A_n \cap B_n$ i.o.$) = 1$

Therefore we can state the following theorem: Let $X_1, X_2, \ldots$ be martingale difference sequence with $E(|X_n|^{\max(2,\beta)} | F_{n-1}) \leq C < \infty$ where $2\alpha > \beta > 0$. If $Y_n = C_n (S_n^+)^\beta$, $C_n P_k(S_n)$ $(\alpha > k/2)$, $n^{-\alpha} \log^+ |S_n|$ are the reward sequences with $\limsup n^\alpha C_n < \infty$ and P_k is a polynomial of degree k with positive lead coefficient, then FER σ_y is optimal in each case provided $P(|S_n| > Kn^{\frac{1}{2}}$ i.o.$) = 1$ for some $K > 0$. In the last case we require only $E(X_n^2 | F_{n-1}) \leq C < \infty$.

(Since the proof is exactly the same as Siegmund, Simmons, and Feder [21] we omit the proof.)

2. We can state Chow's result [7] in more general form when the r.v. are not necessarily martingale difference sequence. The Theorem goes as follows:

Theorem (Chow):

Let $(S_n, F_n, n \geq 0)$ be a stochastic sequence with

$E((S_n - S_{n-1})^2 | F_{n-1}) = \sigma_n^2$, $S_0 = 0$, $X_n = S_n / s_n$ where

$$0 < s_n = \sum_1^n \sigma_k^2 \to \infty .$$ Moreover let $X_n \to 0$, $\{X_n\}$ is uniformly integrable,

$$E(1/s_1^{\frac{1}{2}}) < \infty ,$$ and

$$E((S_t - S_n)/s_t \, I_{(t < \infty)} \mid F_n) \leq 1/s_n^{\frac{1}{2}} \text{ a.e. on } (t > n) \text{ then}$$

$$V = E(X_\sigma) = \int_{\sigma < \infty} S_\sigma / s_\sigma \, dP$$

Moreover if for some $K > 1$

$$P(S_n \geq K s_n^{1/2} \text{ i.o.}) = 1 \quad \text{then } P(\sigma < \infty) = 1.$$

CHAPTER II

HIGHER MOMENTS OF RENEWAL STOPPING TIME

2.1 Introduction

Let (X_n) be independent r.v. with $E(X_n) = \mu$, $E(X_n-\mu)^2 = \sigma^2 < \infty$ and $0 < \mu < \infty$.

Define $N_c=N= \inf(n \geq 1 \ ; \ Z_n > c-n\mu)= \inf\{ n \geq 1 | S_n > C|\}$ where $Z_n=S_n- n\mu$ and $S_n=X_1+X_2+\ldots+ X_n$ and $\infty > c > 0$

Then it is known that when (X_n) obey the Lindeberg condition. (Siegmund [22]); $E(N)=c/\mu+o(c^{\frac{1}{2}})$, $E(N-c/_{\mu})^2=c\sigma^2/\mu^3+o(c)$

According to Brown [1] the sequence of independent r.v. (X_n) with $E(X_n) = \mu$, $s_n^2 = E(S_n^2) \to \infty$, n=1,2,... is said to obey a Lindeberg condition of order $k \geq 2$ (i.e. L_k holds) if

(1) $$\sum_{j=1}^{n} \int_{|X_j-\mu| \geq \epsilon s_n} |X_j-\mu|^k = o(s_n^k) \text{ as } n \to \infty \text{ for all } \epsilon > 0$$

Brown has given equivalent condition for $k > 2$ as follows

(2) $$\sum_{j=1}^{n} \int_{|X_j-\mu| \geq \epsilon s_j} |X_j-\mu|^k = o(s_n^k) \text{ as } n \to \infty, \text{ for all } \epsilon > 0; \text{ and}$$

(3) $$\sum_{j=1}^{n} E|X_j - \mu|^k = o(s_n^k) \text{ as } n \to \infty$$

The conditions (1) and (2) are equivalent even in the case k=2, which was due to Gundy and Siegmund. We shall prove that if L_{2k} holds then $(N-c/\mu)^{2k}$ =

$$\frac{(2k)!}{2^k k!}\ \frac{\sigma^{2k}}{\mu^{3k}}\, c^k \;+\; o(c^k) \quad \text{as } c \to \infty \text{ for } k=1,2,3,\ldots$$

Recently Heyde [13] in the i.i.d. case and Siegmund [23] in the independent case when the r.v. satisfy C.L.T. proved that $\frac{N-c/\mu}{(\sigma^2 c/\mu^3)^{\frac{1}{2}}} \sim N(0,1)$ but they have not proved convergence of moments in the central limit theorem which is our particular interest here. The case k=2 is being discussed by Siegmund [22] and his result is slightly better than ours. Siegmund's result is stated as follows; Let $X_1, X_2, \ldots$ be independent r.v. with $E(X_n) = 0$,

$E(X_n - \mu)^2 = \sigma^2$. Let $0 \leq \alpha < 1$ and

$M = \inf(n \geq 1;\ S_n > n^{\alpha})$ and if $\lim P(\frac{S_n - n\mu}{\sigma n^{1/2}} \leq X) = \Phi(X)$

then $\lim\limits_{c \to \infty} \quad P((M-L)((1-\alpha)^{-1} L^{\frac{1}{2}} \sigma\mu)^{-1} \leq X) = \Phi\ (X)$

where $\Phi\ (X)$ is standard normal d.f. and $L = (c/\mu)^{1/(1-\alpha)}$

Our result implies that the normalized variable $\frac{N-c/\mu}{(\sigma^2 c/\mu^3)^{\frac{1}{2}}}$

has asymptotic mean of order 2k, k=1,2,... as that of standard variable N(0,1) as $c \to \infty$. If we take into account Siegmund's result [23] then even all odd moments of the normalized variable

$\dfrac{N-c/\mu}{(\sigma^2 c/\mu^3)^{\frac{1}{2}}}$ tends to zero as $c \to \infty$.

Our method depends heavily on the techniques developed by Brown [1].

2.2 Lemmas, and Main Theorems

Lemma 2.1 If (X_n) be independent r.v. with $E(x_n^{2k}) \leq l_{2k} < \infty$ then they obey L_{2k}, the Lindeberg condition of order 2k, k= 2,3,... .

Proof.

$$\text{For } k > 1, \quad \sum_{j=1}^{n} E|X_j-\mu|^{2k} \leq n.2^{k-1}(l_{2k}+\mu^{2k})$$

(C_r-inequality)

$$\text{which implies } \sum_{j=1}^{n} E(X_j-\mu)^{2k} = o(n^k)$$

Lemma 2.2 If (X_n) satisfy L_{2k} then $E(X_N^{2k}) = o(E(N^k))$ and $E(X_N-\mu)^{2k}=o(E(N^k))$ where $0 < \mu < \infty$ and $E(X_n) = \mu$.

We shall state a lemma due to Brown [1]

Lemma 2.3

Let $a \geq 0$, $b > 0$ be integers with $a+b/2 = k$ and let $(N_c\ , c > 0)$ be a class of stopping rules such that $E(N_c^k) < \infty$ and $E(N_c^k) \uparrow \infty$ as $c \to \infty$. If (X_n) obey L_{2k}, then

$$E\left(N_c^a\ |X_{N_c}-\mu|^b\right) = o\ (E(N_c^k)) \text{ as } c \to \infty$$

(This includes lemma 2.2, when b=2k, a=0)

Theorem 2.1 Let (X_n) be independent r.v. with $E(X_n) = \mu > 0$ $E(X_n - \mu)^2 = \sigma^2$, $E(X_n-\mu)^3 = \gamma$, $E(X_n-\mu)^4 = \beta < \infty$. Then

$$E(N^2) = O(c^2) \text{ and } E(c-N\mu)^4 = O(c^2)$$

In particular $E(C-N\mu)^4 = 3\sigma^4 . c^2/\mu^2 + o(c^2)$.

