A Queue subject to Extraneous Phase Changes*

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Mimeograph Series No. 207

September 1969

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*The research of the author was partly supported by the Office of Naval Research Contract NONR 1100 (26) at Purdue University. He was on sabbatical leave at Cornell University during the academic year 1968-1969.

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ABSTRACT

Many service systems exhibit variations of a random nature in the intensity of the arrival process or of the speed of service or of both. Changes in work shifts, rush hours, interruptions in the arrival process, server breakdowns, etc. all fall into this category.

The present study deals with a generalization of the classical M|G|1 queue by considering an extraneous process of phases which can be in one of the states $\{1,\dots,m\}$. During any interval spent in phase i, the arrivals are according to a homogeneous Poisson process of rate λ_i and any service initiated during such interval has a duration distributed according to $H_i(\cdot)$. The process of phases is assumed to be an irreducible, Markov chain in continuous time and is fully characterized by its initial conditions, by an irreducible stochastic matrix P and by the mean sojourn times $\sigma_1^{-1},\dots,\sigma_m^{-1}$ in each phase.

Independently of the queueing aspects, this arrival process is a generalization of the classical Poisson process which can be of interest in modelling simple point processes with randomly fluctuating "arrival" rate.

Two approaches to the time dependent study of this queue are presented; one generalizes the imbedded semi-Markov process obtained by considering the queue immediately following departure points; the other approach exploits the relationship between this queue and branching processes. The latter is more eloquent from a

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purely theoretical viewpoint and involves iterates of a general type of matrix function introduced by the author. By making extensive use of the Perron-Frobenius theory of positive matrices the equilibrium condition of the queue is obtained. While retaining a similar intuitive interpretation the equilibrium condition is substantially more complicated than for the M|G|1 model.

The recurrence relations which yield the joint distribution of the phase state at time t, the queuelength, the total number served and the virtual waiting-time at t are exhibited in detail. Via transform techniques a number of limiting and marginal distributions are discussed. The discussion relies heavily on the theory of Markov Renewal processes.

Throughout the paper and in a final section the author advocates the use of the structural properties of the queue and the resulting recurrence relations to organize the numerical analysis of complex queueing models such as the present one.

More explicit results for the case of two phases are given and are compared to results obtained by Yechiali and Naor for a closely related two-phase generalization of the M|M|1 queue.

1. Introduction.

The queueing model discussed here is a generalization of a model treated by Uri Yechiali and Paul Naor [24]. These authors, who kindly sent us a prepublication draft of their work, discuss an M|M|1 queue, modified as follows. The queue alternates between two phases I and II. In phase I, the arrival and service rates are λ_1 and μ_1, whereas in phase II they are λ_2 and μ_2. The successive lengths of time during which the queue is in phases I and II are independent, negative-exponential random variables with parameters σ_1 and σ_2 respectively. They examined the equilibrium equations for the queuelength and obtained a necessary and sufficient condition for the queue to be stable.

We propose to discuss the following partial generalization of this model. Consider a single server queue, governed by an extraneous phase process. This phase process is assumed to be an irreducible, m-state Markov chain in continuous time, with only stable states. As is well-known, this Markov chain is fully characterized by its state at t=0, by the transition probability matrix P of its imbedded discrete parameter chain, which is irreducible and stochastic and by the parameters $\sigma_1, \sigma_2, \ldots, \sigma_m$ of the negative exponential sojourn times in the states $1, \ldots, m$ respectively - Pyke [19].

The arrival process to the queue is assumed to be a homogeneous Poisson process of rate $\lambda_i \geq 0$, during any interval of time that the phase process is in the state i, $i=1,\ldots,m$. This process, an interesting generalization of the ordinary Poisson process, is discussed in some detail below.

The successive service times are assumed to be conditionally independent given the phase process. A customer whose service time is initiated during a phase of type i has a service time distribution $H_i(\cdot)$ of finite mean α_i. For most considerations in this paper the queue discipline is immaterial.

There are many potential applications of this model. Many queues naturally exhibit random fluctuations in their arrival rates or service characteristics or both. Traffic queues typically oscillate between periods of heavy, medium and light traffic.

Many service mechanisms are operated by personnel of varying skills and working speed. This succession of work shifts can in some cases be modeled in the terms defined here.

Interruptions in the arrival process correspond to phases during which the arrival rate is zero. They may hence also be treated within the present framework.

A further special case arises when one or more (but not all) of the service time distributions $H_i(\cdot)$ are degenerate at zero. This is interpreted as a shut-down of the server with dismissal of all customers who have not yet begun service.

Before embarking upon the mathematical exploration of this model, we discuss one assumption in some detail. The service time distribution of a customer depends only on the state of the phase process at the time his service begins. Ours is therefore not a direct generalization of the Naor-Yechiali model, since these authors assume that the service rate of the Markovian queue, treated, changes as soon as the state of the phase process changes. To extend the Naor-Yechiali model directly to general service times introduces very considerable analytic complications. In all but the case of negative-exponential service times, one has to make assumptions describing how a service which straddles one or more phase changes is handled.

In a great majority of potential applications our assumption, on the relation between the phase and the service time distribution is not a serious limitation. Whenever the phases are long compared to individual service times, the practical importance of this issue is little. In a traffic situation, such as an intersection, it amounts to assuming that a vehicle finding itself in the intersection when the light turns red (or amber) continues to cross at the same speed.

In this paper the phase process is a continuous time Markov chain, so that the sojourn times in each state have a negative exponential distribution. This is clearly a limitation and it may be desirable to develop a generalization of the present model by introducing a semi-Markovian process of phases. Given the present state of the art in Queueing Theory, this semi-Markovian phase model could be discussed in terms of imbedded Markov processes but the computational details are truly forbidding. The present technique, akin to the theory of Branching Processes, does not carry over to this case.

Should one wish to study a phase process in which the time intervals in each phase have generalized Erlang distributions, i.e. distributions which are convolutions of finitely many negative exponentials; this case can be handled by our technique. It suffices, in principle, to augment the number of phases sufficiently and appropriately so that the sojourn times all become negative exponential. This technique, due to A. K. Erlang [7], is classical. We refer to the monograph by D. R. Cox and W. L. Smith [6, p 110 ff], where it is very readably discussed.

2. The Arrival Process.

The process of phases, an m-state irreducible Markov chain in continuous time (with all stable states), may be studied equivalently in terms of the semi-Markov sequence $\left\{ \left(\tilde{J}_n, \tau_n \right), n \geq 0, \tau_0 = 0 \right\}$ of the successive states $\tilde{J}_n$ visited and the successive sojourn times $\tau_n, n \geq 1$. The transition probability matrix for the semi-Markov sequence $\left\{ \left(\tilde{J}_n, \tau_n \right), n \geq 0 \right\}$ is given by:

$$(1) \qquad P\left\{ \tilde{J}_{n+1} = j, \tau_{n+1} \leq x \,\middle|\, \tilde{J}_n = i \right\} = p_{ij}\left(1-e^{-\sigma_i x} \right),$$

for $1 \leq i, j \leq m$, $\sigma_i > 0$, $p_{ii} = 0$, $i = 1,\ldots,m$. The matrix $P = \{p_{ij}\}$ is stochastic - Pyke [19].

The sojourn times τ_n, $n \geq 0$ are conditionally independent random variables given the Markov chain $\left\{\tilde{J}_n\right\}$ - Pyke [18].

In the arrival process considered here, customers join the queue according to a Poisson process of rate λ_i during any sojourn interval spent in the phase i. Let $N\left(t_1, t_2 \right), t_1 \leq t_2$ be the number of arrivals during the interval $\left(t_1, t_2 \right]$, then the assumptions imply that the random pairs $\left[\tau_n, N(\tau_o + \ldots + \tau_{n-1}, \tau_o + \ldots + \tau_n) \right]$, $n \geq 1$, are conditionally independent, given the Markov chain $\{ \tilde{J}_n \}$. Moreover:

(2) $$P\left\{ \tilde{J}_{n+1} = j,\ \tau_{n+1} \leq x,\ N(\tau_o + \ldots + \tau_n, \tau_o + \ldots + \tau_{n+1}) = \nu \mid \tilde{J}_n = i \right\}$$

$$= p_{ij} \int_o^x e^{-\sigma_i u - \lambda_i u} \frac{(\lambda_i u)^\nu}{\nu!} \sigma_i \, du ,$$

for $1 \leq i, j \leq m$, $\nu \geq 0$, $x \geq 0$.

The main properties of the counting process $N(\cdot,\cdot)$ are summarized in the following lemma.

Lemma 1.

a. For all $0 \leq t_1 \leq \ldots \leq t_M$ the random variables $N(0, t_1)$, $N(t_1, t_2), \ldots,$ $N(t_{M-1}, t_M)$ are conditionally independent, given the random variables $J^*_o, J^*_{t_1}, \ldots, J^*_{t_M}$. J^*_t represents the phase at time t.

b. The process $N(t_1, t_2)$ is conditionally stationary in the sense that the conditional distribution of $N(t_1, t_2)$, given $J^*_{t_1}$ and $J^*_{t_2}$, is the same as the conditional distribution of $N(0, t_2 - t_1)$, given J^*_o and $J^*_{t_2 - t_1}$. Since we shall only consider these conditional distributions in the sequel we shall henceforth write N_t for $N(0,t)$, $t \geq 0$.

c. The conditional probabilities:

(3) $$P_{ij}(n,t) = P\left\{ J^*_t = j,\ N_t = n \mid J^*_o = i \right\}$$

for $n \geq 0$, $1 \leq i, j \leq m$, $t \geq 0$, satisfy the equations:

$$(4)\quad P_{ij}(n,t) = \delta_{ij} e^{-(\lambda_i+\sigma_i)t} \frac{(\lambda_i t)^n}{n!}$$

$$+\sum_{\rho=1}^{m} \sigma_i \, p_{i\rho} \sum_{\nu=o}^{n} \int_o^t e^{-(\lambda_i+\sigma_i)\tau} \frac{(\lambda_i \tau)^\nu}{\nu!} P_{\rho j} (t-\tau, n-\nu) \, d\tau$$

Proof

The random variable $N(t_k, t_{k+1})$, $0 \leq k < M$, depends only on the path of the continuous parameter Markov chain J_t^* during the time interval (t_k, t_{k+1}). These paths in the nonoverlapping intervals $(o, t_1) \ldots (t_{M-1}, t_M)$ are themselves conditionally independent given the random variables $J_o^*, \ldots, J_{t_M}^*$, as may be shown by repeated application of the Markov property.

One may also prove a. by showing directly, but tediously, that the joint conditional distribution of $N(o, t_1), \ldots, N(t_{M-1}, t_M)$ given $J_o^* = i_o, \ldots, J_{t_M}^* = i_M$ factors for all choices of M, io $, \ldots, i_M$, and $t_1, \ldots, t_M$.

The proof of part b. is elementary and follows directly from the fact that:

$$(5)\qquad P\left\{ J_{t_2}^* = j \mid J_{t_1}^* = i \right\} = P\left\{ J_{t_2-t_1}^* = j \mid J_o^* = i \right\}$$

since the Markov chain has stationary transition probabilities.

The probabilistic argument leading to (4) is standard. Either the Markov chain remains in the state i during $(o,t]$ and n arrivals occur in the corresponding Poisson process of rate λ_i - or - at some time τ, $0 < \tau < t$, the Markov chains enters some state ρ and some number ν, $0 \leq \nu \leq n$, arrivals have occurred in $(o,\tau]$. During the remaining interval $(\tau,t]$ $n-\nu$ additional arrivals must occur and the Markov chain must go from state ρ to state j.

Lemma 2.

The equations (4) have a unique bounded solution $P_{ij}(n,t)$, $n \geq 0$, $1 \leq i$, $j \leq m$, $t \geq 0$.

Proof:

We define the generating functions:

$$P_{ij}(z,t) = \sum_{n=0}^{\infty} P_{ij}(n,t)\, z^n, \quad |z| \leq 1, \tag{6}$$

and the Laplace transforms:

$$T_{ij}(z,\xi) = \int_0^{\infty} e^{-\xi t}\, P_{ij}(z,t)\, dt, \tag{7}$$

for $|z| \leq 1$, Re $\xi > 0$ or $|z| < 1$, Re $\xi \geq 0$.

The equations (4) may then be written equivalently as:

$$P_{ij}(z,t) = \delta_{ij} e^{-(\lambda_i + \sigma_i - \lambda_i z)t} + \sum_{\rho=1}^{m} \sigma_i p_{i\rho} \int_0^t e^{-(\lambda_i + \sigma_i - \lambda_i z)\tau} \cdot P_{\rho j}(z, t-\tau)\, d\tau, \tag{8}$$

for $t \geq 0$, $1 \leq i$, $j \leq m$. Upon taking Laplace transforms we further obtain:

$$T_{ij}(z,\xi) = \delta_{ij}(\xi + \lambda_i + \sigma_i - \lambda_i z)^{-1} + \sum_{\rho=1}^{m} \sigma_i\, p_{i\rho}\, (\xi + \lambda i + \sigma i - \lambda i\, z)^{-1}\, T_{\rho j}(z,\xi), \tag{9}$$

for $1 \leq i$, $j \leq m$, $|z| \leq 1$, Re $\xi > 0$ or $|z| < 1$, Re $\xi \geq 0$.

It suffices to show that the equations (9) have a unique solution matrix, whose entries $T_{ij}(z,\xi)$ are analytic in the (z,ξ) - region of interest.

Let Δ_o be the diagonal matrix with $(\Delta_o)_{ij} = \sigma_i \delta_{ij}$ and Λ the diagonal matrix with $\Lambda_{ij} = \delta_{ij} \lambda_i$ and let I be the identity matrix of order m then the equations (9) may be written as:

(10)
$$T(z,\xi) = (\xi I + \Delta_o + \Lambda - \Lambda z)^{-1} + (\xi I + \Delta_o + \Lambda - \Lambda z)^{-1} \Delta_o P T(z,\xi),$$

where $T(z,\xi) = \left\{T_{ij}(z,\xi)\right\}$.

The matrix $I - (\xi I + \Delta_o + \Lambda - \Lambda z)^{-1} \Delta_o P$ is nonsingular in the region of interest as the spectral radius of the matrix $(\xi I + \Delta_o + \Lambda - \Lambda z)^{-1} \Delta_o P$ is strictly less than one. This follows from the Perron-Frobenius theory of nonnegative matrices [8], since we have:

(11)
$$\left| \left[\left(\xi I + \Delta_o + \Lambda - \Lambda z\right)^{-1} \Delta_o P\right]_{ij} \right| = \frac{\sigma_i \, p_{ij}}{|\xi + \sigma_i + \lambda_i - \lambda_i z|} \leq p_{ij},$$

with strict equality for some pairs (i,j). This implies that the matrix of interest has a smaller spectral radius than P.

It follows that:

(12)
$$T(z,\xi) = \left[I - \left(\xi I + \Delta_o + \Lambda - \Lambda z\right)^{-1} \Delta_o P\right]^{-1} \cdot \left(\xi I + \Delta_o + \Lambda - \Lambda z\right)^{-1};$$

and this may be rewritten as:

(13)
$$T(z,\xi) = \left[\xi I + \Delta_o + \Lambda (1-z) - \Delta_o P \right]^{-1},$$

which implies also that the entries $T_{ij}(z,\xi)$ are analytic functions of z and ξ in the region of interest Re $\xi \geq 0$, $|z| < 1$, or Re $\xi > 0$, $|z| \leq 1$.

