Maxima Of A Sequence Of Random Variables Defined On A Markov Chain

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CHAPTER I

INTRODUCTION

We discuss various problems related to the maxima of a sequence of random variables defined on a Markov chain (M.C.), which are conditionally independent given the chain.

Let $\{J_n, X_n, n \geq 0\}$ be a two-dimensional stochastic process such that

$$X_0 = -\infty \quad \text{a.s.},$$

$$P[J_0 = k] = p_k, \quad k = 1, \ldots, m; \quad \sum_{k=1}^{m} p_k = 1,$$

and

$$\begin{aligned} P\{J_n = j, X_n \leq x \mid X_0, J_0, X_1, J_1, \ldots, X_{n-1}, J_{n-1} = i\} \\ = P\{J_n = j, X_n \leq x \mid J_{n-1} = i\} \\ = p_{ij} H_i(x) = Q_{ij}(x) \end{aligned}$$

for $i, j = 1, \ldots, m$. The distributions $H_i(x)$, $i = 1, \ldots, m$ are nondegenerate and honest $(H_i(+\infty) = 1)$.

Immediate consequences:

(i) The marginal sequence $\{J_n, n \geq 0\}$ is an m-state M.C. with $P\{J_n = j \mid J_{n-1} = i\} = p_{ij}$. The transition matrix $\underset{\sim}{P} = \{p_{ij}\}$ $i, j = 1, \ldots, m$ is assumed to be stochastic, irreducible and aperiodic. The stationary probabilities associated with $\underset{\sim}{P}$ are $(\pi_1, \ldots, \pi_m)$; $\underset{\sim}{P}^n \to \underset{\sim}{\Pi}$ where $\Pi_{ij} = \pi_j$.

(ii) $P\{X_n \le x | J_{n-1} = i\} = H_i(x)$

(iii) $P\{X_1 \le x_1, \ldots, X_n \le x_n | J_0, J_1, \ldots, J_{n-1}\}$

$$= \prod_{i=1}^{n} P\{X_i \le x_i | J_{i-1}\} .$$

The random variables $\{X_n\}$ are conditionally independent given the chain in precisely the sense given by (iii).

Remarks: (1) We adopt the convention $X_0 = -\infty$ instead of the more usual $X_0 = 0$ because we deal with maxima rather than sums.

(2) There is no loss of generality in allowing the distribution of X_n to depend on J_{n-1} only, rather than J_n and J_{n-1} - Pyke [12, p. 1751] . The case where the distribution of X_n depends on the pair (J_{n-1}, J_n) can be reduced to this case.

(3) The theory of semi-Markov processes employs the same formulation except that the random variables X_n are required to be non-negative.

Let $M_n = \max\{X_1, \ldots, X_n\}$ and $\underset{\sim}{Q}(x) = \{p_{ij}H_i(x)\}$. The distribution of M_n is obtained from:

$$P\{J_n = j, M_n \le x | J_0 = i\} = Q_{ij}^n(x) , \tag{1.1}$$

where $\underset{\sim}{Q}^n(x) = \{Q_{ij}^n(x)\}$ is the nth power of the Q-matrix.

(Here we are not concerned with matrix-convolution powers.)

To prove (1.1) we use the conditional independence of $\{X_n\}$:

$$P\{J_n = j,\ M_n \le x | J_0 = i\} =$$

$$\sum_{v=1}^{n-1} \sum_{j_v=1}^{m} P\{J_n = j,\ J_1 = j_1, \ldots, J_{n-1} = j_{n-1},\ M_n \le x \mid J_o = i\}$$

$$= \sum_{v=1}^{n-1} \sum_{j_v=1}^{m} P\{M_n \le x | J_0 = i,\ J_1 = j_1, \ldots, J_{n-1} = j_{n-1},\ J_n = j\}$$

$$\cdot P\{J_n = j,\ J_1 = j_1, \ldots, J_{n-1} = j_{n-1} | J_0 = i\}$$

$$= \sum_{v=1}^{n-1} \sum_{j_v=1}^{m} H_i(x)\, H_{j_1}(x) \ldots H_{j_{n-1}}(x)\, p_{ij_1} p_{j_1 j_2} \ldots p_{j_{n-1} j}$$

$$= \sum_{v=1}^{n-1} \sum_{j_v=1}^{m} Q_{ij_1}(x)\, Q_{j_1 j_2}(x) \ldots Q_{j_{n-1} j}(x)$$

$$= Q^n_{ij}(x) .$$

There are several classes of questions concerning maxima of a sequence of random variables:

i) <u>Limit Laws for M_n</u> . When do normalizing constants $a_n > 0$, b_n, $n \ge 1$, exist such that:

$$P[a_n^{-1}(M_n - b_n) \le x] \overset{c}{\to} \Phi(x)$$

as $n \to \infty$ with $\Phi(x)$ a nondegenerate distribution? What is the class of possible nondegenerate limit laws $\Phi(x)$? What are necessary and sufficient conditions for convergence to a particular member of this class? What are the properties of the normalizing constants a_n, b_n ?

For iid random variables $\{X_n, n \geq 1\}$, these questions were exhaustively answered by B. V. Gnedenko [5]. The possible limit laws are the so-called extreme value distributions. Precisely, if $\{X_n, n \geq 1\}$ is a sequence of i.i.d. random variables with distribution function $F(\cdot)$, and there exist normalizing constants $a_n > 0$, b_n such that

$$P[a_n^{-1}(M_n - b_n) \leq x] = F^n(a_n x + b_n) \overset{c}{\rightarrow} \Phi(x)$$

where $\Phi(x)$ is a nondegenerate distributinn, then $\Phi(x)$ belongs to the type of one of the following distributions:

$$\Lambda(x) = \exp\{-e^{-x}\} \qquad -\infty < x < \infty \tag{1.2}$$

$$\Phi_\alpha(x) = \begin{cases} 0 & x < 0 \\ \exp\{-x^{-\alpha}\} & x \geq 0 \end{cases} \tag{1.3}$$

$$\Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^{\alpha}\} & x < 0 \\ 1 & x \geq 0 \end{cases} \tag{1.4}$$

α is a positive constant. Gnedenko also gave complete domain of attraction criteria and a specification of the norming constants.

We extend Gnedenko's results to the case that $\{X_n, n \geq 1\}$ is a sequence of random variables defined on a M.C. and conditionally independent given the chain. The possible limit laws are again precisely the types of the extreme value distributions. We give criteria for convergence to each type and a specification of the norming constants. We also concern ourselves with the existence of normalizing constants $a_{ijn} > 0$, b_{ijn}, $i, j = 1, \ldots, m$, $n \geq 1$ such that the expressions

$$P\{J_n = j,\ a_{ijn}^{-1}(M_n - b_{ijn}) \leq x \mid J_0 = i\} = Q_{ij}^n(a_{ijn}\, x + b_{ijn})$$

converge to nondegenerate mass functions $U_{ij}(x)$, at all continuity points of the latter and such that $\sum_{j=1}^{m} U_{ij}(x)$, $i = 1, \ldots, m$ is an honest distribution function. We specify the possible limit matrices $\{U_{ij}(\cdot)\}$ and give basic properties of the normalizing constants a_{ijn}, b_{ijn}.

ii) <u>"When" problems:</u> X_n is a record value of the sequence $\{X_k, k \geq 1\}$ if $X_n > \max\{X_1, \ldots, X_{n-1}\}$. Asking when the largest of the values $X_1, \ldots, X_n$ occurred - or when was M_n achieved - is equivalent to asking when records occurred. Let V_k be the index of the kth record and $\Delta_k = V_k - V_{k-1}$, the kth inter-record time. In the i.i.d. case basic properties of $\{V_k\}$ [13] and $\{\Delta_k\}$ [6,7,11,14] were established such as the calculation of distributions and moments. Also Renyi [13] proved $\frac{\log V_k}{k} \to 1$ a.s.,

$$\lim_{k \to \infty} P\left[\frac{\log V_k - k}{\sqrt{k}} \leq t\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{1}{2}x^2}\, dx\,,$$

and an iterated logarithm theorem for $\{V_k\}$. Identical results have been proved for $\{\Delta_k\}$ [6,7,11].

iii) "Where" problems. These questions do not even make sense for i.i.d. random variables X_n. Suppose $\{X_n, n \geq 1\}$ is a sequence of random variables defined on a M.C., conditionally independent given the chain. We inquire where the maximum M_n was achieved. In what state I_n was the M.C. when M_n was realized? Then $I_n = j$ iff $J_{k-1} = j$ and $X_k = M_n$ for some $k = 1, \ldots, n$. We give necessary and sufficient conditions for the weak limits $\lim_{n\to\infty} P[I_n = j]$ to exist and to have value $\ell_j \geq 0$. Existence of weak limits is not a class property as an example shows. How often is the maximum achieved on state j ? State J is maximum-transient or maximum-recurrent according as $P([I_n = j]\text{i.o.}) = 0$ or 1. It turns out that a state must be either maximum-transient or maximum-recurrent. We give necessary and sufficient conditions for a state to be one or the other.

The behavior of sums of random variables is often determined by moments or truncated moments. For maxima the asymptotic behavior of M_n depends on the amount of probability contained in the tails of the distributions. For example, if two distribution functions $F(\cdot)$ and $G(\cdot)$ are tail equivalent $(1 - F(x) \sim 1 - G(x)$ as $x \to \infty)$, then for $\Phi(x)$ an extreme value distribution:

$$F^n(a_n x + b_n) \to \Phi(x)$$

$$\text{iff} \qquad G^n(a_n x + b_n) \to \Phi(x) \quad .$$

The behavior of the maxima of random variables defined on a M.C. depends also on the amounts of time the M.C. spends in each state after reaching equilibrium as reflected by the stationary probabilities $\pi_1, \ldots, \pi_m$.

One expects the maxima of the $\underset{\sim}{Q}$-system to have the same asymptotic properties as the maxima of the $\hat{\underset{\sim}{Q}}$-system, $\hat{\underset{\sim}{Q}}(x) = \{\pi_j H_i(x)\}$. If comparison theorems asserting the identical asymptotic behavior of the two systems can be proven, one need only consider the $\hat{\underset{\sim}{Q}}$-system - a relatively easy task in view of the simple spectral structure of the $\hat{\underset{\sim}{Q}}$-matrix.

CHAPTER II

SEMI-MARKOV MATRICES

A semi-Markov matrix (S.M.M.) $\underset{\sim}{Q}(x) = \{Q_{ij}(x)\}$ is a matrix whose entries $Q_{ij}(x)$, $i, j = 1, \ldots, m$ are mass functions such that $\sum_{j=1}^{m} Q_{ij}(+\infty) \le 1$. A S.M.M. is honest if for all $i = 1, \ldots, m$ equality holds, otherwise it is dishonest. Unless otherwise specified all distribution functions and S.M.M.'s are honest.

Let $\{{}_n\underset{\sim}{Q}(x)\}$ be a sequence of S.M.M.'s. The sequence of S.M.M.'s converges completely to a limit matrix $\underset{\sim}{Q}(x)$ iff $\underset{\sim}{Q}(x)$ is honest and for each i, j ${}_nQ_{ij}(\cdot) \overset{w}{\to} Q_{ij}(\cdot)$. We write ${}_n\underset{\sim}{Q}(\cdot) \overset{c}{\to} \underset{\sim}{Q}(\cdot)$:

A matrix analogue of the classical weak compactness theorem for distribution functions holds for S.M.M.s: Given a sequence of S.M.M.'s ${}_n\underset{\sim}{Q}(x)$, there exists a subsequence n_k and a limit S.M.M. $\underset{\sim}{Q}(x)$ (not necessarily honest) such that ${}_{n_k}\underset{\sim}{Q}(\cdot) \overset{w}{\to} \underset{\sim}{Q}(\cdot)$; that is, for $i, j = 1, \ldots, m$, ${}_{n_k}Q_{ij}(\cdot) \overset{w}{\to} Q_{ij}(\cdot)$.

Two S.M.M.'s $\underset{\sim}{U}(x)$, $\underset{\sim}{V}(x)$ are of the same type if there exist constants $A > 0$ and B such that for each i, j $V_{ij}(x) = U_{ij}(A\,x + B)$. The following lemma of Khintchin is useful [4, p. 246] :

Lemma 2.1. Let $U(\cdot)$ and $V(\cdot)$ be two non-degenerate distribution functions. If for a sequence $\{F_n(\cdot)\}$ of distribution functions and constants $a_n > 0$, b_n and $\alpha_n > 0$, β_n :

$$(2.2) \qquad F_n(a_n x + b_n) \xrightarrow{c} U(x) \ , \quad F_n(\alpha_n x + \beta_n) \xrightarrow{c} V(x) \ .$$

Then:

$$(2.3) \qquad \frac{\alpha_n}{a_n} \to A \neq 0 \ , \quad \frac{\beta_n - b_n}{a_n} \to B$$

and then

$$(2.4) \qquad V(x) = U(Ax + B) \ .$$

Conversely if (2.3) holds, then each of the two relations (2.2) implies the other and (2.4).

The set of normalizing constants $a_n > 0$, b_n, $n \geq 1$ is <u>asymptotically equivalent</u> to the set of normalizing constants $\alpha_n > 0$, β_n, $n \geq 1$ iff

$$\frac{\alpha_n}{a_n} \to 1 \ , \quad \frac{\beta_n - b_n}{a_n} \to 0 \ .$$

A S.M.M. $\underset{\sim}{Q}(x)$ is a non-negative matrix for every x; hence the Perron-Frobenius theory is applicable. For a matrix $\underset{\sim}{A}$ with real entries, we write $\underset{\sim}{A} \geq \underset{\sim}{0} (> \underset{\sim}{0})$ if $a_{ij} \geq 0$ $(a_{ij} > 0)$ for each i, j. For a complex matrix $\underset{\sim}{B} = \{b_{ij}\}$, $|\underset{\sim}{B}|$ denotes the matrix $\{|b_{ij}|\}$.

A square matrix $\underset{\sim}{B} = \{b_{ij}\}$, $i, j = 1, 2, \ldots, m$, is <u>reducible</u> if the index set $1, 2, \ldots, m$ can be split into two disjoint complementary sets $i_1, i_2, \ldots, i_\mu$; $k_1, k_2, \ldots, k_\nu$ $(\mu + \nu = m)$ such that $b_{i_\alpha k_\beta} = 0$ $(\alpha = 1, 2, \ldots, \mu; \beta = 1, 2, \ldots, \nu)$.

Otherwise $\underset{\sim}{B}$ is <u>irreducible</u>.

We use the following theorem [15, p. 30]:

<u>Theorem 2.5</u> Let $\underset{\sim}{A} \geq \underset{\sim}{0}$ be an irreducible $m \times m$ matrix. Then:

1. $\underset{\sim}{A}$ has a simple positive eigenvalue equal to its spectral radius $\rho_{\underset{\sim}{A}}$.
2. To the eigenvalue $\rho_{\underset{\sim}{A}}$ corresponds a positive eigenvector $\underset{\sim}{X} > 0$.
3. $\rho_{\underset{\sim}{A}}$ increases when any entry of $\underset{\sim}{A}$ increases. (If $\underset{\sim}{A}$ is reducible, then $\rho_{\underset{\sim}{A}}$ does not decrease when any entry of $\underset{\sim}{A}$ increases.)

If $\underset{\sim}{A}$ is an $m \times m$ non-negative irreducible matrix, let k be the number of eigenvalues of $\underset{\sim}{A}$ of modulus ρ_A . If $k = 1$, then $\underset{\sim}{A}$ is <u>primitive</u>. If $k > 1$, then $\underset{\sim}{A}$ is <u>cyclic of index k</u> .

A stochastic matrix is irreducible iff its associated M.C. is irreducible. It is primitive iff the M.C. is aperiodic.

<u>Theorem 2.6</u> [15, pp. 28, 47]: Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be two $m \times m$ matrices with $\underset{\sim}{0} \leq |\underset{\sim}{B}| \leq \underset{\sim}{A}$. Then $\rho_{\underset{\sim}{B}} \leq \rho_{\underset{\sim}{A}}$. If $\underset{\sim}{A}$ is irreducible then $\rho_{\underset{\sim}{B}} = \rho_{\underset{\sim}{A}}$ implies that $|\underset{\sim}{B}| = \underset{\sim}{A}$.

<u>Theorem 2.7</u> [15, p. 13]: If $\underset{\sim}{A}$ is an $m \times m$ complex matrix, then $\underset{\sim}{A}^n \to \underset{\sim}{0}$ entrywise iff $\rho_{\underset{\sim}{A}} < 1$.

For fixed x , $\underset{\sim}{Q}(x)$ is a positive matrix whose spectral radius we denote by $\rho(x)$. <u>$\rho(x)$ is a distribution function.</u>

$\underset{\sim}{Q}(+\infty)$ is stochastic; hence $\rho(+\infty) = 1$. $\underset{\sim}{Q}(-\infty) = \underset{\sim}{0}$; hence $\rho(-\infty) = 0$. $\rho(x)$ is nondecreasing by Theorem (2.5-3). Furthermore:

<u>Lemma 2.8</u> (1) If $\underset{\sim}{Q}(x)$ is (right, left) continuous at x_0 , then $\rho(x)$ is (right, left) continuous at x_0 .

(2) If $\rho(x)$ is right continuous at x_0 and $\underset{\sim}{Q}(x_0)$ is irreducible, then $\underset{\sim}{Q}(x)$ is right continuous at x_0 . If $\rho(x)$ is left continuous at x_0 and $\underset{\sim}{Q}(x)$ is irreducible for $x > x_0 - \epsilon$ for some $\epsilon > 0$, then $\underset{\sim}{Q}(x)$ is left continuous at x_0.

<u>Proof</u>: (1) If $Q(x)$ is left continuous at x_0 , select a sequence $x_n \uparrow x_0$. Then $\underset{\sim}{Q}(x_n) \uparrow \underset{\sim}{Q}(x_0)$ and hence $\rho(x_n) \uparrow \rho(x)$. Hence $\rho(x)$ is left continuous at x_0 . Similarly for right continuity.

(2) Suppose $\rho(x)$ is left continuous at x_0 . Choose a sequence $\{x_n\}$ such that $x_0 - \epsilon < x_n \uparrow x_0$. Then $\underset{\sim}{Q}(x_n) \to \underset{\sim}{Q}(x_0-) \leq \underset{\sim}{Q}(x_0)$. If there exists (i,j) such that $Q_{ij}(x_0-) < Q_{ij}(x_0)$ then $\rho(x_0-) < \rho(x_0)$ by Theorem (2.5-3), contradicting the left continuity of $\rho(x)$ at x_0 . Similarly for right continuity.

<u>Lemma 2.9</u> Let $\{{}_n\underset{\sim}{Q}(\cdot)\}$ be a sequence of S.M.M.'s and ${}_n\underset{\sim}{Q}(\cdot) \overset{c}{\to} \underset{\sim}{Q}(\cdot)$. Then $\rho_n(\cdot) \overset{c}{\to} \rho(\cdot)$ where $\rho(x)$ and $\rho_n(x)$ are the spectral radii of $\underset{\sim}{Q}(x)$ and ${}_n\underset{\sim}{Q}(x)$ respectively.

<u>Proof</u>: Weak convergence of distribution functions is equivalent to pointwise convergence on a set everywhere dense

on the real line, so ${}_n\underset{\sim}{Q}(\cdot) \overset{c}{\to} \underset{\sim}{Q}(\cdot)$ implies that for $x \in D$, ${}_n\underset{\sim}{Q}(x) \to \underset{\sim}{Q}(x)$; D is an everywhere dense subset of R. Hence for $x \in D$, $\rho_n(x) \to \rho(x)$ and hence $\rho_n(\cdot) \overset{w}{\to} \rho(\cdot)$. But $\underset{\sim}{Q}(\cdot)$ is honest, so $\underset{\sim}{Q}(+\infty)$ is stochastic. Thus $\rho(+\infty) = 1$ and $\rho_n(\cdot) \overset{c}{\to} \rho(\cdot)$.

We can say more about the spectral properties of a S.M.M. $\underset{\sim}{Q}(x)$. Suppose there exists $x_0 < \infty$ such that for $x > x_0$ $\underset{\sim}{Q}(x)$ is irreducible. Now let $\underset{\sim}{r}(x) = (r_1(x), \ldots, r_m(x))$, $\underset{\sim}{\ell}(x) = (\ell_1(x), \ldots, \ell_m(x))$ be right and left eigenvectors of $\underset{\sim}{Q}(x)$ corresponding to $\rho(x)$. The components of $\underset{\sim}{r}(x)$ and $\underset{\sim}{\ell}(x)$ can be chosen to be non-negative and for $x > x_0$ all components are then strictly positive (2.5-2). As functions of x, $\underset{\sim}{r}(x)$ and $\underset{\sim}{\ell}(x)$ are only determined up to arbitrary factors, since for any scalar functions $k_1(x)$ and $k_2(x)$, $k_1(x)\, \underset{\sim}{r}(x)$ and $k_2(x)\, \underset{\sim}{\ell}(x)$ are also eigenvectors. In order to discuss continuity properties and limiting behavior of $\underset{\sim}{r}(x)$ and $\underset{\sim}{\ell}(x)$ we must specify a version of the eigenvectors.

<u>Lemma 2.10</u> Let $\underset{\sim}{Q}(x)$, $\underset{\sim}{r}(x)$, $\underset{\sim}{\ell}(x)$ be as above. Restrict attention to the domain $x > x_0$ where $\underset{\sim}{Q}(x)$ is irreducible. We normalize $r(x)$ and $\underset{\sim}{\ell}(x)$ by: $\sum_{i=1}^{m} r_i(x) = \sum_{i=1}^{m} \ell_i(x) = 1$. Suppose $\underset{\sim}{P} = \underset{\sim}{Q}(+\infty)$ is primitive. We have:

(1) $\lim_{x \to \infty} \underset{\sim}{r}(x) = (m^{-1}, \ldots, m^{-1})$

$\lim_{x \to +\infty} \underset{\sim}{\ell}(x) = (\pi_1, \ldots, \pi_m)$ where $(\pi_1, \ldots, \pi_m)$ are the stationary probabilities associated with $\underset{\sim}{P}$.

(2) If $\underset{\sim}{Q}(x)$ is (right, left) continuous at $x_1 > x_0$, then $\underset{\sim}{r}(x)$ and $\underset{\sim}{\ell}(x)$ are (right, left) continuous at x_1.

Proof: (1) $\underset{\sim}{r}(x)$ is in a compact set. For any sequence $x_n \uparrow +\infty$, $\{\underset{\sim}{r}(x_n)\}$ must have a convergent subsequence, say $\{\underset{\sim}{r}(x_{n_k})\}$. Suppose $\lim_{k\to\infty} \underset{\sim}{r}(x_{n_k}) = \underset{\sim}{r} = (r_1, \ldots, r_m)$. Since $\sum_{i=1}^{m} r_i = 1$, not all components of $\underset{\sim}{r}$ can vanish. Then $\lim_{k\to\infty} \underset{\sim}{Q}(x_{n_k})\, \underset{\sim}{r}(x_{n_k}) = \lim_{k\to\infty} \rho(x_{n_k})\, \underset{\sim}{r}(x_{n_k})$, so $\underset{\sim}{P}\underset{\sim}{r} = \underset{\sim}{r}$. Since $\underset{\sim}{P}$ is stochastic and irreducible, its right eigenvector corresponding to Perron-Frobenius eigenvalue 1 is uniquely determined up to a factor and hence $r_i = m^{-1}$, $i = 1, \ldots, m$. Since every convergent subsequence of $\{\underset{\sim}{r}(x_n)\}$ converges to the same limit, $\lim_{n\to\infty} \underset{\sim}{r}(x_n) = (m^{-1}, \ldots, m^{-1})$. Similarly for $\underset{\sim}{\ell}(x)$.

(2) Suppose $\underset{\sim}{Q}(x)$ is left continuous at x_1. Pick any sequence $\{x_n\}$ such that $x_0 < x_n \uparrow x_1$. Then $\underset{\sim}{Q}(x_n) \to \underset{\sim}{Q}(x_1)$ and $\rho(x_n) \to \rho(x_1)$. By compactness, there exists a subsequence n_k and $\underset{\sim}{s} = (s_1, \ldots, s_m)$ such that $\sum_{i=1}^{m} s_i = 1$ and $\lim_{k\to\infty} r(x_{n_k}) = \underset{\sim}{s}$. Hence $\lim_{k\to\infty} \underset{\sim}{Q}(x_{n_k})\, \underset{\sim}{r}(x_{n_k}) = \lim_{k\to\infty} \rho(x_{n_k})\, \underset{\sim}{r}(x_{n_k})$, i.e. $\underset{\sim}{Q}(x_1)\, \underset{\sim}{s} = \rho(x_1)\, \underset{\sim}{s}$. But since $\underset{\sim}{Q}(x_1)$ is irreducible $\underset{\sim}{s} = \underset{\sim}{r}(x_1)$. All convergent subsequences have the same limit; hence $\lim_{n\to\infty} \underset{\sim}{r}(x_n) = \underset{\sim}{r}(x_1)$. Similarly for $\underset{\sim}{\ell}(x)$ and for right continuity.

Now let $\underset{\sim}{Q}(x) = \{p_{ij}\, H_i(x)\}$ $i,j = 1, \ldots, m$ where $\underset{\sim}{P} = \{p_{ij}\}$ is an irreducible, aperiodic, stochastic matrix and $\underset{\sim}{P}^n \to \underset{\smile}{\Pi}$ and $H_1(\cdot), \ldots, H_m(\cdot)$ are nondegenerate distribution functions.

There exists a real number x_0, such that for $x > x_0$ $\min\{H_1(x), \ldots, H_m(x)\} > 0$. We may limit ourselves to the domain $x > x_0$ where $\underset{\sim}{Q}(x)$ is irreducible.

The conditions $\sum_{i=1}^{m} \ell_i(x)\, r_i(x) = 1$ and $\sum_{i=1}^{m} \ell_i(x) = 1$ determine a version of the right and left eigenvectors possessing the continuity properties and limiting behavior discussed in Lemma 2.10. This version can be obtained from the one satisfying $\sum_{i=1}^{m} r_i(x) = \sum_{i=1}^{m} \ell_i(x) = 1$ through the transformations $r_i(x) \to \dfrac{r_i(x)}{\sum_{i=1}^{m} r_i(x)\, \ell_i(x)}$, $i = 1, \ldots, m$.

We assume henceforth that $\underset{\sim}{r}(x)$ and $\underset{\sim}{\ell}(x)$ are so normalized.

