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ON SUBSET SELECTION RULES WITH CERTAIN OPTIMALITY PROPERTIES

A Thesis

Submitted to the Faculty

of

Purdue University

by

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In Partial Fulfillment of the

Requirements for the Degree

of

Doctor of Philosophy

August 1970

Meinen Eltern gewidmet

ACKNOWLEDGMENTS

I thank Professor S.S. Gupta for his continual interest in my work and for the guidance and encouragement he has given me since he first introduced me into the field of selection and ranking problems in two seminar courses in 1965-1966. I appreciate the invitation to work under him which he extended to me as well as his provision of research support via the Office of Naval Research Contract NONR 1100(26) and the Aerospace Research Laboratories Contract AF 33(615)C1244.

I express my thanks to the Operations Research Center and to the Computer Center of the University of California which provided the facilities to continue my work, while I was at Berkeley during Professor Gupta's visit in the academic year 1968-69.

To Dr. Panchapakesan I extend my appreciation for many helpful discussions and suggestions.

I thank Myrna Harrison for the excellent typing of this thesis.

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ABSTRACT

Nagel, Klaus Richard O., Ph.D., Purdue University, August 1970. On Subset Selection Rules with Certain Optimality Properties. Major Professor: Shanti S. Gupta.

Suppose $\pi_i = (X_i, A_i, P_i)$, i=1,2,...,k, are k probability spaces called populations. Assume that $X_i = X$ and $A_i = A$ for all i and P_i belongs to a family $\mathcal{P}$ possessing an order relation $\succ$. Let $\Omega = \{(P_1, P_2, \ldots, P_k)\}$ denote the space of joint probability measures and $\Omega_o \subseteq \Omega$ be the subspace where all P_i's are identical. The theory of selection and ranking is concerned with the determination-partial or complete- of the order among the populations based on the observed random events $X_i \in A_i$, i=1,2,...,k. The investigations in this thesis are made under the subset selection formulation of Gupta [11], where the objective is to define a rule R which selects a subset of the k populations with the guarantee that the probability of a correct selection, i.e. the selection of a subset which includes the population which is associated with $P_i \succ P_j$ for all j, called the best population, is at least a pre-assigned value P*. In case of ties, one of the populations is arbitrarily tagged as the best. Letting $P_\omega(CS|R)$ denote the probability of a correct selection for $\omega \in \Omega$, the P*-condition above can be written as $\inf_{\omega \in \Omega} P_\omega(CS|R) \geq P^*$. Chapter I deals with the central problem of evaluating $\inf_{\Omega} P_\omega(CS|R)$. Subset selection rules are formally defined and it is shown in Section 1.4 that, for R restricted to the class of rules invariant under permutation, certain

monotonicity requirements on the rules guarantee that $\inf_{\Omega} P_\omega(CS|R) = \inf_{\Omega_0} P_\omega(CS|R)$ and also that the rule selects better populations with higher probability. These requirements which are, in general, hard to verify are shown in Section 1.5 to be satisfied by "just" rules, which are defined for stochastically ordered families as follows: If $p_i(x_1,x_2,\ldots x_k)$ denotes the probability of selecting π_i based on the observation vector $(x_1,x_2,\ldots,x_k)$ then $p_i(x_1,x_2,\ldots,x_k) \leq p_i(y_1,y_2,\ldots,y_k)$, where $y_i \geq x_i$ and $y_j \leq x_j$ for $j \neq i$. These results are true for a generalized definition of stochastic ordering given in Section 1.5. In the next section many of the rules investigated in the literature are shown to be just and a procedure defined in [28] based on ranks is shown not to be just, which throws some light on the difficulties encountered in [28]. In Section 1.8, two just and translation invariant rules are constructed to be used in the normal means case, the first rule maximizing $P_\omega(CS)$ and the second minimizing the expected subset size $E_\omega(S)$ for some predetermined ω. Since Ω_0 itself is translation invariant, the translation invariant rules yield constant $P_\omega(CS)$ for $\omega\epsilon\Omega_0$ and thus $P_{\omega_0}(CS|R) \geq P^*$ for some fixed $\omega_0 \epsilon \Omega_0$ implies the P*-condition. Chapter II deals with the case where there is no transformation requiring invariance naturally. In this case, constant $P_\omega(CS)$ for $\omega\epsilon\Omega_0$ can be realized by imposing the condition $P_\omega(CS|T=t)=P^*$ for all t, where $\omega=(\theta,\theta,\ldots,\theta)\epsilon\Omega_0$ and T is sufficient for θ. Applications are made to the case of gamma populations and various discrete distributions and tables are provided in the appendix. Testing against slippage alternatives which in the normal case leads to Gupta's rule is extended to the Koopman-Darmois family, yielding selection rules with constant $P_\omega(CS)$ in Ω_0. Chapter III contains discussions on a

locally optimal selection rule based on ranks and a rule for the problem of selecting the multinomial cell with the largest probability which minimizes P(CS) when the cell probabilities are equal and which maximizes $P_{\omega}(CS)$ for all slippage configurations.

INTRODUCTION

The rejection of the hypothesis that the k tested populations are identically distributed is in most cases not satisfactory enough to the experimenter. What he really wants to know is not whether the populations are different, but which ones are significantly different from the others and how big the deviations are. For the case where all but one are identical an answer was found by Mosteller [29] testing homogeneity against slippage alternatives. The general problem has been approached in two different ways. The first, known as indifferent zone approach, states the goal as selecting the best population with a predetermined guaranteed probability provided that this best one exceeds the others by a preassigned amount in terms of a suitable defined distance function. This formulation is due to Bechhofer [4]. The second, which is the one used in the investigations of the present thesis, is the subset selection approach formulated by Gupta [11], where the goal is the selection of a subset containing the best population with a preassigned probability P^*. Here the number of populations retained in the selected subset is a random variable. The best population may be the one with highest (lowest) parameter value if the distributions come from a one-parameter family, in general it is the one which precedes the others with respect to some (partial) order relation. In case of ties one of the qualifying populations is arbitrarily tagged as the best one. The selection of a subset including the best population is called a correct selection (CS)

so that the above P*-condition can be written as $\inf_{\omega\in\Omega} P_\omega(CS) \geq P^*$ where Ω denotes the space of joint distribution functions under consideration. The P*-condition alone does not necessarily lead to useful rules, e.g. the rule which selects all populations regardless of the observations satisfies it on any level P*. For a bibliography on these two approaches see [14] and [19].

Using a maximum likelihood test for slippage hypotheses Gupta [11] derived in the case of means from normal populations the following rule: Include every population in the selected subset whose observed sample mean is at a distance less than D from the maximum of the sample means where the constant D is determined to guarantee the P*-condition. Procedures of this type have successfully been applied to many other families of distributions [12,16,22]. Deverman [8] uses a generalized procedure of this kind for selecting a collection of fixed size subsets one of which has to contain at least c of the t best populations. Barron [3] investigates a sequential subset selection procedure which on each stage uses a rule of this type. In most cases it could be shown that the worst configuration lies in Ω_o, the subset of Ω where all populations have identical distributions. This reduces the evaluation of $\inf_{\Omega} P_\omega(CS)$ to the problem of minimizing univariate function so that the constants necessary to guarantee the P*-condtion can be computed. Moreover one expects a reasonable rule to yield the smallest probability of a correct selection when all populations are identically distributed, because in that case no statistical information can be employed to select the one which has arbitrarily been tagged as the best one. Gupta and Nagel [18] showed that this is not true for the problem of selecting the cell with highest (lowest) probability from k multinomial cells. McDonald [28]

studies rules of this type based on rank sums where it could not be shown that the worst configuration lies in Ω_o, while Rizvi and Woodworth [12] constructed a bimodal distribution for which it is actually not the case.

One way to overcome these difficulties would be finding sufficient conditions on the distributions which would guarantee that the worst configuration lies in Ω_o. Alternatively, we can achieve this goal, as in this thesis, by imposing certain conditions on the rules. In Section 1.4 monotonicity requirements are stated which yield this desired property regarding the worst configuration, and which further guarantee that better populations are selected with higher probabilities for rules invariant under permutations. In general these conditions are hard to verify, but in the case of stochastically ordered families they are satisfied for "just" rules which are defined as follows: If $p_i(x)$ denotes the probability of selecting π_i on observing $x = (x_1, x_2, \ldots, x_k)$ then $p_i(y) \geq p_i(x)$ holds for any vector $y = (y_1, y_2, \ldots, y_k)$ with $y_i \geq x_i$, $y_j \leq x_j$ for $j \neq i$. These results remain valid for a generalized definition of stochastic ordering defined in Section 1.5. Examples in Section 1.6 show that many of the rules considered in literature are just. The difficulties in [28] with a rule based on rank sums are partially due to the fact that the rule is not just. In Section 1.8 an attempt is made to construct two just and translation invariant rules to be used in the normal means case, the first rule maximizing $P_\omega(CS)$ and the second minimizing the expected subset size $E_\omega(S)$ for some alternative ω. The conjectured solution of the second problem is numerically shown to be closely approximated by Gupta's rule. Since Ω_o itself is translation invariant, translation

invariant rules yield constant $P_\omega(CS)$ for $\omega\epsilon\Omega_o$ and thus $P_{\omega_o}(CS) \geq P^*$ for some fixed $\omega_o \epsilon\ \Omega_o$ implies the P*-condition. For the case where there is no transformation for which invariance could be required naturally it is shown in Section 2.1 that the condition $P_\omega(CS\,|T = t) = P^*$ for all t guarantees constant probability of a correct selection in Ω_o, where T is a sufficient statistic for θ in $\omega = (\theta,\theta,\ldots,\theta)\epsilon\Omega_o$. In the remainder of Chapter II rules of this type are investigated for gamma populations and for binomial, Poisson, negative binomial and Fisher's logarithmic distributions. Tables for applications are provided in the Appendix. In Section 2.7 the above mentioned method of Gupta [11] is applied to the Koopman-Darmois family of distributions, yielding selection rules with constant $P_\omega(CS)$ in Ω_o.

In Chapter III a just and locally optimal selection rule based on ranks is derived. For the problem of selecting a subset containing the multinomial cell with largest probability a rule is given in Section 3.2 which minimizes $P_\omega(CS)$ when all cells have equal probabilities and maximizes if for all slippage configurations.

CHAPTER I

SUBSET SELECTION RULES

1.1. Subset Selection as Part of Multiple Decision Theory

The theory of subset-selection is part of the general decision theory obtained from it by a specific choice of the decision space.

The Space of Observations X^k.

We are given k probability spaces $(X_i, \mathcal{B}_i, P_i)$, i=1,2,...,k, henceforth referred to as populations π_i, i=1,2,...,k. We assume that the X_i's and $\mathcal{B}_i$'s are identical: $X_i = X$, $\mathcal{B}_i = \mathcal{B}$, i=1,2,...,k, and that all probability measures P_i belong to some family P. From each population we observe a random element X_i. The space of observation is:

$$X^k = \{(x_1,\ldots,x_k),\ x_i \in X\}$$

In most applications X^k will be a real vector space.

The Decision Space $\mathcal{D}$.

The decision space $\mathcal{D}$ consists of the 2^k subsets d of the set {1,2,...,k}:

$$\mathcal{D} = \{d \mid d \subseteq \{1,2,\ldots,k\}\}\ .$$

A decision is the selection of a subset of the k populations. $i \in d$ indicates that π_i is included in the selected subset if decision d is made.

Remark 1.1.1. Sometimes authors exclude $\emptyset$ from $\mathcal{D}$ to guarantee the selection of a nonempty subset. If one does so, one should consequently remove

$\{1,2,\ldots,k\}$ from $\mathcal{D}$ also, since that decision does not lead to a partition into selected and not selected populations.

The Loss Function.

There is a great variety of reasonable loss functions for the subset selection problem. We will restrict ourselves to loss functions based on two criteria:

a) Correct selection (CS)

b) Subsetsize (S)

We assume that a partial order relation ($\succ$) is given in P :

$P_i \succ P_j$ saying P_i is better than or equal to P_j or P_i is preferred to P_j. $\pi_i \succ \pi_j$ is used equivalently. e.g. If P is a one-parameter family $P_i(x)=P(\theta_i,x)$, we may define: $P_i \succ P_j$ iff $\theta_i \geq \theta_j$. In a great part of this thesis $P_i \succ P_j$ will denote stochastic ordering.

Def. 1.1.1. Assume $\pi_j \succ \pi_i$ for all i. The decision $d \in \mathcal{D}$ is called a correct selection (CS) if $j \in d$, i.e. if the best population is included in the selected subset. If there are several π_{j_ℓ}, $\ell \leq k$, such that $\pi_{j_\ell} \succ \pi_i$ for all i, then one of the π_{j_ℓ}'s is arbitrarily tagged as the best one and its inclusion in the selected subset is called a correct selection.

Throughout this thesis we will be concerned only with selection rules R, which guarantee a correct selection with a probability at least equal to a predetermined level P^*, that is to say

$$\inf_{\Omega} P_\omega\{CS|R\} \geq P^*, \tag{1.1.1}$$

where $\Omega \subseteq P^k$ is the space of joint probability measures under consideration. The points of Ω are denoted by $\omega=(P_1,P_2,\ldots,P_k)$, $P_i \in P$, and P_ω

denotes the corresponding product measure with respect to $\mathcal{B}^k$. In the case of parameter families $\mathcal{P}=\{P_\theta\}, \Omega$ is used for the space of parameter vectors equivalently. $\Omega_o \subseteq \Omega$ denotes the subset of Ω where all P_i's are identical.

Remark 1.1.2. The reason for tagging one population as the best one in the case of ties is done to guarantee

$$\lim_{\omega \to \omega_o} P_\omega(CS|R) = P_{\omega_o}(CS|R) \qquad \text{for } \omega_o \in \Omega_o, \tag{1.1.2}$$

whenever a meaningful convergence can be defined in Ω.

(1.1.1) will be referred to as the basic probability requirement or P*-condition. It should be pointed out that the P*-condition alone does not necessarily lead to reasonable rules, e.g. the degenerate rule select all populations no matter what the observations are, always satisfies (1.1.1).

Therefore, the expected subset size is introduced as a second criterion which evaluates the performance of a selection procedure.

Def. 1.1.2. Expected subset size $E_\omega(S|R)$ where S is the random number of populations included in the selected subset using rule R and $\omega \in \Omega$.

Obviously $0 \leq E_\omega(S|R) \leq k$ holds. Frequently, instead of $E_\omega(S|R)$ we will use the

Def. 1.1.3. Expected proportion

$$\lambda_\omega(R) = \frac{1}{k} E_\omega(S|R).$$

The range is $0 \leq \lambda_\omega(R) \leq 1$.

In general our objective will be to construct rules yielding a large probability of a correct selection and a small value for the expected proportion, but unfortunately we cannot control these two quantities simultaneously. An increase of $P(CS|R)$ will be accompanied by an increase of $E_\omega(S|R)$ (or $\lambda_\omega(R)$). Hence the ratio

$$\eta_\omega(R) = \frac{P_\omega(CS|R)}{\lambda_\omega(R)} \tag{1.1.3}$$

is a reasonable measure for the performance of R.

Def. 1.1.4. $\eta_\omega(R)$ is called the efficiency of R at ω. It is obvious that a random rule - i.e. one which makes a random selection disregarding the observations-has efficiency one.

For each of the following two goals optimal rules will be investigated:

Goal 1: Maximize $P_\omega(CS|R)$ with $\inf_{\omega'\in\Omega} P_{\omega'}(CS|R) = P^*$

Goal 2: Minimize $E_\omega(S|R)$ or equivalently $\lambda_\omega(R)$.

The maximization and minimization for both 1) and 2) are for a fixed alternative $\omega\in\Omega$ and subject to the P^*-condition, which for 1) is to hold with the equal sign: $\inf_{\omega\in\Omega} P_\omega(CS|R) = P^*$.

1.2. Definition of Selection Procedures

Def. 1.2.1. A measurable function δ defined on $X^k \times \mathcal{D}$ is called a selection procedure (or selection rule) provided that for each observation $x\in X^k$

$$\delta(x,d) \geq 0 \quad \text{and} \tag{1.2.1}$$

$$\int_{\mathcal{D}} \delta(x,d) = 1 \quad \text{hold.}$$

$\delta(x,d)$ denotes the probability that subset d is selected when x is observed.

Def. 1.2.2. $p_i(x) = \sum_{d \ni i} \delta(x,d)$ (Summation over all subsets containing i) is the probability that π_i is included in the selected subset. $p_1(x),\ldots,p_k(x)$ are called the individual selection probabilities.

If the p_i's take on the values 0 and 1 only then

$$\delta(x,d)=\begin{cases}1 & \text{if } d=\{i \mid p_i(x) = 1\} \\ 0 & \text{otherwise}\end{cases} \tag{1.2.2}$$

i.e. A nonrandomized procedure is completely determined by its individual selection probabilities. The following example shows that this is not true in general.

Example 1.2.1. k=2. For some fixed x selection rules δ and δ' are defined by

$$\begin{aligned} \delta(x,\emptyset) &= \tfrac{1}{4} & \delta'(x,\emptyset) &= 0 \\ \delta(x,\{1\}) &= \tfrac{1}{4} & \delta'(x,\{1\}) &= \tfrac{1}{2} \\ \delta(x,\{2\}) &= \tfrac{1}{4} & \delta'(x,\{2\}) &= \tfrac{1}{2} \\ \delta(x,\{1,2\}) &= \tfrac{1}{4} & \delta'(x,\{1,2\}) &= 0 . \end{aligned}$$

Both rules have the same individual selection probabilities $p_1(x)=p_2(x)=p_1'(x)=p_2'(x)=\frac{1}{2}$.

We can now express the probability of a correct selection and the expected subset size in terms of the selection rule δ. Assume that π_k is the best population, then

$$P_\omega(CS|\delta) = E_\omega \sum_{d \ni k} \delta(X,d) = E_\omega p_k(X). \tag{1.2.3}$$

We see that $P_\omega(CS|\delta)$ depends on δ only through the individual selection probabilities. The same is true for the expected subset size: Let $s(d)$ denote the number of elements in d

$$(1.2.4)\quad E_\omega(S|\delta) = E_\omega \sum_{d\in\mathcal{D}} s(d)\delta(X,d) = E_\omega \sum_{d\in\mathcal{D}} \sum_{i\in d} \delta(X,d) = E_\omega \sum_{i=1}^{k} \sum_{d\ni i} \delta(X,d) =$$

$$E_\omega \sum_{i=1}^{k} p_i(X).$$

Since we are only concerned with $P(CS|\delta)$ and $E(S|\delta)$ we can consider two rules with equal individual selection probabilities as equivalent. Therefore we will from now on use the following simplified definition of a selection rule.

Def. 1.2.3. A subset selection rule R is a measurable mapping from X^k into R^k (the k-dimensional Euclidean space) $R: x \to (p_1(x),\ldots,p_k(x))$, $0 \le p_i(x) \le 1$.

Def. 1.2.4. If the p_i's are restricted to the values 0 and 1 then the rule is called nonrandomized.

In that case R can also be defined by the sets $A_i = \{x\in X | p_i(x)=1\}$, $i=1,2,\ldots,k$. A_i is the set of observations for which π_i is selected.

In the sequel the use of the letter R will indicate that the modified definition 1.2.3 was used. Let $P_{\omega,i} = E_\omega p_i(X)$ denote the probability that π_i will be included in the selected subset. For nonrandomized rules we get

$$(1.2.5)\qquad P_{\omega,i} = P_\omega(A_i).$$

Def. 1.2.5. R is unbiased iff $\pi_j \succ \pi_i$, i=1,2,...,k, implies $P_{\omega,j} \geq P_{\omega,i}$ for all $\omega \in \Omega$.

Def. 1.2.6. R is monotone iff $\pi_j \succ \pi_i$ implies $P_{\omega,j} \geq P_{\omega,i}$ for all i,j and for all $\omega \in \Omega$.

Obviously monotonicity implies unbiasedness.

Remark 1.2.1. The efficiency of an unbiased rule R is at least 1:

$$(1.2.6) \qquad \eta_\omega(R) = \frac{kP_\omega(CS|R)}{E_\omega(S|R)} = \frac{kP_{\omega,j}}{\sum_i P_{\omega,i}} \geq \frac{kP_{\omega,j}}{\sum_i P_{\omega,j}} = 1,$$

assuming that π_j is the best population.

1.3. Invariance Under Permutation

We will restrict ourselves to selection procedures which are invariant under permutations. Let G denote the group of permutations g of the integers 1,2,...,k:

$$(1.3.1) \qquad g(1,2,\ldots,k) = (g1,g2,\ldots,gk).$$

Let h be the inverse permutation of g, $h=g^{-1}$, and define

$$(1.3.2) \qquad g(x_1,\ldots,x_k) = (x_{h1},\ldots,x_{hk})$$

and for $d \in \mathcal{D}$

$$(1.3.3) \qquad gd = \{i|hi \in d\}.$$

Let G(i,j) be the following subset of G

(1.3.10) $$\lambda_\omega = \frac{1}{k!} \sum_{g \in G} E_{g\omega} p_k(X),$$

which in the case of nonrandomized rules, becomes

(1.3.11) $$\lambda_\omega = \frac{1}{k!} \sum_{g \in G} P_{g\omega}(A_k).$$

1.4. A Sufficient Condition for $\inf_{\Omega} P(CS|R) = \inf_{\Omega_o} P(CS|R)$

To satisfy the basic probability requirement (1.1.1)

(1.4.1) $$\inf_{\Omega} P(CS|R) \geq P^*$$

the left hand side of this inequality has to be evaluated, a task which often proves to be tedious or which could not be solved as in the case of rule R_1 in [28, 17]. On the other hand one expects a reasonable rule to yield the smallest probability of a correct selection in the case, where all populations are identically distributed, because in that case no statistical information can be employed to select that population which arbitrarily has been tagged as the best one. In this section a sufficient condition will be given for

(1.4.2) $$\inf_{\Omega} P(CS|R) = \inf_{\Omega_o} P(CS|R) \quad \text{to hold}$$

where $\Omega_o \subseteq \Omega$ denotes the subset of the space of distributions where all P_i's are identical. In the case of stochastically ordered families this sufficient condition will lead to the introduction of "just" selection rules.

In order to find the best population it is natural to base one's decision on such events from X, whose probabilities increase if a population is replaced by a better one, or in the case of randomized rules one expects the individual selection probability to go up when the population is improved.

Let r be a partial order relation in P, denoted by $P \succ Q$, meaning P is better than or preferred to Q. Let p(x) be a measurable function on X with $0 \leq p(x) \leq 1$ for all $x \in X$. Let Ψ denote the set of all such critical functions.

Def. 1.4.1. $R_r^+ = \{p \in \Psi \mid \int_X p dP \geq \int_X p dQ \quad \text{for all } P \succ Q;\ P,Q \in P\}$

$R_r^- = \{p \in \Psi \mid \int_X p dP \leq \int_X p dQ \quad \text{for all } P \succ Q;\ P,Q \in P\}$

For regions in the case of nonrandomized rules we get the corresponding

Def. 1.4.2. $A_r^+ = \{A \in \mathcal{B} \mid P(A) \geq Q(A) \quad \text{for all } P \succ Q;\ P,Q \in P\}$.

$A_r^- = \{A \in \mathcal{B} \mid P(A) \leq Q(A) \quad \text{for all } P \succ Q;\ P,Q \in P\}$

Remark 1.4.1. $p \in R_r^+$ iff $(1-p) \in R_r^-$

$A \in A_r^+$ iff $A^c \in A_r^-$

where A^c denotes the complement of A.

Remark 1.4.2. If r' ($\succ'$) is another partial order relation in P such that $P \succ' Q$ implies $P \succ Q$ then $R_r^+ \subseteq R_{r'}^+$ and $A_r^+ \subseteq A_{r'}^+$ hold.

Example 1.4.1. Let X be the real line and let P consist of all probability distributions on X. Let r be the stochastic order relation and define r' by

r': $P \succ' Q$ if P and Q are normal $N(\mu_P,1)$ and $N(\mu_Q,1)$ and $\mu_P \geq \mu_Q$.

Then A_r^+ consists of all intervals of the form $[a,\infty)$ or (a,∞), $-\infty \leq a \leq \infty$.

These intervals are also contained in $A_{r'}^{+}$, which in addition includes sets of the type $[a,\infty)-N$ where N has Lebesgue measure zero. Moreover, certain intervals of positive measure can be removed from $[a,\infty)$ and the remainder will still belong to $A_{r'}^{+}$, e.g.

(1.4.3) $$A = [-1,0] \cup [1,\infty) \in A_{r'}^{+}$$

To prove this we have to show that $P(A)$ increases in μ if P is normal $N(\mu,1)$.

(1.4.4) $$P(A) = \Phi(-\mu)-\Phi(-1-\mu)+1-\Phi(1-\mu)$$

(1.4.5) $$\frac{\partial P(B)}{\partial \mu} = \frac{1}{\sqrt{2\pi}} e^{\frac{\mu^2}{2}} [-1+e^{\frac{1}{2}} (e^{\mu}+e^{-\mu})]$$

$$\geq \frac{1}{\sqrt{2\pi}} e^{\frac{\mu^2}{2}} [-1+2e^{\frac{1}{2}}] > 0.$$

This example shows that it might be difficult to find $A_{r'}^{+}$ for a particular partially ordered family of distributions, while the structure of A_{r}^{+} for the stochastic ordering is simple. Hence we will look for sufficient conditions which guarantee $A \in A_{r}^{+}$ (or $P \in R_{r}^{+}$ for the randomized case).

If P is a one-parametric absolutely continuous family of cumulative distribution functions:

(1.4.6) $$P = \{F_{\lambda}, \lambda \in I \subseteq R^{1}\}$$

where an order is defined by $F_{\lambda} \succ F_{\lambda'}$ if $\lambda \geq \lambda'$ then in [30] the condition

(1.4.7) $$\frac{\partial}{\partial\lambda} F_\lambda(x) \frac{\partial}{\partial x} p(x) \leq 0$$

is shown to be sufficient for $\int_{-\infty}^{\infty} p(x)\, dF_\lambda(x)$ to be nondecreasing in λ.