Remarks

1. It is easy to verify that the bivariate process (J_t^*, N_t) is a Markov chain with state space $\{1,\ldots,m\} \times \{0,1,\ldots\}$. The functions $P_{ij}(n,t)$ are its transition probabilities.

2. There is little difficulty in studying the counting process assuming that the underlying phase process is a finite Markov renewal process. This was done by Kshirsagar [11], who assumed that the arrival rate does not depend on the state of the Markov renewal process. His discussion can be carried out without this restriction.

In the sequel we need the quantities $A_{ij}(z,\xi)$, $1 \leq i, j \leq m$, defined by:

$$(14) \qquad A_{ij}(z,\xi) = \int_0^\infty e^{-\xi u} P_{ij}(z,u)\, d\, H_i(u),$$

for $\operatorname{Re} \xi > 0, |z| \leq 1$ or $\operatorname{Re} \xi \geq 0, |z| < 1$. $H_i(\cdot)$ is the service time distribution of a customer commencing service during a phase in state i.

We note that $A_{ij}(z,\xi)$ is the transform with respect to x and n of the probability:

$$(15) \qquad \int_0^x P_{ij}(n,u)\, d\, H_i(u), \quad n \geq 0,\ x \geq 0,\ 1 \leq i, j \leq m.$$

which is the probability that a service time starting during a phase in state i, lasts for a length x at most, ends during a phase in state j and that n arrivals occur during this service time. Clearly:

$$(16) \qquad A_{ij}(z,\xi) = \sum_{n=0}^\infty z^n \int_0^\infty e^{-\xi x}\, d\left[\int_0^x P_{ij}(n,u)\, d\, H_i(u)\right],$$

Since the functions $P_{ij}(z,u)$ are known, the transforms $A_{ij}(z,\xi)$ are also known in principle. The fact that we cannot always write down explicit expressions for the $A_{ij}(z,\xi)$ does not limit their usefulness in the theoretical discussion. Actual computations should follow a different path altogether. This point is elaborated in the final section of this paper.

The matrix $A(z,\xi) = \{ A_{ij}(z,\xi) \}$ is also the transform of a <u>bivariate semi-Markov matrix</u>, since the expression:

$$(17) \qquad \frac{\sum_{n=o}^{\infty} U(y-n) \int_{o}^{x} P_{ij}(n,u) \, d H_i(u)}{\sum_{n=o}^{\infty} \int_{o}^{\infty} P_{ij}(n,u) \, d H_i(u)}$$

where $U(\cdot)$ is the distribution degenerate at zero, is a bivariate distribution function for $1 \leq i, j \leq m$. It is a distribution continuous in x and concentrating on the nonnegative integers for y. A discussion of the basic properties of multivariate semi-Markov matrices may be found in [17].

The following are some properties of $A(z,\xi)$ used in the sequal.

<u>Lemma 3</u>

a. For $i = 1,\ldots,m$:

$$(18) \qquad \sum_{j=1}^{m} A_{ij}(1,\xi) = h_i(\xi) \quad ,$$

where $h_i(\xi)$ is the Laplace-Stieltjes transform of $H_i(\cdot)$. It follows that the means of the row sum distributions corresponding to the continuous variable are α_i, $i = 1,\ldots,m$, the mean service times.

b. The matrix $A(1,0+) = A(1-,0)$ is an irreducible stochastic matrix. Unless some of the distributions $H_i(\cdot)$ are degenerate at zero, all its entries:

$$(19) \qquad A_{ij}(1,0+) = \int_{o}^{\infty} P_{ij}(1,u) \, d H_i(u) \ ,$$

are strictly positive. $A_{ij}(1,0+)$ is the probability that a service starting in phase state i ends during a phase in state j.

<u>Proof</u>:

Setting $z = 1$, we note that $P_{ij}(1,t)$ is simply the transition probability

$P\left\{J_t^* = j \mid J_o^* = i\right\}$ of the Markov chain of phases. It follows that for $i = 1,\ldots, m$

(20) $$\sum_{j=1}^{m} P_{ij}(1,t) = 1 , \quad t \geq 0 ,$$

Formula (18) follows by integration. Since the mean service times are assumed to be finite the next statement is obvious.

The equality of the two matrices in b. is immediate from the properties of transforms of bivariate distributions or from Abel's theorem.

It is well-known that for a finite Markov chain with stable states only, $P_{ij}(1,u) > 0$ for $u > 0$. - Chung [3]. Furthermore $P_{ij}(1,0+) = \delta_{ij}$. This implies that $A_{ij}(1,0+)$ is strictly positive, whenever $H_i(\cdot)$ is not degenerate at zero. The interpretation of $A_{ij}(1,0+)$ is evident.

Let us denote by $\pi_1, \ldots, \pi_m$ the stationary probabilities corresponding to the stochastic matrix $A(1,0+)$, i.e. the unique solution to the system of equations:

(21) $$\sum_{\rho=1}^{m} \pi_\rho \int_0^\infty P_{\rho j}(1,u)\, d H_\rho(u) = \pi_j ,$$

(22) $$\sum_{\rho=1}^{m} \pi_\rho = 1 ,$$

A quantity of fundamental importance is ρ^* defined below. The integral in (23) is the mean number of arrivals during a service initiated in state i of the phase process. ρ^* can be considered as the mean number of arrivals during an "average" service. As we shall see ρ^* is the <u>traffic intensity</u> for the model under study.

(23) $$\rho^* = \sum_{i=1}^{m} \pi_i \int_0^\infty \tilde{\kappa}_i(t)\, d H_i(t) ,$$

where:

$$\tilde{\kappa}_i(t) = \sum_{j=1}^{m} \kappa_{ij}(t), \tag{24}$$

$$\kappa_{ij}(t) = \lim_{z \to 1-} \frac{d}{dz} P_{ij}(z,t), \quad t \geq 0 .$$

Clearly $\tilde{\kappa}_i(t)$ is the expected number of arrivals in a time interval of length t, which begins during a phase in the state i. These expectations exist as is shown by the following argument. The expected number of arrivals during any interval of time is clearly an increasing function of each of the arrival rates $\lambda_1, \ldots, \lambda_m$, so that $\tilde{\kappa}_i(t)$ is majorized by the function $t \max(\lambda_1, \ldots, \lambda_m)$, which is the expected number of arrivals during an interval of length t in an ordinary Poisson process of rate $\max(\lambda_1, \ldots, \lambda_m)$. Since the distributions $H_i(\cdot)$ have finite means, this same argument shows that all the integrals appearing in (23) are finite.

Differentiating with respect to z in (8) and taking limits as $z \to 1-$, we obtain:

$$\kappa_{ij}(t) = \delta_{ij} \lambda_i t e^{-\sigma_i t} + \sigma_i \sum_{\rho=1}^{m} p_{i\rho} \int_0^t e^{-\sigma_i \tau} \left[\lambda_i \tau P_{\rho j}(1, t-\tau) + \kappa_{\rho j}(t-\tau) \right] d\tau , \tag{25}$$

for $1 \leq i, j \leq m$.

Summing over j, keeping in mind that $\sum_{\rho=1}^{m} p_{i\rho} = 1$ and $\sum_{\rho=1}^{m} P_{i\rho}(1,t) \equiv 1$, we obtain:

$$(26)\quad \tilde{\kappa}_i(t) = \lambda_i t\, e^{-\sigma_i t} + \sum_{\rho=1}^{m} \sigma_i p_{i\rho} \int_0^t e^{-\sigma_i \tau}\, \tilde{\kappa}_\rho(t-\tau)\, d\tau$$

$$= \left(\frac{\lambda_i}{\sigma_i}\right)\left(1-e^{-\sigma_i t}\right) + \sigma_i \sum_{\rho=1}^{m} p_{i\rho} \int_0^t \tilde{\kappa}_\rho(\tau)\, e^{-\sigma_i(t-\tau)}\, d\tau \quad ,$$

This equation can be simplified by setting:

$$(27)\qquad \tilde{\kappa}_i(t)\, e^{\sigma_i t} = \varphi_i(t)\ , \qquad i = 1,\dots,m\ ;\ t \geq 0.$$

Upon differentiation with respect to t, we obtain that the m functions $\varphi_i(t)$, $i = 1,\dots,m$ are given by the unique solution to the system of linear differential equations:

$$(28)\qquad \frac{d}{dt}\varphi_i(t) = \sigma_i \sum_{\rho=1}^{m} p_{i\rho}\, \varphi_\rho(t) + \lambda_i\, e^{\sigma_i t}\ , \qquad i = 1,\dots,m\ .$$

with the initial conditions $\varphi_i(0) = 0$, $i = 1,\dots,m$.

The functions $\tilde{\kappa}_i(t)$, $i = 1,\dots,m$ are therefore known in principle, so that the quantity ρ^* defined in (23) may be computed.

3. A first Imbedded Markov Renewal Process.

We now consider an imbedded Markov renewal sequence for the queueing process described in the introductory section. Let $t_0 = 0$, t_1, $t_2,\dots$ be the successive epochs in which departures from the queue occur; let ξ_0, $\xi_1,\dots$ and J_0, $J_1,\dots$ denote respectively the queuelengths and the states of the phase process, immediately following t_0, $t_1\dots$.

As usually:

$$(29)\qquad \xi_{n+1} = (\xi_n - 1 + \nu_n)^+,$$

where ν_n is the number of customers joining the queue during the service time of the n-th customer. The random variable ν_n depends on the past only through the state J_n of the phase process at time t_n+. It follows from this and from the definitions that the sequence of triples:

(30) $$\left\{ \left(J_n, \xi_n, \tau_n - \tau_{n-1} \right), n \geq 0 \right\} \qquad \left(\tau_{-1} = 0 \right)$$

is a Markov renewal sequence, defined on the state space $\{1,\ldots,m\} \times \{0,1,\ldots\}$. For definitions and fundamental properties of Markov renewal processes, we refer to [18] and other basic references listed in [15].

The transition probability matrix of the sequence (30) is defined by its entries:

(31) $$Q(i,k; j,k'; x) =$$
$$P\{ J_{n+1} = j, \xi_{n+1} = k', \tau_{n+1} - \tau_n \leq x \mid J_n = i, \xi_n = k \}$$

which have the following explicit forms:

(32) For $k' \geq k-1 \geq 0$:

$$Q(i,k; j,k'; x) = \int_0^x P_{ij}(k' - k+1, u)\, d H_i(u) ,$$

For $k' < k-1$:

$$Q(i,k; j,k'; x) = 0,$$

For $k = 0$:

$$Q(i,0; j,k'; x) = \sum_{\rho=1}^{m} \lambda_\rho \int_0^x P_{i\rho}(0,u)\, Q(\rho,1; j,k'; x-u)\, du ,$$

As in the theory of the $M|G|1$ queue, important probabilities in expressing the time dependence of the queue are those associated with the busy periods and with the paths of the queuelength process during the busy periods. We first consider these by extending a classical first passage argument to the present case.

Transitions within a busy period.

Let there be $k \geq 1$ customers at $t = 0$, i.e. $\xi_0 = k$, let $J_0 = i$ and let $t_0 = 0$ be the beginning of a service. We define ${}_0Q^{(n)}(i,k; j,k'; x)$ as the probability that the initial busy period involves at least n services, that the n-th service is completed not later than time x, that at the time of the n-th departure there are k' customers in the queue, that at the time of the n-th departure the phase process is in the state j; all this, given that $J_o = i$, $\xi_o = k$. For convenience we set ${}_oQ^{(o)}(i,k; j,k'; x) = \delta_{ij}\,\delta_{kk'}\,U(x)$, where $U(\cdot)$ is the distribution degenerate at zero. We have the following recurrence relation:

$$(33)\quad {}_oQ^{(n+1)}(i,k; j,k'; x) = \sum_{h=1}^{m}\sum_{\nu=1}^{k'+1}\int_0^x {}_oQ^{(n)}(i,k; h,\nu; x-u)\, d\,Q(h,\nu; j,k'; u),$$

for $n \geq 0$, $1 \leq i, j \leq m$, $k \geq 1$, $k' \geq 0$.

Formula (33) appears to be quite well suited for numerical computation as it stands. It yields more easily to a theoretical discussion however after we have lowered the slightly ill-reputed "Laplacian curtain" over it.

Let us write the Laplace-Stieltjes transforms of the mass-functions ${}_oQ^{(n)}(i,k; j,k'; x)$ by ${}_oq^{(n)}(i,k; j,k'; \xi)$, then (33) may be written as:

$$(34)\quad {}_oq^{(n+1)}(i,k; j,k'; \xi) = \sum_{h=1}^{m}\sum_{\nu=1}^{k'+1} {}_oq^{(n)}(i,k; h,\nu; \xi)\int_0^\infty e^{-\xi u}\, P_{hj}(k'-\nu+1, u)\, d\,H_h(u),$$

Furthermore we introduce the generating functions:

$$(35)\quad {}_oW_{ij}(z,\xi,w) = \sum_{n=o}^{\infty}\sum_{k'=o}^{\infty} {}_oq^{(n)}(i,k; j,k'; \xi)\, z^{k'}\, w^{n},$$

defined for $\operatorname{Re}\xi > 0$, $|z| \leq 1$, $|w| \leq 1$, or $\operatorname{Re}\xi \geq 0$, $|z| < 1$, $|w| \leq 1$ or $\operatorname{Re}\xi \geq 0$, $|z| \leq 1$, $|w| < 1$ and for $1 \leq i, j \leq m$.

After a number of routine manipulations, we obtain that the system of equations (34) may be replaced by the equivalent system of linear equations in terms of the generating functions ${}_oW_{ij}\ (z,\xi,w)$.:

$$(36)\quad z\ {}_oW_{ij}\ (z,\xi,w) = \delta_{ij}\ z^{k+1} + w \sum_{h=1}^{m} A_{h_j}\ (z,\xi) \left[{}_oW_{ih}\ (z,\xi,w) - {}_oW_{ih}\ (o,\xi,w) \right] ,$$

In matrix notation, (36) may be written as:

$$(37)\quad {}_oW\ (z,\xi,w) \left[z\ I - w\ A\ (z,\xi) \right] = z^{k+1}\ I - w\ {}_oW\ (o,\xi,w)\ A\ (z,\xi) ,$$

where ${}_oW\ (z,\xi,w) = \{ {}_oW_{ij}\ (z,\xi,w) \}$ and I is the m x m identity matrix.

The matrix equation (37) is similar to an equation occurring in the theory of queues with semi-Markovian service times or interarrival times Çinlar [4,5], Neuts [12, 13]. The following lemma is the analogue of Theorem 1 in [12].

Lemma 4.

For every pair (ξ, w) with Re $\xi > 0$, $|w| \leq 1$, or Re $\xi \geq 0$, $|w| < 1$, the equation:

$$(38)\qquad \det \left[z\ I - w\ A\ (z,\xi) \right] = 0 ,$$

has exactly m roots in the unit disk $|z| < 1$.

Proof:

Each entry of the matrix A (z,ξ) satisfies:

$$(39)\qquad |\ w\ A_{ij}\ (z,\xi)\ | \leq |\ w\ |\ A_{ij} \left[\ |z|,\ \text{Re}\ \xi \ \right] \leq |w|\ p_{ij} ,$$

with strict inequality holding for some pairs (i,j). The spectral radius of $w\,A\,(z,\xi)$ is therefore strictly less than one for $|z| \leq 1$, Re $\xi > 0$, $|w| \leq 1$, or $|z| \leq 1$, Re $\xi \geq 0$, $|w| < 1$, or $|z| < 1$, Re $\xi \geq 0$, $|w| \leq 1$, so that the eigenvalues $w\,\eta_\rho\,(z,\xi)$ of $w\,A\,(z,\xi)$ are all less than one. $\rho = 1,\ldots,m$.

