Statistical Process Controls and Acceptance Sampling Methods to Control the Distribution Of Combined Output Characteristics

by

Richard Eugene James
Purdue University

Department of Statistics
Division of Mathematical Sciences
Mimeograph Series #226

August 1970

CHAPTER I

INTRODUCTION

1.1 Historical Background and Field of Application

Maximum and minimum specification limits for parts dimensions are in extremely wide use today. The following stages of development of dimensional control have occurred in industrial history. Shewhart(1939) gives a good discussion of them.

1). Each part was made individually, with its dimensions gradually shaped until it would fit the particular part(s) with which it was to be assembled.

2.) Interchangeable parts needed for large scale production, giving rise to the aim of making all parts "exactly alike."

3.) As technology improved it became apparent that it was not possible to make parts exactly alike. Thus, there came about, first, single limits, and later maximum and minimum limits.

4.) All parts should lie between these limits.

5.) Since in a large lot, or in the output of a machine, it is not possible to guarantee that all parts are between given limits, an acceptable percent outside the limits may be set; that is, an AQL or acceptable quality level.

The last is in wide use at present.

In order to be precise in our terminology, it is desirable to distinguish between three sometimes confused terms: specification limits, tolerances, and tolerance limits. Specification limits are limits that are set somewhat arbitrarily, say, by the design engineer or the consumer, usually without regard to what the process can actually achieve. If we denote the lower and upper specification limits by L and U, respectively, then T = U - L is often referred to as the tolerance of the specification limits. Tolerance limits, however, are defined as limits estimated from data, so as to have a specified probability of including at least a given percentage of individuals of the process population or distribution. Wilks (1942) points out that the population as used here is the output of a controlled production process, since tolerance limits apply to future product as well as current product. In this research we will be considering specification limits, or tolerances, as opposed to tolerance limits. Ideally, of course, we would like to have the tolerance limits lie wholly within the specification limits L and U.

Having noted briefly the development of maximum and minimum specification limits, as well as defined the term tolerance, we now list the general conditions which, for the purposes of this thesis, define the type of industrial manufacturing with which we are concerned.

1.) Individual piece parts are manufactured.

2.) The dimensions or characteristics of these parts can be measured.

3.) The parts are manufactured for assembly with other parts and therefore typically subject to maximum and minimum limits.

4.) The dimensions or characteristics of the component parts combine in a known fashion, determining characteristics of the assemblies.

It is believed that the methods and techniques developed here can be used for other general processes, but the foregoing is the broad field with which we are now concerned.

1.2. Problems of Specification Limits

The current application of specification limits has created several problems, such as not giving the engineer what he really wants and/or needs, or further increasing production and inspection costs. Design engineers tend to be rather conservative and commonly give specification limits for parts so that, as long as all parts lie within their respective limits, the assembly characteristics will lie between whatever limits are set for them. Thus, this philosophy says in effect that "any distribution of part dimensions between the specification limits is entirely satisfactory, as long as all or a very large percentage of the dimensions lie between the limits." Burr (1967a) gives a good discussion of this problem, and demonstrates that this leads to undesirable kinds of distributions such as shown in Figure 1.1. Distribution (a) has its middle section removed by sorting out parts for another customer who has a tighter tolerance than we do. Distribution (b) is from a process in bad control but heavily sorted. Distributions (c) and (d) are typical of the inside and outside diameters, say, for a bearing and its fitting shaft. For the shaft, the process level is set to the high side because metal can be taken off but cannot be put back on. Then the shafts are sorted, reworked and sorted again. Meanwhile, for the bearing the process has been set to the low side, then they are sorted and some reworked toward larger inside diameters. Distribution (e) is run with much better process capability than called for by the tolerance, and set low, perhaps to save material.

Other authors have recognized this problem: Colley (1959) demonstrates a Monte Carlo technique for deriving correct tolerances for mating parts; Breipohl (1960) differentiates between out-of-tolerance parts and parts which are complete failures; Osuga (1964) developed a "process capability index" which indicates whether or not the specified tolerance is realistic.

It becomes clear then from Burr's work that when the underlying lot distribution is unknown (as is usually the case), accepting or rejecting a lot solely on the basis of specification limits can be quite uneconomic. This is because it forces the design engineer to be ultra-conservative; that is, to specify tighter tolerances than would be necessary if distributions are controlled. A much better

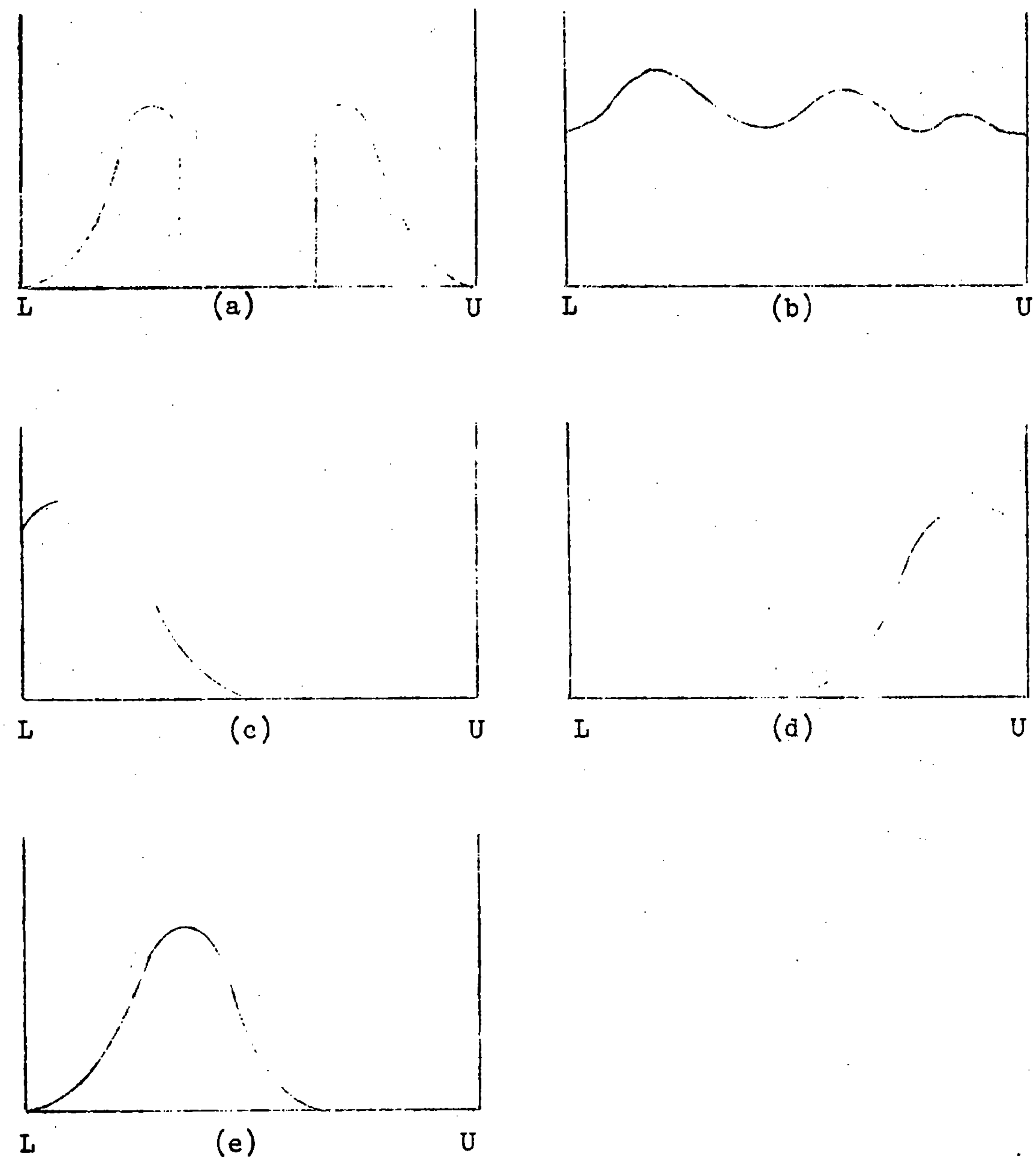

Figure 1.1. Five Undesirable Distributions Meeting Specification Limits.

criterion for acceptance of a given lot would be one in which the design engineer could specify what distributions of part dimensions or characteristics will satisfy the design, so that the assembly will be satisfactory. He thus sets the desired distributions of the component part dimensions in light of how they will be combined, making use of the statistics of combinations, and with knowledge of relative difficulty of manufacture. This provides initial specifications and is a first approximation. As process capabilities for manufacturing the parts become better known, a more economical allocation of distributions can be made.

1.3. Tolerance Combinations

For an assembly composed of k component parts, let

Ta = assembly tolerance, or the total permissible variation for the assembly,

Ti = tolerance for the i^{th} component part, i = 1,..., k

The relationship between Ta and the Ti, i=1,...,k is often expressed by using either(1.3.1) or (1.3.2) given below.

$$T_a = T_1 + T_2 + \dots + T_k = \sum_{i=1}^{k} T_i \tag{1.3.1}$$

$$T_a = \sqrt{T_1^2 + T_2^2 + \dots + T_k^2} = \sqrt{\sum_{i=1}^{k} T_i^2} \tag{1.3.2}$$

It is easy to show that if T_a is known, using (1.3.2) will in almost all cases considerably loosen up on the permissible variabilities of the k parts, as compared to using (1.3.1). The gain is significant even with only two component parts in the assembly.