Form the matrix $\underset{\sim}{M}(x) = \{r_i(x)\, \ell_j(x)\}$, $i, j = 1, \ldots, m$. Then:

(2.11) $$\lim_{x\to\infty} \underset{\sim}{M}(x) = \underset{\sim}{\Pi}$$

(2.12) $$\underset{\sim}{M}^2(x) = \underset{\sim}{M}(x)$$

(2.13) For any vector $\underset{\sim}{V} = (V_1, \ldots, V_m)$ we have:

$$\underset{\sim}{M}(x)\, \underset{\sim}{V} = (\underset{\sim}{V}, \underset{\sim}{\ell}(x))\, \underset{\sim}{r}(x) \quad \text{and} \quad \underset{\sim}{V}\underset{\sim}{M}(x) = (\underset{\sim}{V}, \underset{\sim}{r}(x))\, \underset{\sim}{\ell}(x) .$$

(2.14) $$\underset{\sim}{Q}(x)\, \underset{\sim}{M}(x) = \underset{\sim}{M}(x)\, \underset{\sim}{Q}(x) = \rho(x)\, \underset{\sim}{M}(x)$$

(2.15) $$\lim_{n\to\infty} \frac{\underset{\sim}{Q}^n(x)}{\rho^n(x)} = \underset{\sim}{M}(x)$$

The proof of (2.11) follows from $M_{ij}(x) = r_i(x)\, \ell_j(x)$ and $\ell_i(x) \to \pi_i$, $r_i(x) \to 1$ as $x \to \infty$. The proof of (2.12) - (2.15) is given in [9, p. 248]

We examine (2.15) in detail. Set $\underset{\sim}{B}(x) = \underset{\sim}{Q}(x) - \rho(x)\, \underset{\sim}{M}(x)$. Then by (2.12) and (2.14) , we have $\underset{\sim}{B}^n(x) = \underset{\sim}{Q}^n(x) - \rho^n(x)\, \underset{\sim}{M}(x)$.

<u>Theorem 2.15</u>: Let $\underset{\sim}{Q}(x) = \{p_{ij}\, H_i(x)\}$, $\underset{\sim}{M}(x)$, $\underset{\sim}{B}(x)$ be as above. There exists a real number M such that
$\lim_{n\to\infty} \underset{\sim}{B}^n(x) = \lim_{n\to\infty} [\underset{\sim}{Q}^n(x) - \rho^n(x)\, \underset{\sim}{M}(x)] = \underset{\sim}{0}$ uniformly in $x > M$.
Equivalently:

$$(2.17) \quad \underset{\sim}{Q}^n(x) = \rho^n(x)\, \underset{\sim}{M}(x) + \underset{\sim}{o}(1) \quad \text{where} \quad \lim_{n\to\infty} \underset{\sim}{o}(1) = \underset{\sim}{0} \quad \text{uniformly in } x > M .$$

<u>Proof</u>: We can show by induction that $|\underset{\sim}{B}^n| \leq |\underset{\sim}{B}|^n$ for integral n . Let $\underset{\sim}{E}$ be the $m \times m$ matrix $E_{ij} = 1$ and $\underset{\sim}{B}(x) = \{B_{ij}(x)\}$. Fix N , a positive integer such that $\max_{i,j} |p_{ij}^N - \pi_j| < m^{-1}$. Set $\alpha = \max_{i,j} |p_{i,j}^N - \pi_j|$. Pick $\epsilon > 0$ such that $\alpha + \epsilon < m^{-1}$. Since $\lim_{x\to\infty} \underset{\sim}{B}^n(x) = \underset{\sim}{P}^n - \underset{\sim}{\Pi}$, there exists M_N such that for $x > M_N$, $|B_{ij}^N(x)| \leq \alpha + \epsilon, i,j=1,\ldots,m.$ Then $|\underset{\sim}{B}^N(x)| = \{|B_{ij}^N(x)|\} \leq (\alpha + \epsilon)\, \underset{\sim}{E} < m^{-1}\, E$. The spectral radius of $\underset{\sim}{E}$ is m so the spectral radius of $(\alpha + \epsilon)\, \underset{\sim}{E}$ is is strictly less than 1; hence $((\alpha + \epsilon)\, \underset{\sim}{E})^n \to \underset{\sim}{0}$ as $n \to \infty$ by Theorems (2.6), (2.7) . So for $x > M_N$, $|\underset{\sim}{B}^N(x)|^n \to \underset{\sim}{0}$ uniformly in x and since $|\underset{\sim}{B}^N(x)|^n \geq |\underset{\sim}{B}^{nN}(x)|$ we have that $|\underset{\sim}{B}^{nN}(x)| \xrightarrow{n\to\infty} \underset{\sim}{0}$ uniformly in $x > M_N$.

Now for any n , write

$$|\underset{\sim}{B}^n(x)| = |\underset{\sim}{B}^{[\frac{n}{N}]N}(x)\,\underset{\sim}{B}^{n-[\frac{n}{N}]}(x)| \le |\underset{\sim}{B}^{[\frac{n}{N}]N}(x)|\;|\underset{\sim}{B}^{n-[\frac{n}{N}]N}(x)| .$$

For any n, the second factor is one of the following: $|\underset{\sim}{B}^0(x)|$, $|\underset{\sim}{B}^1(x)|, \ldots, |\underset{\sim}{B}^{N-1}(x)|$. For $k = 1,2,\ldots, N-1$ there exist real numbers $M_1,\ldots, M_{N-1}$ such that $x > M_k$ implies $|\underset{\sim}{B}^k(x)| < \underset{\sim}{E}$. So for $x > M = \max\{M_1,\ldots, M_{N-1}, M_N\}$ the second factor is bounded by $\underset{\sim}{E}$; the first factor approaches $\underset{\sim}{0}$ uniformly in $x > M$. This completes the proof.

We use the following lemma [3]:

Lemma 2.18: Let $\underset{\sim}{P} = \{p_{ij}\}$ be an $m \times m$, irreducible, aperiodic, stochastic matrix such that $\lim_{n\to\infty} \underset{\sim}{P}^n = \underset{\sim}{\Pi}$. Suppose there are constants c_{ijn} with $0 \le c_{ijn} \le 1$, $n \ge 1$, $i, j = 1,2,\ldots, m$, such that $\lim_{n\to\infty} (c_{ijn})^n = \Phi_{ij}$. Then:

$$\lim_{n\to\infty} \{c_{ijn}\, p_{ij}\}^n = \Big[\prod_{i,j=1}^{m} \Phi_{ij}^{\pi_i p_{ij}}\Big]\, \underset{\sim}{\Pi}$$

where $\Phi_{ij}^{p_{ij}}$ is interpreted as 1 if $\Phi_{ij} = 0$ and $p_{ij} = 0$.

The matrix $\hat{\underset{\sim}{Q}}(x) = \{\pi_j\, H_i(x)\}$ $1 \le i,j \le m$ often arises in conjunction with the system governed by $\underset{\sim}{Q}(x) = \{p_{ij}\, H_i(x)\}$. We note some spectral properties of $\hat{\underset{\sim}{Q}}(x)$:

(2.19) $$\hat{\rho}(x) = \sum_{i=1}^{m} \pi_i\, H_i(x)$$

(2.20) $$\hat{\underset{\sim}{r}}(x) = \hat{\rho}(x)^{-1}\, (H_1(x),\ldots, H_m(x))$$

(2.21) $$\hat{\underset{\sim}{\ell}}(x) = (\pi_1,\ldots, \pi_m)$$

$$\hat{\underset{\sim}{Q}}^n(x) = \hat{\rho}^{n-1}(x)\, \hat{\underset{\sim}{Q}}(x) \quad (2.22)$$

Direct computation shows (2.22) in the form $\hat{\underset{\sim}{Q}}^n(x) = (\sum_{i=1}^{m} \pi_i H_i(x))^{n-1} \hat{\underset{\sim}{Q}}(x)$. The formula in [9], p. 249 for the spectral radius of a matrix and (2.22) combine to give (2.19). $\underset{\sim}{r}(x)$ and $\underset{\sim}{\ell}(x)$ as given by (2.20) and (2.21) satisfy the appropriate eigenvalue equations.

$\rho(x)$ may be conveniently represented as follows: We have that $\sum_{i=1}^{m} \ell_i(x)\, Q_{ij}(x) = \rho(x)\, \ell_j(x)$ for all j . Summing over j gives:

$$\rho(x) = \sum_{i=1}^{m} \ell_i(x)\, H_i(x) \quad . \quad (2.23)$$

The moments of the distribution function $\rho(x)$ can be studied. Although it is not used in the sequel, we give the following representative result:

<u>Theorem 2.24</u>: Let $\underset{\sim}{Q}(x) = \{p_{ij} H_i(x)\}$ and suppose that for each i , $H_i(\cdot)$ concentrates on $[0, \infty)$. Let $\eta_i = \int_0^\infty x\, dH_i(x)$. Then $\rho(x) = 0$ for $x \leq 0$ and $\int_0^\infty x\, d\,\rho(x) < \infty$ iff $\max_{1\leq i\leq m} \eta_i < \infty$.

<u>Proof</u>: Given $\max_{1\leq i\leq m} \eta_i < \infty$. From (2.23) for each $x \geq 0$:

$$\min_{1\leq i\leq m} H_i(x) \leq \rho(x) \leq \max_{1\leq i\leq m} H_i(x) \; .$$

Then

$$\int_0^t (1 - \rho(x))\,dx \leq \int_0^t (1 - \min_i H_i(x))\,dx$$

$$= \int_0^t \max_i (1 - H_i(x))\,dx$$

$$\leq \int_0^t \sum_{i=1}^m (1 - H_i(x))\,dx$$

$$\leq \sum_{i=1}^m \eta_i < \infty \quad .$$

This holds for all t so that $\int_0^\infty (1 - \rho(x))\,dx = \int_0^\infty x\,d\,\rho(x) < \infty$.

For the converse, note that for every ϵ, $0 < \epsilon < \min_{1 \leq j \leq m} \pi_j$, there exists M_ϵ such that $x > M_\epsilon$ implies $|\ell_j(x) - \pi_j| < \epsilon$ for $j = 1, \ldots, m$. Then given $\int_0^\infty (1 - \rho(x))\,dx < \infty$ we have:

$$\infty > \int_M^\infty (1 - \rho(x))\,dx$$

$$= \int_M^\infty \sum_{j=1}^m \ell_j(x)(1 - H_j(x))\,dx$$

$$\geq \sum_{j=1}^m (\pi_j - \epsilon) \int_M^\infty (1 - H_j(x))\,dx \; .$$

So for each j, $\int_M^\infty (1 - H_j(x))\,dx < \infty$ and hence $\eta_j < \infty$.

CHAPTER III

LAW OF LARGE NUMBERS AND RELATIVE STABILITY

The results of this section give basic information about the asymptotic behavior of $\{M_n\}$ and preview the methods and the kinds of conditions that will be necessary for the proofs of subsequent more general theorems. Kolmogorov and Gnedenko [10] give related results for sums of independent random variables. For the case of maxima of a sequence of random variables see Gnedenko [5] for the i.i.d. case and Juncosa [8] for the independent case.

Definition 3.1: The sequence of successive maxima $\{M_n\}$ of a sequence of random variables $\{X_n\}$ satisfies the Law of Large Numbers (L.L.N.) iff there exist constants $\{A_n\}$ such that

$$P[|M_n - A_n| < \epsilon] \to 1$$

as $n \to \infty$ for all $\epsilon > 0$.

For random variables $\{X_n\}$ defined on a M.C., $\{M_n\}$ satisfies the L.L.N. iff there exist constants $\{A_n\}$ such that

$$(3.2) \quad \sum_{i=1}^{m} \sum_{j=1}^{m} Q_{ij}^{n} (A_n + \epsilon) p_i - \sum_{i=1}^{m} \sum_{j=1}^{m} Q_{ij}^{n} (A_n - \epsilon) p_i \to 1$$

as $n \to \infty$ for all $\epsilon > 0$. $p_1, \ldots, p_m$ are the initial probabilities for the M.C.

Definition 3.3: The sequence of maxima $\{M_n\}$ is relatively stable if there exists positive constants $\{B_n\}$ such that

$$P[|\frac{M_n}{B_n} - 1| < \epsilon] \to 1$$

as $n \to \infty$ for all $\epsilon > 0$.

For random variables defined on a M.C., $\{M_n\}$ is relatively stable iff there exists $B_n > 0$ such that for all $\epsilon > 0$

$$(3.4) \quad \sum_{i=1}^{m} \sum_{j=1}^{m} Q_{ij}^{n} (B_n(1 + \epsilon))\, p_i - \sum_{i=1}^{m} \sum_{j=1}^{m} Q_{ij}^{n} (B_n(1 - \epsilon))\, p_i \to 1$$

as $n \to \infty$.

We seek necessary and sufficient conditions for each property to hold. For random variables X_n, uniformly bounded above, the results are easy to state:

Theorem 3.5: Suppose there exist $y_i < \infty$ such that $H_i(y_i) = 1$, $H_i(y_i - \epsilon) < 1$, for all $\epsilon > 0$ and $i = 1, \ldots, m$. Let $\max_{1 \leq i \leq m} \{y_i\} = x_0 < \infty$. Then:

(i) The sequence $\{M_n\}$ satisfies the L.L.N. and $A_n = x_0$, $n \geq 1$. Hence $M_n \xrightarrow{P} x_0$ as $n \to \infty$.

(ii) Suppose $x_0 > 0$. Then the sequence $\{M_n\}$ is relatively stable and $B_n = x_0$.

Proof: (i) We have that

$$P[|M_n - x_0| < \epsilon] = \sum_{i=1}^{m} \sum_{j=1}^{m} p_i\, Q_{ij}^{n}(x_0 + \epsilon) - \sum_{i=1}^{m} \sum_{j=1}^{m} p_i\, Q_{ij}^{n}(x_0 - \epsilon).$$

Because of the definition of x_0, $\sum_{i=1}^{m} \sum_{j=1}^{m} p_i \, Q_{ij}^n(x_0 + \epsilon) = 1$.

It suffices to show that for all i and j, $Q_{ij}^n(x_0 - \epsilon) \to 0$ as $n \to \infty$. But $\underset{\sim}{Q}(x_0 - \epsilon) \leq \underset{\sim}{P}$ and strict inequality holds for the components of at least one row. By (2.6) $\rho(x_0 - \epsilon) < 1$ and hence $\underset{\sim}{Q}^n(x_0 - \epsilon) \to \underset{\sim}{0}$ as $n \to \infty$ by (2.7). Similarly for (ii).

If at least one of the distributions $H_1(\cdot), \ldots, H_m(\cdot)$ has support unbounded above, then the results are deeper. In this case $\rho(x) < 1$ for all x.

<u>Theorem 3.6</u>: Suppose there exists i_0 such that $H_{i_0}(x) < 1$ for all x. Then $\{M_n\}$ satisfies the L.L.N. iff for all $\epsilon > 0$ one of the following equivalent conditions holds:

(3.7) $$\lim_{x \to \infty} \frac{1 - \rho(x + \epsilon)}{1 - \rho(x)} = 0 ,$$

(3.8) $$\lim_{x \to \infty} \frac{1 - \prod_{i=1}^{m} H_i^{\pi_i}(x + \epsilon)}{1 - \prod_{i=1}^{m} H_i^{\pi_i}(x)} = 0 ,$$

(3.9) $$\lim_{x \to \infty} \frac{\sum_{i=1}^{m} \pi_i (1 - H_i(x + \epsilon))}{\sum_{i=1}^{m} \pi_i (1 - H_i(x))} = 0 .$$

<u>Remark:</u> The sequence $\{M_n\}$ satisfies the L.L.N. iff the sequence of maxima drawn from the distribution function $\rho(x)$ (or equivalently from either $\prod_{i=1}^{m} H_i^{\pi_i}(x)$ or $\sum_{i=1}^{m} \pi_i H_i(x)$)

satisfies the L.L.N. [5, p. 426] .

Proof: $\{M_n\}$ satisfies the L.L.N. iff there exist constants $\{A_n\}$ such that

$$\sum_{i=1}^{m} \sum_{j=1}^{m} Q_{ij}^{n} (A_n + \epsilon) p_i - \sum_{i=1}^{m} \sum_{j=1}^{m} Q_{ij}^{n} (A_n - \epsilon) p_i \to 1 \text{ as } n \to \infty ,$$

i.e. iff
$$\sum_{i=1}^{m} \sum_{j=1}^{m} Q_{ij}^{n}(A_n + \epsilon) p_i \to 1 \quad \text{and}$$

$$\sum_{i=1}^{m} \sum_{j=1}^{m} Q_{ij}^{n}(A_n - \epsilon) p_i \to 0$$

as $n \to \infty$. Because $\underset{\sim}{Q}^{n}(x) = \rho^{n}(x) \underset{\sim}{M}(x) + \underset{\sim}{o}(1)$ and $A_n \to \infty$, the above conditions are equivalent to

$$\sum_{i=1}^{m} \sum_{j=1}^{m} \rho^{n}(A_n + \epsilon) M_{ij}(A_n + \epsilon) p_i \to 1 \quad \text{and}$$

$$\sum_{i=1}^{m} \sum_{j=1}^{m} \rho^{n}(A_n - \epsilon) M_{ij}(A_n - \epsilon) p_i \to 0 .$$

Since $\lim_{n\to\infty} M_{ij}(A_n \pm \epsilon) = \pi_j$ we have that the L.L.N. holds iff there exists $\{A_n\}$ such that

$$(3.10) \qquad \rho^{n}(A_n + \epsilon) \to 1 \qquad \text{and}$$

$$(3.11) \qquad \rho^{n}(A_n - \epsilon) \to 0$$

as $n \to \infty$. These conditions are equivalent to (3.7), [5, p. 426] .

Because $A_n \to \infty$ and $\underset{\sim}{Q}^{n}(x) = \rho^{n}(x) \underset{\sim}{M}(x) + \underset{\sim}{o}(1)$, (3.10) and (3.11) are equivalent to

$$3.12) \qquad \underset{\sim}{Q}^n(A_n + \epsilon) \to \underset{\sim}{\Pi} \quad \text{and}$$

$$(3.13) \qquad \underset{\sim}{Q}^n(A_n - \epsilon) \to \underset{\sim}{0} \; .$$

These conditions are in turn equivalent to

$$(3.14) \qquad \left[\prod_{i=1}^{m} H_i(A_n + \epsilon)\right]^n \to 1 \quad \text{and}$$

$$(3.15) \qquad \left[\prod_{i=1}^{m} H_i(A_n - \epsilon)\right]^n \to 0$$

as we will now show. Given (3.14), we have that for $i = 1, \ldots, m$ $H_i^n(A_n + \epsilon) \to 1$ as $n \to \infty$. This and Lemma (2.18) give (3.12). Given (3.12) we focus attention on any i'. Select any convergent subsequence and suppose $H_{i'}^{n_k}(A_{n_k} + \epsilon) \to \Phi_{i'}$. To identify $\Phi_{i'}$ we select a further subsequence n_k' such that for all i, $H_i^{n_k'}(A_{n_k'} + \epsilon) \to \Phi_i$, with Φ_i the limit of the convergent subsequence. By (2.18), we have that

$\underset{\sim}{Q}^{n_k'}(A_{n_k'} + \epsilon) \to \left(\prod_{i,j=1}^{m} \Phi_i^{\pi_i p_{ij}}\right) \underset{\sim}{\Pi} = \left(\prod_{i=1}^{m} \Phi_i^{\pi_i}\right) \underset{\sim}{\Pi}$ and since also $\underset{\sim}{Q}^{n_k'}(A_{n_k'} + \epsilon) \to \underset{\sim}{\Pi}$ we have $\prod_{i=1}^{m} \Phi_i^{\pi_i} = 1$ and hence for each i $\Phi_i = 1$. In particular $\Phi_{i'} = 1$ and since every convergent subsequence converges to 1, $H_{i'}^n(A_n + \epsilon) \to 1$. The index i' was arbitrarily selected so the result holds for all i and hence for the product giving (3.14).

Given (3.13) we pick any convergent subsequence n_k and suppose $\prod_{i=1}^{m} H_i^{n_k}(A_{n_k} - \epsilon) \to \Phi$. Select further subsequences n_k' such that $H_i^{n_k'}(A_{n_k'} - \epsilon) \to \Phi_i$. Then by (2.18) $\underset{\sim}{Q}^{n_k'}(A_{n_k} - \epsilon) \to (\prod_{i=1}^{m} \Phi_i^{\pi_i}) \underset{\sim}{\Pi}$ and also $\underset{\sim}{Q}^{n_k'}(A_{n_k} - \epsilon) \to \underset{\sim}{0}$ so that $\prod_{i=1}^{m} \Phi_i^{\pi_i} = 0$. Since $\prod_{i=1}^{m} \Phi_i = \Phi$, we have that $\Phi = 0$. All convergent subsequences of $\prod_{i=1}^{m} H_i^n(A_n - \epsilon)$ converge to zero and hence the full sequence converges to zero.

Conversely suppose (3.15). There exists a subsequence n_k and a matrix $\underset{\sim}{U} = \{U_{ij}\}$ such that $\underset{\sim}{Q}^{n_k}(A_{n_k} - \epsilon) \to \underset{\sim}{U}$. In order to identify $\underset{\sim}{U}$ select a further subsequence n_k' such that $H_i^{n_k'}(A_{n_k'} - \epsilon) \to \Phi_i$ for $i = 1, \ldots, m$. Because of (3.15), $\prod_{i=1}^{m} \Phi_i = 0$ and at least one of the Φ_i's is zero. By (2.18) we have that $\underset{\sim}{Q}^{n_k'}(A_{n_k'} - \epsilon) \to \underset{\sim}{0}$ and hence $\underset{\sim}{U} = \underset{\sim}{0}$. Thus for all convergent subsequences n_k , $\underset{\sim}{Q}^{n_k}(A_{n_k} - \epsilon) \to \underset{\sim}{0}$ and hence the full sequence converges to zero giving (3.13) . This completes the demonstration of the equivalence of (3.12, 3.13) to (3.14, 3.15) .

(3.14) and (3.15) hold iff

(3.16) $$\left[\prod_{i=1}^{m} H_i^{\pi_i}(A_n + \epsilon)\right]^n \to 1 \quad \text{and}$$

(3.17) $$\left[\prod_{i=1}^{m} H_i^{\pi_i}(A_n - \epsilon)\right] \to 0$$

as $n \to \infty$. Thus the L.L.N. holds iff (3.16) and (3.17) hold and these conditions are equivalent to (8) [5, p. 426].

(3.16) and (3.17) hold iff

$$n \sum_{i=1}^{m} \pi_i \log H_i(A_n + \epsilon) \to 0 \quad \text{and}$$

$$n \sum_{i=1}^{m} \pi_i \log H_i(A_n - \epsilon) \to -\infty .$$

Since $A_n \to \infty$, these are equivalent to

$$(3.18) \qquad n \sum_{i=1}^{m} \pi_i(1 - H_i(A_n + \epsilon)) \to 0$$

$$(3.19) \qquad n \sum_{i=1}^{m} \pi_i(1 - H_i(A_n - \epsilon)) \to \infty$$

as $n \to \infty$. So the L.L.N. holds iff there exist constants $\{A_n\}$ such that (3.18) and (3.19) hold. The proof that (3.18) and (3.19) are equivalent to (3.9) is exactly the proof given by Gnedenko for the i.i.d. case [5, pp. 426-7] and will be omitted. This completes the proof of Theorem 3.6.

The results concerning relative stability are completely analogous to those for the L.L.N. We will only sketch the proofs.

<u>Theorem 3.20</u>: Suppose there exists i_0 such that $H_{i_0}(x) < 1$ for all x. Then the sequence $\{M_n\}$ is relatively stable iff for all $k > 1$ one of the following equivalent conditions holds:

$$(3.21) \qquad \lim_{x \to \infty} \frac{1 - \rho(kx)}{1 - \rho(x)} = 0 ,$$

(3.22) $$\lim_{x\to\infty} \frac{1 - \prod_{i=1}^{m} H_i^{\pi_i}(kx)}{1 - \prod_{i=1}^{m} H_i^{\pi_i}(x)} = 0 ,$$

(3.23) $$\lim_{x\to\infty} \frac{\sum_{i=1}^{m} \pi_i(1 - H_i(kx))}{\sum_{i=1}^{m} \pi_i(1 - H_i(x))} = 0 .$$

<u>Proof</u>: The sequence $\{M_n\}$ is relatively stable iff there exist $B_n > 0$ such that

$$\sum_{i=1}^{m} \sum_{j=1}^{m} Q_{ij}^{n}(B_n(1 + \epsilon))\, p_i - \sum_{i=1}^{m} \sum_{j=1}^{m} Q_{ij}^{n}(B_n(1 - \epsilon))\, p_i \to 1$$

as $n \to \infty$; i.e. iff

$$\sum_{i=1}^{m} \sum_{j=1}^{m} Q_{ij}^{n}(B_n(1 + \epsilon))\, p_i \to 1 \quad \text{and}$$

$$\sum_{i=1}^{m} \sum_{j=1}^{m} Q_{ij}^{n}(B_n(1 - \epsilon))\, p_i \to 0 .$$

Since $B_n \to \infty$, these conditions are equivalent to

(3.24) $$\rho^{n}(B_n(1 + \epsilon)) \to 1$$

(3.25) $$\rho^{n}(B_n(1 - \epsilon)) \to 0$$

as $n \to \infty$ and (3.24) and (3.25) are equivalent to (3.21) [5, p. 428] .

Now (3.24) and (3.25) are equivalent to

(3.26) $$\underset{\sim}{Q}^n(B_n(1+\epsilon)) \to \underset{\sim}{\Pi}$$

(3.27) $$\underset{\sim}{Q}^n(B_n(1-\epsilon)) \to \underset{\sim}{0}$$

These conditions hold iff

(3.28) $$\prod_{i=1}^{m} H_i^n(B_n(1+\epsilon)) \to 1$$

(3.29) $$\prod_{i=1}^{m} H_i^n(B_n(1-\epsilon)) \to 0$$

which hold iff

(3.30) $$\left[\prod_{i=1}^{m} H_i^{\pi_i}(B_n(1+\epsilon))\right]^n \to 1$$

(3.31) $$\left[\prod_{i=1}^{m} H_i^{\pi_i}(B_n(1-\epsilon))\right]^n \to 0$$

and these conditions are equivalent to (3.22) [5, p. 428] .

Taking logarithms and utilizing the fact that $B_n \to \infty$ shows that (3.30) and (3.31) are equivalent to

(3.32) $$n \sum_{i=1}^{m} \pi_i(1 - H_i(B_n(1+\epsilon)) \to 0 \qquad \text{and}$$

$$n \sum_{i=1}^{m} \pi_i(1 - H_i(B_n(1-\epsilon)) \to \infty$$

and these conditions are equivalent to (3.23) by a proof which is completely analogous to the one given by Gnedenko for the i.i.d. case [5, p. 429] .

CHAPTER IV

LIMIT LAWS

Throughout this chapter we let $\underset{\sim}{Q}(x) = \{p_{ij} H_i(x)\}$ where $\underset{\sim}{P} = \{p_{ij}\}$ is irreducible, aperiodic, stochastic, $\lim_{n\to\infty} \underset{\sim}{P}^n = \underset{\sim}{\Pi}$ and $H_1(\cdot), \ldots, H(\cdot)$ are nondegenerate, honest distribution functions.