If these m eigenvalues are all distinct throughout the region of interest, then for each (suitable) fixed ξ and w the equation:

(40) $$z = w\,\eta_\rho\,(z,\xi)\,, \qquad \rho = 1,\ldots,m\,,$$

has a unique root in the unit disk by Rouché's theorem. It then follows immediately that (38) has m roots in $|z| < 1$.

If the m eigenvalues are not all distinct, then we can still show that (38) has m roots in $|z| < 1$ by a perturbation argument and by appealing to Hurwitz's theorem. For the latter, see Titchmarsh [23].

Using lemma 4, we could proceed to discuss the equation (37) as in references [4,5,12,13]. We would obtain an additional system of equations for the unknown term ${}_{o}W\,(o,\xi,w)$ by noting that the entries of the matrix:

(41) $$\left[\, z^{k+1}\, I - w\; {}_{o}W\,(o,\xi,w) \right] \left[\, z\,I - w\,A\,(z,\xi) \right]^{-1}$$

can only have removable singularities in $|z| \leq 1$ for each (ξ,w).

We choose not to present this argument in detail as we consider the following alternate approach more elegant and illuminating.

4. A second Imbedded Markov Renewal Process - The Markov Renewal Branching Process.

The queueing process under consideration lends itself particularly well to an argument suggested for the M|G|1 model by David Kendall [9] and developed in detail in Neuts [14]. It was a pleasant discovery that the matrix formalism introduced in [16] can also be put to use here. Before defining the imbedded process used in this section, we recall the definition introduced there.

Let $F(z) = \{ F_{ij}(z) \}$ be an m x m matrix whose entries are analytic functions of z in $|z| \leq 1$, with Maclaurin series $\sum_{n=0}^{\infty} F_{ij}^{(n)} z^n = F_{ij}(z)$.

By $C = \{C_{ij}\}$ we denote an m x m complex matrix such that $|| C || = \max_i \sum_{j=1}^{m} |C_{ij}| \leq 1$. The matrix function $F[\cdot]$ is defined over the set C of matrices C by:

$$(42) \quad F[C] = \sum_{n=0}^{\infty} F^{(n)} C^n \quad ,$$

where $F^{(n)} = \{F_{ij}^{(n)}\}$ and C^n is the ordinary n-th power of the matrix C.

It is easy to verify that $F[\cdot]$ is a continuous mapping of C into itself provided $F(z) \in C$ for all $|z| \leq 1$. If this is the case, the functional iterates

$$(43) \qquad F_n(z) = F_{n-1}\left[F(z) \right] = F\left[F_{n-1}(z) \right], \quad n \geq 1 ,$$

$$F_o(z) = z I ,$$

are also well defined.

We now define an imbedded discrete parameter process in the queue as follows. Starting at t = 0, with $k \geq 1$ unprocessed customers and in phase i, we define T_1' as the time until all customers present at t = 0 have completed service. By ξ_1' we understand the number of customers arriving during $(0, T_1')$ and by I_1 we represent the state of the phase process at time T_1' +. For notational convenience we set $T_o' = 0$, $I_o = i$, $\xi_o' = k$. If $\xi_1' = 0$, then T_1' is the end of the initial busy period. If $\xi_1' > 0$, then T_2' is the time at which all customers present at T_1' have departed and I_2, ξ_2' are respectively the state of the phase process and the queuelength at that time. Equivalently, ξ_2' is the number of arrivals to the system in $(T_1', T_2']$.

Continuing in this manner, we define the triples $(I_n, \xi_n', T_n' - T_{n-1}')$, $n \geq 1$, recursively and it follows routinely from lemma 1 and the basic definitions that the sequence $\left\{ (I_n, \xi_n', T_n' - T_{n-1}') , n \geq 0, T_0' = T_{-1}' = 0 \right\}$ is a Markov renewal sequence on the state space $\{ 1,\ldots, m \} \times \{ 0, 1, \ldots \}$ with the states $(j, 0)$, $j = 1 ,\ldots, m$, absorbing.

The transition probabilities $\tilde{Q}(i,k; j,k'; x)$ defined by:

$$(44) \quad \tilde{Q}(i,k; j,k'; x) = P\left\{ I_{n+1} = j, \xi_{n+1}' = k', T_{n+1}' - T_n' \leq x \mid I_n = i, \xi_n' = k \right\}$$

may be written in terms of the probabilities whose definitions preceed (33) as:

$$(45) \qquad \tilde{Q}(i,k; j,k'; x) = {}_0Q^{(k)}(i,k; j,k'; x),$$

The following lemma expresses the generating function of the Laplace-Stieltjes transforms $\tilde{q}(i,k; j,k'; \xi)$ of the mass functions $\tilde{Q}(i,k; j,k'; x)$ in terms of the matrix $A(z,\xi)$ defined in equation (14).

<u>Lemma 5</u>

For $1 \leq i, j \leq m$, $k \geq 1$ and $\text{Re}\, \xi > 0$, $|z| \leq 1$ or $\text{Re}\, \xi \geq 0$, $|z| < 1$, we have:

$$(46) \qquad \sum_{k'=0}^{\infty} \tilde{q}(i,k; j,k'; \xi) z^{k'} = \left[A^k(z,\xi) \right]_{ij},$$

<u>Proof</u>:

By virtue of (45), we have that:

$$(47) \quad \sum_{k'=0}^{\infty} \tilde{q}(i,k; j,k'; \xi) z^{k'} = \sum_{k'=0}^{\infty} {}_0q^{(k)}(i,k; j,k', \xi) z^{k'},$$

The recurrence relation (33) yields upon taking transforms and recalling (32) that:

$$(48)\quad \sum_{k'=0}^{\infty} {}_oq^{(n+1)}(i,k;\ j,k';\ \xi)\ z^{k'} =$$

$$\sum_{h=1}^{m}\sum_{\nu=0}^{\infty}\sum_{k'=\nu}^{\infty} z^{k'}\int_0^{\infty}\int_0^{x} e^{-\xi x}\, P_{hj}(k'-\nu,\ u)\ d\,H_h(u)\ d\,{}_oQ^{(n)}(i,k;\ h,\ \nu+1,\ x-u)$$

$$= \sum_{h=1}^{m} A_{hj}(z,\xi)\,\frac{1}{z}\left[\sum_{\nu=0}^{\infty} {}_oq^{(n)}(i,k;\ h,\nu;\ \xi)\ z^{\nu} - {}_oq^{(n)}(i,k;\ h,0,\xi)\right]$$

We note that for $n \leq k-1$, the quantity ${}_oq^{(n)}(i,k;\ h,0,\xi)$ is zero, since with k customers present initially, the initial busy period involves at least k services. Since also:

$$(49)\quad \sum_{k'=0}^{\infty} {}_oq^{(1)}(i,k;\ j,k',\xi)\ z^{k'} = z^{k-1}\ A_{ij}(z,\xi)\ ,$$

we obtain recursively from (48) that:

$$(50)\quad \sum_{k'=0}^{\infty} {}_oq^{(n)}(i,k;\ j,k';\ \xi)\ z^{k'} = z^{k-n}\left[A^n(z,\xi)\right]_{ij}\ ,$$

for $k \leq n$. In particular for $n = k$, we obtain the desired result.

We now obtain a full characterization of the matrix ${}_oW(0,\xi,w)$ appearing in equation (37). To this effect we first define the following random variable of independent interest.

Let θ_n denote the total number of customers served up to time T_n', provided T_n' is defined. In other words, θ_n is the total number of customers served up to and including the n-th generation of customers in the queue.

If we let ${}_o\tilde{Q}^{(n)}(i,k;\ j,k';\ r;\ x)$ with Laplace-Stieltjes transform ${}_o\tilde{q}^{(n)}(i,k;\ j,k';\ r;\ \xi)$ be the probability:

(51) $P\left\{ I_n = j,\ \xi_n' = k',\ \theta_n = r,\ T_n' \leq x,\ \xi_\nu' \neq 0, \nu=1,\ldots,n-1 \mid I_o=i,\ \xi_o' = k \right\}$

then we have:

(52) $_o\tilde{q}^{(n+1)}\ (i,k;\ j,k';\ r;\ \xi) =$

$$\sum_{h=1}^{m} \sum_{\nu=1}^{r} {}_o\tilde{q}^{(n)}\ (i,k;\ h,\nu;\ r-\nu;\ \xi)\ \tilde{q}\ (h,\nu;\ j,k';\ \xi)\ ,$$

for $n \geq 0$, provided we set:

(53) $${}_o\tilde{q}^{(o)}\ (i,k;\ j,k';\ r,\ \xi) = \delta_{ij}\ \delta_{kk'}\ \delta_{or}\ ,$$

where the deltas are Kronecker deltas.

Introducing the generating functions:

(54) $${}_o\Phi_{ij}^{(n)}\ (z,\xi,w) = \sum_{r=1}^{\infty} \sum_{k'=o}^{\infty} {}_o\tilde{q}^{(n)}\ (i,k;\ j,k';\ r,\ \xi)\ z^{k'}\ w^{r}\ ,$$

(52) may be written compactly as:

(55) $${}_o\Phi_{ij}^{(n+1)}\ (z,\xi,w) = \sum_{h=1}^{m} \sum_{\nu=1}^{\infty} \sum_{r=1}^{\infty} \tilde{q}^{(n)}\ (i,k;h,\nu;r,\xi) w^{r} \left[w^{\nu}\ A^{\nu}\ (z,\xi) \right]_{hj}\ ,$$

by (46).

Finally let $_o\Phi^{(n)}\ (z,\xi,w)$ be the m x m matrix with entries $_o\Phi_{ij}^{(n)}\ (z,\xi,w)$ and let the matrixfunctional iterates $A_{[n]}\ (z,\xi,w)$ be defined by:

(56) $$A_{[o]}\ (z,\xi,w) = z\ I$$

$$A_{[n+1]}\ (z,\xi,w) = w\ A\ \left[A_{[n]}\ (z,\xi,w)\ ,\ \xi \right],\quad n \geq 0.$$

for $|z| \leq 1$, Re $\xi \geq 0$, $|w| < 1$ or $|z| \leq 1$, Re $\xi > 0$, $|w| \leq 1$, or $|z| < 1$, Re $\xi \geq 0$, $|w| \leq 1$. The equation (55) can then be written as:

$$(57)\quad {}_{o}\Phi^{(n+1)}(z,\xi,w) = {}_{o}\Phi^{(n)}\left[w\,A\,(z,\xi),\ \xi,\ w \right] - \Phi^{(n)}(0,\xi,w)$$

for $n \geq 0$, with:

$$(58)\quad {}_{o}\Phi^{(o)}(z,\xi,w) = z^{k}\ I\ ,$$

Using a simple induction argument and (56) we further obtain:

$$(59)\quad {}_{o}\Phi^{(o)}(z,\xi,w) = z^{k}\,I\ ,$$

$${}_{o}\Phi^{(n)}(z,\xi,w) = A^{k}_{[n]}(z,\xi,w) - A^{k}_{[n-1]}(o,\xi,w)\ ,$$

for $n \geq 1$, $k \geq 1$.

Remarks

Formula (59) expresses the matrices ${}_{o}\Phi^{(n)}(z,\xi,w)$, $n \geq o$, in terms of suitable matrixfunctional iterates of the known "fundamental" matrix A (z,ξ). While these functional iterates correspond in the real time domain to extremely complicated successive substitutuons, they exhibit a remarkable simplicity in terms of the formalism developed here. A number of interesting conclusions may be drawn from (59).

Before doing so, we recall what each entry of ${}_{o}\Phi^{(n)}(z,\xi,w)$ corresponds to. Specifically ${}_{o}\Phi^{(n)}_{ij}(z,\xi,w)$ is the transform of the probability mass functions, which express the chance that:

a. The initial busy period starting in a phase-state i, with k customers involves at least n generations of customers and is in a phase-state j at the end of the service of the n-th generation customers.

b. the queuelength at the end of the n-th generation's servicetime is k' and the total number of customers served by that time is r.

c. the n-th generation customers complete service no later than time x.

Clearly this complicated joint probability itself is of little immediate interest, but by particularisation the many results may be obtained.

<u>The joint distribution of the initial busy period and the number of customers served during it.</u>

Setting z = o in (59), we easily show that:

$$\sum_{n=1}^{N} {}_{o}\Phi^{(n)} (0,\xi,w) = A^{k}_{[N]} (0,\xi,w) , \tag{60}$$

If we denote by ψ_N (i,k; j; r; x) the probability that the initial busy period with k customers and in phase-state i initially lasts for N generations and ends before time x during a phase of type j and involves exactly r services, then it is clear that:

$$\sum_{r=k}^{\infty} w^r \int_0^{\infty} e^{-\xi x} d\,\psi_N (i,k; j; r; x) = \tag{61}$$

$$\sum_{n=1}^{N} {}_{o}\Phi^{(n)}_{ij} (0,\xi,w) = \left[A^{k}_{[N]} (0,\xi,w)\right]_{ij} ,$$

If N tends to infinity ψ_N (i,k; j; r; x) converges to the probability ψ (i,k; j; r; x) that the busy period with initial conditions $\xi'_o = k$, $I_o = i$, ends before time x during a phase of type j and involves a total of r services. These probabilities converge to the stated limits because the corresponding events converge. Actually for N > r, if not earlier, ψ_N (i,k; j; r; x) ceases to depend on N.

Let us set:

(62) $$\sum_{r=1}^{\infty} w^r \int_0^{\infty} e^{-\xi x}\, d\Psi\ (i,k;\ j;\ r;\ x) = \gamma_{ij}^{(k)}(\xi,w)\ ,$$

then the m x m matrix $\gamma^{(k)}(\xi,w) = \left\{ \gamma_{ij}^{(k)}(\xi,w) \right\}$ is given by:

(63) $$\gamma^{(k)}(\xi,w) = \lim_{N\to\infty} A^{k}_{[N]}(0,\xi,w) = \left[\gamma^{(1)}(\xi,w)\right]^k\ ,$$

for $k \geq 1$. Henceforth we shall write $\gamma^{(1)}(\xi,w)$ simply as $\gamma(\xi,w)$.

We state this result formally as a theorem:

Theorem 1

The probabilities $\psi\ (i,k;\ j;\ r;\ x)$ which express the joint conditional distribution of the duration of and the total number of customers served during the initial busy period as well as the state of the phase process at the end of the initial busy period, given the initial conditions $I_o = i$, $\xi_o' = k$, have a generating function, defined in (62), which is given by the limit of the iterates of formula (63).

Corollary 1

The transform (62) is equal to ${}_oW_{ij}\ (0,\xi,w)$, since both refer to the same set of probabilities. We may write:

(64) $${}_oW\ (0,\xi,w) = \gamma^{k}(\xi,w)\ ,$$

Corollary 2

In view of Cor. 1, formula (37) may be written as:

(65) $_{o}W\ (z,\xi,w) =$

$$\left[z^{k+1}\, I - w\, \gamma^{k}\, (\xi,w)\, A\, (z,\xi) \right] \left[z\, I - w\, A\, (z,\xi) \right]^{-1} ,$$

Corollary 3

The matrix $\gamma\ (\xi,w)$ must satisfy the equation:

(66) $$\gamma\ (\xi,w) = w\, A \left[\gamma\ (\xi,w)\ ,\ \xi \right],$$

for Re $\xi > o$, $|w| \leq 1$ or Re $\xi \geq o$, $|w| < 1$.

Proof:

This follows from (56) and (63) by a routine argument establishing the continuity of the matrix function $A\ [Z,\xi]$ in Z for $||\ Z\ || \leq 1$.

