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ON A MONOTONICITY PROPERTY RELATING TO THE GAMMA DISTRIBUTIONS*

by

S. Panchapakesan

Department of Statistics

Division of Mathematical Sciences

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1. Introduction and Summary. Let $\{F_\lambda\}$, $\lambda \geq 1$, be the family of gamma distributions with density function f_λ given by

$$(1.1) \qquad f_\lambda(x) = \begin{cases} \frac{\alpha^\lambda}{\Gamma(\lambda)} e^{-x} x^{\lambda-1} & , \ x \geq 0 \\ 0 & \text{elsewhere,} \end{cases}$$

where the shape parameter $\alpha > 0$ is same for all distributions of the family. We define

$$(1.2) \qquad A(\lambda) = \int_0^\infty F_\lambda^{k-1} (\tfrac{x}{b}) \, dF_\lambda(x),$$

where $k \geq 2$ is an integer and $0 < b \leq 1$. Gupta [2] has discussed the subset selection problem for gamma populations with the same degrees of freedom in terms of their shape parameters and his numerical computations indicate the monotonic behavior of $A(\lambda)$ in $\lambda \geq 1$. This monotonic behavior of $A(\lambda)$ can be easily proved by appealing to the fact that the gamma distributions are convex ordered, i.e., for $0 < \lambda < \lambda'$, $F_\lambda^{-1} F_{\lambda'}(x)$ is

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convex on $[0, \infty)$ (see Barlow and Gupta [1]). However, the proof of this convex ordering property of the gamma family is very tedious (see Van Zwet [5]). McDonald [3] gives a direct proof of the monotonicity of $A(\lambda)$ with the purpose of avoiding recourse to the convex ordering property, but only for the case of $k = 2$ and integer-valued λ. We first note that a general result of the author [4] is relevant here. This result states that, if $\{F_\lambda\}$, $\lambda \in \Lambda$, an interval on the real line, is a family of absolutely continuous distributions and $\Psi(x,\lambda)$ is a real valued function differentiable in x and λ, then, for any positive integer t, $B(\lambda) = \int \Psi^t(x,\lambda)dF_\lambda(x)$ is non-decreasing in λ provided that $\Psi(x,\lambda) \geq 0$ for all x and λ and

$$(1.3) \qquad f_\lambda(x) \frac{\partial}{\partial\lambda} \Psi(x,\lambda) - \frac{\partial}{\partial x} \Psi(x,\lambda) \frac{\partial}{\partial\lambda} F_\lambda(x) \geq 0.$$

In the case of $A(\lambda)$ defined in (1.2), the condition (1.3) reduces to

$$(1.4) \qquad f_\lambda(x) \frac{\partial}{\partial\lambda} F_\lambda\left(\frac{x}{b}\right) - \frac{1}{b} f_\lambda\left(\frac{x}{b}\right) \frac{\partial}{\partial\lambda} F_\lambda(x) \geq 0.$$

However, verification of this condition in the case of the gamma family when $\lambda \in [1, \infty)$ presents difficulties. The aim of this note is to obtain sufficient conditions for $B(\lambda)$ to be nondecreasing in λ where $\lambda \in \Lambda_d = (\lambda_1 < \lambda_2 < \ldots\ldots)$. This forms the content of the next section. The last section applies this result to establish the monotonicity of $A(\lambda)$ defined in (1.2) for $k \geq 2$ and $\Lambda_d = (1,2,3,\ldots\ldots)$.

2. <u>The Main Result</u>. Let $\{F_\lambda\}$, $\lambda \in \Lambda_d = (\lambda_1 < \lambda_2 < \ldots\ldots)$, be a family of absolutely continuous distributions on the real line all having the same support and $\Psi(x, \lambda)$ be a non-negative function differentiable in x. Then, for any positive integer t, $B(\lambda) = \int \Psi^t(x,\lambda)dF_\lambda(x)$ is nondecreasing

in λ over Λ_d, provided that, for $i = 1,2,\ldots.,$

$$(2.1) \qquad \Delta_\Psi(x, \lambda_i)\, f_{\lambda_j}(x) - \Delta F_{\lambda_i}(x)\, \Psi'(x\ \lambda_j) \geq 0,\ j = i,\ i+1,$$

where $\Delta\Psi(x, \lambda_i) = \Psi(x, \lambda_{i+1}) - \Psi(x, \lambda_i)$, $\Delta F_{\lambda_i}(x) = F_{\lambda_{i+1}}(x) - F_{\lambda_i}(x)$ and the prime over Ψ denotes the derivative w.r.t. x.