We begin with a lemma:

Lemma 4.1: If there exist normalizing constants a_n, b_n, $n \geq 1$ and an index pair (i_0, j_0), $1 \leq i_0, j_0 \leq m$, such that

$$Q^n_{i_0 j_0}(a_n x + b_n) \xrightarrow{w} U_{i_0 j_0}(x) \qquad \text{and}$$

$$Q^n_{i_0 j_0}(\infty) = P^n_{i_0 j_0} \to \pi_{j_0} = U_{i_0 j_0}(\infty), \quad \text{and}$$

$U_{i_0 j_0}(x)$ is a nondegenerate mass function, then

$$\underset{\sim}{Q}^n(a_n x + b_n) \xrightarrow{c} \{U_{ij}(x)\}$$

where

$$U_{ij}(x) = \pi_j \pi^{-1}_{j_0} U_{i_0 j_0}(x) .$$

Proof: Focus attention on any $(i, j) \neq (i_0, j_0)$. By the weak compactness theorem for S.M.M.'s we may pick a convergent subsequence n_k and suppose

$$Q_{ij}^{n_k}(a_{n_k} x + b_{n_k}) \xrightarrow{w} U_{ij}(x) .$$

We wish to identify $U_{ij}(x)$ and so we select a further subsequence n_k' such that

$H_i^{n_k'}(a_{n_k'} x + b_{n_k'}) \xrightarrow{w} \Phi_i(x)$, $1 \leq i \leq m$; $\Phi_i(x)$ is a mass function. Hence $\underset{\sim}{Q}^{n_k'}(a_{n_k'} x + b_{n_k'}) \xrightarrow{w} [\prod_{i=1}^{m} \Phi_i^{\pi_i}(x)]\underset{\sim}{\Pi}$ by Lemma (2.18) , which identifies $U_{ij}(x) = [\prod_{i=1}^{m} \Phi_i^{\pi_i}(x)]\pi_j$. But $[\prod_{i=1}^{m} \Phi_i^{\pi_i}(x)]\pi_{j_0} = U_{i_0 j_0}(x)$ and therefore $[\prod_{i=1}^{m} \Phi_i^{\pi_i}(x)] = \pi_{j_0}^{-1} U_{i_0 j_0}(x)$; this is a nondegenerate, honest probability distribution function. So $\lim_{k\to\infty} Q_{ij}^{n_k}(a_{n_k} x + b_{n_k}) = U_{ij}(x) = [\pi_{j_0}^{-1} U_{i_0 j_0}(x)]\pi_j$. Since this holds for any convergent subsequence

$$\lim_{n\to\infty} Q_{ij}^{n}(a_n x + b_n) = [\pi_{j_0}^{-1} U_{i_0 j_0}(x)]\pi_j .$$

The pair (i,j) is arbitrary, which completes the proof.

<u>Theorem 4.2</u>: Limit Laws for the Q-Matrix.

If there exist $a_{ijn} > 0$, b_{ijn}, $i,j = 1,2,\ldots, m$, $n \geq 1$, such that

$$\{P[J_n = j,\ a_{ijn}^{-1}(M_n - b_{ijn}) \leq x | J_0 = i]\} = \{Q_{ij}^{n}(a_{ijn} x + b_{ijn})\}$$

$$\xrightarrow{c} \{U_{ij}(x)\}$$

where $U_{ij}(x)$ is nondegenerate, then

(1) $U_{ij}(x)$ is independent of i and is given by $\rho_U(x)\pi_j$; $\rho_U(x)$ is an honest, nondegenerate distribution function, the Perron-Frobenius eigenvalue of $\{U_{ij}(x)\}$.

(2) $\rho_U(x)$ is an extreme value distribution. In fact for all i,j $\rho^n(a_{ijn}\, x + b_{ijn}) \overset{c}{\rightarrow} \rho_U(x)$

(3) a_{ijn} and b_{ijn} may be chosen independently of i,j . $\rho_U(x)$ is of the form $\prod_{i=1}^{m} \Phi_i^{\pi_i}(x)$ where $\Phi_i^{\pi_i}(x)$ is an honest distribution function such that $H_i^{n_k}(a_{n_k}\, x + b_{n_k})$ $\overset{c}{\rightarrow}$ $\Phi_i(x)$ for some subsequence n_k .

(4) The domain of attraction of $\rho_U(x)$ includes also $\prod_{i=1}^{m} H_1^{\pi_i}(x)$.

The proof of part (2) requires a lemma. We state it now but defer its proof until after the proof of Theorem (4.2). Recall the representation $\underset{\sim}{Q}^n(x) = \rho^n(x)\, \underset{\sim}{M}(x) + \underset{\sim}{o}(1)$ where $\lim_{n\to\infty} \underset{\sim}{o}(1) = \underset{\sim}{0}$ uniformly in $x \in [K, \infty]$ for a suitably chosen K .

<u>Lemma 4.3</u>: If $\rho_U(x) > 0$ then: $\lim_{n\to\infty} M_{ij}(a_{ijn}\, x + b_{ijn}) = \pi_j$ for all i,j . We can show more. If $\rho_U(x) > 0$ then:

(a) $\lim_{n\to\infty} H_i(a_{ijn}\, x + b_{ijn}) = 1$

(b,1) If there exists some i_0 such that $H_{i_0}(x) < 1$ for all x , then $\lim_{n\to\infty} a_{ijn}\, x + b_{ijn} = +\infty$ for all i,j .

(b,2) If $H_i(x_i) = 1$ and $H_i(x_i - \varepsilon) < 1$ for all $\varepsilon > 0$, $i = 1,2,\ldots, m$ and $x_0 = \max\{x_1,\ldots, x_m\} < \infty$, then for x fixed either

(b,2,i) $a_{ijn} x + b_{ijn} > x_0$ for finitely many n and $\lim_{n\to\infty} a_{ijn} x + b_{ijn} = x_0$

or (b,2,ii) $a_{ijn} x + b_{ijn} > x_0$ infinitely often and $\underset{\sim}{Q}^n(a_{ijn} x + b_{ijn}) \to \underset{\sim}{\Pi}$ and $\rho_U(x) = 1$. (Note in $\underset{\sim}{Q}^n(a_{ijn} x + b_{ijn})$ we evaluate each component $Q^n_{k\ell}(\cdot)$ at $a_{ijn} x + b_{ijn}$ for $k, \ell = 1,2,\ldots, m$).

<u>Proof or Theorem 4.2</u>: (1) We have $\{Q_{ij}(a_{ijn} x + b_{ijn})\}^n = \{p_{ij} H_i(a_{ijn} x + b_{ijn})\}^n \overset{c}{\to} \{U_{ij}(x)\} = \underset{\sim}{U}(x)$.

There exists a subsequence n_k such that for all i, j:

$$H_i(a_{ijn_k} x + b_{ijn_k})^{n_k} \overset{w}{\to} \Phi_{ij}(x)$$

for mass functions $\Phi_{ij}(x)$ by the weak compactness theorem. It follows from (2.18) that:

$$\{p_{ij} H_i(a_{ijn_k} x + b_{ijn_k})\}^{n_k} \overset{w}{\to} \left[\prod_{i,j=1}^{m} \Phi_{ij}^{\pi_i p_{ij}}(x)\right] \underset{\sim}{\Pi}$$

$(0^0 = 1)$. Therefore at all continuity points of $\underset{\sim}{U}(x)$,

$$\underset{\sim}{U}(x) = \left[\prod_{i,j=1}^{m} \Phi_{ij}^{\pi_i p_{ij}}(x)\right] \underset{\sim}{\Pi} .$$

Hence $U_{ij}(x)$ is independent of i. Further since the Perron-Frobenius eigenvalue of $\underset{\sim}{\Pi}$ is 1, the Perron-frobenius eigenvalue of $\underset{\sim}{U}(x)$, say $\rho_U(x)$, is $\prod_{i,j=1}^{m} \Phi_{ij}^{\pi_i p_{ij}}(x)$ and $\underset{\sim}{U}(x) = \rho_U(x)\underset{\sim}{\Pi}$. $\rho_U(x)$ is independent of the choice of subsequence n_k since

$$\rho_U(x) = \sum_{j=1}^{m} U_{ij}(x) \text{ for all } i .$$

By the definition of complete convergence of S.M.M.'s $\rho_U(x)$ is an honest, nondegenerate distribution function.

Also:

$$(4.4) \qquad \rho_U(x) > 0 \text{ implies that } \Phi_{ij}(x) > 0$$

for all (i,j) such that $p_{ij} > 0$.

If $p_{ij} > 0$, then $\Phi_{ij}(x)$ cannot be dishonest. Hence whenever $p_{ij} > 0$, $H_i(a_{ijn_k} x + b_{ijn_k}) \overset{c}{\to} \Phi_{ij}(x)$.

If $p_{ij} = 0$, $\Phi_{ij}^{p_{ij}}(x) \equiv 1$. At least one $\Phi_{ij}(\cdot)$ is nondegenerate for which $p_{ij} > 0$ since otherwise $\rho_U(x)$ would be degenerate.

Finally:

$$(4.5) \qquad \left[\prod_{i,j=1}^{m} H_i^{\pi_i p_{ij}}(a_{ijn} x + b_{ijn})\right]^n \overset{c}{\to} \rho_U(x)$$

since every convergent subsequence will converge to $\rho_U(x)$.

(2) By Lemma (4.1) we have for each (i,j) :

$$\underset{\sim}{Q}^n(a_{ijn}\, x + b_{ijn}) \overset{c}{\rightarrow} \underset{\sim}{U}(x) .$$

If there are x such that $\rho_U(x) = 0$, then for each (i,j)

$$\underset{\sim}{Q}^n(a_{ijn}\, x + b_{ijn}) \rightarrow \underset{\sim}{0} .$$

For any ϵ , there exists $N_{\epsilon,x}$ such that $n > N_{\epsilon,x}$ implies that

$$\underset{\sim}{Q}^n(a_{ijn}\, x + b_{ijn}) \leq \epsilon \underset{\sim}{E} .$$

Hence by (2.6), for $n > N_{\epsilon,x}$:

$$\rho^n(a_{ijn}\, x + b_{ijn}) \leq \epsilon\, m .$$

Therefore $\rho^n(a_{ijn}\, x + b_{ijn}) \rightarrow 0$

as $n \rightarrow \infty$ for every i,j.

If x is such that $\rho_U(x) > 0$, then Lemma (4.3) assures us that for large n $a_{ijn}\, x + b_{ijn}$ will be large enough for Theorem (2.15) to be applicable. We have

$$\lim_{n\to\infty} Q^n_{ij}(a_{ijn}\, x + b_{ijn})$$

$$= \lim_{n\to\infty}[\rho^n(a_{ijn}\, x + b_{ijn})\, M_{ij}(a_{ijn}\, x + b_{ijn}) + o(1)] .$$

Therefore:

$$\rho_U(x)\,\pi_j = \lim_{n\to\infty} \rho^n(a_{ijn}\,x + b_{ijn})\,\pi_j$$

by Lemma (4.3). We have shown that for all x and for all i,j:

(4.6) $$\rho_U(x) = \lim_{n\to\infty} \rho^n(a_{ijn}\,x + b_{ijn}) .$$

Therefore $\rho_U(x)$ is an extreme value distribution [5].

(3) Because of (4.6) and (2.1), the sequences $\{a_{ijn}, b_{ijn}\}$ $1 \leq i,j \leq m$ are asymptotically equivalent. Let $\{a_n, b_n\}$ be any sequence asymptotically equivalent to these sequences. Choose a subsequence n_k such that for all i,j : $H_i(a_{ijn_k}\,x + b_{ijn_k})^{n_k} \overset{w}{\to} \Phi_{ij}(x)$ for mass functions $\Phi_{ij}(x)$. For each i, there is a j_0 such that $p_{ij_0} > 0$ and as in the proof of (1):

$$H_i(a_{ij_0n_k}\,x + b_{ij_0n_k})^{n_k} \overset{c}{\to} \Phi_{ij_0}(x) .$$

Set $\Phi_i(x) = \Phi_{ij_0}(x)$ and (2.1) gives

$$H_i(a_{n_k}\,x + b_{n_k})^{n_k} \overset{c}{\to} \Phi_i(x) .$$

Again by (2.1):

$$H_i(a_{ijn_k}\,x + b_{ijn_k})^{n_k} \overset{c}{\to} \Phi_i(x)$$

so for all i,j, $\Phi_{ij}(x) = \Phi_i(x)$. Therefore:

$$\prod_{i=1}^{m} \Phi_i^{\pi_i}(x) = \prod_{i,j=1}^{m} \Phi_{ij}^{\pi_i p_{ij}}(x) = \rho_U(x) .$$

Hence by (2.18):

$$\{p_{ij} H_i(a_{n_k} x + b_{n_k})\}^{n_k} \overset{c}{\rightarrow} \rho_U(x) \underset{\sim}{\Pi} .$$

Since every convergent subsequence has the same limit,

$$\{p_{ij} H_i(a_n x + b_n)\}^n \overset{c}{\rightarrow} \rho_U(x) \underset{\sim}{\Pi}$$

and $\{a_{ijn}, b_{ijn}\}$ may be chosen independently of (i,j) .

(4) As in (3), let $\{a_n, b_n\}$ be any sequence asymptotically equivalent to $\{a_{ijn}, b_{ijn}\}$, $1 \leq i,j \leq m$. Then from (4.5):

$$\left[\prod_{i,j=1}^{m} H_i^{\pi_i p_{ij}}(a_{ijn} x + b_{ijn})\right]^n \overset{c}{\rightarrow} \rho_U(x) .$$

From (2.1):

$$\left[\prod_{i,j=1}^{m} H_i^{\pi_i p_{ij}}(a_n x + b_n)\right] = \left[\prod_{i=1}^{m} H_i^{\pi_i}(a_n x + b_n)\right]^n$$

$$\overset{c}{\rightarrow} \rho_U(x) .$$

Hence $\prod_{i=1}^{m} H_i^{\pi_i}(x)$ is in the domain of attraction of $\rho_U(x)$ and by (2.1):

$$\left[\prod_{k=1}^{m} H_k^{\pi_k}(a_{ijn}\, x + b_{ijn})\right]^n \stackrel{c}{\to} \rho_U(x)$$

for $1 \leq i, j \leq m$. It only remains to prove Lemma (4.3):

Proof of Lemma 4.3: (a) We fix x such that $\rho_U(x) > 0$ and pick a subsequence n_k such that $H_i(a_{ijn_k}\, x + b_{ijn_k})$ converges. Suppose that $\lim_{k\to\infty} H_i(a_{ijn_k}\, x + b_{ijn_k}) = \ell$. There exists a further subsequence n_k' such that

$$H_i(a_{ijn_k'}\, x + b_{ijn_k'})^{n_k'} \to \psi_{ij}(x)$$

and because of (4.4) and the assumption that $\rho_U(x) > 0$ we have $\psi_{ij}(x) > 0$ whenever $p_{ij} > 0$. So taking logarithms:

$$n_k' \log H_i(a_{ijn_k'}\, x + b_{ijn_k'}) \to \log \psi_{ij}(x)$$

and therefore

$$\log H_i(a_{ijn_k'}\, x + b_{ijn_k'}) \to 0$$

and

$$H_i(a_{ijn_k'}\, x + b_{ijn_k'}) \to 1 \quad \text{whenever} \quad p_{ij} > 0 .$$

This identifies $\ell = 1$ and since any convergent subsequence must converge to 1 we have $H_i(a_{ijn}\, x + b_{ijn}) \to 1$ whenever $p_{ij} > 0$. The restriction that $p_{ij} > 0$ can be dropped as will be shown in (b).

(b,1) If $H_{i_0}(x) < 1$ for all x then $\rho(x) < 1$ for all x by (2.6) and for all x, $\lim_{n\to\infty} \underset{\sim}{Q}^n(x) = \underset{\sim}{0}$ by (2.7).

Suppose $a_{ijn} x + b_{ijn}$ does not converge to $+\infty$. Then there is a subsequence n_k and a real number K^0 such that

$a_{ijn_k} x + b_{ijn_k} \le K^0 < +\infty$ for all k.

Then:

$$\underset{\sim}{Q}^{n_k}(a_{ijn_k} x + b_{ijn_k}) \le Q^{n_k}(K^0) \to 0 \quad \text{as} \quad k \to \infty .$$

In particular

$$Q_{ij}^{n_k}(a_{ijn_k} x + b_{ijn_k}) \to 0 . \quad \text{Since}$$

$$Q_{ij}^{n_k}(a_{ijn_k} x + b_{ijn_k}) \to \rho_U(x)\, \pi_j > 0 \quad \text{we have a}$$

contradiction.

For this case, since $\lim_{n\to\infty} a_{ijn} x + b_{ijn} = +\infty$, we have immediately from (2.11): $\lim_{n\to\infty} M_{ij}(a_{ijn} x + b_{ijn}) = \pi_j$.

Also $\lim_{n\to\infty} H_i(a_{ijn} x + b_{ijn}) = 1$, $1 \le i,j \le m$.

(b,2,i) If $a_{ijn} x + b_{ijn} > x_0$ for only finitely many n then there exists a positive integer N_x such that if $n > N_x$ then $a_{ijn} x + b_{ijn} \le x_0$. Pick a convergent subsequence n_k and suppose $a_{ijn_k} x + b_{ijn_k} \to x' \le x_0$ as $k \to \infty$. If $x' < x_0$ then there is an $\varepsilon > 0$ such that $x' < x_0 - \varepsilon$. Then for all n_k sufficiently large $\underset{\sim}{Q}^{n_k}(a_{ijn_k}\, x + b_{ijn_k}) \leq \underset{\sim}{Q}^{n_k}(x_0 - \epsilon) \to \underset{\sim}{0}$ as $k \to \infty$ but also $Q_{ij}^{n_k}(a_{ijn_k}\, x + b_{ijn_k}) \to \rho_U(x)\, \pi_j > 0$ which gives a contradiction. Hence $x' = x_0$. Since any convergent subsequence converges to x_0, the sequence converges to x_0.

Hence for $n > N_x$, $x_0 \geq a_{ijn}\, x + b_{ijn} \to x_0$, $n \to \infty$; we have $H_i(a_{ijn}\, x + b_{ijn}) \to H_i(x_0-)$. Since for a fixed i, there is some j such that $p_{ij} > 0$, it follows from (a) using this j that

$$H_i(a_{ijn}\, x + b_{ijn}) \to 1\,, \tag{4.9}$$

whence $H_i(x_0-) = 1 = H_i(x_0)$. So $H_i(\cdot)$, $i = 1, \ldots, m$ are continuous at x_0 and hence so is $\underset{\sim}{Q}(\cdot)$. By Lemma (2.10) $\rho(\cdot)$, $\underset{\sim}{r}(\cdot)$, $\underset{\sim}{\ell}(\cdot)$ and hence $\underset{\sim}{M}(\cdot)$ are all continuous at x_0. Therefore $\lim\limits_{n\to\infty} M_{ij}(a_{ijn}\, x + b_{ijn}) = M_{ij}(x_0) = \pi_j$. Also $\lim\limits_{n\to\infty} H_i(a_{ijn}\, x + b_{ijn}) = 1$, $1 \leq i, j \leq m$.

(b,2,ii) If $a_{ijn}\, x + b_{ijn} > x_0$ for infinitely many n, then for infinitely many n, $Q_{ij}^n(a_{ijn}\, x + b_{ijn}) = P_{ij}^n$. By supposition $Q_{ij}^n(a_{ijn}\, x + b_{ijn}) \to U_{ij}(x)$ so we must have $Q_{ij}^n(a_{ijn}\, x + b_{ijn}) \to \pi_j$. By Lemma (4.1) this suffices for $\underset{\sim}{Q}^n(a_{ijn}\, x + b_{ijn}) \to \underset{\sim}{\Pi}$. Hence $\rho^n(a_{ijn}\, x + b_{ijn}) \to 1$. If there are also infinitely many n , say $\{n_k\}$, such that $a_{ijn_k} x + b_{ijn_k} < x_0$, then as above, $x_0 \geq a_{ijn_k} x + b_{ijn_k} \to x_0$ as $k \to \infty$ and $H_1(\cdot), \ldots, H_m(\cdot)$ are continuous at x_0 .

Whether or not such a sequence $\{n_k\}$ exists,

$\lim_{n\to\infty} H_i(a_{ijn} x + b_{ijn}) = 1$, $1 \leq i,j \leq m$, and

$\underline{\lim}_{n\to\infty} a_{ijn} x + b_{ijn} \geq x_0$. Hence theorem (2.15) is applicable

and $\lim_{n\to\infty} Q^n_{ij}(a_{ijn} x + b_{ijn})$

$$= \lim_{n\to\infty}[\rho^n(a_{ijn} x + b_{ijn}) M_{ij}(a_{ijn} x + b_{ijn}) + o(1)] ,$$

whence $\pi_j = \lim_{n\to\infty} M_{ij}(a_{ijn} x + b_{ijn})$. The lemma is completely proved.

Without loss of generality we henceforth assume that normalizing constants are chosen independently of i and j . That this can be done is not surprising in view of Lemma (4.1). Also, when we take the nth power of the Q-matrix we sum over all paths of length n starting at i and ending at j . This entails sufficient mixing of the distributions involved so that the effects of the endpoints i and j become negligible for large n .

Corollary 4.7 Convergence to Types: If for given constants $\alpha_n > 0$, β_n and $a_n > 0$, b_n :

$$\{Q^n_{ij}(\alpha_n x + \beta_n)\} \overset{c}{\to} \underset{\sim}{V}(x) = \{V_{ij}(x)\} \text{ and}$$

$$\{Q_{ij}^n(a_n x + b_n) \xrightarrow{c} \underset{\sim}{U}(x) = \{U_{ij}(x)\}$$

where $U_{ij}(x)$, $V_{ij}(x)$ are nondegenerate for each (i,j) , then $\underset{\sim}{U}(x)$ and $\underset{\sim}{V}(x)$ are of the same type. There exist $A > 0$ and B such that

$$A = \lim_{n\to\infty} \alpha_n^{-1} a_n \text{ and } B = \lim_{n\to\infty} a_n^{-1}(\beta_n - b_n) \text{ and}$$

$\{V_{ij}(x)\} = \underset{\sim}{V}(x) = \underset{\sim}{U}(Ax + B) = \{U_{ij}(Ax + B)\}$. Furthermore $\underset{\sim}{U}(x) = \rho_U(x) \underset{\sim}{\Pi}$, where $\rho_U(x)$ is an extreme value distribution and $\underset{\sim}{V}(x) = \rho_U(Ax + B) \underset{\sim}{\Pi}$.

<u>Corollary 4.8</u> Asymptotic Independence: Given

$$\{P[J_n = j, a_n^{-1}(M_n - b_n) \le x | J_0 = i]\} \to \{U_{ij}(x)\} = \rho_U(x) \underset{\sim}{\Pi}$$

then $P[a_n^{-1}(M_n - b_n) \le x] \xrightarrow{c} \rho_U(x)$ and

$$\lim_{n\to\infty} P[J_n = j, a_n^{-1}(M_n - b_n) \le x] =$$

$$\lim_{n\to\infty} P[J_n = j] \lim_{n\to\infty} P[a_n^{-1}(M_n - b_n) \le x] .$$

<u>Proof</u>: We have that

$$\lim_{n\to\infty} P[J_n = j,\ a_n^{-1}(M_n - b_n) \le x | J_0 = i] = \rho_U(x)\, \pi_j \quad \text{so}$$

$$\lim_{n\to\infty} P[a_n^{-1}(M_n - b_n) \le x | J_0 = i] = \rho_U(x) \quad \text{and}$$

$$\lim_{n\to\infty} P[a_n^{-1}(M_n - b_n) \le x] = \rho_U(x) .$$ Therefore M_n has

a limiting distribution which is an extreme value distribution. Next we have that

$$\lim_{n\to\infty} P[J_n = j,\ a_n^{-1}(M_n - b_n) \le x] =$$

$$\lim_{n\to\infty} P[J_n = j,\ a_n^{-1}(M_n - b_n) \le x \mid J_0 = i] = \pi_j\ \rho_U(x) =$$

$$\lim_{n\to\infty} P[J_n = j] \lim_{n\to\infty} P[a_n^{-1}(M_n - b_n) \le x]$$ which completes the proof.

Our results are related to those of Gnedenko by the following theorem.

Theorem 4.9: There exist norming constants $a_n > 0$, b_n, $n \ge 1$ such that $P[a_n^{-1}(M_n - b_n) \le x] \overset{c}{\to} \rho_U(x)$ where $\rho_U(x)$ is a nondegenerate distribution function iff $\underset{\sim}{Q}^n(a_n x + b_n) \overset{c}{\to} \rho_U(x) \underset{\sim}{\Pi}$. Hence $\rho_U(x)$ is an extreme value distribution and the only possible limiting distributions for the sequence $\{M_n\}$ are the extreme value types.

Proof: Given the convergence of the Q-matrix, the desired result follows from (4.2) and (4.7).

Now we suppose that $\lim_{n\to\infty} P[a_n^{-1}(M_n - b_n) \le x] = \rho_U(x)$.

For some initial distribution (p_i), $i = 1, \ldots, m$ we have from (1.1) that

$$(4.10)\quad \lim_{n\to\infty} P[a_n^{-1}(M_n - b_n) \le x] = \lim_{n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{m} Q_{ij}^n(a_n x + b_n)\ p_i = \rho_U(x).$$

By the weak compactness theorem for S.M.M.'s we can select a subsequence n_k such that, for some limit: $\underset{\sim}{U}(x) = \{U_{ij}(x)\}$, $\lim_{k\to\infty} \{Q_{ij}^{n_k}(a_{n_k} x + b_{n_k})\} = \{U_{ij}(x)\}$. We will identify $\{U_{ij}(x)\}$.

From (4.10) we have:

$$(4.11) \qquad \sum_{k=1}^{m} \sum_{\ell=1}^{m} U_{k\ell}(x)\, p_k = \rho_U(x) .$$

There exists a further subsequence n_k' such that

$H^{n_k'}(a_{n_k'} x + b_{n_k'}) \overset{w}{\to} \Phi_i(x)$ with the $\Phi_i(x)$ mass functions.

We have $\underset{\sim}{Q}^{n_k'}(a_{n_k'} x + b_{n_k'}) \to \underset{\sim}{U}(x)$ and also

$\underset{\sim}{Q}^{n_k'}(a_{n_k'} x + b_{n_k'}) \to [\prod_{i=1}^{m} \Phi_i^{\pi_i}(x)]\, \underset{\sim}{\Pi}$ by (2.18). So

$U_{ij}(x) = [\prod_{i=1}^{m} \Phi_i^{\pi_i}(x)]\, \pi_j$ and from (4.11)

$$\rho_U(x) = \sum_{k=1}^{m} \sum_{\ell=1}^{m} [\prod_{i=1}^{m} \Phi_i^{\pi_i}(x)]\, \pi_\ell p_k = \prod_{i=1}^{m} \Phi_i^{\pi_i}(x) .$$

Therefore $U_{ij}(x) = \rho_U(x)\, \pi_j$ and $\{Q_{ij}^{n_k}(a_{n_k} x + b_{n_k})\} \to \rho_U(x)\, \underset{\sim}{\Pi}$.