It is clear, then, that the design engineer would prefer to use (1.3.2) over (1.3.1), if this could be done safely. One can show that by assuming

(a) normal distributions for the part dimensions,

(b) independence between parts,

(c) a state of statistical control existing in each part production process, and

(d) the distributions of the part dimensions all properly centered about respective nominal values,

then the design engineer is safe in using (1.3.2). It is not presently known, however, whether or not (1.3.2) can be safely used whenever one or more of the above four assumptions is violated. Many authors have advocated the use of (1.3.2) over (1.3.1): Fielden (1960), Ogden (1961), Sandquist and Enrick (1963), Enrick (1964), and Mouradian (1966) to mention a few, but little attempt has been made to justify its use. It is to this basic problem that we will attempt to provide an answer: under what conditions will the underlying lot distribution be "good enough" to allow the safe use of (1.3.2)?

We now have mentioned two basic problems, both stemming from the fact that the underlying lot or process distribution is usually unknown:

1.) Accepting or rejecting a process or lot solely on the basis of conformance to specification limits can lead to difficulties, and

2.) The criteria under which (1.3.2),

$$T_a = \sqrt{\sum_{i=1}^{k} T_i^2}$$

can be safely used are unknown.

We propose to solve both problems by replacing the standard specification limits with two statistical tests which will judge the acceptability of an underlying lot distribution, and enable safe use of (1.3.2) through acceptance and rejection-rectification. In order to do this we must provide the design engineer with two basic assurances:

a.) The process or lot mean is sufficiently close to the middle of the specified range, and

b.) The variability of the process is not excessive.

We will be more specific about what we mean by (a) and (b) later in this thesis when the statistical tools necessary to provide such assurance are given.

1.4. Characteristics for an Assembly

Burr (1967) gives an excellent discussion of the characteristics for a process assembly. We shall be using the following notations:

Let X_1 represent a measurable characteristic of a component part,

μ_{X_1} = the process mean for X_1

σ_{X_1} = the process standard deviation for X_1

For a sample of size n from a given lot of component parts, say $(X_{11}, X_{12}, \ldots, X_{1n})$, define

$$\bar{X}_1 = \text{sample mean} = \sum_{i=1}^{n} X_{1i}/n \qquad (1.4.1)$$

$$R_1 = \text{sample range} =$$

$$\max (X_{11},\ldots,X_{1n}) - \min (X_{11},\ldots,X_{1n}) \qquad (1.4.2)$$

Let Y be the characteristic of the assembly. If there are k component characteristics X_1, X_2, ..., X_k, then we will assume for the purposes of this thesis that our assembly characteristic has the form:

$$Y = X_1 \pm X_2 \pm \ldots \pm X_k \ . \qquad (1.4.3)$$

Other more complex relationships can also be handled. See, for example, Burr (1961) and Tukey (c1958).

Assuming assembly characteristic (1.4.3), one can easily show that the following is true:

$$\mu_Y = \mu_{X_1} \pm \mu_{X_2} \pm \ldots \pm \mu_{X_k} \ , \qquad (1.4.4)$$

whether or not the X_i are independent. If the X_i are independent, however, then:

$$\sigma_Y = (\sigma^2_{X_1} + \sigma^2_{X_2} + \ldots + \sigma^2_{X_k})^{\frac{1}{2}} \ . \qquad (1.4.5)$$

Such independence is achieved in practice if the processes are in control, or if assembly is done by random choice out of the respective lots. If the parts characteristics are not independent then we could have, at least theoretically, the two extremes

$$\sigma_Y = 0 \ , \qquad (1.4.6)$$

or

$$\sigma_Y = \sigma_{X_1} + \sigma_{X_2} + \dots + \sigma_{X_k} \quad (1.4.7)$$

The right side of (1.4.7) is in general much larger than that of (1.4.5), and is like what the design engineer is assuming if he uses the additive tolerance formula (1.3.1).

If we consider the $X_1, X_2, \dots, X_k$ as random variables, the well-known Central Limit Theorem implies that Y will be approximately normally distributed regardless of the distribution of the X_i, $i=1,\dots,k$, if k is sufficiently large and no one or few of the X_i predominate. The theorem is valid regardless of whether or not the random variables are identically distributed, provided that their variances are not too different. Although the theorem will not help us much when considering a single component part, as k increases it favors the use of tolerance formula (1.3.2) over (1.3.1).

Formulas (1.4.4) and (1.4.5) work very well for relationship (1.4.3) for the assembly characteristic Y, when we have large lots of articles to be assembled at random, or processes in good statistical control. Unfortunately processes often do not remain in good statistical control. Thus the process mean μ_X will commonly have a tendency to vary, and the process standard deviation σ_X may also change. Because of this we shall want to extend (1.4.4) and (1.4.5) somewhat to take account of lack-of-controlness and thereby facilitate use of tolerance allocation formula (1.3.2).

1.5. Characteristic Variation for a Single Part

Consider again the measurable characteristic of a single component part X, with the corresponding process mean μ_X and standard deviation σ_X. Suppose, as indicated above, that the process mean varies. With this variability comes a mean and standard deviation of the process mean. Let

μ_μ = mean of the process means $\{\mu_X\}$

σ_μ = standard deviation of the process means $\{\mu_X\}$

We will further assume that σ_X is independent of μ_X, and remains constant for a given X. Let

μ_o = desired nominal mean (commonly the middle of a tolerance range under current practice)

Although we would hope that $\mu_\mu = \mu_o$, we know in general this is not always true. We are interested in determining the total variability; that is $E(X - \mu_o)^2$, where E is an expected value operator as usually defined. The following lemma and proof are given by Burr (1967).

<u>Lemma 1</u>: With the above definitions for

$X, \mu_o, \mu_X, \sigma_X, \mu_\mu$, and σ_μ,

$$E(X - \mu_o)^2 = \sigma_X^2 + \sigma_\mu^2 + (\mu_\mu - \mu_o)^2. \qquad (1.5.1)$$

<u>Proof</u>: Call $\mu_X = \mu$ for ease of notation. We have distributions for random variables X and μ, so using subscripts on E to denote the random variables with respect to which the expectation is taken, we seek:

$$E_{X,\mu}(X - \mu_o)^2$$

$$E_{X,\mu}(X - \mu_o)^2 = E_{X,\mu}[(X-\mu) + (\mu-\mu_\mu) + (\mu_\mu - \mu_o)]^2$$

$$= E_{X,\mu}[(X-\mu)^2 + (\mu_\mu - \mu_o)^2 + (\mu - \mu_\mu)^2 \qquad (1.5.2)$$

$$+ 2(X-\mu)(\mu-\mu_\mu) + 2(X-\mu)(\mu_\mu-\mu_o) + 2(\mu-\mu_\mu)(\mu_\mu-\mu_o)]$$

Taking first the expectation with respect to X gives

$$E_{X,\mu}(X-\mu_o)^2 = E_\mu[\sigma_X^2 + (\mu - \mu_\mu)^2 + (\mu_\mu - \mu_o)^2 \qquad (1.5.3)$$

$$+ 2E_X[(X-\mu)(\mu-\mu_\mu)] + 0 + 2(\mu-\mu_\mu)(\mu_\mu-\mu_o)]$$

since $E_X(X-\mu) = 0$, $E_X(X-\mu)^2 = \sigma_X^2$. Now, define $f(X,\mu)$ as the joint density function of (X,μ). Then:

$$E_X[(X - \mu)(\mu - \mu_\mu)] \qquad (1.5.4)$$

$$= \int_\mu \int_X (X - \mu)(\mu - \mu_\mu) f(X,\mu)\, dX\, d\mu$$

$$= \int_\mu (\mu - \mu_\mu) \left[\int_X (X - \mu) f(X,\mu)\, dX \right] d\mu$$

But $\int_X (X - \mu) f(X,\mu)\, dX$

$$= \int_X X f(X,\mu)\, dx - \int_X \mu f(X,\mu)\, dX$$

$$= E(X|\mu)\, g(\mu) - \mu\, g(\mu) \qquad (1.5.5)$$

where $\quad g(\mu) = \int_X f(X,\mu)\, dX$

But since $E(X|\mu) = \mu$,

$$\int_X (X-\mu) f(X,\mu)\, dX = 0$$

Therefore $E_X[(X-\mu)(\mu - \mu_\mu)] = 0$

Taking next the expectation with respect to μ gives

$$E_{X,\mu}(X - \mu_o)^2 = \sigma_X^2 + \sigma_\mu^2 + (\mu_\mu - \mu_o)^2 + 0, \qquad (1.5.6)$$

since $E_\mu(\mu - \mu_\mu) = 0,\ E_\mu(\mu - \mu_\mu)^2 = \sigma_\mu^2.$

Hence $E_{X,\mu}(X - \mu_o)^2 = \sigma_X^2 + \sigma_\mu^2 + (\mu_\mu - \mu_o)^2$.

Thus in controlling the total variability, we must consider three sources:

σ_X^2 , the instantaneous process variance

σ_μ^2 , the process mean variance, and

$(\mu_\mu - \mu_o)^2$, a bias term.

In the past when considering the variability of X around the nominal mean μ_o, some authors have made insufficient allowance for σ_μ^2 , and too little attention has been paid to μ_μ vs. μ_o. One good discussion of this problem is given by Freund (1957).

1.6. Variation of the Assembly Characteristic

We now wish to extend the above procedure to Y. Let

μ_{oY} = nominal mean for the assembly Y

μ_{oi} = nominal mean for the i^{th} component part characteristic X_i.