__Proof__. For $\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_{t+1}} \in \Lambda_d$, define

$$(2.2) \quad A_r(\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_{t+1}}) = \int \prod_{\substack{j=1 \\ j\neq r}}^{t+1} \Psi_j(x)\, dF_j(x),\ r = 1,2,\ldots,t+1,$$

and

$$(2.3) \quad A(\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_{t+1}}) = \sum_{r=1}^{t+1} A_r(\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_{t+1}}),$$

where $F_j \equiv F_{\lambda_j}$ and $\Psi_j = \Psi(x, \lambda_j)$. For any positive integer s,

$$(2.4) \qquad A(\lambda_{s+1}, \lambda_{s+1}, \ldots, \lambda_{s+1}) - A(\lambda_s, \lambda_s, \ldots, \lambda_s)$$
$$= (A_o - A_1) + (A_1 - A_2) + \ldots. + (A_t - A_{t+1}),$$

where A_m stands for A with the first m arguments equalto λ_{s+1} and the remaining t+1-m equal to λ_s. Let us consider a typical term, namely, $A_m - A_{m+1}$, in (2.4). We can see that

$$(2.5) \quad A_m - A_{m+1} = m \int \Psi_s^{m-1} \Psi_{s+1}^{t+1-m}\, dF_s(x) + (t+1-m) \int \Psi_s^m \Psi_{s+1}^{t-m}\, dF_{s+1}(x)$$

$$- (m+1) \int \Psi_s^m \Psi_{s+1}^{t-m}\, dF_s(x) - (t-m) \int \Psi_s^{m+1} \Psi_{s+1}^{t-m-1}\, dF_{s+1}(x).$$

Using integration by parts, we obtain

$$(2.6) \qquad \int \Psi_s^m \Psi_{s+1}^{t-m} \, dF_{s+1}(x) - \int \Psi_s^m \Psi_{s+1}^{t-m} \, dF_s(x)$$

$$= \int (F_s - F_{s+1}) \Psi_s^{m-1} \Psi_{s+1}^{t-m-1} [m \Psi_{s+1} \Psi_s' + (t-m) \Psi_s \Psi_{s+1}'] \, dx .$$

Using (2.6) in (2.5) and regrouping the terms, we can write (2.5) as

$$(2.7) \quad A_m - A_{m+1} = m \int \Psi_s^{m-1} \Psi_{s+1}^{t-m} [f_s \Delta \Psi_s - \Psi_s' \Delta F_s] dx$$

$$+ (t-m) \int \Psi_s^m \Psi_{s+1}^{t-m-1} [f_{s+1} \Delta \Psi_s - \Psi_{s+1}' \Delta F_s] \, dx,$$

which is non-negative if (2.1) is satisfied. This completes the proof of the main result.

Remark. The non-negativity of $\Psi(x,\lambda)$ is essentially needed when $t > 1$.

3. Application to the Gamma family. Let F_λ be the distribution function with density f_λ given by (1.1) and let $\lambda \in [1,2,3,\ldots.]$. In order to establish the monotonicity of $A(\lambda)$, we need to show that for any positive integer i,

$$(3.1) \qquad f_i(x) \, \Delta F_j \left(\frac{x}{b}\right) - \frac{1}{b} \quad f_j\left(\frac{x}{b}\right) \Delta F_i(x) \geq o \quad \text{for } j=i,i+1.$$

It is well-known that $\Delta F_i(x) = -\frac{1}{\alpha} f_{i+1}(x)$. Hence, the left hand side of (3.1)

$$= - \frac{1}{\alpha} f_{i+1} \left(\frac{x}{b}\right) \quad f_j(x) + \frac{1}{b\,\alpha} f_{i+1} \,(x) \; f_j \left(\frac{x}{b}\right)$$

$$= \frac{e^{-\alpha(\frac{1}{b} + 1)x}}{\Gamma(j)\,\Gamma(i+1)} \quad \alpha^{i+j} \, x^{i+j-1} \left(\frac{1}{b^j} - \frac{1}{b^i}\right)$$

≥ 0 for $j = i, i+1$, since $b \leq 1$.

REFERENCES

[1] Barlow, R. E. and S. S. Gupta (1969). Selection procedures for restricted families of probability distributions. Ann. Math. Statist. 40, 905-917.

[2] Gupta, S. S. (1963). On a selection and ranking procedure for gamma populations. Ann. Inst. Statist. Math. (Tokyo). 14, 199-216.

[3] McDonald, G. C. (1969) A note on monotcnicity in the gamma distributions. Research Publication GMR-930, General Motors Corporation, Warren, Michigan.

[4] Panchapakesan, S. (1969). Some contributions to multiple decision (selection and ranking) procedures. Mimeo. Series. No. 192, Purdue University, Lafayette, Indiana.

[5] Van Zwet (1964). Convex Transformations of random variables. Mathematisch Centrum Amsterdam.