Since this holds for any convergent subsequence we have

$\underset{\sim}{Q}^{n}(a_n x + b_n) \overset{c}{\to} \rho_U(x)\, \underset{\sim}{\Pi}$. By (4.2) $\rho_U(x)$ is an extreme value distribution.

Criteria for the existence of a limiting distribution for $\{M_n\}$ are given in

Theorem 4.12: There exist constants $a_n > 0$, b_n, $n \geq 1$ such that:

$$(4.13) \qquad \underset{\sim}{Q}^n(a_n x + b_n) \overset{c}{\rightarrow} \rho_U(x) \underset{\sim}{\Pi}$$

where $\rho_U(x)$ is a nondegenerate (extreme value) distribution function

$$(4.14) \qquad \text{iff } \rho^n(a_n x + b_n) \overset{c}{\rightarrow} \rho_U(x) \quad ,$$

or:

$$(4.15) \qquad \text{iff } \left[\prod_{i=1}^{m} H_i^{\pi_i}(a_n x + b_n)\right]^n \overset{c}{\rightarrow} \rho_U(x) \; .$$

It follows that M_n has a limiting extreme value distribution $\rho_U(x)$ iff $\rho(x)$ or equivalently $\prod_{i=1}^{m} H_i^{\pi_i}(x)$ are in the domain of attraction of $\rho_U(x)$.

Proof: Given (4.13), the latter two statements follow from theorem (4.2).

Assuming (4.14) there are two cases:

Case I: If $\rho(x) < 1$ for all $x < \infty$. For all x such that $\rho_U(x) > 0$, (4.14) implies $\rho(a_n x + b_n) \rightarrow 1$, $n \rightarrow \infty$, [5, p. 439]. For such x, $a_n x + b_n \rightarrow \infty$ and therefore

$$\lim_{n\to\infty} M_{ij}(a_n x + b_n) = \pi_j \; .$$

Since $a_n x + b_n \rightarrow \infty$, Theorem (2.15) is applicable and:

$$\lim_{n\to\infty} \underset{\sim}{Q}^n(a_n x + b_n) = \lim_{n\to\infty} [\rho^n(a_n x + b_n) \underset{\sim}{M}(a_n x + b_n) + \underset{\sim}{o}(1)]$$

so that

$$\lim_{n\to\infty} \underset{\sim}{Q}^n(a_n x + b_n) = \rho_U(x) \underset{\sim}{\Pi} .$$

If there are x such that $\rho_U(x) = 0$ then we proceed as follows: $\rho_U(x)$ is an extreme value distribution and hence is continuous. For any ϵ, there is a z such that $0 < \rho_U(z) < \epsilon$. Then $z > x$ and

$$0 \leq \overline{\lim_{n\to\infty}} \underset{\sim}{Q}^n(a_n x + b_n) \leq \lim_{n\to\infty} \underset{\sim}{Q}^n(a_n z + b_n) = \rho_U(z) \underset{\sim}{\Pi} \leq \epsilon \underset{\sim}{\Pi} .$$

Since ϵ is arbitrary we must have $\lim_{n\to\infty} \underset{\sim}{Q}^n(a_n x + b_n) = \underset{\sim}{0} = \rho_U(x) \underset{\sim}{\Pi}$.

So for all x, $\lim_{n\to\infty} \underset{\sim}{Q}^n(a_n x + b_n) = \rho_U(x) \underset{\sim}{\Pi}$.

Case II: There exists $x_0 < \infty$ such that $\rho(x_0) = 1$ and $\rho(x_0 - \epsilon) < 1$ for all $\epsilon > 0$. For a fixed x such that $\rho_U(x) > 0$, suppose $a_n x + b_n > x_0$ for only finitely many n, then for n sufficiently large $a_n x + b_n \leq x_0$. In fact $a_n x + b_n \to x_0$ as $n \to \infty$. To show this, suppose there is a subsequence n_k with $a_{n_k} x + b_{n_k} \to x' < x_0$ as $k \to \infty$.

Then for some $\epsilon > 0$, $x' < x_0 - \epsilon$. Now $\lim_{n\to\infty} \rho(a_n x + b_n) = 1$ [5, p. 439] and $\lim_{k\to\infty} \rho(a_{n_k} x + b_{n_k}) = 1$. But

$\lim_{k\to\infty} \rho(a_{n_k} x + b_{n_k}) \leq \rho(x') \leq \rho(x_0 - \epsilon) < 1$ yielding a contradiction. There are no subsequential limits less than x_0 and hence $a_n x + b_n \to x_0$. Thus $\rho(a_n x + b_n) \to \rho(x_0-)$ and since also $\rho(a_n x + b_n) \to 1$, $\rho(x_0-) = 1 = \rho(x_0)$ and $\rho(\cdot)$ is continuous at x_0. So $Q(\cdot)$, $r(\cdot)$, $\Lambda(\cdot)$, $M(\cdot)$ are all continuous at x_0 (2.8-2), (2.10-2), and $\lim_{n\to\infty} M_{ij}(a_n x + b_n) = \pi_j$. Therefore since $a_n x + b_n \to x_0$, Theorem (2.15) is applicable and:

$$\lim_{n\to\infty} \underset{\sim}{Q}^n(a_n x + b_n) = \lim_{n\to\infty} [\rho^n(a_n x + b_n)\, \underset{\sim}{M}(a_n x + b_n) + \underset{\sim}{o}(1)] \text{ and}$$

$$\lim_{n\to\infty} \underset{\sim}{Q}(a_n x + b_n) = \rho_U(x)\, \underset{\sim}{\Pi}\,.$$

Suppose $a_n x + b_n > x_0$ for infinitely many n, then $\rho_U(x) = 1$ and $\underset{\sim}{Q}^n(a_n x + b_n) = \underset{\sim}{P}^n$ for such n. If $a_n x + b_n \leq x_0$ for only finitely many n, then $\lim_{n\to\infty} \underset{\sim}{Q}^n(a_n x + b_n) = \underset{\sim}{\Pi}$, as was to be proved. If $a_n x + b_n < x_0$ for infinitely many n then we partition the set of positive integers into sets $\{n_1\}$ and $\{n_2\}$ such that $a_{n_1} x + b_{n_1} \leq x_0$ for all n_1 and $a_{n_2} x + b_{n_2} > x_0$ for all n_2. As above $a_{n_1} x + b_{n_1} \to x_0$ as $n_1 \uparrow +\infty$ and $M(\cdot)$ is continuous at x_0, so

$$\lim_{n_1\to\infty} \underset{\sim}{Q}^{n_1}(a_{n_1} x + b_{n_1}) = \lim_{n_1\to\infty} [\rho^{n_1}(a_{n_1} x + b_{n_1})\, \underset{\sim}{M}(a_{n_1} x + b_{n_1}) + \underset{\sim}{o}(1)]$$

and $\lim_{n_1\to\infty} \underset{\sim}{Q}^{n_1}(a_{n_1} x + b_{n_1}) = \underset{\sim}{\Pi}$. Since $\underset{\sim}{Q}^{n_2}(a_{n_2} x + b_{n_2}) = \underset{\sim}{\Pi}$

for all n_2 we have

$$\lim_{n\to\infty} \underset{\sim}{Q}^n(a_n x + b_n) = \underset{\sim}{\Pi}$$ as was to be shown.

If there are x such that $\rho_U(x) = 0$, we proceed as in Case I.

Now assume (4.15). By the weak compactness theorem for S.M.M.'s we can select a convergent subsequence n_k such that

$$\{Q_{ij}^{n_k}(a_{n_k} x + b_{n_k})\} \xrightarrow{w} \{U_{ij}(x)\}.$$ To identify $U_{ij}(x)$ as

$\rho_U(x)\, \pi_j$, we select a further subsequence n_k' such that for

$$1 \le i \le m, \quad H_i^{n_k'}(a_{n_k'} x + b_{n_k'}) \xrightarrow{w} \Phi_i(x)$$ with

the $\Phi_i(x)$ mass functions, and therefore

$$\underset{\sim}{Q}^{n_k'}(a_{n_k'} x + b_{n_k'}) \to \left[\prod_{i=1}^{m} \Phi_i^{\pi_i}(x)\right] \underset{\sim}{\Pi}$$ by (2.18). But

$$\left[\prod_{i=1}^{m} H_i^{\pi_i}(a_{n_k'} x + b_{n_k'})\right]^{n_k'} \to \prod_{i=1}^{m} \Phi_i^{\pi_i}(x)$$ and also

$$\left[\prod_{i=1}^{m} H_i^{\pi_i}(a_{n_k'} x + b_{n_k'})\right]^{n_k'} \to \rho_U(x) \quad \text{so} \quad \prod_{i=1}^{m} \Phi_i^{\pi_i}(x) = \rho_U(x)$$

and $$\{Q_{ij}^{n_k}(a_{n_k} x + b_{n_k})\} \to \{U_{ij}(x)\} = \rho_U(x)\, \underset{\sim}{\Pi}.$$

This holds for all convergent subsequences, and hence for the full sequence.

CHAPTER V

TAIL EQUIVALENCE AND ITS APPLICATIONS

It is intuitively clear that the properties of the successive maxima of a sequence of random variables are determined by the quantity of probability contained in the right hand tails of the distribution functions. In this chapter we make this intuition precise.

Although it is not necessary to do so, it is simpler to assume here that all distribution functions are right continuous.

Convention: For $F(\cdot)$ a distribution function set $x_0 = \inf\{y | F(y) = 1\}$. If $F(y) < 1$ for all y then $x_0 = \infty$. If two distribution functions are involved in a discussion we write x_0^F , x_0^G . If no distinction by superscripts is made, it is to be understood that $x_0^F = x_0^G = x_0$.

Definition 5.1: Two distribution functions $F(\cdot)$ and $G(\cdot)$ are tail equivalent iff $x_0^F = x_0^G = x_0$ and $1 - F(x) \sim 1 - G(x)$ as $x \to x_0-$; i.e. iff

$$\lim_{x \to x_0-} \frac{1-F(x)}{1-G(x)} = 1 .$$

We shall often speak of two distribution functions whose tails have a ratio approaching α where $0 \le \alpha \le \infty$.

Remark 5.2: For two arbitrary distributions the ratio of the tails need not have a limit as $x \to x_0-$. As an example, let $F(x)$ be any continuous, strictly increasing distribution function. Pick x_0 such that $F(x_0) < 1$ and set x_n to be the (unique) solution of the equation $1 - F(x) = 2^{-n}(1 - F(x_0))$. Define $G(x)$ as follows: $G(x) = F(x)$ for $x \leq x_0$, $1 - G(x_{2n-1}) = 1 - G(X_{2n}) = 1 - F(X_{2n})$. For other values of x, define $G(\cdot)$ by linear interpolation. Then $\lim_{n\to\infty} \frac{1-F(x_{2n})}{1-G(x_{2n})} = 1$ and $\lim_{n\to\infty} \frac{1-F(x_{2n+1})}{1-G(x_{2n+1})} = 2$ which shows $\frac{1-F(x)}{1-G(x)}$ does not have a limit as $x \to \infty$.

Theorem 5.3: $F(\cdot)$ and $G(\cdot)$ are distribution functions such that

$$\lim_{x\to x_0-} \frac{1-F(x)}{1-G(x)} = \alpha, \qquad 0 < \alpha < \infty .$$

If there exist normalizing constants $a_n > 0$, b_n, $n \geq 1$, such that

$$F^n(a_n x + b_n) \to \Phi(x),$$

$\Phi(x)$ nondegenerate, then

$$G^n(a_n x + b_n) \to \Phi^{\frac{1}{\alpha}}(x) .$$

Proof: Since $\frac{1-F(x)}{1-G(x)} \to \alpha$ we have that as $x \to x_0-$, $1 - G(x) = \alpha^{-1}(1 - F(x)) + o(1 - F(x))$, and

$G(x) = 1 - \alpha^{-1}(1-F(x)) - o(1 - F(x))$. Fix x such that $0 < \Phi(x) < 1$. Then since $n[1 - F(a_n x + b_n)] \to -\log \Phi(x)$ [5, p. 438] and $F(a_n x + b_n) \to 1$ as $n \to \infty$ we have $a_n x + b_n \to x_0-$. Therefore we can write $G^n(a_n x + b_n) =$

$\{1 - \alpha^{-1}(1 - F(a_n x + b_n)) - o(1 - F(a_n x + b_n))\}^n$ as $n \to \infty$.

Set $y_n = F(a_n x + b_n)$. We first show that $o(1 - F(a_n x + b_n))$ can be neglected by showing that $d_n = |\{1 - \alpha^{-1}(1-y_n) - o(1-y_n)\}^n - \{1 - \alpha^{-1}(1-y_n)\}^n| \to 0$ as $n \to \infty$. Partition the positivd integers into sets N_1 and N_2 such that for $n_1 \in N_1$, $o(1-y_{n_1}) \geq 0$ and for $n_2 \in N_2$ $-o(1-y_{n_2}) \geq 0$. If either set is finite it can be neglected. Otherwise

$d_{n_1} \leq n_1[1 - \alpha^{-1}(1-y_{n_1})]^{n_1-1} o(1-y_{n_1})$ which follows from the inequality $(t - a)^n \geq t^n - nt^{n-1} a$ for $t \geq 0, a \geq 0$. Given any $\epsilon > 0$, for n_1 sufficiently large we have

$o(1-y_{n_1}) \leq \epsilon(1-y_{n_1})$ so that

$$d_{n_1} \leq \epsilon[1 - \alpha^{-1}(1-y_{n_1})]^{n_1} n_1(1-y_{n_1})$$

$$= \epsilon[1 - \frac{\alpha^{-1}}{n_1} n_1(1-y_{n_1})]^{n_1} n_1(1-y_{n_1})$$

$$\to \epsilon\, e^{\alpha^{-1}\log\Phi(x)} (-\log \Phi(x)) .$$

ϵ can be chosen arbitrarily small so that we must have

$$\lim_{n_1 \to \infty} d_{n_1} = 0 .$$

The procedure for $n_2 \in N_2$ is similar: For any ϵ, if n_2 is sufficiently large: $|o(1-y_{n_2})| = -o(1-y_{n_2}) \leq \epsilon(1-y_{n_2})$.

Then:

$$d_{n_2} \leq (1 - \alpha^{-1}(1-y_{n_2}) + \epsilon(1-y_{n_2}))^{n_2} - (1 - \alpha^{-1}(1-y_{n_2}))^{n_2}$$

$$= (1 - \frac{(\alpha^{-1} - \epsilon)}{n_2} n_2(1-y_{n_2}))^{n_2} - (1 - \frac{\alpha^{-1}}{n_2} n_2(1-y_{n_2}))^{n_2}$$

$$\to e^{(\alpha^{-1}-\epsilon)\log\Phi(x)} - e^{\alpha^{-1}\log\Phi(x)}$$

$$\leq \Phi^{-\epsilon}(x) - 1 .$$

Since ϵ can be chosen arbitrarily small, we must have $\lim_{n_2\to\infty} d_{n_2} = 0$. This terminates the proof that $\lim_{n\to\infty} d_n = 0$ and shows that $o(1 - F(a_n x + b_n))$ can be neglected.

Therefore for all x such that $1 > \Phi(x) > 0$:

$$\lim_{n\to\infty} G^n(a_n x + b_n) = \lim_{n\to\infty} (1 - \alpha^{-1}(1 - F(a_n x + b_n)))^n$$

$$= \lim_{n\to\infty} (1 - (\frac{\alpha^{-1}}{n})n (1 - F(a_n x + b_n)))^n$$

$$= e^{\alpha^{-1}\log\Phi(x)} = \Phi^{\frac{1}{\alpha}}(x) .$$

If there are x such that $\Phi(x) = 0$ we proceed as follows: $\Phi(x)$ is an extreme value distribution and therefore is continuous. Hence for any $\epsilon > 0$ there exists z such that $0 < \Phi^{\frac{1}{\alpha}}(z) < \epsilon$. Then:

$$0 \leq \overline{\lim_{n\to\infty}} G^n(a_n x + b_n) \leq \overline{\lim_{n\to\infty}} G^n(a_n z + b_n) = \Phi^{\frac{1}{\alpha}}(z) < \epsilon .$$

Since ϵ can be chosen arbitrarily small $\lim_{n\to\infty} G^n(an\ x + bn) = 0$.

A similar procedure works if there are x such that $\Phi(x) = 1$ so that the proof is complete.

The converse will be given in Theorem (5.26).

An extreme value distribution raised to a positive power, is an extreme value distribution of the same type. From (1.2, 1.3, 1.4) we have for all x and $\gamma > 0$:

(5.4) $$\Lambda(x)^\gamma = \Lambda(x - \log \gamma)$$

(5.5) $$\Phi_\alpha(x)^\gamma = \Phi_\alpha(\gamma^{-\frac{1}{\alpha}} x)$$

(5.6) $$\Psi_\alpha(x)^\gamma = \Psi_\alpha(\gamma^{\frac{1}{\alpha}} x) \quad .$$

As an application of theorem (5.3) we prove the following particularization of a result by Barndorff-Nielson [1] .

<u>Proposition 5.7</u>: $\{X_n, n \geq 1\}$ is a sequence of i.i.d. random variables with distribution function $F(\cdot)$. $\{\tau_n, n \geq 1\}$ is a sequence of positive integervalued i.i.d. random variables with density $p_k = P[\tau_1 = k]$, $k \geq 1$. $\{\tau_n\}$ and $\{X_n\}$ are independent of each other, $\sum_{k=1}^{\infty} p_k = 1$, $E\tau_1 = \sum_{k=1}^{\infty} k\ p_k$, and $S_n = \sum_{j=1}^{n} \tau_j$. Set

$\chi_n = \max \{X_{S_{n-1}+1}, \ldots, X_{S_n}\}$ and $\overline{M_n} = \max \{\chi_1, \ldots, \chi_n\}$.

$\{\overline{M_n}\}$ has a limiting distribution iff $\{M_n\}$ has.

There exist norming constants $a_n > 0$, b_n , $n \geq 1$, such that

$$F^n(a_n\ x + b_n) = P[M_n \leq a_n\ x + b_n] \to \Phi(x)$$

with $\Phi(x)$ nondegenerate, iff

$$P[\overline{M}_n \le a_n x + b_n] \to [\Phi(x)]^{E\tau_1}$$

Proof: The χ_n's are i.i.d. and

$$P[\chi_n \le x] = \sum_{k=1}^{\infty} P[\chi_n \le x\ ,\ \ \tau_n = k]$$

$$= \sum_{k=1}^{\infty} P[\max\{X_{S_{n-1}+1}, \dots, X_{S_{n-1}+k}\} \le x]p_k$$

$$= \sum_{k=1}^{\infty} F^k(x)\ p_k\ .$$

We have:

$$\lim_{x \to x_0-} \frac{1 - P[X_1 \le x]}{1 - F(x)} = \lim_{x \to x_0-} \sum_{k=1}^{\infty} \frac{p_k(1 - F^k(x))}{1-F(x)}$$

$$= \lim_{x \to x_0-} \frac{\sum_{k=1}^{\infty} p_k(1 - F(x))(\sum_{j=0}^{k-1} F^j(x))}{1 - F(x)}$$

$$= \sum_{k=1}^{\infty} k\ p_k = E\ \tau_1\ .$$

The conclusions of the proposition follow from Theorem (5.3).

Theorem 5.8: The following are tail equivalent:

(i) $\rho(x)$ and $\sum_{i=1}^{m} \pi_i H_i(x)$

(ii) $\sum_{i=1}^{m} \pi_i H_i(x)$ and $\prod_{i=1}^{m} H_i^{\pi_i}(x)$.

Since this property is an equivalence relation, $\rho(x)$ and $\prod_{i=1}^{m} H_i^{\pi_i}(x)$ are also tail equivalent .

<u>Proof</u>: (i) Let $x_0 = x_0^\rho$. We have $\sum_{i=1}^{m} \ell_i(x) = 1$, $\lim_{x\to x_0-} \ell_i(x) = \pi_i$ and from (2.23) $\rho(x) = \sum_{i=1}^{m} \ell_i(x) H_i(x)$.

For any ϵ , $0 < \epsilon < \min_{1\le j\le m} \pi_j$, there exists a real number $M_\epsilon < x_0$ such that $x > M_\epsilon$ implies $|\ell_i(x) - \pi_i| < \epsilon$ for $i = 1,\ldots, m$. For such x , $0 < \pi_i - \epsilon \le \ell_i(x) \le \pi_i + \epsilon$ so that for $x_0 > x > M_\epsilon$ we have

$$\frac{1 - \rho(x)}{1 - \sum_{i=1}^{m} \pi_i H_i(x)} = \frac{\sum_{j=1}^{m} \ell_j(x)(1 - H_j(x))}{\sum_{j=1}^{m} \pi_j(1 - H_j(x))}$$

$$\le \frac{\sum_{j=1}^{m} (\pi_j + \epsilon)(1 - H_j(x))}{\sum_{j=1}^{m} \pi_j(1 - H_j(x))}$$

$$= 1 + \epsilon \frac{\sum_{j=1}^{m} (1 - H_j(x).}{\sum_{j=1}^{m} \pi_j(1 - H_j(x).}$$

$$\le 1 + \epsilon(\min_j \pi_j)^{-1} \quad .$$

Similarly we can get a reverse inequality so that for

$x_0 > x > M_\epsilon$ we have

$$1 - \epsilon(\max_j \pi_j)^{-1} \le \frac{1 - \rho(x)}{1 - \sum_{i=1}^{m} \pi_i H_i(x)} \le 1 + \epsilon(\min_j \pi_j)^{-1}$$

and since ϵ is arbitrary, we have

$$\lim_{x \to x_0 -} \frac{1 - \rho(x)}{1 - \sum_{i=1}^{m} \pi_i H_i(x)} = 1 .$$

(ii) From (2.5), (2.6) we have

$$x_0 = \max\{x_0^{H_1}, \ldots, x_0^{H_m}\} .$$

We use the following:

(5.9) $$1 - z \sim \log z \qquad \text{as} \quad z \to 1-$$

Then $$\lim_{x \to x_0 -} \frac{1 - \prod_{i=1}^{m} H_i^{\pi_i}(x)}{1 - \sum_{i=1}^{m} \pi_i H_i(x)} = \lim_{x \to x_0 -} \frac{-\log \prod_{i=1}^{m} H_i^{\pi_i}(x)}{\sum_{i=1}^{m} \pi_i (1 - H_i(x))}$$

$$= \lim_{x \to x_0 -} \frac{-\sum_{i=1}^{m} \pi_i \log H_i(x)}{\sum_{i=1}^{m} \pi_i (1 - H_i(x))}$$

$$= 1 .$$

Theorems (5.3) and (5.8) explain why there are three equivalent conditions which are necessary and sufficient for the L.L.N. (3.6); likewise for relative stability (3.20). They also explain why $\{M_n\}$ has a limiting distribution iff $\rho(x)$ OR $\prod_{i=1}^{m} H_i^{\pi_i}(x)$ is in the domain of attraction of an extreme value distribution (4.13).

Using the dissection principle [2, p. 83] and Theorem (5.3) we achieve the obvious extension of Proposition (5.7) and the natural reduction to the i.i.d. case of the limit law problem for maxima of random variables defined on a M.C.

Pick an arbitrary state j and suppose τ_0 is the time of the first visit to state j and $\{\tau_n , \; n \geq 1\}$ the waitingtimes between visits to j. $\{\tau_n , \; n \geq 1\}$ is an i.i.d. sequence and $S_n = \sum_{k=0}^{n} \tau_k$ is the time of the $(n + 1)$st visit to state j. Let $N_i^{(n)}$ be the number of visits to state i which occur between the nth and $(n + 1)$st visits to state j, $i = 1, \ldots, m$; i.e. the number of times $J_k = i$, $k = S_{n-1} + 1, \ldots, S_n$. Then:

$$\sum_{i=1}^{m} N_i^{(n)} = \tau_n , \qquad n \geq 0$$

$$EN_i^{(n)} = \pi_i \, \pi_j^{-1} \qquad n \geq 1 .$$

Set $\chi_0 = \max \{X_1, \ldots, X_{\tau_0+1}\}, \ldots, \; \chi_n = \max \{X_{S_{n-1}+2}, \ldots, X_{S_n+1}\}$.

We calculate the distribution of χ_n :

$$P[X_n \leq x \mid N_\nu^{(n)} = n_\nu\ , \quad \nu = 1, \ldots, m]$$
$$= \prod_{\nu=1}^{m} H_\nu^{n_\nu}(x) \quad .$$

Therefore:

$$\sum_{n_1=0}^{\infty} \cdots \sum_{n_m=0}^{\infty} P[X_n \leq x\ , \quad N_\nu^{(n)} = n_\nu\ , \quad \nu = 1, \ldots, m]$$

$$= \sum_{n_1=0}^{\infty} \cdots \sum_{n_m=0}^{\infty} \prod_{\nu=1}^{m} H_\nu^{n_\nu}(x)\ P[N_\nu^{(n)} = n_\nu\ , \ \nu = 1, \ldots, m]$$

and so:

$$P[X_n \leq x] = E \prod_{i=1}^{m} H_i^{N_i^{(n)}}(x)$$

where "E" is mathematical expectation. For $n \geq 1$, the distribution of X_n is independent of n . (Alternative formulations using taboo probabilities are available. They are not used and hence we omit them.)

