An Approximation to the Distribution of the Largest Root

of a Matrix in Complex Multivariate Analysis*

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1. Introduction

Let $\underset{\sim}{X}(q\text{x}r)$ $(r \geq q)$ be a complex valued random matrix whose columns are independent and have the q-variate complex normal distribution $N_c(\underset{\sim}{M},\underset{\sim}{\Sigma})$ (Wooding [10], Goodman [3]). The distribution of $\underset{\sim}{X}\,\underset{\sim}{\bar{X}}'$ is then complex Wishart $W_c(q,r,\underset{\sim}{\Sigma})$ (Goodman [3], Khatri [7]). If $\underset{\sim}{M}=\underset{\sim}{0}$ and $\underset{\sim}{\Sigma}=\underset{\sim}{I}_q$ (the qxq identity matrix), the distribution of $0 \leq f_1 \leq f_2 \leq \cdots \leq f_q < \infty$, the characteristic roots of $\underset{\sim}{X}\,\underset{\sim}{\bar{X}}'$, is given by (Khatri [7], James [4]):

$$C_1\left(\prod_{j=1}^{q} f_j^m\right) \exp\left(-\sum_{j=1}^{q} f_j\right) \prod_{j>k} (f_j - f_k)^2, \tag{1.1}$$

where

$$C_1 = 1/\left(\prod_{j=1}^{q} \Gamma(m+j)\,\Gamma(j)\right) \quad \text{and} \quad m=r-q. \tag{1.2}$$

This distributional form also arises in another manner. Suppose that $X(q\text{x}r)$ is distributed $N_c(\underset{\sim}{0},\underset{\sim}{I}_q)$ and that $\underset{\sim}{S}(q\text{x}q)$ is independent of $\underset{\sim}{X}$ with distribution $W_c(q,s,\underset{\sim}{I}_q)$. Then the distribution of the characteristic roots

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of $[\underset{\sim}{X}\,\underset{\sim}{\bar{X}}'(\underset{\sim}{S}+\underset{\sim}{X}\,\underset{\sim}{\bar{X}}')^{-1}]$, say $0 \le w_1 \le w_2 \le \dots \le w_q \le 1$, has the form:

(1.3) $$C_2 \prod_{j=1}^{q} [w_j^m (1-w_j)^n] \prod_{j>k} (w_j - w_k)^2,$$

where

(1.4) $$C_2 = \prod_{j=1}^{q} \Gamma(m+n+q+j)/(\Gamma(m+j)\Gamma(n+j)\Gamma(j)),$$

$m=r-q$ and $n=s-q$.

By making the transformation $f_j = nw_j$, $j=1,\dots,q$, and allowing $n\to\infty$, the distribution of $0 \le f_1 \le f_2 \le \dots \le f_q < \infty$ is that given by (1.1) (Khatri [7]).

Because of the similarity of handling the classical problem of point estimation and hypothesis testing for normal populations in the complex case with that in the real case, the largest (or smallest) characteristic root has been proposed as a test criterion by Khatri [7], [8].

The distribution of the largest characteristic root $(f_q$ or $w_q)$ has been given by Khatri [6] as follows:

(1.5) $$P\{f_q \le x;\ m\} = C_1 |(\gamma_{i+j-2})| ,$$

where C_1 is defined in (1.2),

(1.6) $$\gamma_{i+j-2} = \int_0^x z^{m+i+j-2} \quad e^{-z}\, dz, \quad i,j=1,\dots,q,$$

and (γ_{i+j-2}) is a $q\text{x}q$ matrix; and

(1.7) $$P\{w_q \le x;\ m\} = C_2 |(\beta_{i+j-2})|,$$

where C_2 is defined in (1.4),

(1.8) $$\beta_{i+j-2} = \int_0^x w^{m+i+j-2}(1-w)^n dw, \quad i,j=1,\ldots,q,$$

and (β_{i+j-2}) is a $q\times q$ matrix.