Proof. By corollary 1 and 3 of [6] $E(N^2) < \infty$ and

$$\lim_{c \to \infty} E(N^\alpha/c^\alpha) = \mu^{-\alpha} \text{ for } 0 \leq \alpha \leq 2 \ldots (1)$$

which implies $E(N^2) = O(c^2)$

Therefore, by Theorem 7 of [5]

$$E(Z_n^4) = 6\sigma^2 E(N Z_N^2) + 4\gamma E(NZ_N) + \beta E(N) - 3\sigma^4 E(N(N+1)) \quad (2)$$

Now $E(Z_N) = 0$, $E(Z_N^2) = \sigma^2 E(N)$ (by Theorem 2 of [5] ... (3)

Therefore, $E(Z_N^2) = \sigma^2 C/\mu$ ... (4)

By (2) and (1),

$$E(Z_N^4) \leq 6\sigma^2 E^{\frac{1}{2}}(N^2).E^{\frac{1}{2}}(Z_N^4) + 4\gamma E^{\frac{1}{2}}(N^2)E^{\frac{1}{2}}(Z_N^2) + 3E((N) - 3(E(N^2) + E(N))\sigma^4$$

$$\leq 6\sigma^2 . 1/_\mu\ O(c)E^{\frac{1}{2}}(Z_N^4) + 4\gamma 1/_\mu\ O(c).\ ^\sigma/\mu^{\frac{1}{2}}\ O(c^{\frac{1}{2}}) +$$

$$^3/\mu\ O(c) - 3\sigma^4(1/\mu\ O(c) + 1/\mu^2\ O(c^2))$$

$$\ldots (5)$$

Therefore, $E(Z_N^4) = O(c^2)$ as $c \to \infty$

Therefore, from (2), (5), and (1) $E(NZ_N^2) = {}^{\sigma^2}/\mu^2 . c^2$ as $c \to \infty$

Therefore, $E(Z_N^4) = 3\sigma^4 \dfrac{c^2}{\mu^2} + o(c^2)$ as $c \to \infty$

Since, $0 \leq Z_N - (c-N\mu) \leq X_N$, We have $E(Z_N - (c-N\mu))^4 \leq E(X_N^4)$

Therefore, $E(Z_N/c^{\frac{1}{2}} - (c^{\frac{1}{2}} - N/c^{\frac{1}{2}}\mu))^4 \leq E(X_N^4)/c^2 = o(1)$ as $c \to \infty$

Therefore $\{E(Z_N^4)/c^2 - E(c^{\frac{1}{2}} - N/c^{\frac{1}{2}}\mu)^4\} \to 0$.

Therefore $E(c-N\mu)^4 \sim E(Z_N^4)$ as $c \to \infty$

Therefore $E(c-N\mu)^4 = 3\sigma^4 \dfrac{c^2}{\mu^2} + o(c^2)$ as $c \to \infty$

Therefore $E(N-c/\mu)^4/c^2 = 3\sigma^4/\mu^6$ as $c \to \infty$. Q E.D.

The proof becomes very complicated in case moments higher than 4th. But in the i.i.d. case the proof is simpler due to the following observation:

Let $M=\sup(n: S_n \leq c)$, then $M \geq N-1$

But Heyde proved that in the i.i.d. case $F(M^k) < \infty$ if $E(X^-)^{k+1} < \infty$ ($k \geq 1$ integers) and $E(|X|) < \infty$ and $E(X) = \mu > 0$

Therefore $E(N^k) < \infty$ if $E(X^-)^{k+1} < \infty$ and $E(|X|) < \infty$.

Also if $M_n=\max(0,S_1,S_2,\ldots,S_n)$, $n \geq 1$

$$\tau=\max (n\geq 1: M_n \leq c) \quad \text{then } \tau+1 = N$$

and it follows from theorem 3 of Heyde (1966) that for positive integral r, $E(\tau^r) < \infty$ if $E(X^-)^r < \infty$. By theorem 6 of Heyde (1966)

$$\lim_{c\to\infty} \frac{E(\tau^k)}{c^k} = 1/\mu^k \text{ if } E(x^-)^{k+1} < \infty$$

Therefore, $E(N^k) < \infty$ and $\dfrac{E(N^k)}{c^k} \to 1/\mu^k$ as $c \to \infty$ if $E(X^-)^{k+1} < \infty$

for all $k \geq 1$ and $E|X| < \infty$. Theorem 2.1 is a special case of theorem 2.2 but to understand the more complicated proof of Theorem 2.2 better we have given a slightly different separate proof of Theorem 2.1.

<u>Theorem</u> 2.2

Let $X_1, X_2, \ldots$ be independent identical r.v. with $E(X_n) = \mu > 0$, $E(X_n^2) = \sigma^2 + \mu^2$ then, $E(N-c/\mu)^{2k} < \infty$, $E(N^{2k}) = O(c^{2k})$, and

$E(Z_N^{2k}) = O(c^k)$, where $Z_n = X_1+\ldots+X_n - n\mu$

Proof.

Since $E(N^2) = O(c^2)$, $E(N-c/\mu)^2 = O(c)$ and $E(Z_N^2) = \sigma^2 c/\mu + o(c)$ the theorem is true for k=1. We shall prove by induction. Without loss of generality we snall assume $\sigma^2 = 1$.

Define $t_c = \min(N_c, [c])$, then t_c is a bounded stopping rule, and hence Moment Identity holds (see Brown [1]).

Now suppose that the theorem is true for k-1 i.e.

$$E(N-c/\mu)^{2m} < \infty \ , \ E(N^{2m}) = O(c^{2m}) \ , \text{ and } E(Z_N^{2m}) = O(c^m) \qquad (1)$$

for $m=1,2,\ldots,k-1$.

By lemme (1) of page 21 (Brown [1]), if t is a stopping rule with $E(t^k) < \infty$ and the (X_n) obey L_{2k} then

$$0 = E(Z_t^{2k}) + \sum_{r=2}^{2k} (2k)!/(2k-r)!\ E(Z_t^{2k-r}\ t^{r/2}\ A(t,r))$$

where $A(n,r) = n^{-r/2} \sum_{Q_r} \frac{(-1)^\ell}{w_1!w_2!\ldots w_\ell!} \sum_{1\le i_1\le\ldots\le i_\ell\le n} E(X_{i_1}-\mu)^{w_1}\ldots E(X_{i_\ell}-\mu)^{w_\ell}$

where $Q_r = (\ (w_1,\ldots,w_\ell\ ):$ each w_j is an integer ≥ 2, $w_1+\ldots w_\ell=r)$

$$\text{Therefore, } E(Z_t^{2k}) = -\sum_{r=2}^{2k} (2k)!/(2k-r)!\ E(Z_t^{2k-r}\ t^{r/2}\ A_1(t,r))$$

$$+ \sum_{j=1}^{k} (-1)^{j+1} \frac{(2k)!}{(2k-2j)!\ 2^j}\ E(Z_t^{2k-2j} \sum_{1\le i_1\le\ldots\le i_j\le t}\qquad)$$

where $A_1=A(n,r)$ if r is odd

$$A_1 = n^{-r/2} \sum_{Q'_r} \frac{(-1)^j}{w_1!\dots w_j!} \sum_{1\le i_1\le\dots\le i_j\le n} E(X_{i_1}-\mu)^{w_1}\dots E(x_{i_j}-\mu)^{w_j} \quad \text{if } r \text{ is even}$$

where $Q'_r = Q_r - \{\underbrace{(2,2,\dots,2)}_{r/2 \text{ entries}}\}$

By Brown [1] page 25 $A_1(n,r)=o(1)$ as $n\to\infty$, if $\{X_n\}$ obey L_{2k}.