Remark:

Equation (66) is the analogue for this model of the classical Takács functional equation for the transform of the joint distribution of the busy period and the number of customers served during it, which occurs in the theory of the $M|G|1$ queue.

As in the simpler case, equation (66) corresponds to a fairly simple system of recurrence relations for $w \neq 1$, whereas for $w = 1$, it corresponds to a system of nonlinear matrix integral equations of Volterra type. To show this, we note that the equation (63) implies that:

(67) $\psi\ (i,k;\ j;\ r;\ x) =$

$$\sum_{E_k} \sum_{i_1, i_2, \ldots, i_{k-1}} \psi\,(i,1;\, i_1;\, r_1;\, .) * \psi\,(i_1,1;\, i_2;\, r_2;\, .) * \ldots$$

$$* \psi\,(i_{k-1},1;\, j;\, r_k;\, x)\ ,$$

for $k > 1$. E_k is the set of all k-tuples $(r_1,\ldots,r_k)$ with $r_1 \geq 1, \ldots\ r_k \geq 1$ and $r_1 + \ldots + r_k = r$. The indices $i_1,\ldots,i_{k-1}$ all range from 1 to m.

In view of (67) it suffices to show that the $\psi\,(i,1;\, j;\, r;\, x)$ can be calculated recursively. By considering all possible situations at the end of the first service we obtain:

$$(68)\quad \psi\,(i,1;\, j;\, r;\, x) = \delta_{r1} \int_o^x P_{ij}\,(o,u)\, d\, H_i\,(u)$$

$$+ \sum_{k=1}^{r-1} \sum_{\rho=1}^{m} \int_o^x P_{i\rho}\,(k;u)\, \psi\,(\rho,k;\, j;\, r-1;\, x-u)\, d\, H_i\,(u)\ ,$$

for $r \geq 1,\ 1 \leq i,\ j \leq m,\ x \geq o$.

Formulae (67) and (68) display a set of recurrence relations from which the probabilities $\psi\,(i,k;\, j;\, r;\, x)$ may in principle be calculated for all values of i, k, j, r and x. The recurrence is initiated by:

$$(69)\qquad \psi\,(i,1;\, j;\, r;\, x) = \delta_{r1} \int_o^x P_{ij}\,(o,u)\, d\, H_i\,(u)\ ,$$

The reader may verify that upon taking the suitable transforms the equations (67), (68) and (69) lead to (66).

<u>Corollary 4</u>

The equation:

(70) $$Z = w\,A\,[Z,\xi]\ , \qquad \|\,Z\,\| \leq 1\ ,$$

has a unique solution $\gamma\,(\xi,w) = Z$, with entries analytic in w and ξ for Re $\xi \geq 0$, $|w| < 1$.

Proof:

The functional equation (70) may first be discussed in exactly the same manner as the basic equation (40) in [16]. We summarize this approach without repeating all details.

a. There exists a solution, namely the limit of the sequence of iterates in (63).

b. If Z is any solution of (70) and if $\eta\,(\xi,w)$ and $\underline{u}\,(\xi,w)$ are respectively an eigenvalue and the corresponding right eigenvector of Z, then $\eta\,(\xi,w)$ is a root of the equation (38) in the unit disk. Moreover the equation:

(71) $$\left[\,\eta\,(\xi,w)\,I - w\,A\,(\eta\,(\xi,w)\ ,\ \xi)\,\right]\underline{u}\,(\xi,w) = 0\ ,$$

is satisfied.

If we can show that the m roots of (38) and the corresponding right eigenvectors determine a matrix uniquely, then this matrix is clearly the only solution to equation (70). This is however an exceedingly difficult question. Çinlar [5] has given general sufficient conditions for this to be the case, but his conditions cannot be checked in specific cases as they require explicit knowledge of the roots of (38).

The uniqueness of a solution analytic in ξ and w for Re $\xi \geq 0$, $|w| < 1$ can be shown directly as follows. By the uniqueness theorem for generating functions and Laplace-Stieltjes transforms every analytic solution to (70) must have coefficients of w^r, $r \geq 1$ satisfying the recurrence relations which we obtain upon taking Laplace-Stieltjes transforms in (68) and (69). The latter are true recurrences which express terms for higher values of r uniquely in terms corresponding to lower values of r. Given any set of initial terms corresponding to $r = 1$, $1 \leq i, j \leq m$, we may therefore calculate in principle the coefficients of w^r for $r \geq 2$.

This argument uses the assumption of analyticity of the solution crucially. Finally it suffices to show that we can determine the constant term and the coefficient matrix of w uniquely from (70).

Setting first $w = 0$ in (70) we find $Z(\xi, o) = 0$ as expected since the busy period involves at least one service.

Using this, dividing both sides of (70) by w and letting $w \to 0$, we see that the coefficient of w in $Z(\xi, w)$ is given by $A(o, \xi)$. But this is precisely the matrix of Laplace-Stieltjes transforms of the quantities given in (69).

The initial terms are therefore determined and hence the coefficients of w^r, $r \geq 2$ by the analyticity assumption.

Remark

This argument cannot be applied to the equation:

(72) $$Z = A[Z, \xi] \qquad ||Z|| \leq 1,$$

obtained for $w = 1$. While the uniqueness of a solution to (72) remains a difficult open question, it is as far as queueing theory is concerned an academic one. The probabilities of interest may be calculated recursively, whereas on the other hand an explicit solution of (72) followed by inversion of the Laplace-Stieltjes transforms is beyond any practical reach.

5. The idle times and the busy cycles.

The initial busy period is followed by an idle period which lasts until the arrival of a new customer. Because of the regenerative nature of the queueing process, the epoch of arrival of a new customer is the beginning of a new busy period, this time with one initial customer. Conceivably during the idle period one or more changes in phase-state may occur. The probability that an idle period starting in phase-state i, lasts at least for a length of time x and that the phase-state at time x is j is equivalent to the probability that during an interval of length x the phase process changes from state i to state j and not a single arrival occurs; this is however exactly $P_{ij}(o;x)$.

We define a busy cycle as the time between two successive epochs in which the queue becomes empty. Each busy cycle consists of an idle period and a busy period with one customer initially. An exception to this is the initial busy cycle, which we interpret as identical with the initial busy period with k customers initially.

We now show that the successive busy cycles are the sojourn times of a finite-state Markov Renewal process. Consider the Markov Renewal Sequence

(30) $\left\{ \left(J_n, \xi_n, \tau_n - \tau_{n-1} \right), n \geq 0 \right\}$ discussed in section 3. Let T_1, $T_1 + T_2$, $T_1 + T_2 + T_3$, ... be the times of successive visits to the set of states $E = \left\{ (1,0), \ldots, (m,o) \right\}$ in the corresponding semi-Markov process. Furthermore let θ_1, θ_2, ... be the first indices (or equivalently the states of the phase process) of the states in E visited at these times. We also set $T_o = 0$, $\theta_o = i$.

It is clear that the random variables T_1, T_2, ... are the durations of the successive busy cycles and that θ_o is the initial phase-state and θ_1, θ_2, ... the phase-states at the beginning of each of the subsequent busy cycles.

Since the random variables T_n, $n \geq 1$ are first passage times between the states in E for the Markov Renewal sequence (30), they could be infinite with positive probability when the semi-Markov process corresponding to (30) is transient.

Since the Markov Renewal sequence (30) is clearly regular and irreducible it follows that the successive states (θ_ν, o), $\nu = o, 1, \ldots$ of the set E which are visited, along with the times $T_1, T_2, \ldots$ between such visits form again a Markov Renewal sequence, denoted by:

(73) $$\left\{ \left(\theta_n, T_n \right), n \geq o \right\}$$

This m-state Markov Renewal sequence will be referred to as the sequence of busy cycles. The transition probability matrix of the sequence of busy cycles could be improper in the sense that the sojourn times could be infinite with positive probability.

However since we have interpreted the random variables T_n, $n \geq 1$ as first passage times in the regular and irreducible Markov renewal sequence (30) it follows that:

the sequence of busy cycles (73) is (positive) recurrent if and only if the Markov renewal sequence (30) is (positive)recurrent.

We now write down the transition probabilities for the sequence of busy cycles. Firstly, since T_1 is just the length of the initial busy period, we have:

(74) $$\int_o^\infty e^{-\xi x} \, d\, P\left\{ \theta_1 = j, T_1 \leq x \mid \theta_o = i, \xi_o = k \right\} = \left[\gamma^k (\xi, 1) \right]_{ij} ,$$

Secondly, for each of the subsequent busy cycles, the probability $\chi_{ij}(\nu;x)$ that a busy cycle beginning in the phase-state i, lasts for a length of time not exceeding x, involves exactly $\nu \geq 1$ services and ends in the phase-state j is given by:

(75) $$\chi_{ij}(\nu;x) = \sum_{\rho=1}^{m} \lambda_\rho \int_o^x P_{i\rho}(o;u) \, \psi(\rho, 1; j; \nu; x-u) \, du,$$

where the ψ's are given by formulae (67), (68) and (69).

If we denote by N_n, $n \geq 1$, the number of services dispensed during the n-th busy cycle, we actually have:

$$(76) \quad \Xi_{ij}(\xi,w) = \sum_{\nu=1}^{\infty} w^{\nu} \int_0^{\infty} e^{-\xi x}\, d\, P\left\{ \theta_n = j,\ T_n \leq x,\ N_n = \nu \mid \theta_{n-1} = i \right\}$$

$$= \sum_{\nu=1}^{\infty} w^{\nu} \int_0^{\infty} e^{-\xi x}\, d\, \chi_{ij}(\nu;\, x) \quad ,$$

for $n \geq 2$.

Using (75) and matrix notation, we obtain:

$$(77) \quad \Xi(\xi,w) = T(o,\xi)\,.\Lambda.\, \gamma(\xi,w) \,,$$

where $\Xi(\xi,w) = \left\{ \Xi_{ij}(\xi,w) \right\}$, $T(o,\xi)$ is given by formula (10) and Λ is the diagonal matrix with $\Lambda_{ij} = \delta_{ij}\, \lambda_j$.

Formula (74) can also be written more generally as:

$$(78) \quad \sum_{\nu=k}^{\infty} w^{\nu} \int_0^{\infty} e^{-\xi x}\, d\, P\left\{ \theta_1 = j,\ T_1 \leq x,\ N_1 = \nu \mid \theta_o = i,\ \xi_o = k \right\} = \left[\gamma^{k}(\xi,w) \right]_{ij} \,,$$

where $\gamma(\xi,w)$ is the fundamental matrix, studied in section 4.

6. The Equilibrium Condition.

As seen in section 5 the imbedded sequence (30) is recurrent if and only if the sequence of busy cycles is recurrent. Since the latter is an m-state Markov renewal sequence it is recurrent if and only if the transition matrices with transforms $\gamma^{k}(\xi,1)$ and $\Xi(\xi,1)$ are proper semi-Markov matrices.

This is the case if and only if both $\gamma(0+,\, 1)$ and $\Xi(o+,\, 1)$ are stochastic matrices.

The following result simplifies the investigation of this issue.

Theorem 2

The matrix Ξ (o+, 1) is stochastic if and only if γ (o+, 1) is.

Proof:

We first show that:

(79) $$T\,(o,o+)\ .\Lambda.\ \underline{e} = \underline{e}$$

where $\underline{e}$ is an m-vector with all its components equal to one. Equation (9) implies trivially that:

(80) $$\sum_{j=1}^{m} T_{ij}\,(o,o+)\,\lambda_j = \frac{\lambda_i}{\lambda_i+\sigma_i} + \frac{\sigma_i}{\lambda_i+\sigma_i} \sum_{\rho=1}^{m} p_{i\rho} \left[\sum_{j=1}^{m} T_{\rho j}\,(o,o+)\,\lambda_j \right]$$

for i = 1 ,..., m .

This is a system of nonhomogeneous linear equations in the unknowns $\sum_{j=1}^{m} T_{ij}\,(o,o+)\,\lambda_j$, i = 1 ,..., m. Since $P = \left\{ p_{ij} \right\}$ is stochastic it follows that the values:

(81) $$\sum_{j=1}^{m} T_{ij}\,(o,o+)\,\lambda_j = 1, \qquad i = 1\,,\ldots,\,m$$

satisfy the system (80) and are therefore a unique solution.

It is now obvious that:

(82) $$\gamma\,(o+,\,1)\ \underline{e} = \underline{e}\ ,$$

implies:

(83) $$\Xi\,(o+,\,1)\ \underline{e} = T(o,o+)\ .\Lambda.\ \gamma(o+,1)\ \underline{e} = T(o,o+)\ \Lambda\ \underline{e} = \underline{e},$$ by (77) and (79).

If γ (o+, 1) is substochastic, $\gamma(o+,1)\ \underline{e} \leq \underline{e}$ componentwise with strict inequality for at least one component. By (79) this implies $\Xi\,(o+,1)\ \underline{e} \leq \underline{e}$, componentwise with strict inequality for at least one component, since $T\,(o,o+)\ \Lambda$ is a positive (in fact, stochastic) matrix.

It remains to investigate when γ (o+, 1) is stochastic. The following discussion relies heavily on the Perron-Frobenius theory of positive matrices and on the results on multivariate semi-Markov matrices established in [17]. We first observe that these results were proved using multiple Laplace-Stieltjes transforms. Here we are dealing with multiple transforms which are mixed probability generating functions and Laplace-Stieltjes transforms. There is no difficulty however in applying the results of [17] since probability generating functions (in w) become Laplace-Stieltjes transforms via a change of variable $w = e^{-\xi'}$. Any limit obtained as $\xi' \to o+$ is equivalent to a limit as $w \to 1-$ here.

Let us consider the matrices $\gamma(\xi,w)$ and $A(z,\xi)$ for $\text{Re}\ \xi \geq 0$, $0 \leq z \leq 1$ and $0 \leq w \leq 1$.

Both are then irreducible, nonnegative, (sub)stochastic matrices over these domains of z, ξ, w. That $\gamma(\xi,w)$ is irreducible follows from the elementary observation that the Markov renewal sequence (30) is irreducible. There is hence always positive probability of eventually reaching (j,o) from(i,o) in (30)for i, j = 1 ,..., m.

We denote the Perron-Frobenius eigenvalue of $A(z,\xi)$ by $\eta(z,\xi)$, $0 \leq z \leq 1$, $\xi \geq 0$, and the Perron-Frobenius eigenvalue of $\gamma(\xi,w)$ by $\chi(\xi,w)$, $\xi \geq 0$, $0 \leq w \leq 1$.

The following properties follow immediately from [17].

<u>Lemma 6</u>

a. For $0 \leq z \leq 1$, $\xi \geq 0$, $\eta(z,\xi)$ is uniquely determined and is analytic in (z,ξ) for $0 \leq z < 1$, $\xi \geq 0$ or $0 \leq z \leq 1$, $\xi > 0$.

b. The function $\eta(z,\xi)$ is a convex function jointly of z and ξ. For every z, $0 \leq z \leq 1$, it is a strictly decreasing function of $\xi \geq 0$ and for every $\xi \geq 0$ it is a strictly increasing function of z in $0 \leq z \leq 1$.

c. Since A (1,0) is stochastic, $\eta(1,0) = 1$. Moreover since the means of the distributions $H_i(\cdot)$ are finite, $i = 1, \ldots, m$, the derivatives:

$$\lim_{\substack{z \to 1- \\ \xi \to 0+}} \frac{\partial}{\partial z} \eta(z,\xi), \qquad \lim_{\substack{z \to 1- \\ \xi \to 0+}} \frac{\partial}{\partial \xi} \eta(z,\xi),$$

exist.

d. The derivatives in c. are given by:

$$\lim_{\substack{z \to 1- \\ \xi \to 0+}} \frac{\partial}{\partial z} \eta(z,\xi) = \rho^{*}, \tag{84}$$

and:

$$\lim_{\substack{z \to 1- \\ \xi \to 0+}} \frac{\partial}{\partial \xi} \eta(z,\xi) = -\alpha^{*} = -\sum_{j=1}^{m} \pi_j \alpha_j, \tag{85}$$

where the quantity ρ^{*} is given by equation (23).