Jouris [5], using an approach due to Pillai [9], has suggested an approximation to the distribution (1.7) and has obtained various percentage points. The purpose here is to suggest an approximation to the distribution of f_q (1.5) using a similar approach and to tabulate upper tail percentage points using this approximation.

2. Approximation to the C.D.F. of f_q.

By using integration by parts for integral values of m, (1.6) can be written as:

(2.1) $$\gamma_k = \int_0^x z^{m+k} e^{-z} dz = (m+k)! - T_k$$

where

$$T_k = (m+k)!\, e^{-x} \sum_{j=0}^{m+k} x^j/j! \quad .$$

By definition

(2.2) $$|(\gamma_{i+j-2})| = \sum_{\underset{\sim}{j}} \text{sign } (\underset{\sim}{j}) \prod_{k=1}^{q} (\gamma_{k+j_k-2}),$$

where $\sum_{\underset{\sim}{j}}$ denotes the summation over the permutation $\underset{\sim}{j} = (j_1, j_2, \ldots, j_q)$ of $(1,2,\ldots,q)$. Using the expansion of γ_k in (2.1) and neglecting terms of the type $T_\ell T_k$ (that is, all terms involving e^{-bx} for $b \geq 2$) we find that :

$$\gamma_{j_1-1}\gamma_{j_2} \doteq (m+j_1-1)!\, \gamma_{j_2} + (m+j_2)!\, \gamma_{j_1-1} - (m+j_1-1)!\, (m+j_2)!,$$

$$\gamma_{j_1-1}\gamma_{j_2}\gamma_{j_3+1} \doteq (j_1-1)!j_2!\gamma_{j_3+1} + (j_1-1)!(j_3+1)!\gamma_{j_2} + j_2!(j_3+1)!\gamma_{j_1-1} - 2(j_1-1)!j_2!(j_3+1)!$$

and in general

$$\text{(2.3)} \qquad \prod_{k=1}^{q} \gamma_{k+j_k-2} \doteq \sum_{\alpha=1}^{q} \Big(\prod_{\substack{k=1 \\ k\neq\alpha}}^{q} (m+k+j_k-2)! \Big)\, \gamma_{\alpha+j_\alpha-2} - (q-1) \prod_{k=1}^{q} (m+k+j_k-2)!\,.$$

Upon using (2.3) in the definition of $|(\gamma_{i+j-2})|$ given in (2.2), we can approximate (1.5) by

$$C_1|(\gamma_{i+j-2})| \doteq C_1 \sum_{\underset{\sim}{j}} \sum_{\alpha=1}^{q} \operatorname{sign}(\underset{\sim}{j}) \Big(\prod_{\substack{k=1 \\ k\neq\alpha}}^{q} (m+k+j_k-2)!\Big)\, \gamma_{\alpha+j_\alpha-2} - C_1(q-1)|((m+i+j-2)!)|$$

$$= C_1 \sum_{k=0}^{2q-2} G'_k \gamma_k - (q-1),$$

since $|((m+i+j-2)!)| = C_1^{-1}$ and where G'_k is the sum of the cofactors of $(m+k)!$ in the $q \times q$ matrix

$$G = \begin{bmatrix} m! & (m+1)! & \cdots & (m+q-1)! \\ (m+1)! & (m+2)! & \cdots & (m+q)! \\ \vdots & & & \\ \vdots & & & \\ (m+q-1)! & (m+q)! & \cdots & (m+2q-2)! \end{bmatrix} .$$

Thus for $q \geq 2$ we have

(2.4) $$P\{f_q \leq x;\ m\} \doteq c_1 \sum_{k=0}^{2q-2} G_k' \gamma_k - (q-1).$$

Explicit simplified expressions for (2.4) when $q=2,3,4$ and 5 are given below in (2.5), (2.6), (2.7) and (2.8) respectively.