Example 1.4.2. Let $F_\lambda(x) = F(\lambda x)$, $\lambda > 0$, where F is a given absolutely continuous cumulative distribution function, i.e. $\{F_\lambda\}$ is a scale parameter family. Define $\succ$ by $F_\lambda \succ F_{\lambda'}$ if $\lambda \geq \lambda'$. (1.4.7) becomes

(1.4.8) $$xF'(\lambda x)\, p'(x) \leq 0$$

(where $F' = \frac{d}{dx} F$, $p' = \frac{d}{dx} p$) hence

$$p'(x) \begin{cases} \geq 0 & \text{if } x < 0 \\ \leq 0 & \text{if } x > 0 \end{cases}$$

i.e. R_r^+ contains the critical functions p which are nonincreasing for positive x and nondecreasing for negative x.

Lemma 1.4.1. If the critical function $p(x_1,\ldots,x_k)$ as a function of x_i only belongs to R_r^+ for all $x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_k$, then $P_i \succ Q_i$ implies

(1.4.9) $$\int_\chi \cdots \int_\chi p\, dP_1,\ldots,dP_k \geq \int_\chi \cdots \int_\chi p\, dP_1,\ldots,dQ_i,\ldots,dP_k$$

Proof: Evident by definition of R_r^+.

Theorem 1.4.1. Let R be a subset selection rule, defined by its individual selection probabilities $p_i(x)$, $i=1,2,\ldots,k$. Consider $p_i(x_1,\ldots,x_k)$ as a function of x_j, for all $j \neq i$. If $p_i(x_1,\ldots,x_k) \in R_r^-$, then

(1.4.10) $$\inf_{\Omega} P(CS|R) = \inf_{\Omega_o} P(CS|R).$$

Proof: Without loss of generality we can assume $P_k \succ P_i$, i=1,2,...,k-1. Hence

(1.4.11) $$P_\omega(CS|R) = \int_X \cdots \int_X p_k(x_1,\ldots,x_k)dP_1,\ldots,dP_k.$$

By Lemma 1.4.1 the right hand side of (1.4.11) does not increase if $P_1,\ldots P_{k-1}$ are replaced by $Q_1,\ldots,Q_{k-1}$ with $Q_i \succ P_i$, i=1,...,k-1. Thus for every $\omega\epsilon\Omega$ there exists an $\omega_o\epsilon\ \Omega_o$ such that $P_{\omega_o}(CS|R) \leq P_\omega(CS|R)$ holds, which proves (1.4.10).

Theorem 1.4.2. Let R be a subset selection rule, defined by its individual selection probabilities $p_i(X_1,\ldots,X_k)$, i=1,2,...,k. If R is invariant under permutation and if

(i) $p_i\epsilon R^+$ as a function of x_i and any fixed set of x_j's, $j\neq i$

(ii) $p_i\epsilon R^-$ as a function of x_j, for all $j\neq i$ and any fixed choice of the other arguments hold, then R is monotone.

Proof: To prove monotonicity it has to be shown that $P_i \succ P_j$ implies $Ep_i = P\{\pi_i \text{ is selected}\} \geq P\{\pi_j \text{ is selected}\} = Ep_j$. Without loss of generality we can assume i=1,j=2. Then

(1.4.12) $$Ep_1 = \int_x \cdots \int_x p_1(x_1,\ldots,x_k)dP_1(x_1)\ldots dP_k(x_k) \geq$$

$$\int_x \cdots \int_x p_1(x_1,\ldots,x_k)\ dP_2(x_1)dP_2(x_2)\ldots dP_k(x_k)$$

because $p_1\epsilon R^+$ as a function of x_1 and $P_1 \succ P_2$, and changing the order of integration for the two innermost integrals, (1.4.12) becomes

(1.4.13) $$Ep_1 \geq \int\cdots\int p_1(x_1,\ldots,x_k)dP_2(x_2)dP_2(x_1)dP_3(x_3)\ldots dP_k(x_k)$$

and applying lemma 1.4.1 once more using the fact that $p_1 \varepsilon R^-$ as a function of x_2, we get:

(1.4.14) $$Ep_1 \geq \ldots \geq \int\cdots\int p_1(x_1,\ldots,x_k)dP_1(x_2)dP_2(x_1)dP_3(x_3)\ldots dP_k(x_k)$$

$$= \int\cdots\int p_2(x_1,\ldots,x_k)dP_1(x_1)dP_2(x_2)\ldots dP_k(x_k) = Ep_2,$$

where the identity follows from the invariance under permutation.

Theorem 1.4.1 allows the restriction to Ω_o to determine the infimum of the probability of a correct selection if R satisfies (1.4.10). By theorem 1.4.2 we can establish the monotonicity of a rule by verifying that the individual selection probabilities belong to R^+ or R^- as functions of one argument only. However, this might still be difficult for particular families as example 1.4.1 indicates. On the other hand many order relations imply stochastic ordering (st) and the structure of R^+_{st} is very simple. Stochastically ordered families in a generalized sense shall be investigated in the next section.

1.5. Just Selection Rules

Let $(X,\mathcal{B},P)$ be a probability space, where P belongs to a family $\mathcal{P}$ of probability measures. A partial order $\succ$ is given in X and we say y is the better than x if $y \succ x$ (or equivalently $x \prec y$).

Def. 1.5.1. A subset $A \subseteq X$ is increasing iff $x \varepsilon A$ and $y \succ x$ imply, $y \varepsilon A$.

Def. 1.5.2. Let $P,Q \varepsilon \mathcal{P}$. P is stochastically better than Q ($P \underset{st}{\succ} Q$) iff $P(A) \geq Q(A)$ for all increasing sets $A \varepsilon \mathcal{B}$.

If X is the real line and $\succ$ stands for $>$ (or $\geq$) then the increasing sets are the intervals $[a,\infty)$ and (a,∞), which induce the usual stochastic ordering on the distribution functions.

If $X = R^k$ and $\succ$ stands for component wise larger, then the increasing sets are the ones introduced in [26].

Our objective will be to include the stochastically best population in the selected subset. The following example illustrates the advantage of this extension of the concept of stochastic ordering.

Example 1.5.1. Let π_i be normal distributed $N(0,\sigma_i^2)$, $i=1,\ldots,k$. n independent observations $x_{i1},\ldots,x_{in}$ are taken from each of the k populations. Normal distributions with common mean are not stochastically ordered in the usual sense, with respect to their variance σ^2. Now, if we replace $\geq$ in X by the partial order relation $\succ$ defined by

(1.5.1) $$(x_{i1},\ldots,x_{ik}) \succ (x_{j1},\ldots,x_{jk}) \text{ iff}$$

$$\sum_{\ell} (x_{i\ell}-\bar{x}_i)^2 \geq \sum_{\ell}(x_{j\ell} - x_j)^2$$

and denote the generalized stochastic ordering by $\underset{st}{\succ}$ then

(1.5.2) $$N(0,\sigma^2) \underset{st}{\succ} N(0,\tau^2) \quad \text{iff } \sigma^2 \geq \tau^2.$$

This result is based on the fact that the sample variances in (1.5.1) have χ^2-distributions which are stochastically ordered with respect to their scale parameter. For a more illustrative example see (1.6.5).

The following lemma generalizes a result stated in [25].

Lemma 1.5.1. Let g be an integrable function on X such that $y \succ x$ implies $g(y) \geq g(x)$. If $P \underset{st}{\succ} Q$, then

$$E_P g = \int_X g dP \geq \int_X g dQ = E_Q g. \tag{1.5.3}$$

Proof: Let g be an indicator function $g(x) = X_A(x)$. From $g(y) \geq g(x)$ for $y \succ x$ follows that A is an increasing set and hence (1.5.3) holds by the definition of stochastic ordering. From indicator functions this result can be extended to integrable functions in the usual way.

Def. 1.5.3. A selection rule R, defined by its individual selection probabilities $p_i(x_1,\ldots,x_k)$, i=1,2,...,k, is called just iff

$$\text{(1.5.5)} \qquad \text{(i)} \quad x_i \prec y_i$$

$$\text{(ii)} \quad x_j \succ y_j \quad \text{for all } j \neq i$$

imply

$$p_i(y_1,\ldots,y_k) \geq p_i(x_1,\ldots,x_k)$$

For nonrandomized rules, determined by acceptance regions $A_1,\ldots,A_k$ we get the corresponding definition

Def. 1.5.4. R is just iff

$$\text{(i)} \quad x \in A_i$$

$$\text{(1.5.6)} \qquad \text{(ii)} \quad x_i \prec y_i$$

$$\text{(iii)} \quad x_j \succ y_j \quad \text{for all } j \neq i$$

imply $y \in A_i$

For symmetric rules it suffices to verify these conditions for one of the A_i's say A_k. In plain words just means that the probability of selecting population π_i does not decrease if the observation becomes more

favorable with respect to this population. The word "consistent" would describe that property better, but it was not used here to avoid confusion, since consistent normally describes asymptotic properties. The importance of just rules was recognized in [35, Lemma 3.1 (iv)]. There it was shown that an optimal rule for selecting the best of k populations with distributions from an exponential family under slippage configuration has this property.

Some of the results on just rules can be proved more easily using randomized rules, but we will see later that for the construction of optimal rules, we can restrict ourselves to nonrandomized procedures, if the probability measures are atomless.

Theorem 1.5.1. Let R be a just selection rule. Then

$$\inf_{\Omega} P(CS|R) = \inf_{\Omega_o} P(CS|R) \qquad \text{holds.} \tag{1.5.7}$$

Proof: Since R is just p_i is nondecreasing in x_i and hence by Lemma 1.5.1

$$\int_X p_i dP(x_i) \geq \int_X p_i dQ(x_i) \quad \text{if } P \underset{st}{\succ} Q$$

i.e. $p_i \in R^+_{st}$ as function of x_j, $j \neq i$.

Therefore Theorem (1.4.1) can be applied, yielding the assertion (1.5.7).

Theorem 1.5.2. Let R be just and symmetric. Then R is monotone.

Proof: In the proof of theorem 1.5.1 it was shown, that $p_1 \in R^-_{st}$ holds, when p_i is considered as a function of x_j, $j \neq i$, only. In the same way we get $p_i \in R^+_{st}$ as a function of x_i. Hence Theorem 1.4.2 can be applied proving the theorem.

Remark 1.5.1. The class of just procedures is closed under the formation of unions and intersections, i.e. if R and S are just rules, then $R \cap S$ and $R \cup S$ are just, where $R \cap S$ denotes the procedure, which selects that populations that are selected by both rules R and S, while $R \cup S$ includes a population in the selected subset, if at least one of the rules R or S selects it.

Lemma 1.5.2. Let (X,A,P) be a probability space, and let $\succ$ be a partial order relation in X which induces the stochastic ordering $\underset{st}{\succ}$ in P. Let (X^n,A^n,P^n) be the corresponding product space, and let $\underset{c}{\succ}$ (c stands for component-wise) be defined by $(x_1,\ldots,x_n) \underset{c}{\succ} (y_1,\ldots,y_n)$ iff $x_i \succ y_i$, $i=1,2,\ldots,n$. $\underset{c}{\succ}$ induces the stochastic order relation $\underset{cst}{\succ}$ in P^n. Then

$$P \underset{st}{\succ} Q \Leftrightarrow P^n \underset{cst}{\succ} Q^n.$$

Proof: $\Rightarrow$) Assume $P \underset{st}{\succ} Q$. Let $A^n \in A^n$ be an increasing set with respect to $\underset{c}{\succ}$. The indicator function $X_A(x_1,\ldots,x_n)$ satisfies the conditions of lemma 1.4.1 for each argument. Repeated application of this lemma with $P_i = P$, $Q_i = Q$, $i=1,2,\ldots,n$ yields

$$P^n(A^n) = \int_{X^n} X_A \, dP^n \geq \int_{X^n} X_A \, dQ^n = Q^n(A^n)$$

i.e. $P^n \underset{cst}{\succ} Q^n$.

$\Leftarrow$) Assume $P^n \underset{cst}{\succ} Q^n$. Let $A \in A$ be increasing with respect to $\succ$. Define $A^n = \{x \in X^n \mid x_1 \in A\}$. A^n is obviously increasing with respect to $\underset{c}{\succ}$, hence $P(A) = P(A^n) \geq Q(A^n) = Q(A)$, i.e. $P \underset{st}{\succ} Q$.

Remark 1.5.2. This result can be extended to infinite product spaces, since it is true on cylinder sets.

The definition of just rules and the results of this chapter show that the class of just rules possesses some very desirable properties. Another advantage is that this class is defined by a natural basic property of selection rules, which usually can be verified very easily. The definition does not depend on the specific form in which the rule is presented. On the other hand the class of just rules is not too restrictive, it includes most of the selection procedures considered in the literature. Hence it seems justifiable to construct optimal procedures in the subclass of just procedures.

1.6. Examples of Just Rules

Example 1.6.1. Seal's class C

Assume that the populations $\pi_1,\ldots,\pi_k$ are stochastically ordered in the usual sense and that we observe the random variables $X_1,\ldots,X_k$. Let $X_{[1]},\ldots,X_{[k-1]}$ be the ordered set of the first k-1 components of the observation-vector x. In [34] the following class C of symmetric selection rules is defined:

Select π_k iff

$$X_k \geq \sum_{j=1}^{k-1} c_j X_{[j]} - t \tag{1.6.1}$$

where the c_j's are given constants satisfying

$$\sum_{i=1}^{k-1} c_i = 1, \quad c_i \geq 0, \quad i=1,2,\ldots,k-1. \tag{1.6.2}$$

Any rule from C is just: Let $(x_1,\ldots,x_k)$ satisfy (1.6.1) and let $(y_1,\ldots,y_k)$ be such that $y_i \leq x_i$ for $i < k$, $y_k \geq x_k$ holds. Then y satisfies (1.6.1) because the left hand side of (1.6.1) is nondecreasing and the right hand side is nonincreasing, when the x's are replaced by the y's.

Example 1.6.2. Gupta's Rules

The specific choice $c_i=0$ $i=1,2,\ldots,k-2$, and $c_{k-1}=1$ in (1.6.1) yields the rule:

Select π_k, iff

$$X_k \geq X_{[k-1]} - t. \tag{1.6.3}$$

For $t \geq 0$ $X_{[k-1]}$ can be replaced by $\max_{i=1,\ldots,k} X_i = X_{max}$, without changing the rule, which now can be written as

$$\text{Select } \pi_i \quad \text{iff} \quad X_i \geq X_{max} - d. \tag{1.6.4}$$

In example (1.6.1) the condition $\Sigma c_i=1$ is not needed to prove that the rules of the class C are just. Therefore the same proof applies to the following rule:

$$\text{Accept } \pi_i \quad \text{if } X_i \geq c\ X_{max}. \tag{1.6.5}$$

These rules were proposed by Gupta [11] originally for selecting a subset containing the population with highest mean (resp. variance) in the case of normal distributions, and have later been applied to other distributions [8,11,16,18,20,22,24,28,30].

Example 1.6.3. The following generalization of Gupta's rule was proposed in [11] and more thoroughly investigated in [30]:

R_h: Select π_i iff

$$h(x_i) \geq x_{max} \tag{1.6.6}$$

where h satisfies some additional requirements which are irrelevant in this context. These rules are obviously just if h is nondecreasing.

Remark 1.6.1. In [30] the monotonicity of the rule (1.6.6) was shown. Also

$$\inf_{\Omega} P(CS|R) = \inf_{\Omega_o} P(CS|R)$$

was shown without the restriction that h is nondecreasing i.e. for not necessarily just rules of this type. In the proof of theorem 1.5.1 property (1.5.5 (i)) (resp. (1.5.6 (ii)) of just rules was not used. Thus the assumptions of theorem (1.5.1) can be weakened, such that theorem (1.5.1) would imply this result in [30].

Example 1.6.4. In [28] the following rank sum procedure, called R_3 was considered: Take from each population π_i the n observations $x_{i1},\ldots,x_{in}$ and let R_{ij} denote the rank of x_{ij} in the pooled sample $x_{11},\ldots,x_{kn}$. Define

$$H_i = \sum_{j=1}^{n} a(R_{ij})$$

where the scores $a_j = a(j)$, $j=1,\ldots,kn$ are given and nondecreasing in j. R_3 is defined as follows. Select π_i if $H_i \geq c$ where c is to be determined to satisfy the P*-condition.

R_3 is just,because for any fixed i, an increase in X_{ij} results in no decrease of H_i, neither does an decrease in $X_{\ell j}$ for $\ell \neq i$.

Example 1.6.5. Barron [6] considers a sequential subset selection rule of the following type: Take a sequence of observations $X_i^{(\ell)}$, i=1,...,k, ℓ=1,2,..., from k normal populations. (The $X_i^{(\ell)}$ may be sample means from n independent observations on the ith populations). Define a sequence of binomial random variables $Y_i^{(\ell)}$, where $Y_i^{(\ell)} = 1$ if population π_i would be selected using Gupta's rule $X_i^{(\ell)} \geq X_{max}^{(\ell)} - d$ on the observations of the ℓth stage, while $Y_i^{(\ell)} = 0$ denotes the rejection. Define

$$S_{im} = \sum_{\ell=1}^{m} Y_i^{(\ell)} .$$

The procedure is: Select π_i iff $S_{im} > b_m$ for the smallest value of m for which $S_{im} \notin [a_m, b_m]$, where $[a_m, b_m]$ is a predetermined sequence of intervals.

If we drop the assumption that the populations are normal and require only stochastic ordering and if we replace Gupta's rule by an arbitrary single stage selection rule R, then we can construct a sequential rule $R^S(a,b)$ in the same way as in [3]. (a,b) denotes the sequence of intervals $[a_\ell, b_\ell]$.

Lemma 1.6.1. If R is just with respect to the partial order relation $\succ$, then R^S is just with respect to the corresponding component wise ordering $\underset{c}{\succ}$.

Proof: Let $(X_k^{(1)}, X_k^{(2)}, \ldots,) \underset{c}{\succ} (\tilde{X}_k^{(1)}, \tilde{X}_k^{(2)}, \ldots,)$ and $(X_i^{(1)}, X_i^{(2)}, \ldots,) \underset{c}{\prec} (\tilde{X}_i^{(1)}, \tilde{X}_i^{(2)}, \ldots,)$ for all $i \neq k$ hold. Since R is just the selection of π_k if $\tilde{X}_k^{(\ell)}$ is observed at the ℓth stage will imply the selection of π_k if $X_k^{(\ell)}$

is observed, for every stage ℓ . Hence $y_k^{(\ell)} \geq \tilde{y}_k^{(\ell)}$ and $S_{i\ell} \geq \tilde{S}_{i\ell}$ hold for all ℓ, where $y_k^{(\ell)}$ and $\tilde{y}_k^{(\ell)}$ are the values of the binomial random variable corresponding to $X_k^{(\ell)}$ and $\tilde{X}_k^{(\ell)}$, therefore R^S will select π_k under X, if it does so under $\tilde{X}$.

This lemma shows that the sequential procedure so constructed is monotone with respect to the stochastic ordering in P^∞ induced by $\succ_c$ and that further

$$\inf_{\Omega} P(CS|R^S) = \inf_{\Omega_Q} P(CS|R^S) \quad \text{holds,}$$

where a correct selection now consists in including the best population with respect to the order relation $\succ_{cst}$ in P^∞. But, by Lemma 1.5.2 and Remark 1.5.2, we know that this is equivalent with the order relation $\succ_{st}$ in P.

Example 1.6.6. A rule, which is not just. Let H_i be defined as in example 1.6.4 with a(j)=j. In [4] the rule R_1 was introduced by:

Select π_i iff $H_i \geq \max_{j=1,\ldots,k} H_j$-D.

Choose k = 3 n = 2 D = 7.

Compare the following two sets of observations and the corresponding values of H_i, i=1,2,3.

	X_{11}	X_{12}	X_{21}	X_{22}	X_{31}	X_{32}	H_1	H_2	H_3	H_{max}
1st:	1	3	6	12	9	11	3	9	9	9
2nd:	2	4	5	7	8	10	3	7	11	11

X_{11} and X_{12} obtain higher values in the second set, while all other values are less in the second set, i.e. the observations are strictly more favorable for population π_1 in the second set. However R_1 selects π_1 in the first case and excludes it in the second.

This example makes it somewhat understandable that in [28] the infimum of the probability of a correct selection could not be found, even in the case of location parameters for normal distributions, and that for less well behaved distributions examples have been constructed with

$$\inf_{\Omega} P(CS|R_1) < \inf_{\Omega_o} P(CS|R_1).$$

1.7. Elimination of Randomization

We will now see that randomization is not necessary in order to construct optimal just selection rules that are invariant under permutation if all the probability measures under consideration are atomless. Let R be a just rule specified by its individual selection probabilities $p(x) = (p_1(x),\ldots,p_k(x))$, and since we assume symmetry the knowledge of one of its components say p_k is sufficient. If we assume that π_k is the best population, then P*-condition becomes

$$\text{(1.7.1)} \qquad \inf_{\omega\in\Omega_o} P_\omega(CS|R) = \inf_{\omega\in\Omega_o} \int_X p_k(x_1,\ldots,x_k)\,dP_\omega \geq P^*$$

(with the equal sign for goal 1). The requirement that R is just yields the condition

(1.7.2) $p_k(x_1,\ldots,x_k)$ is nondecreasing in x_k and nonincreasing in $x_1,x_2,\ldots,x_{k-1}$, both with respect to the partial order in X.

The goal of maximizing $P_\omega(CS|R)$ or minimizing $E_\omega(S|R)$ for some alternative $\omega\in\Omega$ is equivalent to obtaining the extremum of

$$(1.7.3) \qquad \Psi(R) = \int_X p_k(x_1,\ldots,x_k)\, dQ,$$

where Q denotes the probability measure P_ω or the one corresponding to $E_\omega(S|R)$, see 1.3.7.

At first we notice that the set of p_k's satisfying 1.7.1 and 1.7.2 is convex: Let p' and p'' satisfy 1.7.1 and 1.7.2. Define $p=\alpha p'+(1-\alpha)p''$ for $0 \leq \alpha \leq 1$.

(i) $0 \leq p \leq 1$

(ii) $\inf_\Omega \int p\,dP_\omega \leq \alpha \inf_\Omega \int p'\,dP_\omega + (1-\alpha) \inf_\Omega \int p''\,dP_\omega \geq P^*$

(iii) p decreases resp. increases in the same coordinates as p' and p''.

Using the arguments of [9,10] it follows that the linear functional $\Psi(R)$ cannot have any extrema in the interior of this convex region. The only difference arising from the requirement that R is just, is that the boundary of that convex region contains in addition to the nonrandomized rules, i.e. zero-one functions p_k, also the random rule

$$(1.7.4) \qquad p_k(x) \equiv P^*,$$

as an extreme point. This rule, which makes no use of the observation, is of no interest. Hence we can restrict ourselves to nonrandomized rules, if the probability measures are atomless.

1.8. Two Selection Rules for Normal Populations

Let $\pi_1,\pi_2,\ldots,\pi_k$ be normal populations with common known variance σ^2 which, without loss of generality, can be assumed to be 1. The density for X_i from π_i is given by

$$f(x,\theta_i) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\theta_i)^2} \tag{1.8.1}$$

We are interested in selecting a subset containing the population with highest mean θ_i. Since the θ_i's are location parameters it is natural to require the selection rule to be invariant under shift in location. This can be achieved by basing the rule on the differences

$$Z_i = X_i - X_k, \quad i=1,2,\ldots,k, \tag{1.8.2}$$

which have the joint density function

$$h_\omega(z_1,\ldots,z_{k-1}) = \int_{-\infty}^{\infty}\Big[\prod_{i=1}^{k-1} f(z_i+t,\theta_i)\Big]\, f(t,\theta_k)\,dt = \tag{1.8.3}$$

$$(2\pi)^{\frac{k-1}{2}}\sqrt{k}\; e^{-\frac{1}{2}\sum_{i=1}^{k}[(z_i-\theta_i)^2-\bar{z}^2]}$$

with $\bar{z} = \frac{1}{k}\sum_{i=1}^{k-1}(z_i-\theta_i)$, $z_k=0$ and $\omega = (\theta_1,\theta_2,\ldots,\theta_k)$.

Since the distributions are continuous we do not need randomization and a selection rule will be determined by the sets $A_i=\{(x_1,\ldots,x_k) \mid \pi_i$ is selected$\}$. Because of the symmetry we can restrict ourselves to the

determination of one of the A_i's, A_k say. Let A_k' denote the corresponding set of $(z_1,\ldots,z_{k-1})$'s.

(1.8.4) $$A_k' = \{(z_1,\ldots,z_{k-1}') \mid \pi_k \text{ is selected}\}$$

The condition that A_k is just:

(1.8.5) $(x_1,\ldots,x_k)\in A_k$, $y_k \geq x_k$, $y_i \leq x_i$, $i\neq k$ implies $(y_1,\ldots,y_k)\in A_k$,

reduces for A_k' to the condition:

(1.8.6) $(z_1,\ldots,z_{k-1})\in A_k'$, $v_i \leq z_i$, $i=1,\ldots,k-1$ implies $(v_1,\ldots,v_{k-1})\in A_k'$,

i.e. A_k' is a decreasing set with respect to the greater or equal relation. (1.8.3), as a function of ω, is invariant under shift of the parameter vector, hence, for the convenience of notation, we can assume

(1.8.7) $$\sum_{i=1}^{k} \theta_i = 0$$

Denote $\omega_o = (0,\ldots,0)$

$\omega = (\theta_1,\ldots,\theta_k)$, $\Sigma\theta_i = 0$.