Similarly for $\gamma(\xi,w)$, $\xi \geq 0$, $0 \leq w \leq 1$, we have:

Lemma 7

a. For $\xi \geq 0$, $0 \leq w \leq 1$, the Perron-Frobenius eigenvalue $\chi(\xi,w)$ is uniquely determined and is analytic in (ξ,w) for $\xi \geq 0$, $0 \leq w < 1$ and $\xi > 0$, $0 \leq w \leq 1$.

b. The function $\chi(\xi,w)$ is jointly convex in (ξ,w). For every $\xi \geq 0$, it is a strictly increasing function of w in $0 \leq w \leq 1$ and for every w, $0 \leq w \leq 1$, it is a strictly decreasing function of ξ.

c. The limit $\chi(0+, 1-)$ exists and is the Perron-Frobenius eigenvalue of $\gamma(0+, 1-)$.

Corollary 5

The matrix $\gamma(0+, 1-)$ is stochastic if and only if $\chi(0+, 1-) = 1$.

Proof:

This is an obvious consequence of c.

The following theorem establishes the relationship between $\chi(\xi, w)$ and $\eta(z, \xi)$. The equilibrium condition for the queue is then an easy consequence.

Theorem 3

a. For every w, $0 \leq w \leq 1$, and $\xi > 0$, the quantity $\chi(\xi, w)$ is the unique root in the interval (0,1) of the equation:

(86) $$z = w\,\eta(z, \xi),$$

b. As $\xi \to 0+$, $\chi(0+, w)$ tends to the smallest positive root of:

(87) $$z = w\,\eta(z, 0), \qquad 0 \leq w \leq 1.$$

For $0 \leq w < 1$, equation (87) has a unique root in $0 \leq z < 1$.

c. The queue is in equilibrium or equivalently $\gamma(0+, 1-)$ is stochastic if and only if:

(88) $$\rho^* \leq 1,$$

Proof:

For every fixed z, $0 \leq z \leq 1$, the entries $A_{ij}(z, \xi)$ of the matrix $A(z, \xi)$ are decreasing functions of $\xi \geq 0$. This implies that $\eta(z, \xi)$ is decreasing in $\xi \geq 0$ for every z in [0,1].

Furthermore $\eta(0, \xi) > 0$ and $\eta(1, \xi) \leq 1$, with strict inequality for $\xi > 0$, and the function $\eta(z, \xi)$ is convex increasing in z. Hence for every w, $0 < w \leq 1$, there is a point of intersection between the curves representing the functions $w\,\eta(z, \xi)$ and z.

For fixed w and for $0 \leq \xi_1 < \xi_2$, the curve of $w\,\eta(z, \xi_1)$ lies entirely above the curve of $w\,\eta(z, \xi_2)$, considered as functions of z in [0, 1]. Likewise for fixed $\xi \geq 0$, the curve of $w_1\,\eta(z, \xi)$ lies entirely above the curve of $w_2\,\eta(z, \xi)$ for $w_1 > w_2$.

It follows that for each ξ and w with $\xi > 0$, $0 < w \leq 1$ and $\xi \geq 0$, $0 < w < 1$, there is a unique abcissa $z = \chi_o(\xi,w)$, such that:

(89) $$\chi_o(\xi,w) = \eta\left[\chi_o(\xi,w), \xi\right], \qquad 0 < \chi_o(\xi,w) < 1,$$

For $\xi = 0$, $w = 1$, the value $z = 1$, is always a solution of the equation $z = \eta(z,0)$. By continuity $\chi_o(0+, 1-)$ is also a solution of this equation which may or may not be identical with $z = 1$. This depends on the derivative of $\eta(z,0)$ at $z = 1-$. If this derivative, whose value is ρ^* by Lemma 6, exceeds one then the convex increasing curve $\eta(z,0)$ has two distinct points of intersection with the curve of the function z. Hence for $\rho^* > 1$, we have that $\chi_o(0+, 1-) < 1$. If $\rho^* \leq 1$, then the curve of $\eta(z,0)$ lies entirely above that of the function z, so that $\chi_o(0+, 1-) = 1$.

Next, we show that $\chi_o(\xi,w)$, $\xi \geq 0$, $0 \leq w \leq 1$, is identical with the Perron-Frobenius eigenvalue $\chi(\xi,w)$ of $\gamma(\xi,w)$. To do so, we use the important property that the Perron-Frobenius eigenvalue of an irreducible nonnegative matrix is the one and only eigenvalue with associated left and right eigenvectors <u>all</u> of whose components can be chosen to be strictly positive.

Let $\underline{v}(z,\xi)$, $0 \leq z \leq 1$, $\xi \geq 0$ be a vector with all its components positive, which is the eigenvector of $A(z,\xi)$ corresponding to $\eta(z,\xi)$, then we have for $0 \leq z \leq 1$, $\xi \geq 0$ that:

(90) $$\left[w\eta(z,\xi) I - wA(z,\xi)\right] \underline{v}(z,\xi) = 0,$$

If we set $z = \chi_o(\xi,w)$, then (90) yields by (89):

(91) $$\left[\chi_o(\xi,w) I - w A[\chi_o(\xi,w), \xi]\right] \underline{v}[\chi_o(\xi,w), \xi] = 0,$$

The equation (91) implies, since $\underline{v}[\chi_o(\xi,w), \xi] \neq \underline{0}$, that $\chi_o(\xi,w)$ must be a root of the equation (38) and hence an eigenvalue of $\gamma(\xi,w)$. Moreover, a corresponding eigenvector $\underline{v}\,[\,\chi_o\,(\xi,w),\,\xi]$ has all its components positive, so that $\chi_o\,(\xi,w)$ must be the Perron-Frobenius eigenvalue $\chi\,(\xi,w)$ of $\gamma\,(\xi,w)$.

This completes the proof of the theorem.

Theorem 4

a. For $\rho^* < 1$, the derivative:

$$(92)\quad \lim_{\substack{\xi\to 0+\\ w\to 1-}} \frac{\partial}{\partial\xi}\,\chi\,(\xi,w) = \chi_\xi'\,(o+,\,1-) = -\,\alpha^*\,(1-\rho^*)^{-1} < \infty\,,$$

b. For $\rho^* < 1$, the sequence of busy cycles is positive recurrent. Equivalently the Markov renewal sequence (30) is positive recurrent.

c. For $\rho^* = 1$, the queue is null-recurrent. The unconditional means of all busy periods are infinite.

Proof:

Since $\eta\,(z,\xi)$ is a differentiable function of z and ξ in its domain of definition, the function $\chi\,(\xi,w)$ is likewise differentiable for all points in its domain of definition by virtue of equation (89), with the possible exception of $\xi = 0$, $w = 1$.

We have:

$$(93)\quad \frac{\partial}{\partial\xi}\,\chi\,(\xi,w) = w\left[\frac{\partial}{\partial z}\,\eta\,(z,\xi)\right]_{z=\chi\,(\xi,w)}\cdot\frac{\partial}{\partial\xi}\,\chi\,(\xi,w) + w\left[\frac{\partial}{\partial\xi}\,\eta\,(z,\xi)\right]_{z=\chi\,(\xi,w)}\,,$$

, The limit as $w\to 1-$, $\xi\to 0+$, must satisfy:

$$(94)\quad (\,1-\rho^*\,)\,\lim_{\substack{\xi\to 0+\\ w\to 1-}}\frac{\partial}{\partial\xi}\,\chi\,(\xi,w) = -\,\alpha^*\,,$$

If $\rho^* < 1$, this implies (92).

If a finite Markov renewal process is irreducible and if all its sojourn times have finite first moments, then the derivative at 0+ of the Perron-Frobenius eigenvalue of its matrix of Laplace-Stieltjes transforms is finite. Conversely if at least one of the sojourn time distributions has an infinite first moment, the derivative described above must also be infinite. It follows from (92), that when $\rho^* < 1$, the Markov renewal sequence of busy cycles has finite expected sojourn times. Since the sojourn times of this sequence are first passage times of the process (30) and since positive recurrence is a class property. - Pyke [18] - it follows that both these Markov renewal processes are positive recurrent.

If $\rho^* = 1$, it follows from (94) that $\lim_{\xi \to 0+} \frac{\partial}{\partial \xi} \chi(\xi, 1)$ cannot be finite. This implies that at least one state of the sequence of busy cycles must have an infinite mean sojourn time. However, the sojourn times of this process are first passage times of the process (30). If in the process (30) there is a first passage time with infinite mean, all first passage times must have infinite means since null-recurrence is a class property.

Therefore all the busy cycles must have infinite expected lengths, when $\rho^* = 1$.

Remark

At this stage we have obtained all the essential ingredients needed to discuss the transient behavior of the queue. The basic imbedded process is the sequence of busy cycles, which is completely known in terms of the fundamental matrix of transforms $\gamma(\xi, w)$.

By using the regenerative nature of the queueing process and the results obtained in sections 3 and 4, all other processes of interest such as the queue-length at time t, the virtual waitingtime at time t, etc. may be treated routinely. Finally, by appealing to a general result of Smith [20] for regenerative stochastic processes the limiting distributions for these same quantities may be written down. The ergodic properties of these continuous parameter processes follow immediately from those of the m-state Markov renewal sequence of busy cycles.

7. The Queuelength, the Virtual Waitingtime and the Total Number served.

We now consider the queue at time $t \geq 0$. Among the many features of the queue at t, the following are of particular interest: (a) the state J_t of the phase process (b) the queuelength $\xi(t)$, i.e. the number of customers in the system counting the one being served. (c) the virtual waitingtime $\eta(t)$, i.e. the length a customer arriving at t has to wait before his service begins. (d) the total number $N(t)$ of services completed in $(0,t]$]

We propose to discuss the recurrence relations which govern the joint distribution of J_t, $\xi(t)$, $\eta(t)$ and $N(t)$ and to obtain, where possible, closed form expressions for appropriate transforms. A variety of results concerning the marginal and limiting distributions then follow.

Define the following probabilities:

(95) $\theta_{ij}(k,k'; x; \nu; t) =$

$$P\left\{ J_t = j,\ \xi(t) = k',\ \eta(t) \leq x,\ N(t) = \nu \mid J_0 = i,\ \xi(0) = k \right\}$$

and:

(96) $${}_0\theta_{ij}(k,k'; x; \nu; t) =$$

$$P\left\{ J_t=j,\ \xi(t)=k',\ \eta(t) \leq x,\ N(t) = \nu,\ \xi(\tau) \neq 0.\ 0\leq\tau\leq t \mid J_0=i,\ \xi(0)=k \right\}$$

with all the variables ranging over the appropriate sets of values.

Before we write the recurrence relations which determine the quantities in (95) and (96), we consider the following important probabilities. Consider the interval $(0,t]$. One or more busy cycles may have ended during this interval. On the other hand, if t is included in the initial busy period, then no busy cycles were completed in (0,t]. Let $K_{ij}^{(n)}(\nu;t)$ be the probability that before t at least $n \geq 1$ busy cycles have ended and during them ν customers were served and at the end of the n-th cycle the phase process is in state j, given that it was in state i initially.

By an elementary recurrence argument we obtain that:

$$(97) \quad \sum_{\nu=k}^{\infty} w^{\nu} \int_0^{\infty} e^{-\xi t} \, d\, K_{ij}^{(n)}(\nu;t) = \left\{ \gamma^{k}(\xi,w) \, . \, \Xi^{n-1}(\xi,w) \right\}_{ij} ,$$

for $n \geq 1$.

In order to bypass a lengthy formal argument, we observe heuristically that the "differential" $\sum_{n=1}^{\infty} d\, K_{ij}^{(n)}(\nu,t)$ is the elementary probability that in (t,t+dt) there is a transition in the process of busy cycles, that the phase process is in state j and a total of ν customers have been served in all the completed busy cycles, given the usual initial conditions.

From (97) we obtain:

$$(98) \quad \sum_{n=1}^{\infty} \sum_{\nu=k}^{\infty} w^{\nu} \int_0^{\infty} e^{-\xi t} \, d\, K_{ij}^{(n)}(\nu;t) =$$

$$\sum_{n=1}^{\infty} \left\{ \gamma^{k}(\xi,w) \, \Xi^{n-1}(\xi,w) \right\}_{ij} = \left\{ \gamma^{k}(\xi,w) \left[I - \Xi(\xi,w) \right]^{-1} \right\}_{ij}$$

for Re $\xi > 0$ $|w| \leq 1$ or Re $\xi \geq 0$, $|w| < 1$.

The matrix $I - \Xi(\xi,w)$ is nonsingular since $\Xi(\xi,w)$ is the transform of an irreducible bivariate semi-Markov matrix. The spectral radius of $\Xi(\xi,w)$ in the domain of interest is less than unity,

For brevity we write:

$$(99) \quad K_{ij}(\nu;t) = \sum_{n=1}^{\infty} K_{ij}^{(n)}(\nu;t) ,$$

The empty queue

We have that:

$$(100) \qquad \theta_{ij} (k,0;\ x;\ \nu;\ t) = \sum_{\rho=1}^{m} \int_0^t P_{\rho j} (o;\ t-u)\ d\ K_{i\rho} (\nu;u)\ ,$$

independently of x for all $x > 0$. The probability argument leading to (100) is the following. At some time u prior to t a busy cycle ends and no new customers arrive in the interval (u,t). We also keep track of the state of the phase process and the number of customers served up to u.

In terms of transforms:

$$(101) \qquad \sum_{\nu=0}^{\infty} w^{\nu} \int_0^{\infty} e^{-\xi t}\ \theta_{ij} (k;\ 0;\ 0+;\ \nu;\ t)\ dt =$$

$$\left\{ \gamma^{k} (\xi,w) \left[I - \Xi (\xi,w) \right]^{-1} T (0,\xi) \right\}_{ij} \ ,$$

The non-empty queue

Next we express the probabilities $\theta_{ij} (k,k';\ x;\ \nu;\ t)$, $k' > 0$, in terms of the probabilities ${}_o\theta_{ij} (k,k';x;\ \nu;\ t)$. In order to do so observe that in (0,t] either the queue has never emptied out or alternatively one or more beginnings of busy cycles precede t. In the latter case, as we are dealing with a regular Markov Renewal process there is a last beginning of a busy cycle before t. Moreover since we have already dealt with $k' = 0$, we may assume $k' > 0$, so that there must be at least one arrival between this last beginning and t. In other words we are in the busy period portion of the busy cycle which includes the instant t.