(2.5) $$P\{f_2 \leq x;m\} \doteq \frac{1}{(m+1)!} [(m+1)_2\gamma_0 - 2(m+1)\gamma_1 + \gamma_2] -1,$$

where $(a)_k = a(a-1) \ldots (a-k+1)$.

(2.6) $$P\{f_3 \leq x;m\} \doteq \frac{1}{2(m+2)!} [(m+2)(m+1)_3\gamma_0 - 4(m+1)_3\gamma_1 + 6(m+1)_2\gamma_2 - 4(m+2)\gamma_3 + \gamma_4] -2.$$

(2.7) $$P\{f_4 \leq x;m\} \doteq \frac{1}{6(m+3)!} [(m+1)_2(m+2)_2(m+3)_2\gamma_0 - 6(m+1)(m+2)_2(m+3)_2 \gamma_1$$

$$+ 15(m+2)(m+3)_2(m+9/5)\gamma_2 - 20(m+2)(m+3)(m+17/5) \gamma_3$$

$$+ 15(m+3)(m+14/5) \gamma_4 - 6(m+3) \gamma_5 + \gamma_6] - 3.$$

$$(2.8)\quad P\{f_5 \le x;m\} \doteq \frac{1}{24(m+4)!}[(m+1)_2(m+2)_2(m+3)_2(m+4)_2\,\gamma_0$$

$$- 8(m+1)(m+2)_2(m+3)_2(m+4)_2\gamma_1 + 28(m+2)(m+3)_2(m+4)_2(m+12/7)\gamma_2$$

$$- 56(m+2)(m+3)(m+4)_2(m+22/7)\gamma_3 + 70(m+3)(m+4)(m^2+51/7m+86/7)\gamma_4$$

$$- 56(m+3)(m+4)(m+29/7)\gamma_5 + 28(m+4)(m+26/7)\gamma_6 - 8(m+4)\gamma_7 + \gamma_8] - 4.$$

The approximation is thus a linear combination of incomplete gamma functions and is simpler than the exact C.D.F. which involves products of q incomplete gamma functions.

3. Computation of Percentage Points

By using the approximation obtained in the previous section, upper 10%, 5%, 2.5%, 1% and .5% points were obtained for the C.D.F. of f_q for $q=2,3,\ldots,11$. The computations were carried out on the CDC 6500 computer at the Purdue University Computer Sciences Center using double precision arithmetic. The percentage points are given to five significant digits for $m=0(1)20(2)30(5)50(10)100$.

Some exact percentage points were also tabulated for comparisons with the approximate ones. Table 1 below displays some representative values of both the exact and approximate percentage points. As can be seen from this table, the approximate and exact percentage points usually agree through five significant digits. This same degree of accuracy has been found in the approximation suggested by Jouris [5] to the distribution (1.3).

Table 1

Comparison of the Approximate and Exact Percentage Points for the C.D.F. of the Largest Root f_q.

		1%		5%	
q	m	Approximate	Exact	Approximate	Exact
2	15	32.6968	32.6968	28.9562	28.9561
3	30	58.6083	58.6083	53.8994	53.8992
4	60	103.5274	103.5274	97.5795	97.5791
5	100	160.1230	160.1230	153.0122	153.0118
6	20	59.6795	59.6795	55.2224	55.2221
7	10	48.2632	48.2632	44.2295	44.2293
8	70	139.8957	139.8956	133.5577	133.5572
9	30	88.6527	88.8526	83.5302	83.5298
10	5	52.0095	52.0094	47.9146	47.9143
11	18	78.7225	78.7205	73.9153	73.9145

4. Applications

The complex multivariate normal and related distributions have been found useful in such areas as physics and time series analysis. Under certain basic assumptions Bronk [2] has found that the distribution (1.1) is that of the energy levels of atomic nuclei. Goodman [3] has noted several applications of complex multivariate theory to time series analysis. Brillinger [1] has shown that the asymptotic distributions of the matrix of second-order periodograms and the matrix of spectral densities of a strictly stationary time series are complex Wishart (the distribution of whose characteristic roots is given by (1.1)). It has been noted in Section 1 that many hypothesis testing problems in the complex case can be handled as in the real case. It is hoped that the findings here will be useful in the areas mentioned above as well as in other fields.