Goal (i). Construct a symmetric just subset selection rule R such that

(1.8.8) $$P_{\omega_o}(CS|R) = P^* \text{ holds and}$$

(1.8.9) $$\frac{1}{k!}\sum_{g\in G} P_{g\omega}(CS|R) \text{ is maximized.}$$

$G = \{g\}$ is the permutation group of k! elements (compare 1.3). For just and symmetric rules (1.8.8) guarantees the P*-condition. (1.8.9) is the requirement that the expected probability of a correct selection is maximized when the k! alternatives $g\omega$, $g\in G$, are equally likely. If we assume $\theta_k \geq \theta_i$ $i=1,\ldots,k$, goal (i) is equivalent to finding a set A_k' such that

(1.8.10) $$P_{\omega_o}(A_k') = P^*,$$

(1.8.11) $$\frac{1}{(k-1)!} \sum_{g\in G(k,k)} P_{g\omega}(A_k') \quad \text{maximal and}$$

(1.8.12) A_k' is a decreasing set.

By (1.8.3) and (1.8.7) conditions (1.8.10) and (1.8.11) become

(1.8.13) $$\int_{A_k'}\ldots\int dP_{\omega_o} = (2\pi)^{-\frac{k-1}{2}} k^{-\frac{1}{2}} \int_{A_k'}\ldots\int e^{-\frac{1}{2}\sum_{i=1}^{k}(z_i^2-\bar{z}^2)} dz_1,\ldots,dz_{k-1} = P^*$$

(1.8.14) $$\int_{A_k'}\ldots\int d\tilde{P}_{\omega} = (2\pi)^{-\frac{k-1}{2}} k^{-\frac{1}{2}} \int_{A_k'}\ldots\int \frac{1}{(k-1)!} \sum_{g\in G(k,k)} e^{-\frac{1}{2}\sum_{i=1}^{k}[(z_i-\theta_{gi})^2-\bar{z}^2]} dz_1,\ldots,dz_{k-1}$$

large, where $\bar{z} = \frac{1}{k}\sum_{i=1}^{k} z_i$.

Without condition (1.8.12) a solution would be supplied by the lemma of Neyman and Pearson. This suggests the investigation of the density ratio

$$(1.8.15) \qquad r(z) = \frac{d\tilde{P}_{\omega}(z)}{dP_{\omega_o}(z)} = \frac{e^{-\frac{1}{2}\sum_{i=1}^{k}\theta_i^2}}{(k-1)!} \sum_{g\in G(k,k)} e^{\sum_{i=1}^{k-1}\theta_{gi}z_i} .$$

$r(z)$ is a convex function as a sum of convex functions of linear-and hence convex-functions. $r(z_1+t,\ldots,z_{k-1}+t)$ is a decreasing function in t:

$$r(z_1+t,\ldots,z_{k-1}+t) = \frac{e^{-\frac{1}{2}\sum_{i=1}^{k}\theta_i^2}}{(k-1)!} \sum_{g\in G(k,k)} e^{t\sum_{i=1}^{k-1}\theta_{gi}+\sum_{i=1}^{k-1}\theta_{gi}z_i} =$$

$$e^{-\theta_k t}\,\frac{dP_{\omega}(z)}{dP_{\omega_o}(z)},$$

and from (1.8.7) and $\theta_k \geq \theta_i$, i=1,...,k follows $\theta_k \geq 0$.

For k=3, Figure 1 indicates the shape of the curves of constant density ratio and the conjectured sets A_k' in the z_1z_2-plane.

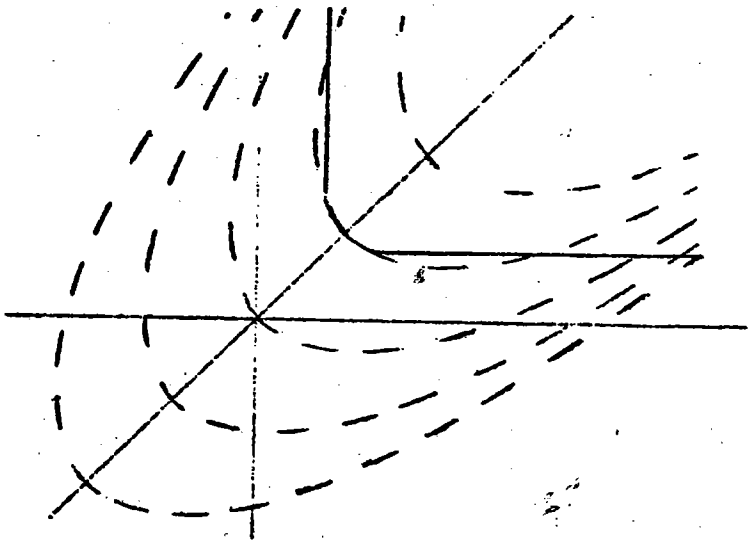

Figure 1. Curves of Constant Density Ratio and One A_k' for Goal (i)

A_k' is the region to the left and below the bold line.

A set A_k' can only be a solution if $P_\omega(B') \leq P_\omega(C')$ holds for any pair of sets B', C' such that $B' \cap A_k' = \emptyset$, $C' \subseteq A_k'$, $P_{\omega_o}(B') = P_{\omega_o}(C')$ hold and such that $(A_k' - C') \cup B'$ is a decreasing set. Choosing sets B' and C' with $P_{\omega_o}(B') = P_{\omega_o}(C')$ tending to zero, this condition leads to the:

Conjecture 1.8.1. $A_k' = \{\{z | r(z) \leq c\}^* - \{z | \min_{i=1,\ldots,k-1} z_i \leq d\}\}^c$

is a solution for (1.8.10), (1.8.11), (1.8.12) where B^c denotes the complement of B, and B^* denotes the smallest increasing set containing B and where c and d are determined to satisfy

(1.8.16) $$\int_{A_k'} \cdots \int dP_{\omega_o} = P^* \qquad \text{and}$$

(1.8.17) $$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int \cdots \int_{\{d \leq z_{\min} \leq d+\varepsilon\} \cap A_k'} dP_\omega = c \lim_{\varepsilon \to 0} \int \cdots \int_{\{d \leq z_{\min} \leq d+\varepsilon\} \cap A_k'} dP_{\omega_o},$$

This region A_k' is essentially determined by $\{z | \min z_i \leq d\}$, which for the x_i's would lead to the selection rule

(1.8.18) $$\text{Select } \pi_i \text{ iff } x_i \leq \min_{j \neq i} x_j + D$$

This rule has two disadvantages: First, it selects an empty subset if $D > 0$ (i.e. for $P^* < \frac{k-1}{k}$) and the range of the x_i is less than D (compare Remark 1.1.1). Second, if the configuration of the θ_i's is such that the smallest one is essentially smaller than the others, this rule will tend to select all the other $k-1$ populations no matter how far their means

are apart. This shows that goal (i) does not necessarily lead to useful rules.

Goal (ii). Construct a symmetric just subset selection rule R such that

(1.8.19) $$P_{\omega_o}(CS|R) = P^* \quad \text{holds and}$$

(1.8.20) $$E_{\omega}(S|R) \quad \text{is minimized.}$$

Again the rule will be determined by the set A_k for the X_i's or by A_k' for the shift invariant variables $Z_i = X_i - X_k$, i=1,2,...,k, so that (1.8.19) and (1.8.20) become

(1.8.21) $$\int_{A_k'} \cdots \int dP_{\omega_o} \equiv (2\pi)^{-\frac{k-1}{2}} k^{-\frac{1}{2}} \int_{A_k'} \cdots \int e^{-\frac{1}{2}\Sigma[z_i^2 - \bar{z}^2]} dz_1, \ldots, dz_{k-1} = P^*$$

(1.8.22) $$\int_{A_k'} \cdots \int dQ_{\omega} \equiv (2\pi)^{-\frac{k-1}{2}} k^{-\frac{1}{2}} \int_{A_k'} \cdots \int \frac{1}{(k-1)!} \sum_{g\in G} e^{-\frac{1}{2}\sum_{i=1}^{k}[(z_i - \theta_{gi})^2 - \bar{z}^2]} dz_1, \ldots, dz_{k-1}$$

small, where

(1.8.23) $$A_k' \quad \text{is a decreasing set.}$$

A solution would be provided by the lemma of Neyman and Pearson if condition (1.8.23) was not present. The density ratio in this case is

(1.8.24) $$r(z) = \frac{dQ_{\omega}(z)}{dP_{\omega_o}(z)} = \frac{e^{-\frac{1}{2}\sum_{i=1}^{k}\theta_i^2}}{(k-1)!} \sum_{g\in G} e^{\sum_{i=1}^{k}\theta_i z_i}$$

The sets $\{z|r(z) \leq c\}$ are still convex, but $r(z_1+t,\ldots,z_{k-1}+t)$ is no longer a decreasing function of t.

Figure 2 shows the shape of the lines of constant density ratio for k=3 and the conjectured solution A_k' for various values of P*.

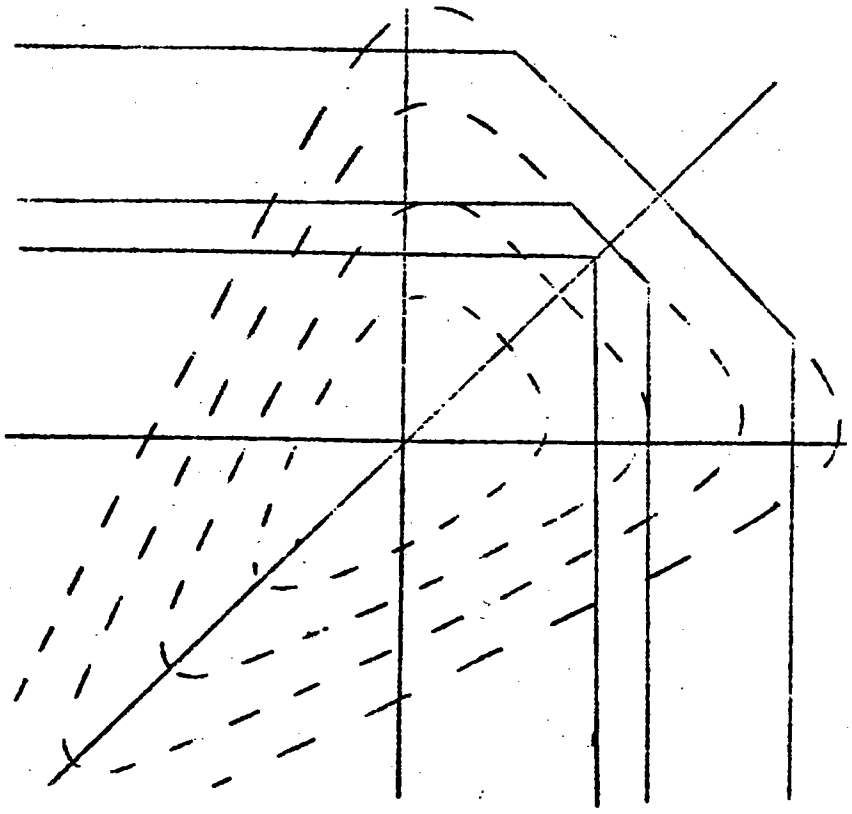

Figure 2. Curves of Constant Density Ratio and Three A_k''s for Goal (ii)

By the same arguments as in goal (i) we get the

Conjecture 1.8.2.

(1.8.25) $$A_k' = \{z|r(z) \leq c\}_* - \{z|\max_{i=1,\ldots,k-1} z_i \geq d\}$$

is a solution for goal (ii), where B_* denotes the smallest decreasing set containing B and where c and d are determined to satisfy

(1.8.26) $$\int_{A_k'} \cdots \int dP_{\omega_o} = P^*$$

(1.8.27) $$\lim_{\varepsilon \to o} \frac{1}{\varepsilon} \int \cdots \int_{\{d \le z_{max} \le d+\varepsilon\}} \int_{A_k'} dQ_\omega = c \lim_{\varepsilon \to o} \frac{1}{\varepsilon} \int \cdots \int_{\{d \le z_{max} \le d+\varepsilon\}} \int_{A_k'} dP_{\omega_o}$$

The corresponding rule will be denoted by R_{opt}. For k=3 and various configurations $(\theta_1, \theta_2, \theta_3)$ the values of c and d have been determined numerically and it turned out that A_k' is essentially given by the set $\{z \mid \max_i z_i \le d\}$. For sufficiently small values of P* the complement of the second term of (1.8.25) is contained in the first term so that

(1.8.28) $$A_k' = \{z \mid \max z_i \le d\} \quad \text{holds.}$$

For the x_i's this leads to Gupta's selection rule:

(1.8.29) $$R_G: \text{ Select } \pi_i \text{ if } x_i \ge \max_{j=1,\ldots,k} x_j - d.$$

Table 1 gives the values d and the corresponding values of P* up to which Gupta's rule is identical with the conjectured solution for goal (ii).

R_{opt} is difficult to apply because the density ratio cannot be evaluated easily. Another disadvantage is that this rule depends on a fixed alternative ω and it is usually not known how to choose it. On the other hand Table 1 shows that R_G is optimal up to rather high values of P*. Table 2 compares the expected proportion ($\frac{1}{k}$ ES) of Gupta's rule with that of the optimal one for three parameter configurations with P*-values above the ones given in Table 2. It also gives the values of c and d for R_{opt}

TABLE 1. Values of d and the Corresponding P* up to Which the Rule R_G Minimizes $E_\omega S$

ω			d	P*
-10^{-6}	-10^{-6}	2.10^{-6}	2.033	.8713
-1/6	-1/6	1/3	11.937	1.0000
-10^{-4}	0.	10^{-4}	2.057	.8749
-0.5	0.	0.5	2.123	.8849
-1.	0.	1.	2.329	.9125
-2.	0.	2.	3.218	.9787
-1.	0.5	0.5	1.579	.7839
-2.	1.	1.	1.836	.8369
-3.	1.5	1.5	2.388	.9194

(1.8.25). For slippage alternatives R_G is optimal up to extremely high values of P* and for equally spaced θ_i's no ω was found for which the ratio $E_\omega(S|R_{opt})/E_\omega(S|R_G)$ was less than .9999. R_{opt} yields the maximum improvement in the case $\theta_1 < \theta_2 = \theta_3$ but even there the ratio $E_\omega(S|R_{opt})/E_\omega(S|R_G)$ was never found to be less than .97.

TABLE 2. Comparison of R_{opt} and R_G. Constants for R_{opt}

ω	-1,0.5,0.5	-2,1.,1.	-2.4,1.2,1.2
p^*	.8577	.9887	.9928
c	1.0374	4.4899	5.9383
d	1.9694	4.5202	4.9458
$E_\omega(S\|R_{opt})/3$	.7642	.8107	.7704
$E_\omega(S\|R_G)/3$	.7643	.8294	.7926
$E_\omega(S\|R_{opt})/E_\omega(S\|R_G)$	.9998	.9775	.9720

CHAPTER II

RULES WITH CONSTANT P(CS|R) IN Ω_o

2.1. A Sufficient Condition for P(CS|R) to be Constant in Ω_o

In the last section invariance under certain transformations was used for the construction of optimal just rules. Its importance lies not so much in the fact that it allows us to consider a smaller class of selection rules, but that these invariant rules yield constant probability of a correct selection in Ω_o:

$$\int_\chi p_i(x)\, dP_{\omega_o} = \int_\chi p_i(x)\, dP_{\omega_o'}, \quad i=1,2,\ldots,k, \tag{2.1.1}$$

for all $\omega_o, \omega_o' \in \Omega_o$ and for all invariant rules. By Theorem 1.5.1 this guarantees the P*-condition for just rules.

For an arbitrary family of distributions in general there exists no transformation for which one could naturally require invariance for selection rules. However, invariance under those transformations is only a sufficient condition for (2.1.1). Therefore, we can replace the invariance condition by the requirement, that (2.1.1) is satisfied, and we can try to find other sufficient conditions to guarantee (2.1.1). It is reasonable to require that the probability of a correct selection is constant in Ω_o, because in stating the P*-condition one expresses that one is content if a correct selection is guaranteed with a certain probability and one is not interested in exceeding that probability, at least not in Ω_o

where it can be achieved only by increasing the expected subset size. The following example illustrates this situation.

Example 2.1.1. Let $\pi_1,\ldots,\pi_k$ be binomial with success probabilities $p_1,\ldots,p_k$. Take n observations from each and denote by x_i the number of successes. For selecting a subset containing the population with highest p_i the following rule R was proposed and investigated in [22].

(2.1.2) R: Select π_i iff $x_i \geq x_{max}-d$ where $d=d(k,n,P^*)$ is the smallest integer such that the P*-condition is satisfied.

This rule is just (see example 1.6.2), hence the minimum of $P(CS|R)$ takes place in Ω_o, but depends on the common value p. It is minimized for some value of p close to $\frac{1}{2}$. If p approaches 0 or 1 the probability of selecting all k populations tends to 1. This disadvantage was mitigated by using the transformation

(2.1.3) $$Z_i = 2 \text{ arc sin } \sqrt{X_i/n}$$

and applying Gupta's rule to the Z_i's, which has the same effect as using smaller values of d in (2.1.2) if the observations indicate that the common p lies near the ends of the unit interval.

In section 2.3 a selection rule will be proposed which has constant $P_{\omega_o}(CS|R)$ for $\omega_o \epsilon \Omega_o$.

The following lemma solves the problem of how to construct a rule with constant $P_{\omega_o}(CS|R)$ for $\omega_o \epsilon \Omega_o$ for families of distributions, permitting sufficient statistics.

(2.2.2) $$f_{\omega_o}(x_1,x_2) = \Gamma^{-2}(\theta)e^{(\theta-1)[\log x_1+\log x_2]}e^{-(x_1+x_2)}$$

It belongs to the Koopman-Darmois family and shows that $T=X_1X_2$ is a sufficient statistic for θ.

Our objective is to construct a just symmetric rule for selecting a subset containing the population with the highest θ_i. Having continuous distributions we can restrict ourselves to non-randomized rules. Since the rule R is to be just it follows that the region of acceptance for one of the populations, π_1 say, is of the form:

(2.2.3) $$A_1 = \{x_1 \geq c(P^*, T)\}.$$

(2.1.4) yields the condition

(2.2.4) $$\int_{c(P^*,T)}^{\infty} \frac{1}{x} f(x,\frac{T}{x})dx = \int_{0}^{\infty}\frac{1}{x} f(x,\frac{T}{x})dx \cdot P^*$$

from which $c(P^*,T)$ can be evaluated.

Although by construction this rule is just on each set where T is constant, it does not follow that R is just. It will now be shown that R is just by proving that $x_1(T)$ and $x_2(T)$ are increasing functions, where $(x_1(T),x_2(T))$ are the coordinates of the boundary of A_1. Substituting (2.2.2) in (2.2.4) yields

(2.2.5) $$\int_{c(P^*,T)}^{\infty} e^{-(x+\frac{T}{x})}dx = P^* \int_0^{\infty} e^{-(x+\frac{T}{x})}dx$$

from which we get by differentiation with respect to T

Lemma (2.1.1). Let $X_1,\ldots,X_k$ be independently and identically distributed random variables with the joint distribution P_θ. Let $T(X_1,\ldots,X_k)$ be a sufficient statistic for θ.

(2.1.4) (i) If $E(\delta(X_1,\ldots,X_k)|\,T)=P^*$ for all T then $E_\theta\delta = P^*$ for all θ.

(ii) If T is complete with respect to $\{P_\theta(X)\}$, then $E_\theta(\delta(X_1,\ldots,X_k)\,|T)=P^*$ is also necessary for $E_\theta\delta = P^*$ for all θ.

Proof. (i) $E_\theta\delta = \int_X \delta \; dP_\theta = \int_T E(\delta|T)\; dP_\theta^T = P^*$ where T and P_θ^T are the range and the distribution of T. (ii) $\int_T E(\delta|T)\; dP_\theta^T = P^*$ hence $\int_T (E(\delta|T)-P^*)\; dP_\theta^T = 0$ and because of the completeness $E(\delta|T)=P^*$ for all T q.e.d.

In the next sections this lemma will be applied to various families of distributions.

2.2. Shape Parameter for Two Gamma Populations

Let π_1, π_2 have gamma distributions with common known variances, $\sigma^2 = 1$ say, the densities given by

(2.2.1)
$$f_i(x) = \begin{cases} 0 & , \; x \leq 0 \\ \\ \dfrac{1}{\Gamma(\theta_i)} x^{\theta_i - 1} e^{-x}, & x > 0 \end{cases} \quad , \; i=1,2$$

Take $\omega_o = (\theta,\theta) \in \Omega_o$. The joint density of x_1, x_2 is

TABLE 3. C_T for Two Gamma Distributions

T \ P*	0.75	0.90	0.95	0.99
.1	.1518	.0827	.0596	.0347
.2	.2364	.1383	.1030	.0628
.3	.3050	.1857	.1410	.0883
.4	.3648	.2282	.1756	.1121
.5	.4186	.2672	.2077	.1346
.6	.4681	.3037	.2380	.1561
.7	.5142	.3381	.2668	.1768
.8	.5575	.3708	.2943	.1968
.9	.5987	.4021	.3208	.2162
1.0	.6379	.4322	.3463	.2351
2.0	.9634	.6889	.5679	.4031
3.0	1.2213	.8988	.7525	.5478
4.0	1.4428	1.0823	.9158	.6783
5.0	1.6404	1.2481	1.0644	.7987
6.0	1.8207	1.4008	1.2021	.9115
7.0	1.9877	1.5434	1.3312	1.0181
8.0	2.1441	1.6777	1.4533	1.1197
9.0	2.2917	1.8051	1.5695	1.2170
10.0	2.4320	1.9267	1.6807	1.3106
20.0	3.5814	2.9387	2.6152	2.1114
30.0	4.4788	3.7428	3.3659	2.7684
40.0	5.2428	4.4342	4.0154	3.3437
50.0	5.9204	5.0515	4.5979	3.8639
60.0	6.5361	5.6152	5.1314	4.3434
70.0	7.1045	6.1376	5.6272	4.7911
80.0	7.6353	6.6271	6.0925	5.2130
90.0	8.1352	7.0893	6.5328	5.6135
100.0	8.6091	7.5285	6.9517	5.9958
200.0	12.4655	11.1317	10.4062	9.1794
300.0	15.4523	13.9481	13.1221	11.7108
400.0	17.9837	16.3473	15.4433	13.8883
500.0	20.2219	18.4762	17.5074	15.8330
600.0	22.2508	20.4111	19.3866	17.6092
700.0	24.1206	22.1980	21.1243	19.2558
800.0	25.8640	23.8668	22.7489	20.7985
900.0	27.5039	25.4388	24.2806	22.2555
1000.0	29.0569	26.9294	25.7341	23.6402

$$(2.2.6)\qquad \left[\frac{\partial}{\partial T}\, c(P^*,T)\right] e^{-\left(c+\frac{T}{c}\right)} = P^* \int_0^{\infty} \frac{1}{x^2} e^{-\left(x+\frac{T}{x}\right)} dx - \int_{c(P^*,T)}^{\infty} \frac{1}{x^2} e^{-\left(x+\frac{T}{x}\right)} dx$$

and since $\frac{1}{x^2}$ is decreasing it follows from (2.2.5) that the right hand side of (2.2.6) is non-negative proving that $x_1(T,P^*) = c(P^*,T)$ is non-decreasing in T and the same is true for

$$(2.2.8)\qquad x_2(T,P^*) = x_1(T,1-P^*)$$

which completes the proof.

Table 3 gives the values of $c(P^*,T)$ for $P^* = 0.75, 0.90, 0.95, 0.99$ and $T = i \cdot 10^j$, $i = 1(1)10$, $j = -1(1)2$.

For $k > 2$ condition (2.1.4) can be satisfied in many ways, so that further conditions are necessary to determine a rule uniquely.

2.3. Selection Rules for Binomial Populations

Let $\pi_1,\ldots,\pi_k$ have binomial distributions: $X_i \sim B(\theta_i,n)$, $i=1,2,\ldots,k$. Denote $\omega = (\theta_1,\ldots,\theta_k)$, $\Omega = \{\omega\}$ and $\Omega \supseteq \Omega_o = \{\omega | \theta_i = \theta,\ i=1,\ldots,k\}$. We will construct a just selection rule R for including the population with highest θ_i such that

$$(2.3.1)\qquad P_{\omega_o}(CS|R) = P^* \quad \text{for all } \omega_o \in \Omega_o \text{ holds.}$$

It is clear that this goal can be achieved only by the use of randomized procedures, because in the case $\omega = (0,\ldots,0)$ or $\omega = (1,\ldots,1)$ the observation will be $x = (0,\ldots,0)$ or $x = (n,\ldots,n)$ with probability one, requiring the use of individual selection probabilities $p_i = P^*$, $i=1,\ldots,k$ in those cases.

The density with respect to the counting measure of the binomial distribution with success probability θ and n trials is

$$(2.3.2) \qquad p_\theta(x) = \binom{n}{x} \theta^x (1-\theta)^{n-x}$$

$$= (1-\theta)^n \exp[x \log \frac{\theta}{1-\theta}] \binom{n}{x}.$$

Hence the joint density for $\omega \epsilon \Omega_o$ is

$$(2.3.3) \qquad f_\omega(x_1,\ldots,x_k) = \prod_{i=1}^{k} p_\theta(x_i) = (1-\theta)^{nk} \exp[\log \frac{\theta}{1-\theta} \sum_{i=1}^{k} x_i] \prod_{i=1}^{k} \binom{n}{x_i}$$

showing that $T = \sum_{i=1}^{k} X_i$ is a sufficient statistic for θ. $\theta' = \log \frac{\theta}{1-\theta}$ is an increasing function of θ, thus selecting the population with highest θ' is the same as selecting the one with highest θ. Hence we have to determine individual selection probabilities p_i- and again because of the symmetry it suffices to find one, say p_k- which satisfy.

$$(2.3.4) \qquad E(p_k(X) \mid \Sigma X_i = T) = P^* \quad \text{for } T = 0,1,\ldots,kn.$$

and if

$$(2.3.5) \qquad y_i \leq x_i,\ i=1,2,\ldots,k-1,\ y_k \geq x_k \quad \text{then}$$

$$p_k(x_1,\ldots,x_k) \leq p_k(y_1,\ldots,y_k).$$

Figure 3 shows the partial ordering induced by (2.3.5) among the observation vectors for the case k=3, n=2. A_3 leads to a just rule if it contains with any observation x all observations ranking above x,

indicated by arrows in the figure. Because of the symmetry only one of the permutations (x_1,x_2,x_3) and (x_2,x_1,x_3) is plotted. The numbers under the observation vectors denote the corresponding $T(x) = \sum_{i=1}^{3} x_i$ values.