The design engineer should arrange that

$$\mu_{oY} = \mu_{o1} \pm \mu_{o2} \pm \ldots \pm \mu_{ok} \qquad (1.6.1)$$

Hence

$$Y - \mu_{oY} = (X_1 - \mu_{o1}) \pm (X_2 - \mu_{o2}) \pm \ldots \pm (X_k - \mu_{ok}) \qquad (1.6.2)$$

by combining (1.4.3) and (1.6.1).

Lemma 2:

$$E(Y - \mu_{oY})^2 = \sigma^2_{X_1} + \sigma^2_{\mu_1} + \ldots \sigma^2_{X_k} + \sigma^2_{\mu_k}$$

$$+ \left[(\mu_{\mu_1} - \mu_{o1}) \pm \ldots \pm (\mu_{\mu_k} - \mu_{ok})\right]^2 \qquad (1.6.3)$$

Indication of Proof: The proof is completely analogous to that of Lemma 1, except it involves 2k distributions instead of two. Burr (1967) indicates the proof through a special case, i.e. k=2, and gives the general result (1.6.3). Basically the proof involves evaluating 2k multiple integrals with independence. After adding and subtracting the appropriate variables in (1.6.2) and then squaring, one can integrate out variable by variable, noting that the cross products all drop out through independence. Following complete integration, the remaining terms are collected and form (1.6.3).

The last term in (1.6.3) reflects the biases in the process controls being used. The right hand side of (1.6.3) is very similar to the sum of the right hand sides of k equations like (1.5.1). The difference lies in the fact that

$$\left[(\mu_{\mu_1} - \mu_{o1}) \pm \ldots \pm (\mu_{\mu_k} - \mu_{ok})\right]^2 \qquad (1.6.4)$$

is not precisely

$$(\mu_{\mu_1} - \mu_{o1})^2 + \ldots + (\mu_{\mu_k} - \mu_{ok})^2 , \qquad (1.6.5)$$

since the former would include not only the latter terms, but also many cross product terms, too. Burr (1967) discusses this and indicates that unless nearly all biases are in one direction, the cross product terms will largely cancel out. Thus the right side of (1.5.1) is a good indication of the contribution of the i^{th} part of the right side of (1.6.3). In addition, the suggested controls given in this thesis should usually force the biases to be negligibly small.

For a single component part characteristic X_i of the assembly, equation (1.5.1) indicates that in order to control the total variability of X_i about μ_{oi}, we must consider three different sources of variability: $\sigma^2_{X_i}$, $\sigma^2_{\mu_i}$, and $(\mu_{\mu_i} - \mu_{oi})^2$. Burr (1967) chose the following as being practical working values, which can be used to justify the use of (1.3.2). Let T_i = tolerance allocated for part characteristic X_i. Then, if

$$\sigma_{X_i} = T_i/8 ,$$

$$\sigma_{\mu_i} = T_i/10$$

$$| \mu_{\mu_i} - \mu_{oi} | = T_i/20$$

(1.5.1) gives

$$E(X_i - \mu_{oi})^2 = \left(\frac{T_i}{8}\right)^2 + \left(\frac{T_i}{10}\right)^2 + \left(\frac{T_i}{20}\right)^2$$

$$= .028125\ T_i^2$$

Hence

$$\sqrt{E(X_i - \mu_{oi})^2} = .1677\ T_i ,$$

or $$T_i \doteq 6\sqrt{E(X_i - \mu_{oi})^2} , \tag{1.6.6}$$

Comparing (1.5.1) and (1.6.3), we see that the i^{th} contribution to the assembly variability, $E(Y - \mu_{oY})^2$, is approximately $E(X_i - \mu_{oi})^2$ if the biases are small. Thus

$$E(Y - \mu_{oY})^2 = \sum_{i=1}^{k} E(X_i - \mu_{oi})^2 \tag{1.6.7}$$

We now define

$$T_a = 6\sqrt{E(Y - \mu_{oY})^2} , \tag{1.6.8}$$

and multiplying both sides of (1.6.7) by $6^2 = 36$, and recalling equation (1.6.6), yields

$$T_a^2 = 36 \sum_{i=1}^{k} E(X_i - \mu_{oi})^2 \doteq \sum_{i=1}^{k} T_i^2 .$$

It is clear of course that σ_{X_i}, σ_{μ_i}, and the bias term can be <u>less</u> than the stated values, and the proof still holds, with

$T_a \doteq \sqrt{\sum_{i=1}^{k} T_i^2}$, or less.

Not every practical situation will fit these three ratios with respect to T_i, however; for example, σ_{μ_i} may be larger than σ_{X_i} in some cases. But if the total variability is not excessive, one should still be able to show that tolerance formula (1.3.2) holds. Rather than choosing specific values for our three sources of variability, suppose $\sigma_{X_i} = a\ T_i$, $\sigma_{\mu_i} = b\ T_i$, and $|\ \mu_{\mu_i} - \mu_o\ | = c\ T_i$, for constants a, b, and c. Equations (1.5.1) and (1.6.6) then combine to give

$$(a\ T_i)^2 + (b\ T_i)^2 + (c\ T_i)^2 \leq .0278\ T_i^2 ,$$

or

$$a^2 + b^2 + c^2 \leq .0278 , \tag{1.6.9}$$

with the inequality added because the right hand side is actually an upper bound on the variability.

We now have a basis by which we can judge whether or not the total variability of each component part characteristic X_i is controlled well enough to allow the usage of tolerance formula (1.3.2). We will make frequent use of this later in this thesis. Normally σ_X, σ_μ, and the biases are unknown and often can only be roughly estimated. By developing statistical tests based on a sample from our underlying lot distribution, however, we will be able to provide assurances that inequality (1.6.9) is satisfied, thus allowing the safe use of tolerance formula (1.3.2).

1.7. General Discussion of New Statistical Tests

In the foregoing we have briefly described the problem we wish our statistical tests to solve. We will now discuss in general the type of tests in which we are interested, as well as some general requirements to be met.

We are given a component part characteristic with an unknown distribution F(X). In order to estimate the general characteristics of F(X), we will take a random sample of size N and compute two simple statistics:

Let $(X_1, X_2, \ldots, X_N)$

be a random sample from $F(X)$; then find

a) $\bar{X}_N = \sum_{i=1}^{N} X_i/N$, and either (1.7.1)

b) $R_N = \max(X_1, \ldots, X_N) - \min(X_1, \ldots, X_N)$, or

$$S_{nm} = \sum_{j=1}^{m} R_{nj} ,$$

the sum of m ranges, each of size n, so that $N = nm$.

The question as to how a sample of size N should best be subdivided into m ranges from the point of view of smallest variance has been studied by Pearson (1932) and Grubbs and Weaver (1947). It has been found that subgroups of eight are best, with very little efficiency lost in subgroups from five to ten. Accordingly, we will limit the size of a single range to 10, and use the sum-range $S_{nm} = \sum_{j=1}^{m} R_{nj}$ as our statistic whenever $N > 10$ for $N = nm$.

We are using the range as an estimate of the variability in $F(X)$ rather than the sample standard deviation for the sake of simplicity. In order to encourage wide implementation of our tests, it was felt that the form of the tests should be simple to understand, calculate, and interpret.

In addition to the criteria already mentioned, we want our $\bar{X}$ and R tests to satisfy two additional requirements:

a) The tests should be as robust as possible. That is, we would hope that the tests would perform substantially well regardless of the actual form of the unknown distribution $F(X)$.

b) We want to develop a series of tests to cover a wide variety of process types where this general approach can be applied. The various classifications of such processes will be discussed briefly below, and at length in later chapters.

In developing our $\overline{X}$ and R tests, we want to provide assurance that the process mean will be reasonably close to the nominal mean, and that the variability will not be excessive. In addition, we would like to express our tests in terms of known values: μ_o, the nominal mean, and T, the tolerance range for component part characteristic X. Thus we use the following test form:

Accept the distribution F(X) if <u>both</u> of the following criteria are met:

a) $\mu_o - c_1 T \leq \overline{X} \leq \mu_o + c_1 T$

b) R_N or $S_{nm} \leq c_2 T$

Reject F(X) if either/or both of (a) and (b) is not met. In some cases we will then sort the rejected material.

Here $\overline{X}$ and R are computed as given above, μ_o and T are known, and c_1, c_2 are constants dependent on the general classification of the process under consideration.

1.8. Process Classification

Situations for which tolerance ranges are important can be classified into two general areas:

a) process control, and

b) acceptance sampling, where one is generally not close to the process.

Two types of process behavior has been postulated; namely

a1) where there is a consistent tendency for μ_X to move in one direction or the other, as in tool wear, for example, and

a2) where the process, if left alone, makes erratic random jumps to various levels at least some of which are undesirable.

Two approaches were used in controlling the tool wear process, (a1), depending on the method of reset following a test rejection. Details are given in Chapter V. Control tests to handle these cases and to provide assurance of the combined distribution proving satisfactory have been developed.

The program followed in the research has been to propose general probabilistic models which closely describe the physical characteristics of each of the cases mentioned above. We then see how the tests under study perform on these models. By also considering badly controlled or highly erratic models, and applying the tests, assurance against undesirable process conditions is obtained.

No attempt has been made to try to optimize the tests for all situations since the field of application of the methods of this paper is so broad. The author feels, however, that most commonly occurring processes can be satisfactorily controlled using these tests. A summary of the step-by-step procedures recommended to implement the proposed tests will be given following a complete discussion of the general models.