This leads to a natural decomposition of the event corresponding to $\theta_{ij} (k,k';\ X;\ \nu;\ t)$. In the latter case let the busy cycle covering t start at time $u < t$ and the busy period covering t at time τ, $u < \tau < t$. Let the states of the phase process at u and τ be ρ and ρ' respectively, then we obtain from the law of total probability that:

(102) $$\theta_{ij}(k,k';\ x;\ \nu;\ t) = {}_o\theta_{ij}(k,k';\ x;\ \nu;\ t) + \sum_{\rho'=1}^{m} \sum_{\rho=1}^{m} \sum_{\nu'=k}^{\nu} \int_o^t \int_o^{\tau} {}_o\theta_{\rho'j}(1,k';\ x;\ \nu-\nu';\ t-\tau)$$

$$P_{\rho\rho'}(0;\ \tau-u)\ \lambda_{\rho'}\ d\ K_{i\rho}(\nu';u)\ d\tau\ ,$$

It remains to study the probabilities ${}_o\theta_{ij}(k,k';\ x;\ \nu;\ t)$. Consider the corresponding event. In order that at time t $J_t = j$; $\xi(t) = k'$; $\eta(t) \leq x$, and $N(t) = \nu$ without the queue being empty in $(0,t]$, given $J_o = i$, $\xi(0) = k$, the last departure prior to t must be the ν-th. Let it occur at some time $\tau \leq t$. Let there be k'', $0 < k'' \leq k'$ customers at time τ + and let the phase process be in some state $\rho \in \{1,\ldots,m\}$.

In order that $\xi(t)$ be equal to k', $k'-k''$ new arrivals should occur during $(\tau,t]$. Moreover the phase process must change from the state ρ to the state j during that same interval. Let the customer in service at time t depart at time t + v. In order to satisfy $\eta(t) \leq x$, the $k' - 1$ individuals in the system at t + v who arrived prior to the virtual customer must all leave before time t + x. Therefore we must have $0 \leq v \leq x$ and the total service time of the $k' - 1$ customers should not exceed x - v.

This verbal discussion again corresponds to a natural decomposition of the event into mutually exclusive or regenerative sets of events. Applying the law of total probability and the regenerative properties of the queue we obtain:

(103) $${}_o\theta_{ij}(k,k';\ x;\ \nu;\ t) =$$

$$\sum_{k''=1}^{k'} \sum_{\rho=1}^{m} \sum_{\sigma=1}^{m} \int_o^t \int_o^x d\ {}_oQ^{(\nu)}(i,k;\ \rho,k'';\ \tau).\ P_{\rho j}(k'-k'';\ t-\tau).$$

$$\cdot\left[\sum_{r=o}^{\infty} P_{j\sigma}(r;v)\right]\cdot d\ H_\rho(t+v-\tau)\left[\sum_{n=o}^{\infty} \sum_{\sigma'=1}^{m} {}_oQ^{(k'-1)}(\sigma,k'-1;\ \sigma';\ n;\ x-v)\right]$$

where the mass-functions ${}_{o}Q^{(\nu)}(i,k;\ \rho;\ k'';\ \tau)$ and ${}_{o}Q^{(k'-1)}(\sigma,\ k'-1;\ \sigma';\ n;\ \tau)$ were defined in Section 3. Note that $\sum_{r=o}^{\infty} P_{j\sigma}(r;\ v)$ is merely the probability that the phase process changes from state j to σ during a time interval of length v. It is hence also equal to $P_{j\sigma}(z;v)$, z = 1, so that we may write the sum as $P_{j\sigma}(1;\ v)$ provided the reader correctly interpret this symbol.

(Formula (103) relates ${}_{o}\theta_{ij}(k,k';\ x;\ \nu;\ t)$ to quantities already expressed in terms of the fundamental parameters of the queue. Should one attempt to perform explicit numerical computations leading to results on the timedependence of the queue one would have to do so using (103) and the auxiliary functions which were discussed earlier in the paper.

Pursuing the theoretical discussion, one may wish to obtain closed form expressions for the equations (103) via the use of transforms. The author attempted to calculate the transform:

$$(104)\quad {}_{o}\theta_{ij}^{(k)}(z,s,w,\xi) = \sum_{k'=1}^{\infty}\ \sum_{\nu=o}^{\infty}\ \int_{o}^{\infty}\int_{o}^{\infty} e^{-\xi t-sx}\, d_{x}\, {}_{o}\theta_{ij}(k,k';\ x;\ \nu;\ t)\, dt.\ z^{k'}\, w^{\nu},$$

from equation (103) but his efforts were frustrated by the noncommutativity of the matrix product. However, this only affected the portions of the calculation related to the virtual waitingtime $\eta(t)$. If we content ourselves with a discussion of the (marginal) distribution of J_t, $\xi(t)$ and N(t), then further progress in the discussion may be obtained.

Letting now x tend to + ∞ and defining the transform:

$$(105)\quad {}_o\theta_{ij}^{(k)}(z,w,\xi) =$$

$$\sum_{k'=1}^{\infty} \sum_{\nu=o}^{\infty} \int_o^{\infty} e^{-\xi t}\, {}_o\theta_{ij}(k,k';+\infty;\nu;t)\,dt.\; z^{k'} w^{\nu},$$

we obtain successively that:

$$(106)\quad {}_o\theta_{ij}(k,k';+\infty;\nu;t) =$$

$$\sum_{k''=1}^{k'} \sum_{\rho=1}^{m} \int_o^{t} P_{\rho j}(k'-k'';t-\tau)\left[1-H_\rho(t-\tau)\right] d_oQ^{(\nu)}(i,k;\rho;k'';\tau),$$

and

$$(107)\quad {}_o\theta_{ij}^{(k)}(z,w,\xi) =$$

$$\sum_{\rho=1}^{m} \sum_{\nu=o}^{\infty} \sum_{k'=1}^{\infty} \sum_{k''=1}^{k'} z^{k'} w^{\nu} \int_o^{\infty} e^{-\xi t}\,dt \int_o^{t} P_{\rho j}(k'-k'';t-\tau)\left[1-H_\rho(t-\tau)\right]$$

$$d_oQ^{(\nu)}(i,k;\rho,k'';\tau)$$

$$= \sum_{\rho=1}^{m} \sum_{\nu=o}^{\infty} \sum_{k''=1}^{\infty} \sum_{k_1=o}^{\infty} z^{k_1+k''} w^{\nu} \int_o^{\infty}\int_o^{\infty} e^{-\xi(\tau+t')} P_{\rho j}(k_1;t')\left[1-H_\rho(t')\right]$$

$$dt.\; d_oQ^{(\nu)}(i,k;\rho,k'';\tau),$$

after interchanging two summations and two integrations.

We now have:

$$(108)\quad {}_o\theta_{ij}^{(k)}(z,w,\xi) = \sum_{\rho=1}^{m}\left\{\sum_{\nu=o}^{\infty}\sum_{k''=1}^{\infty} z^{k''} w^{\nu} \int_o^{\infty} e^{-\xi\tau}\, d_oQ^{(\nu)}(i,k;\rho,k'';\tau)\right\}.$$

$$.\left\{\sum_{k_1=o}^{\infty} z^{k_1} \int_o^{\infty} e^{-\xi t'} P_{\rho j}(k_1;t')\left[1-H_\rho(t')\right] dt'\right\},$$

The expression between the first two curly brackets is seen to be equal to ${}_oW_{ij}(z,\xi,w) - {}_oW_{ij}(o,\xi,w)$ upon consideration of formula (35). Recalling (64) - (65), it is therefore also the (i,j)-entry of the matrix:

$$(109)\quad \left[z^{k+1} I - w\gamma^{k}(\xi,w) \right] \left[zI - wA(z,\xi) \right]^{-1} - \gamma^{k}(\xi,w)$$

$$= z\left[z^{k} I - \gamma^{k}(\xi,w) \right]\left[zI - wA(z,\xi) \right]^{-1} ,$$

The expression between the latter two curly brackets is equal to:

$$(110)\quad \sum_{k_1=o}^{\infty} z^{k_1} \int_o^{\infty} e^{-\xi t'} P_{\rho j}(k_1;t')\,dt' \int_{t'}^{\infty} dH_\rho(u) =$$

$$\int_o^{\infty} dH_\rho(u) \int_o^{u} e^{-\xi t'} P_{\rho j}(z;t')\,dt' =$$

$$\int_o^{\infty} dH_\rho(u) \,.\, \int_o^{u} \left\{ \exp\left(-\xi I - \Lambda - \Delta_o + \Lambda_z + \Delta_o P\right) t' \right\}_{\rho j} dt' =$$

$$\int_o^{\infty} dH_\rho(u) \left\{ \left[I - \exp\left(-\xi I - \Lambda - \Delta_o + \Lambda_z + \Delta_o P\right) u \right] \,.\right.$$

$$\left. \left[\xi + \Lambda + \Delta_o - \Lambda_z - \Delta_o P \right]^{-1} \right\}_{\rho j}$$

One may verify by writing out the entries that the latter expression is the (ρ,j) - entry of the matrix:

$$(111)\quad \left[I - A(z,\xi) \right]\left[\xi + \Lambda + \Delta_o - \Lambda_z - \Delta_o P \right]^{-1}$$

Only the mere outline of the matrix manipulations involved is given here. The details are easy to fill in, but are lengthy. Crucial is the representation:

$$(112)\qquad P(z,t) = \exp\left(-\Lambda - \Delta_o + \Lambda_z - \Delta_o P\right) t, \quad t \geq 0$$

While this follows directly from Markov Chain theory, we may also verify directly that:

(113) $T(z,\xi) = \int_0^\infty e^{-\xi t} P(z,t)\, dt = \left[\xi I + \Lambda + \Delta_0 - \Lambda z - \Delta_0 P \right]^{-1}$,

both from (112) and from (13).

Returning to (108), we see that the m x m matrix ${}_0\theta^{(k)}(z,w,\xi) = \left\{ {}_0\theta_{ij}^{(k)}(z,w,\xi) \right\}$ is given by:

(114) $z \left[z^k I - \gamma^k(\xi,w) \right] \left[z I - w A(z,\xi) \right]^{-1} \left[I - A(z,\xi) \right] T(z,\xi)$,

The formal analogy between (114) and the corresponding expression for the much simpler M|G|1 queue should be noted.

The final step is now routine. Recalling (101) - (102) and applying (97) and (114), the transform $\theta_{ij}^{(k)}(z,w,\xi)$ defined by:

(115) $\int_0^\infty e^{-\xi t} E\left\{ z^{\xi(t)} w^{N(t)} I\{J_t = j\} \,\middle|\, \xi(o) = k, J_o = i \right\} dt$,

is the (i,j) entry of the matrix:

(116) $\theta^{(k)}(z,w,\xi) =$

$$z \left[z^k I - \gamma^k(\xi,w) \right] \left[z I - w A(z,\xi) \right]^{-1} \left[I - A(z,\xi) \right] T(z,\xi)$$
$$+ \gamma^k(\xi,w) \left[I - \Xi(\xi,w) \right]^{-1} T(o,\xi)$$
$$+ \gamma^k(\xi,w) \left[I - \Xi(\xi,w) \right]^{-1} T(o,\xi) \Lambda z \left[z I - \gamma(\xi,w) \right] \cdot$$
$$\left[z I - w A(z,\xi) \right]^{-1} \left[I - A(z,\xi) \right] T(z,\xi)$$

The first matrix is the "contribution" of those paths where t is included in the initial busy period, the second one corresponds to t included in an idle period and the third term corresponds to t included in a busy period other than the initial one. Formula (116) allows for rather substantial simplifications, based on the following observation:

$$(117)\ \left[I - \Xi(\xi,w) \right]^{-1} T(o,\xi) = \left[T^{-1}(o,\xi) - T^{-1}(o,\xi)\,\Xi(\xi,w) \right]^{-1}$$

$$= \left[\xi I + \Lambda + \Delta_o - \Delta_o P - \Lambda \gamma(\xi,w) \right]^{-1},$$

by formulae (77) and (113).

After a number of routine matrix operations, we obtain:

$$(118)\ \theta^{(k)}(z,w,\xi) =$$

$$z^{k+1}\left[zI - wA(z,\xi) \right]^{-1}\left[I - A(z,\xi) \right]\left[\xi I + \Lambda + \Delta_o - \Lambda z - \Delta_o P \right]^{-1}$$

$$+ \gamma^{k}(\xi,w)\left[\xi I + \Lambda + \Delta_o - \Delta_o P - \Lambda\gamma(\xi,w) \right]^{-1} \cdot$$

$$\left\{ I - z\left[\xi I + (1-z)\Lambda + \Delta_o - \Delta_o P \right]\left[zI - wA(z,\xi) \right]^{-1} \left[I - A(z,\xi) \right]\left[\xi I + (1-z)\Lambda + \Delta_o - \Delta_o P \right]^{-1} \right\}$$

$$= z^{k+1}\left[zI - wA(z,\xi) \right]^{-1}\left[I - A(z,\xi) \right]\left[\xi I + \Lambda + \Delta_o - \Lambda z - \Delta_o P \right]^{-1}$$

$$+ \gamma^{k}(\xi,w)\left[\xi I + \Lambda + \Delta_o - \Delta_o P - \Lambda\gamma(\xi,w) \right]^{-1}\left[\xi I + (1-z)\Lambda + \Delta_o - \Delta_o P \right]$$

$$(z-w)\left[zI - wA(z,\xi) \right]^{-1} A(z,\xi)\left[\xi I + (1-z)\Lambda + \Delta_o - \Delta_o P \right]^{-1}$$

This formula is the complete analogue of the corresponding result for the M|G|1 queue. In fact, interpreting the latter as the case $m = 1$, $\sigma = 0$, we obtain successively:

$$(119)\qquad A(z,\xi) = h(\xi + \lambda - \lambda z), \qquad \Lambda = \lambda,\ \Delta_o = 0,\ P = 1,$$

and:

$$(120)\qquad \theta_1^{(k)}(z,w,\xi) = \frac{z^{k+1}}{\xi + \lambda - \lambda z} \cdot \frac{1 - h(\xi + \lambda - \lambda z)}{z - w\,h(\xi + \lambda - \lambda z)}$$

$$+ \frac{\gamma^{k}(\xi,w)}{\xi + \lambda - \lambda\gamma(\xi,w)} \cdot \frac{(z-w)\,h(\xi + \lambda - \lambda z)}{z - w\,h(\xi + \lambda - \lambda z)},$$

which is identical to formula (65) in Neuts [14], where a derivation of the corresponding result for the M|G|1 queue was given.

The Limiting Distribution of $\left(J_t, \xi (t) \right)$.

Setting w = 1 in (118) we obtain the matrix of transforms $\theta_{ij}(z,1,\xi)$:

(121) $$\int_0^\infty e^{-\xi t} E \left\{ z^{\xi(t)} I \{J_t = j\} \,\Big\|\, \xi(o) = k, J_o = i \right\},$$

Let us consider the process of busy cycles, which is as we have seen an m-state Markov Renewal process. This finite Markov Renewal process has a stationary version if and only if it is positive recurrent, i.e. if and only if $\rho^* < 1$. [19]. We may connect the processes J_t, $\xi(t)$ and $N(t)$ to the stationary version (which corresponds to an appropriate choice of initial conditions [19]) in exactly the same way as done in the preceding section, where these three processes were related to a particular non-stationary version of the process of busy cycles. If we perform the detailed calculations as before, we find an expression for the joint distribution identical to the limit given below.

Without presenting the details, which are by now standard in Queueing theory, we conclude from this argument that the limiting expressions below are indeed transforms of joint conditional probability distributions.

Finally we must argue that the expression obtained from relating the processes J_t, $\xi(t)$, $N(t)$ to the stationary version of the process of busy cycles is indeed the limiting distribution as t tends to infinity in the case $\rho^* < 1$.

In order to see this, we first let x tend to infinity in formula (102), obtaining thus the marginal conditional distribution of J_t, $\xi(t)$ and $N(t)$.