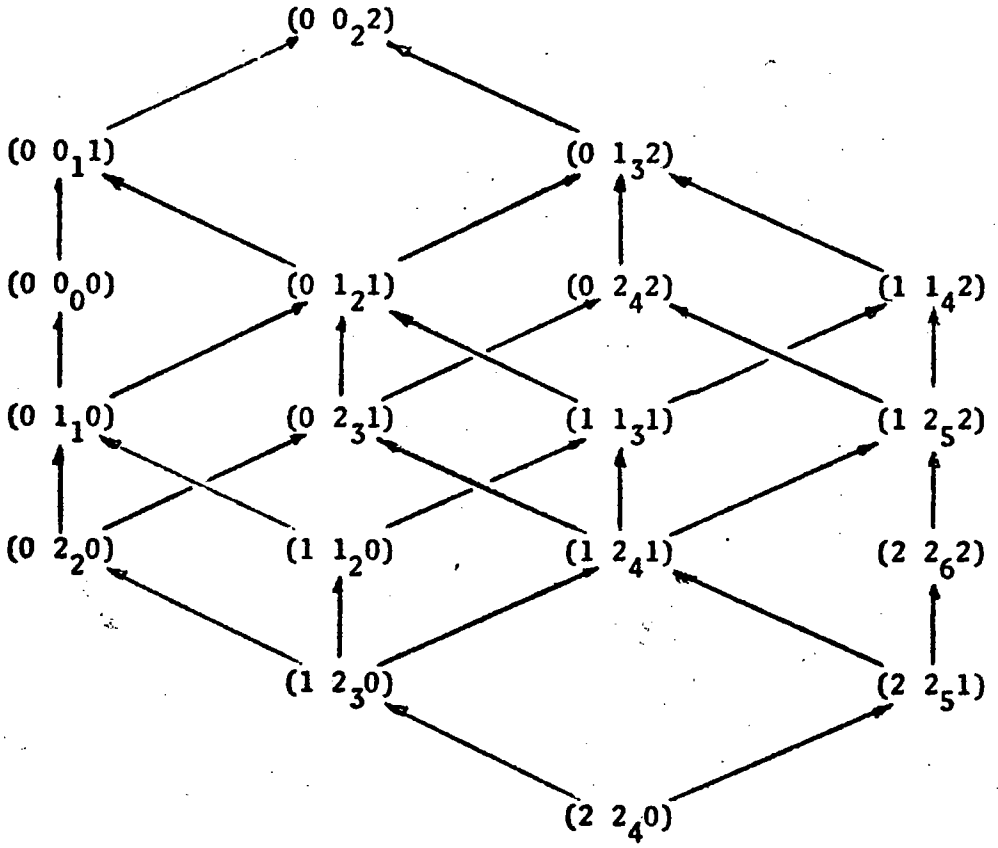

Figure 3. Partial Ordering for Binomial Observations k=3, n=2.

Conditions (2.3.4) and (2.3.5) do not determine a procedure uniquely. We propose the following rule R_o:

$$p_k(x) = \begin{cases} 1 & \text{if } x_k > c_T \\ \rho & \text{if } x_k = c_T \\ 0 & \text{if } x_k < c_T \end{cases}, \tag{2.3.6}$$

where $\rho = \rho(T,P^*)$ and $c_T = c_T(P^*)$, $(T = \sum_{i=1}^{k} x_i)$, are determined to satisfy (2.3.4):

$$E(p_k(X)|T) = P\{X_k > c_T|T\} + \rho P\{X_k = c_T|T\} = P^*. \tag{2.3.7}$$

The conditional distribution of X_k given T is hypergeometric:

$$P\{X_k = i|T\} = \frac{\binom{n}{i}\binom{(k-1)n}{T-i}}{\binom{kn}{T}}. \tag{2.3.8}$$

To simplify the notation let us introduce a random variable Z_T which is distributed as $(X_k|T)$:

$$P\{Z_T = i\} = P\{X_k = i|T\} \quad \text{for } i=0,1,2,\ldots, . \tag{2.3.9}$$

(2.3.7) then becomes

$$P\{Z_T > c_T\} + \rho P\{Z_T = c_T\} = P^* \tag{2.3.10}$$

and the integers c_T can be determined from the inequalities

$$P\{Z_T > c_T\} \leq P^* \tag{2.3.11}$$

(2.3.12) $$P\{Z_T \geq c_T\} > P^*.$$

(2.3.10) yields

(2.3.13) $$\rho = \frac{P^* - P\{Z_T > c_T\}}{P\{Z_T = c_T\}}$$

Lemma 2.3.1. R_o is just.

Proof: Let $x=(x_1,\ldots,x_k)$, $y=(y_1,\ldots,y_k)$ and denote the preference relation (2.3.5) by $y \underset{k}{\succ} x$. Define $T_x = \sum_{i=1}^{k} x_i$.

Case 1. $T_x = T_y$.

In this case $y \underset{k}{\succ} x$ implies $p_k(y) \geq p_k(x)$ and the assertion follows from (2.3.6).

Case 2. $T_x \neq T_y$.

It suffices to show $p_k(y) \geq p_k(x)$ for those pairs (x,y) where y ranks immediately above x i.e. $y \underset{k}{\succ} x$ and there is no y' such that $y \underset{k}{\succ} y' \underset{k}{\succ} x$ holds.

There are two types of such y's for each x.

Type 1.

$y_k = x_k+1$, $y_i = x_i$ for $i \neq k$, hence $T_y = T_x+1$. If $p_k(x)=1$ then by (2.3.11) $P\{Z_T > x_k\} \leq P^*$ holds, therefore

(2.3.14) $$P\{Z_{T_y} > y_k\} = P\{Z_{T_x+1} > x_k+1\} \leq P\{Z_{T_x} > x_k\} \leq P^*,$$

hence $p_k(y) = 1 \geq p_k(x)$. If $p_k(x) = \rho > 0$ then by (2.3.11), (2.3.12)

(2.3.15) $P\{Z_{T_x} > x_k\} \leq P^*$ and

$P\{Z_{T_x} \geq x_k\} > P^*$ follow,

and by (2.3.14)

(2.3.16) $P\{Z_{T_y} > y_k\} \leq P^*$ holds.

If also

(2.3.17) $P\{Z_{T_y} \geq y_k\} \leq P^*$ holds, then

$$p_k(y) = 1 \geq p_k(x).$$

It remains the case, where (2.3.16), but not (2.3.17) holds:

(2.3.18) $$p_k(y) = \frac{P^*-P\{Z_{T_y} > y_k\}}{P\{Z_{T_y} = y_k\}} = \frac{P^*-P\{Z_{T_y} > y_k\}}{P\{Z_{T_y} > y_k-1\}-P\{Z_{T_y} > y_k\}} \geq$$

$$\frac{P^*-P\{Z_{T_x} > x_k\}}{P\{Z_{T_x} > x_k-1\}-P\{Z_{T_x} > x_k\}} = p_k(x),$$

where the inequality is of the kind $\frac{P^*-a}{b-a} \geq \frac{P^*-A}{B-A}$ with $0<a\leq A\leq P^*<b\leq B$ and is seen to be true as follows: $\frac{P^*-a}{b-a} \geq \frac{P^*-a}{B-a} \geq \frac{P^*-A}{B-A}$ where the second inequality holds because the expression in the middle is a decreasing function in a.

The third possibility $p_k(x)=0$ is trivial.

Type 2.

$y_k=x_k$, $y_j=x_j-1$ for some $j\neq k$, hence $T_y = T_x-1$. The proof is analogous

to that for type 1, with (2.3.14) replaced by

$$P\{Z_{T_y} > y_k\} = P\{Z_{T_x-1} > x_k\} < P\{Z_{T_x} > x_k\}. \tag{2.3.19}$$

This concludes the proof of Lemma 2.3.1.

Table A1 gives the values for c_T and ρ for various values of P*, k and n. Since T can take on the values 0,1,2,...,kn these tables become very extensive for large values of k and n. Therefore it is desirable to find approximations for c_T and ρ. Normal approximation for the hypergeometric distribution gives good results when n is large and T is not extreme (close to 0 or kn). Expectation and variance of Z_T are

$$\mu = \frac{T}{k}, \tag{2.3.19}$$

$$\sigma^2 = \frac{(kn-T)\ T(k-1)}{(kn-1)\ k^2}$$

yielding the normal approximation

$$(X_k|T) \sim N(\mu,\sigma^2) \tag{2.3.20}$$

from which an approximate $\tilde{c}_T$ can be evaluated as follows:

$$\hat{c}_T = \mu - \sigma\Phi^{-1}(P^*) \tag{2.3.21}$$

$$\tilde{c}_T = [\hat{c}_T + \tfrac{1}{2}],$$

where Φ^{-1} is the inverse of the standard normal distribution and [x] denotes largest integer not greater than x. For ρ we get the approximation

TABLE 4. Exact and Approximate Values for c_T and ρ_T

(Binomial Populations)

k	n	T	p* = 0.75		0.90		0.95		0.99	
2	5	4	1	.05	1	.68	1	.89	0	.58
			1	.05	1	.55	1	.84	0	.40
	10	6	2	.26	2	.88	1	.31	1	.93
			2	.21	2	.85	1	.23	1	.95
	20	10	4	.49	3	.35	3	.83	2	.79
			4	.44	3	.28	3	.78	2	.73
3	5	5	1	.53	1	.95	0	.40	0	.88
			1	.43	1	.98	0	.30	0	.91
	10	15	4	.44	3	.23	3	.74	2	.67
			4	.39	3	.18	3	.66	2	.56
	20	50	16	.77	15	.69	14	.18	13	.15
			16	.76	15	.59	14	.09	13	.03
5	5	5	0	.14	0	.66	0	.83	0	.97
			0	.20	0	.66	0	.93	0	1.00
	10	25	4	.52	3	.41	3	.88	2	.85
			4	.46	3	.33	3	.85	2	.82
	20	90	17	.33	16	.11	16	.71	15	.72
			17	.31	16	.05	16	.48	15	.31
10	5	20	1	.29	1	.87	0	.26	0	.85
			1	.21	1	.85	0	.23	0	.94
	10	50	4	.57	3	.52	3	.96	2	.94
			4	.52	3	.43	3	.88	2	.91
	20	30	2	.62	1	.47	1	.86	0	.69
			2	.52	1	.45	1	1.00	0	1.00

To every k,n,T,P* the table shows the integer c_T and the probability ρ_T (upper lines) and their approximations $\tilde{c}_T$ and $\tilde{\rho}_T$ (lower lines).

(2.3.22) $$\tilde{\rho} = \tilde{c}_T + 0.5 - \hat{c}_T$$

Table 4 gives the exact and approximate values of c_T and ρ_T for some selected values of k,n,T and P*.

We see that the deviation of the approximate values from the exact ones is small. The non-randomized version R_o' of R_o:

R_o': Select π_i if $x_i \geq c_T$ is conservative, i.e. the P*-condition will be satisfied with a P*-value not less than that for R_o. However, R_o' may not be just and it selects large subsets if the p_i's are close to zero or one. The performance of rule R (2.2.1) was studied in [22] and a table was given for the expected proportion $\eta(\omega)$ under various slippage configurations. A comparison of R_o and R is difficult, because the location of the minimum of a correct selection is not known. Since it takes place near $p = \frac{1}{2}$, the P* for R_o was chosen to satisfy

$$P^* = P_\omega(CS|R) \quad \text{with} \quad \omega = (\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2})$$

which makes the comparison slightly more favorable for R. Table 5 compares the expected proportion retained in the selected subset for slippage configurations and different values of k and n. We see that R_o yields better results for small values of δ, while R becomes better if δ is large. Hence R_o should be applied if only small differences in the success probabilities of the populations are expected. This disadvantage of R_o becomes more evident in the case of equally spaced configurations, where almost surely more than half of the populations will be retained in the selected subset if the number of observations is increased indefinitely, whereas R will eventually select only the best one.

TABLE 5. The Expected Proportion for Rules R (Upper Line) and R_o (Lower Line)

		k=2		k=3			k=5		k=10
δ	x+δ	N=5	N=10	N=5	N=10	N=25	N=10	N=25	N=10
0	0.5	.945	.942	.904	.962	.942	.934	.902	.961
		.945	.942	.904	.962	.942	.934	.902	.961
	0.75	.945	.942	.904	.962	.942	.934	.902	.961
		.999	.966	.950	.984	.971	.974	.951	.990
	0.95	.945	.942	.904	.962	.942	.934	.902	.961
		.945	.999	.999	1.000	1.000	1.000	1.000	1.000
	1.00	.945	.942	.904	.962	.942	.934	.902	.961
		1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
.1	0.5	.938	.926	.898	.954	.916	.928	.883	.959
		.938	.925	.895	.951	.902	.923	.861	.956
	0.75	.938	.923	.897	.953	.913	.927	.881	.958
		.952	.939	.922	.965	.919	.946	.887	.974
	0.95	.937	.914	.893	.946	.887	.921	.865	.956
		.989	.983	.984	.995	.979	.994	.973	.999
	1.00	.938	.911	.890	.941	.863	.916	.850	.954
		.996	.994	.993	.999	.994	.999	.992	1.000
.25	0.5	.901	.838	.868	.915	.795	.900	.805	.951
		.896	.838	.850	.883	.682	.853	.615	.926
	0.75	.901	.838	.869	.917	.807	.902	.816	.951
		.896	.838	.849	.882	.682	.849	.614	.919
	0.95	.896	.793	.859	.902	.765	.893	.796	.948
		.934	.870	.908	.928	.696	.912	.634	.968
	1.00	.896	.771	.852	.893	.737	.888	.784	.947
		.948	.888	.931	.948	.707	.937	.649	.982
.5	0.5	.755	.549	.774	.707	.336	.815	.239	.939
		.750	.586	.667	.585	.338	.502	.206	.661
	0.75	.777	.627	.782	.799	.526	.833	.637	.933
		.737	.607	.646	.588	.352	.505	.223	.625
	0.95	.762	.581	.783	.801	.534	.839	.659	.934
		.744	.595	.657	.585	.343	.502	.212	.645
	1.00	.755	.549	.779	.794	.523	.836	.655	.933
		.750	.586	.667	.585	.388	.502	.206	.661

In Section 2.6 a rule will be proposed, which combines the advantages of both R and R_o'.

2.4. Selection Rules for Poisson Distributions

Since there is a close relation between the Binomial and the Poisson distribution, it is clear that a selection rule similar to R_o (2.3.6) can be constructed in that case. Let $X_1, X_2, \ldots, X_k$ have Poisson distribution

$$P(X_i = x) = e^{-\theta_i} \frac{\theta_i^x}{x!} \qquad i=1,2,3,\ldots,k \quad x=0,1,2,\ldots \tag{2.4.1}$$

and for $\omega_o = (\theta, \ldots, \theta) \in \Omega_o$ the joint distribution is

$$P(X_1=x_1, \ldots, X_k=x_k) = e^{-k\theta} \frac{\theta^{\Sigma x_i}}{\prod_{i=1}^{k} x_i!} \tag{2.4.2}$$

indicating that $T = \Sigma X_i$ is a sufficient statistic for θ. As in (2.3.6) we propose a rule and again call it R_o, which yields constant probability of a correct selection in Ω_o. R_o is defined by its individual selection probabilities

$$p_i(x) = \begin{cases} 1 & \text{if } x_i > c_T \\ \rho & \text{if } x_i = c_T \\ 0 & \text{if } x_i < c_T \end{cases} \qquad i=1,2,\ldots,k \tag{2.4.3}$$

where $c_T = c_T(P^*)$ and $\rho = \rho(T,P^*)$ satisfy (2.3.7). The conditional distribution of X_k given T under $\omega_o \in \Omega_o$ is:

$$(2.4.4) \qquad P_{\omega_o}\{X_k=x|T\} = \frac{\frac{1}{x!} \sum_{x_1+\ldots+x_{k-1}=T-x} \frac{1}{x_1!\ldots x_{k-1}!}}{\sum_{x_1+\ldots+x_k=T} \frac{1}{x_1!\ldots x_k!}}$$

and applying the relation

$$(2.4.5) \qquad \sum_{x_1+\ldots+x_k=T} \frac{T!}{x_1!\ldots x_k!} \left(\frac{1}{k}\right)^T = 1$$

yields

$$(2.4.6) \qquad P_{\omega_o}\{X_k=x|T\} = \binom{T}{x} \frac{(k-1)^{T-x}}{k^T}$$

As in (2.3.7), (2.3.8) the integer c_T can be determined from

$$(2.4.7) \qquad P_{\omega_o}\{X_k > c_T|T\} \leq P^*$$

$$P_{\omega_o}\{X_k \geq c_T|T\} > P^*$$

which do not depend on the particular value ω_o one chooses from Ω_o. ρ again can be evaluated by the formula

$$\rho = \frac{P^*-P\{X_k > c_T|T\}}{P\{X_k = c_T|T\}}.$$

Table A2 gives the values of c_T and ρ for k=2,3,4,5,10, T=0(1)50 and P*=0.75,0.90,0.95,0.99. The proof that R_o is just for Poisson distributions is essentially the same as in the Binomial case. The remark, that R_o should preferably be applied if the ϑ_i's are close together, holds for the Poisson case as well.

2.5. The Negative Binomial Distribution

Procedure R_o (2.3.6) can also be applied in the case of negative binomial distributions (binomial waitingtime distribution). Let $\pi_1,..,\pi_k$ be populations with negative binomial distributions:

$$P\{X_i = x\} = \binom{x-1}{r-1} \theta_i^r (1-\theta_i)^{x-r} \tag{2.5.1}$$

for some fixed positive integer r, $x=r,r+1,\ldots,0 \leq \theta \leq 1$, $i=1,2,\ldots,k$. If $\theta_1=\ldots=\theta_k=\theta$, the joint distribution becomes

$$P\{X_1 = x_1,\ldots,X_k = x_k\} = \left(\frac{\theta}{1-\theta}\right)^{kr} e^{\log(1-\theta)\Sigma x_i} \prod_{i=1}^{k} \binom{x_i-1}{r-1} \tag{2.5.2}$$

Since $\log(1-\theta)$ is a decreasing function of θ, and since we are interested in selecting the populations with large θ, we will now select a population if small values of x_i are observed. Again $T = \sum_{i=1}^{k} X_i$ is a sufficient statistic for θ.

Define R_o by the individual selection probabilities

$$p_i(x) = \begin{cases} 1 & \text{if } x_i < c_T \\ \rho & \text{if } x_i = c_T \\ 0 & \text{if } x_i > c_T \end{cases} \tag{2.5.3}$$

where $c_T = c_T(P^*,k,r)$ and $\rho = \rho(T,P^*,k,r)$ satisfy

$$P_{\omega_o}\{X_k < c_T | T\} < P^* \tag{2.5.4}$$

$$P_{\omega_o}\{X_k \leq c_T | T\} \geq P^* \tag{2.5.5}$$

$$(2.5.6) \qquad P_{\omega_o}\{X_k < c_T|T\} + \rho\, P\{X_k = c_T|T\} = P^* \qquad \text{for all } T \text{ and } \omega_o \epsilon \Omega_o.$$

Since T is a sufficient statistic for the common value θ, the conditional probabilities (2.5.4), (2.5.5) and (2.5.6) do not depend on the choice of $\omega_o \epsilon \Omega_o$:

$$(2.5.7) \qquad P_{\omega_o}\{X_k = x|T\} = \frac{\binom{x-1}{r-1}\binom{T-x-1}{(k-1)r-1}}{\binom{T-1}{kr-1}}$$

(2.5.4) through (2.5.7) enable us to evaluate the c_T and ρ values in a similar way as it was done for binomial and Poisson distributions. The proof that R_o is just is essentially the same as in the binomial case. Table A3 gives the values of c_T and ρ for k=2,3,5,10, r=5,10 and P*=0.75, 0.9, 0.95, 0.99.

2.6. Fisher's Logarithmic Distribution

R_o can also be applied for selecting a subset containing the one with highest θ from k populations which are distributed according to Fisher's logarithmic distribution

$$(2.6.1) \qquad P\{X_i = x\} = \frac{1}{\log(1-\theta)} \frac{\theta^x}{x} \qquad x = 1,2,\ldots \quad 0 < \theta < 1$$

Under $\omega_o = (\theta,\ldots,\theta)$ the joint distribution is

$$(2.6.2) \qquad P\{X_1=x_1,\ldots,X_k=x_k\} = \left[\frac{-1}{\log(1-\theta)}\right]^k \frac{\theta^{\sum_{i=1}^{k} x_i}}{\prod_{i=1}^{k} x_i}$$

$T = \sum_{i=1}^{k} X_i$ is a sufficient statistic for θ and the conditional probability of X_1 given T is

$$(2.6.3) \qquad P\{X_1=x|T\} = \frac{\frac{1}{x}\sum_{x_2+\ldots+x_k=T-x} \frac{1}{x_2\cdots x_k}}{\sum_{x_1+\ldots+x_k=T} \frac{1}{x_1\cdots x_k}}$$

The sums in (2.6.3) can be evaluated by using the recursion relation

$$(2.6.4) \qquad A_{k,T} \equiv \sum_{x_1+\ldots+x_k=T} \frac{1}{x_1\cdots x_k} = \sum_{x_1=1}^{T+1-k} \frac{1}{x_1} \sum_{x_2+\ldots+x_k=T-x_1} \frac{1}{x_2\cdots x_k} =$$

$$\sum_{x_1=x}^{T+1-k} \frac{1}{x_1} A_{k-1,T-x_1}$$

with the initial values $A_{1,T} = \frac{1}{T}$, $T=1,2,\ldots$. R_o is defined as in the Poisson case and the necessary constants can be evaluated in the same way. However R_o will not work satisfactorily in this case for the following reason: Unless θ is extremely close to one, there is a very high probability concentrated at one, e.g. $P_{0.5}(X=1)=0.72$, $P_{0.9}(X=1)=0.39$, $P_{0.99}(X=1)=0.21$. Hence even for large θ small values of X are very likely to be observed, which leads to small values of c_T and hence the rule with high probability includes all populations in the selected subset. This situation can be improved by taking repeated observations $X_{i1},\ldots,X_{in}$, $i=1,\ldots,k$ from each population. $T_i = \sum_{j=1}^{n} X_{ij}$ is a sufficient statistic for θ_i, but unlike the Poisson distribution the logarithmic

distribution is not reproductive, i.e. T_i does not have a logarithmic distribution so that for each value of n a different table must be used. Let

$$(2.6.5) \qquad Z_i^{(n)} = \sum_{j=1}^{n} X_{ij}, \qquad X_{ij} \text{ i.i.d. according to (2.6.1).}$$

Hence

$$(2.6.6) \qquad P\{Z_i^{(n)}=z\} = \left[\frac{-1}{\log(1-\theta)}\right]^z \theta^z \sum_{\substack{n \\ \sum_{i=1} x_i=z}} \frac{1}{x_1 \cdots x_n} =$$

$$\left[\frac{-\theta}{\log(1-\theta)}\right]^z A_{n,z} \qquad z=kn,kn+1,\ldots$$

For $\omega_o=(\theta,\ldots,\theta)\in\Omega_o$ the joint distribution is given by

$$(2.6.7) \qquad P\{Z_1^{(n)}=z_1,\ldots,Z_k^{(n)}=z_k\}=\left[\frac{-\theta}{\log(1-\theta)}\right]^{\sum_{i=1}^{k} z_i} \prod_{i=1}^{k} A_{n,z_i}$$

Again $T=\Sigma Z_i$ is a sufficient statistic for θ and the conditional distribution of Z_1 given T is

$$(2.6.8) \qquad P\{Z_1=z|T\} = \frac{A_{n,z} \sum_{z_2+\ldots+z_k=T-z} \prod_{i=2}^{k} A_{n,z_i}}{\sum_{z_1+\ldots+z_k=T} \prod_{i=1}^{k} A_{n,z_i}}, \quad z=n,\ldots,T-(k-1)n$$

Define

$$(2.6.9) \qquad B_{k,n,T} = \sum_{z_1+\ldots+z_k=T} \prod_{i=1}^{k} A_{n,z_i}$$

then the $B_{k,n,T}$'s can be evaluated from the recursion formula

$$B_{k,n,T} = \sum_{\nu=1}^{T-(k-1)n} A_{n,\nu} B_{k-1,n,T-\nu} \tag{2.6.10}$$

with the initial values

$$B_{1,n,T} = A_{n,T} . \tag{2.6.11}$$

Now we can determine the values for c_T and ρ analogously to the other cases.

Table A4 gives the values for c_T and ρ for k=2,3,5,10 , n=5 and P^*=0.75,0.90,0.95,0.99.

2.7. Rules of Gupta Type with Constant P(CS|R) in Ω_0.

Gupta's rule (1.6.4) for selecting a subset containing the one with highest mean from several normal populations was derived in [11] from a likelihood ratio test under slippage hypotheses. This derivation can be generalized for Koopman-Darmois familes and more general hypotheses.