CHAPTER II

METHODS AND TECHNIQUES

2.1 Introduction

Before investigating the models in depth, it is appropriate to briefly discuss several of the more frequent procedures, techniques, and ideas used throughout the research.

2.2 Burr Distribution

It should be clear by now that a technique is needed to describe various lot distributions F(X). We would like to not only consider the normal distribution, but also departures from the normal. It would appear to be rather cumbersome to consider various classes of well-known distributions, particularly when it is necessary to implement them on a computer. But consider the following class of distribution functions:

$$F(x) = 1 - (1 + x^c)^{-k} \quad , x \geq 0,\ c,\ k \geq 0 \qquad (2.2.1)$$

$$= 0 \quad , x < 0$$

where F(X) gives the probability of variable $X \leq x$. Often called the Burr distribution, it and many of its properties were developed by its namesake (1942). It was further developed by Hatke (1949) who also computed more extensive tables of values of the parameters (c,k) of the function based on values of the coefficients of skewness and kurtosis, α_3 and α_4, respectively. The Burr distribution function is widely applicable as an excellent approximation to many observed distributions. Not only can it give an excellent approximation to the normal distributions, (Burr, 1967), but simply by varying the parameters c and k , a very wide range of skewness and kurtosis combinations are obtainable. Since the functional form does not contain an integral, the distribution is quite easy to implement on a computer.

Using the Burr distribution, probabilities for an interval (a,b) take on the simple form $(1 + a^c)^{-k} - (1 + b^c)^{-k}$. Moments around the origin are by definition

$$\mu_i = E(x^i) = \int_0^\infty ck\, x^{c-1+i} (1 + x^c)^{-k-1}\, dx \qquad (2.2.2)$$

Using the transformation $v = (1 + x^c)^{-1}$, one obtains

$$\mu_i = k \int_0^1 v^{k-(i/c)-1} (1-v)^{i/c}\, dv = k\, \beta \left[k - i/c, 1 + i/c\right], \qquad (2.2.3)$$

form which central moments μ_2, μ_3, μ_4, and standardized central moments $\alpha_{3:x}$, $\alpha_{4:x}$ are obtained. For $\alpha_{4:x}$ to exist it is sufficient that $ck > 4$ (Burr and Cislak, 1968).

Because of its ease of computation and versatility in providing $\alpha_{3:x}$, $\alpha_{4:x}$ combinations, the author used theBurr distribution quite frequently in his research. Function H(u) given below is a close approximation to the normal distribution with mean zero, variance one:

$$H(u) = .5\left[G(u) + 1 - G(-u)\right], \text{ where}$$
$$G(u) = 1 - \left[1 + (.644693 + .161984\, u)^{4.874}\right]^{-6.158} \qquad (2.2.4)$$

Equation (2.2.5) is an example of a typical non-normal distribution:

$$F(X) = 1 - (1 + x^{3.077})^{-5.00} \qquad (2.2.5)$$

Distribution (2.2.5) has a mean = .55457, standard deviation = .21832, $\alpha_3 = + .527$, and $\alpha_4 = 3.542$. Tables of the mean, standard deviation, α_3, and α_4 for various levels of c and k are given by both Burr (1942) and Hatke (1949).

Burr distributions were used frequently to represent the distributions of a component part characteristic X, the sample mean $\overline{X}$, and occasionally for the population mean μ_X.

2.3. Distribution of the Sum-Range

In random samples from a normal distribution, it is well-known that the efficiency of the range as an estimator of σ decreases as the sample size N increases. For this reason it is often preferable to randomly divide the sample into a number of equal-sized subgroups and to find the mean or the sum of the several group ranges. We will work with the sum-range, denoted by

$$S_{nm} = \sum_{i=1}^{m} R_{ni} \qquad (2.3.1)$$

where m refers to the number of subgroups and n to the subgroup size. The total sample size is then N = nm.

As mentioned in Chapter I, it has been found that sample subgroups of size eight are optimum when computing the range, with very little efficiency lost in subgroups from five to ten. Resnikoff (1954) considered sample subgroup sizes of five. Because of their usefulness in this paper, subgroup sizes of eight and ten only are considered here, with $m = 2,3$.

The exact distribution of the sum-range cannot in general be obtained in simple form. Hence is necessary to tabulate the distribution numerically, using successive convolutions on the numerical probability density function of a single range.

Let $x_1, x_2, \ldots, x_n$ be a random sample of n observations drawn from a normal population with arbitrary mean and variance $\sigma^2 = 1$. If there are m such samples, the sum of the m ranges will be called the sample sum-range. We will derive the approximate numerical sum-range distribution for $n = 8, 10$, $m = 2, 3$. The techniques given below can be used for any n, m combination, however.

We first will solve for the numerical density function of the sum-range:

$$S_{nm} = \sum_{i=1}^{m} R_{ni}$$

Assume increments in the range of $\Delta = .05$. Let

$$P_{nj} = P\left[.05j - \frac{\Delta}{2} \leq R_n \leq .05j + \frac{\Delta}{2}\right], \; j = 0,1,\ldots \qquad (2.3.2)$$

$$P_{nj}^{(2)} = P\left[.05j - \Delta \leq R_{n1} + R_{n2} \leq .05j + \Delta\right], \; j = 0,1,\ldots \qquad (2.3.3)$$

$$p_{nj}^{(3)} = P\left[.05j - \frac{3\Delta}{2} \leq R_{n1} + R_{n2} + R_{n3} \leq .05j + \frac{3\Delta}{2}\right], \; j=0,1,\ldots, \qquad (2.3.4)$$

Values for (2.3.2) can be computed from tables given by Harter and Clemm (1959). We wish to solve for (2.3.3) and (2.3.4). It is easy to show that

$$p_{nj}^{(2)} \doteq \sum_{k=0}^{j} p_{nk}\, p_{n,j-k} \;, \; j = 0,1,2,\ldots$$

and

$$p_{nj}^{(3)} \doteq \sum_{\ell=0}^{j} \sum_{k=0}^{j-\ell} p_{n\ell}\, p_{nk}\, p_{n,j-\ell-k}$$

$$= \sum_{\ell=0}^{j} p_{n\ell}\, p_{n,j-\ell}^{(2)} \;, \; j = 0,1,\ldots$$

In general,

$$p_{nj}^{(m)} \doteq \sum_{\ell=0}^{j} p_{n\ell}\, p_{n,j-\ell}^{(m-1)} \;, \quad j = 0,1,\ldots \qquad (2.3.5)$$

The numerical density function is then:

$$f_{nj}^{(m)} = p_{nj}^{(m)}/\Delta \;, \quad j = 0,1,\ldots \qquad (2.3.6)$$

The numerical distribution function can then by computed by setting $F_n^{(m)}(0) = 0.0$, and solving recursively

$$F_n^{(m)}(j) = F_n^{(m)}(j-1) + \frac{\Delta}{2}\left[f_n^{(m)}(j) + f_n^{(m)}(j-1)\right] \qquad (2.3.7)$$

for $j = 1,2,\ldots.$

The computations for the numerical density and distribution functions for the sum-range were carried out on a computer using double precision throughout. Table 2.1 gives the results for n = 8, 10, m = 2,3, rounded to four places.

It was decided to check the results of Table 2.1 by using Burr distributions to approximate the four distributions given above. In order to do this the first four moments of each sum-range distribution must be matched. Let $\alpha_{3:R_n}$ and $\alpha_{4:R_n}$ represent the skewness and kurtosis of range R_n, respectively. Then

$$E(S_{nm}) = \sum_{i=1}^{m} E(R_{ni}) = m\,E(R_n)$$

$$\mathrm{Var}\,(S_{nm}) = \sum_{i=1}^{m} \mathrm{Var}\,(R_{ni}) = m\,\sigma^2_{R_n}$$

$$\alpha_{3:S_{nm}} = \frac{\alpha_{3:R_n}}{\sqrt{m}}$$

$$\alpha_{4:S_{nm}} = \frac{\alpha_{4:R_n} - 3.0}{m} + 3.0$$

Table 2.2 gives $E(R_n)$, $\sigma^2_{R_n}$, $\alpha_{3:R_n}$, and $\alpha_{4:R_n}$ for n = 8, 10, as given by Harter and Clemm (1959).

Table 2.2

Moments of R_n, n = 8, 10.

n	$E(R_n)$	$\sigma^2_{R_n}$	$\alpha_{3:R_n}$	$\alpha_{4:R_n}$
8	2.8472	.6721	.4073	3.1838
10	3.0775	.6353	.3976	3.1998

Table 2.1. Numerical Density and Distribution Functions for the Sum-Range.