Upper α Points of the Largest Root

m \ α	q=2 .10	q=2 .05	q=2 .025	q=2 .01	q=2 .005	q=3 .10	q=3 .05	q=3 .025	q=3 .01	q=3 .005
0	5.912	6.900	7.839	9.031	9.904	9.649	10.829	11.931	13.308	14.305
1	7.664	8.753	9.778	11.068	12.007	11.465	12.722	13.890	15.324	16.391
2	9.287	10.461	11.156	12.933	13.928	13.177	14.502	15.729	17.250	18.344
3	10.830	12.081	13.243	14.691	15.736	14.817	16.204	17.485	19.067	20.203
4	12.319	13.637	14.858	16.374	17.464	16.403	17.848	19.178	20.818	21.992
5	13.766	15.147	16.422	18.000	19.133	17.947	19.445	20.822	22.515	23.727
6	15.181	16.620	17.946	19.582	20.755	19.456	21.005	22.425	24.170	25.416
7	16.570	18.064	19.438	21.129	22.339	20.937	22.534	23.996	25.788	27.067
8	17.937	19.483	20.902	22.645	23.891	22.394	24.036	25.538	27.376	28.687
9	19.286	20.881	22.343	24.136	25.415	23.830	25.515	27.055	28.938	30.278
10	20.619	22.261	23.764	25.605	26.916	25.248	26.975	28.551	30.477	31.846
11	21.938	23.626	25.168	27.054	28.397	26.649	28.417	30.028	31.995	33.392
12	23.244	24.976	26.556	28.486	29.859	28.036	29.843	31.488	33.494	34.919
13	24.540	26.313	27.930	29.903	31.305	29.411	31.255	32.933	34.978	36.428
14	25.826	27.640	29.292	31.306	32.736	30.773	32.654	34.364	36.446	37.922
15	27.102	28.956	30.642	32.697	34.154	32.125	34.041	35.782	37.900	39.401
16	28.371	30.263	31.983	34.076	35.560	33.467	35.418	37.189	39.342	40.867
17	29.323	31.562	33.314	35.445	36.954	34.800	36.785	38.585	40.773	42.321
18	30.886	32.852	34.636	36.804	38.338	36.124	38.142	39.972	42.193	43.764
19	32.134	34.136	35.950	38.154	39.713	37.441	39.491	41.349	43.603	45.196
20	33.376	35.412	37.257	39.496	41.079	38.751	40.832	42.717	45.003	46.619
22	35.844	37.947	39.850	42.157	43.787	41.351	43.493	45.431	47.779	49.438
24	38.292	40.460	42.419	44.791	46.465	43.926	46.127	48.116	50.525	52.224
26	40.724	42.953	44.966	47.401	49.118	46.481	48.738	50.777	53.243	54.912
28	43.140	45.428	47.494	49.990	51.748	49.016	51.328	53.415	55.937	57.714
30	45.542	47.888	50.004	52.559	54.357	51.535	53.899	56.032	58.608	60.422
35	51.495	53.978	56.214	58.908	60.802	57.765	60.256	62.498	65.203	67.106
40	57.385	59.997	62.343	65.168	67.152	63.917	66.524	68.870	71.696	73.681
45	63.224	65.956	68.408	71.356	73.423	70.002	72.721	75.164	78.104	80.168
50	69.018	71.865	74.416	77.481	79.628	76.032	78.857	81.392	84.441	86.578
60	80.499	83.559	86.297	89.580	91.876	87.955	90.977	93.685	96.936	99.213
70	91.865	95.121	98.032	101.52	103.95	99.730	102.93	105.80	109.24	111.64
80	103.14	106.58	109.65	113.32	115.88	111.39	114.76	117.78	121.39	123.92
90	114.34	117.95	121.17	125.02	127.70	122.94	126.48	129.64	133.42	136.05
100	125.47	129.24	132.61	136.62	139.42	134.42	138.11	141.40	145.33	148.08