We can then sum over ν from k to $+\infty$, yielding the marginal distribution of J_t and $\xi(t)$. Finally appealing to Smith's key theorem for regenerative processes [20] we obtain that the limits $\theta_j^*(k') = \lim_{t\to\infty} \left\{ \sum_{\nu=k}^{\infty} \theta_{ij}(k,k'; +\infty; \nu; t) \right\}$ exist and must satisfy:

$$(122)\quad \overset{*}{\theta}_j (k') = \sum_{\rho=1}^{m} \frac{1}{M_\rho} \sum_{\rho'=1}^{m} \lambda_\rho \int_0^\infty P_{\rho\rho'} (o;\ \tau) \left\{ \sum_{\nu=1}^{\infty} {}_o\theta_{\rho' j} (1, k';\ +\infty;\ \nu;\ \tau) \right\} d\tau$$

in which M_ρ is the expected recurrence time of the state ρ in the process of busy cycles. Actually the transform of limiting joint distribution of J_t, $\xi(t)$ can be obtained directly from (118) by evaluating the limit matrix

$$(123)\qquad \lim_{\xi \to o+} \xi\, \theta^{(k)} (z, 1, \xi)\ ,$$

We shall show that this limit exists and is independent of k. Moreover the limit is an m x m matrix with constant columns. We now exhibit the details of this argument.

$$(124)\quad \lim_{\xi \to o+} \xi\, z^{k+1} \left[z\, I - A\, (z, \xi) \right]^{-1} \left[I - A\, (z, \xi) \right] \left[\xi\, I + \Lambda + \Delta_o - \Lambda z - \Delta_o P \right]^{-1}$$

$$= 0\ ,$$

for all $|z| < 1$, $k \geq 0$.

Moreover by an important result [19] from the theory of Markov Renewal processes, we have:

$$(125)\quad \lim_{\xi \to o+} \xi\, \gamma^{k} (\xi, 1) \left[I - \Xi\, (\xi, 1) \right]^{-1} = N$$

where $N_{ij} = 0$, when $\rho^* \geq 1$ and $N_{ij} = M_j^{-1}$, when $\rho^* < 1$. M_j is the mean recurrence time of the state j in the process of busy cycles. It follows that for $\rho^* < 1$, we have:

$$(126)\quad \overset{*}{\theta} (z) = N \,.\, \left[(1-z)\, \Lambda + \Delta_o - \Delta_o\, P \right] \left[z\, I - A\, (z, o) \right]^{-1} A\, (z, o)\ .$$

$$\left[(1-z)\, \Lambda + \Delta_o - \Delta_o\, P \right]^{-1} (z-1)\ ,$$

and it is easy to see that the right hand side does not depend on k and is a matrix with constant columns. Also

(127) $$\overset{*}{\theta}_{ij}(z) = \sum_{k'=o}^{\infty} \overset{*}{\theta}_{j}(k')\, z^{k'} ,$$

We may verify that (122) leads also to (126) upon taking generating functions with respect to k'.

If the queue is unstable, $\overset{*}{\rho} \geq 1$, then the limits $\overset{*}{\theta}_j(k')$ are all zero, by Smith's theorem [20] and the fact that N = 0.

The main difficulty in applying (126) is to determine the mean recurrence times M_j, j = 1,...,m.

This is considerably simplified upon using the following theorem.

<u>Theorem 5.</u>

Let $\Psi(\xi)$, $\xi \geq o$ be the Perron-Frobenius eigenvalue of the matrix $\Xi(\xi,1) = T(o,\xi)\,\Lambda\,\gamma(\xi,1)$, for $\xi \geq o$, then in the stable queue we have:

(128) $$\overset{*}{u}_j\, M_j = -\Psi'(o+) , .$$

where the $\overset{*}{u}_j$ are the stationary probabilities associated with $\Xi(o+, 1)$.

<u>Proof:</u>

We use the known fact that the matrix N, with entries $N_{ij} = M_j^{-1}$ is given by:

$$\lim_{\xi \to o+} \xi \left[I - \Xi(\xi,1) \right]^{-1} ,$$

and that $\Xi(o+, 1)$ is an irreducible stochastic matrix in the stable case.

Moreover let $\underline{u}(\xi)$ be the left eigenvector of $\Xi(\xi,1)$ normalized so that as $\xi \to o+$, $\underline{u}(\xi)$ tends to the vector of stationary probabilities $\overset{*}{\underline{u}}$ associated with $\Xi(o+,1)$, i.e. $\overset{*}{\underline{u}} = \overset{*}{\underline{u}}\,\Xi(o+,1)$ and $\overset{*}{u}_1 + \ldots + \overset{*}{u}_m = 1$.

Rewriting:

(129) $$\underline{u}(\xi)\;\Xi(\xi,1) = \Psi(\xi)\;\underline{u}(\xi) ,$$

as:

(130) $$\underline{u}(\xi) = \frac{1 - \Psi(\xi)}{\xi} \quad \underline{u}(\xi) \left[\frac{I - \Xi(\xi,1)}{\xi} \right]^{-1},$$

for $\xi > o$ and letting ξ tend to o+, we obtain in the limit:

(131) $$\overset{*}{\underline{u}} = - \Psi'(o+) \cdot \overset{*}{\underline{u}} \; N ,$$

However since N has constant columns and since $\overset{*}{u}_1 + \ldots + \overset{*}{u}_m = 1$, we obtain:

(132) $$\overset{*}{u}_i = \frac{- \Psi'(o+)}{M_i}, \qquad i = 1,\ldots,m .$$

This theorem has a number of interesting implications.

Corollary 6: In the stable queue:

(133) $$- \Psi'(o+) = \left[\sum_{i=1}^{m} \frac{1}{M_i} \right]^{-1},$$

Proof:

This is immediate upon summing over i in (132).

Corollary 7:

Let:

(134) $$\lim_{\xi \to o+} \frac{- \Xi(\xi,1) \; \underline{e} + \underline{e}}{\xi} = \overset{*}{\underline{v}}, \qquad \underline{e} = \begin{pmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix},$$

then it is easy to see that $\overset{*}{v}_j$ is the mean sojourn time in state j in the process of busy cycles. Equivalently $\overset{*}{v}_j$ is the expected duration of a busy cycle starting in the phase state j.

Then:

(135) $$- \Psi'(o+) = \overset{*}{\underline{u}} \cdot \overset{*}{\underline{v}}$$

Proof:

This is immediate from theorem 2 in [17].

Corollary 8

(136) $$\sum_{j=1}^{m} \frac{v_j^*}{M_j^*} = 1 ,$$

Proof:

Substitution of (132) in (135).

This result is not surprising to those familiar with Markov renewal processes. If we consider the probability that at time t the finite Markov renewal process of busy cycles is in the state j, then theorem 7.1, p. 1254 of Pyke [19] identifies the limit of this probability as $v_j^* M_j^{-1}$. So the latter are the limiting probabilities of m mutually exclusive events, rendering (136) obvious.

Corollary 9

The relationship between the means v_j^* and expected durations of busy periods is the following:

Let:

(137) $$-\xi^{-1} \left[\gamma(\xi,1)\, \underline{e} - \underline{e} \right] \rightarrow \underline{v}^{o} ,$$

as $\xi \rightarrow o+$, then v_j^o is the expected duration of a busy period which starts with the phase process in the state j at its start.

From (134) and (77) we obtain:

(138) $$\underline{v}^* = \lim_{\xi \to o+} \xi^{-1} \left[I - \Xi(\xi,1) \right] \underline{e} =$$

$$\lim_{\xi \to o+} \xi^{-1} \left[I - T(0,\xi)\, \Lambda\, \gamma(\xi,1) \right] \underline{e} =$$

$$\lim_{\xi \to 0+} \xi^{-1} \left\{ \left[I - T(o,\xi)\, \Lambda \right] \underline{e} + T(o,\xi)\, \Lambda . \left[I - \gamma(\xi,1) \right] \underline{e} \right\} =$$

$$\lim_{\xi \to 0+} \xi^{-1} \left[I - T(o,\xi)\Lambda \right] \underline{e} + T(o,o+)\Lambda \underline{v}^{o} ,$$

It is elementary to show that the first limit exists and is a vector $\underline{t}^{*}$, where t_j^{*} is the expected duration of an idle period starting in the phase state j.

Likewise $\left[T(o,o+)\Lambda \right]_{ij}$ is easily seen to be the probability that an idle period which starts in the phase state i ends in the phase state j.

The final result:

$$\underline{v}^{*} = \underline{t}^{*} + T(o,o)\Lambda \underline{v}^{o} , \tag{139}$$

is therefore also highly intuitive.

<u>Corollary 10</u>

The matrix $\theta^{*}(z)$ of (126) is a matrix with constant columns and each row is therefore given in particular by $\underline{u}^{*}\theta^{*}(z)$. However:

$$\underline{u}^{*}\theta^{*}(z) = \left(M_1^{-1}, \ldots, M_m^{-1} \right) \cdot \left[(1-z)\Lambda + \Delta_o - \Delta_o P \right] \left[zI - A(z,o) \right]^{-1} \cdot A(z,o) \left[(1-z)\Lambda + \Delta_o - \Delta_o P \right]^{-1} (z-1) , \tag{140}$$

In particular:

$$\underline{u}^{*} \cdot \theta^{*}(o) = \left(M_1^{-1}, \ldots, M_m^{-1} \right) , \tag{141}$$

<u>Proof</u>:

Immediate from the definition of the matrix N. The particular choice of $\underline{u}^{*}$ is immaterial. We see that $\frac{1}{M_j}$ is the limiting value as t tends to infinity of the probability that at time t the server is idle and the phase state is j. $j = 1, \ldots, m$. It follows also, using (133) that $\left[-\Psi'(o+) \right]^{-1}$ is the limit of the probability that the server is <u>idle</u> at time t.

8. The Case of Two Phases

The matrix formalism, though highly useful in obtaining the theoretical results given above, tends to obscure the rather formidable manipulations involved in obtaining the timedependent solutions in actual cases. While the author would strongly recommend the development of numerical solutions from the recurrence relations rather than from transform solutions, more explicit results in the simpler case of two phases retain some interest. In particular we wish to make comparisons with the results obtained by Naor and Yechiali for a closely related but not identical two-phase model.

If m = 2, i.e. there are only two phases 1 and 2, then the matrices of interest particularize as follows:

$$(142) \qquad P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \Delta_o = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix},$$

and hence:

$$\Lambda - \Lambda z + \Delta_o - \Delta_o P =$$

$$\begin{pmatrix} \lambda_1 + \sigma_1 - \lambda_1 z & -\sigma_1 \\ -\sigma_2 & \lambda_2 + \sigma_2 - \lambda_2 z \end{pmatrix}$$

It simplifies matters if we use the spectral decomposition of the latter matrix. Its eigenvalues are given by the roots of:

$$(143) \qquad \left[\lambda_1 + \sigma_1 - \lambda_1 z - \eta \right] \left[\lambda_2 + \sigma_2 - \lambda_2 z - \eta \right] - \sigma_1\sigma_2 = 0 ,$$

or:

$$\eta^2 - \eta \left[(\lambda_1 + \lambda_2)(1-z) + \sigma_1 + \sigma_2 \right] + \left[(\lambda_1+\sigma_1-\lambda_1 z)(\lambda_2+\sigma_2-\lambda_2 z) - \sigma_1\sigma_2 \right] = 0 ,$$

The discriminant of the latter equation is:

$$(144)\quad \left[(\lambda_1-\lambda_2)(1-z) + \sigma_1 - \sigma_2 \right]^2 + 4\sigma_1\sigma_2 = \nabla(z) ,$$

and is real for $-1 \leq z \leq +1$ and strictly positive there except in the trivial case $\lambda_1 = \lambda_2$, $\sigma_1 = \sigma_2 = 0$, which we exclude henceforth from consideration. The discriminant vanishes only at the points: $1 + (\lambda_1-\lambda_2)^{-1} \left(\sqrt{\sigma_1} \pm i\sqrt{\sigma_2}\right)^2$.

This implies that the matrix $\Lambda - \Lambda z + \Delta_o - \Delta_o P$ has two distinct eigenvalues in the region $|z| < R$, where R is the smaller of the moduli of the two points where the discriminant vanishes.

The two eigenvalues may be written as:

$$(145)\quad \eta_1(z) = \frac{1}{2}\left\{ (\lambda_1+\lambda_2)(1-z)+\sigma_1+\sigma_2 + \left\{\left[(\lambda_1-\lambda_2)(1-z)+\sigma_1-\sigma_2\right]^2 + 4\sigma_1\sigma_2\right\}^{\frac{1}{2}} \right\} ,$$

and:

$$\eta_2(z) = \frac{1}{2}\left\{ (\lambda_1+\lambda_2)(1-z)+\sigma_1+\sigma_2 - \left\{\left[(\lambda_1-\lambda_2)(1-z)+\sigma_1-\sigma_2\right]^2 + 4\sigma_1\sigma_2\right\}^{\frac{1}{2}} \right\} ,$$

Calculating a pair of corresponding <u>right</u> eigenvectors of the form $\begin{pmatrix}1\\x_1\end{pmatrix}$ and $\begin{pmatrix}1\\x_2\end{pmatrix}$ we obtain:

$$(146)\qquad x_1 = \frac{1}{\sigma_1}\left\{ \frac{(\lambda_1-\lambda_2)(1-z) + \sigma_1 - \sigma_2}{2} - \frac{1}{2}\nabla^{\frac{1}{2}}(z) \right\} ,$$

corresponding to $\eta_1(z)$ and:

$$x_2 = \frac{1}{\sigma_1}\left\{ \frac{(\lambda_1-\lambda_2)(1-z) + \sigma_1 - \sigma_2}{2} + \frac{1}{2}\nabla^{\frac{1}{2}}(z) \right\} ,$$

corresponding to $\eta_2(z)$. The spectral representation of the matrix $\Lambda - \Lambda z + \Delta_o - \Delta_o P$ is now:

$$(147)\qquad \begin{pmatrix} 1 & 1 \\ x_1 & x_2 \end{pmatrix} \begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix} \begin{pmatrix} x_2 & -1 \\ -x_1 & +1 \end{pmatrix} \frac{1}{x_2 - x_1} ,$$

Using formula (112) we obtain:

(148) P (z,t) =

$$\begin{pmatrix} 1 & 1 \\ x_1 & x_2 \end{pmatrix} \begin{pmatrix} e^{-\eta_1 t} & 0 \\ 0 & e^{-\eta_2 t} \end{pmatrix} \begin{pmatrix} x_2 & -1 \\ -x_1 & +1 \end{pmatrix} \frac{1}{x_2 - x_1},$$

where η_1, η_2, x_1, x_2 are given by (145) and (146). Noting that $x_2 - x_1 = \sigma_1^{-1} \nabla^{\frac{1}{2}}(z)$ and performing the matrix products in (148), we obtain:

(149) $\sigma_1^{-1} \nabla^{\frac{1}{2}}(z)$. P (z,t) =

$$\begin{pmatrix} x_2 e^{-\eta_1 t} - x_1 e^{-\eta_2 t} & - e^{-\eta_1 t} + e^{-\eta_2 t} \\ x_1 x_2 \left(e^{-\eta_1 t} - e^{-\eta_2 t}\right) & x_2 e^{-\eta_2 t} - x_1 e^{-\eta_1 t} \end{pmatrix},$$

This immediately yields an expression for the matrix A (z,ξ):

(150) $\sigma_1^{-1} \nabla^{\frac{1}{2}}(z)$. A (z,ξ) =

$$\begin{pmatrix} x_2 h_1 (\xi+\eta_1) - x_1 h_1 (\xi+\eta_2) & h_1 (\xi+\eta_2) - h_1 (\xi+\eta_1) \\ x_1 x_2 \left[h_2 (\xi+\eta_1) - h_2 (\xi+\eta_2)\right] & x_2 h_2 (\xi+\eta_2) - x_1 h_2 (\xi+\eta_1) \end{pmatrix}$$

The matrix A (1,o +) is given by:

(151) A (1,o) =

$$\frac{\sigma_1}{\sigma_1+\sigma_2}\begin{pmatrix} \frac{\sigma_2}{\sigma_1} + h_1(\sigma_1+\sigma_2) & 1 - h_1(\sigma_1+\sigma_2) \\ \frac{\sigma_2}{\sigma_1} - \frac{\sigma_2}{\sigma_1} h_2(\sigma_1+\sigma_2) & 1 + \frac{\sigma_2}{\sigma_1} h_2(\sigma_1+\sigma_2) \end{pmatrix}$$

where:

$$(152) \qquad h_1(\sigma_1+\sigma_2) = \int_0^\infty e^{-(\sigma_1+\sigma_2)t}\, d H_1(t) ,$$

$$h_2(\sigma_1+\sigma_2) = \int_0^\infty e^{-(\sigma_1+\sigma_2)t}\, d H_2(t) ,$$

For purposes of comparison with the Naor-Yechiali model we note that if $H_\nu(t) = 1 - e^{-\mu_\nu t}$, $\nu = 1,2$, then:

$$(153) \qquad h_1(\sigma_1+\sigma_2) = \mu_1 (\mu_1 + \sigma_1 + \sigma_2)^{-1} ,$$

$$h_2(\sigma_1+\sigma_2) = \mu_2 (\mu_2 + \sigma_1 + \sigma_2)^{-1} ,$$

We note in (151) that A (1,o) is stochastic as it should be.