S_{nm}	$f_8^{(2)}$	$F_8^{(2)}$	$f_8^{(3)}$	$F_8^{(3)}$	$f_{10}^{(2)}$	$F_{10}^{(2)}$	$f_{10}^{(3)}$	$F_{10}^{(3)}$
1.70	0	0	0	0	0	0	0	0
1.80	.0001							
1.90	.0001							
2.00	.0003							
2.10	.0004	.0001						
2.20	.0007	.0001						
2.30	.0012	.0002			.0001			
2.40	.0018	.0004			.0001			
2.50	.0028	.0006			.0002			
2.60	.0041	.0009			.0004	.0001		
2.70	.0058	.0014			.0007	.0001		
2.80	.0082	.0021			.0011	.0002		
2.90	.0113	.0031			.0017	.0004		
3.00	.0152	.0044			.0025	.0006		
3.10	.0202	.0062			.0037	.0009		
3.20	.0263	.0085			.0054	.0013		
3.30	.0337	.0115			.0077	.0020		
3.40	.0425	.0153			.0106	.0029		
3.50	.0527	.0201	.0001		.0144	.0041		
3.60	.0645	.0259	.0001		.0193	.0058		
3.70	.0778	.0330	.0001		.0253	.0080		
3.80	.0926	.0415	.0002		.0326	.0109		
3.90	.1089	.0516	.0003	.0001	.0414	.0146		
4.00	.1264	.0633	.0004	.0001	.0518	.0193		
4.10	.1449	.0769	.0006	.0001	.0638	.0250		
4.20	.1643	.0924	.0008	.0002	.0774	.0321		
4.30	.1843	.1098	.0011	.0003	.0927	.0406	.0001	
4.40	.2044	.1292	.0015	.0004	.1095	.0507	.0001	
4.50	.2245	.1507	.0021	.0006	.1278	.0625	.0001	
4.60	.2440	.1741	.0028	.0009	.1473	.0763	.0002	
4.70	.2627	.1994	.0037	.0012	.1677	.0920	.0003	.0001
4.80	.2802	.2266	.0048	.0016	.1887	.1098	.0004	.0001
4.90	.2961	.2554	.0062	.0021	.2100	.1298	.0006	.0002
5.00	.3102	.2857	.0080	.0029	.2312	.1518	.0009	.0002
5.10	.3221	.3174	.0101	.0038	.2519	.1760	.0012	.0003
5.20	.3318	.3501	.0127	.0049	.2717	.2022	.0016	.0005
5.30	.3389	.3836	.0157	.0063	.2901	.2303	.0022	.0007
5.40	.3435	.4178	.0194	.0081	.3068	.2602	.0030	.0009
5.50	.3455	.4522	.0236	.0102	.3214	.2916	.0040	.0013
5.60	.3448	.4868	.0284	.0128	.3338	.3244	.0052	.0017
5.70	.3417	.5211	.0340	.0159	.3435	.3582	.0067	.0023
5.80	.3362	.5550	.0403	.0196	.3505	.3929	.0086	.0031
5.90	.3284	.5882	.0474	.0240	.3546	.4282	.0109	.0041
6.00	.3187	.6206	.0552	.0291	.3559	.4638	.0137	.0053
6.10	.3072	.6519	.0639	.0351	.3544	.4993	.0171	.0068

S_{nm}	$f_8^{(2)}$	$F_8^{(2)}$	$f_8^{(3)}$	$F_8^{(3)}$	$f_{10}^{(2)}$	$F_{10}^{(2)}$	$f_{10}^{(3)}$	$F_{10}^{(3)}$
6.20	.2942	.6820	.0733	.0419	.3501	.5345	.0210	.0087
6.30	.2799	.7107	.0834	.0498	.3433	.5692	.0256	.0111
6.40	.2647	.7379	.0943	.0587	.3342	.6031	.0309	.0139
6.50	.2488	.7636	.1058	.0687	.3229	.6360	.0370	.0173
6.60	.2324	.7877	.1178	.0798	.3099	.6676	.0438	.0213
6.70	.2159	.8101	.1303	.0922	.2953	.6979	.0516	.0261
6.80	.1994	.8309	.1431	.1059	.2796	.7266	.0601	.0316
6.90	.1831	.8500	.1562	.1209	.2629	.7538	.0695	.0381
7.00	.1672	.8675	.1693	.1371	.2457	.7792	.0798	.0456
7.10	.1519	.8834	.1824	.1547	.2281	.8029	.0908	.0541
7.20	.1372	.8979	.1952	.1736	.2106	.8248	.1026	.0638
7.30	.1234	.9109	.2076	.1938	.1932	.8450	.1150	.0746
7.40	.1103	.9226	.2195	.2151	.1762	.8635	.1279	.0868
7.50	.0981	.9330	.2307	.2376	.1597	.8803	.1413	.1002
7.60	.0869	.9423	.2411	.2612	.1440	.8955	.1550	.1151
7.70	.0766	.9504	.2505	.2858	.1291	.9091	.1689	.1313
7.80	.0671	.9576	.2589	.3113	.1151	.9213	.1827	.1488
7.90	.0586	.9639	.2660	.3375	.1021	.9322	.1964	.1678
8.00	.0509	.9694	.2719	.3644	.0901	.9418	.2097	.1881
8.10	.0440	.9741	.2764	.3919	.0791	.9502	.2225	.2097
8.20	.0379	.9782	.2795	.4197	.0691	.9576	.2346	.2326
8.30	.0325	.9817	.2813	.4477	.0600	.9641	.2458	.2566
8.40	.0277	.9847	.2816	.4759	.0519	.9697	.2561	.2817
8.50	.0235	.9873	.2805	.5040	.0447	.9745	.2651	.3078
8.60	.0199	.9894	.2781	.5319	.0383	.9786	.2729	.3347
8.70	.0168	.9913	.2744	.5596	.0326	.9822	.2793	.3623
8.80	.0141	.9928	.2695	.5868	.0277	.9852	.2842	.3905
8.90	.0118	.9941	.2634	.6134	.0234	.9877	.2877	.4191
9.00	.0098	.9952	.2563	.6394	.0197	.9899	.2896	.4480
9.10	.0081	.9961	.2483	.6647	.0165	.9917	.2899	.4770
9.20	.0067	.9968	.2394	.6890	.0137	.9932	.2887	.5059
9.30	.0055	.9974	.2299	.7125	.0114	.9945	.2861	.5346
9.40	.0045	.9979	.2198	.7350	.0094	.9955	.2820	.5631
9.50	.0037	.9983	.2093	.7565	.0078	.9964	.2767	.5910
9.60	.0030	.9987	.1984	.7768	.0064	.9971	.2700	.6184
9.70	.0024	.9989	.1873	.7961	.0052	.9976	.2623	.6450
9.80	.0020	.9991	.1761	.8143	.0042	.9981	.2536	.6708
9.90	.0016	.9993	.1650	.8313	.0034	.9985	.2440	.6957
10.00	.0013	.9995	.1539	.8473	.0028	.9988	.2337	.7195
10.10	.0010	.9996	.1430	.8621	.0022	.9990	.2228	.7424
10.20	.0008	.9997	.1324	.8759	.0018	.9992	.2115	.7641
10.30	.0006	.9997	.1221	.8886	.0014	.9994	.1999	.7847
10.40	.0005	.9998	.1122	.9003	.0011	.9995	.1880	.8041
10.50	.0004	.9998	.1027	.9111	.0009	.9996	.1762	.8223
10.60	.0003	.9999	.0936	.9209	.0007	.9997	.1643	.8393
10.70	.0002	.9999	.0851	.9298	.0006	.9998	.1527	.8551
10.80	.0002	.9999	.0771	.9379	.0004	.9998	.1413	.8698
10.90	.0001	.9999	.0695	.9452	.0003	.9999	.1302	.8834

S_{nm}	$f_8^{(2)}$	$F_8^{(2)}$	$f_8^{(3)}$	$F_8^{(3)}$	$f_{10}^{(2)}$	$F_{10}^{(2)}$	$f_{10}^{(3)}$	$F_{10}^{(3)}$
11.00	.0001	1.0000	.0625	.9518	.0003	.9999	.1195	.8959
11.10	.0001		.0560	.9578	.0002	.9999	.1092	.9073
11.20	.0001		.0500	.9631	.0002	.9999	.0995	.9178
11.30	.0001		.0445	.9678	.0001	1.0000	.0902	.9272
11.40	.0000		.0395	.9720	.0001		.0815	.9358
11.50			.0349	.9757	.0001		.0734	.9436
11.60			.0308	.9790	.0001		.0658	.9505
11.70			.0270	.9819	.0000		.0588	.9568
11.80			.0237	.9844			.0524	.9623
11.90			.0207	.9866			.0465	.9673
12.00			.0180	.9886			.0411	.9716
12.10			.0156	.9902			.0362	.9755
12.20			.0135	.9917			.0318	.9789
12.30			.0116	.9930			.0278	.9819
12.40			.0100	.9940			.0243	.9845
12.50			.0086	.9950			.0211	.9867
12.60			.0073	.9958			.0183	.9887
12.70			.0063	.9964			.0158	.9904
12.80			.0053	.9970			.0136	.9919
12.90			.0045	.9975			.0117	.9932
13.00			.0038	.9979			.0100	.9942
13.10			.0032	.9983			.0085	.9952
13.20			.0027	.9986			.0072	.9959
13.30			.0023	.9988			.0061	.9966
13.40			.0019	.9990			.0052	.9972
13.50			.0016	.9992			.0044	.9977
13.60			.0013	.9993			.0037	.9981
13.70			.0011	.9995			.0031	.9984
13.80			.0009	.9996			.0026	.9987
13.90			.0007	.9996			.0021	.9989
14.00			.0006	.9997			.0018	.9991
14.10			.0005	.9998			.0015	.9993
14.20			.0004	.9998			.0012	.9994
14.30			.0003	.9998			.0010	.9995
14.40			.0003	.9999			.0008	.9996
14.50			.0002	.9999			.0007	.9997
14.60			.0002	.9999			.0006	.9997
14.70			.0001	.9999			.0004	.9998
14.80			.0001	.9999			.0004	.9998
14.90			.0001	1.0000			.0003	.9999
15.00			.0001				.0002	.9999
15.10			.0001				.0002	.9999
15.20			.0000				.0002	.9999
15.30							.0001	.9999
15.40							.0001	1.0000
15.50							.0001	
15.60							.0001	
15.70							.0001	
15.80							.0000	

Table 2.3 gives $E(S_{nm})$ $\sigma_{S_{nm}}$, $\alpha_{3:S_{nm}}$, and $\alpha_{4:S_{nm}}$ for n = 8, 10, m = 2, 3.