$$\text{Therefore, } E(Z_{t_c}^{2k}) = -\sum_{r=2}^{2k} (2k)!/(2k-r)!\; E(Z_{t_c}^{2k-r}\, t_c^{\,r/2}\; A_1(t_c,r))$$

$$+ \sum_{j=1}^{k} \frac{(-1)^{j+1}(2k)!}{(2k-2j)!j!2^j}\; E(Z_{t_c}^{2k-2j}\; \frac{(t_c+j-1)!}{(t_c-1)!}\,) \tag{2}$$

Now for large c, $Z_{t_c}^{2k} \approx Z_N^{2k}$ and since $\{Z_{t_c}^{2k},\ c\geq 0\}$ is a non-negative submartingale we have (by Doob [8] pp 324-25) $E(Z_{tc}^{2k}) \approx E(Z_N^{2k})$ for large c. Taking lim $c\to\infty$

$$E(Z_N^{2k}) \approx \sum_{j=1}^{k} \frac{(-1)^{j+1}(2k)!}{(2k-2j)!j!2^j}\; E(Z_{t_c}^{2k-2j} t_c^{\,j}) + o(Et_c^k) \text{ for large c.}$$

$$\leq \sum_{j=1}^{k} a_j E^{j/k}(t_c^k)\; E^{1-j/k}\,(Z_{t_c}^{2k}) \quad \text{(by Holder's)}$$

where $a_j\geq 0$, are constants.

$$\text{Therefore, } E(Z_N^{2k}) \leq M_k E(t_c^k) \leq M_k E(N^k) \quad \text{for large c.} \tag{3}$$

where M_k is a positive constant.

Now $E(Z_{t_c}^{2k}) = E(S_{tc} - c + c - \mu t_c)^{2k} = \sum_{j=0}^{2k} \frac{(2k)!}{j!(2k-j)!} E(S_{t_c} - c)^j (c - \mu t_c)^{2k-j}$

By Holder's inequality

$$\mu^{2k} (E(t_c - c/\mu)^{2k} \leq \sum_{j=1}^{2k-1} \frac{(2k)!}{j!(2k-j)!} (E(S_{t_c} - c)^{2k})^{j/2k}$$

$$\mu^{2k-j} (E(t_c - c/\mu)^{2k})^{1-j/2k} + E(S_{t_c} - c)^{2k} + O(c^k) \qquad (4)$$

Now by lemma 2.2 (since L_{2k} holds)

$$E(S_{t_c} - c)^{2k} \leq E(X_{t_c})^{2k} = o(Et_c^k) \qquad (5)$$

So by (1),(3),(4), and (5) taking lim $c \to \infty$, $E(N - c/\mu)^{2k} < \infty$ (6)

Repeating the same argument and remembering that

$E(Z_N^{2k}) < \infty$ iff $E(N^k) < \infty$ We get

$$E(Z_N^{2k}) = \sum_{j=1}^{k} \frac{(-1)^{j+1}(2k)!}{(2k-2j)!j!2^j} E(Z_N^{2k-2j} N^j) + o(E(N^k)) \text{ as } c \to \infty \qquad (7)$$

Now $E(N^{2k}) \leq 2^{k-1} ((c/\mu)^{2k} + E(N - c/\mu)^{2k}) < \infty$ (8)

Strong Law implies $\lim_{c \to \infty} N/c = 1/\mu$ a.e.

Therefore, expanding the function y^b in Taylor's series about 1 where $y = N.\mu/c$ and remembering that

$E(|X_N|^b) = o(E(N^{b/2}))$, we get by (1) and (8) $E(N^{2k}) = O(c^{2k})$. (9)

Therefore by (1), (7) and (8) We get

$$E(Z_N^{2k}) = O(c^k) \tag{10}$$

Corollary 2.1

Let $X_1, X_2, \ldots$ be independent identical r.v. with $E(X_n) = \mu > 0$,

$E(X_N^2) = \sigma^2 + \mu^2$, then, $E(Z_N^{2k}) = \frac{(2k)!\,c^k\sigma^{2k}}{k!\,2^k\mu^k} + o\,(c^k)$, and

$E(N-c/\mu)^{2k} = \frac{\sigma^{2k}c^k(2k)!}{k!\,2^k\mu^{3k}} + o\,(c^k)$ as $c \to \infty$, for all $k=1,2,3,\ldots$

Proof.

Now by (7) of Theorem 2.2

$$E(Z_N^{2k}) = \sum_{j=1}^{k} \frac{(-1)^{j+1}(2k)!}{(2k-2j)!\,j!\,2^j} E(Z_N^{2k-2j} N^j) + o\,(E(N^k)) \tag{1}$$

As in theorem 2.2 We shall assume $\sigma^2 = 1$ and We shall prove by induction.

Expanding the function $y^b(1-\mu y)^a$ in the neighborhood of 1, where $y = N/c$, remembering that $E(|X_N|^b) = o(E(N^{b/2}))$, and assuming $E(Z_N^{2m}) = \frac{(2m)!\,c^m}{m!\,2^m\mu^m} + o(c^m)$ for $m=1,\ldots,k-1$ (since it is true for $m=1$ (Siegmund [22])), We get from (1)

$$E(Z_N^{2k}) = \sum_{j=1}^{k} \frac{(-1)^{j+1}(2k)!}{(2k-2j)!\,j!\,2^j} \frac{(2k-2j)!}{(k-j)!\,2^{k-j}\mu^{k-j}} c^{k-j}\, c^j/\mu^j + o(c^k)$$

$$= \frac{(2k)!c^k}{k!2^k \mu^k} \sum_{j=0}^{k} \frac{(-1)^{j+k}k!}{(k-j)!j!} + \frac{(2k)!c^k}{k!2^k\mu^k} + o(c^k) \text{ as } c \to \infty$$

$$= \frac{(2k)!c^k}{k!2^k\mu^k} + o(c^k) \text{ as } c \to \infty \qquad (2)$$

Since $0 \leq Z_N - (c - \mu N) \leq X_N$

We have $E(Z_N - (c-\mu N))^{2k} \leq E(X_N^{2k})$

Therefore, $E(Z_N/c^{\frac{1}{2}} - (c^{\frac{1}{2}} - N/c^{\frac{1}{2}}\mu))^{2k} \leq E(X_N^{2k})/c^k = o(1)$

Therefore, $\{E(Z_N^{2k})/c^k - E(c^{\frac{1}{2}} - N/c^{\frac{1}{2}}\mu)^{2k}\} \to 0$ as $c \to \infty$ (3)

Therefore by (2) and (3)

$$E(c^{\frac{1}{2}} - N/c^{\frac{1}{2}}\mu)^{2k} \to \frac{(2k)!}{k!2^k\mu^k} \quad \text{as } c \to \infty$$

Therefore, $E(N - c/\mu)^{2k} = \dfrac{(2k)!c^k}{k!2^k\mu^{3k}} + o(c^k)$ as $c \to \infty$

Remark: We conjecture that the theorem 2.2 and Corollary 2.1 are true for if $\{X_n\}$ are independent r.v. satisfying L_{2k}.