Table 2 Continued

m	α: q=4 .10	q=4 .05	q=4 .025	q=4 .01	q=4 .005	q=5 .10	q=5 .05	q=5 .025	q=5 .01	q=5 .005
0	13.441	14.768	15.997	17.520	18.615	17.265	18.715	20.050	21.694	22.871
1	15.293	16.683	17.967	19.552	20.689	19.139	20.644	22.025	23.724	24.938
2	17.058	18.506	19.839	21.482	22.659	20.940	22.494	23.920	25.669	26.918
3	18.758	20.260	21.640	23.337	24.550	22.682	24.284	25.751	27.548	28.830
4	20.408	21.960	23.383	25.131	26.378	24.378	26.025	27.531	29.374	30.686
5	22.017	23.616	25.081	26.876	28.156	26.035	27.725	29.267	31.153	32.495
6	23.592	25.236	26.740	28.581	29.892	27.659	29.389	30.968	32.895	34.265
7	25.138	26.825	28.366	30.250	31.591	29.254	31.024	32.636	34.603	36.001
8	26.659	28.387	29.964	31.890	33.259	30.824	32.632	34.277	36.282	37.706
9	28.158	29.926	31.537	33.503	34.890	32.373	34.216	35.893	37.936	39.384
10	29.638	31.444	33.088	35.093	36.516	33.901	35.780	37.487	39.566	41.039
11	31.101	32.944	34.620	36.662	38.111	35.412	37.325	39.062	41.175	42.672
12	32.548	34.427	36.134	38.213	39.686	36.907	38.853	40.619	42.766	44.286
13	33.982	35.894	37.632	39.746	41.244	38.388	40.365	42.159	44.339	45.882
14	35.402	37.349	39.116	41.264	42.785	39.855	41.864	43.685	45.897	47.462
15	36.811	38.790	40.586	42.767	44.312	41.309	43.349	45.197	47.440	49.026
16	38.209	40.220	42.044	44.258	45.824	42.752	44.822	46.696	48.970	50.577
17	39.597	41.639	43.490	45.736	47.325	44.185	46.284	48.183	50.487	52.115
18	40.976	43.048	44.926	47.203	48.813	45.608	47.735	49.660	51.992	53.640
19	42.346	44.448	46.352	48.660	50.290	47.022	49.176	51.126	53.487	55.154
20	43.708	45.840	47.768	50.106	51.757	48.426	50.609	52.582	54.971	56.658
22	46.410	48.598	50.576	52.972	54.663	51.212	53.448	55.468	57.912	59.636
24	49.085	51.328	53.354	55.806	57.535	53.970	56.256	58.321	60.818	62.579
26	51.736	54.031	56.104	58.610	60.376	56.701	59.037	61.145	63.694	65.489
28	54.366	56.712	58.829	61.387	63.189	59.409	61.793	63.943	66.541	68.371
30	56.976	59.371	61.532	64.141	65.978	62.095	64.526	66.718	69.364	71.226
35	63.426	65.939	68.203	70.933	72.853	68.729	71.271	73.561	76.322	78.263
40	69.785	72.409	74.769	77.613	79.611	75.263	77.910	80.292	83.161	85.177
45	76.069	78.797	81.248	84.200	86.272	81.713	84.460	86.928	89.900	90.987
50	82.288	85.115	87.654	90.708	92.851	88.092	90.933	93.485	96.555	98.708
60	94.566	97.579	100.28	103.53	105.80	100.67	103.69	106.40	109.65	111.93
70	106.67	109.86	112.71	116.13	118.53	113.06	116.24	119.10	122.52	124.91
80	118.64	121.98	124.98	128.57	131.08	125.29	128.63	131.62	135.20	137.70
90	130.49	133.99	137.12	140.86	143.48	137.40	140.88	143.99	147.72	150.34
100	142.25	145.89	149.14	153.04	155.76	149.39	153.01	156.25	160.12	162.83