Table 2.3.

Moments of the Sum-Range S_{nm}, n = 8, 10, m = 2, 3.

Set	n	m	$E(S_{nm})$	$\sigma_{S_{nm}}$	$\alpha_{3:S_{nm}}$	$\alpha_{4:S_{nm}}$
1	8	2	5.6944	1.1594	.2880	3.0919
2	8	3	8.5416	1.4200	.2351	3.0613
3	10	2	6.1550	1.1272	.2811	3.0999
4	10	3	9.2325	1.3805	.2295	3.0666

Table 2.4 gives the parameters c and k necessary for a Burr distribution to match the above four sets of α_3 and α_4. The mean M and standard deviation S of the Burr distributions are also given.

Table 2.4.

Burr Distribution Parameters to Approximate the Sum-Range Distributions

Set	c	k	M	S
1	3.4720	6.6382	.5368	.1838
2	3.7001	6.3773	.5625	.1822
3	3.5383	6.3561	.5499	.1857
4	3.7527	6.1976	.5715	.1831

From the above tables it should be clear that, except for location, (2.3.5) and (2.3.6) below approximate the four sum-range distribution and density functions, respectively, simply by substituting in values from the above tables:

$$F(x) = 1 - \left[1 + \left(\frac{Sx}{\sigma_S}\right)^c\right]^{-k} \quad , x \geq 0 \qquad (2.3.5)$$

$$= 0 \quad , x < 0$$

$$f(x) = \frac{c\,k\,S\left(\frac{Sx}{\sigma_S}\right)^{c-1}}{\sigma_S\left[1 + \left(\frac{Sx}{\sigma_S}\right)^c\right]^{k+1}} \quad , x \geq 0 \qquad (2.3.6)$$

$$f(x) = 0 \quad , x < 0$$

To correct for differences in location, let

$$u = x + E(S) - \frac{M\,\sigma_S}{S}$$

Then

$$P[S_{nm} \leq u] = F(x).$$

The Burr approximations for the sum-range distributions and densities were quite close to the numerical distributions and densities obtained by the convolution technique. For example, the maximum difference between the two methods for $S_{10,3}$ was .004 for the distribution function and .007 for the density function. Tables for the Burr approximations will not be given, however, since Table 2.1 gives a good approximation to the numerical sum-range distribution. The Burr distributions did provide an excellent check, however, on the validity of the convolution results.

2.4 Form of the Range Test

Whenever m ranges are computed from a sample of size N = mn, a problem arises as to the actual form of the range test. The two logical possibilities are:

a) Let the range test be of the form $R_n \leq c_2 T$, and require all m ranges to be acceptable;

b) Let $S_{nm} = \sum_{i=1}^{m} R_{ni}$, and require $S_{nm} \leq c_2' T$ as the range test.

It is not clear without further investigation as to whether plan (a) or plan (b) is "best" in some sense. Since the range test is a measure of the variability of our process, we will define "best" to be that plan which has the lowest type I and type II errors, for a given sample size. That is, we want to accept with high probability processes with low variability, and to reject with high probability those processes with high variability. Variability will be classified in terms of the process tolerance T, with "low" and "high" to be determined shortly.

Because the m groups of n samples are chosen at random,

$$P[R_{n1} \leq c_2 T] = \dots = P[R_{nm} \leq c_2 T]$$

Hence

$$\begin{aligned} & P[(R_{n1} \leq c_2 T) \cap \dots \cap (R_{nm} \leq c_2 T)] \\ &= P[R_{n1} \leq c_2 T] \cdot \dots \cdot P[R_{nm} \leq c_2 T] \\ &= \{P[R_{n1} \leq c_2 T]\}^m \\ &= \{P[R_n \leq c_2 T]\}^m \qquad (2.4.1) \end{aligned}$$

Hence the probability of acceptance under plan (a) is (2.4.1). Tables of the sample range are given by Harter and Clemm (1959). The probability of acceptance under plan (b) can be obtained from table 2.1, which we computed for this purpose.

Since the variability of our process will be classified in terms of the tolerance T, we will assume T = 1 without loss of generality. It was shown in Chapter I that an acceptable standard deviation σ could be taken as T/8 or less, while $\sigma \geq T/6$ would be excessive. We wish to subject our range test to distributions with variability from quite low to excessive. Hence a reasonable range for σ would be from, say, T/15 to T/4, since T/15 provides very low variability, while $\sigma = T/4$ is excessive. Therefore, let $\sigma_j = .055 + .01j$, $j = 1,2,\ldots,20$. Each σ_j represents the standard deviation of a hypothetical process. By considering various levels of σ_j, we can compare the acceptance probabilities of plans (a) and (b). At the same time, since we know whether or not the standard deviation σ_j can be considered excessive, we can estimate and compare types I and II errors for the two plans.

Equation (2.4.2) below gives the probability of acceptance for the sample range, given σ_j .

$$P[R_n \leq c_2 T \mid \sigma = \sigma_j] \tag{2.4.2}$$

$$(T = 1) = P[R_n \leq c_2 \mid \sigma = \sigma_j]$$

$$= P[R_n \leq c_2/\sigma_j \mid \sigma = 1]$$

This is necessary since the Harter and Clemm tables assume $\sigma = 1$. Since the author has already assumed $\sigma = 1$, $P[S_{nm} \leq c_2']$ can be read directly from table 2.1, interpolating when necessary. Again, we will let n = 8, 10, m = 2, 3.

Numerous values for c_2 and c_2' were considered for various combinations of (n,m). Only those which seemed of interest (i.e., gave maximum reduction in error types I and II for a given sample size, or differentiated between plans (a) and (b)), however, will be given here. In later chapters three ranges of size ten each is recommended in most cases, so we will emphasize this combination. Table 2.5 gives a direct comparison of the OC curves of plans (a) and (b) for the four combinations of (n,m). A single range of size ten is also given to indicate the improvement gained by considering m subgroups. Five decimal places were carried throughout the computations, although only two are given in the table.

In order to check on type I error and type II error, a rule is needed to distinguish between "good" and "bad" σ_j values. After considering the various possible upper bounds on σ, it was clear that the set $A = \{ \sigma_X \mid \sigma_X \leq .125\ T\}$ includes the σ_j values which generally could be classified as acceptable. On the other hand, set $B = \{ \sigma_X \mid \sigma_X \geq .185T\}$ clearly contains unacceptable σ_j values. For set $C = \{ \sigma_X \mid .125\ T \leq \sigma_X \leq .185\ T\}$, our primary interest is that the OC curve reach its maximum slope within this interval, and drop from a high acceptance probability to a low acceptance probability at the respective interval end points. By dividing each OC curve into three areas according to sets A, B, and C, we may conclude after careful study

Table 2.5. Comparison of the OC Curves of Plans (a) and (b) for various n,m Combinations.

		m = 3, n = 10				m=3, n=8		m=2, n=10		m=2, n=8		m=1 n=10
Set	σ_j	$c_2' =$ 1.35	$c_2' =$ 1.40	$c_2' =$ 1.45	$c_2 =$ .58	$c_2' =$ 1.30	$c_2 =$ .55	$c_2' =$ .933	$c_2 =$ .55	$c_2' =$ .867	$c_2 =$ .55	$c_2 = c_2' =$ .50
	.065	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	.075	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	.085	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
A	.095	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.99
	.105	.99	1.00	1.00	.99	.99	.98	.99	.98	.98	.99	.97
	.115	.96	.98	.99	.96	.97	.95	.95	.95	.94	.97	.93
	.125	.87	.92	.95	.90	.90	.89	.87	.89	.86	.92	.87
	.135	.72	.80	.86	.80	.78	.79	.76	.79	.74	.85	.79
	.145	.54	.63	.72	.67	.63	.66	.61	.67	.63	.76	.70
C	.155	.37	.46	.55	.52	.47	.53	.47	.54	.48	.65	.60
	.165	.23	.30	.39	.38	.33	.43	.34	.42	.37	.57	.50
	.175	.13	.19	.25	.26	.22	.29	.24	.31	.27	.44	.42
	.185	.07	.11	.16	.17	.14	.20	.16	.22	.20	.34	.34
	.195	.04	.06	.09	.11	.09	.14	.11	.16	.14	.26	.27
	.205	.02	.03	.05	.06	.05	.09	.07	.11	.10	.20	.22
	.215	.01	.02	.03	.04	.03	.06	.04	.07	.07	.15	.17
B	.225	.01	.01	.02	.02	.02	.04	.03	.05	.05	.11	.14
	.235	.00	.00	.01	.01	.01	.02	.02	.03	.03	.08	.11
	.245	.00	.00	.00	.01	.01	.01	.01	.02	.02	.06	.09
	.255	.00	.00	.00	.00	.00	.01	.01	.01	.02	.04	.07

that regardless of the (n,m) combination, plan (b) is superior to plan (a).

By emphasizing the OC curves for the four (n,m) combinations under plan (b), we note the decrease in type I and II errors as the total sample size increases. From a practical viewpoint, the increased efficiency gained with larger sample sizes must be weighed against the cost of additional sampling. This problem will not be considered in this thesis, although general guidelines and recommendations will be presented in later chapters.