CHAPTER III

THE LAW OF ITERATED LOGARITHM

3.1 Introduction

In this chapter law of iterated logarithm results are proved when the r.v. are not necassarily bounded. Let $(X_n, F_n, n \geq 1)$ be a stochastic sequence with $S_n = X_1 + X_2 + \ldots + X_n$. We say that the law of iterated logarithm holds for the sequence (X_n) with norming constant $c_n \uparrow \infty$ if $P(\overline{\lim}\ S_n/C_n = 1) = 1$.

In theorem 3.1 we proved the law of iterated log when the independent r.v. (X_n) do not have moments but they satisfy some Berry-Essen type of bound as done by Petrov [19]. In Theorem 3.2 we have tried to get law of iterated log for martingale difference sequence which are not bounded like classical Kolmogorov type but under some regularity conditions to be stated later (c.f. Stout [25]). Later we tried to get some Berry-Essen type of Bounds to justify conditions imposed in Theorem 3.1.

Finally we got some one-sided law of iterated log type of results when the r.v. are martingale difference sequence satisfying some boundedness conditions on all moments.

3.2 The Iterated Logarithm without Assumptions about the Existence of Moments

Lemma 3.1 Let $\{U_j\}$ and $\{V_j\}$, $1 \leq k \leq n < \infty$, be two sequences of events. Suppose that for each k, the events $\{U_1, U_2, \ldots, U_k, V_k\}$ are independent, and there exists a constant $q > 0$ such that $P(V_k) \geq q$ for every k. Then $P(\bigcup_{k=1}^{n} U_k V_k) \geq q\, p(\bigcup_{k=1}^{n} U_k)$.

Proof.

$$P(\bigcup_{k=1}^{n} U_k V_k) = P(\bigcup_{k=1}^{n} [(U_1 V_1)^c \ldots (U_{k-1} V_{k-1})^c \quad (U_k V_k])$$

$$\geq P(\bigcup_{k=1}^{n} [U_1^c \ldots U_{k-1}^c U_k V_k]) = \sum_{k=1}^{n} P(U_1^c \ldots U_{k-1}^c U_k) P(V_k)$$

$$\geq \sum_{k=1}^{n} P(U_1^c \ldots U_{k-1}^c U_k) q = q^P(\bigcup_{k=1}^{n} U_k)$$

Theorem 3.1 Suppose that (X_n) be independent r.v. There exists a sequence of positive numbers (B_n) such that $B_n \uparrow \infty$ and $B_{n+1}/B_n \to 1$ (1)

$$P(S_n - S_k \geq -(B_n)^{\frac{1}{2}}) \geq q > 0 \text{ for all } 1 \leq k \leq n \qquad (2)$$

and

$$\sup_x \left| P(S_n < B_n^{\frac{1}{2}} x) - (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} \exp(-t^2/2)\, dt \right| = O(\log B_n)^{-(1+\mu)} \text{ for some } \mu > 0 \qquad (*)$$

Then $P(\overline{\lim}\, S_n (2 B_n \log \log B_n)^{-\frac{1}{2}} = 1) = 1$.

Proof. From the estimate

$$\int_x^\infty \exp(-t^2/2)\,dt \sim 1/x \exp(-x^2/2) \quad \text{as } x \to \infty$$

we get

$$(\log B_n)^{-(1+\delta)a^2} < P(S_n \geq a\, B_n^{\frac{1}{2}}\, t_n) < (\log B_n)^{-a^2} \tag{3}$$

for $\delta > 0$ and $a < (1+\mu)^{\frac{1}{2}}$ and n sufficiently large. where

$t_n = (2\log\log B_n)^{\frac{1}{2}}$ (1) implies that for each $c > 0$ there exists (n_k)

such that $B_{n_{k-1}} \leq (1+c)^k < B_{n_k}$ $K = 1,2,\ldots$

assume $B_o = 0$

Hence (1) implies $B_{n_k} \sim (1+c)^k$ and

$$B_{n_k} - B_{n_{k-1}} = B_{n_k}(1 - B_{n_{k-1}} / B_{n_k}) \sim B_{n_k}\, c/(1+c) \quad \text{as } k \to \infty \tag{4}$$

Let $U_n = \{S_n > (1+\epsilon)^{\frac{1}{2}} t_n B_n^{\frac{1}{2}}\}$, $\epsilon > o$

We shall prove that $P(U_n \text{ i.o.}) = 0$ (5)

Now for each k, consider the range of j below

$$n_k \leq j < n_{k+1} \tag{6}$$

Put $P(V_j) = P\{(S_{n_{k+1}} - S_j) \geq -(B_{n_{k+1}})^{\frac{1}{2}}\} \geq q > 0$ for large k (by (2))

By lemma 3.1

we get $P(\bigcup_{j=n_k}^{n_{k+1}-1} U_j V_j) \geq q\, P(\bigcup_{j=n_k}^{n_{k+1}-1} U_j)$

It is clear that $U_j \cap V_j$ implies

$$S_{n_{k+1}} > S_j - (B_{n_{k+1}})^{\frac{1}{2}} > (1+\epsilon)^{\frac{1}{2}} B_j^{\frac{1}{2}} t_j - (B_{n_{k+1}})^{\frac{1}{2}}$$

which by (4) and (6) asympotically greater than

$$(1+\epsilon)^{\frac{1}{2}}/(1+c)^{\frac{1}{2}} t_{n_{k+1}} B^{\frac{1}{2}}_{n_{k+1}}$$

Choose $c > 0$ close to 0 such that $(1+\epsilon)/(1+c) > 1 + \epsilon/2$

and put $A_k = \{S_{n_{k+1}} > (1 + \epsilon/2)^{\frac{1}{2}} t_{n_{k+1}} B^{\frac{1}{2}}_{n_{k+1}}\}$

This implies that $U_j \cap V_j \subset A_k$ for large k.

It follows from (3) that

$$\sum_k P(A_k) < \sum_k 1/(\log B_{n_k})^{-(1+\epsilon/2)} = \sum_k (K(\log (1+c))^{-(1+\epsilon/2)} < \infty$$

Therefore, by Borel-Cantelli limma

$$P\left(\bigcup_{j=n_k}^{n_{k+1}-1} U_j \text{ i.o.}\right) = 0$$

this is equivalent to (5).

Similar reasoning applied to $-S_n$ shows

$$P(|S_n| > (1+\epsilon) B_n^{\frac{1}{2}} t_n \text{ i.o.}) = 0 \quad \text{for any } \epsilon > 0 \tag{7}$$

The rest of the proof follows inthe same line as Petrov [19], but for the sake of completeness we are giving the proof.

Let $\Psi(n_k) = (2(B_{n_k} - B_{n_{k-1}}) \log \log (B_{n_k} - B_{n_{k-1}}))^{\frac{1}{2}}$

From (4) we get, $\log (B_{n_k} - B_{n_{k-1}}) < \log (B_{n_k}) < 2k \log (1+c)$ for large k.