Table 2 Continued

	q=6					q=7				
m \ α	.10	.05	.025	.01	.005	.10	.05	.025	.01	.005
0	21.111	22.667	24.093	25.843	27.093	24.973	26.623	28.130	29.975	31.289
1	23.001	24.605	26.073	27.871	29.154	26.875	28.568	30.113	32.002	33.345
2	24.827	26.476	27.983	29.827	31.141	28.720	30.454	32.635	33.965	35.337
3	26.601	28.292	29.837	31.724	33.067	30.518	32.291	33.906	35.876	37.275
4	28.331	30.064	31.643	33.572	34.944	32.275	34.086	35.734	37.742	39.167
5	30.025	31.796	33.410	35.379	36.777	33.998	35.845	37.524	39.569	41.019
6	31.686	33.496	35.142	37.149	38.573	35.691	37.572	39.282	41.362	42.837
7	33.321	35.166	36.844	38.888	40.337	37.356	39.271	41.010	43.125	44.623
8	34.930	36.810	38.519	40.598	42.072	39.998	40.946	42.713	44.861	46.382
9	36.518	38.432	40.170	42.283	43.781	40.619	42.598	44.393	46.573	48.116
10	38.086	40.033	41.799	43.946	45.467	42.220	44.229	46.051	48.263	49.828
11	39.637	41.615	43.409	45.589	47.131	43.803	45.843	47.691	49.933	51.519
12	41.171	43.180	45.001	47.212	48.777	45.370	47.439	49.312	51.585	53.191
13	42.690	44.729	46.577	48.819	50.404	46.922	49.019	50.917	53.219	54.845
14	44.195	46.264	48.137	50.410	52.016	48.460	50.585	52.508	54.838	56.484
15	45.688	47.785	49.684	51.986	53.612	49.985	52.137	54.084	56.442	58.107
16	47.169	49.294	51.218	53.548	55.195	51.499	53.677	55.647	58.033	59.716
17	48.639	50.792	52.739	55.098	56.764	53.001	55.206	57.198	59.611	61.313
18	50.099	52.279	54.249	56.636	58.321	54.493	56.723	58.738	61.177	62.897
19	51.549	53.755	55.749	58.163	59.866	55.974	58.230	60.267	62.737	64.469
20	52.991	55.222	57.239	59.680	61.401	57.447	59.727	61.786	64.276	66.031
22	55.848	58.130	60.191	62.683	64.441	60.366	62.694	64.795	67.334	69.124
24	58.675	60.006	63.093	65.652	67.444	63.254	65.628	67.770	70.357	72.180
26	61.474	63.852	65.997	68.589	70.414	66.113	68.532	70.714	73.348	75.202
28	64.249	66.673	68.858	71.497	73.354	68.946	71.410	73.629	76.309	78.194
30	67.001	69.470	71.694	74.378	76.267	71.756	74.262	76.519	79.242	81.158
35	73.794	76.368	78.686	81.480	83.445	78.688	81.296	83.643	86.472	88.460
40	80.479	83.154	85.560	88.458	90.494	85.508	88.212	90.643	93.572	95.629
45	87.075	89.845	92.334	95.330	97.434	92.232	95.028	97.540	100.56	102.68
50	93.594	96.454	99.023	102.11	104.28	98.875	101.76	104.35	107.46	109.64
60	106.44	109.47	112.19	115.12	117.74	111.96	115.00	117.74	121.02	123.32
70	119.07	122.26	125.12	128.55	130.95	124.81	128.01	130.88	134.32	136.74
80	131.54	134.88	137.86	141.45	143.96	137.49	140.83	143.83	147.42	149.93
90	143.87	147.34	150.45	154.18	156.79	150.02	153.50	156.61	160.34	162.95
100	156.07	159.68	162.91	166.78	169.48	162.42	166.02	169.25	173.11	175.82