2.5 A Priori Distributions of μ and σ

In the acceptance sampling model (Chapter III), it was desirable to see how the $\bar{X}$ and R tests operated against various a priori distributions of μ and of σ, the mean and standard deviation of the process under consideration. In order to make meaningful comparisons among the several $\bar{X}$ and R tests considered, care must be taken in selecting appropriate a priori distributions. By considering all possible combinations of the a priori distributions, we want to include a wide variety of (μ, σ) distributions. These should run from highly satisfactory a priori distributions to undesirable ones. We are interested in how well $\bar{X}$ and R tests discriminate between these various a priori distributions. In constructing the distributions, however, emphasis has been placed on considering distributions which are in some sense out of control. The reason for this is quite simple: once assurance has been given that desirable distributions are readily accepted by the $\overline{X}$ and R tests, we are primarily interested in how well undesirable distributions are recognized by the tests, and corrective action taken.

In all, nine a priori distributions of μ and seven of σ were developed, or 63 possible combinations. Four general types of distributions were used: uniform, triangular, normal, and non-normal Burr. Discrete levels of μ and σ were chosen, with μ levels symmetric about $\mu_o = 0$ (bias will be considered in Chapter III), and σ levels corresponding to those discussed earlier in this chapter:

$$\mu_i = .02\,i - .01, \quad i = 1,\ldots,18$$

$$= .02\,i + .01, \quad i = -1,\ldots,-18$$

and

$$\sigma_j = .055 + .01j, \quad j = 1,\ldots,20$$

Table 2.6 gives seven of the discrete distributions considered for μ. Distributions (8) and (9) were computed directly from a Burr distribution, given in (2.5.1), which has mean = 0, variance = 1, $\alpha_3 = .401$, and $\alpha_4 = 3.106$.

$$\Phi^*(t) = 1 - [\,1 + (.40786 + .16293\,t)^{2.857}]^{-10.00} \qquad (2.5.1)$$

By setting

$$t_i = \frac{\mu_i}{\sigma}, \qquad i \pm 1,\ldots, \pm 18,$$

and letting

$$\sigma = .10\ , \ .20$$

two discrete non-normal distributions were constructed. The discrete

Table 2.6. p_i=P (level μ_i occurring) for Seven a Priori μ Distributions, i = $\pm$1,...,$\pm$18.

<u>μ Distributions</u>

		1	2	3	4	5	6	7
i	μ_i	uniform	triangle	N(0,σ=.10)	triangle	uniform	N(0,σ=.20)	uniform
± 1	±.01	.10	.115	.07935	.06250	.05	.04290	.03333
± 2	±.03	.10	.100	.07615	.05833	.05	.04255	.03333
± 3	±.05	.10	.085	.07045	.05417	.05	.04160	.03333
± 4	±.07	.10	.070	.06235	.05000	.05	.04040	.03333
± 5	±.09	.10	.055	.05320	.04583	.05	.03980	.03333
± 6	±.11	0	.040	.04360	.04167	.05	.03695	.03333
± 7	±.13		.025	.03430	.03750	.05	.03470	.03333
± 8	±.15		.010	.02600	.03333	.05	.03240	.03333
± 9	±.17		0	.01890	.02917	.05	.02995	.03333
±10	±.19			.01320	.02500	.05	.02735	.03333
±11	±.21			.00880	.02083	0	.02480	.03333
±12	±.23			.00570	.01667		.02220	.03333
±13	±.25			.00350	.01250		.01970	.03333
±14	±.27			.00210	.00833		.01725	.03333
±15	±.29			.00130	.00417		.01510	.03333
±16	±.31			.00060	0		.01290	0
±17	±.33			.00040			.01100	
±18	±.35	↓	↓	.00010	↓	↓	.00935	↓

Table 2.7. q_j = P (level σ_j occurring) for Seven a Priori σ Distributions.

		σ Distributions						
		q_j						
		1	2	3	4	5	6	7
j	σ_j	uniform	triangle	normal	uniform	triangle	normal	uniform
1	.065	.20	.23	.1587	.10	.100	.0834	.05
2	.075	.20	.20	.1523	.10	.095	.0828	.05
3	.085	.20	.17	.1409	.10	.090	.0809	.05
4	.095	.20	.14	.1247	.10	.085	.0786	.05
5	.105	.20	.11	.1064	.10	.080	.0756	.05
6	.115	0	.08	.0872	.10	.075	.0719	.05
7	.125		.05	.0686	.10	.070	.0675	.05
8	.135		.02	.0520	.10	.065	.0631	.05
9	.145		0	.0378	.10	.060	.0582	.05
10	.155			.0264	.10	.055	.0532	.05
11	.165			.0176	0	.050	.0482	.05
12	.175			.0114		.045	.0431	.05
13	.185			.0070		.040	.0383	.05
14	.195			.0042		.035	.0335	.05
15	.205			.0026		.030	.0293	.05
16	.215			.0012		.025	.0251	.05
17	.225			.0008		0	.0214	.05
18	.235			.0002			.0182	.05
19	.245			0			.0151	.05
20	.255	↓	↓	↓	↓	↓	.0126	.05

Table 2.9. A Priori Values for σ_E for 49 (μ,σ) Distributions.

Dist. for μ	σ	σ_E	Dist. for μ	σ	σ_E	Dist. for μ	σ	σ_E
1	1	.1036	3	4	.1513	5	6	.1826
1	2	.1063	3	5	.1641	5	7	.2055
1	3	.1191	3	6	.1732	6	1	.1876
1	4	.1274	3	7	.1972	6	2	.1891
1	5	.1424	4	1	.1529	6	3	.1966
1	6	.1527	4	2	.1548	6	4	.2017
1	7	.1795	4	3	.1639	6	5	.2115
2	1	.1091	4	4	.1700	6	6	.2186
2	2	.1117	4	5	.1815	6	7	.2381
2	3	.1239	4	6	.1897	7	1	.1934
2	4	.1319	4	7	.2119	7	2	.1949
2	5	.1464	5	1	.1440	7	3	.2021
2	6	.1565	5	2	.1460	7	4	.2071
2	7	.1827	5	3	.1550	7	5	.2166
3	1	.1319	5	4	.1619	7	6	.2236
3	2	.1341	5	5	.1740	7	7	.2427
3	3	.1445						

CHAPTER III

ACCEPTANCE SAMPLING

3.1 Introduction

One of the fundamental problems in quality control is: for a given process, how does one decide systematically whether to accept or reject the output of that process? Short of sorting the output 100 percent, a sampling procedure is usually implemented, with the process output accepted or rejected on the basis of some criteria applied to the sample. When the actual process is not close to those deciding whether to accept or reject, the procedure is often called "acceptance sampling". For example, the problem may be to determine whether a shipment of parts manufactured by another concern is acceptable to you. The aim of the sampling procedure should be to provide assurance that the overall distribution of parts will be satisfactory.

Ideally we would like to develop a technique which would precisely determine any underlying process distribution. Because of the multitude of different process distributions which can occur, however, determining a general distribution covering all possible types of process distributions and yet specific enough to be of some practical use seemed improbable at best. Instead emphasis has been placed on controlling the moments of the a posteriori process distribution. Evans (1967) discusses the use of numerical integration techniques for obtaining the moments of a system in order to set statistical tolerances. The author's research to date has been limited to controlling the first four moments, which is sufficient in most practical situations. How one controls the moments will be discussed in detail following a brief discussion of a useful method of finding the best compromise among the various $\bar{X}$ and R tests considered for a given sample size.

3.2 Probability of Acceptance

In addition to controlling the first four moments, another criterion used in finding the best compromise among the various $\bar{X}$ and R test combinations considered, for a given sample size, was to consider the probability of making an acceptance decision. Such a decision may be considered at two levels. For a particular lot, with lot mean μ and standard deviation σ, we may find the probability of accepting that particular lot. By comparing the OC curves for various $\bar{X}$ and R test combinations and various sample sizes, we can study the risks of types I and II error involved, for a particular lot.

A second acceptance decision criterion would be an average probability of acceptance over many lots. This allows one to judge whether the process distribution itself is in some sense desirable. Such a judgment should of course emphasize the outgoing quality of the process after testing and possible sorting of rejected material has occurred. We will investigate the average probability of acceptance over many lots, assuming various process distributions, each with a known mean distribution $f_1(\mu)$ and standard deviation distribution $f_2(\sigma)$. Using this criterion along with an investigation of the first four outgoing moments, we can then find the best compromise among the various $\overline{X}$ and R test combinations considered, for a given sample size. We of course want to reduce to a minimum both the average probability of rejecting a well-controlled process distribution and the average probability of accepting a badly-controlled or erratic distribution. We will do this by subjecting several $\overline{X}$ and R test combinations to the 63 a priori distributions of μ and σ given in Chapter II, and then selecting an $\overline{X}$ and R combination on the basis of satisfactory types I and II error. To formulate this, let:

$f_1(\mu)$ = density function of the process mean μ

$f_2(\sigma)$ = density function of the process standard deviation σ

P_a = average probability of making an acceptance decision, for a process distribution with known $f_1(\mu)$, $f_2(\sigma)$.

$P(a \mid \mu,\sigma)$ = probability of accepting a process, given that it has mean μ, standard deviation σ

Then, in general

$$P_a = \int_{\mu} \int_{\sigma} P(a \mid \mu,\sigma)\, f_1(\mu)\, f_2(\sigma)\, d\sigma\, d\mu \qquad (3.2.1)$$

Rather than solve (3.2.1) for P_a in general, we will consider only the discrete case:

$$P_a = \sum_{all_i} \sum_{all_j} P(a \mid \mu_i, \sigma_j)\, p_i\, q_j , \qquad (3.2.2)$$

where p_i = probability of the i^{th} μ level occurring,

q_j = probability of the j^{th} σ level occurring.