Therefore, $\Psi(n_k)/B^{\frac{1}{2}}_{n_{k-1}} t_{n_{k-1}} \sim c^{\frac{1}{2}}$

By (3), for $0 < \gamma < 1$, $P(S_{n_k} - S_{n_{k-1}} > (1-\gamma)\,\Psi(n_k))$

$$\geq P([S_{n_k} > (1-{}^{\gamma}/2)\,\Psi(n_k)] \cap [S_{n_{k-1}} < {}^{\gamma}/2\,\Psi(n_k)])$$

$$\geq P(S_{n_k} > (1-{}^{\gamma}/2)\,\Psi(n_k)) - P(S_{n_{k-1}} \geq {}^{\gamma}/2\,\Psi(n_k))$$

$$> \log(B_{n_k})^{-(1+\delta)\,(1-{}^{\gamma}/2)^2} - (\log B_{n_{k-1}})^{-\gamma^2 c/5}$$

$$\geq A.(k^{-(1+\delta)\,(1-{}^{\gamma}/2)^2} - k^{-\gamma^2 c/5}) > A/2\; k^{-(1+\delta)\,(1-{}^{\gamma}/2)^2}$$

for k and c large where A is a constant independent of k. Choose δ small enough such that $(1+\delta)\,(1-{}^{\gamma}/2)^2 < 1$

then $\sum_k P(S_{n_k} - S_{n_{k-1}} > (1-\gamma)\,\Psi(n_k)) = \infty$

since $(S_{n_k} - S_{n_{k-1}})$ are independent, by Borel-Cantelli lemma

$$P(S_{n_k} - S_{n_{k-1}} > (1-\gamma)\,\Psi(n_k) \text{ i.o. }) = 1 \text{ for } 0 < \gamma < 1 \qquad (8)$$

Now (7) implies $|S_n(w)| \leq 2B_n^{\frac{1}{2}}\, t_n$ for $n > n_o(w)$ a.e. Hence

(4) implies $(1-\gamma)\,\Psi(n_k) - 2B_{n_{k-1}}^{\frac{1}{2}}\, t_{n_{k-1}}$

$$\sim ((1-\gamma)\,(c/1+c)^{\frac{1}{2}} - 2/(1+c)^{\frac{1}{2}})\, B_{n_k}^{\frac{1}{2}}\, t_{n_k} \text{ as } k \to \infty$$

since $\epsilon > 0$ is arbitrary, choose $\gamma > 0$ and $c > 0$ such that

$$(1-\gamma)\,(c/1+c)^{\frac{1}{2}} - 2/(1+c)^{\frac{1}{2}} > 1 - \epsilon$$

Hence (8) implies $P(S_{n_k} > (1-\epsilon)\, B_{n_k}^{\frac{1}{2}}\, t_{n_k} \text{ i.o.})$

$$\geq P(S_{n_k} > (1-\gamma)\,\Psi(n_k) = 2B^{\frac{1}{2}}_{n_{k-1}}\, t_{n_{k-1}} \text{ i.o.})$$

$$\geq P(S_{n_k} - S_{n_{k-1}} > (1-\gamma)\,\Psi(n_k) \text{ i.o.}) = 1$$

this proves the theorem.

Remarks:

1. According to Khintchine a distribution belongs to class L iff it is the limit distribution of a sequence $\bar{S}_n = (S_n - a_n)/B_n^{\frac{1}{2}}$ satisfying $B_n \uparrow \infty$ and $B_{n+1}/B_n \to 1$. So our theorem can be stated in loose terms as follows: If independent sequence of r.v. belongs to the class L distribution and satisfy some Berry-Essen type of bound and also if they take both positive and negative values with positive probability then they obey law of iterated log.

It is to be noted that the usual case $B_n = \sigma^2 s_n^2$ is proved by Petrov [19]. The main interest is when the r.v. do not possess finite variances. In order to verify whether the relation (*) holds we can use the estimates of the remainder in the central limit theorem given by Hertz [14]. According to Hertz [14] if $c > 0$

$$U_i(c) = \int_{|X| \leq c} X^2 \, dF_i \quad \text{and} \quad A_n(c) = \sum_1^n c \int_{|X| > c} |X| \, dF_i$$

where F_i is the d.f. of the independent r.v. X_i. Assume

$B_n^2 = \sum_1^n U_i(B_n) > 0$, then Hertz [14] proved that if the r.v. are i.i.d. continuous and in the domain of attraction of the normal law, then for sufficiently large n, there exists solutions (B_n) Of the last equation so that $B_n \to \infty$

Let $\Delta_n = \sup_x |P(S_n < B_n^{\frac{1}{2}}x) - \Phi(x)| \leq k' B_n^{-3} \int_0^{B_n} A_n(u)\, du$

(theorem 5 of Hertz)

Let $b_n(c) = 1/c \sum_1^n \int_{|x| \leq c} |X|^3\, dF_i$ Integrating by parts

Now $c^{-1} \int_0^c A_n(u)\, du = \frac{1}{2}(A_n(c) + b_n(c))$

Therefore, $\Delta_n \leq k/B_n^2\, (A_n(B_n) + b_n(B_n))$

$$= k/B_n^2 \Big(\sum_1^n B_n \int_{|X|>B_n} |X|\, dF_i + 1/B_n \sum_1^n \int_{|X|\leq B_n} |X|^3 dF_i \Big)$$

for $\delta > 0$,

$$\leq k/B_n^2 \; ((\log|B_n|)^{-(1+\delta)} \sum_1^n B_n \int_{|X|>B_n} |X| (\log|X|)^{(1+\delta)}\, dF_i$$

$$+ 1/B_n \sum_1^n \int_{|X|\leq B_n} \frac{X^2 |X| (\log|X|)^{1+\delta}}{(\log|X|)^{1+\delta}})\, dF_i)$$

$$\leq k/B_n^2 ((\log|B_n|)^{-(1+\delta)} \sum_1^n B_n \int_{|X|>B_n} |X| (\log|X|)^{1+\delta} dF_i$$

$$+ (\log B_n)^{-(1+\delta)} \sum_1^n \int_{|X|\leq B_n} X^2 (\log|X|)^{(1+\delta)}\, dF_i))$$

$$= k(B_n^2 (\log B_n)^{1+\delta})^{-1} \sum_1^n (B_n \int_{|X|>B_n} |X| (\log|X|)^{1+\delta} dF_i + \int_{|X|\leq B_n} X^2 (\log|X|)^{1+\delta}\, dF_i)$$

$$= k/(\log|B_n|)^{1+\delta} \; .1/B_n^2 \; . \; L_n$$

where $L_n = \sum_{1}^{n} (B_n \int_{|X|>B_n} |X| (\log|X|)^{1+\delta} \, dF_i + \int_{|X|>B_n} X^2 (\log|X|)^{1+\delta} \, dF_i)$

So if $L_n/B_n^2 \leq b$, a finite constant we get relation [*].