Table 2 Continued

m	q=8 .10	q=8 .05	q=8 .025	q=8 .01	q=8 .005	q=9 .10	q=9 .05	q=9 .025	q=9 .01	q=9 .005
0	28.847	30.582	32.163	34.094	35.466	32.731	34.544	36.192	38.202	39.628
1	30.758	32.532	34.148	36.119	37.518	34.650	36.499	38.197	40.226	41.677
2	32.619	34.430	36.079	38.088	39.514	36.522	38.406	40.117	42.199	43.675
3	34.435	36.283	37.963	40.009	41.460	38.355	40.272	42.013	44.129	45.628
4	36.215	38.097	39.808	41.884	43.365	40.152	42.102	43.871	46.021	47.543
5	37.961	39.878	41.618	43.733	45.232	41.919	43.900	45.697	47.879	49.423
6	39.679	41.628	43.396	45.546	47.067	43.658	45.670	47.493	49.707	51.272
7	41.371	43.352	45.148	47.329	48.872	45.372	47.414	49.263	51.507	53.094
8	43.040	45.051	46.874	49.087	50.652	47.064	49.135	51.009	53.283	54.890
9	44.688	46.729	48.578	50.821	52.407	48.736	50.834	52.734	55.036	56.663
10	46.317	48.387	50.261	52.534	54.140	50.388	52.515	54.438	56.769	58.415
11	47.928	50.026	51.925	54.227	55.853	52.024	54.177	56.124	58.482	60.148
12	49.523	51.649	53.572	55.902	57.548	53.644	55.823	57.793	60.178	61.862
13	51.104	53.256	55.203	57.561	59.226	55.248	57.453	59.446	61.858	63.560
14	52.670	54.849	56.819	59.204	60.887	56.839	59.069	61.084	63.522	65.242
15	54.224	56.428	58.420	60.832	62.534	58.418	60.672	62.709	65.172	66.909
16	55.765	57.995	60.009	62.447	64.167	59.984	62.262	64.320	66.809	68.563
17	57.295	59.550	61.586	64.050	65.787	61.538	63.841	65.919	68.432	70.204
18	58.815	61.094	63.151	65.640	67.395	63.082	65.408	67.507	70.044	71.832
19	60.324	62.627	64.706	67.219	68.991	64.616	66.965	69.084	71.645	73.449
20	61.824	64.150	66.250	68.788	70.576	66.141	68.512	70.651	73.235	75.055
22	64.798	67.170	69.310	71.895	73.716	69.163	71.578	73.756	76.386	78.237
24	67.739	70.156	72.335	74.966	76.819	72.152	74.610	76.826	79.500	81.382
26	70.651	73.111	75.328	78.004	79.887	75.111	77.611	79.863	82.580	84.492
28	73.536	76.038	78.292	81.102	82.925	78.043	80.584	82.871	85.630	87.571
30	76.397	78.940	81.230	83.992	85.935	80.950	83.530	85.852	88.653	90.621
35	83.454	86.095	88.472	91.335	93.348	88.120	90.795	93.200	96.099	98.135
40	90.394	93.128	95.586	98.546	100.63	95.169	97.933	100.42	103.41	105.51
45	97.234	100.06	102.59	105.64	107.79	102.11	104.96	107.52	110.61	112.77
50	103.99	106.90	109.51	112.64	114.85	108.97	119.90	114.54	117.70	119.92
60	117.28	120.35	123.10	126.40	128.72	122.44	125.55	128.32	131.64	133.97
70	130.34	133.56	136.44	139.90	142.32	135.68	138.93	141.83	145.31	147.74
80	143.21	146.56	149.57	153.17	155.69	148.77	152.12	155.13	158.75	161.28
90	155.92	159.41	162.53	166.27	168.89	161.64	165.12	168.24	172.01	174.62
100	168.49	172.09	175.32	179.19	181.91	174.38	177.98	181.21	185.09	187.80