Let X_i ,i=1,2,...,k have the probability densities

$$f(\theta_i, x_i) = c(\theta_i)\, e^{Q(\theta_i)T(x_i)}\, h(x_i). \tag{2.7.1}$$

If we make the usual assumption that $Q(\theta_i)$ is strictly monotonic, then $Q(\theta_i)$ can be assumed as increasing, so that we can consider $Q(\theta_i)$ as the parameter and rename it θ_i simplifying (2.7.1) to

$$f(\theta_i, x_i) = c(\theta_i)\, e^{\theta_i T(x_i)}\, h(x_i) \tag{2.7.2}$$

Let us assume we know that the θ_i take on the values $\theta_1' \leq \theta_2' \leq \ldots \leq \theta_k'$, but that the correct pairing is not known. Consider the hypotheses

$$
\begin{aligned}
&H_1: \quad \theta_1 = \theta_k' \\
&\vdots \\
&H_i: \quad \theta_i = \theta_k' \\
&\vdots \\
&H_k: \quad \theta_k = \theta_k'
\end{aligned}
\tag{2.7.3}
$$

i.e. H_i is the hypothesis that θ_i corresponds to θ_k' without specifying the parameters of the remaining (k-1) populations. If Ω_i, i=1,2,...,k denotes the subset of Ω where H_i is true, then the likelihood ratio test of hypothesis H_k against the alternatives $H_1,\ldots,H_{k-1}$ yields the region of acceptance:

$$
\lambda \equiv \frac{\max\limits_{\omega\in\Omega} \prod_{i=1}^{k} f(\theta_i', x_i)}{\max\limits_{\omega\in\Omega} \prod_{i=1}^{k} f(\theta_i', x_i)} = e^{\Sigma\theta_i'[T_{[i]}' - T_{[i]}]} \geq c, \tag{2.7.4}
$$

where the $T_{[i]}$ are the ordered values of $T_i = T(X_i)$, i=1,2,...,k and $T_{[i]}'$ are the ordered values of T_i, i=1,2,...,k-1, $T_{[k]}' = T_k$. Let r be the rank of T_k among the T_i's, i.e. $T_{[r]} = T_i$. Then (2.7.4) becomes

$$
\sum_{i=1}^{k} \theta_i (T_{[i]}' - T_{[i]}) = \sum_{j=r+1}^{k} (\theta_{j-1} - \theta_j) T_{[j]} + (\theta_k - \theta_r) T_k \geq c_1. \tag{2.75}
$$

Under slippage configuration $\omega_k = (\theta_1', \ldots, \theta_k') = (\theta, \ldots, \theta, \theta+\delta)$ (2.7.5) simplifies to

$$
-\delta\, T_{[k]} + \delta T_k \geq c_1 \tag{2.7.6}
$$

or

(2.7.7) $$T_k \geq T_{[k]} - c_2$$

If θ and δ are known this gives rise to a selection rule.

(2.7.8) Select π_i, iff $T_i \geq T_{[k]} - c_2$

where $c_2 = c_2(P^*, \theta, \delta)$ is determined by the P*-condition

(2.7.9) $$P_{\omega_k}\{T_k \geq T_{[k]} - c_2\} = P^*.$$

(2.7.8) is Gupta's rule which we know is just from Example 1.6.2, hence if we keep θ fixed the minimum of P(CS) takes place if $\delta = 0$, in which case (2.7.9) becomes

(2.7.10) $$\int_{-\infty}^{\infty} F_\theta^{k-1}(t+c_2)\,dF_\theta(t) = P^*,$$

where F_θ is the cumulative distribution function of T. For normal distributions with θ as location parameter, c_2 in (2.7.10) does not depend on θ. In general c_2 does depend on θ and if θ is not known an estimator for θ may be used in (2.7.9). Since ΣT_i is a sufficient statistic for θ, this leads to a selection rule of the form:

(2.7.11) Select π_i iff $T_i \geq T_{[k]} - c(\Sigma T_i, P^*)$

By (2.1.4) this rule has constant P(CS) in Ω_o, if $c(\Sigma T_i, P^*)$ is determined to satisfy:

(2.7.12) $$P_{\omega_o}\{T_i \geq T_{[k]} - c(\Sigma T_j, P^*) \mid \Sigma T_j\} = P^* \quad \text{for all } \Sigma T_i, \omega_o \epsilon \Omega_o.$$

However, it is not known whether (2.7.11) is a just rule.

Another configuration which brings (2.7.4) into a simple form is the case where the θ_i are equally spaced: $\theta_i = \theta_o + i\delta$, $i=1,2,\ldots,k$. (2.7.4) then becomes

(2.7.13) $$-\delta \sum_{j=r+1}^{k} T_{[j]} + \delta(k-r)T_k \geq c_1'$$

or

$$T_k \geq \frac{1}{k-r} \sum_{j=r+1}^{k} T_{[j]} - c_2' .$$

The same methods as in the slippage case lead to the selection rule:

(2.7.14) $$\text{Select } \pi_i \text{ iff } T_i \equiv T_{[r]} \geq \frac{1}{k-r} \sum_{j=r+1}^{k} T_{[j]} - c'\left(\sum_{j=1}^{k} T_j, P^*\right)$$

i.e. T_i, is now compared with the average of the T_j's better than T_i. $c'\left(\sum_{j=1}^{k} T_j, P^*\right)$ can be determined from

(2.7.15) $$P_{\omega_o}\{T_{[r]} \geq \frac{1}{k-r} \sum_{j=r+1}^{k} T_{[j]} - c''(\Sigma T_j, P^*) \mid \Sigma T_j\} = P^*$$

for all ΣT_j and $\omega_o \epsilon \Omega_o$.

It is not known whether this rule is just either.

CHAPTER III

RULES BASED ON DEPENDENT OBSERVATIONS

3.1. Selection Rules Based on Ranks

In practice it frequently happens that the actual values of a random variable can only be observed under great cost or not at all, while their ordering is readily available. This occurs for instance in life testing when one only observes the order in which the parts under investigation fail without recording the actual time of the failure. Problems of this type suggest the investigation of subset selection rules based on ranks. Although the distributions of rank statistics are usually very involved, the resulting rules are often simple. Another advantage of rank procedures is that under the null hypothesis-i.e. under the hypothesis that all distributions are identical-the distribution of the ranks does not depend on the underlying distribution. For this reason rank procedures are sometimes referred to as non-parametric rules.

McDonald [28] investigated several subset selection rules based on ranks. In this section a rule will be derived which satisfies the P*-condition for some ω_o in Ω_o and which among all rank procedures yields the fastest increase of P(CS) in the neighborhood of ω_o.

From each of the populations π_i, i=1,2,...,k we take n observations $X_{i1},\ldots,X_{in}$. Let R_{ij} denote the rank of X_{ij} in the pooled sample of the N=kn observations $(X_{11},\ldots,X_{1n},X_{2n},\ldots,X_{kn})$.

Def. 3.1.1. A rank configuration is a N-tuple $\Delta = (\Delta_1,\ldots,\Delta_N)$, $\Delta_i \in \{1,2,\ldots,k\}$, where $\Delta_i=j$ indicates that the ith smallest observation in the pooled sample comes from π_j i.e. there exists an ℓ such that $R_{j\ell}= i$ holds.

Let $\triangle = \{\Delta\}$ denote the set of all rank configurations for a pair k,n which is kept fixed in these considerations. Δ_x denotes the rank configuration of $x = (x_{11},\ldots,x_N)$. For a fixed Δ let $X_\Delta = \{x \in X \mid \Delta_x=\Delta\}$.

Def. 3.1.2. A rank selection rule is a measurable function δ defined on $\triangle x \mathcal{D}$, where $\mathcal{D}$ is the decision space introduced in (1.1), provided that for each $\Delta \in \triangle$ (i) $\delta(\Delta,d) \geq 0$ and (ii) $\sum_{d\in\mathcal{D}} (\Delta,d)=1$ hold.

$\delta(\Delta,d)=p$ indicates that the decision d is made with probability p, if the rank configuration Δ is observed.

As in Section 1.2 it can be shown that the probability of a correct selection and the expected subset size depend on the individual selection probabilities only. Hence, corresponding to (1.2.3), we can use the following simplified definition of a rank selection rule.

Def. 3.1.3. A rank selection rule R is a measurable mapping from $\triangle$ into R^k: R: $\Delta \to (p_1(\Delta),\ldots,p_k(\Delta))$, $0 \leq p_i(\Delta) \leq 1$. Let the distribution of π_i, i=1,...,k, be given by a density function $f(x_i,\theta_i)$ from a one-parametric family with the θ_i's belonging to some interval θ which, without loss of generality, can be assumed to contain 0. Furthermore let the family $\{f(x,\theta)\}$ have the following properties.

(3.1.1) $f(x,\theta)$ is absolutely continuous in θ for almost every x,

(3.1.2) the limit $\dot{f}(x,0)=\lim_{\theta\to 0} \frac{1}{\theta} [f(x,\theta)-f(x,0)]$ exists for almost every x,

(3.1.3) $$\lim_{\theta \to 0} \int_{-\infty}^{\infty} |\dot{f}(x,\theta)|\,dx = \int_{-\infty}^{\infty} |\dot{f}(x,0)|\,dx < \infty$$

holds with $\dot{f}(x,\theta)$ denoting the partial derivative with respect to θ. Compare [24, p. 64].

Our goal is to construct a selection rule based on ranks such that

(3.1.4) $$P_{\omega_o}(CS) \geq P^* \quad \text{with } \omega_o = (0,0,\ldots,0)$$

holds and that

(3.1.5) $$\frac{1}{k!} \sum_{g \in G} \left. \frac{\partial P_{\alpha(g\omega)}(CS)}{\partial \alpha} \right|_{\alpha=0}$$

is as large as possible for some $\omega=(\theta_1,\ldots,\theta_k)$; $G=\{g\}$ denotes the permutations of $\{1,2,\ldots,k\}$. For notation see Section 1.3. Since in Ω_o the distribution of the ranks does not depend on the underlying distribution of the X_i's, (3.1.4) implies that $P_\omega(CS)$ is constant for $\omega \in \Omega_o$. Hence for just rank selection rules (3.1.4) is equivalent to the P*-condition. Condition (3.1.5) yields the fastest possible expected increase of P(CS) at ω_o when the parameter vector is changed toward $\omega=(\theta_1,\ldots,\theta_k)$ assuming that all permutations $g\omega=(\theta_{h1},\ldots,\theta_{hk})$ where $h=g^{-1}$, $g \in G$, are equally likely.

The probability that rank configuration Δ is observed under $\alpha\omega$ is

(3.1.6) $$P_{\alpha\omega}(\Delta) = \int_{-\infty}^{\infty} \int_{-\infty}^{x_N} \cdots \int_{-\infty}^{x_2} \prod_{i=1}^{N} f(x_i, \alpha\theta_{\Delta_i})\,dx_1,\ldots,dx_N$$

Because of (3.1.1), (3.1.2) and (3.1.3) the differentiation with respect to α can be carried out under the integral sign:

$$(3.1.7)\qquad \dot{P}_{\alpha\omega}(\Delta) = \frac{\partial P_{\alpha\omega}(\Delta)}{\partial \alpha} = \int_{-\infty}^{\infty}\int_{-\infty}^{x_N}\cdots\int_{-\infty}^{x_2} \sum_{j=1}^{N} \dot{f}(x_j,\alpha\theta_{\Delta_j})\theta_{\Delta_j} \prod_{\substack{i=1\\ i\neq j}}^{N} f(x_i,\alpha\theta_{\Delta_i})dx_1,\ldots,dx_N$$

$$(3.1.8)\qquad \dot{P}_o(\Delta) = \int_{-\infty}^{\infty}\int_{-\infty}^{x_N}\cdots\int_{-\infty}^{x} \sum_{j=1}^{N} \theta_{\Delta_j} \prod_{\substack{i=1\\ i\neq j}}^{N} f(x_i)\dot{f}(x_j)dx_1,\ldots,dx_N =$$

$$\sum_{j=1}^{N} \theta_{\Delta_j} A_j = \sum_{j=1}^{k} \theta_j \sum_{\substack{i\\ \Delta_i=j}} A_i = \sum_{j=1}^{k} \theta_j B_j(\Delta)$$

where $f(x) = f(x,0)$,

$$(3.1.9)\qquad A_j = \int_{-\infty}^{\infty}\int_{-\infty}^{x_N}\cdots\int_{-\infty}^{x_2} \dot{f}(x_j) \prod_{\substack{i=1\\ i\neq j}}^{N} f(x_i)\,dx_1,\ldots,dx_N,\ i=1,2,\ldots,N$$

and

$$(3.1.10)\qquad B_j(\Delta) = \sum_{\substack{i\\ \Delta_i=j}} A_i$$

Thus if we assume that π_k is the best population (3.1.5) becomes

$$(3.1.11)\qquad \frac{1}{(k-1)!} \sum_{g\in G(k,k)} \sum_{j=1}^{k} \theta_{gj} B_j(\Delta) = \frac{1}{(k-1)!} \sum_{j=1}^{k} \sum_{g\in G(k,k)} \theta_{gj} B_j(\Delta) =$$

$$\frac{1}{(k-1)!}\{(k-2)! \sum_{\nu=1}^{k-1} \theta_\nu \sum_{i=1}^{k-1} B_i(\Delta) + (k-1)!\,\theta_k B_k(\Delta)\} =$$

$$\frac{1}{(k-1)}\{(U-\theta_k)V + (k\theta_k - U)B_k(\Delta)\}$$

large, where $U = \sum_{i=1}^{k} \theta_k$ and $V = \sum_{i=1}^{k} B_i(\Delta) = n \sum_{i=1}^{k} A_i$, independent of Δ. V is zero if the underlying distribution is symmetric. Since $\theta_k \geq \theta_i$, $i=1,2,\ldots,k$, it follows $k\theta_k - U \geq 0$. Hence (3.1.11) is a nondecreasing function in $B_k(\Delta)$ and by the lemma of Neyman and Pearson the following subset rule solves (3.1.4) and (3.1.5):

$$p_k(\Delta) = \begin{cases} 1 & \text{if } B_k(\Delta) > c \\ \rho & \text{if } B_k(\Delta) = c \\ 0 & \text{if } B_k(\Delta) < c \end{cases} \tag{3.1.12}$$

employing the fact that $P_{\omega_o}(\Delta)$ is the same for all Δ. c and ρ have to be determined to satisfy

$$\sum_{\substack{\Delta \\ B_k(\Delta)>c}} P_{\omega_o}(\Delta) + \rho \sum_{\substack{\Delta \\ B_k(\Delta)=c}} P_{\omega_o}(\Delta) = P^* \tag{3.1.13}$$

This rule is based on weighted rank sums [28] using the scores

$$A_i = \int_{-\infty}^{\infty} u^{i-1}(1-u)^{N-i} \varphi(u,f)\, du \tag{3.1.14}$$

with

$$\varphi(u,f) = \frac{\dot{f}(F^{-1}(u,0),0)}{f(F^{-1}(u,0),0)} \tag{3.1.15}$$

This result could be expected since these A_i's are the scores which yield locally optimal rank tests [24]. In section 1.6 it was shown that rules of this type are just provided that the A_i's are non-decreasing in i,

which for location parameters is always true iff f(x) is strongly unimodal i.e. if $-\log g(x)$ is a convex function [24, p. 20]. If the assumption $\omega_o=(0,\ldots,0)$ is replaced by the more general one $\omega_o=(\theta,\ldots,\theta)$, then (3.1.15) becomes

$$\varphi(u,f) = \frac{\dot{f}(F^{-1}(u,\theta),\theta)}{f(F^{-1}(u,\theta),\theta)} \tag{3.1.16}$$

which in general depends on θ. However it is independent of θ if θ is a location or scale parameter. If $f(x,\theta)$ is the normal density with mean θ then

$$\varphi(u,f) = \Phi^{-1}(u), \tag{3.1.17}$$

where Φ is the standardized normal cumulative distribution function, so that the scores can be evaluated as

$$A_i^{(N)} = \int_0^1 u^{i-1}(1-u)^{N-i}\,\Phi^{-1}(u)\,du \tag{3.1.18}$$

For references on tables for these scores see [24]. If f comes from a logistic family $f(x,\theta) = e^{-(x-\theta)}/(1+e^{-(x-\theta)})^2$, then $\varphi(u,f) = 2u-1$ which leads to equally spaced scores: $A_i^{(N)} = a_N+ib_N$ where the actual values of a_N and $b_N > 0$ are irrelevant. Hence the rule R_3 [28],

R_3: Select π_i iff $\Sigma R_{ij} \geq c$, is locally optimal on the respective P^* level if the underlying distributions are logistic with location parameter θ.

For k=3, n=2 the probability of a correct selection, the expected subset size and the efficiency using this rule for various values of ω

TABLE 6. Comparison of the Rank Procedures R_1 and R_3

			Normal		Logistic	
	ω		R_1	R_3	R_1	R_3
.0	.0	.1	.9444	.9452	.9452	.9461
			2.7989	2.7990	2.7987	2.7988
			1.0122	1.0130	1.0131	1.0141
.0	.0	.5	.9754	.9771	.9773	.9791
			2.7726	2.7790	2.7684	2.7764
			1.0554	1.0548	1.0590	1.0580
.0	.0	1.	.9929	.9938	.9939	.9947
			2.6980	2.7386	2.6857	2.7344
			1.1040	1.0887	1.1102	1.0914
.0	.1	.1	.9390	.9396	.9395	.9402
			2.7989	2.7989	2.7987	2.7987
			1.0065	1.0071	1.0070	1.0078
.0	.5	.5	.9585	.9622	.9601	.9644
			2.7726	2.7989	2.7684	2.7631
			1.0371	1.0425	1.0404	1.0471
.0	1.	1.	.9767	.9824	.9788	.9847
			2.6980	2.6681	2.6857	2.6468
			1.0860	1.1046	1.0934	1.1161

The three values in each block are from top to bottom: $P_\omega(CS|R_i)$, $E_\omega(S|R_i)$ and $\frac{1}{3} P_\omega(CS|R_i)/E_\omega(S|R_i)$, i=1,3 for normal and logistic populations.

on normal and logistic populations have been computed. For comparison the corresponding results for Rule R_1 [28] are also caluclated:

R_1: Select π_i iff $B_i \geq \max_{j=1,\ldots,k} B_j - D$

R_3: Select π_i iff $B_i \geq C$

with $B_i = \sum_{j=1}^{n} R_{ij}$, $i=1,2,\ldots,k$.

The values D=6 and C=4 were chosen because they yield the same value $P_{\omega_o}(CS|R_1) = P_{\omega_o}(CS|R_3) = \frac{14}{15} = 0.9\overline{3}$ for $\omega_o \epsilon \Omega_o$ so that a fair comparison can be made without need for randomization. Table 6 shows these comparisons for six different values of ω.

Remark 3.1.1. The differentiation with respect to α is more tedious, if it can not be done under the integral sign, but otherwise the same methods apply, e.g. for rectangular distributions and shift parameter θ one gets the scores $A_1 = -1$, $A_N = 1$, $A_i = 0$, $i=2,3,\ldots,N-1$.

Remark 3.1.2. If condition (3.1.5) is replaced by the requirement that the expected subset size $E_{\alpha\omega} S$ decreases as fast as possible as a function of α at $\alpha = 0$, minimizing the first derivative would not work because

$$\frac{\partial}{\partial \alpha} E_{\alpha\omega} S \Big|_{\alpha=0} = 0 \qquad \text{holds for all } \omega.$$

Hence second derivatives would have to be used and that would not lead to rules of a simple type like R_3.

3.2. A Selection Rule for Multinomial Cells

The theory of just rules as developed in Chapter I cannot be applied directly to rank procedures taking the rank configurations with their probabilities as observation space, because the rank statistics $(R_{i,1},\ldots,R_{i,n})$, $i=1,\ldots,k$, are obviously not independent. This fact accounts for the difficulties which occurred in [28] for the rule R_1, which in example 1.6 was shown to be not necessarily just. The concept of just rules has not yet been extended to the case of dependent populations. Similar difficulties were encountered in [18] for selecting a subset containing the cell with highest (lowest) probability from k multinomial cells. Taking N observations with cell frequencies $X_1,\ldots,X_k$, $\sum_{i=1}^{k} X_i=N$ it is shown that the rule of Gupta type:

(3.2.1) $$\text{Select cell } i \quad \text{iff } X_i \geq \max_{j=1,\ldots,k} X_j - D$$

does not attain its minimum of $P_\omega(CS)$ in Ω_o, which in this case consists of the single point $\omega_o = (\frac{1}{k},\ldots,\frac{1}{k})$.

We will now construct a selection rule which yields a minimum of $P_\omega(CS)$ for $\omega = \omega_o$ and which maximizes $P_\omega(CS)$ for $\omega = (\theta,\ldots,\theta,\theta+\delta)$, $\delta>0$ $k\theta+\delta = 1$.

Let x_i, $i=1,2,\ldots,k$ be the observed value in the ith cell and let $\sum_{i=1}^{k} x_i=N$. Let θ_i, $i=1,2,\ldots,k$ denote the cell probabilities, $\omega=(\theta_1,\ldots,\theta_k)$. We have

(3.2.2) $$P_\omega(X_i = x_i,\ i=1,2,\ldots,k) = \binom{N}{x_1,\ldots,x_k} \prod_{i=1}^{k} \theta_i^{x_i}$$

$$(3.2.3) \qquad P_{\omega_o}(X_i=x_i,\ i=1,2,\ldots,k) = \binom{N}{x_1,\ldots,x_k} \frac{1}{k^N}$$

Let $p_i(x)$ denote the individual selection probabilities and let us assume that $\theta_k \geq \theta_i$, $i=1,\ldots,k$. Again, we are interested in symmetric rules only (compare section 1.3), hence

$$(p_1(gx),\ldots,p_k(gx)) = g(p_1(x),\ldots,p_k(x)) \quad \text{for all } g\in G.$$

In particular we get the condition

$$(3.2.4) \qquad p_k(x) = \frac{1}{(k-1)!} \sum_{g\in G(k,k)} p_k(gx)$$

Hence we get the conditions

$$(3.2.5) \qquad P_{\omega_o}(CS) = \sum_{\substack{x \\ x_1+\ldots+x_k=N}} p_k(x) P_{\omega_o}(x) = \sum p_k(x) \binom{N}{x_1,\ldots,x_k} \left(\frac{1}{k}\right)^N = P^*$$

$$(3.2.6) \qquad P_{\omega}(CS) = \sum_{\substack{x \\ x_1+\ldots+x_k=N}} p_k(x) \binom{N}{x_1,\ldots,x_k} \theta^{N-x_k} (\theta+\delta)^{x_k} \text{ large.}$$

The lemma of Neyman and Pearson yields the solution

$$(3.2.7) \qquad p_k(x) = \begin{cases} 1 & > \\ \rho \text{ if } \theta^{N-x_k}(\theta+\delta)^{x_k} = c\left(\frac{1}{k}\right)^N & \\ 0 & < \end{cases}$$

for some constants c and ρ such that $P_{\omega_o}(CS) = P^*$ holds. Since for $\delta > 0$ $\theta^{N-x_k}(\theta+\delta)^{x_k}$ is an increasing function in x_k this rule reduces to

$$(3.2.8) \qquad p_k(x) = \begin{cases} 1 & > \\ \rho & \text{if } x_k = d \\ 0 & < \end{cases}$$

The integer d is easily determined from the conditions

$$(3.2.9) \qquad (\tfrac{1}{k})^N \sum_{i=d+1}^{N} \binom{N}{i} (k-1)^{N-i} < P^*$$

$$(3.2.10) \qquad (\tfrac{1}{k})^N \sum_{i=d}^{N} \binom{N}{i} (k-1)^{N-i} \geq P^*$$

from which follows

$$(3.2.11) \qquad \rho = \frac{P^* k^N - \sum_{i=d+1}^{N} \binom{N}{i} (k-1)^{N-i}}{(k-1)^{N-d}} .$$

For general $\omega = (\theta_1, \theta_2, \ldots, \theta_k)$ assuming $\theta_k \geq \theta_i$, $i=1,2,\ldots,k-1$, we get

$$(3.2.12) \quad P_\omega(CS) = \sum_{\substack{x_1+\ldots+x_k=N \\ x_k > d}} \binom{N}{x_1,\ldots,x_k} \prod_{i=1}^{k} \theta_i^{x_i} +$$

$$+ \rho \sum_{\substack{x_1+\ldots+x_k=N \\ x_k = d}} \binom{N}{x_1,\ldots,x_1} \prod_{i=1}^{k} \theta_i^{x_i} =$$

$$\sum_{j=d+1}^{N} \binom{N}{j} \theta_k^j \sum_{x_1+\ldots+x_{k-1}=N-j} \binom{N}{x_1,\ldots,x_{k-1}} \prod_{i=1}^{k-1} \theta_i^{x_i} +$$

$$\rho\binom{N}{d}\theta_k^d \sum_{x_1+\ldots+x_{k-1}=N-d} \binom{N-d}{x_1,\ldots,x_{k-1}} \prod_{i=1}^{k-1} \theta_i^{x_i} =$$

$$\sum_{j=d+1}^{N} \binom{N}{j}\theta_k^j(1-\theta_k)^{N-j} + \rho\binom{N}{d}\theta_k^d(1-\theta_k)^{N-d} =$$

$$(1-\rho)\sum_{j=d+1}^{N}\binom{N}{j}\theta_k^j(1-\theta_k)^{N-j} + \rho\sum_{j=d}^{N}\binom{N}{j}\theta_k^j(1-\theta_k)^{N-j},$$

which increases as a convex combination of two increasing functions in θ_k as can be seen by the incomplete beta function representation

$$(3.2.13)\qquad \sum_{j=d+1}^{N}\binom{N}{j}\theta^j(1-\theta)^{N-j} = \frac{1}{B(N-d,d+1)}\int_{1-\theta}^{1} t^{N-d-1}(1-t)^d dt$$

Hence the probability of a correct selection is minimized when θ_k is as small as possible i.e. for $\omega = (\frac{1}{k},\ldots,\frac{1}{k})$.