2. If the r.v. are symmetric they satisfy condition (2) of theorem 3.1

3.3 Law of Iterated Logarithm for Martingale Sequence

Lemma 3.2 Let $(D_n, F_n, n \geq 1)$ be a stochastic sequence. Let (a_n), and (c_n) be F_{n-1} measurable positive random variables such that $a_n < c_n, c_n \uparrow \infty$

Let

$$Y_n = D_n \text{ if } |D_n| < a_n$$
$$= 0 \quad \text{otherwise}$$

let $P(\overline{\lim}\, 1/c_n \sum_{k=1}^{n} (Y_k - E(Y_k|F_{k-1}) = a) = 1$ (1)

If $\sum_{n=1}^{\infty} E(D_n^{2r}(D_n^{2r} + c_n^{2r})^{-1} I(|D_n| \geq a_n)|F_{n-1}) < \infty$ a.e. (2)

for $\frac{1}{2} \leq r \leq 1$

Then $P(\overline{\lim}\, 1/c_n \sum_{k=1}^{n} (D_k - g_k) = a) = 1$ (3)

where $g_k = E(D_k I(|D_k| < c_k). |F_{k-1})$

Proof. Let $Z_n = D_n$ if $|D_n| \geq a_n$

$= 0$ otherwise

then $D_n = Y_n + Z_n$, Let $\overline{Z}_n = D_n$ if $a_n \leq |D_n| < c_n$

$= 0$ otherwise

then $\overline{Z}_n = Z_n$ if $|Z_n| < c_n$

$= 0$ otherwise

Now applying corollary 3.1 of theorem 3.1 of Stout [25] (The proof goes through if constants a_n and c_n are replaced by F_{n-1} measurable r.v.) to the sequence $(Z_n, F_n, n \geq 1)$ we get by Kronecker's lemma

$$c_n^{-1} \sum_{k=1}^{n} (Z_k - E(\overline{Z}_k | F_{k-1})) \rightarrow 0 \quad \text{a.e.} \tag{4}$$

Now $D_k - g_k = Y_k - E(Y_k | F_{k-1}) + Z_k - E(\overline{Z}_k | F_{k-1})$ (5)

therefore, (4), (5), and (1) implies (3).

<u>Theorem</u> 3.2 Let $(D_n, F_n, n \geq 1)$ be a martingale difference sequence with $s_n^2 \sum_{k=1}^{n} E(D_k^2 | F_{k-1}) \rightarrow \infty$ where $E(D_k^2 | F_{k-1}) =$ constant.

Let $a_n = o(s_n (\log_2 s_n^2)^{-\frac{1}{2}})$, $c_n = (2s_n^2 \log\log s_n^2)^{\frac{1}{2}}$. Moreover if

$1/s_n^2 \sum_{k=1}^{n} E(D_k^2 \, I_{(|D| \geq a_k)}) \rightarrow 0$ and either, (a)$E(\sup |D_n|/c_n) < \infty$ and

(b) $\sum_{n=1}^{\infty} E(D_n^{2r}(D_n^{2r} + c_n^{2r})^{-1}) < \infty$, or $\sum_{n=1}^{\infty} P(|D_n| \geq a_n) < \infty$, then the law of iterated log holds for $\{D_n\}$.

Proof. Follows easily from lemma 3.2., corollary 3.4. and theorem 4.2. of Stout [25].

Remark: It is interesting to compare Theorem 3.2. with the recently proved result of Stout [26]: let $(X_n, F_n, n\geq 1)$ be a martingale with difference sequence $D_n = X_n - X_{n-1}$,

$$s_n^2 = \sum_1^n E(D_j^2 | F_{j-1}) \to \infty \text{ and}$$

$$\sum_{n=1}^{\infty} (K_n s_n)^{-2} u_n^2 E(D_n^2 I_{(D_n^2 > s_n^2 K_n^2 / u_n^2)} | F_{n-1}) < \infty$$

where K_n are F_{n-1} measurable, $K_n \to 0$ and $u_n = (2\log_2 s_n^2)^{\frac{1}{2}}$ then lim sup $X_n / s_n u_n = 1$.

3.4 Some Berry-Essen Type of Bound for Independent Random Variables

Let $X_1, X_2, \ldots$ be independent r.v. with $E(X_i) = 0$, $E(X_i^2) = \sigma_i^2$,

$$s_n^2 = \sum_1^n \sigma_i^2, \; S_n = X_1 + X_2 + \ldots + X_n \text{ and } Z_n = S_n / s_n$$

$$\text{Let } g_n(\epsilon) = 1/s_n^2 \sum_{k=1}^n \int_{|X| \geq \epsilon s_n} X^2 \, dF_k \,, \; \epsilon > 0$$

Let $\overline{F}_n(x) = P(Z_n < x)$ and $\Phi(x)$ be the standard normal d.f.

B.V. Gnedenko conjectured that $\sup_X |\overline{F}_n(x) - \Phi(x)| \leq c\, g_n(\epsilon)$ for large n where c is a positive constant.

Ibragimov and Osipov [15] gave counter example to show that it is false if the absolute moments of order $2+\delta$ ($\delta>0$ is arbitrary) are infinite.

However it is possible to find out a bound which is a function of $g_n(\epsilon)$ and ϵ.

Theorem 3.3 Let (X_n) be independent r.v. with mean zero and satisfy Lindeberg condition then

$$\sup_x |\bar{F}_n(x) - \Phi(x)| \leq C_{1/s_n} + C_2\, g_n(\epsilon_n) + C_3\, g_n(\epsilon_n)/\epsilon_n + g_n(\epsilon_n)/\epsilon_n^2$$

$$\to 0 \text{ as } n \to \infty$$

Proof. By hypothesis $g_n(\epsilon) \to 0$ for every $\epsilon>0$, following Loeve [16] there exists a sufficiently slowly decreasing sequence $\epsilon_n > 0$ such that (i) $g_n(\epsilon_n)/\epsilon_n^2 \to 0$, (ii) $g_n(\epsilon_n)/\epsilon_n \to 0$, (iii) $g_n(\epsilon_n) \to 0$, and $\sup_{n\geq 1} \epsilon_n\, s_n \leq \gamma < \infty$.

Define $\bar{X}_k = X_k$ if $|X_k| \leq \epsilon_n\, s_n$

$= 0$ otherwise $k = 1,2, \ldots n$

$$\bar{a}_j = E(\bar{X}_j),\ \bar{\sigma}_j^2 = E(\bar{X}_j^2) - \bar{a}_j^2,\ \bar{s}_n^2 = \bar{\sigma}_1^2 + \bar{\sigma}_2^2 + \ldots + \bar{\sigma}_n^2$$

Now $\bar{\sigma}_j^2 \leq \sigma_j^2$

$$s_n^2 - \bar{s}_n^2 = \sum_1^n \sigma_j^2 - \sum_1^n \bar{\sigma}_j^2 = \sum_1^n \int_{|X|>\epsilon_n s_n} X^2\, dF_j + \sum_1^n \Big(\int_{|X|\leq\epsilon_n s_n} X\, dF_j \Big)^2$$

Since, $E(X_j) = 0$ for all j, by Holder's inequality

$$s_n^2 - \bar{s}_n^2 \leq 2g_n(\epsilon_n)s_n^2 \tag{1}$$

Therefore, $1 \leq 8/3\ g_n(\epsilon_n)$ if $\bar{s}_n \leq s_n/2$ (2)

So Gnedenko's conjecture is trivially true in this case with $C = 8/3$.

$$(s_n - \bar{s}_n) / \bar{s}_n \leq \frac{s_n^2 - \bar{s}_n^2}{s_n \bar{s}_n} \leq 2(s_n^2 - \bar{s}_n^2)/s_n^2 \quad \text{if } \bar{s}_n > s_n/2$$