<u>Theorem 5.10:</u> The distribution function $E \prod_{i=1}^{m} H_i^{N_i^{(n)}}(x)$ is in the domain of attraction of an extreme value distribution $\Phi(x)$ iff $\prod_{i=1}^{m} H_i^{\pi_i}(x)$ (equivalently $\rho(x)$) is in the domain of attraction of an extreme value distribution of the same type. There exist normalizing constants $a_n > 0$, b_n , $n \geq 1$ such that

$$P[M_n \leq a_n x + b_n] \to \Phi(x)$$

iff

$$P[\max\{X_0, X_1, \ldots, X_n\} \le a_n x + b_n] \to \Phi(x)^{\frac{1}{\pi_j}} = \Phi(x)^{E\tau_1} .$$

Proof: As previously, $x_0 = \max\{x_o^{H_1}, \ldots, x_0^{H_m}\}$. We have that:

$$\frac{1 - E\prod_{\nu=1}^{m} H_\nu^{N_\nu^{(1)}}(x)}{1 - \prod_{\nu=1}^{m} H_\nu^{\pi_\nu}(x)} = \frac{E[-\log \prod_{\nu=1}^{m} H_\nu^{N_\nu^{(1)}}(x) - \sum_{k=2}^{\infty} k^{-1}(1 - \prod_{\nu=1}^{m} H_\nu^{N_\nu^{(1)}}(x))^k]}{1 - \prod_{\nu=1}^{m} H_\nu^{\pi_\nu}(x)}$$

$$= \frac{-\pi_j^{-1} \sum_{\nu=1}^{m} \pi_\nu \log H_\nu(x) - E\sum_{k=2}^{\infty} k^{-1}(1 - \prod_{\nu=1}^{m} H_\nu^{N_\nu^{(1)}}(x))^k}{1 - \prod_{\nu=1}^{m} H_\nu^{\pi_\nu}(x)}$$

so that by (5.9) :

$$\lim_{x \to x_0-} \frac{1 - E\prod_{\nu=1}^{m} H_\nu^{N_\nu^{(1)}}(x)}{1 - \prod_{\nu=1}^{m} H_\nu^{\pi_\nu}(x)}$$

$$= \frac{1}{\pi_j} - \lim_{x \to x_0-} \frac{E\sum_{k=2}^{\infty} k^{-1}(1 - \prod_{\nu=1}^{m} H_\nu^{N_\nu^{(1)}}(x))^k}{1 - \prod_{\nu=1}^{m} H_\nu^{\pi_\nu}(x)} .$$

To show that the last term is zero observe that

$$\frac{\sum_{k=2}^{\infty} k^{-1}\left(1 - \prod_{\nu=1}^{m} H_\nu^{N_\nu^{(1)}}(x)\right)^k}{1 - \prod_{\nu=1}^{m} H_\nu^{\pi_\nu}(x)} \leq \frac{-\log \prod_{\nu=1}^{m} H_\nu^{N_\nu^{(1)}}(x)}{1 - \prod_{\nu=1}^{m} H_\nu^{\pi_\nu}(x)}$$

$$\leq \tau_1 \left(\frac{-\log \prod_{\nu=1}^{m} H_\nu(x)}{1 - \prod_{\nu=1}^{m} H_\nu^{\pi_\nu}(x)} \right) = \tau_1\, O(1)$$

so that

$$\frac{\sum_{k=2}^{\infty} k^{-1}\left(1 - \prod_{\nu=1}^{m} H_\nu^{N_\nu^{(1)}}(x)\right)^k}{1 - \prod_{\nu=1}^{m} H_\nu^{\pi_\nu}(x)} \quad \text{is bounded by}$$

an integrable function and we can apply Fatou's Lemma:

$$0 \leq \overline{\lim_{x \to x_0^-}} \; E \; \frac{\sum_{k=2}^{\infty} k^{-1}\left(1 - \prod_{\nu=1}^{m} H_\nu^{N_\nu^{(1)}}(x)\right)^k}{1 - \prod_{\nu=1}^{m} H_\nu^{\pi_\nu}(x)}$$

$$\leq E \; \overline{\lim_{x \to x_0^-}} \; \frac{\sum_{k=2}^{\infty} k^{-1}\left(1 - \prod_{\nu=1}^{m} H_\nu^{N_\nu^{(1)}}(x)\right)^k}{1 - \prod_{\nu=1}^{m} H_\nu^{\pi_\nu}(x)} = 0$$

since

$$0 \leq \overline{\lim_{x\to x_0^-}} \frac{\sum_{k=2}^{\infty} k^{-1}\left(1 - \prod_{\nu=1}^{m} H_\nu^{N_\nu^{(1)}}(x)\right)^k}{1 - \prod_{\nu=1}^{m} H_\nu^{\pi_\nu}(x)}$$

$$\leq \overline{\lim_{x\to x_0^-}} \frac{\left(1 - \left(\prod_{\nu=1}^{m} H_\nu(x)\right)^{\tau_1}\right)^2}{\left(\prod_{\nu=1}^{m} H_\nu(x)\right)^{\tau_1}\left(1 - \prod_{\nu=1}^{m} H_\nu^{\pi_\nu}(x)\right)}$$

$$= \overline{\lim_{x\to x_0^-}} \frac{\left(1 - \prod_{\nu=1}^{m} H_\nu(x)\right)^2 \left[\sum_{\ell=0}^{\tau_1 - 1}\left(\prod_{\nu=1}^{m} H_\nu(x)\right)^{\ell}\right]^2}{\left(1 - \prod_{\nu=1}^{m} H_\nu^{\pi_\nu}(x)\right)\left(\prod_{\nu=1}^{m} H_\nu(x)\right)^{\tau_1}}$$

$$< \lim_{x\to x_0^-} \frac{\left(1 - \prod_{\nu=1}^{m} H_\nu(x)\right)^2 \tau_1^2}{\left(1 - \prod_{\nu=1}^{m} H_\nu^{\pi_\nu}(x)\right)\left(\prod_{\nu=1}^{m} H_\nu(x)\right)^2}$$

$$= 0 \quad \text{a.s.}$$

Therefore we have that

$$\lim_{x\to x_0^-} \frac{1 - E\prod_{\nu=1}^{m} H_\nu^{N_\nu^{(1)}}(x)}{1 - \prod_{\nu=1}^{m} H_\nu^{\pi_\nu}(x)} = \frac{1}{\pi_j}$$

and an application of Theorem (5.3) gives

$$\left[\prod_{i=1}^{m} H_i^{\pi_i}(a_n x + b_n)\right]^n \to \Phi(x)$$

iff

$$\left[E\prod_{i=1}^{m} H_i^{N_i^{(1)}}(a_n x + b_n)\right]^n \to \Phi(x)^{\frac{1}{\pi_j}} .$$

For all x such that $0 < \Phi(x) < 1$ $a_n x + b_n \to x_0-$. For such x

$$P[X_0 \leq a_n x + b_n] =$$

$$\sum_{k=0}^{\infty} \{ \sum_{j_0 \neq j} \sum_{j_1 \neq j} \cdots \sum_{j_k \neq j} Q_{j_0 j_1}(a_n x + b_n) \cdots Q_{j_{k-1} j}(a_n x + b_n)\}$$

$$H_j(a_n x + b_n) \to 1$$

as $n \to \infty$. Hence for all x

$$\{\prod_{i=1}^{m} H_i^{\pi_i} (a_n x + b_n)\}^n \to \Phi(x)$$

iff

$$P[\max\{X_0, \ldots, X_n\} \leq a_n x + b_n]$$
$$= P[X_0 \leq a_n x + b_n] \{E \prod_{i=1}^{m} H_i^{N_i^{(i)}} (a_n x + b_n)\}^n$$
$$\to \Phi(x)^{\frac{1}{\pi_j}} .$$

<u>Convention:</u> If $\Phi(x)$ is an extreme value distribution and $F(x)$ is a distribution function in the domain of attraction of $\Phi(x)$, we write:

$$F(x) \in \Phi(x) .$$

If $F(x) \in \Phi(x)$, there exist $a_n > 0$, b_n, $n \geq 1$, such that $F^n(a_n x + b_n) \to \Phi(x)$. Gnedenko [5] has shown that $\{a_n\}$ and $\{b_n\}$ can always be chosen in a precise way. Any other choice of normalizing constants is asymptotically equivalent to this choice which we will tabulate.

For this purpose, set $F^{-1}(x) = \inf\{y|F(y) = x\}$ for any distribution function $F(\cdot)$ and put $\mu_n = F^{-1}(1 - \frac{1}{n})$. If in a discussion more than one distribution function are involved we write $\mu_n^F = F^{-1}(1 - \frac{1}{n})$. Observe that $\mu_n < \mu_{n+1}$. Clearly $\mu_n \leq \mu_{n+1}$ and if $\mu_n = \mu_{n+1}$ were true we could set $A = \{y|F(y) = 1 - \frac{1}{n}\}$, $B = \{y|F(y) = 1 - \frac{1}{n+1}\}$ so that $\mu_n = \inf A = \mu_{n+1} = \inf B$. Because of right continuity $\mu_n \in A \cap B$ and so $F(\mu_n) = 1 - \frac{1}{n+1}$ which gives a contradiction and shows that strict inequality must hold.

The extreme value distributions $\Phi_\alpha(x)$, $\Psi_\alpha(x)$, $\Lambda(x)$ were given in (1.2), (1.3), (1.4).

Gnedenko's results:

(5.11) If $F(x) \in \Phi_\alpha(x)$, then $F(x) < 1$ for all x,

and $\lim_{x\to\infty} \frac{1 - F(x)}{1 - F(kx)} = k^\alpha$ for all $k > 0$.

We can set $a_n = \mu_n$ and $b_n = 0$.

(5.12) If $F(x) \in \Psi_\alpha(x)$, then $x_0 < \infty$. Also

$\lim_{x\to 0-} \frac{1 - F(kx + x_o)}{1 - F(x + x_o)} = k^\alpha$ for all $k > 0$ and we can set

$a_n = x_0 - \mu_n$, $b_n = x_0$.

(5.13). If $F(x) \in \Lambda(x)$, then we can set $b_n = \mu_n$

and $a_n = F^{-1}(1 - \frac{1}{ne}) - \mu_n$.

For an extreme value distribution $\Phi(x)$ let $\mathcal{F}_\Phi = \{F|F \in \Phi\}$ be its domain of attraction. Partition $\mathcal{F}_\Phi$ by the equivalence relation: $F(\cdot)$ is equivalent to $G(\cdot)$ iff $\lim_{x \to x_0-} \frac{1 - F(x)}{1 - G(x)} = \alpha$, $0 < \alpha < \infty$. Suppose $\mathcal{N}_\Phi$ is the class of all sets of normalizing constants $(a_n, b_n, n \geq 1)$ such that for some $F \in \Phi$ and some constants $C > 0$, D: $F^n(a_n x + b_n) \to \Phi(C x + D)$. Partition N_Φ by the equivalence relation: (a_n, b_n) is equivalent to (α_n, β_n) iff $\alpha_n a_n^{-1} \to A > 0$, $a_n^{-1}(\beta_n - b_n) \to B$ (cf. 2.1). The partitions of $\mathcal{N}_\Phi$ and $\mathcal{F}_\Phi$ are in 1-1 correspondence as shown by the following lemmas which lead up to Theorem (5.26), the promised converse of Theorem (5.3).

<u>Lemma 5.14</u>: $F(\cdot)$ and $G(\cdot)$ are distribution functions. Suppose for normalizing constants $a_n > 0$, b_n, $n \geq 1$, $F^n(a_n x + b_n) \to \Phi_\alpha(x)$. Then $G^n(a_n x + b_n) \to \Phi_\alpha(Ax + B)$ and $A > 0$

iff $B = 0$ and

$$\lim_{x \to \infty} \frac{1 - F(x)}{1 - G(x)} = A^\alpha .$$

<u>Proof</u>: If $\lim_{x \to \infty} \frac{1 - F(x)}{1 - G(x)} = A^\alpha$, then by Theorem (5.3) we have that

$$G^n(a_n x + b_n) \to \{\Phi_\alpha(x)\}^{A^{-\alpha}} = \Phi_\alpha(Ax) \quad \text{from (5.5)} .$$

For the converse, we can let $a_n = \mu_n^F$, $b_n = 0$ so that we are given that $G^n(\frac{\mu_n^F}{A} x - \frac{\mu_n^F}{A} B) \to \Phi_\alpha(x)$ as $n \to \infty$. But since $G(x) \in \Phi_\alpha(x)$ we have that $G^n(\mu_n^G x) \to \Phi_\alpha(x)$ and therefore by (2.1)

$$\frac{\mu_n^G}{\mu_n^F} \to \frac{1}{A} \tag{5.15}$$

$$\frac{-\mu_n^F A^{-1} B}{\mu_n^G} \to 0 \tag{5.16}$$

as $n \to \infty$. Since $A > 0$, (5.15) and (5.16) can both hold iff $B = 0$.

Given any $\epsilon > 0$, there exists because of (5.15) an integer N_ϵ such that for $n > N_\epsilon$ we have

$$\left|\frac{\mu_n^G}{\mu_n^F} - \frac{1}{A}\right| < \epsilon ;$$

i.e.
$$\mu_n^F(A^{-1} - \epsilon) < \mu_n^G < (A^{-1} + \epsilon)\, \mu_n^F .$$

Since $\mu_n^G < \mu_{n+1}^G \to \infty$, we have that for every x **sufficiently** large there exists an integer $n = n(x)$ such that

$x \in [\mu_n^G , \mu_{n+1}^G]$. Then

$$\frac{1 - F(x)}{1 - G(x)} \le \frac{1 - F(\mu_n^G)}{1 - G(\mu_{n+1}^G)}$$

$$\le [1 - F(\mu_n^F(A^{-1} - \epsilon))]\,(n + 1)$$

$$= \frac{1 - F(\mu_n^F(A^{-1} - \epsilon))}{1 - F(\mu_n^F)}\, [(n+1)(1 - F(\mu_n^F))]$$

$$\to (A^{-1} - \epsilon)^{-\alpha} \text{ as } n \to \infty \text{ by (5.11) .}$$

Therefore $\overline{\lim_{x\to\infty}} \frac{1 - F(x)}{1 - G(x)} \leq \frac{1}{(A^{-1} - \epsilon)^{\alpha}}$.

Similarly $\underline{\lim_{x\to\infty}} \frac{1-F(x)}{1-G(x)} \geq \frac{1}{(A^{-1} + \epsilon)^{\alpha}}$

and since ϵ is arbitrary

$$\lim_{x\to\infty} \frac{1 - F(x)}{1 - G(x)} = A^{\alpha} .$$

Lemma 5.17: $F(\cdot)$ and $G(\cdot)$ are distribution functions and $(a_n > 0 , \; b_n , \; n \geq 1)$ are normalizing constants such that $F^n(a_n x + b_n) \to \Psi_\alpha(x)$. Then $G^n(a_n x + b_n) \to \Psi_\alpha(Ax + B)$ and $A > 0$

$$\text{iff} \qquad B = 0$$

$$x_0^F = x_0^G = x_0 \quad \text{and}$$

$$\lim_{x\to x_0-} \frac{1 - F(x)}{1 - G(x)} = A^{-\alpha} .$$

Proof: If $\lim_{x\to x_0-} \frac{1 - F(x)}{1 - G(x)} = A^{-\alpha}$ then by (5.3) we have $G^n(a_n x + b_n) \to \{\Psi_\alpha(x)\}^{A^\alpha} = \Psi_\alpha(Ax)$ by (5.6) .

For the converse we can suppose $a_n = x_0^F - \mu_n^F$, $b_n = x_0^F$, so that we are given that

$$G^n((x_0^F - \mu_n^F) A^{-1} x + x_0^F - (x_0^F - \mu_n^F) A^{-1} B) \to \Psi_\alpha(x) .$$

This means that $G(x) \in \Psi_\alpha (x)$. Therefore $x_0^G < \infty$ and $G^n((x_0^G - \mu_n^G) x + x_0^G) \to \Psi_\alpha(x)$.

By (2.1)

(5.18) $$\frac{x_0^F - \mu_n^F}{x_0^G - \mu_n^G} \to A \quad \text{and}$$

$$(5.19) \qquad \frac{x_0^G - (x_0^F - (x_0^F - \mu_n^F)\frac{B}{A})}{x_0^G - \mu_n^G} \to 0$$

as $n \to \infty$. Combining (5.18) and (5.19) we have that

$$\frac{x_0^G - x_0^F}{x_0^G - \mu_n^G} + B \to 0 \quad \text{and since} \quad x_0^G - \mu_n^G \to 0 , \quad \text{we have} \quad x_0^G = x_0^F = x_0$$

and $B = 0$.

From (5.18) for any $\epsilon > 0$, there exists N_ϵ such that for $n > N_\epsilon$, $\left|\frac{x_0 - \mu_n^G}{x_0 - \mu_n^F} - A^{-1}\right| < \epsilon$; i.e. :

$$(5.20) \qquad x_0 - (A^{-1} + \epsilon)(x_0 - \mu_n^F) < \mu_n^G < x_0 - (A^{-1} - \epsilon)(x_0 - \mu_n^F) .$$

For any $x < x_0$ but sufficiently close to x_0, there exists an integer $n = n(x)$ such that $x \in [\mu_n^G, \mu_{n+1}^G]$. Then

$$\frac{1 - F(x)}{1 - G(x)} \leq \frac{1 - F(\mu_n^G)}{1 - G(\mu_{n+1}^F)}$$

$$\leq (n + 1) \{1 - F(x_0 - (A^{-1} + \epsilon)(x_0 - \mu_n^F))\}$$

$$= (n + 1) \{1 - F(a_n[-(\epsilon + A^{-1})] + b_n)\}$$

$$\to - \log \Psi_\alpha(x)\Big|_{x = -(A^{-1} + \epsilon)}$$

$$= (A^{-1} + \epsilon)^\alpha \quad .$$

So $\overline{\lim}_{x \to x_0^-} \frac{1 - F(x)}{1 - G(x)} \leq (A^{-1} + \epsilon)^{\alpha}$. Similarly

$\underline{\lim}_{x \to x_0^-} \frac{1 - F(x)}{1 - G(x)} \geq (A^{-1} - \epsilon)^{\alpha}$ and since ϵ is arbitrary

$$\lim_{x \to x_0^-} \frac{1 - F(x)}{1 - G(x)} = A^{-\alpha} \quad .$$

Corollary 5.21: Let $F(\cdot)$ and $G(\cdot)$ be distribution functions.

(i) If $F^n(\mu_n^F x) \to \Phi_\alpha(x)$ and $\mu_n^G/\mu_n^F \to A^{-1}$ then

$\lim_{x \to \infty} \frac{1 - F(x)}{1 - G(x)} = A^{\alpha}$.

(ii) If $x_0^F = x_0^G < \infty$ and if $F^n((x_0 - \mu_n^F) x + x_0)$ $\to \Psi_\alpha(x)$, and $\lim_{n \to \infty} \frac{x_0 - \mu_n^G}{x_0 - \mu_n^F} = A^{-1}$, then $\lim_{x \to x_0^-} \frac{1 - F(x)}{1 - G(x)} = A^{-\alpha}$.

Lemma 5.22: $F(\cdot)$ and $G(\cdot)$ are distribution functions. Suppose for normalizing constants $a_n > 0$, b_n , $n \geq 1$, $F^n(a_n x + b_n) \to \Phi_\alpha(x)$. If $G^n(a_n x + b_n) \to \Phi(x)$, $\Phi(x)$ nondegenerate, then $\Phi(x) = \Phi_\alpha(Ax)$ for some $A > 0$ and $\lim_{x \to \infty} \frac{1 - F(x)}{1 - G(x)} = A^{\alpha}$.

Proof: We have $F(x) < 1$ for all x . Without loss of generality suppose $a_n = \mu_n^F$ and $b_n = 0$. Then $G^n(\mu_n^F x)$ $\to \Phi(x)$ and since $\mu_n^F \uparrow \infty$ we have $\Phi(x) = 0$ for all $x < 0$.

The only possibility is that $\Phi(x) = \Phi_\beta(Ax)$ for some $\beta > 0$.

To show $\beta = \alpha$: We have $G^n(A^{-1} \mu_n^F x) \to \Phi_\beta(x)$ and $G^n(\mu_n^G x) \to \Phi_\beta(x)$ so that by (2.1) $\mu_n^G/\mu_n^F \to A^{-1}$.

By Corollary (5.21)

$\lim_{x\to\infty} \frac{1 - F(x)}{1 - G(x)} = A^{\alpha}$. But

$$n\{1 - F(\mu_n^F x)\} \to -\log \Phi_\alpha(x) = x^{-\alpha}$$

and

$$n\{1 - G(\mu_n^F x)\} \to -\log \Phi_\beta(Ax) = (Ax)^{-\beta}$$

as $n \to \infty$. Dividing gives:

$$\lim_{n\to\infty} \frac{1 - F(\mu_n^F x)}{1 - G(\mu_n^F x)} = A^{\beta} x^{\beta-\alpha} .$$

Since $\lim_{x\to\infty} \frac{1 - F(x)}{1 - G(x)} = A^{\alpha}$ and $\mu_n^F \to \infty$ we must have $\beta = \alpha$ which completes the proof.

<u>Lemma 5.23</u>: $F(\cdot)$ and $G(\cdot)$ are distribution functions. Suppose for normalizing constants $a_n > 0$, b_n , $n \geq 1$, $F^n(a_n x + b_n) \to \Psi_\alpha(x)$. If $G^n(a_n x + b_n) \to \Phi(x)$, $\Phi(x)$ nondegenerate, then $\Phi(x) = \Psi_\alpha(Ax)$ for some $A > 0$, $x_0^F = x_0^G = x_0$, and $\lim_{x\to x_0-} \frac{1 - F(x)}{1 - G(x)} = A^{-\alpha}$.

<u>Proof</u>: Since $F(x) \in \Psi_\alpha(x)$, $x_0^F < \infty$. Without loss of generality suppose that $a_n = x_0^F - \mu_n^F$ and $b_n = x_0^F$. We show that $x_0^F = x_0^G$. Suppose $G(x) < 1$ for all x . For

any x there exists a positive integer N_x and a real number M_x such that for $n > N_x$, $(x_0^F - \mu_n^F)\, x + x_0^F \leq M_x$

and therefore $G^n((x_0^F - \mu_n^F)\, x + x_0^F) \leq G^n(M_x) \to 0$ as $n \to \infty$. If $G(x) < 1$ for all x, then $\Phi(x) \equiv 0$ so that we must have $x_0^G < \infty$.

Similar arguments show that $x_0^G = x_0^F$: If $x_0^G > x_0^F$, then there exists $\epsilon > 0$ such that $x_0^G - \epsilon > x_0^F$ so that for any fixed x, if n is sufficiently large $(x_0^F - \mu_n^F)\, x + x_0^F < x_0^G - \epsilon$. Consequently, $G^n((x_0^F - \mu_n^F)\, x + x_0^F) \leq G^n(x_0^G - \epsilon) \to 0$. So we must have $x_0^G \leq x_0^F$. If strict inequality holds then for any fixed x, if n is sufficiently large we have that $(x_0^F - \mu_n^F)\, x + x_0^F > x_0^G$. For such n $G^n((x_0^F - \mu_n^F)\, x + x_0^F) = 1$ and therefore $\Phi(x) \equiv 1$. This shows that $x_0^F = x_0^G$.

For $x > 0$, $G^n((x_0 - \mu_n^F)\, x + x_0) = 1$ for all n so that $\Phi(x) = 1$, $x > 0$. We can only have $\Phi(x) = \Psi_\beta(Ax)$ for $A > 0$, $\beta > 0$.

To show $\beta = \alpha$: We have $G^n((x_0 - \mu_n^F)\, A^{-1} x + x_0) \to \Psi_\beta(x)$, $G^n((x_0 - \mu_n^G)\, x + x_0) \to \Psi_\beta(x)$ so that by (2.1) $\lim\limits_{n\to\infty} \dfrac{x_0 - \mu_n^G}{x_0 - \mu_n^F} = A^{-1}$. By Corollary (5.21-ii) $\lim\limits_{x\to x_0-} \dfrac{1 - F(x)}{1 - G(x)} = A^{-\alpha}$.

Also $n\{1 - F((x_0 - \mu_n^F) x + x_0)\} \to (-x)^\alpha$ and

$n\{1 - G((x_0 - \mu_n^F) x + x_0)\} \to (-Ax)^\beta$ so that

$$\lim_{n\to\infty} \frac{1 - F((x_0 - \mu_n^F) x + x_0)}{1 - G((x_0 - \mu_n^F) x + x_0)} = \frac{x^{\alpha-\beta}}{A^\beta} .$$

Since $\lim_{x\to x_0-} \frac{1 - F(x)}{1 - G(x)} = A^{-\alpha}$ we must have for all $x < 0$

that $A^{-\beta} x^{\alpha-\beta} = A^{-\alpha}$. This requires $\alpha = \beta$ and the proof is complete.

<u>Corollary 5.24</u>: Let $F(\cdot)$, $G(\cdot)$ be distribution functions. Suppose there exist $a_n > 0$, b_n, $n \geq 1$ such that $F^n(a_n x + b_n) \to \Lambda(x)$. If $G^n(a_n x + b_n) \to \Phi(x)$, $\Phi(x)$ nondegenerate, then $\Phi(x) = \Lambda(Ax + B)$ for some $A > 0$, B.

<u>Proof</u>: Suppose $\Phi(x) = \Phi_\alpha(Ax + B)$ $(\Psi_\alpha(Ax + B))$. Then $G^n(\frac{a_n}{A}x + b_n - a_n \frac{B}{A}) \to \Phi_\alpha(x)$ $(\Psi_\alpha(x))$ and so by the previous lemmas $F^n(\frac{a_n x}{A} + b_n - a_n \frac{B}{A}) \to \Phi_\alpha(A'_x)$ $(\Psi_\alpha(A'_x))$ for some $A' > 0$ which gives a contradiction to the fact that $F(x) \in \Lambda(x)$ since a distribution function can be in the domain of attraction of at most one extreme value distribution.

<u>Lemma 5.25:</u> $F(\cdot)$ and $G(\cdot)$ are distribution functions, and $a_n > 0$, b_n, $n \geq 1$ are normalizing constants such that $F^n(a_n x + b_n) \to \Lambda(x)$. Then $G^n(a_n x + b_n) \to \Lambda(A_x + B)$

and $A > 0$ iff $A = 1$,

$$x_0^F = x_0^G = x_0 \ , \quad \text{and}$$

$$\lim_{x \to x_0 -} \frac{1 - F(x)}{1 - G(x)} = e^B \ .$$

<u>Proof</u>: If $\lim_{x \to x_0 -} \frac{1 - F(x)}{1 - G(x)} = e^B$ then by Theorem (5.3) we have that $G^n(a_n x + b_n) \to \Lambda(x)^{e^{-B}} = \Lambda(x + B)$ by (5.4).

Conversely if $G^n(a_n x + b_n) \to \Lambda(Ax + B)$, we can without loss of generality suppose $b_n = \mu_n^F$ so that

$G^n(\frac{a_n}{A} x + \mu_n^F - a_n \frac{B}{A}) \to \Lambda(x)$. But also

$G^n([G^{-1}(1 - \frac{1}{ne}) - \mu_n^G] x + \mu_n^G) \to \Lambda(x)$ so that by (2.1) we have that

$$\frac{\mu_n^F - a_n \frac{B}{A} - \mu_n^G}{a_{n/A}} \to 0 \quad \text{as} \quad n \to \infty \ ; \quad \text{i.e. :}$$

$$\frac{\mu_n^F - \mu_n^G}{a_n} \to B/A \ .$$

Given ϵ there exists N_ϵ such that for $n > N_\epsilon$:

$$\mu_n^F - (\frac{B}{A} + \epsilon) a_n < \mu_n^G < \mu_n^F - (\frac{B}{A} - \epsilon) a_n \ .$$

For any x sufficiently large, there exists $n = n(\bar{x})$ such that $x \in [\mu_n^G, \mu_{n+1}^G]$. Therefore:

$$\frac{1 - F(x)}{1 - G(x)} \leq \frac{1 - F(\mu_n^G)}{1 - G(\mu_{n+1}^G)}$$

$$\leq (n + 1)[1 - F(a_n[-(\frac{B}{A} + \epsilon) + \mu_n^F)]$$

$$\to \quad e^{-x}\Big|_{x = -(\frac{B}{A} + \epsilon)} = e^{B/A}\, e^{\epsilon} \quad .$$

So $\overline{\lim}_{x \to x_0^G -} \frac{1 - F(x)}{1 - G(x)} \leq e^{B/A}\, e^{\epsilon}$. Similarly

$\underline{\lim}_{x \to x_0^G} \frac{1 - F(x)}{1 - G(x)} \geq e^{B/A}\, e^{-\epsilon}$ and since ϵ is arbitrary we have

$\lim_{x \to x_0^G} \frac{1 - F(x)}{1 - G(x)} = e^{B/A}$. Since this limit is finite and positive we must have $x_0^F = x_0^G$.