3.3. Conclusion

The idea of just selection rules as introduced in Chapter I is a very basic concept which can be applied in many areas of multiple decision theory as the various examples indicate. Hence this thesis cannot be expected to give an exhaustive treatise of all problems related to just subset selection rules. The most urgent question concerns the extension of just rules to dependent variables so that the methods of this thesis can be applied among others to the multinomial and multivariate normal case, for selecting the component with highest cell-probability and highest mean, respectively. The conjectures in Section 1.8 should

be investigated and it would also be of interest to construct optimal just rules for other than normal populations. An upper bound should be derived for the ratio of the expected size of the subset in Gupta's rule and in the optimal rule. If this bound turns out to be as high as the evaluations for k=3 indicate, this would be a strong argument for Gupta's rule which is easier to apply and has the advantage that it does not depend on a specified alternative. Results on the existence of just rules with constant probability of a correct selection in Ω_o would also be useful, in particular it should be determined if the rules of the type (2.7.11) are just.

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APPENDIX

TABLE A1. c_T and ρ_T for Binomial Distributions

K = 2 N = 5

T	P=.75		P=.90		P=.95		P=.99	
0	0	.75	0	.90	0	.95	0	.99
2	1	.95	0	.55	0	.78	0	.96
4	1	.05	1	.68	1	.89	0	.58
6	2	.05	2	.68	2	.89	1	.58
8	4	.95	3	.55	3	.78	3	.96
10	5	.75	5	.90	5	.95	5	.99

T	P=.75		P=.90		P=.95		P=.99	
1	0	.50	0	.80	0	.90	0	.98
3	1	.60	1	.96	0	.40	0	.88
5	2	.63	1	.03	1	.54	1	.94
7	3	.60	3	.96	2	.40	2	.88
9	4	.50	4	.80	4	.90	4	.98

K = 3 N = 5

T	P=.75		P=.90		P=.95		P=.99	
0	0	.75	0	.90	0	.95	0	.99
2	0	.42	0	.77	0	.88	0	.98
4	1	.78	0	.35	0	.68	0	.94
6	1	.17	1	.77	1	.97	0	.76
8	2	.54	1	.00	1	.54	1	.97
10	3	.79	2	.45	2	.78	1	.42
12	4	.98	3	.65	3	.87	2	.55
14	4	.25	4	.70	4	.85	4	.97

T	P=.75		P=.90		P=.95		P=.99	
1	0	.63	0	.85	0	.93	0	.98
3	0	.05	0	.62	0	.81	0	.96
5	1	.53	1	.95	0	.40	0	.88
7	2	.83	1	.50	1	.81	0	.46
9	2	.15	2	.78	2	.99	1	.82
11	3	.48	3	.93	2	.37	2	.91
13	4	.68	4	.99	3	.48	3	.90
15	5	.75	5	.90	5	.95	5	.99

TABLE A1 (Continued)

K = 4 N = 5

T	P=.75		P=.90		P=.95		P=.99	
0	0	.75	0	.90	0	.95	0	.99
2	0	.55	0	.82	0	.91	0	.98
4	0	.11	0	.65	0	.82	0	.96
6	1	.69	0	.23	0	.61	0	.92
8	1	.22	1	.81	0	.02	0	.80
10	2	.72	1	.38	1	.75	0	.38
12	2	.19	2	.82	1	.14	1	.88
14	3	.66	2	.27	2	.69	1	.29
16	4	1.00	3	.69	3	.92	2	.71
18	4	.50	4	.88	3	.05	3	.81
20	5	.75	5	.90	5	.95	5	.99

T	P=.75		P=.90		P=.95		P=.99	
1	0	.67	0	.87	0	.93	0	.99
3	0	.37	0	.75	0	.87	0	.97
5	1	.87	0	.48	0	.74	0	.95
7	1	.48	1	.95	0	.40	0	.88
9	2	.93	1	.63	1	.89	0	.66
11	2	.49	2	.99	1	.53	1	.98
13	3	.89	2	.61	2	.89	1	.71
15	3	.40	3	.91	2	.33	2	.92
17	4	.76	3	.31	3	.69	3	.99
19	4	.00	4	.60	4	.80	4	.96

K = 5 N = 5

T	P=.75		P=.90		P=.95		P=.99	
0	0	.75	0	.90	0	.95	0	.99
2	0	.61	0	.84	0	.92	0	.98
4	0	.35	0	.74	0	.87	0	.97
6	1	.93	0	.54	0	.77	0	.95
8	1	.63	0	.14	0	.57	0	.91
10	1	.25	1	.83	0	.12	0	.82
12	2	.82	1	.53	1	.84	0	.59
14	2	.45	2	.98	1	.53	1	.99
16	3	.95	2	.69	2	.95	1	.80
18	3	.58	2	.13	2	.63	1	.19
20	3	.01	3	.71	3	.94	2	.77
22	4	.62	4	.98	3	.48	3	.94
24	5	.94	4	.50	4	.75	4	.95

T	P=.75		P=.90		P=.95		P=.99	
1	0	.69	0	.87	0	.94	0	.99
3	0	.50	0	.80	0	.90	0	.98
5	0	.14	0	.66	0	.83	0	.97
7	1	.78	0	.38	0	.69	0	.94
9	1	.46	1	.94	0	.39	0	.88
11	2	.99	1	.70	1	.94	0	.73
13	2	.65	1	.30	1	.71	0	.33
15	2	.22	2	.85	1	.24	1	.91
17	3	.77	2	.47	2	.81	1	.60
19	3	.34	3	.89	2	.31	2	.93
21	4	.81	3	.44	3	.78	2	.39
23	4	.35	4	.80	4	.95	3	.70
25	5	.75	5	.90	5	.95	5	.99

TABLE A1 (Continued)

K = 10 N = 5

T	P=.75		P=.90		P=.95		P=.99	
0	0	.75	0	.90	0	.95	0	.99
2	0	.69	0	.88	0	.94	0	.99
4	0	.61	0	.85	0	.92	0	.98
6	0	.51	0	.80	0	.90	0	.98
8	0	.38	0	.75	0	.88	0	.98
10	0	.20	0	.68	0	.84	0	.97
12	1	.97	0	.58	0	.79	0	.96
14	1	.81	0	.44	0	.72	0	.94
16	1	.66	0	.24	0	.62	0	.92
18	1	.49	1	.98	0	.47	0	.89
20	1	.29	1	.87	0	.26	0	.85
22	1	.04	1	.75	1	.98	0	.78
24	2	.85	1	.59	1	.89	0	.68
26	2	.68	1	.39	1	.77	0	.50
28	2	.49	1	.09	1	.61	0	.20
30	2	.26	2	.90	1	.38	1	.96
32	3	.98	2	.74	1	.01	1	.87
34	3	.79	2	.54	2	.87	1	.73
36	3	.60	2	.24	2	.70	1	.47
38	3	.36	3	.93	2	.44	2	.99
40	3	.04	3	.75	3	.99	2	.87
42	4	.82	3	.50	3	.83	2	.62
44	4	.62	3	.03	3	.57	3	.99
46	4	.33	4	.82	4	.98	3	.81
48	5	.93	4	.50	4	.77	4	.99
50	5	.75	5	.90	5	.95	5	.99

T	P=.75		P=.90		P=.95		P=.99	
1	0	.72	0	.89	0	.94	0	.99
3	0	.65	0	.86	0	.93	0	.99
5	0	.57	0	.83	0	.91	0	.98
7	0	.45	0	.78	0	.89	0	.98
9	0	.29	0	.72	0	.86	0	.97
11	0	.08	0	.63	0	.82	0	.96
13	1	.89	0	.51	0	.76	0	.95
15	1	.74	0	.35	0	.67	0	.93
17	1	.58	0	.11	0	.55	0	.91
19	1	.40	1	.93	0	.38	0	.88
21	1	.18	1	.81	0	.11	0	.82
23	2	.94	1	.68	1	.94	0	.74
25	2	.77	1	.50	1	.83	0	.60
27	2	.59	1	.25	1	.70	0	.37
29	2	.38	2	.97	1	.51	1	.00
31	2	.12	2	.82	1	.22	1	.92
33	3	.88	2	.65	2	.94	1	.81
35	3	.70	2	.40	2	.80	1	.62
37	3	.49	2	.03	2	.59	1	.25
39	3	.22	3	.85	2	.24	2	.94
41	4	.92	3	.64	3	.92	2	.77
43	4	.73	3	.31	3	.72	2	.38
45	4	.49	4	.92	3	.32	3	.92
47	4	.10	4	.70	4	.89	3	.59
49	5	.83	4	.00	4	.50	4	.90

TABLE A1 (Continued)

K = 2 N = 10

T	P=.75		P=.90		P=.95		P=.99	
0	0	.75	0	.90	0	.95	0	.99
2	1	.98	0	.58	0	.79	0	.96
4	1	.17	1	.77	1	.97	0	.77
6	2	.26	2	.88	1	.31	1	.93
8	3	.31	3	.94	2	.47	2	1.00
10	4	.33	4	.96	3	.51	2	.14
12	5	.31	5	.94	4	.47	4	1.00
14	6	.26	6	.88	5	.31	5	.93
16	7	.17	7	.77	7	.97	6	.77
18	9	.98	8	.58	8	.79	8	.96
20	10	.75	10	.90	10	.95	10	.99

T	P=.75		P=.90		P=.95		P=.99	
1	0	.50	0	.80	0	.90	0	.98
3	1	.63	0	.05	0	.53	0	.91
5	2	.72	1	.38	1	.75	0	.38
7	3	.77	2	.51	2	.85	1	.69
9	4	.79	3	.57	3	.90	2	.77
11	5	.79	4	.57	4	.90	3	.77
13	6	.77	5	.51	5	.85	4	.69
15	7	.72	6	.38	6	.75	5	.38
17	8	.63	7	.05	7	.53	7	.91
19	9	.50	9	.80	9	.90	9	.98

K = 3 N = 10

T	P=.75		P=.90		P=.95		P=.99	
0	0	.75	0	.90	0	.95	0	.99
2	0	.43	0	.77	0	.89	0	.98
4	1	.82	0	.43	0	.72	0	.94
6	1	.29	1	.87	0	.23	0	.85
8	2	.68	1	.41	1	.78	0	.54
10	2	.00	2	.80	1	.22	1	.93
12	3	.43	2	.18	2	.70	1	.56
14	4	.75	3	.60	3	.96	2	.89
16	4	.05	4	.87	3	.42	2	.26
18	5	.42	4	.19	4	.73	3	.66
20	6	.70	5	.53	5	.91	4	.85
22	7	.94	6	.75	5	.13	5	.94
24	7	.24	7	.90	6	.39	6	.98
26	8	.50	8	.98	7	.52	7	.97
28	9	.68	8	.03	8	.52	8	.90
30	10	.75	10	.90	10	.95	10	.99

T	P=.75		P=.90		P=.95		P=.99	
1	0	.63	0	.85	0	.93	0	.98
3	0	.11	0	.64	0	.82	0	.96
5	1	.58	0	.08	0	.54	0	.91
7	2	.94	1	.67	1	.94	0	.74
9	2	.38	2	1.00	1	.57	0	.15
11	3	.73	2	.54	2	.91	1	.80
13	3	.05	3	.86	2	.38	1	.11
15	4	.44	3	.23	3	.74	2	.67
17	5	.74	4	.60	4	.96	3	.90
19	5	.00	5	.84	4	.36	3	.15
21	6	.35	5	.04	5	.63	4	.50
23	7	.63	6	.37	6	.79	5	.67
25	8	.84	7	.57	7	.88	6	.72
27	8	.01	8	.68	8	.91	7	.66
29	9	.25	9	.70	9	.85	9	.97

TABLE A1 (Continued)

K = 4 N = 10

T	P=.75	P=.90	P=.95	P=.99	T	P=.75	P=.90	P=.95	P=.99
0	0 .75	0 .90	0 .95	0 .99	1	0 .67	0 .87	0 .93	0 .99
2	0 .55	0 .82	0 .91	0 .98	3	0 .39	0 .76	0 .88	0 .98
4	0 .17	0 .67	0 .83	0 .97	5	1 .92	0 .54	0 .77	0 .95
6	1 .74	0 .35	0 .68	0 .94	7	1 .56	0 .08	0 .54	0 .91
8	1 .34	1 .91	0 .34	0 .87	9	1 .08	1 .78	0 .04	0 .81
10	2 .85	1 .62	1 .91	0 .72	11	2 .65	1 .41	1 .80	0 .58
12	2 .43	1 .14	1 .65	0 .35	13	2 .18	2 .91	1 .44	1 1.00
14	3 .91	2 .75	1 .15	1 .93	15	3 .71	2 .55	2 .93	1 .83
16	3 .48	2 .30	2 .78	1 .69	17	3 .22	3 .98	2 .59	1 .47
18	4 .94	3 .81	2 .33	1 .13	19	4 .72	3 .61	3 .99	2 .93
20	4 .49	3 .35	3 .83	2 .79	21	4 .22	3 .00	3 .63	2 .59
22	5 .93	4 .82	3 .36	2 .27	23	5 .71	4 .60	4 .99	3 .95
24	5 .46	4 .32	4 .82	3 .80	25	5 .17	5 .97	4 .60	3 .57
26	6 .89	5 .77	4 .28	3 .17	27	6 .66	5 .53	5 .93	4 .90
28	6 .40	5 .19	5 .73	4 .70	29	6 .08	6 .89	5 .45	4 .34
30	7 .82	6 .66	6 1.00	5 .94	31	7 .57	6 .35	6 .80	5 .74
32	7 .28	7 .95	6 .53	5 .34	33	8 .94	7 .73	6 .04	6 .92
34	8 .70	7 .42	7 .81	6 .66	35	8 .42	8 .96	7 .50	7 1.00
36	8 .03	8 .73	8 .96	7 .80	37	9 .77	8 .36	8 .72	7 .18
38	9 .50	9 .89	8 .13	8 .83	39	9 .00	9 .60	9 .80	9 .96
40	10 .75	10 .90	10 .95	10 .99					

TABLE A1 (Continued)

K = 5 N = 10

T	P=.75		P=.90		P=.95		P=.99	
0	0	.75	0	.90	0	.95	0	.99
2	0	.61	0	.84	0	.92	0	.98
4	0	.37	0	.75	0	.87	0	.97
6	1	.98	0	.59	0	.79	0	.96
8	1	.69	0	.30	0	.65	0	.93
10	1	.37	1	.93	0	.39	0	.88
12	2	.96	1	.72	1	.98	0	.78
14	2	.64	1	.41	1	.80	0	.60
16	2	.27	2	.97	1	.54	0	.22
18	3	.86	2	.72	1	.11	1	.92
20	3	.51	2	.36	2	.82	1	.75
22	3	.08	3	.92	2	.51	1	.41
24	4	.70	3	.61	3	1.00	2	.95
26	4	.31	3	.16	3	.73	2	.72
28	5	.87	4	.78	3	.32	2	.27
30	5	.49	4	.39	4	.87	3	.86
32	5	.00	5	.90	4	.50	3	.50
34	6	.63	5	.52	5	.95	4	.93
36	6	.17	6	.96	5	.59	4	.57
38	7	.74	6	.59	6	.97	5	.93
40	7	.30	7	.98	6	.59	5	.50
42	8	.81	7	.60	7	.94	6	.86
44	8	.36	8	.94	7	.48	6	.09
46	9	.82	8	.49	8	.81	7	.56
48	9	.35	9	.81	9	.96	8	.73
50	10	.75	10	.90	10	.95	10	.99

T	P=.75		P=.90		P=.95		P=.99	
1	0	.69	0	.88	0	.94	0	.99
3	0	.50	0	.80	0	.90	0	.98
5	0	.20	0	.68	0	.84	0	.97
7	1	.84	0	.46	0	.73	0	.95
9	1	.54	0	.08	0	.54	0	.91
11	1	.17	1	.83	0	.19	0	.84
13	2	.80	1	.58	1	.90	0	.71
15	2	.46	1	.20	1	.69	0	.44
17	2	.04	2	.85	1	.36	1	.98
19	3	.69	2	.56	2	.94	1	.85
21	3	.31	2	.12	2	.69	1	.61
23	4	.88	3	.78	2	.29	1	.13
25	4	.52	3	.41	3	.88	2	.85
27	4	.07	4	.94	3	.56	2	.54
29	5	.68	4	.60	3	.01	3	.97
31	5	.27	4	.11	4	.71	3	.71
33	6	.82	5	.73	4	.22	3	.16
35	6	.42	5	.26	5	.79	4	.78
37	7	.93	6	.80	5	.31	4	.22
39	7	.53	6	.33	6	.81	5	.77
41	8	1.00	7	.81	6	.28	5	.01
43	8	.60	7	.30	7	.75	6	.62
45	8	.04	8	.75	8	.99	7	.87
47	9	.62	9	.99	8	.52	8	.96
49	10	.94	9	.50	9	.75	9	.95

TABLE A1 (Continued)

K = 16 N = 16

T	P=.75		P=.90		P=.95		P=.99		T	P=.75		P=.90		P=.95		P=.99	
0	0	.75	0	.90	0	.95	0	.99	1	0	.72	0	.89	0	.94	0	.99
2	0	.69	0	.88	0	.94	0	.99	3	0	.66	0	.86	0	.93	0	.99
4	0	.62	0	.85	0	.92	0	.98	5	0	.57	0	.83	0	.91	0	.98
6	0	.52	0	.81	0	.90	0	.98	7	0	.46	0	.79	0	.89	0	.98
8	0	.40	0	.76	0	.88	0	.98	9	0	.33	0	.73	0	.87	0	.97
10	0	.24	0	.70	0	.85	0	.97	11	0	.15	0	.66	0	.83	0	.97
12	0	.04	0	.62	0	.81	0	.96	13	1	.95	0	.57	0	.78	0	.96
14	1	.88	0	.51	0	.76	0	.95	15	1	.81	0	.45	0	.72	0	.94
16	1	.73	0	.37	0	.69	0	.94	17	1	.66	0	.29	0	.64	0	.93
18	1	.59	0	.19	0	.60	0	.92	19	1	.51	0	.08	0	.54	0	.91
20	1	.42	1	.98	0	.47	0	.89	21	1	.33	1	.93	0	.40	0	.88
22	1	.23	1	.88	0	.31	0	.86	23	1	.13	1	.83	0	.21	0	.84
24	1	.01	1	.77	0	.09	0	.82	25	2	.93	1	.71	1	.99	0	.79
26	2	.85	1	.65	1	.95	0	.76	27	2	.77	1	.58	1	.91	0	.72
28	2	.69	1	.50	1	.86	0	.68	29	2	.61	1	.41	1	.81	0	.63
30	2	.52	1	.32	1	.76	0	.56	31	2	.43	1	.21	1	.70	0	.49
32	2	.33	1	.09	1	.64	0	.40	33	2	.23	2	.98	1	.56	0	.30
34	2	.11	2	.92	1	.48	0	.18	35	3	.99	2	.86	1	.39	0	.03
36	3	.91	2	.79	1	.28	1	.98	37	3	.83	2	.72	1	.16	1	.95
38	3	.74	2	.64	1	.02	1	.91	39	3	.65	2	.56	2	.95	1	.88
40	3	.56	2	.47	2	.90	1	.83	41	3	.47	2	.37	2	.84	1	.79
42	3	.37	2	.25	2	.78	1	.73	43	3	.26	2	.13	2	.71	1	.66
44	3	.15	3	.99	2	.63	1	.59	45	3	.03	3	.93	2	.54	1	.50
46	4	.93	3	.86	2	.45	1	.39	47	4	.84	3	.78	2	.33	1	.26
48	4	.75	3	.70	2	.21	1	.11	49	4	.66	3	.62	2	.06	2	.98

TABLE A1 (Continued)

K = 10 N = 10

T	P=.75		P=.90		P=.95		P=.99	
50	4	.57	3	.52	3	.96	2	.94
52	4	.37	3	.30	3	.83	2	.84
54	4	.13	3	.03	3	.68	2	.71
56	5	.91	4	.86	3	.48	2	.52
58	5	.73	4	.70	3	.23	2	.25
60	5	.54	4	.50	4	.96	3	.97
62	5	.32	4	.25	4	.81	3	.85
64	5	.07	5	.97	4	.64	3	.70
66	6	.86	5	.81	4	.40	3	.47
68	6	.67	5	.62	4	.09	3	.13
70	6	.46	5	.39	5	.89	4	.91
72	6	.22	5	.08	5	.71	4	.76
74	7	.96	6	.87	5	.48	4	.52
76	7	.77	6	.68	5	.15	4	.15
78	7	.56	6	.44	6	.90	5	.91
80	7	.33	6	.11	6	.71	5	.72
82	7	.04	7	.87	6	.44	5	.41
84	8	.83	7	.67	6	.03	6	.97
86	8	.62	7	.39	7	.84	6	.79
88	8	.38	8	.99	7	.60	6	.48
90	8	.06	8	.80	7	.19	7	.97
92	9	.83	8	.54	8	.88	7	.75
94	9	.62	8	.09	8	.61	7	.22
96	9	.33	9	.83	9	1.00	8	.84
98	10	.93	9	.50	9	.77	9	.99
100	10	.75	10	.90	10	.95	10	.99

T	P=.75		P=.90		P=.95		P=.99	
51	4	.47	3	.42	3	.90	2	.90
53	4	.25	3	.17	3	.76	2	.78
55	4	.00	4	.94	3	.59	2	.62
57	5	.82	4	.78	3	.36	2	.40
59	5	.64	4	.60	3	.07	2	.07
61	5	.43	4	.38	4	.89	3	.92
63	5	.20	4	.11	4	.73	3	.78
65	6	.95	5	.89	4	.53	3	.59
67	6	.77	5	.72	4	.26	3	.32
69	6	.57	5	.51	5	.96	4	.98
71	6	.35	5	.24	5	.80	4	.84
73	6	.09	6	.95	5	.60	4	.65
75	7	.87	6	.78	5	.33	4	.36
77	7	.67	6	.57	6	.98	5	.97
79	7	.45	6	.29	6	.81	5	.82
81	7	.19	7	.96	6	.59	5	.59
83	8	.93	7	.78	6	.26	5	.17
85	8	.73	7	.54	7	.93	6	.89
87	8	.51	7	.21	7	.73	6	.66
89	8	.23	8	.90	7	.43	6	.20
91	9	.93	8	.69	8	.97	7	.88
93	9	.73	8	.36	8	.76	7	.56
95	9	.49	9	.93	8	.38	8	.95
97	9	.09	9	.70	9	.90	8	.63
99	10	.83	9	.00	9	.50	9	.90

TABLE A2. c_T and ρ_T for Poisson Distributions

K = 2

T	P=.75		P=.90		P=.95		P=.99	
0	0	.75	0	.90	0	.95	0	.99
2	0	.00	0	.60	0	.80	0	.96
4	1	.25	1	.85	0	.20	0	.84
6	2	.40	1	.10	1	.63	0	.36
8	3	.52	2	.41	2	.86	1	.81
10	4	.62	3	.61	2	.11	1	.08
12	5	.71	4	.78	3	.43	2	.58
14	6	.79	5	.92	4	.65	3	.84
16	7	.87	5	.08	5	.83	3	.07
18	8	.94	6	.27	6	.97	4	.47
20	8	.01	7	.43	6	.21	5	.72
22	9	.10	8	.57	7	.42	6	.91
24	10	.18	9	.69	8	.59	6	.17
26	11	.25	10	.80	9	.74	7	.46
28	12	.32	11	.91	10	.87	8	.68
30	13	.38	11	.00	11	.99	9	.85
32	14	.44	12	.15	11	.17	9	.00
34	15	.50	13	.27	12	.34	10	.28
36	16	.55	14	.39	13	.48	11	.50
38	17	.61	15	.50	14	.62	12	.69
40	18	.66	16	.60	15	.74	13	.84
42	19	.71	17	.69	16	.85	14	.98
44	20	.76	18	.78	17	.95	14	.20
46	21	.80	19	.86	17	.08	15	.41
48	22	.85	20	.94	18	.22	16	.58
50	23	.90	20	.03	19	.35	17	.73

T	P=.75		P=.90		P=.95		P=.99	
1	0	.50	0	.80	0	.90	0	.98
3	1	.67	0	.20	0	.60	0	.92
5	2	.80	1	.56	1	.88	0	.68
7	3	.91	2	.77	1	.23	1	.96
9	3	.02	3	.94	2	.57	1	.54
11	4	.15	3	.16	3	.79	2	.85
13	5	.26	4	.38	4	.96	2	.13
15	6	.35	5	.56	4	.22	3	.55
17	7	.43	6	.70	5	.46	4	.80
19	8	.51	7	.83	6	.65	5	.98
21	9	.58	8	.94	7	.80	5	.34
23	10	.65	8	.09	8	.94	6	.61
25	11	.72	9	.24	8	.12	7	.81
27	12	.78	10	.38	9	.32	8	.97
29	13	.83	11	.50	10	.48	8	.26
31	14	.89	12	.62	11	.63	9	.50
33	15	.95	13	.72	12	.76	10	.70
35	16	1.00	14	.82	13	.88	11	.86
37	16	.06	15	.91	14	.99	11	.01
39	17	.12	16	1.00	14	.15	12	.26
41	18	.18	16	.12	15	.30	13	.47
43	19	.23	17	.23	16	.44	14	.65
45	20	.29	18	.34	17	.56	15	.80
47	21	.34	19	.43	18	.68	16	.94
49	22	.39	20	.53	19	.78	16	.11

TABLE A2 (Continued)