Therefore, $(s_n - \bar{s}_n)/\bar{s}_n \leq 4g_n(\epsilon_n)$ if $\bar{s}_n > s_n/2$ (3)

let $Y_n = 1/s_n \sum_1^n \bar{X}_j$, $\bar{Z}_n = 1/\bar{s}_n \sum_1^n (\bar{X}_j - \bar{a}_j)$, $Z_n = 1/s_n \sum_1^n X_j$

$$((Z_n < x) \subset (Y_n < x) \cup (|X_1| > \epsilon_n s_n) \cup \ldots \cup (|X_n| > \epsilon_n s_n)$$

Therefore, $P(Z_n < x) \leq P(Y_n < x) + \sum_{j=1}^n P(|X_j| > \epsilon_n s_n)$

Similiarly $P(Y_n < x) \leq P(Z_n < x) + \sum_{j=1}^n P(|X_j| > \epsilon_n s_n)$

Therefore, $|P(Z_n < x) - P(Y_n < x)| \leq \sum_{j=1}^n P(|X_j| > \epsilon_n s_n)$

$$= \sum_{j=1}^n \int_{|X| > \epsilon_n s_n} dF_j(x) \leq 1/\epsilon_n^2\, g_n(\epsilon_n)$$

Therefore, for all x, $|P(Z_n < x) - \Phi(x)|$

$$\leq \sup_x |P(\bar{Z}_n < (xs_n - \sum_{j=1}^n \bar{a}_j)/\bar{s}_n) - \Phi((xs_n - \sum_{j=1}^n \bar{a}_j)/\bar{s}_n)|$$

$$+ \sup |\Phi((xs_n - \sum_1^n \bar{a}_j)/\bar{s}_n) - \Phi(x)| + 1/\epsilon_n^2 g_n(\epsilon_n)$$

$$= T_1 + T_2 + 1/\epsilon_n^2 g_n(\epsilon_n) \tag{4}$$

Now $E((|\bar{X}_j - \bar{a}_j|)^3)/\bar{\sigma}_j^2 \leq 2\epsilon_n s_n/\bar{\sigma}_j^2 \, E(\bar{X}_j - \bar{a}_j)^2 = 2\epsilon_n s_n \leq 2\gamma$

Now applying Berry-Essen's theorem to $\bar{X}_1, \bar{X}_2, \ldots$

(see Feller vol II pp 521 [12])

$$T_1 \leq \bar{C}_1/\bar{s}_n \leq 2\bar{C}_1/s_n \qquad \text{if } \bar{s}_n > s_n/2 \tag{5}$$

$$T_2 \leq 1/(2\pi)^{\frac{1}{2}} \left((s_n - \bar{s}_n)/\bar{s}_n + 1/\bar{s}_n \left|\sum_{j=1}^n \bar{a}_j\right|\right)$$ (by an estimate of Petrov [18])

$$\sum_{j=1}^n |\bar{a}_j| \leq \sum_{j=1}^n \left|\int_{|X| \leq \epsilon_n s_n} X \, dF_j(x)\right| = \sum_{j=1}^n \left|\int_{|X| > \epsilon_n s_n} X \, dF_j(x)\right| \leq 1/\epsilon_n s_n \sum_1^n \int_{|X| > \epsilon_n s_n} X^2 dF_j(x)$$

Therefore, $1/\bar{s}_n \left|\sum_{j=1}^n \bar{a}_j\right| \leq 2/(\epsilon_n s_n^2) \sum_{j=1}^n \int_{|x| > \epsilon_n s_n} X^2 dF_j$ if $\bar{s}_n > s_n/2$

$$= 2g_n(\epsilon_n)/\epsilon_n \tag{6}$$

Therefore, by (3) and (6)

$$T_2 \leq C_3(g_n(\epsilon_n) + g_n(\epsilon_n)/\epsilon_n) \tag{7}$$

Therefore, by (4), (5), and (7) we get the theorem.

3.5 One-sided Law of Iterated Log in the Martingale Case When the Random Variables are not Bounded

Recently Stout [26] proved if $s_n^2 \to \infty$ and $(Y_n, F_n, n \geq 1)$ be a martingale difference sequence with $|Y_n| \leq K_n s_n / u_n$ for all $n \geq 1$ where K_n are F_{n-1} measurable with $K_n \to 0$, $s_n^2 = \sum_1^n E(Y_j^2 | F_{j-1})$ and $u_n = (2 \log_2 s_n^2)^{\frac{1}{2}}$. then $\lim\sup X_n/s_n u_n = 1$. where $X_n = Y_1 + \ldots + Y_n$

Relaxing boundedness condition on the r.v. by the same type of boundedness condition on moments we are able to prove one-sided result namely $\lim\sup X_n/s_n u_n \leq 1$

Let $\sigma_j^2 = E(Y_j^2 | F_{j-1})$, $X_o = 0$, T_k = first time $s_{n+1}^2 \geq p^{2k}$, $p > 1$, since $s_n^2 \to \infty$, T_k is a stopping rule.

<u>Lemma</u> 3.3 Let t be a stopping rule such that $\max_{\ell \leq j \leq t} E(Y_j^k | F_{j-1}) \sigma_j^2 \leq C^{k-2}$ where ℓ is an integer and C F_ℓ measurable. Then for all $\ell \geq 0$,

$$E(\exp(\lambda(X_t - X_\ell)\exp(-\lambda^2/2)(1 + \lambda C/2) \sum_{j=\ell+1}^{t} \sigma_j^2 \mid F_\ell) \leq 1$$

On $t \geq \ell$ provided λ is F_ℓ measurable and $0 \leq \lambda C \leq 1$. (This lemma essentially like lemma 1 of Stout [26] but for sake of completeness we are giving the proof.)

Proof. By Fatou lemma for conditional expection, it suffices to prove the result for $t(N)=\min(t,N)$ with $N>\ell$.

Assume $0 \leq \lambda C \leq 1$ and $I_{(t(N) \geq \ell)} = 1$.

$$E(\exp(\lambda(X_{t(N)}-X_\ell))|\ F_{N-1}) = E(\exp(\lambda \sum_{j=\ell+1}^{N} I(t(N) \geq j)y_j|F_{N-1})$$

$$= \exp(\lambda \sum_{j=\ell+1}^{N-1} I_{(t(N) \geq j)}Y_j)E(\exp(\lambda Y_N\ I_{(t(N) \geq N)}|F_{N-1})$$

For any j such that $t(N) \geq j > \ell$

$$E(\exp(\lambda Y_j I_{(t(N) \geq j)})|F_{j-1})$$

$$= 1 + (\lambda^2/2\sigma_j^2 + \lambda^3/3!\sigma_j^2\ C + \lambda^4/4!\sigma_j^2\ C^2 + \ldots\ldots.)I_{(t(N) \geq j)}$$

$$= 1 + \lambda^2/2\sigma_j^2/2\ (1+\lambda C/3 + \lambda^2 C^2/3.4 + \ldots\ldots.\ )I_{(t(N) \geq j)}$$

$$\leq \exp(\lambda^2\sigma_j^2/2\ (1 + \lambda C/2)\ I_{(t(N) \geq j)})$$

setting j = N and combining we get

$$E(\exp(\lambda(X_{t(N)}-X_\ell)\ \exp(-\lambda^2/2)\ \ (1+C\lambda/2)\ \sigma_N^2)\ I_{(t(N) \geq N)}|F_{N-1})$$

$$\leq \exp(\lambda \sum_{j=\ell+1}^{N-1} I_{(t(N) \geq j)}\ Y_j)$$

If $\ell = N + 1$, we are done. Otherwise, proceeding by backward induction we assume that

$$E(\exp(\lambda(X_{t(N)}- X_\ell)\ \exp(-\lambda^2/2)(1+\lambda C.2) \sum_{j=n}^{N} I_{(t(N) \geq j)}\sigma_j^2|F_{N-1}) \qquad (3)$$

$$\leq \exp(\lambda \sum_{j=\ell+1}^{n-1} I_{(t(N) \geq j)}Y_j) \quad \text{for } N \geq n \geq \ell+2.$$