To show that $A = 1$, observe that $n\{1 - F(a_n x + b_n)\} \to e^{-x}$ and $n\{1 - G(a_n x + b_n)\} \to e^{-(Ax+B)}$. Therefore

$\lim_{n\to\infty} \frac{1 - F(a_n x + b_n)}{1 - G(a_n x + b_n)} = e^{-x+Ax+B}$. For all x ,

$0 < \Lambda(x) < 1$ so that $a_n x + b_n \to x_0-$. Since

$\lim_{x \to x_0-} \frac{1 - F(x)}{1 - G(x)} = e^{B/A}$ we have that for all x , $-x + Ax + B = B/A$.

If $B = 0$, then $-x + Ax = 0$ requires $A = 1$. If $B \neq 0$, then setting $x = 0$ gives $A = 1$ and the proof is complete.

We have proved:

<u>Theorem 5.26</u>: Let $F(\cdot)$, $G(\cdot)$ be distribution functions and let $\Phi(x)$ be an extreme value distribution. Suppose $F(x) \in \Phi(x)$ and that $F^n(a_n x + b_n) \to \Phi(x)$ for normalizing constants $a_n > 0$, b_n, $n \geq 1$. Then $G^n(a_n x + b_n) \to \Phi'(x)$, $\Phi'(\cdot)$ nondegenerate, iff for some $A > 0$, B:

$$\Phi'(x) = \Phi(Ax + B),$$

$$x_0^F = x_0^G = x_0,$$

$$\lim_{x \to x_0 -} \frac{1 - F(x)}{1 - G(x)} \quad \text{exists}$$

and if

(i) $\Phi(x) = \Phi_\alpha(x)$, then $B = 0$ and $\lim_{x \to \infty} \frac{1 - F(x)}{1 - G(x)} = A^\alpha$

(ii) $\Phi(x) = \Psi_\alpha(x)$, then $B = 0$ and $\lim_{x \to x_0 -} \frac{1 - F(x)}{1 - G(x)} = A^{-\alpha}$

(iii) $\Phi(x) = \Lambda(x)$, then $A = 1$ and $\lim_{x \to x_0 -} \frac{1 - F(x)}{1 - G(x)} = e^B$.

In Chapter II we showed that $\{M_n\}$ had a limit distribution iff the distribution $\prod_{i=1}^{m} H_i^{\pi_i}(x)$ was in the domain of attraction of an extreme value distribution. It is natural to ask domain of attraction questions for products of distribution functions. The following theorem offers some information about this class of problems and is an application of Theorem (5.26).

Theorem 5.27: $F(\cdot)$ and $G(\cdot)$ are distribution functions and $\Phi(\cdot)$ an extreme value distribution. Suppose $F^n(a_n x + b_n) \to \Phi(x)$ for normalizing constants $a_n > 0$, b_n, $n \geq 1$. Then $(F\,G)^n(a_n x + b_n) = F^n(a_n x + b_n)\,G^n(a_n x + b_n) \to \Phi(Ax + B)$

iff (i) $\Phi(x) = \Phi_\alpha(x)$: $B = 0$, $0 < A \leq 1$ and

$$\lim_{x\to\infty} \frac{1 - F(x)}{1 - G(x)} = \frac{1}{A^{-\alpha}-1}$$

(ii) $\Phi(x) = \Psi_\alpha(x)$: $B = 0$, $\infty > A \geq 1$, and

$$\lim_{x\to x_0^F-} \frac{1 - F(x)}{1 - G(x)} = \frac{1}{A^{\alpha}-1}$$

(iii) $\Phi(x) = \Lambda(x)$: $A = 1$, $B < 0$ and

$$\lim_{x\to x_0-} \frac{1 - F(x)}{1 - G(x)} = \frac{1}{e^{-B}-1} .$$

Proof: Sufficiency follows in each case from Theorem (5.3) (5.4, 5.5, 5.6).

Necessity: (i) By Theorem (5.26-i) (replacing $G(x)$ by $F\,G(x)$ we have that $B = 0$ and $\lim_{x\to\infty} \frac{1 - F(x)}{1 - FG(x)} = A^\alpha$.

For $x > 0$, $F^n(a_n x + b_n)\,G^n(a_n x + b_n) \to \Phi_\alpha(Ax)$ and $F^n(a_n x + b_n) \to \Phi_\alpha(x)$, so that, since $F\,G^n(a_n x + b_n) \leq F^n(a_n x + b_n)$, we have $\Phi_\alpha(Ax) \leq \Phi_\alpha(x)$. Therefore,

$Ax \leq x$ and $A \leq 1$. Also for $x > 0$ $G^n(a_n x + b_n)$

$$\rightarrow \frac{\Phi_\alpha(Ax)}{\Phi_\alpha(x)} = \Phi_\alpha\left((A^{-\alpha}-1)^{-\frac{1}{\alpha}} x\right)$$ and by Theorem (5.26-i) we have:

$$\lim_{x\to\infty} \frac{1 - F(x)}{1 - G(x)} = \frac{1}{A^\alpha - 1} .$$

(ii) As in (i), $B = 0$ and $F^n(a_n x + b_n) \rightarrow \Psi_\alpha(x)$; $(F\,G)^n\,(a_n x + b_n) \rightarrow \Psi_\alpha(Ax)$ imply that $G^n(a_n x + b_n)$

$$\rightarrow \frac{\Psi_\alpha(Ax)}{\Psi_\alpha(x)} = \Psi_\alpha\left((A^\alpha-1)^{\frac{1}{\alpha}} x\right).$$ By Theorem (5.26-ii)

$$\lim_{x\to x_0^F-} \frac{1 - F(x)}{1 - G(x)} = \frac{1}{A^\alpha-1} .$$

Also $(FG)^n\,(a_n x + b_n) \leq F^n(a_n x + b_n)$ so that $\Psi_\alpha(Ax) \leq \Psi_\alpha(x)$ and for $x < 0$ $Ax \leq x$ so that $A \geq 1$.

(iii) As above, applying Theorem (5.26-iii) gives $A = 1$. Then $(F\,G)^n\,(a_n x + b_n) \rightarrow \Lambda(x + B)$ and $F^n(a_n x + b_n) \rightarrow \Lambda(x)$ implies that $G^n(a_n x + b_n) \rightarrow \frac{\Lambda(x + B)}{\Lambda(x)} = \Lambda(x - \log(e^{-B} - 1))$.

and applying Theorem (5.26-iii) gives $\lim_{x\to x_0^F-} \frac{1 - F(x)}{1 - G(x)} = \frac{1}{e^{-B}-1}$.

Since $(F\,G)^n\,(a_n x + b_n) \leq F^n(a_n x + b_n)$ we have $\Lambda(x + B) \leq \Lambda(x)$ so that $B < 0$. This completes the proof.

CHAPTER VI

WEAK LIMITS AND RECURRENCE PROPERTIES

We now change our point of view. Instead of investigating limit laws for M_n, we ask where the maximum M_n was achieved. We are interested in the state of the M.C. when the maximum was achieved. Also how often the maximum occurs in a particular state. These questions are concerned with the degree of intimacy between the maximum term and the underlying M.C.

Let I_n be the state in which M_n is achieved; i.e. $I_n = j$ iff for some $k = 0,1,\dots, n-1$, $J_k = j$ and $X_{k+1} = M_n$. In order to insure that I_n is well defined, we must preclude the possibility of ties. In this chapter, we assume that $H_1(\cdot),\dots, H_m(\cdot)$ are continuous.

We calculate the distribution of I_n using the conditional independence of the random variables $\{X_n\}$:

$$P[I_n = j \mid J_0 = i] = \sum_{k=0}^{n-1} P[J_k = j,\ X_{k+1} = M_n \mid J_0 = i]$$

$$= \sum_{k=0}^{n-1} \sum_{\substack{\alpha=1 \\ \alpha\neq k}}^{n-1} \sum_{j_\alpha=1}^{m} P[J_k = j,\ J_\alpha = j_\alpha,\ \alpha \neq k,\ 1 \le \alpha \le n-1,\ X_{k+1} = M_n \mid J_0 = i]$$

$$= \sum_{k=0}^{n-1} \sum_{\substack{\alpha=1 \\ \alpha \neq k}}^{n-1} \sum_{j_\alpha=1}^{m} P[X_{k+1} > \max_{\substack{1 \leq \ell \leq n \\ \ell \neq k+1}} \{X_\ell\} | J_0 = i,\ J_\alpha = j_\alpha,\ 1 \leq \alpha \leq n-1,$$

$$\alpha \neq k,\ J_k = j] \cdot P[J_\alpha = j_\alpha,\ 1 \leq \alpha \leq n-1,\ \alpha \neq k,\ J_k = j | J_0 = i]$$

$$= \sum_{k=0}^{n-1} \sum_{\substack{\alpha=1 \\ \alpha \neq k}}^{n-1} \sum_{j_\alpha=1}^{m} \int H_i(x)\, H_{j_1}(x) \cdots H_{j_{k-1}}(x)\, H_{j_{k+1}}(x) \cdots H_{j_{n-1}}(x)\ dH_j(x)$$

$$\cdot\ p_{ij_1}\, p_{j_1 j_2} \cdots p_{j_{k-1} j}\, p_{i j_{k+1}} \cdots p_{j_{n-2} j_{n-1}}\ .$$

Introducing matrix notation gives:

$$(6.1) \qquad P[I_n = j | J_0 = i] = \sum_{k=0}^{n-1} \int_{-\infty}^{\infty} Q_{ij}^{k}(x) \sum_{\alpha=1}^{m} p_{j\alpha} \sum_{\ell=1}^{m} Q_{\alpha\ell}^{n-k-1}(x)\ dH_j(x)\ .$$

We wish to study the limiting behavior of expression (6.1) .

First a remark: Consider a two-state alternating M.C. $\{J_n^A,\ n \geq 1\}$ on which are defined random variables X_n^A with $P\{X_n^A \leq x | J_n^A = 1\} = F_1(x)$, $P\{X_n^A \leq x | J_n^A = 2\} = F_2(x)$.

Quantities superscripted by "A" are defined in the alternating system. The transition matrix is $\underset{\sim}{P}^A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and suppose $J_1^A = 1$ a.s. Then:

$$P[M_{2n}^A \leq x] = F_1^n(x)\, F_2^n(x)\ , \quad P[M_{2n+1}^A \leq x] = F_1^{n+1}(x)\, F_2^n(x)$$

and

$$(6.2) \qquad P[I_{2n}^A = 1] = \int_{-\infty}^{\infty} F_2^n(y)\ d\, F_1^n(y)\ ,$$

(6.3) $$P[I_{2n}^{A} = 2] = \int_{-\infty}^{\infty} F_1^n(y)\, d\, F_2^n(y)$$

(6.4) $$P[I_{2n+1}^{A} = 1] = \int_{-\infty}^{\infty} F_2^n(y)\, d\, F_1^{n+1}(y)$$

(6.5) $$P[I_{2n+1}^{A} = 2] = \int_{-\infty}^{\infty} F_1^{n+1}(y)\, d\, F_2^n(y) .$$

The limiting behavior of (6.2) - (6.5) is the same as the limiting behavior of $\int_{-\infty}^{\infty} F_1^n(y)\, d\, F_2^n(y)$. The study of the limiting behavior of (6.1) reduces to an examination of the limiting behavior of the integral $\int_{-\infty}^{\infty} F_1^n(y)\, d\, F_2^n(y)$, so so that this simple alternating scheme contains all the difficulties of the more complicated general scheme. This will be made precise in the Comparison Theorem (6.11).

Recall that for a distribution function $F(\cdot)$, $x_0^F = \inf \{y | F(y) = 1\}$.

We begin a study of $\lim_{n\to\infty} \int_{-\infty}^{\infty} F_1^n(x)\, d\, F_2^n(x)$ by considering the case where $-\infty < x_0^{F_1} < x_0^{F_2} \leq \infty$. Set $x_1 = x_0^{F_1}$, $x_2 = x_0^{F_2}$. Then:

$$\int_{-\infty}^{\infty} F_1^n(x)\, d\, F_2^n(x) = \int_{-\infty}^{x_1} F_1^n(x)\, d\, F_2^n(x) + \int_{x_1}^{x_2} d\, F_2^n(x) .$$

But $\int_{-\infty}^{x_1} F_1^n(x)\, d\, F_2^n(x) \leq F_2^n(x_1) \to 0$ as $n \to \infty$ and

$$\int_{x_1}^{x_2} d\, F_2^n(x) = 1 - F_2^n(x_1) \to 1 \quad \text{as} \quad n \to \infty \quad \text{so}$$

$$\int_{-\infty}^{\infty} F_1^n(x)\, d\, F_2^n(x) \to 1 \,.$$

The interesting cases are when either $F_1(x) < 1$, $F_2(x) < 1$ for all x, or $X_0^{F_1} = x_0^{F_2} < \infty$.

The following lemma is very useful:

<u>Lemma 6.6</u>: Let $F_1(\cdot)$, $F_2(\cdot)$, $G(\cdot)$, $H(\cdot)$ be distribution functions.

i) If $F_1(\cdot)$ and $G(\cdot)$ are tail equivalent and $\lim_{n\to\infty} \int_{-\infty}^{\infty} F_1^n(x)\, d\, F_2^n(x) = \ell$, then $\lim_{n\to\infty} \int_{-\infty}^{\infty} G^n(x)\, d\, F_2^n(x) = \ell$.

ii) $$\lim_{n\to\infty} \int_{-\infty}^{\infty} F_1^n(x)\, d\, F_2^n(x) = \lim_{n\to\infty} \int_{-\infty}^{\infty} F_1^n(x)\, H(x)\, d\, F_2^n(x) \,.$$

Here "lim" is understood in the sense that the limit of one side exists iff the limit of the other side exists in which the case the limits are equal.

<u>Proof</u>: i) We suppose for simplicity $F_1(x) < 1$, $F_2(x) < 1$ for all x. If not, the proof will still go through after trivial modifications.

Observe that for any M, $\int_{-\infty}^{M} F_1^n(x)\, d\, F_2^n(x) \to 0$ as $n \to \infty$, so that $\int_M^{\infty} F_1^n(x)\, d\, F_2^n(x) \to \ell$. We will show that for any ϵ, there is an M such that

$$\int_M^{\infty} |G^n(x) - F_1^n(x)|\, d\, F_2^n(x) \leq 2\,\epsilon \,.$$

Because of tail equivalence:

$$G(x) = F_1(x) - o(1 - F_1(x))$$

$$F_1(x) = G(x) - o(1 - G(x))$$

as $x \to \infty$. Given ϵ, pick M so that

$$|o(1 - F_1(x))| \leq \epsilon\ (1 - F_1(x))$$

$$|o(1 - G(x)| \leq \epsilon\ (1 - G(x))\ .$$

Since all distribution functions are assumed continuous, the sets $B^+ = \{x > M | G(x) > F_1(x)\}$, $B^- = \{x > M | G(x) < F_1(x)\}$ are measurable . Then:

$$\int_M^\infty |G^n(x) - F_1^n(x)| d\ F_2^n(x)$$

$$= \int_{[M,\infty)B^+} (G^n(x) - F_1^n(x))\ d\ F_2^n(x)$$

$$+ \int_{[M,\infty)B^-} (F_1^n(x) - G^n(x))\ d\ F_2^n(x)$$

$$= \int_{[M,\infty)B^+} G^n(x) - (G(x) - o(1 - G(x)))^n\ d\ F_2^n(x)$$

$$+ \int_{[M,\infty)B^-} F_1^n(x) - (F_1(x) - o(1 - F_1(x))^n\ \ d\ F_2^n(x)$$

$$\leq n\ \epsilon \int_M^\infty G^{n-1}_{(x)}\ (1 - G(x))\ d\ F_2^n(x)$$

$$+ n\ \epsilon \int_M^\infty F_1^{n-1}(x)\ (1 - F_1(x))\ d\ F_2^n(x)\ \ .$$

This last step follows from the inequality $(t - a)^n \geq t^n - n\, a\, t^{n-1}$, for $t \geq 0$, $a \geq 0$, and from the fact that on B^+, $o(1 - G(x)) \geq 0$ and on B^-, $o(1 - F_1(x)) \geq 0$. Integrating by parts shows that the above is bounded by

$$\epsilon \int_M^\infty F_2^n(x)\, d\, G^n(x) + \epsilon \int_M^\infty F_2^n(x)\, d\, F_1^n(x) \leq 2\, \epsilon\ .$$

We have that

$$\ell - 2\,\epsilon \leq \underline{\lim}_{n\to\infty} \int_M^\infty G^n(x)\, d\, F_2^n(x) \leq \overline{\lim}_{n\to\infty} \int_M^\infty G^n(x)\, d\, F_2^n(x)$$

$$\leq \ell + 2\,\epsilon\ .$$

But $\lim_{n\to\infty} \int_{-\infty}^M G^n(x)\, d\, F_2^n(x) = 0$ so that

$$\ell - 2\,\epsilon \leq \underline{\lim}_{n\to\infty} \int_{-\infty}^\infty G^n(x)\, d\, F_2^n(x) \leq \overline{\lim}_{n\to\infty} \int_{-\infty}^\infty G^n(x)\, d\, F_2^n(x) \leq \ell + 2\,\epsilon\ .$$

This requires $\lim_{n\to\infty} \int_{-\infty}^\infty G^n(x)\, d\, F_2^n(x) = \ell$.

ii) For any ϵ, choose M so large that for $x > M$, $|1 - H(x)| < \epsilon$. Then:

$$0 \leq \overline{\lim}_{n\to\infty} \left| \int_{-\infty}^\infty F_1^n(x)\, d\, F_2^n(x) - \int_{-\infty}^\infty F_1^n(x)\, H(x)\, d\, F_2^n(x) \right|$$

$$\leq \overline{\lim}_{n\to\infty} \int_{-\infty}^\infty F_1^n(x) |1 - H(x)|\, d\, F_2^n(x)$$

$$= \overline{\lim_{n\to\infty}} \int_M^\infty F_1^n(x) |1 - H(x)| \, d\, F_2^n(x)$$

$$\leq \overline{\lim_{n\to\infty}} \, \epsilon \int_M^\infty F_1^n(x) \, d\, F_2^n(x) \leq \epsilon \; .$$

Since ϵ is arbitrary, the proof is complete.

For the following theorem we suppose for simplicity that $F_1(x) < 1$, $F_2(x) < 1$ for all x . Only minor changes are necessary when $x_0^{F_1} = x_0^{F_2} < \infty$.

<u>Theorem 6.7</u>: $F_1(\cdot)$, $F_2(\cdot)$ are distribution functions such that $F_1(x) < 1$, $F_2(x) < 1$ for all x . Then

$$\lim_{x\to\infty} \frac{1 - F_1(x)}{1 - F_2(x)} = L \tag{6.8}$$

for $0 \leq L \leq \infty$ iff

$$\lim_{n\to\infty} \int_{-\infty}^\infty F_1^n(x) \, d\, F_2^n(x) = (1 + L)^{-1} \; . \tag{6.9}$$

<u>Remark</u>: Lemma (6.6) and the fact that for all real M , $\lim_{n\to\infty} \int_{-\infty}^M F_1^n(x) \, d\, F_2^n(x) = 0$ show that $\lim_{n\to\infty} \int_{-\infty}^\infty F_1^n(x) \, d\, F_2^n(x)$ depends only on the tails of the distributions.

From (6.8) , $F_1(x) = 1 - (1 - F_2(x)) \, L - o(1 - F_2(x))$ as $x \to \infty$. Hence $\lim_{n\to\infty} \int_{-\infty}^\infty F_1^n(x) \, d\, F_2^n(x) =$ $\lim_{n\to\infty} \int_M^\infty (1-(1 - F_2(x)) L - o(1 - F_2(x)))^n \, d\, F_2^n(x)$. After showing that $o(1 - F_2(x))$ can be neglected, substitute $y = F_2(x)$. The resulting Beta-integral can be calculated and the limit on n evaluated to be $(1 + L)^{-1}$.

This is the rationale behind Theorem (6.7). A Tauberian argument supplied by Professor H. Pollard is strong enough to prove the theorem in both directions.

<u>Proof of Theorem 6.7:</u> We make a series of substitutions designed to bring the integral into a form where the Karamata Tauberian Theorem is applicable. At each stage we keep track of how the substitutions affect $\lim\limits_{x\to\infty} \dfrac{1 - F_1(x)}{1 - F_2(x)}$. If

$\lim\limits_{n\to\infty} \int_{-\infty}^{\infty} F_1^n(x)\, d\, F_2^n(x) = (1 + L)^{-1}$, then setting $y = F_2(x)$,

$G(y) = F_1(F_2^{-1}(y))$, we have $\lim\limits_{n\to\infty} \int_0^1 n\, G^n(y)\, y^{n-1}\, dy = (1 + L)^{-1}$.

Also $\lim\limits_{x\to\infty} \dfrac{1 - F_1(x)}{1 - F_2(x)} = L$ iff $\lim\limits_{y\to 1} \dfrac{1 - G(y)}{1 - y} = L$.

Set $H(y) = y\, G(y)$ and we get $\int_0^1 \dfrac{H^n(y)}{y}\, dy \sim \dfrac{1}{n(1+L)}$

as $n \to \infty$ and $\lim\limits_{x\to\infty} \dfrac{1 - F_1(x)}{1 - F_2(x)} = L$ iff $\lim\limits_{y\to 1} \dfrac{y - H(y)}{1 - y} = L$.

Putting $y = e^{-v}$ gives $\int_0^{\infty} H^n(e^{-v})\, dv \sim \dfrac{1}{n(1+L)}$ and setting

$K(v) = H(e^{-v})$ gives $\int_0^{\infty} K^n(v)\, dv \sim \dfrac{1}{n(1+L)}$. Then $K(0) = 1$,

$K(\infty) = 0$, and $\lim\limits_{x\to\infty} \dfrac{1 - F_1(x)}{1 - F_2(x)} = L$ iff

$$\lim_{v \to 0+} \frac{e^{-v} - K(v)}{1 - e^{-v}} = L \text{ iff } \lim_{v \to 0+} \frac{e^{-v} - K(v)}{v} = L ,$$

$$\text{iff} \quad \frac{1 - K(v)}{v} = \frac{1 - e^{-v}}{v} + \frac{e^{-v} - K(v)}{v} \to 1 + L \text{ as } v \to 0+ .$$

If $\log K(v) = - S(v)$, then $\int_0^\infty e^{-nS(v)} \, dv \sim \frac{1}{n(1 + L)}$.

Substitute $u = S(v)$ and set $\Phi(u) = S^{-1}(u)$ so that $\int_0^\infty e^{-nu} \, d\Phi(u) \sim \frac{1}{n(1 + L)}$. Also $\lim_{x\to\infty} \frac{1 - F_1(x)}{1 - F_2(x)} = L$ iff $\Phi(u) \sim \frac{u}{1 + L}$ as $u \to 0$.

Observe that $\int_0^\infty e^{-nu} \, d\Phi(u) \sim \frac{1}{n(1 + L)}$ iff $\lim_{x\to\infty} x \int_0^\infty e^{-xu} \, d\Phi(u) = (1 + L)^{-1}$. This is shown by the inequalities:

$$[x] \int_0^\infty e^{-([x]+1)u} \, d\Phi(u) \leq x \int_0^\infty e^{-xu} \, d\Phi(u) \leq ([x]+1) \int_0^\infty e^{-[x]u} d\Phi(u)$$

and by multiplying and dividing on the right by $[x]$ and on the left by $[x] + 1$.

We have shown that:

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} F_1^n(x) \, d \, F_2^n(x) = (1 + L)^{-1}$$

$$\text{iff} \quad \lim_{x\to\infty} x \int_0^\infty e^{-xu} \, d\Phi(u) = (1 + L)^{-1} ,$$

and also that:

$$\lim_{x\to\infty} \frac{1 - F_1(x)}{1 - F_2(x)} = L$$

iff $\Phi(u) \sim \frac{u}{1 + L}$ as $u \to 0$.

By the Karamata Tauberian Theorem [4, p. 422]

$$\Phi(u) \sim \frac{u}{1 + L} \quad u \to 0$$

iff

$$\int_0^\infty e^{-xu} \, d\Phi(u) \sim \frac{1}{x(1 + L)} , \qquad x \to \infty .$$

This completes the proof.

Before proving the Comparison Theorem we need a lemma:

Lemma (6.10): (Cf. Theorem (2.15)): There exists a real number K such that for $x > K$, $\underset{\sim}{Q}^n(x) = \rho^n(x) \underset{\sim}{M}(x) + \underset{\sim}{o}(1)$. For $x > K$, $\underset{\sim}{o}(1) \to 0$ uniformly in x at a geometric rate as $n \to \infty$. There exist constants $c > 0$ and $0 < \lambda < 1$ such that for $x > K$, $|\underset{\sim}{o}(1)| \leq c \lambda^n \underset{\sim}{E}$, $n = 1,2,\ldots$.

Proof: From Theorem (2.15) we have that $\underset{\sim}{o}(1) = \underset{\sim}{B}^n(x) = \underset{\sim}{Q}^n(x) - \rho^n(x) \underset{\sim}{M}(x)$. Also there exists a positive integer N such that for $x > K$, $|\underset{\sim}{B}^N(x)| \leq (\alpha + \epsilon) \underset{\sim}{E} < m^{-1} \underset{\sim}{E}$. Since $\underset{\sim}{E}^n = m^{n-1} \underset{\sim}{E}$ we have

$$|\underset{\sim}{B}^{nN}(x)| \leq (\alpha + \epsilon)^n E^n = (\alpha + \epsilon)^n m^{n-1} \underset{\sim}{E}$$

$$\leq \{(\alpha + \epsilon) m\}^{n-1} \underset{\sim}{E} .$$

Therefore:

$$|\underset{\sim}{B}^n(x)| \leq |\underset{\sim}{B}^{[\frac{n}{N}]N}(x)|\,|\underset{\sim}{B}(x)^{n-[\frac{n}{N}]N}|$$

$$\leq |B^{[\frac{n}{N}]N}(x)|\underset{\sim}{E}$$

$$\leq \{((\alpha + \epsilon)m)^{[\frac{n}{N}]-1}\underset{\sim}{E}\}\,\underset{\sim}{E}$$

$$\leq m[(\alpha + \epsilon)m]^{[\frac{n}{N}]-1}\underset{\sim}{E}$$

and setting $\lambda = (\alpha + \epsilon)m$ gives $\alpha < \lambda < 1$ and

$$|B^n(x)| \leq c\lambda^n \underset{\sim}{E} .$$

<u>Theorem 6.11 Comparison Theorem:</u> We have:

$$\lim_{n\to\infty} P[I_n = j | J_0 = i] = \lim_{n\to\infty} P[I_n = j]$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n-1} \int_{-\infty}^{\infty} Q_{ij}^{k}(x) \sum_{\alpha=1}^{m} p_{j\alpha} \sum_{\ell=1}^{m} Q_{\alpha\ell}^{n-k-1}(x)\, dH_j(x)$$

$$= \lim_{n\to\infty} \int_{-\infty}^{\infty} [\prod_{k\neq j} H_k^{\pi_k}(x)]^n \, d\, H_j^{\pi_j n} .$$

Here and in the sequel the equalities are to be understood in the sense that the limit of any of the quantities exists iff the limit of all the quantities exists and then all the limits are equal.