Table 2 Continued

m	α: q=10 .10	q=10 .05	q=10 .025	q=10 .01	q=10 .005	q=11 .10	q=11 .05	q=11 .025	q=11 .01	q=11 .005
0	36.624	38.507	40.219	42.301	43.777	40.523	42.473	44.243	46.394	47.916
1	38.548	40.466	42.207	44.324	45.824	42.452	44.435	46.233	48.416	49.960
2	40.431	42.381	44.151	46.301	47.824	44.343	46.357	48.181	50.396	51.962
3	42.276	44.258	46.055	48.238	49.783	46.200	48.243	50.093	52.339	53.926
4	44.089	46.101	47.925	50.140	51.706	48.025	50.097	51.973	54.248	55.856
5	45.872	47.915	49.765	52.009	53.596	49.823	51.923	53.824	56.128	57.756
6	47.629	49.701	51.576	53.851	55.458	51.596	53.724	55.648	57.981	59.628
7	49.363	51.462	53.362	55.666	57.293	53.346	55.501	57.449	59.808	61.474
8	51.075	53.202	55.126	57.457	59.104	55.075	57.256	59.227	61.614	63.298
9	52.767	54.920	56.868	59.227	60.892	56.785	58.991	60.985	63.398	65.100
10	54.440	56.620	58.591	60.977	62.661	58.477	60.708	62.724	65.163	66.883
11	56.097	58.302	60.296	62.708	64.410	60.153	62.408	64.445	66.909	68.647
12	57.738	59.969	61.984	64.422	66.142	61.813	64.092	66.151	68.639	70.394
13	59.365	61.620	63.656	66.120	67.857	63.458	65.762	67.841	70.354	72.125
14	60.977	63.257	65.314	67.803	69.557	65.091	67.417	69.516	72.053	73.841
15	62.577	64.880	66.959	69.472	71.243	66.710	69.059	71.178	73.739	75.542
16	64.165	66.491	68.590	71.127	72.915	68.318	70.689	72.827	75.410	77.231
17	65.742	68.091	70.210	72.770	74.574	69.914	72.308	74.466	77.073	78.907
18	67.308	69.679	71.818	74.402	76.221	71.500	73.915	76.093	78.722	80.571
19	68.864	71.257	73.415	76.022	77.857	73.075	75.511	77.707	80.358	82.224
20	70.410	72.825	75.002	77.637	79.482	74.641	77.097	79.311	81.986	83.867
22	73.475	75.933	78.147	80.820	82.701	77.747	80.245	82.500	85.210	87.121
24	76.508	79.006	81.257	83.973	85.885	80.818	83.355	85.641	88.398	90.336
26	79.510	82.048	84.335	87.092	89.032	83.859	86.336	88.756	91.552	93.519
28	82.484	85.062	87.382	90.179	92.149	86.872	89.487	91.840	94.676	96.669
30	85.433	88.050	90.405	93.242	95.246	89.859	92.510	94.897	97.771	99.791
35	92.701	95.414	97.848	100.78	102.84	97.226	99.966	102.45	105.40	107.48
40	99.853	102.65	105.16	108.18	110.31	104.46	107.29	109.83	112.88	115.03
45	106.87	109.77	112.36	115.47	117.65	111.60	114.50	117.11	120.25	122.46
50	113.85	116.80	119.46	122.65	124.90	118.63	121.62	124.30	127.52	129.77
60	127.52	130.63	133.42	136.76	139.11	132.47	135.60	138.41	141.78	144.15
70	140.94	144.18	147.10	150.59	153.03	146.05	149.32	152.24	155.76	158.22
80	154.14	157.52	160.55	164.19	166.72	159.41	162.80	165.85	169.49	172.05
90	167.18	170.68	173.82	177.58	180.21	172.59	176.10	179.26	183.03	185.68
100	180.06	183.68	186.93	190.82	193.54	185.61	189.25	192.50	196.40	199.12

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