K = 3

T	P=.75		P=.90		P=.95		P=.99	
0	0	.75	0	.90	0	.95	0	.99
2	0	.44	0	.77	0	.89	0	.98
4	1	.87	0	.49	0	.75	0	.95
6	1	.38	1	.95	0	.43	0	.89
8	2	.80	1	.61	1	.93	0	.74
10	2	.25	1	.05	1	.62	0	.42
12	3	.68	2	.64	1	.09	1	.95
14	3	.07	2	.07	2	.71	1	.73
16	4	.51	3	.62	2	.21	1	.30
18	5	.90	3	.02	3	.75	2	.88
20	5	.32	4	.57	3	.24	2	.53
22	6	.72	5	.97	4	.76	3	.97
24	6	.11	5	.48	4	.24	3	.66
26	7	.52	6	.90	5	.74	3	.13
28	8	.89	6	.38	5	.20	4	.74
30	8	.30	7	.81	6	.70	4	.25
32	9	.69	7	.25	6	.13	5	.79
34	9	.06	8	.70	7	.64	5	.32
36	10	.47	8	.10	7	.05	6	.83
38	11	.84	9	.57	8	.57	6	.36
40	11	.23	10	.95	9	.96	7	.84
42	12	.62	10	.42	9	.48	7	.38
44	13	.98	11	.83	10	.89	8	.85
46	13	.38	11	.26	10	.37	8	.37
48	14	.75	12	.69	11	.80	9	.84
50	14	.13	12	.08	11	.25	9	.35

T	P=.75		P=.90		P=.95		P=.99	
1	0	.63	0	.85	0	.93	0	.98
3	0	.16	0	.66	0	.83	0	.97
5	1	.64	0	.24	0	.62	0	.92
7	1	.07	1	.80	0	.15	0	.83
9	2	.54	1	.37	1	.80	0	.62
11	3	.93	2	.84	1	.40	0	.14
13	3	.39	2	.39	2	.89	1	.85
15	4	.79	3	.84	2	.49	1	.55
17	4	.21	3	.35	3	.93	2	.99
19	5	.62	4	.80	3	.52	2	.73
21	6	.99	4	.28	4	.95	2	.27
23	6	.43	5	.74	4	.52	3	.84
25	7	.81	5	.18	5	.94	3	.43
27	7	.21	6	.66	5	.49	4	.91
29	8	.61	6	.05	6	.92	4	.53
31	9	.97	7	.55	6	.44	5	.96
33	9	.39	8	.94	7	.87	5	.58
35	10	.77	8	.42	7	.37	6	.99
37	10	.15	9	.83	8	.81	6	.62
39	11	.55	9	.27	8	.28	6	.03
41	12	.91	10	.70	9	.74	7	.63
43	12	.31	10	.10	9	.17	7	.05
45	13	.69	11	.56	10	.65	8	.63
47	13	.05	12	.94	10	.04	8	.05
49	14	.45	12	.40	11	.54	9	.62

TABLE A2 (Continued)

K = 4

T	P=.75		P=.90		P=.95		P=.99	
0	0	.75	0	.90	0	.95	0	.99
2	0	.56	0	.82	0	.91	0	.98
4	0	.21	0	.68	0	.84	0	.97
6	1	.80	0	.44	0	.72	0	.94
8	1	.44	0	.00	0	.50	0	.90
10	2	.98	1	.77	0	.11	0	.82
12	2	.61	1	.46	1	.86	0	.68
14	2	.17	1	.01	1	.61	0	.44
16	3	.75	2	.73	1	.25	0	.00
18	3	.33	2	.37	2	.89	1	.87
20	4	.87	3	.93	2	.62	1	.68
22	4	.46	3	.61	2	.23	1	.37
24	5	.98	3	.20	3	.86	2	.97
26	5	.57	4	.81	3	.56	2	.78
28	5	.11	4	.44	3	.13	2	.50
30	6	.67	5	.98	4	.79	2	.07
32	6	.22	5	.64	4	.44	3	.82
34	7	.77	5	.21	5	.99	3	.54
36	7	.33	6	.81	5	.68	3	.12
38	8	.86	6	.42	5	.28	4	.83
40	8	.42	7	.96	6	.87	4	.53
42	9	.94	7	.59	6	.52	4	.11
44	9	.51	7	.14	6	.07	5	.81
46	9	.03	8	.74	7	.72	5	.49
48	10	.59	8	.33	7	.33	5	.04
50	10	.12	9	.88	8	.90	6	.76

T	P=.75		P=.90		P=.95		P=.99	
1	0	.67	0	.87	0	.93	0	.99
3	0	.41	0	.76	0	.88	0	.98
5	1	.97	0	.58	0	.79	0	.96
7	1	.63	0	.25	0	.63	0	.93
9	1	.22	1	.89	0	.33	0	.87
11	2	.80	1	.63	1	.95	0	.76
13	2	.40	1	.26	1	.75	0	.58
15	3	.94	2	.87	1	.45	0	.25
17	3	.54	2	.56	1	.00	1	.94
19	3	.09	2	.14	2	.76	1	.78
21	4	.67	3	.78	2	.44	1	.54
23	4	.23	3	.42	3	.99	1	.16
25	5	.78	4	.97	3	.72	2	.88
27	5	.35	4	.64	3	.36	2	.65
29	6	.88	4	.22	4	.94	2	.31
31	6	.46	5	.82	4	.63	3	.93
33	7	.98	5	.44	4	.23	3	.69
35	7	.55	6	.97	5	.84	3	.35
37	7	.08	6	.62	5	.49	4	.95
39	8	.65	6	.19	5	.04	4	.69
41	8	.19	7	.78	6	.71	4	.34
43	9	.73	7	.38	6	.32	5	.93
45	9	.28	8	.93	7	.89	5	.66
47	10	.81	8	.55	7	.54	5	.29
49	10	.37	8	.09	7	.09	6	.90

TABLE A2 (Continued)

K = 5

T	P=.75		P=.90		P=.95		P=.99	
0	0	.75	0	.90	0	.95	0	.99
2	0	.61	0	.84	0	.92	0	.98
4	0	.39	0	.76	0	.88	0	.98
6	0	.05	0	.62	0	.81	0	.96
8	1	.75	0	.40	0	.70	0	.94
10	1	.47	0	.07	0	.53	0	.91
12	1	.12	1	.85	0	.27	0	.85
14	2	.79	1	.64	1	.96	0	.77
16	2	.46	1	.36	1	.81	0	.64
18	2	.12	2	.99	1	.61	0	.44
20	3	.79	2	.77	1	.33	0	.13
22	3	.46	2	.51	2	.98	1	.94
24	3	.09	2	.18	2	.79	1	.81
26	4	.76	3	.87	2	.55	1	.64
28	4	.42	3	.61	2	.24	1	.40
30	4	.04	3	.29	3	.93	1	.06
32	5	.71	4	.94	3	.70	2	.88
34	5	.36	4	.67	3	.42	2	.71
36	6	.98	4	.36	3	.06	2	.47
38	6	.65	5	.99	4	.81	2	.14
40	6	.29	5	.72	4	.55	3	.90
42	7	.91	5	.41	4	.22	3	.71
44	7	.57	5	.03	5	.90	3	.46
46	7	.20	6	.75	5	.64	3	.12
48	8	.84	6	.43	5	.33	4	.88
50	8	.49	6	.06	6	.96	4	.66

T	P=.75		P=.90		P=.95		P=.99	
1	0	.69	0	.87	0	.94	0	.99
3	0	.51	0	.80	0	.90	0	.98
5	0	.24	0	.69	0	.85	0	.97
7	1	.89	0	.52	0	.76	0	.95
9	1	.62	0	.25	0	.63	0	.93
11	1	.31	1	.94	0	.42	0	.88
13	2	.94	1	.75	0	.09	0	.82
15	2	.64	1	.51	1	.89	0	.72
17	2	.31	1	.19	1	.71	0	.56
19	3	.94	2	.89	1	.48	0	.31
21	3	.63	2	.65	1	.16	1	.98
23	3	.28	2	.36	2	.89	1	.68
25	4	.91	3	.99	2	.68	1	.74
27	4	.59	3	.74	2	.41	1	.54
29	4	.24	3	.46	2	.05	1	.25
31	5	.87	3	.10	3	.82	2	.95
33	5	.54	4	.81	3	.57	2	.80
35	5	.17	4	.52	3	.25	2	.60
37	6	.81	4	.18	4	.93	2	.32
39	6	.47	5	.86	4	.69	3	.98
41	6	.09	5	.57	4	.39	3	.81
43	7	.74	5	.23	4	.02	3	.59
45	7	.39	6	.89	5	.77	3	.30
47	7	.01	6	.60	5	.49	4	.96
49	8	.67	6	.26	5	.14	4	.78

TABLE A2 (Continued)

K = 10

T	P=.75		P=.90		P=.95		P=.99	
0	0	.75	0	.90	0	.95	0	.99
2	0	.69	0	.88	0	.94	0	.99
4	0	.62	0	.85	0	.92	0	.98
6	0	.53	0	.81	0	.91	0	.98
8	0	.42	0	.77	0	.88	0	.98
10	0	.28	0	.71	0	.86	0	.97
12	0	.11	0	.65	0	.82	0	.96
14	1	.94	0	.56	0	.78	0	.96
16	1	.80	0	.46	0	.73	0	.95
18	1	.67	0	.33	0	.67	0	.93
20	1	.52	0	.18	0	.59	0	.92
22	1	.37	1	.99	0	.49	0	.90
24	1	.20	1	.90	0	.37	0	.87
26	1	.01	1	.81	0	.23	0	.85
28	2	.86	1	.71	0	.04	0	.81
30	2	.71	1	.59	1	.95	0	.76
32	2	.55	1	.46	1	.87	0	.71
34	2	.39	1	.31	1	.79	0	.64
36	2	.22	1	.14	1	.70	0	.56
38	2	.02	2	.97	1	.59	0	.45
40	3	.86	2	.86	1	.46	0	.32
42	3	.71	2	.75	1	.32	0	.16
44	3	.55	2	.62	1	.15	1	.99
46	3	.38	2	.48	2	.98	1	.95
48	3	.20	2	.33	2	.89	1	.89
50	3	.00	2	.15	2	.79	1	.83

T	P=.75		P=.90		P=.95		P=.99	
1	0	.72	0	.89	0	.94	0	.99
3	0	.66	0	.86	0	.93	0	.99
5	0	.58	0	.83	0	.92	0	.98
7	0	.48	0	.79	0	.90	0	.98
9	0	.35	0	.74	0	.87	0	.97
11	0	.20	0	.68	0	.84	0	.97
13	0	.02	0	.61	0	.80	0	.96
15	1	.87	0	.51	0	.76	0	.95
17	1	.74	0	.40	0	.70	0	.94
19	1	.60	0	.26	0	.63	0	.93
21	1	.45	0	.09	0	.54	0	.91
23	1	.29	1	.95	0	.44	0	.89
25	1	.11	1	.86	0	.30	0	.86
27	2	.93	1	.76	0	.14	0	.83
29	2	.78	1	.65	1	.98	0	.79
31	2	.63	1	.53	1	.91	0	.74
33	2	.47	1	.39	1	.83	0	.68
35	2	.31	1	.23	1	.74	0	.60
37	2	.12	1	.04	1	.64	0	.51
39	3	.94	2	.92	1	.53	0	.39
41	3	.79	2	.81	1	.39	0	.25
43	3	.63	2	.69	1	.24	0	.07
45	3	.46	2	.55	1	.05	1	.97
47	3	.29	2	.41	2	.94	1	.92
49	3	.10	2	.24	2	.84	1	.86

TABLE A3. c_T and ρ_T for Negative Binomial Distributions

K = 2 R = 5

T	P=.75		P=.90		P=.95		P=.99	
10	5	.75	5	.90	5	.95	5	.99
12	7	.08	7	.63	7	.82	7	.96
14	8	.38	8	.99	9	.49	9	.90
16	9	.61	10	.54	10	.94	11	.76
18	10	.82	11	.91	12	.56	13	.51
20	12	.04	13	.37	13	.99	15	.08
22	13	.28	14	.77	15	.60	16	.84
24	14	.51	16	.17	17	.05	18	.57
26	15	.72	17	.60	18	.62	20	.20
28	16	.93	18	.97	20	.08	21	.87
30	18	.16	20	.41	21	.63	23	.59
32	19	.39	21	.80	23	.10	25	.23
34	20	.60	23	.20	24	.63	26	.88
36	21	.82	24	.62	26	.11	28	.59
38	23	.03	25	.99	27	.64	30	.23
40	24	.26	27	.42	29	.12	31	.88
42	25	.48	28	.81	30	.64	33	.58
44	26	.69	30	.21	32	.12	35	.22
46	27	.90	31	.62	33	.64	36	.88
48	29	.12	32	1.00	35	.12	38	.57
50	30	.34	34	.42	36	.64	40	.21
52	31	.56	35	.81	38	.12	41	.87
54	32	.77	37	.21	39	.64	43	.55
56	33	.99	38	.62	41	.12	45	.20
58	35	.21	40	.00	42	.64	46	.85
60	36	.43	41	.42	44	.12	48	.53

T	P=.75		P=.90		P=.95		P=.99	
11	6	.50	6	.80	6	.90	6	.98
13	7	.73	8	.37	8	.69	8	.94
15	8	.95	9	.79	10	.21	10	.84
17	10	.22	11	.22	11	.77	12	.65
19	11	.45	12	.66	13	.31	14	.32
21	12	.67	14	.03	14	.81	15	.94
23	13	.88	15	.49	16	.35	17	.71
25	15	.11	16	.87	17	.84	19	.40
27	16	.34	18	.30	19	.37	20	.98
29	17	.56	19	.70	20	.85	22	.74
31	18	.77	21	.09	22	.38	24	.42
33	19	.98	22	.52	23	.86	26	.01
35	21	.21	23	.90	25	.38	27	.74
37	22	.43	25	.31	26	.87	29	.42
39	23	.65	26	.71	28	.39	31	.02
41	24	.86	28	.10	29	.87	32	.74
43	26	.08	29	.52	31	.39	34	.41
45	27	.30	30	.91	32	.87	36	.02
47	28	.52	32	.32	34	.39	37	.73
49	29	.73	33	.72	35	.87	39	.40
51	30	.94	35	.11	37	.39	41	.01
53	32	.16	36	.52	38	.87	42	.71
55	33	.39	37	.91	40	.39	44	.38
57	34	.60	39	.32	41	.87	45	1.00
59	35	.82	40	.71	43	.38	47	.70

TABLE A3 (Continued)

K = 3 R = 5

T	P=.75		P=.90		P=.95		P=.99	
15	5	.75	5	.90	5	.95	5	.99
17	6	.70	7	.20	7	.60	7	.92
19	7	.58	8	.33	8	.76	9	.56
21	8	.43	9	.38	9	.88	10	.86
23	9	.26	10	.40	10	.98	12	.18
25	10	.08	11	.41	12	.14	13	.58
27	10	.92	12	.42	13	.28	14	.83
29	11	.76	13	.42	14	.40	16	.09
31	12	.60	14	.42	15	.50	17	.45
33	13	.43	15	.41	16	.60	18	.72
35	14	.26	16	.41	17	.68	19	.94
37	15	.07	17	.40	18	.77	21	.25
39	15	.90	18	.39	19	.85	22	.55
41	16	.74	19	.38	20	.93	23	.80
43	17	.57	20	.37	22	.00	25	.02
45	18	.39	21	.36	23	.11	26	.34
47	19	.21	22	.35	24	.20	27	.61
49	20	.03	23	.33	25	.30	28	.85
51	20	.86	24	.32	26	.39	30	.10
53	21	.69	25	.31	27	.47	31	.40
55	22	.52	26	.29	28	.56	32	.66
57	23	.34	27	.28	29	.64	33	.89
59	24	.16	28	.27	30	.72	35	.16
61	24	.98	29	.25	31	.80	36	.44
63	25	.81	30	.24	32	.88	37	.70
65	26	.64	31	.22	33	.95	38	.93

T	P=.75		P=.90		P=.95		P=.99	
16	6	.25	6	.70	6	.85	6	.97
18	7	.10	7	.78	8	.03	8	.81
20	7	.95	8	.83	9	.35	10	.08
22	8	.83	9	.85	10	.51	11	.60
24	9	.68	10	.87	11	.63	12	.86
26	10	.53	11	.88	12	.73	14	.16
28	11	.36	12	.89	13	.82	15	.52
30	12	.18	13	.89	14	.91	16	.79
32	12	.99	14	.89	15	.99	17	1.00
34	13	.84	15	.89	17	.10	19	.36
36	14	.67	16	.88	18	.22	20	.64
38	15	.50	17	.87	19	.32	21	.87
40	16	.33	18	.87	20	.41	23	.14
42	17	.15	19	.86	21	.50	24	.45
44	17	.96	20	.85	22	.59	25	.71
46	18	.80	21	.84	23	.67	26	.94
48	19	.63	22	.83	24	.75	28	.22
50	20	.46	23	.82	25	.83	29	.51
52	21	.28	24	.81	26	.91	30	.76
54	22	.10	25	.80	27	.99	31	.98
56	22	.92	26	.78	29	.08	33	.28
58	23	.75	27	.77	30	.17	34	.55
60	24	.58	28	.76	31	.26	35	.80
62	25	.41	29	.75	32	.35	37	.04
64	26	.23	30	.73	33	.43	38	.32

TABLE A3 (Continued)

K = 4 R = 5

T	P=.75		P=.90		P=.95		P=.99	
20	5	.75	5	.90	5	.95	5	.99
22	6	.50	6	.92	7	.30	7	.86
24	7	.10	7	.84	8	.29	8	.96
26	7	.78	8	.70	9	.20	10	.17
28	8	.45	9	.51	10	.06	11	.34
30	9	.04	10	.27	10	.96	12	.42
32	9	.71	11	.00	11	.87	13	.48
34	10	.35	11	.82	12	.76	14	.53
36	10	.95	12	.62	13	.64	15	.56
38	11	.60	13	.39	14	.50	16	.59
40	12	.22	14	.13	15	.34	17	.61
42	12	.84	14	.90	16	.18	18	.62
44	13	.48	15	.69	17	.00	19	.64
46	14	.09	16	.45	17	.88	20	.65
48	14	.73	17	.20	18	.74	21	.66
50	15	.36	17	.95	19	.60	22	.67
52	15	.96	18	.73	20	.45	23	.68
54	16	.60	19	.50	21	.28	24	.69
56	17	.22	20	.25	22	.11	25	.70
58	17	.84	20	.99	22	.95	26	.70
60	18	.48	21	.77	23	.81	27	.71
62	19	.09	22	.54	24	.67	28	.71
64	19	.72	23	.29	25	.51	29	.71
66	20	.34	24	.03	26	.35	30	.72
68	20	.95	24	.80	27	.18	31	.72
70	21	.59	25	.57	28	.01	32	.72

T	P=.75		P=.90		P=.95		P=.99	
21	5	1.00	6	.60	6	.80	6	.96
23	6	.79	7	.47	7	.81	8	.56
25	7	.49	8	.27	8	.77	9	.72
27	8	.08	9	.01	9	.70	10	.80
29	8	.75	9	.86	10	.60	11	.85
31	9	.40	10	.68	11	.47	12	.89
33	9	.99	11	.46	12	.33	13	.93
35	10	.66	12	.21	13	.17	14	.96
37	11	.29	12	.96	13	1.00	15	.98
39	11	.90	13	.76	14	.89	17	.01
41	12	.55	14	.54	15	.76	18	.05
43	13	.16	15	.30	16	.63	19	.08
45	13	.79	16	.03	17	.48	20	.10
47	14	.42	16	.82	18	.32	21	.12
49	15	.02	17	.60	19	.15	22	.14
51	15	.67	18	.36	19	.98	23	.15
53	16	.29	19	.10	20	.85	24	.16
55	16	.90	19	.86	21	.71	25	.17
57	17	.54	20	.64	22	.56	26	.18
59	18	.15	21	.40	23	.40	27	.19
61	18	.78	22	.14	24	.24	28	.19
63	19	.41	22	.90	25	.06	29	.20
65	20	.02	23	.67	25	.91	30	.20
67	20	.65	24	.43	26	.77	31	.21
69	21	.27	25	.18	27	.62	32	.21

TABLE A3 (Continued)

K = 5 R = 5

T	P=.75		P=.90		P=.95		P=.99	
25	5	.75	5	.90	5	.95	5	.99
27	6	.34	6	.83	6	.99	7	.78
29	6	.84	7	.59	7	.92	8	.81
31	7	.42	8	.22	8	.77	9	.78
33	7	.90	8	.87	9	.55	10	.72
35	8	.46	9	.57	10	.24	11	.63
37	8	.93	10	.18	10	.93	12	.52
39	9	.48	10	.82	11	.71	13	.37
41	9	.95	11	.48	12	.42	14	.21
43	10	.49	12	.06	13	.08	15	.02
45	10	.96	12	.73	13	.81	15	.91
47	11	.50	13	.36	14	.54	16	.79
49	11	.97	13	.94	15	.21	17	.66
51	12	.51	14	.60	15	.90	18	.51
53	12	.98	15	.21	16	.62	19	.34
55	13	.51	15	.83	17	.31	20	.15
57	13	.99	16	.46	17	.97	20	.97
59	14	.51	17	.05	18	.70	21	.84
61	14	.99	17	.70	19	.40	22	.69
63	15	.51	18	.31	20	.05	23	.53
65	15	.99	18	.91	20	.77	24	.36
67	16	.51	19	.55	21	.47	25	.18
69	16	.99	20	.15	22	.14	25	.99
71	17	.52	20	.77	22	.83	26	.84
73	17	.00	21	.40	23	.54	27	.69
75	18	.51	21	.99	24	.21	28	.53

T	P=.75		P=.90		P=.95		P=.99	
26	5	.94	6	.50	6	.75	6	.95
28	6	.62	7	.14	7	.63	8	.16
30	7	.10	7	.87	8	.40	9	.24
32	7	.67	8	.60	9	.07	10	.18
34	8	.17	9	.21	9	.86	11	.05
36	8	.70	9	.86	10	.64	11	.96
38	9	.20	10	.53	11	.35	12	.89
40	9	.72	11	.12	11	.99	13	.79
42	10	.22	11	.78	12	.76	14	.68
44	10	.73	12	.42	13	.48	15	.54
46	11	.24	12	.99	14	.15	16	.38
48	11	.74	13	.67	14	.86	17	.20
50	12	.25	14	.28	15	.58	18	.01
52	12	.75	14	.89	16	.26	18	.89
54	13	.25	15	.54	16	.93	19	.75
56	13	.75	16	.13	17	.66	20	.61
58	14	.26	16	.76	18	.35	21	.44
60	14	.76	17	.39	19	.01	22	.26
62	15	.26	17	.98	19	.74	23	.07
64	15	.76	18	.62	20	.43	23	.91
66	16	.26	19	.23	21	.10	24	.77
68	16	.76	19	.84	21	.80	25	.62
70	17	.26	20	.47	22	.50	26	.45
72	17	.76	21	.07	23	.18	27	.27
74	18	.26	21	.70	23	.86	28	.08

TABLE A3 (Continued)

K = 10 R = 5

T	P=.75		P=.90		P=.95		P=.99	
50	5	.75	5	.90	5	.95	5	.99
52	5	.92	6	.50	6	.78	7	.15
54	6	.31	6	.85	7	.16	7	.92
56	6	.61	7	.23	7	.72	8	.63
58	6	.84	7	.65	8	.00	8	.99
60	7	.09	7	.93	8	.57	9	.69
62	7	.42	8	.34	8	.90	10	.09
64	7	.68	8	.69	9	.34	10	.70
66	7	.90	8	.96	9	.73	11	.10
68	8	.18	9	.36	10	.04	11	.67
70	8	.46	9	.69	10	.51	12	.06
72	8	.71	9	.96	10	.85	12	.63
74	8	.94	10	.35	11	.23	12	.99
76	9	.22	10	.68	11	.64	13	.57
78	9	.49	10	.96	11	.95	13	.95
80	9	.74	11	.33	12	.37	14	.49
82	9	.96	11	.66	12	.74	14	.90
84	10	.25	11	.94	13	.06	15	.40
86	10	.51	12	.29	13	.48	15	.84
88	10	.76	12	.63	13	.82	16	.30
90	10	.98	12	.91	14	.18	16	.76
92	11	.27	13	.25	14	.57	17	.20
94	11	.53	13	.59	14	.90	17	.68
96	11	.77	13	.88	15	.29	18	.08
98	12	.00	14	.21	15	.66	18	.58
100	12	.28	14	.55	15	.98	18	.97

T	P=.75		P=.90		P=.95		P=.99	
51	5	.83	6	.00	6	.50	6	.90
53	6	.07	6	.71	6	.92	7	.72
55	6	.48	6	.97	7	.51	8	.22
57	6	.73	7	.47	7	.88	8	.85
59	6	.94	7	.80	8	.33	9	.41
61	7	.27	8	.10	8	.75	9	.88
63	7	.55	8	.53	9	.07	10	.45
65	7	.79	8	.83	9	.56	10	.89
67	8	.01	9	.15	9	.89	11	.43
69	8	.33	9	.54	10	.30	11	.87
71	8	.59	9	.84	10	.69	12	.38
73	8	.83	10	.16	10	.99	12	.83
75	9	.07	10	.52	11	.45	13	.31
77	9	.36	10	.82	11	.80	13	.78
79	9	.62	11	.14	12	.15	14	.22
81	9	.85	11	.50	12	.56	14	.71
83	10	.10	11	.80	12	.89	15	.13
85	10	.38	12	.11	13	.28	15	.64
87	10	.64	12	.47	13	.66	16	.02
89	10	.87	12	.77	13	.97	16	.55
91	11	.13	13	.07	14	.39	16	.94
93	11	.40	13	.43	14	.75	17	.45
95	11	.65	13	.74	15	.08	17	.87
97	11	.89	14	.02	15	.48	18	.35
99	12	.14	14	.38	15	.82	18	.79

TABLE A3 (Continued)