<u>Proof</u>: We proceed by steps:

(1) Suppose that:

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \Big(\prod_{i\neq j} H_i^{\pi_i}(x)\Big)^n \, d\, H_j^{\pi_j n}(x) = \ell \tag{6.12}$$

By Theorem (6.7), this is true iff

$$\lim_{x\to\infty} \frac{1 - \prod_{i\neq j} H_i^{\pi_i}(x)}{1 - H_j^{\pi_j}(x)} = \frac{1-\ell}{\ell}$$

$$\text{iff} \quad \lim_{x\to\infty} \frac{1 - \prod_{i=1}^{m} H_i^{\pi_i}(x)}{1 - H_j^{\pi_j}(x)} = \frac{1}{\ell}$$

$$\text{iff} \quad \lim_{x\to\infty} \frac{1 - \rho(x)}{1 - H_j^{\pi_j}(x)} = \frac{1}{\ell}$$

iff

$$\lim_{x\to\infty} \frac{1 - \rho(x)}{1 - H_j(x)} = \frac{\pi_j}{\ell} \,, \tag{6.13}$$

where we have used the tail equivalence of $\prod_{i=1}^{m} H_i^{\pi_i}(x)$ and $\rho(x)$. Hence (6.12) holds iff (6.13) holds.

(2) We have that:

$$\lim_{n\to\infty} n \int_{-\infty}^{\infty} \rho^{n-1}(x)\, dH_j(x) = L$$

$$\text{iff} \quad \lim_{x\to\infty} \frac{1 - H_j(x)}{1 - \rho(x)} = L \,.$$

To prove this note that

$$\int_{-\infty}^{\infty} \rho^{n-1}(x)\, d\, H_j(x) = \int_0^1 y^{n-1}\, d\, H_j(\rho^{-1}(y))$$

$$= \int_{\infty}^{0} e^{-nv}\, d\, H_j(\rho^{-1}(e^{-v}))$$

$$= \int_0^{\infty} e^{-nv}\, d(1 - H_j(\rho^{-1}(e^{-v}))) \,, \quad \text{where} \quad 1 - H_j(\rho^{-1}(e^{-v}))$$

is a distribution function. As in the proof of the Theorem (6.7)

$$\lim_{n\to\infty} n \int_{-\infty}^{\infty} \rho^{n-1}(x)\, d\, H_j(x) = L$$

iff

$$\int_0^{\infty} e^{-xv}\, d(1 - H_j(\rho^{-1}(e^{-v}))) \sim \frac{L}{x} \,, \quad x \to \infty$$

iff

$$\lim_{v\to\infty} \frac{1 - H_j(\rho^{-1}(e^{-v}))}{v} = L$$

iff

$$\lim_{y\to 1} \frac{1 - H_j(\rho^{-1}(y))}{-\log y} = L$$

iff

$$\lim_{x\to\infty} \frac{1 - H_j(x)}{-\log \rho(x)} = L$$

iff

$$\lim_{x\to\infty} \frac{1 - H_j(x)}{1 - \rho(x)} = L \,.$$

So (6.12) holds iff $\lim_{x\to\infty} \frac{1-\rho(x)}{1-H_j(x)} = \frac{\pi_j}{\ell}$

and hence iff

(6.14) $$\lim_{n\to\infty} n\,\pi_j \int_{-\infty}^{\infty} \rho^{n-1}(x)\, d\,H_j(x) = \ell\ .$$

At this point we rule out the possibility that there exists an x_0 such that $H_j(x_0) = 1$, $\rho(x_0) < 1$. If this were the case then there is an index k such that $H_k(x_0) < 1$ and hence $\lim_{n\to\infty} P[I_n = j \mid J_0 = i] = 0$.

Eliminating this possibility means that for any ϵ, there exists M such that $H_j(M) < 1$, $\rho(M) < 1$ and for $x > M$:

$$|\pi_k - \ell_k(x)| < \epsilon\ ,$$

$$|1 - r_k(x)| < \epsilon\ ,$$

$k = 1,\ldots, m$.

Observe also that for each k, $r_k(x)$ and $\ell_k(x)$ are continuous functions with limits at $+\infty$ so that for any conveniently chosen T, $r_k(\cdot)$ and $\ell_k(\cdot)$ are uniformly bounded on $[T, \infty]$. We denote these bounds by $||r_k||$, $||\ell_k||$.

(3) We have that (6.14) holds iff

(6.15) $$\lim_{n\to\infty} n\int_{-\infty}^{\infty} \rho^{n-1}(x)\, \ell_j(x)\, d\,H_j(x) = \ell\ .$$

Proof: Given (6.14). Then

$$0 \leq \overline{\lim_{n\to\infty}} \left| n \int_{-\infty}^{\infty} \rho^{n-1}(x)\ \ell_j(x)\ d\ H_j(x) - n \int_{-\infty}^{\infty} \rho^{n-1}(x)\ \pi_j\ d\ H_j(x) \right|$$

$$\leq \overline{\lim_{n\to\infty}}\ n \int_{-\infty}^{\infty} \rho^{n-1}(x) |1 - \ell_j(x)| d\ H_j(x)$$

$$= \overline{\lim_{n\to\infty}}\ n \int_{M}^{\infty} \rho^{n-1}(x) |1 - \ell_j(x)| d\ H_j(x)$$

$$\leq \overline{\lim_{n\to\infty}}\ \epsilon\ n \int_{M}^{\infty} \rho^{n-1}(x)\ d\ H_j(x)$$

$$= \epsilon\ \ell\ \pi_j^{-1} \quad .$$

Since ϵ is arbitrary, the limit of the difference must be zero so that (6.15) follows.

Given (6.15) we have:

$$0 \leq \overline{\lim_{n\to\infty}} \left| n \int_{-\infty}^{\infty} \rho^{n-1}(x)\ \ell_j(x)\ d\ H_j(x) - n \int_{-\infty}^{\infty} \rho^{n-1}(x)\ \pi_j\ d\ H_j(x) \right|$$

$$\leq \overline{\lim_{n\to\infty}}\ \epsilon\ n \int_{M}^{\infty} \rho^{n-1}(x)\ d\ H_j(x)$$

$$\leq \epsilon\ \ell\ \{ \inf_{T \leq x \leq \infty} \ell_j(x) \}^{-1}$$

where T is chosen less than M but large enough so that $\inf_{T \leq x \leq \infty} \ell_j(x) > 0$. As above, (6.15) implies (6.14) .

(4) By the same technique one shows that (6.15) holds iff

$$\lim_{n\to\infty} n \int_{-\infty}^{\infty} \rho^{n-1}(x)\, r_i(x)\, \ell_j(x)\, d\, H_j(x) = \ell \quad . \tag{6.16}$$

(5) (6.16) holds iff

$$\lim_{n\to\infty} n \int_{-\infty}^{\infty} \rho^{n-1}(x)\, r_i(x)\, \ell_j(x)\, \Big(\sum_{\alpha=1}^{m} p_{j\alpha}\, r_\alpha(x)\Big)\, d\, H_j(x) = \ell \; . \tag{6.17}$$

(6) (6.17) holds iff

$$\lim_{n\to\infty} n \int_{-\infty}^{\infty} \rho^{n-1}(x)\, M_{ij}(x) \sum_{k=1}^{m} \sum_{\alpha=1}^{m} p_{j\alpha}\, M_{\alpha k}(x)\, d\, H_j(x) = \ell$$

iff

$$\lim_{n\to\infty} \sum_{\ell=\nu}^{n-\nu} \int_{-\infty}^{\infty} \rho^{n-1}(x)\, M_{ij}(x) \sum_{k=1}^{m} \sum_{\alpha=1}^{m} p_{j\alpha}\, M_{\alpha k}(x)\, d\, H_j(x) = \ell \tag{6.18}$$

since $2\nu \int_{-\infty}^{\infty} \rho^{n-1}(x)\, M_{ij}(x) \sum_{k=1}^{m} \sum_{\alpha=1}^{m} p_{j\alpha}\, M_{\alpha k}(x)\, d\, H_j(x) \to 0$,

$n \to \infty$.

(7) For any M such that $\rho(M) < 1$:

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \int_{-\infty}^{M} Q_{ij}^{k}(x) \sum_{\alpha=1}^{m} p_{j\alpha} \sum_{\ell=1}^{m} Q_{\alpha\ell}^{n-k-1}(x)\, d\, H_j(x) = 0 \; . \tag{6.19}$$

Proof: $\displaystyle \sum_{k=0}^{n-1} \int_{-\infty}^{M} Q_{ij}^{k}(x) \sum_{\alpha=1}^{m} p_{j\alpha} \sum_{\ell=1}^{m} Q_{\alpha\ell}^{n-k-1}(x)\, d\, H_j(x)$

$$\le \sum_{k=0}^{n-1} \sum_{\ell=1}^{m} Q_{ij}^{k}(M) \sum_{\alpha=1}^{m} p_{j\alpha}\, Q_{\alpha\ell}^{n-k-1}(M)\, H_j(M)$$

$$= \sum_{\ell=1}^{m} \sum_{k=0}^{n-1} Q_{ij}^{k}(M)\, Q_{j\ell}^{n-k}(M)$$

$$\leq \sum_{\ell=1}^{m} Q_{i\ell}^{n}(M) \to 0$$

as $n \to \infty$ since $\rho(M) < 1$.

(8) For any fixed integer ν

$$(6.20). \quad \lim_{n\to\infty} \sum_{k=0}^{\nu-1} \int_M^\infty Q_{ij}^{k}(x) \sum_{\alpha=1}^{m} p_{j\alpha} \sum_{\ell=1}^{m} Q_{\alpha\ell}^{n-k-1}(x)\, d\, H_j(x) = 0 \, .$$

Proof: $\displaystyle \sum_{k=0}^{\nu-1} \int_M^\infty Q_{ij}^{k}(x) \sum_{\alpha=1}^{m} p_{j\alpha} \sum_{\ell=1}^{m} Q_{\alpha\ell}^{n-\nu}(x)\, d\, H_j(x)$

$$\leq \sum_{k=0}^{\nu-1} \int_M^\infty Q_{ij}^{k}(x) \sum_{\alpha=1}^{m} p_{j\alpha} \sum_{\ell=1}^{m} Q_{\alpha\ell}^{n-\nu}(x)\, d\, H_j(x)$$

$$\leq \Big(\sum_{k=0}^{\nu-1} p_{ij}^{k} \Big) \sum_{\alpha=1}^{m} p_{j\alpha} \int_M^\infty \sum_{\ell=1}^{m} Q_{\alpha\ell}^{n-\nu}(x)\, d\, H_j(x)$$

$$\to 0 \text{ as } n \to \infty$$

by the Lebesgue Dominated Convergence Theorem.

A similar proof shows that

$$(6.21) \quad \lim_{n\to\infty} \sum_{k=n-\nu+1}^{n-1} \int_M^\infty Q_{ij}^{k}(x) \sum_{\alpha=1}^{m} p_{j\alpha} \sum_{\ell=1}^{m} Q_{\alpha\ell}^{n-k-1}(x)\, dH_j(x) = 0 \, .$$

(9) Given any $\epsilon > 0$, there exists a positive integer ν_0 such that for $\nu > \nu_0$ and n sufficiently large:

$$(6.22) \qquad \Big| \sum_{k=\nu}^{n-\nu} \int_M^\infty Q_{ij}^k(x) \sum_{\alpha=1}^m p_{j\alpha} \sum_{\ell=1}^m Q_{\alpha\ell}^{n-k-1}(x)\, d\, H_j(x)$$

$$- \sum_{k=\nu}^{n-\nu} \int_M^\infty \rho^{n-1}(x)\, M_{ij}(x) \sum_{\ell=1}^m \sum_{\alpha=1}^m p_{j\alpha}\, M_{\alpha\ell}(x)\, d\, H_j(x) \Big|$$

$< \epsilon$ uniformly in M.

To show this we pick M large enough that Theorem (2.15) is applicable. Substitute $\rho^k(x)\, M_{ij}(x) + o(1)$ for $Q_{ij}^k(x)$ in (6.22). After a similar substitution for $Q_{\alpha\ell}^{n-k-1}(x)$ we use Lemma (6.10) and the difference (6.22) is bounded by:

$$\Big| \sum_{k=\nu}^{n-\nu} \int_M^\infty \rho^k(x)\, M_{ij}(x) \sum_{\ell=1}^m \sum_{\alpha=1}^m c\, \lambda^{n-k-1}\, p_{j\alpha}\, d\, H_j(x) \Big|$$

$$+ \Big| \sum_{k=\nu}^{n-\nu} \int_M^\infty \rho^{n-k-1}(x) \, \big(\sum_\alpha p_{j\alpha}\, r_\alpha(x)\big)\, c\, \lambda^k\, d\, H_j(x) \Big|$$

$$+ \Big| \sum_{k=\nu}^{n-\nu} \int_M^\infty \sum_{\ell=1}^m \sum_{\alpha=1}^m c\, \lambda^k\, c\, \lambda^{n-k-1}\, p_{j\alpha}\, d\, H_j(x) \Big|$$

$$\leq m\, c\, \|M_{ij}(x)\| \sum_{k=\nu}^{n-\nu} \lambda^{n-k-1}$$

$$+ c \sup_j \|r_j(x)\| \sum_{k=\nu}^{n-\nu} \lambda^k + c^2\, m \sum_{k=\nu}^{n-\nu} \lambda^{n-1}$$

where $\|r_j(x)\|$ and $\|M_{ij}(x)\|$ are the suprema of these continuous functions with limits at $+\infty$ over any convenient interval $[T, \infty]$ such that $\underset{\sim}{Q}(T)$ is irreducible. Taking suprema over such an interval guarantees that the result will be independent of M. The above expressions are bounded by:

$$c\, m \, ||M_{ij}(x)|| \, \frac{\lambda^{\nu-1}}{1-\lambda} + c \sup_{1 \le j \le m} ||r_j(x)|| \, \frac{\lambda^{\nu}}{1-\lambda}$$

$$+ c^2 \, m \, (n \, \lambda^{n-1})$$

and since $0 < \lambda < 1$ we can choose ν_0 so large that the first two terms are less than $\epsilon/3$. For n sufficiently large the last term will be less than $\epsilon/3$.

(10) If (6.18) holds then by (9)

$$\ell - \epsilon \le \varliminf_{n\to\infty} \sum_{k=\nu}^{n-\nu} \int_M^\infty Q_{ij}^k(x) \sum_{\alpha=1}^m p_{j\alpha} \sum_{\ell=1}^m Q_{\alpha\ell}^{n-k-1}(x) \, d\, H_j(x)$$

$$\le \varlimsup_{n\to\infty} \le \ell + \epsilon \quad .$$

Taking into account (7) and (8) we must have

$$\ell - \epsilon \le \varliminf_{n\to\infty} \sum_{k=0}^{n-1} \int_{-\infty}^\infty Q_{ij}^k(x) \sum_{\alpha=1}^m p_{j\alpha} \sum_{\ell=1}^m Q_{\alpha\ell}^{n-k-1}(x) \, d\, H_j(x)$$

$$\le \varlimsup_{n\to\infty} \le \ell + \epsilon$$

which requires that

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \int_{-\infty}^\infty Q_{ij}^k(x) \sum_{\alpha=1}^m p_{j\alpha} \sum_{\ell=1}^m Q_{\alpha\ell}^{n-k-1}(x) \, d\, H_j(x) = \ell \, . \tag{6.23}$$

Similarly (6.23) implies (6.18). Since (6.18) is equivalent to (6.13) we have completed the proof of the Comparison Theorem.

Studying $\lim_{n\to\infty} P[I_n = j]$ is thus equivalent to studying these probabilities in the alternating case. In fact we can lump all the states $k \neq j$ into a single class, adjust the distribution functions to take into account sojourn times and study the two-state alternating scheme with distribution functions $H_j^{\pi_j}(x)$ and $\prod_{k\neq j} H_k^{\pi_k}(x)$.

The Comparison Theorem (6.11) and Theorem (6.7) combine immediately to give:

Corollary (6.24) Weak Limits Criteria:

For $0 \leq \ell_i \leq 1$,

$$\lim_{n\to\infty} P[I_n = i] = \ell_i$$

iff

$$\lim_{x\to\infty} \frac{1 - \prod_{k\neq i} H_k^{\pi_k}(x)}{1 - H_i^{\pi_i}(x)} = \frac{1-\ell_i}{\ell_i}$$

or equivalently iff:

$$\lim_{x\to\infty} \frac{1 - H_i^{\pi_i}(x)}{1 - \rho(x)} = \ell_i \quad .$$

Remark: In the above, $\rho(x)$ may be replaced by any tail equivalent distribution function such as $\sum_{k=1}^{m} \pi_k H_k(x)$ or $\prod_{k=1}^{m} H_k^{\pi_k}(x)$.

The results obtained in proving the Comparison Theorem (6.11) afford us the following interpretation of Corollary (6.24): If we evaluate (6.1) using the matrix

$\hat{\underset{\sim}{Q}}(x) = \{\pi_j \, H_i(x)\}$, then we obtain:

$$(6.25) \qquad P\{\hat{I}_n = j \mid \hat{J}_0 = i\} = \pi_j (n-1) \int_{-\infty}^{\infty} H_i(x) \left(\sum_{k=1}^{m} \pi_k \, H_k(x) \right)^{n-2} d\, H_j(x) \; .$$

Then

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} \left(\prod_{k \neq j} H_k^{\pi_k}(x) \right)^n d\, H_j^{\pi_j n}(x) = \ell$$

iff $\displaystyle \lim_{x \to \infty} \frac{1 - H_j(x)}{1 - \rho(x)} = \frac{\ell}{\pi_j}$ (6.24)

iff

$$\lim_{x \to \infty} \frac{1 - H_j(x)}{1 - \sum_{k=1}^{m} \pi_k \, H_k(x)} = \frac{\ell}{\pi_j}$$

iff

$$\lim_{n \to \infty} n \int_{-\infty}^{\infty} \left(\sum_{k=1}^{m} \pi_k \, H_k(x) \right)^{n-1} d\, H_j = \frac{\ell}{\pi_j}$$

(Theorem (6.12-2))

iff $\displaystyle \lim_{n \to \infty} n \, \pi_j \int_{-\infty}^{\infty} H_i(x) \left(\sum_{k=1}^{m} \pi_k \, H_k(x) \right)^{n-1} d\, H_j = \ell$

(same proof as Theorem (6.12-3)) .

We have proved:

$$\lim_{n\to\infty} P[I_n = j | J_0 = i] = \ell$$

iff

$$\lim_{n\to\infty} P[\hat{I}_n = j | \hat{J}_0 = i] = \ell .$$

The two systems governed by the S.M.M.'s $\underset{\sim}{Q}(x) = \{p_{ij} H_i(x)\}$ and $\hat{\underset{\sim}{Q}}(x) = \{\pi_j H_i(x)\}$ have the same properties as far as the existence and numerical value of $\lim_{n\to\infty} P[I_n = j]$ is concerned. Likewise with respect to the existence of limiting extreme value distributions (4.12, 5.13). The limitnng behavior of the sequence $\{M_n\}$ is determined by the quantity of probability contained in the tails of the distributions $H_i(\cdot)$, $i = 1,\ldots, m$ and also by the relative amounts of time the Markov chain spends in each state after the chain has reached equilibrium.

We postpone a discussion of solidarity questions till the end of the chapter and proceed to investigate recurrence properties of the sequence $\{I_n\}$.

Let $(\Omega, \mathcal{F}, P)$ be the underlying probability space. Then:

Definition 6.25: State j is maximum-recurrent (max-rec) iff $P\{[I_n = j] \text{ i.o.}\} = 1$; i.e. for any integer N, there exists some $n(\omega) > N$ such that $I_n(\omega) = j$ almost surely.

Definition (6.26): State j is maximum-transient (max-trans) iff $P\{[I_n = j] \text{i.o.}\} = 0$; i.e. iff for almost all ω there exists a positive integer $N(\omega)$ such that for all $n > N(\omega)$ $I_n(\omega) \neq j$.

<u>Definition (6.27)</u>: (Cf. [11, 13, 14]): For a sequence of random variables $\{X_n, n \geq 1\}$, X_j is a <u>record value</u> of the sequence if it is strictly greater than all preceding values, i.e., if $X_j > \max(X_1, \ldots, X_{j-1})$. By convention X_1 is a record value.

For $n \geq 1$ define the events A_n^j by

$$A_n^j = [X_n \text{ is a record, } J_{n-1} = j].$$

A_n^j is the event that a record occurs at time n in state j.

We have that

$$(6.28) \qquad j \text{ is max-trans iff } P\{A_n^j \text{ i.o.}\} = 0$$

$$(6.29) \qquad j \text{ is max-rec iff } P\{A_n^j \text{ i.o.}\} = 1.$$

To calculate $P A_n^j$, let $\{p_\ell\}$, $\ell = 1, \ldots, m$ be some initial distribution. Then

$$P A_n^j = P[X_n > \max\{X_1, \ldots, X_{n-1}\}, \; J_{n-1} = j]$$

$$= \sum_{\ell=1}^m p_\ell \sum_{j_1=1}^m \cdots \sum_{j_{n-2}=1}^m P[X_n > \max\{X_1, \ldots, X_{n-1}\} | J_0 = \ell, J_1 = j_1, \ldots, J_{n-2} = j_{n-2}, J_{n-1} = j]$$

$$\cdot P[J_{n-1} = j, J_{n-2} = j_{n-2}, \ldots, J_1 = j_1 | J_0 = \ell]$$

$$= \sum_{\ell=1}^m p_\ell \sum_{j_1=1}^m \cdots \sum_{j_{n-2}=1}^m p_{\ell j_1} p_{j_1 j_2} \cdots p_{j_{n-2} j} \int_{-\infty}^{\infty} H_\ell(x) H_{j_1}(x) \cdots H_{j_{n-2}}(x) d H_j(x),$$

Introducing matrix notation gives

$$(6.29) \qquad PA_n^j = \sum_{\ell=1}^{m} p_\ell \int_{-\infty}^{\infty} Q_{\ell j}^{n-1}(x)\, d\, H_j(x) \; .$$

For an i.i.d. sequence $\{X_n, n \geq 1\}$, the events $A_n = [X_n$ is a record$]$ are independent - Renyi [13]. Although our events A_n^j are not independent they exhibit some properties of an independent sequence, namely they satisfy a zerO-one law. We will show that the only values for $P\{A_n^j \text{ i.o.}\}$ are 0 or 1. Hence a state must be either max-trans or max-rec. Before a formal statement of these results, we prove a lemma:

Lemma 6.30: For any M such that $\rho(M) < 1$,

$$\sum_{n=1}^{\infty} P\, A_n^j < \infty \quad \text{iff} \quad \int_M^{\infty} \frac{d\, H_j(x)}{1 - \rho(x)} < \infty \, ,$$

$$\sum_{n=1}^{\infty} P\, A_n^j = \infty \quad \text{iff} \quad \int_M^{\infty} \frac{d\, H_j(x)}{1 - \rho(x)} = \infty \, .$$

Proof: From (6.29) we have that $\sum_{n=1}^{\infty} P\, A_n^j = \sum_{\ell=1}^{m} p_\ell \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} Q_{\ell j}^n(x)\, d\, H_j(x)$. For any M such that $\rho(M) < 1$

$$\sum_{\ell=1}^{m} p_\ell \int_{-\infty}^{M} \sum_{n=0}^{\infty} Q_{\ell j}^n\,(x)\, d\, H_j(x) \leq \sum_{\ell=1}^{m} p_\ell \sum_{n=0}^{\infty} Q_{\ell j}^n(M)\, H_j(M) \; .$$

For fixed M, there exists a positive constant k_M so large that $Q_{ij}^n(M) \leq k_M\, \rho^n(M)$ for $1 \leq i, j \leq m$, $n \geq 1$.

So the above is dominated by $\sum_{\ell=1}^{m} p_\ell\, H_j(M) \dfrac{k_M}{1 - \rho(M)} < \infty$.

Therefore $\sum_{n=1}^{\infty} P\, A_n^j$ converges or diverges according as

$\sum_{\ell=1}^{m} p_\ell \int_M^\infty \sum_{n=0}^{\infty} Q_{j\ell}^n(x)\, d\, H_j(x)$ converges or diverges.

Now for all x sufficiently large $\underset{\sim}{Q}^n(x) = \rho^n(x)\, \underset{\sim}{M}(x) + \underset{\sim}{o}(1)$ where $|\underset{\sim}{o}(1)| \leq c\, \lambda^n\, \underset{\sim}{E}$, $0 < \lambda < 1$ (Lemma (6.10)). Hence:

$$\sum_{\ell=1}^{m} p_\ell \int_M^\infty \sum_{n=0}^{\infty} Q_{\ell j}^n\, (x)\, d\, H_j(x)$$

$$= \sum_{\ell=1}^{m} p_\ell \int_M^\infty \sum_{n=0}^{\infty} \rho^n(x)\, M_{\ell j}(x)\, d\, H_j(x) +$$

$$+ \sum_{\ell=1}^{m} p_\ell \int_M^\infty \sum_{n=0}^{\infty} o(1)\, d\, H_j(x) .$$

The last term is dominated by

$$\sum_{n=0}^{\infty} c\lambda^n\, (1 - H_j(M)) = \frac{c(1 - H_j(M))}{1 - \lambda} < \infty .$$

So $\sum_{n=1}^{\infty} P\, A_n^j$ converges or diverges according as

$\sum_{\ell=1}^{m} p_\ell \int_M^\infty \sum_{n=0}^{\infty} \rho^n(x)\, M_{\ell j}(x)\, d\, H_j(x)$. But we have:

$$\min_{1 \leq \ell \leq m}\ \inf_{T \leq x \leq \infty} |M_{\ell j}(x)| \int_M^\infty \frac{d\, H_j(x)}{1 - \rho(x)}$$

$$\leq \sum_{\ell=1}^{m} p_\ell \int_M^\infty \sum_{n=0}^{\infty} \rho^n(x)\, M_{\ell j}\, (x)\, d\, H_j(x)$$

$$\leq \max_{1 \leq \ell \leq m}\ \sup_{T \leq x \leq \infty} |M_{\ell j}(x)| \int_M^\infty \frac{dH_j(x)}{1 - \rho(x)}$$

where T is chosen less than M but large enough so that $\min_{1 \le \ell \le m} \inf_{T \le x \le \infty} |M_{\ell j}(x)| > 0$. Hence $\sum_{n=1}^{\infty} P A_n^j$ and $\int_M^\infty \frac{d H_j}{1 - \rho(x)}$ converge or diverge together and since

$$\int_{-\infty}^{M} \frac{d H_j(x)}{1 - \rho(x)} \le \frac{H_j(M)}{1 - \rho(x)} < \infty ,$$

this suffices to show the desired result.