The stationary probabilities associated with A (1,o) are

$$(154) \qquad \pi_1 = \frac{\sigma_2 - \sigma_2 h_2(\sigma_1+\sigma_2)}{\sigma_1 + \sigma_2 - \sigma_1 h_1(\sigma_1+\sigma_2) - \sigma_2 h_2(\sigma_1+\sigma_2)} ,$$

$$\pi_2 = \frac{\sigma_1 - \sigma_1 h_1(\sigma_1+\sigma_2)}{\sigma_1 + \sigma_2 - \sigma_1 h_1(\sigma_1+\sigma_2) - \sigma_2 h_2(\sigma_1+\sigma_2)}$$

As is to be expected these do not depend on λ_1 and λ_2.

Next, we calculate the functions $\tilde{K}_1(t)$ and $\tilde{K}_2(t)$ as defined in (24).

(155) $$\begin{pmatrix} \tilde{\kappa}_1 (t) \\ \\ \tilde{\kappa}_2 (t) \end{pmatrix} = \lim_{z \to 1-} \cdot \frac{d}{dz} \left\{ P(z,t) \ \underline{e} \right\} =$$

$$\sigma_1 \lim_{z \to 1-} \frac{d}{dz} \cdot \left\{ \nabla^{-\frac{1}{2}} (z) \cdot \begin{pmatrix} (x_2-1) e^{-\eta_1 t} - (x_1-1) e^{-\eta_2 t} \\ \\ (x_1 x_2 - x_1) e^{-\eta_1 t} - (x_1 x_2 - x_2) e^{-\eta_2 t} \end{pmatrix} \right\}$$

We have:

(156) $$\nabla (1-) = (\sigma_1 + \sigma_2)^2, \quad \eta_1 (1-) = \sigma_1 + \sigma_2, \quad \eta_2 (1-) = 0 ,$$

$$x_1 (1-) = - \frac{\sigma_2}{\sigma_1}, \quad x_2 (1-) = 1, \ \nabla' (1-) = -2 (\lambda_1 - \lambda_2)(\sigma_1 - \sigma_2) ,$$

$$x_1' (1-) = - \frac{\sigma_2}{\sigma_1} \frac{\lambda_1 - \lambda_2}{\sigma_1 + \sigma_2} , \qquad x_2' (1-) = - \frac{\lambda_1 - \lambda_2}{\sigma_1 + \sigma_2}$$

$$\eta_1' (1-) = - \frac{1}{\sigma_1 + \sigma_2} (\lambda_1 \sigma_1 + \lambda_2 \sigma_2) , \quad \eta_2' (1-) = - \frac{\lambda_1 \sigma_2 + \lambda_2 \sigma_1}{\sigma_1 + \sigma_2} ,$$

and:

(157) $$\tilde{\kappa}_1 (t) = \sigma_1 \lim_{z \to 1-} \left\{ - \tfrac{1}{2} \nabla^{-\frac{3}{2}} (z) \nabla' (z) \left[(x_2-1) e^{-\eta_1 t} - (x_1-1) e^{-\eta_2 t} \right] \right.$$

$$+ \nabla^{-\frac{1}{2}} (z) \left[x_2' e^{-\eta_1 t} - x_1' e^{-\eta_2 t} \right] + \nabla^{-\frac{1}{2}} (z) \left[- (x_2-1) e^{-\eta_1 t} \eta_1' t \right.$$

$$\left. \left. + (x_1-1) e^{-\eta_2 t} \eta_2' t \right] \right\} =$$

$$\frac{\lambda_1 \sigma_2 + \lambda_2 \sigma_1}{\sigma_1 + \sigma_2} t + \sigma_1 \frac{\lambda_1 - \lambda_2}{(\sigma_1 + \sigma_2)^2} \left[1 - e^{-(\sigma_1 + \sigma_2) t} \right] , \qquad t \geq 0.$$

Similarly:

$$(158)\quad \tilde{\kappa}_2(t) = \frac{\lambda_1\sigma_2 + \lambda_2\sigma_1}{\sigma_1+\sigma_2}\, t - \sigma_2 \frac{\lambda_1 - \lambda_2}{(\sigma_1+\sigma_2)^2}\left[1 - e^{-(\sigma_1+\sigma_2)t}\right],$$

We note that:

$$(159)\quad \tilde{\kappa}_1(t) = \tilde{\kappa}_2(t) = \lambda t, \text{ for } \lambda_1 = \lambda_2 = \lambda .$$

and in general.

$$(160)\quad \frac{\sigma_2\tilde{\kappa}_1(t) + \sigma_1\tilde{\kappa}_2(t)}{\sigma_1+\sigma_2} = \frac{\lambda_1\sigma_2 + \lambda_2\sigma_1}{\sigma_1+\sigma_2}\, t ,$$

The last result is interesting since $\sigma_2(\sigma_1+\sigma_2)^{-1}$ and $\sigma_1(\sigma_1+\sigma_2)^{-1}$ are the stationary probabilities of being in the state 1, respectively 2 in the continuous time Markov chain.

Let α_1 and α_2 be the means of $H_1(\cdot)$ and $H_2(\cdot)$ then using (154) and (158), we obtain that ρ^* defined in (23) is given by:

$$(161)\quad \rho^* = \frac{\lambda_1\sigma_2 + \lambda_2\sigma_1}{\sigma_1+\sigma_2} \cdot \left\{ \frac{[\sigma_2 - \sigma_2 h_2(\sigma_1+\sigma_2)]\alpha_1 + [\sigma_1 - \sigma_1 h_1(\sigma_1+\sigma_2)]\alpha_2}{\sigma_1 - \sigma_1 h_1(\sigma_1+\sigma_2) + \sigma_2 - \sigma_2 h_2(\sigma_1+\sigma_2)} \right.$$

We see that the first factor is an average of the arrival rates λ_1 and λ_2 and the second one an average of the mean service times α_1 and α_2. The interesting part is of course to discover just what the correct averages are.

In the case of negative exponential service times the equilibrium condition is given by:

$$(162)\quad \rho^* = \frac{\lambda_1\sigma_2 + \lambda_2\sigma_1}{\sigma_1+\sigma_2} \cdot \left\{ \frac{\sigma_2}{\mu_2+\sigma_1+\sigma_2}\,\frac{1}{\mu_1} + \frac{\sigma_1}{\mu_1+\sigma_1+\sigma_2}\cdot\frac{1}{\mu_2} \right\} \cdot$$

$$\left\{ \frac{\sigma_2}{\mu_2+\sigma_1+\sigma_2} + \frac{\sigma_1}{\mu_1+\sigma_1+\sigma_2} \right\}^{-1} < 1,$$

This should be compared to the equilibrium condition:

$$(163)\quad \tilde{\rho} = \frac{\lambda_1 \sigma_2 + \lambda_2 \sigma_1}{\mu_1 \sigma_2 + \mu_2 \sigma_1} < 1$$

of the Naor-Yechiali model. We recall that the difference between this model and the two-phase case with negative exponential services treated here is the following:

In our case the service time distribution of a customer depends only on the phase state at the beginning of his service, whereas in the Naor-Yechiali model the rate of service of an individual customer may fluctuate with the changes in phase.

Comparing ρ^* and $\tilde{\rho}$ we obtain:

$$(164)\quad \rho^* = \tilde{\rho} \cdot \left(1 + \frac{\sigma_1}{\mu_2} + \frac{\sigma_2}{\mu_1}\right) \left[1 + \frac{(\sigma_1 + \sigma_2)^2}{\mu_1 \sigma_2 + \mu_2 \sigma_1}\right]^{-1}$$

and we note that if μ_1 and μ_2 are both large compared to σ_1 and σ_2 then ρ^* and $\tilde{\rho}$ are very nearly equal. This is intuitive as it says that the durations of the phases are long when compared to individual services, so that services straddling one or more phase changes will be infrequent. It is still noteworthy that the ratio of ρ^* and $\tilde{\rho}$ does not depend on λ_1 and λ_2.

9. Remarks on Computation

The remarks on numerical computation which come to mind are very similar to those made in Section VI of [16]. We shall therefore not repeat but refer the reader there. It is clear that in the study of more complicated queues there is little hope of developing anything like the elegant combinatorial procedures of Takács [22] to yield explicit expressions for the quantities associated with $M|G|1$ and $GI|M|1$. On the other hand the transform methods as used in this and many other papers are mainly of theoretical importance. If the convolution algebra on the Lebesgue-Stieltjes measures on the Borel subsets of R were as familiar to most as "ordinary" products between complex functions, there would be little or no need for transform methods in Applied Probability.

For an excellent example of such a transform-free approach, the reader may consult the basic paper [10].

Recent papers in Queueing theory have reiterated the "Laplacian curtain." complaint and with a great deal of justification. However, the complaint is justified only when a complicated transform formula is passed off as a "practical" solution. When it becomes necessary to organize the computational work involved in a study of the transient behavior of a queue, the structural properties, such as the recognition of one or more imbedded Markov Renewal processes and the recurrence relations induced by them become far more important than "explicit" formulae in terms of multiple transforms.

Recurrence relations, falling as they do under the general qualitative description of "initial value problems" are ideally suited to the iterative abilities of digital computers [1].

As we have repeatedly indicated in the theoretical discussion, it suffices to analyze the paths of a queueing process with Poisson input during the intial busy period. All the queue features at an arbitrary instant of time t can be expressed in terms of these by making judicious use of the regenerative nature of the queueing process at the ends of busy periods.

This approach was used to date and to this author's knowledge only in the work of U.N. Bhat and Sahin [2] who calculated rather extensive tables for the timedependence of $M|D|1$, $M|E_k|1$, $D|M|1$ and $E_k|M|1$. While this author can only applaud this contribution to practical timedependence-studies of queues, he would personally favor the construction of a library of computer routines for the transient analysis of more general queues such as $E_k|G|1$, $GI|E_k|1$, $M|SM|1$, $SM|M|1$, certain bulk service queues and in view of its potential applicability in Traffic analysis also the present model. In view of the large number of parameters and arbitrary probability distributions involved in these models, tables and graphs are bound to be either too limited in range or too unwieldy for practical purposes.

A final word concerns the relevance of timedependent discussions of queueing models. In unstable queues they are of course the only possible ones. However, even in a stable queue of some complexity one often needs to know the time-dependent properties of some of its simpler components.

Notably such queues as those with alternating priorities or the tandem queues, which have yielded to analysis of their transient behavior fall in this category. One decomposes the multivariate processes associated with these in terms of simpler events associated with $M|G|1$ or similar elementary models. Even to express the limiting distributions of the complex model one needs to know "timedependent" distributions of simpler processes. A concrete instance of this occurs in the present paper. The equilibrium condition parameter ρ^* involves the timedependent functions $\tilde{\kappa}_i(t)$ of the arrival process. If m is larger than about three or four we need to calculate these functions numerically and perform a numerical integration in (23) to get an approximate value of ρ^*.

Acknowledgement

The author expresses sincere appreciation to the Department of Operations Research at Cornell University, where most of this work was done, for its hospitality during his sabbatical leave.

A special word of thanks goes to Dr. Stella Dafermos and Prof. P. Naor with whom he discussed this work extensively.

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Unclassified

Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Purdue University	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

A Queue subject to Extraneous Phase Changes.

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*
Technical Report, September 1969

5. AUTHOR(S) *(Last name, first name, initial)*

Neuts, Marcel F.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
September 1969	66	24

8a. CONTRACT OR GRANT NO. NONR 1100 (26) b. PROJECT NO. c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) Mimeograph Series No. 207 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*

10. AVAILABILITY/LIMITATION NOTICES

Distribution of this document is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Office of Naval Research Washington, D.C.

13. ABSTRACT

Many service systems exhibit variations of a random nature in the intensity of the arrival process or of the speed of service or of both. Changes in work shifts, rush hours, interruptions in the arrival process, server breakdowns, etc. all fall into this category.

The present study deals with a generalization of the classical M|G|1 queue by considering an extraneous process of phases which can be in one of the states $\{1,\dots,m\}$. During any interval spent in phase i, the arrivals are according to a homogeneous Poisson process of rate λ_i and any service initiated during such interval has a duration distributed according to $H_i(.)$. The process of phases is assumed to be an irreducible, Markov chain in continuous time and is fully characterized by its initial conditions, by an irreducible stochastic matrix P and by the mean sojourn times $\sigma_1^{-1},\dots,\sigma_m^{-1}$ in each phase.

Independently of the queueing aspects, this arrival process is a generalization of the classical Poisson process which can be of interest in modelling simple point processes with randomly fluctuating "arrival" rate.

Two approaches to the time dependent study of this queue are presented; one generalizes the imbedded semi-Markov process obtained by considering the queue immediately following departure points; the other approach exploits the relationship between this queue and branching processes. The latter is more eloquent from a purely theoretical viewpoint and involves iterates of a general type of matrix

CONTINUED

DD FORM 1 JAN 64 1473

Unclassified

Security Classification

Unclassified

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
M|G|1 queue Timedependent behavior Extraneous phase changes Markov Renewal processes Branching processes Busy period Queuelength Virtual Waitingtime Traffic analysis						

13. Abstract (continued)

function introduced by the author. By making extensive use of the Perron-Frobenius theory of positive matrices the equilibrium condition of the queue is obtained. While retaining a similar intuitive interpretation the equilibrium condition is substantially more complicated than for the M|G|1 model.

The recurrence relations which yield the joint distribution of the phase state at time t, the queuelength, the total number served and the virtual waitingtime at t are exhibited in detail. Via transform techniques a number of limiting and marginal distributions are discussed. The discussion relies heavily on the theory of Markov Renewal processes.

Throughout the paper and in a final section the author advocates the use of the structural properties of the queue and the resulting recurrence relations to organize the numerical analysis of complex queueing models such as the present one.

More explicit results for the case of two phases are given and are compared to results obtained by Yechiali and Naor for a closely related two-phase generalization of the M|M|1 queue.