K = 2 R = 10

T	P=.75		P=.90		P=.95		P=.99	
20	10	.75	10	.90	10	.95	10	.99
22	12	.05	12	.62	12	.81	12	.96
24	13	.32	13	.92	14	.38	14	.88
26	14	.51	15	.37	15	.81	16	.65
28	15	.68	16	.70	17	.24	18	.09
30	16	.84	17	.97	18	.68	19	.78
32	17	.99	19	.34	19	1.00	21	.33
34	19	.18	20	.65	21	.49	22	.85
36	20	.35	21	.92	22	.84	24	.43
38	21	.52	23	.25	24	.23	25	.90
40	22	.67	24	.56	25	.63	27	.49
42	23	.82	25	.84	26	.96	28	.94
44	24	.97	27	.13	28	.38	30	.52
46	26	.14	28	.45	29	.75	31	.96
48	27	.30	29	.73	31	.10	33	.54
50	28	.46	30	.00	32	.50	34	.98
52	29	.61	32	.32	33	.85	36	.56
54	30	.77	33	.62	35	.23	37	1.00
56	31	.92	34	.89	36	.61	39	.57
58	33	.07	36	.18	37	.94	41	.01
60	34	.23	37	.49	39	.34	42	.57
62	35	.39	38	.77	40	.70	44	.02
64	36	.54	40	.04	42	.04	45	.57
66	37	.70	41	.35	43	.44	47	.02
68	38	.84	42	.64	44	.79	48	.57
70	39	.99	43	.91	46	.15	50	.03

T	P=.75		P=.90		P=.95		P=.99	
21	11	.50	11	.80	11	.90	11	.98
23	12	.70	13	.30	13	.65	13	.93
25	13	.88	14	.69	14	.98	15	.79
27	15	.06	15	.96	16	.57	17	.42
29	16	.27	17	.37	17	.91	18	.92
31	17	.44	18	.68	19	.39	20	.59
33	18	.61	19	.95	20	.77	21	1.00
35	19	.76	21	.30	22	.13	23	.67
37	20	.92	22	.61	23	.57	25	.13
39	22	.08	23	.88	24	.90	26	.72
41	23	.25	25	.19	26	.31	28	.21
43	24	.41	26	.51	27	.69	29	.75
45	25	.57	27	.79	29	.02	31	.25
47	26	.72	29	.07	30	.45	32	.77
49	27	.87	30	.39	31	.80	34	.28
51	29	.02	31	.68	33	.17	35	.78
53	30	.19	32	.95	34	.56	37	.29
55	31	.35	34	.25	35	.90	38	.79
57	32	.50	35	.55	37	.28	40	.31
59	33	.66	36	.83	38	.65	41	.80
61	34	.81	38	.11	39	.99	43	.31
63	35	.96	39	.42	41	.39	44	.80
65	37	.11	40	.70	42	.75	46	.31
67	38	.27	41	.97	44	.10	47	.80
69	39	.43	43	.28	45	.48	49	.31

TABLE A3 (Continued)

K = 3 R = 10

T	P=.75		P=.90		P=.95		P=.99	
30	10	.75	10	.90	10	.95	10	.99
32	11	.69	12	.15	12	.58	12	.92
34	12	.56	13	.23	13	.70	14	.43
36	13	.38	14	.22	14	.76	15	.72
38	14	.17	15	.17	15	.80	16	.87
40	14	.95	16	.10	16	.82	17	.99
42	15	.78	17	.02	17	.83	19	.25
44	16	.59	17	.96	18	.84	20	.45
46	17	.38	18	.90	19	.84	21	.59
48	18	.16	19	.83	20	.83	22	.72
50	18	.94	20	.75	21	.82	23	.82
52	19	.75	21	.67	22	.81	24	.92
54	20	.54	22	.58	23	.79	26	.01
56	21	.33	23	.48	24	.77	27	.16
58	22	.10	24	.38	25	.75	28	.30
60	22	.89	25	.28	26	.72	29	.42
62	23	.68	26	.17	27	.70	30	.53
64	24	.47	27	.06	28	.67	31	.63
66	25	.25	27	.96	29	.64	32	.73
68	26	.02	28	.87	30	.61	33	.82
70	26	.81	29	.78	31	.58	34	.90
72	27	.60	30	.68	32	.55	35	.98
74	28	.38	31	.58	33	.52	37	.09
76	29	.16	32	.48	34	.49	38	.21
78	29	.94	33	.37	35	.45	39	.32
80	30	.73	34	.27	36	.41	40	.42

T	P=.75		P=.90		P=.95		P=.99	
31	11	.25	11	.70	11	.85	11	.97
33	12	.07	12	.75	12	.97	13	.77
35	12	.92	13	.75	14	.17	14	.95
37	13	.77	14	.72	15	.27	16	.26
39	14	.59	15	.68	16	.32	17	.54
41	15	.40	16	.62	17	.35	18	.71
43	16	.18	17	.55	18	.36	19	.83
45	16	.96	18	.48	19	.36	20	.94
47	17	.77	19	.39	20	.35	22	.08
49	18	.57	20	.30	21	.34	23	.26
51	19	.36	21	.20	22	.33	24	.41
53	20	.13	22	.09	23	.31	25	.53
55	20	.92	22	.99	24	.28	26	.65
57	21	.72	23	.91	25	.26	27	.75
59	22	.51	24	.82	26	.23	28	.84
61	23	.29	25	.73	27	.20	29	.93
63	24	.06	26	.64	28	.17	31	.02
65	24	.85	27	.54	29	.14	32	.15
67	25	.64	28	.44	30	.10	33	.27
69	26	.43	29	.34	31	.07	34	.38
71	27	.20	30	.23	32	.03	35	.49
73	27	.98	31	.12	32	.00	36	.58
75	28	.77	32	.00	33	.97	37	.68
77	29	.56	32	.91	34	.94	38	.76
79	30	.34	33	.81	35	.91	39	.85

TABLE A3 (Continued)

K = 4 R = 10

T	P=.75		P=.90		P=.95		P=.99	
40	10	.75	10	.90	10	.95	10	.99
42	11	.50	11	.91	12	.25	12	.85
44	12	.08	12	.80	13	.17	13	.92
46	12	.76	13	.62	13	.99	14	.95
48	13	.40	14	.36	14	.89	15	.95
50	13	.97	15	.02	15	.75	16	.94
52	14	.62	15	.80	16	.57	17	.90
54	15	.22	16	.55	17	.35	18	.85
56	15	.82	17	.24	18	.07	19	.79
58	16	.44	17	.92	18	.87	20	.71
60	17	.00	18	.67	19	.68	21	.61
62	17	.64	19	.37	20	.45	22	.50
64	18	.23	20	.02	21	.18	23	.37
66	18	.82	20	.76	21	.93	24	.24
68	19	.43	21	.47	22	.72	25	.08
70	19	.00	22	.14	23	.48	25	.95
72	20	.62	22	.84	24	.22	26	.86
74	21	.20	23	.55	24	.95	27	.75
76	21	.80	24	.22	25	.73	28	.62
78	22	.40	24	.90	26	.49	29	.49
80	22	.97	25	.61	27	.22	30	.35
82	23	.56	26	.30	27	.95	31	.19
84	24	.17	26	.96	28	.73	32	.03
86	24	.76	27	.68	29	.48	32	.91
88	25	.36	28	.36	30	.21	33	.79
90	25	.94	29	.02	30	.94	34	.66

T	P=.75		P=.90		P=.95		P=.99	
41	11	.00	11	.60	11	.80	11	.96
43	11	.78	12	.44	12	.79	13	.48
45	12	.47	13	.17	13	.70	14	.59
47	13	.03	13	.92	14	.55	15	.60
49	13	.70	14	.72	15	.35	16	.57
51	14	.32	15	.47	16	.10	17	.51
53	14	.90	16	.15	16	.91	18	.44
55	15	.54	16	.87	17	.74	19	.34
57	16	.12	17	.61	18	.53	20	.22
59	16	.73	18	.31	19	.28	21	.09
61	17	.34	18	.97	19	.99	21	.97
63	17	.91	19	.72	20	.81	22	.89
65	18	.54	20	.42	21	.59	23	.80
67	19	.11	21	.08	22	.34	24	.70
69	19	.72	21	.80	23	.06	25	.58
71	20	.32	22	.51	23	.84	26	.45
73	20	.90	23	.18	24	.61	27	.31
75	21	.51	23	.87	25	.36	28	.16
77	22	.09	24	.58	26	.06	28	.99
79	22	.69	25	.26	26	.84	29	.89
81	23	.28	25	.93	27	.61	30	.77
83	23	.87	26	.65	28	.36	31	.65
85	24	.47	27	.33	29	.08	32	.51
87	25	.05	27	.99	29	.84	33	.37
89	25	.65	28	.70	30	.60	34	.21

TABLE A3 (Continued)

K = 5 R = 10

T	P=.75		P=.90		P=.95		P=.99	
50	10	.75	10	.90	10	.95	10	.99
52	11	.34	11	.82	11	.98	12	.77
54	11	.84	12	.56	12	.89	13	.75
56	12	.40	13	.12	13	.69	14	.65
58	12	.87	13	.79	14	.38	15	.48
60	13	.41	14	.43	14	.97	16	.23
62	13	.87	14	.95	15	.72	16	.97
64	14	.40	15	.62	16	.37	17	.82
66	14	.86	16	.16	16	.95	18	.62
68	15	.37	16	.76	17	.66	19	.35
70	15	.84	17	.35	18	.28	20	.02
72	16	.34	17	.89	18	.88	20	.82
74	16	.81	18	.50	19	.55	21	.59
76	17	.30	19	.02	20	.13	22	.30
78	17	.78	19	.63	20	.78	22	.97
80	18	.26	20	.18	21	.41	23	.76
82	18	.74	20	.76	21	.98	24	.49
84	19	.22	21	.32	22	.64	25	.18
86	19	.70	21	.87	23	.24	25	.89
88	20	.17	22	.45	23	.85	26	.64
90	20	.66	22	.97	24	.49	27	.35
92	21	.12	83	.57	25	.06	28	.01
94	21	.61	24	.10	25	.70	28	.77
96	22	.07	24	.68	26	.31	29	.49
98	22	.57	25	.23	26	.90	30	.17
100	23	.08	25	.79	27	.53	30	.88

T	P=.75		P=.90		P=.95		P=.99	
51	10	.94	11	.50	11	.75	11	.95
53	11	.62	12	.10	12	.60	12	1.00
55	12	.08	12	.83	13	.29	13	.99
57	12	.65	13	.52	13	.94	14	.94
59	13	.12	14	.03	14	.72	15	.84
61	13	.65	14	.72	15	.39	16	.69
63	14	.12	15	.31	15	.97	17	.47
65	14	.64	15	.87	16	.70	18	.17
67	15	.10	16	.49	17	.33	18	.92
69	15	.62	17	.00	17	.92	19	.73
71	16	.07	17	.64	18	.61	20	.49
73	16	.58	18	.19	19	.21	21	.18
75	17	.04	18	.77	19	.84	21	.90
77	17	.55	19	.35	20	.49	22	.68
79	17	.00	19	.89	21	.05	23	.41
81	18	.51	20	.48	21	.71	24	.07
83	18	.96	20	1.00	22	.33	24	.83
85	19	.47	21	.61	22	.92	25	.57
87	19	.92	22	.15	23	.57	26	.27
89	20	.42	22	.72	24	.15	26	.95
91	20	.88	23	.28	24	.78	27	.71
93	21	.37	23	.83	25	.40	28	.42
95	21	.84	24	.41	25	.97	29	.09
97	22	.32	24	.94	26	.62	29	.82
99	22	.80	25	.52	27	.21	30	.55

TABLE A3 (Continued)

K = 10 R = 10

T	P=.75		P=.90		P=.95		P=.99	
100	10	.75	10	.90	10	.95	10	.99
102	10	.92	11	.50	11	.78	12	.08
104	11	.32	11	.84	12	.11	12	.89
106	11	.62	12	.19	12	.69	13	.53
108	11	.84	12	.62	12	.96	13	.93
110	12	.08	12	.89	13	.47	14	.52
112	12	.41	13	.24	13	.81	14	.90
114	12	.66	13	.61	14	.13	15	.41
116	12	.86	13	.87	14	.57	15	.82
118	13	.14	14	.19	14	.87	16	.24
120	13	.43	14	.55	15	.23	16	.71
122	13	.67	14	.82	15	.61	17	.00
124	13	.89	15	.10	15	.90	17	.54
126	14	.15	15	.46	16	.26	17	.89
128	14	.43	15	.75	16	.68	18	.33
130	14	.67	15	.99	16	.90	18	.74
132	14	.89	16	.35	17	.26	19	.06
134	15	.14	16	.65	17	.62	19	.55
136	15	.41	16	.91	17	.90	19	.89
138	15	.65	17	.23	18	.25	20	.31
140	15	.88	17	.54	18	.60	20	.71
142	16	.12	17	.81	18	.88	21	.02
144	16	.39	18	.08	19	.22	21	.49
146	16	.63	18	.42	19	.57	21	.84
148	16	.85	18	.70	19	.86	22	.23
150	17	.09	18	.96	20	.18	22	.64

T	P=.75		P=.90		P=.95		P=.99	
101	10	.83	10	1.00	11	.50	11	.90
103	11	.08	11	.71	11	.92	12	.70
105	11	.46	11	.95	12	.47	13	.05
107	11	.73	12	.44	12	.84	13	.77
109	11	.94	12	.76	13	.22	14	.17
111	12	.26	12	1.00	13	.66	14	.74
113	12	.54	13	.44	13	.94	15	.07
115	12	.77	13	.75	14	.38	15	.65
117	12	.98	13	.99	14	.73	15	.96
119	13	.29	14	.38	14	.99	16	.50
121	13	.55	14	.69	15	.44	16	.87
123	13	.79	14	.94	15	.76	17	.31
125	13	1.00	15	.29	16	.03	17	.73
127	14	.29	15	.61	16	.45	18	.05
129	14	.55	15	.87	16	.77	18	.55
131	14	.76	16	.18	17	.05	18	.89
133	14	1.00	16	.51	17	.45	19	.33
135	15	.26	16	.79	17	.77	19	.73
137	15	.53	17	.04	18	.04	20	.05
139	15	.77	17	.39	18	.43	20	.52
141	15	.98	17	.68	18	.75	20	.97
143	16	.26	17	.94	19	.02	21	.27
145	16	.51	18	.26	19	.40	21	.68
147	16	.74	18	.56	19	.72	21	.99
149	16	.96	18	.83	19	.99	22	.45

TABLE A4. c_T and ρ_T for Fisher's Logarithmic Distributions

K = 2 N = 5

T	P=.75		P=.90		P=.95		P=.99	
10	5	.75	5	.90	5	.95	5	.99
14	6	.46	5	.17	5	.59	5	.92
18	7	.06	6	.30	6	.87	5	.74
22	9	.72	7	.44	6	.27	5	.45
26	10	.39	8	.61	7	.68	5	.06
30	11	.08	9	.80	7	.08	6	.83
34	13	.80	9	.02	8	.56	6	.60
38	14	.53	10	.28	8	.01	6	.34
42	15	.28	11	.56	9	.53	6	.06
46	16	.04	12	.84	9	.02	7	.85
50	18	.81	12	.14	10	.56	7	.66
54	19	.60	13	.46	10	.09	7	.46
58	20	.39	14	.79	11	.65	7	.25
62	21	.19	14	.14	11	.21	7	.03
66	22	.00	15	.49	12	.79	8	.86
70	24	.82	16	.86	12	.37	8	.70
74	25	.65	16	.23	13	.96	8	.53
78	26	.49	17	.62	13	.56	8	.36
82	27	.33	17	.00	13	.17	8	.18
86	28	.18	18	.41	14	.78	8	.01
90	29	.03	19	.81	14	.40	9	.86
94	31	.89	19	.23	14	.02	9	.72
98	32	.76	20	.65	15	.66	9	.58
102	33	.63	20	.07	15	.30	9	.43
106	34	.51	21	.51	16	.94	9	.29

T	P=.75		P=.90		P=.95		P=.99	
12	5	.13	5	.65	5	.82	5	.96
16	7	.75	6	.74	5	.22	5	.84
20	8	.39	7	.85	6	.60	5	.61
24	9	.04	8	.99	7	.94	5	.27
28	11	.73	8	.20	7	.39	6	.93
32	12	.44	9	.42	8	.82	6	.72
36	13	.16	10	.66	8	.29	6	.47
40	15	.90	11	.91	9	.77	6	.20
44	16	.66	11	.19	9	.28	7	.95
48	17	.42	12	.49	10	.79	7	.76
52	18	.20	13	.80	10	.33	7	.56
56	20	.99	13	.12	11	.87	7	.35
60	21	.79	14	.47	11	.44	7	.14
64	22	.60	15	.81	12	.99	8	.94
68	23	.41	15	.17	12	.58	8	.78
72	24	.24	16	.55	12	.16	8	.61
76	25	.07	17	.92	13	.76	8	.44
80	27	.91	17	.31	13	.37	8	.27
84	28	.75	18	.71	14	.97	8	.09
88	29	.61	18	.11	14	.59	9	.93
92	30	.46	19	.52	14	.21	9	.79
96	31	.33	20	.94	15	.84	9	.65
100	32	.20	20	.36	15	.48	9	.51
104	33	.07	21	.79	15	.12	9	.36
108	35	.95	21	.23	16	.76	9	.22

TABLE A4 (Continued)

K = 3 N = 5

T	P=.75		P=.90		P=.95		P=.99	
15	5	.75	5	.90	5	.95	5	.99
19	5	.02	5	.61	5	.80	5	.96
23	6	.22	5	.04	5	.52	5	.90
27	7	.39	6	.58	5	.07	5	.81
31	8	.58	6	.03	6	.75	5	.69
35	9	.78	7	.58	6	.41	5	.54
39	10	.99	7	.08	6	.02	5	.35
43	10	.22	8	.63	7	.73	5	.14
47	11	.46	8	.17	7	.42	6	.96
51	12	.70	9	.73	7	.08	6	.85
55	13	.96	9	.30	8	.79	6	.73
59	13	.22	10	.87	8	.51	6	.60
63	14	.50	10	.47	8	.21	6	.47
67	15	.77	10	.05	9	.92	6	.33
71	15	.06	11	.66	9	.66	6	.18
75	16	.35	11	.26	9	.40	6	.03
79	17	.65	12	.87	9	.12	7	.92
83	18	.95	12	.49	10	.87	7	.82
87	18	.26	12	.11	10	.62	7	.72
91	19	.57	13	.74	10	.37	7	.62
95	20	.89	13	.38	10	.12	7	.51
99	20	.21	13	.01	11	.88	7	.41
103	21	.54	14	.66	11	.65	7	.30
107	22	.87	14	.31	11	.42	7	.19
111	22	.21	15	.95	11	.19	7	.07

T	P=.75		P=.90		P=.95		P=.99	
17	5	.46	5	.79	5	.89	5	.98
21	6	.63	5	.36	5	.68	5	.94
25	7	.79	6	.81	5	.32	5	.86
29	8	.97	6	.32	6	.90	5	.76
33	8	.18	7	.81	6	.59	5	.62
37	9	.39	7	.34	6	.22	5	.45
41	10	.61	8	.85	7	.87	5	.25
45	11	.84	8	.41	7	.57	5	.02
49	11	.08	9	.94	7	.25	6	.90
53	12	.33	9	.52	8	.93	6	.79
57	13	.59	9	.08	8	.65	6	.67
61	14	.86	10	.67	8	.36	6	.54
65	14	.13	10	.26	8	.06	6	.40
69	15	.42	11	.85	9	.79	6	.26
73	16	.71	11	.46	9	.53	6	.11
77	17	.00	11	.06	9	.26	7	.97
81	17	.30	12	.68	10	.99	7	.87
85	18	.61	12	.30	10	.74	7	.77
89	19	.92	13	.92	10	.50	7	.67
93	19	.23	13	.56	10	.25	7	.57
97	20	.55	13	.20	11	.00	7	.46
101	21	.88	14	.84	11	.77	7	.35
105	21	.21	14	.48	11	.54	7	.24
109	22	.54	14	.13	11	.31	7	.13
113	23	.86	15	.78	11	.07	7	.02

TABLE A4 (Continued)

K = 5 N = 5

T	P=.75		P=.90		P=.95		P=.99	
25	5	.75	5	.90	5	.95	5	.99
35	6	.74	5	.45	5	.73	5	.95
45	7	.60	6	.73	5	.25	5	.85
55	8	.48	6	.02	6	.75	5	.70
65	9	.38	7	.40	6	.29	5	.49
75	10	.32	8	.76	7	.83	5	.24
85	11	.28	8	.15	7	.43	6	.97
95	12	.26	9	.57	8	.99	6	.83
105	13	.26	10	.99	8	.62	6	.67
115	14	.28	10	.46	8	.24	6	.50
125	15	.32	11	.92	9	.87	6	.32
135	16	.36	11	.40	9	.53	6	.13
145	17	.42	12	.89	9	.18	7	.96
155	18	.50	12	.40	10	.84	7	.84
165	19	.58	13	.90	10	.53	7	.71
175	20	.67	13	.43	10	.21	7	.57
185	21	.77	14	.96	11	.89	7	.44
195	22	.89	14	.50	11	.60	7	.30
205	22	.01	14	.04	11	.30	7	.16
215	23	.13	15	.60	12	1.00	7	.02
225	24	.27	15	.16	12	.72	8	.91
235	25	.41	16	.72	12	.45	8	.80
245	26	.56	16	.30	12	.17	8	.69
255	27	.71	17	.87	13	.90	8	.58
265	28	.87	17	.46	13	.63	8	.47

T	P=.75		P=.90		P=.95		P=.99	
30	5	.34	5	.73	5	.87	5	.97
40	6	.17	5	.04	5	.52	5	.90
50	7	.03	6	.40	6	.95	5	.78
60	9	.92	7	.72	6	.53	5	.60
70	10	.85	7	.06	6	.03	5	.37
80	11	.80	8	.46	7	.63	5	.09
90	12	.77	9	.86	7	.21	6	.90
100	13	.76	9	.29	8	.81	6	.75
110	14	.77	10	.73	8	.43	6	.59
120	15	.80	10	.19	8	.04	6	.41
130	16	.84	11	.66	9	.70	6	.23
140	17	.89	11	.14	9	.35	6	.04
150	18	.96	12	.64	10	1.00	7	.90
160	18	.04	12	.15	10	.69	7	.77
170	19	.12	13	.67	10	.37	7	.64
180	20	.22	13	.19	10	.04	7	.51
190	21	.33	14	.73	11	.74	7	.37
200	22	.45	14	.27	11	.45	7	.23
210	23	.57	15	.82	11	.15	7	.09
220	24	.70	15	.38	12	.86	8	.96
230	25	.84	16	.94	12	.59	8	.85
240	26	.98	16	.51	12	.31	8	.75
250	26	.13	16	.08	12	.03	8	.64
260	27	.29	17	.67	13	.76	8	.53
270	28	.46	17	.25	13	.50	8	.42

TABLE A4 (Continued)

K = 10 N = 5

T	P=.75		P=.90		P=.95		P=.99		T	P=.75		P=.90		P=.95		P=.99	
50	5	.75	5	.90	5	.95	5	.99	55	5	.59	5	.84	5	.92	5	.98
60	5	.38	5	.75	5	.88	5	.98	65	5	.10	5	.64	5	.82	5	.96
70	6	.82	5	.51	5	.75	5	.95	75	6	.55	5	.34	5	.67	5	.93
80	6	.28	5	.15	5	.58	5	.92	85	7	1.00	6	.96	5	.47	5	.89
90	7	.73	6	.82	5	.34	5	.87	95	7	.47	6	.67	5	.20	5	.84
100	7	.20	6	.52	5	.05	5	.81	105	8	.93	6	.36	6	.94	5	.78
110	8	.67	6	.19	6	.85	5	.74	115	8	.41	6	.01	6	.75	5	.70
120	8	.15	7	.87	6	.66	5	.66	125	9	.89	7	.73	6	.55	5	.61
130	9	.64	7	.58	6	.45	5	.57	135	9	.38	7	.43	6	.33	5	.52
140	9	.13	7	.28	6	.22	5	.46	145	10	.88	7	.12	6	.10	5	.41
150	10	.63	8	.97	7	.98	5	.35	155	10	.38	8	.84	7	.90	5	.29
160	10	.14	8	.70	7	.81	5	.23	165	11	.89	8	.56	7	.72	5	.17
170	11	.65	8	.42	7	.62	5	.10	175	11	.41	8	.28	7	.53	5	.03
180	11	.17	8	.14	7	.43	6	.98	185	12	.93	9	.99	7	.33	6	.95
190	12	.69	9	.87	7	.23	6	.92	195	12	.46	9	.74	7	.13	6	.88
200	12	.22	9	.61	7	.03	6	.85	205	13	.99	9	.48	8	.94	6	.82
210	13	.75	9	.34	8	.86	6	.78	215	13	.52	9	.21	8	.78	6	.75
220	13	.29	9	.08	8	.69	6	.71	225	13	.06	10	.95	8	.61	6	.67
230	14	.83	10	.83	8	.52	6	.64	235	14	.61	10	.70	8	.44	6	.60
240	14	.38	10	.58	8	.35	6	.56	245	14	.15	10	.46	8	.26	6	.52
250	15	.93	10	.33	8	.17	6	.48	255	15	.70	10	.21	8	.08	6	.44
260	15	.48	10	.08	9	.99	6	.40	265	15	.26	11	.96	9	.92	6	.36
270	15	.04	11	.84	9	.84	6	.32	275	16	.82	11	.72	9	.76	6	.23
280	16	.60	11	.61	9	.69	6	.24	285	16	.38	11	.49	9	.61	6	.19
290	16	.16	11	.37	9	.53	6	.15	295	17	.94	11	.25	9	.45	6	.11

VITA

VITA

Klaus Richard O. Nagel was born April 29th, 1937 in Wilhelmshaven, Germany. He is a citizen of Germany. In February he graduated from the Paul-Gerhardt-Schule in Dassel, Lower Saxonia, Germany. Thereafter he studied Mathematics at the University of Heidelberg, from which he received the degree of Diplom-Mathematiker in May 1965, working under Professor K. Krickeberg. From September 1965 to June 1966 and again since September 1967 he attended Purdue University for graduate studies in Statistics. From October 1966 through August 1967 he worked as a mathematician with Bull-General-Electric Company in Cologne, Germany.