Let $V_1^j = \inf\{n > 1 | X_n$ is a record, $J_{n-1} = j\}$, i.e. V_1^j is the index of the first non-trivial record in state j.

<u>Theorem (6.31) Recurrence Criteria</u>:

State j is max-trans iff (i) $P\{A_n^j \text{ i.o.}\} = 0$

iff (ii) $\sum_{n=1}^{\infty} P A_n^j < \infty$

iff (iii) $\int_M^\infty \frac{d H_j(x)}{1 - \rho(x)} < \infty$

iff (iv) $P\{V_1^j = \infty | X_1 = y, J_0 = j\} > 0$ for some y.

State j is max-rec iff (v) $P\{A_n^j \text{ i.o.}\} = 1$

iff (vi) $\sum_{n=1}^{\infty} P A_n^j = \infty$

iff (vii) $\int_M^\infty \frac{d H_j(x)}{1 - \rho(x)} = \infty$

iff (viii) $P\{V_1^j = \infty | X_1 = y, J_0 = j\} = 0$ for all y.

Proof: The equivalence of (ii) and (iii), and (vi) and (vii) follows from Lemma (6.30). That (ii) implies (i) is the statement of the Borel-Cantelli Lemma.

We have that:

$$P\{\varliminf_{n\to\infty} (A_n^j)^c\}$$

$$= P[\text{The number of records in state } j \text{ is finite}]$$

$$= \sum_{n=1}^{\infty} P[\text{The last record in state } j \text{ is at index } n]$$

$$= \sum_{n=1}^{\infty} P[X_n \text{ is a record in state } j\text{; there are no records in state } j \text{ among } X_{n+1}, X_{n+2}, \ldots]$$

$$= \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} P\{\text{There are no records in state } j \text{ among } X_{n+1}, X_{n+2}, \ldots | X_n \text{ is a record in } j,\ X_n = y\} \cdot d\, P[X_n \text{ is a record in } j,\ X_n \le y] .$$

Now $\quad P[X_n \text{ is a record in } j,\ X_n \le y]$

$$= P[y \ge X_n > \max\{X_1, \ldots, X_{n-1}\}]$$

$$= \sum_{\ell=1}^{m} p_\ell \int_{-\infty}^{y} Q_{\ell j}^{n-1}(x)\, d\, H_j(x) .$$

Therefore setting

$$P\{\text{There are no records in state } j \text{ among } X_{n+1}, X_{n+2}, \ldots | X_n \text{ is a record in } j,\ X_n = y\}$$

$$= P\{V_1^j = \infty | J_0 = j,\ X_1 = y\}$$

gives:

$$(6.32)\quad P\{\underline{\lim}_{n\to\infty} (A_n^j)^c\}$$

$$= \sum_{\ell=1}^{m} p_\ell \int_{-\infty}^{\infty} P\{V_1^j = \infty | J_0 = j,\ X_1 = y\} \sum_{n=0}^{\infty} Q_{\ell j}^n(y)\, d\, H_j(y)\ .$$

If j is max-trans, then $P\{A_n^j \text{ i.o.}\} = 0$ and $P\{\underline{\lim}_{n\to\infty} (A_n^j)^c\} = 1$ so that (6.32) requires that we have for some y, $P\{V_1^j = \infty | X_1 = y,\ J_0 = j\} > 0$ and (i) implies (iv). Still assuming (i) we note that $P\{V_1^j = \infty | X_1 = y,\ J_0 = j\}$ is non-decreasing in y and hence $\lim_{y\to\infty} P\{V_1^j = \infty | X_1 = y,\ J_0 = j\}$ exists and is strictly positive. Therefore for all ℓ :

$$\lim_{y\to\infty} \frac{P\{V_1^j = \infty | J_0 = j,\ X_1 = y\} \sum_{n=0}^{\infty} Q_{\ell j}^n (y)}{\sum_{n=0}^{\infty} Q_{\ell j}^n (y)}$$

$$= \lim_{y} P\{V_1^j = \infty | J_0 = j,\ X_1 = y\} > 0$$

so that

$$\sum_{\ell=1}^{m} p_\ell \int_{-\infty}^{\infty} P\{V_1^j = \infty | J_0 = j,\ X_1 = y\} \sum_{n=0}^{\infty} Q_{\ell j}^n (y) d\, H_j(y) < \infty$$

$$\text{iff} \quad \sum_{\ell=1}^{m} p_\ell \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} Q_{\ell j}^n (y)\, d\, H_j(y) < \infty$$

and this happens when and only when

$$\int_M^\infty \frac{d\,H_j(x)}{1-\rho(x)} < \infty \quad \text{by Lemma (6.30)} .$$

Therefore (i) implies (iii) .

If state j is max-rec, $P\{A_n^j \text{ i.o.}\} = 1$ and $P[\underline{\lim}_{n\to\infty} (A_n^j)^c] = 0$ so that from (6.32) we have that:

$$0 = \int_{-\infty}^{\infty} P\{V_1^j = \infty | J_0 = j, x_1 = y\} \sum_{\ell=1}^{n} p_\ell \sum_{n=0}^{\infty} Q_{\ell j}^n (y)\, d\, H_j(y) .$$

Let $y_0 = \inf\{y | \min_{1\le k\le m} H_k(y) > 0\}$. Then for $y > y_0$,

$\sum_{\ell=1}^{m} p_\ell \sum_{n=0}^{\infty} Q_{\ell j}^n(y) > 0$ so that if $P\{V_1^j = \infty | J_0 = j, X_1 = y_0\} > 0$,

then $H_j(y_0) = 1$. But by the definition of y_0 and the continuity of the H's, there must be a subscript ℓ_0 such that $H_{\ell_0}(y_0) = 0$ so that j could not possibly be max-rec. Therefore $P\{V_1^j = \infty | J_0 = j, X_1 = y_0\} = 0$. Suppose there

exists $y_1 > y_0$ such that for $y > y_1$. Then $H_j(y_1) = 1$ and if there were a subscript k , $1 \le k \ne j \le m$, such that $H_k(y_1) < 1$ then j could not be max-rec. Therefore for all k , $H_k(y_1) = 1$. In particular if there exists an index α, $1 \le \alpha \le m$ such that $H_\alpha(y) < 1$ for all y, then $P[V_1^j = \infty | J_o = j, X_1 = y] = 0$ for all y. Otherwise we observe that $P\{V_1^j = \infty | J_0 = j, X_1 = y]\} = 0$ for all y such that $\min_{1\le k\le m} H_k(y) < 1$. For other values of y , the conditional probability is not well defined and we can arbitrarily assign it the value zero. Therefore (v) implies (viii). Conversely if $P\{V_i^j = \infty | J_0 = j , X_1 = y\} = 0$ for all y, then $P\{A_n^j \text{ i.o.}\} = 1$ by (6.32) so that (viii) implies (v).

Given that j is max-rec, this implies $P\{V_1^j = \infty | J_0 = j,\ X_1 = y\} = 0$ for all y and therefore j is not max-trans. Hence $\int_M^\infty \frac{d\ H_j(x)}{1 - P(x)} = \infty$ and we have

shown that (v) implies (vii). The proof of Theorem (6.3) will be completed once we show that (iv) implies (i). For this demonstration we assume there is an index k such that $H_k(x) < 1$ for all x. If this is not true, the proof will still go through after trivial modifications.

Given that $P\{V_1^j = \infty | J_0 = j,\ X_1 = y\} > 0$ for some y, there exists an $\eta > 0$ and y_0 such that for $y > y_0$ $P\{V_1^j = \infty | J_0 = j,\ X_1 = y\} \geq \eta$. Since $H_k(x) < 1$ for all x, $M_n \to \infty$ a.s. as $n \to \infty$. By Egoroff's Theorem, given any $\epsilon > 0$, there is a measurable set A_ϵ such that $P\ A_\epsilon < \epsilon$ and $M_n(\omega) \to \infty$ uniformly for $\omega \in \Omega - A_\epsilon$. Therefore there exists n_0 (independent of ω) such that for $n > n_0$ $M_n(\omega) > y_0$ for all $\omega \in \Omega - A_\epsilon$. We have that

$$1 - \epsilon < P\ A_\epsilon^c = P\{A_\epsilon^c[\text{There are infinitely many records in state } j \text{ among } X_{n_0+1}, X_{n_0+2}, \ldots\ldots]\}$$

$$+ P\{A_\epsilon^c[\text{There are finitely many records in state } j \text{ among } X_{n_0+1}, X_{n_0+2}, \ldots]\}.$$

Also:

$P\{A_\epsilon^c$[There are infinitely many records in state j among $X_{n_0+1}, X_{n_0+2}, \ldots$]$\}$

$\leq P\{A_\epsilon^c$[There are k records in state j among $X_{n_0+1}, X_{n_0+2}, \ldots$]$\}$

$< (1 - \eta)^k \to 0$

as $k \to \infty$. This last step follows from the fact that on A_ϵ^c, $M_n > y_0$ for $n > n_0$ and $P\{V_1^j < \infty | X_1 = y, J_0 = j\} < 1 - \eta$ for all $y > y_0$. Therefore

$1 - \epsilon < P A_\epsilon^c = P\{A_\epsilon^c$[There are finitely many records in state j among $X_{n_0+1}, X_{n_0+2}, \ldots$]$\}$

$\leq P$[There are finitely many records in state j] .

Since ϵ can be chosen arbitrarily small, $P\{A_n^j \text{ i.o.}\} = 0$ and j is max-trans. Theorem (6.31) is completely proven.

We now investigate the connections between recurrence properties and weak limits and give solidarity results. For the construction of counterexamples recall the following: If $p_{ij} = \pi_j$, $1 \leq i, j \leq m$, then $\rho(x) = \sum_{i=1}^{m} \pi_i H_i(x)$. It is often convenient to take $p_{ij} = m^{-1}$ so that $\rho(x) = m^{-1} \sum_{i=1}^{m} H_i(x)$. We give our results as a sequence of propositions:

Proposition (6.33): If $\lim_{n\to\infty} P[I_n = j] = \ell_j > 0$ then j is max-rec .

Proof: Observe that $\int_M^\infty \frac{d\,H_j(x)}{1-H_j(x)} = \infty$. Now

$$\lim_{n\to\infty} P[I_n = j] = \ell_j > 0 \quad \text{iff} \quad \lim_{x\to\infty} \frac{1-H_j^{\pi_j}(x)}{1-\rho(x)} = \ell_j \quad \text{(Corollary (6.24))}$$

$$\text{iff} \quad \lim_{x\to\infty} \frac{1-H_j(x)}{1-\rho(x)} = \frac{\ell_j}{\pi_j} > 0 .$$

Therefore the integrals $\int_M^\infty \frac{dH_j(x)}{1-\rho(x)}$ and $\int_M^\infty \frac{dH_j(x)}{1-H_j(x)}$ converge or diverge together and j is max-rec. (Theorem (6.31))

Proposition (6.34): If $\lim_{n\to\infty} P[I_n = j] = 0$. Then j can be either max-rec or max-trans. A weak limit of zero gives no information about the recurrence properties of the state.

Proof: Take a 2 x 2 stochastic matrix with entries $p_{ij} = \frac{1}{2}$, $i,j = 1,2$. Take any two distribution functions $H_1(x)$, $H_2(x)$ such that there exists x_0 with $H_1(x_0) = 1$ but $H_2(x) < 1$ for all x . Then $\rho(x) = \frac{1}{2} H_1(x) + \frac{1}{2} H_2(x)$ and $\frac{1-H_1(x)}{1-\rho(x)} \to 0$ as $x \to \infty$. Also $\int_M^\infty \frac{dH_1(x)}{1-\rho(x)} \le \int_{-\infty}^{x_0} \frac{dH_1(x)}{1-\rho(x)}$

$\le (1 - \rho(x_0))^{-1} < \infty$.

Therefore: $\lim_{n\to\infty} P[I_n = 1] = 0$ and 1 is max-trans.

We now give an example where the weak limit is zero but the state is max-rec.

Consider again the stochastic matrix $p_{ij} = \frac{1}{2}$, $i,j = 1,2$. It suffices to find two distribution functions $H_1(\cdot)$ and $H_2(\cdot)$ such that $\lim_{x\to\infty} \frac{1-H_1(x)}{1-H_2(x)} = 0$ and $\int_M^\infty \frac{dH_1(x)}{1-H_2(x)} = \infty$.

This is sufficient since $\lim_{x\to\infty} \frac{1-H_1(x)}{1-H_2(x)} = 0$ implies $\frac{1-H_1(x)}{1-\rho(x)} =$

$$\frac{1-H_1(x)}{\frac{1}{2}(1-H_1(x)) + \frac{1}{2}(1-H_2(x))} \to 0 \text{ as } x \to \infty .$$

Also $$\frac{1-H_2(x)}{1-\rho(x)} = \frac{1-H_2(x)}{\frac{1}{2}(1-H_1(x)) + \frac{1}{2}(1-H_2(x))} =$$

$$\frac{1}{\frac{1}{2}\left(\frac{1-H_1(x)}{1-H_2(x)}\right) + \frac{1}{2}} \to 2$$ as $x \to \infty$ so that $\int_M^\infty \frac{dH_1(x)}{1-H_2(x)}$ and $\int_M^\infty \frac{dH_1(x)}{1-\rho(x)}$ will converge or diverge together.

It is sufficient to find a continuous function $f(z)$ on $(0,1]$ with the following properties:

i) $f(1) = 1$

ii) $\lim_{z\to 0+} z\, f(z) = 0$

iii) $f(\cdot)$ is decreasing on $(0,1]$

iv) $\lim_{z \to 0} f(z) = \infty$

v) $\int_0^1 f(z)\,dz = \infty$.

Given such a function $f(\cdot)$, select any continuous distribution function $H_1(\cdot)$ such that $H_1(x) < 1$ for all x and set $\frac{1}{1-H_2(x)} = f(1-H_1(x))$. Then $H_2(x) = 1 - \frac{1}{f(1-H_1(x))}$. We have that $H_2(-\infty) = 0$, $H_2(\infty) = 1$, and $H_2(\cdot)$ is non-decreasing so that $H_2(\cdot)$ is a distribution function. Furthermore $\lim_{x\to\infty} \frac{1-H_1(x)}{1-H_2(x)} = \lim_{x\to\infty} (1-H_1(x))\, f(1-H_1(x)) = 0$ and

$$\int_M^\infty \frac{dH_1(x)}{1-H_2(x)} = \int_{1-\delta}^1 f(1-H_1(x))\, dH_1(x) = \int_0^\delta f(z)\, dz = \infty$$

where $0 < \delta < 1$.

To construct the required function we define f as follows:

$$f(1) = 1,$$

$$f(x) = 1 , \qquad \frac{1}{2} \leq x \leq 1 ,$$

$$f(\tfrac{1}{n!}) = (n-2)! \qquad n \geq 2 .$$

For other values of x in $(0,1]$ define $f(\cdot)$ by linear interpolation. $f(\cdot)$ has the required properties. For any $x \in (0,1]$ there exists n such that $x \in [\frac{1}{(n+1)!} , \frac{1}{n!}]$ so that $x\, f(x) \leq \frac{f(\frac{1}{(n+1)!})}{n!} = \frac{1}{n} \to 0$ as $x \to 0$. Also

$\lim_{x\to 0} f(x) = \infty$ and $\int_0^1 f(x)\,dx = \infty$. This follows by a direct summation of the areas of the rectangles and triangles under the curve:

$$\int_0^1 f(x)\,dx = \sum_n \{(\frac{1}{n!} - \frac{1}{(n+1)!})\ (n-2)!$$

$$+ \frac{1}{2}(\frac{1}{n!} - \frac{1}{(n+1)!})((n-1)! - (n-2)!)\}$$

$$= \sum_n \frac{1}{(n+1)(n-1)} + \frac{1}{2}\sum_n \frac{n-2}{(n+1)(n-1)} = \infty .$$

Proposition (6.35): If state j is max-trans, then the weak limit exists and $\lim_{n\to\infty} P[I_n = j] = 0$.

Proof: If state j is max-trans then $P([I_n = j]\text{ i.o.}) = 0$ or equivalently $\lim_{n\to\infty} P(\bigcup_{k\geq n} [I_k=j]) = 0$. Therefore

$$P[I_n = j] \leq P(\bigcup_{k\geq n} [I_k = i]) \to 0 \text{ as } n \to \infty .$$

Proposition (6.36): Suppose j is max-rec. This gives no information about the existence of the weak limit $\lim_{n\to\infty} P[I_n = j]$.

Proof: We construct an example where j is max-rec yet $\lim_{n\to\infty} P[I_n = j]$ does not exist. In Remark (5.2) we showed how to construct two distribution functions $H_1(\cdot)$, $H_2(\cdot)$, such that $\lim_{x\to\infty} \frac{1-H_1(x)}{1-H_2(x)}$ does not exist. If necessary the method of construction can be slightly modified to insure $1-H_1(x) \geq 1-H_2(x)$ for all large x. Let the stochastic matrix $\underset{\sim}{P}$ be defined by $p_{ij} = \frac{1}{2}$, $i,j = 1,2$ so that $\rho(x) = \frac{1}{2} H_1(x) + \frac{1}{2} H_2(x)$. Then $\lim_{n\to\infty} P[I_n = 1]$ does not exist since $\lim_{x\to\infty} \frac{1-H_1(x)}{1-\rho(x)} = \lim_{x\to\infty} (\frac{1}{2} + \frac{1}{2}(\frac{1-H_2(x)}{1-H_1(x)}))^{-1}$ does not exist. But 1 is max-rec since

$$\int_M^\infty \frac{dH_1(x)}{1-\rho(x)} = \int_M^\infty \frac{dH_1(x)}{\frac{1}{2}(1-H_1(x)) + \frac{1}{2}(1-H_2(x))} \geq \int_M^\infty \frac{dH_1(x)}{1-H_1(x)} = \infty .$$

Proposition (6.33) showed how to construct an example where j was max-rec and $\lim_{n\to\infty} P[I_n = j] = \ell_j > 0$ and Proposition (6.34) showed how to construct an example where j was max-rec and $\lim_{n\to\infty} P[I_n = j] = 0$.

<u>Proposition (6.37)</u>: Maximum-transience is not a class property. In fact, it is impossible for all states to be max-trans.

<u>Proof</u>: Suppose all states are max-trans. Setting $A_n^k = [X_n \text{ is a record}, J_{n-1} = k]$ gives $\sum_{k=1}^m P A_n^k = P[X_n \text{ is a record}]$. Therefore $\sum_{n=1}^\infty P[X_n \text{ is a record}] = \sum_{n=1}^\infty \sum_{k=1}^m P A_n^k = \sum_{k=1}^m (\sum_{n=1}^\infty P A_n^k) < \infty$ by Theorem (6.31, ii). Hence by the Borel-Cantelli Lemma $P\{[X_n \text{ is a record}] \text{ i.o.}\} = 0$. It is impossible for there to be only a finite number of records a.s. as the following

dissection argument shows. Pick an arbitrary state j and let τ_0 be the time of the first visit to state j and let τ_n, $n \geq 1$ be the waiting times between visits to j. $\{\tau_n, n \geq 1\}$ is an i.i.d. sequence. Set $S_n = \sum_{k=0}^{n} \tau_k$ and $\chi_0 = \max\{x_1, \ldots, x_{\tau_0+1}\}$, $\chi_1 = \max\{X_{S_0+2}, \ldots, X_{S_1+1}\}, \ldots$, $\chi_n = \max\{X_{S_{n-1}+2}, \ldots, X_{S_n+1}\}$. The sequence $\{\chi_n, n \geq 1\}$ is i.i.d. and χ_n is a record value of the sequence $\{\chi_n, n \geq 0\}$ iff at least one of the random variables $X_{S_{n-1}+2}, \ldots, X_{S_n+1}$ is a record value of the sequence $\{X_n, n \geq 1\}$. But the events $\{[\chi_k$ is a record value of the sequence $(\chi_n, n \geq 1)]\}$ for $k \geq 1$ are independent and have probabilities k^{-1} [13]. Hence $\sum_{k=1}^{\infty} P[\chi_k$ is a record value of the sequence $(\chi_n, n \geq 1)] = \infty$ and by the Borel Zero-One Law:

$$P\{[\chi_k \text{ is a record value of the sequence } (\chi_n, n \geq 1)] \text{i.o.}\} = 1 .$$

With probability 1, the sequence $\{\chi_n, n \geq 1\}$ has infinitely many records and this is true for the sequence $\{\chi_n, n \geq 0\}$ since χ_0 is exceeded a.s. This completes the proof.

<u>Proposition (6.38)</u>: Maximum-recurrence is not a class property. State j max-rec does not necessitate all states being max-rec.

<u>Proof</u>: Pick two distribution functions $H_1(\cdot)$, $H_2(\cdot)$ such that $H_1(x_0) = 1$ for $x_0 < \infty$ and $H_2(x) < 1$ for all x.

Let $p_{ij} = \frac{1}{2}$, $i,j = 1,2$. Then $\rho(x) = \frac{1}{2} H_1(x) + \frac{1}{2} H_2(x)$

and $\int_{-\infty}^{\infty} \frac{dH_1(x)}{1-\rho(x)} \leq \frac{1}{1-\rho(x_0)} < \infty$. Therefore state 1 is

max-trans and by Proposition (6.26), state 2 is max-rec.

Proposition (6.39): Existence of weak limits is not a class property. The existence of $\lim_{n\to\infty} P[I_n = j]$ does not imply $\lim_{n\to\infty} P[I_n = k]$ exists for $k \neq j$. However, if all the weak limits exist, then they form a probability distribution:

$$\sum_{j=1}^{m} \lim_{n\to\infty} P[I_n = j] = 1 .$$

Proof: The last statement is proved by integrating by parts:

$$\int_{-\infty}^{\infty} \Big(\prod_{k\neq j} H_k^{\pi_k}(x)\Big)^n \, dH_j^{\pi_j n}(x) = 1 - \sum_{\alpha\neq j} \int_{-\infty}^{\infty} \Big(\prod_{k\neq\alpha} H_k^{\pi_k}(x)\Big)^n \, dH_\alpha^{\pi_\alpha}(x).$$

Hence:

$$\sum_{\alpha=1}^{m} \int_{-\infty}^{\infty} \Big(\prod_{k\neq\alpha} H_k^{\pi_k}(x)\Big)^n \, dH_\alpha^{\pi_\alpha n}(x) = 1.$$

It is easy to show that $\lim_{n\to\infty} P[I_n = 3] = 0$ does not imply that other states need have weak limits: Take any $H_3(\cdot)$ for which there exists $x_0 < \infty$ and $H_3(x_0) = 1$. As in (5.2) construct two distribution functions $H_1(\cdot)$, $H_2(\cdot)$ such that $H_1(x) < 1$, $H_2(x) < 1$ for all x and $\lim_{x\to\infty} \frac{1-H_1(x)}{1-H_2(x)}$ does not exist. Set $p_{ij} = \frac{1}{3}$, $i \leq i,j \leq 3$ and we have that $\lim_{n\to\infty} P[I_n = 3] = 0$ but neither $\lim_{n} P[I_n = 1]$ nor $\lim_{n\to\infty} P[I_n = 2]$ exist.

One can also construct an example where one state has a positive weak limit but the other states do not possess weak limits. If $1-H_1(x) = 2^{-x}$, $x \geq 0$ then $1-H_1(n) = 2^{-n}$. Define $1-H_2(x)$ as follows:

$$1-H_2(x) = 1 \qquad \text{if } x < 0$$

$$1-H_2(2n) = 2^{-2n}$$

$$1-H_2(2n-1) = 2^{-2n} \quad .$$

For remaining values of x, define $1-H_2(x)$ by linear interpolation so that

$$1-H_2(x) = 2^{-(2n+2)}[1 + 3(2n+1-x)] \quad \text{if } x \epsilon (2n, 2n+1)$$

$$= 2^{-(2n+2)} \qquad \text{if } x \epsilon (2n+1, 2n+2) \; .$$

Then $\frac{1-H_1(2n)}{1-H_2(2n)} = 1$ and $\frac{1-H_1(2n+1)}{1-H_2(2n+1)} = 2$ so that $\lim_{x\to\infty} \frac{1-H_1(x)}{1-H_2(x)}$ does not exist.

Define $a(x) = (1-H_1(x)) - (1-H_2(x))$

$$= 2^{-x} - 2^{-(2n+2)}[1 + 3(2n+1-x)] \quad \text{if} \quad x\in(2n, 2n+1)$$

$$2^{-x} - 2^{-(2n+2)} \qquad \text{if} \quad x\in(2n+1, 2n+2)$$

and set

$$1-H_3(x) = 1-H_1(x) + a(x)$$

$$= 2^{-(x-1)} - 2^{-(2n+2)}[1 + 3(2n+1-x)] \quad \text{if} \quad x\in(2n, 2n+1)$$

$$2^{-(x-1)} - 2^{-(2n+2)} \qquad \text{if} \quad x\in(2n+1, 2n+2).$$

Then $1-H_3(x)$ is the tail of a distribution function and

$\frac{1-H_1(2n)}{1-H_3(2n)} = 1$, $\frac{1-H_1(2n+1)}{1-H_3(2n+1)} = \frac{2}{3}$ so $\lim_{x\to\infty} \frac{1-H_1(x)}{1-H_3(x)}$ does not

exist. Letting $p_{ij} = \frac{1}{3}$, $1 \le i,j \le 3$ gives

$$1-\rho(x) = \tfrac{1}{3}(1-H_1(x)) + \tfrac{1}{3}(1-H_2(x)) + \tfrac{1}{3}(1-H_3(x)) = 1-H_1(x) .$$

Therefore $\frac{1-H_1(x)}{1-\rho(x)} = 1$ but $\lim_{x\to\infty} \frac{1-H_2(x)}{1-\rho(x)}$ and $\lim_{x\to\infty} \frac{1-H_3(x)}{1-\rho(x)}$

do not exist. Hence state 1 has a positive weak limit but states 2,3 do not have weak limits.

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