Then, $E(\exp(\lambda \sum_{\ell+1}^{n-1} Y_j I_{(t(N) \geq j)}|F_{n-2})$

$$\leq \exp(\lambda \sum_{j=\ell+1}^{n-2} I_{(t(N) \geq j)} Y_j) \exp(-\lambda^2/2)(1+\lambda C/2)\sigma_{n-1}^2 I_{(t(N) \geq n-1)})$$

follows by computation. Combining with (3), we get

$$E(\exp(\lambda(X_{t(N)}-X_\ell) \exp(-(\lambda^2/2)(1+\lambda C/2) \sum_{j=n-1}^{N} I_{(t(N) \geq j)} \sigma_j^2 | F_{n-2})$$

$$< \exp(\lambda \sum_{j=\ell+1}^{n-2} I_{(t(N) \geq j)} Y_j)$$

Then by backward induction the lemma is established.

Corollary 3.1 Let $\max E(Y_j^m | F_{j-1})/\sigma_j^2 \leq (Cp^k)^{m-2}$

$$\ell \leq j \leq T_k \qquad \text{for } m = 3,4,\ldots.$$

and C is F_ℓ measurable and $p > 1$, is a constant. Then

$$E(I_{((X_{T_k}-X_\ell)/p^k > \epsilon)} | F_\ell \leq \exp(-\epsilon/2)\ (1-\epsilon C/2) \text{ on } T_k \text{ provided}$$

ϵ is F_ℓ measurable and $0 \leq \epsilon c \leq 1$.

Proof. Letting $t = T_k$ and noting that $s_{t_k}^2 \leq p^{2k}$, by lemma 3.3 we get $E(\exp(t(X_{T_k}-X_\ell)|F_\ell \leq \exp((t^2/2)(1+tCp^k/2)\ p^{2k})$

on $T_k \geq 1$ provided $0 < tC\ \ p^k \leq 1$

$$E((I_{(X_{T_k}-X_\ell)p^k > \epsilon)} | F_\ell) \leq \exp(-\epsilon t)\ E(\exp(t(X_{T_k}-X_\ell)/p^k | F_\ell)$$

$$\leq \exp(-\epsilon_t)\ \exp((t^2/2)(1+tC/2)) \text{ if } 0 < tC \leq 1$$

putting $t = \epsilon$ yields the results.

Lemma 3.4 $(Y_j, F_j,\ j \geq 1)$ be a Martingale difference sequence with

K_n is a constant and $K_n \to 0$. Then $\limsup X_n/s_n u_n \leq 1$ where

$u_n = (2 \log_2 s_n^2)^{\frac{1}{2}}$.

Proof. Since $K_n \to \infty$ there exists $K > 0$ such that $K_n \leq K$ for n sufficiently large and $(1 + \delta') K \leq 1$ where δ' is to be chosen later.

We shall prove $P(X_n > (1+\delta) s_n u_n \text{ i.o.}) = 0$ for all $\delta > 0$.

$$P(X_n > (1+\delta) s_n u_n \text{ i.o.}) \leq P(X^*_{T_k} > (1+\delta) s_{T_{k-1}} u_{T_{k-1}} \text{ i.o.})$$

Now $\left(s^2_{T_{k-1}+1} \cdot u^2_{T_{k-1}+1}\right) / (s^2_{T_k} u^2_{T_k}) \geq p^{-2} \log_2 p^{2(k-1)} / \log_2 p^{2k}$

Thus choosing $\delta' > 0$ and $p > 1$ such that $(1+\delta) > (1+\delta')p$, it follows that $P(X_n > (1+\delta) s_n u_n \text{ i.o.}) \leq)(X^*_{T_k} > (1+\delta') s_{T_k} u_{T_k} \text{ i.o.})$

Now from lemma 3.4 for any $a > 0$

$$E(I_{(X^*_{T_k} \geq x)}) \leq \exp(-ax)\, E(\exp(a X^*_{T_k})) \leq 8\exp(-ax)\, E(\exp(a|X_{T_k}|))$$

$$\leq 8\exp(-ax) \quad (E(\exp(a\, X_{T_k}) + \exp(-aX_{T_k}))$$

By corollary 3.1 with $\ell = 0$

$$P(X^*_{T_k}/p^k > \epsilon) \leq 16 \exp(-\epsilon^2/2\ (1-\epsilon C/2)$$

Therefore, $E(I_{(X^*_{T_k}} \geq (1+\delta'(\ (2p^{2k} \log_2 p^{2k})^{\frac{1}{2}})$ for large K.

$$\leq \exp(-(1+\delta')^2 \log_2 p^{2k}(1-K(1+\delta')/2)) \text{ where } C = K(2 \log_2 p^{2k})^{-\frac{1}{2}},$$

$\epsilon = (1+\delta')(2 \log_2 p^{2k})^{\frac{1}{2}})$ with $(1+\delta')K \leq 1$.

Therefore, for sufficiently large k

$$E(I_{(X^*_{T_k}} > (1+\delta')\ (2p^{2k} \log_2 p^{2k})^{\frac{1}{2}})^0 \leq 2\ (2k \log p)^{-\alpha}$$

for some $\alpha > 1$ by choosing $K > 0$ such that $(1+\delta')^2(1-K(1+\delta')/2 > 1$

Thus $\sum_{k=1}^{\infty} P(X^*_{T_k} > (1+\delta')\ (2p^{2k}\log_2 p^{2k})^{\frac{1}{2}})) < \infty$. By Borel-Cantelli lemma the result follows.

3.6 Concluding Remarks

(1) It is to be noted that i.i.d.r.v. satisfy the condition of our theorem. So that we can get one part of Hartman-Weintner's law of iterated log.

Also when (X_n) are independent $N(0,\sigma_n^2)$

$$E(X^{2n})/\sigma_n^{2n} = \frac{(2n)!}{n!2^n} \approx (2/e)^n\ n^n \text{ (Stirling's approximation)} \qquad (1)$$

$(n/(2\log_2 n)^{(n-2)/2}$ goes to zero less faster than (1).

So if (x_n) are independent $N(0,\sigma_n^2)$ they satisfy condition of our theorem (2) It is interesting to note that if $\{X_n\}$ be independent r.v. with

$E(X_n) = 0,\ E(X_n^2) = \sigma_n^2\ ,\ s_n^2 = \sum_{k=1}^{n} \sigma_n^2 \to \infty$, and $s_{n+1}/s_n \to 1$ as $n \to \infty$,

moreover suppose that for every $t \geq t_o > 0$ there exists $c > 0$ and $\delta_0 > 0$ such that we have

$$\exp(t^2/2\ (1-tc/2)\ \sigma_k^2/s_n^2 \leq E(\exp(tX_k/s_n)) \leq \exp(t^2/2(1+tc/2)\ \sigma_k^2/s_n^2)$$

wherever $tc \leq \delta_o$, then $\{X_n\}$ obey the law of iterated logarithm.

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