Tables 2.6 and 2.7 provide the $\{p_i\}$ and $\{q_j\}$, respectively.

How do we evaluate $P(a \mid \mu_i, \sigma_j)$?

Let $P_{\overline{X}}(a \mid \mu_i, \sigma_j)$ = probability of accepting on the $\overline{X}$ test, given μ_i, σ_j.

$P_R(a \mid \mu_i, \sigma_j)$ = probability of accepting on the range test, given μ_i, σ_j.

Now, the range test is invariant under changes in location, so with fixed σ_j,

$$P_R(a \mid \mu_i, \sigma_j) = P_R(a \mid \sigma_j) .$$

Our requirement is that a process sample pass both tests, so:

$$P(a \mid \mu_i, \sigma_j) = P_{\overline{X}}(a \mid \mu_i, \sigma_j) \cdot P_R(a \mid \sigma_j) , \qquad (3.2.3)$$

where (3.2.3) is based on the assumption that the sample $\overline{X}$ and R are independent. Daly (1946) proved that $\overline{X}$ and R are statistically independent if the underlying process distribution from which the sample is taken is a normal distribution. The validity of (3.2.3) is unknown for non-normal distributions; however, it should give a reasonable approximation for $P(a \mid \mu_i, \sigma_j)$ in most practical situations. One possible area of future research might be to determine the effects of non-normality on (3.2.3).

Consider now the evaluation of $P_{\overline{X}}(a \mid \mu_i, \sigma_j)$. From the Central Limit Theorem it is well known that as the sample size N increases, the distribution of $\overline{X} = \sum_{i=1}^{N} X_i/N$ tends to normality. This is true of almost all underlying process distributions. Since we will be considering sample sizes from ten to thirty, it is reasonable to assume the distribution of $\overline{X}$ to be normal. In a later section of this chapter, however, the effects of non-normality will be investigated with the help of a Burr distribution. For the purposes of illustration now, however, we assume $\overline{X}$ has a normal distribution. Thus

$$P_{\overline{X}}(a \mid \mu_i, \sigma_j) \tag{3.2.4}$$

$$= P\,(\text{acceptance on } \overline{X} \text{ test} \mid \mu_i, \sigma_j)$$

$$= P(\mu_o - c_1 T \leq \overline{X} \leq \mu_o + c_1 T \mid \mu_i, \sigma_j)$$

where μ_o and T are known constants, and c_1 is to be determined. (3.2.4) may be written as:

$$\int_{\mu_o - c_1 T}^{\mu_o + c_1 T} \frac{e^{-\frac{(\overline{X} - \mu_i)^2}{2\sigma_j^2/N}}}{\sqrt{2\pi}\;\; \sigma_j/\sqrt{N}}\, d\overline{X} \tag{3.2.5}$$

let $y = \left(\dfrac{\overline{X} - \mu_i}{\sigma_j} \right) \sqrt{N}$

Then (3.2.5) gives

$$\int_{L}^{U} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, dy \tag{3.2.6}$$

where

$$U = \left(\frac{\mu_o - \mu_i + c_1 T}{\sigma_j} \right) \sqrt{N}$$

$$L = \left(\frac{\mu_o - \mu_i - c_1 T}{\sigma_j} \right) \sqrt{N}$$

Denoting the standardized normal distribution function by $\Phi(t)$, we have

$$P_{\overline{X}}(a \mid \mu_i, \sigma_j) = \tag{3.2.7}$$

$$\Phi\left[\left(\frac{\mu_o - \mu_i + c_1 T}{\sigma_j} \right) \sqrt{N}\right] - \Phi\left[\left(\frac{\mu_o - \mu_i - c_1 T}{\sigma_j} \right) \sqrt{N}\right]$$

Without loss of generality we can specify $\mu_o = 0$ and $T = 1.0$, reducing (3.2.7) to:

$$P_{\overline{X}}(a \mid \mu_i, \sigma_j) = \Phi\left[\left(\frac{-\mu_i + c_1}{\sigma_j} \right) \sqrt{N}\right] - \Phi\left[\left(\frac{-\mu_i - c_1}{\sigma_j} \right) \sqrt{N}\right]$$

which can easily be evaluated for known values of μ_i, σ_j, N, and specified levels of c_1.

In Chapter II the range test was discussed extensively. Depending on the sample size N, the range test has the form $R_N \leq c_2 T$ or $\sum_{i=1}^{m} R_{ni} \leq c_2' T$. Hence

$$P_R(a \mid \sigma_j) = P(R_N \leq c_2 T \mid \sigma_j)$$

$$= P(R_N \leq c_2 T/\sigma_j \mid \sigma = 1) ,$$

values of which may be found in Harter and Clemm (1959); or

$$P_R(a \mid \sigma_j) = P\left(\sum_{i=1}^{m} R_{ni} \leq c_2'T \mid \sigma_j\right)$$

$$= P\left(\sum_{i=1}^{m} R_{ni} \leq \frac{c_2'T}{\sigma_j} \mid \sigma = 1\right) ,$$

values of which may be found in Table 2.1.

It is now possible to compute P_a, the overall probability of accepting the output of a process distribution:

$$P_a = \sum_{\text{all}_i} \sum_{\text{all}_j} P_{\bar{X}_N}(a \mid \mu_i, \sigma_j)\, P_R(a \mid \sigma_j)\, p_i\, q_j ,$$

where $P_{\bar{X}_N}(a \mid \mu_i, \sigma_j)$ and $P_R(a \mid \sigma_j)$ are given above, and p_i, q_j are given in Tables 2.6 and 2.7, respectively. Various levels of c_1 were considered in order to find the best compromise c_1 level for a particular sample size N which minimized the types I and II error. The calculations given by Table 3.1 assume N = 10, the smallest sample size considered; $\bar{X}$ normal; and consider 49 p_i, q_j combinations, although the table includes only 16 combinations. The μ and σ a priori distributions, denoted by p_i and q_j, respectively, are given in Table 3.1 under the headings of μ and σ. $P_R(a \mid \sigma_j) = P(R_N \leq .50T \mid \sigma_j)$, as indicated in the last column of Table 2.5. The c_1 values considered were .12 (.01).17, since earlier related work by Burr (1967a) indicated a c_1 value for N = 10 should lie within that range. Table values are listed for c_1 = .13, .14, and .15 only, since c_1 = .14 was judged on the basis of P_a to provide the best compromise $\bar{X}$ test for a sample size of 10. Thus our tests are:

a) $\mu_o - .14T \leq \bar{X}_{10} \leq \mu_o + .14T$ (3.2.8)

and

b) $R_{10} < .50T$ (3.2.9)

These criteria are slightly tighter than those derived by Burr (1967a).

Table 3.1.

P_a = average probability of acceptance, using test:

(a) $\mu_o - c_1T \leq \bar{X}_{10} \leq \mu_o + c_1T$

(b) $R_{10} \leq .50T$

where sample size N = 10.

Distributions for μ	σ	c_1=.13	c_1=.14	c_1=.15	Distributions for μ	σ	c_1=.13	c_1=.14	c_1=.15
1	1	.98	.98	.99	5	1	.65	.69	.74
1	3	.90	.91	.91	5	3	.60	.64	.69
1	5	.75	.76	.77	5	5	.51	.54	.58
1	7	.53	.54	.55	5	7	.36	.39	.41
3	1	.78	.82	.85	7	1	.43	.46	.50
3	3	.72	.75	.78	7	3	.40	.43	.46
3	5	.61	.64	.66	7	5	.34	.37	.39
3	7	.43	.45	.47	7	7	.24	.26	.28

Although c_1 = .14, c_2 = .50 seemed to be the best compromise test constants for N = 10, the results were still not considered satisfactory in all cases. Errors in accepting poor distributions reached up to 60 percent, to the point that if the loss for type II error is more than minimal, the tests could be quite expensive. The constants c_1 and c_2 could be changed so that the type II error was reduced, but unfortunately this raised the type I error to unacceptable levels. Since this is the best compromise for N = 10, an effort was made to provide a more discriminatory test by increasing the sample size. We have, of course, anticipated this in earlier chapters by deriving results up to N = 30. The reader should keep in mind throughout this discussion that economics may play an important part in selecting a sample size for a particular process. Test recommendations will be given for various sample sizes between 10 and 30. It should be made clear that we are not completely discarding testing procedures when N = 10. As can easily be noted from Table 3.1, for either very good or very poor a priori μ and σ distributions (i.e. $\mu = 1$, $\sigma = 1$ or $\mu = 7$, $\sigma = 7$, respectively), a sample size of ten seems adequate. For a priori distributions which are somewhere between these two extremes, however, a more discriminatory test is needed. Research discussed later in this chapter indicated that N = 30 provided a sufficient level of discrimination, although of course the higher the sample size, the more discriminatory the test will be.

3.3 Fine Grid vs. Regular

In order to provide some assurances that the discrete distributions of μ and σ used in (3.2.2) give a fine enough grid to adequately compute

P_a, a subset of nine of the 63 (μ, σ) distribution combinations were reconsidered, letting:

$$\mu_i = .004i - .002, \ i = \pm 1, \pm 2, \ldots, \pm 75$$

$$\sigma_j = .058 + .004j, \ j = 1, \ldots, 50$$

That is, the interval between μ_i values was reduced from .020 to .004, and between σ_j values from .010 to .004. P_a was then calculated for nine (μ, σ) distribution combinations using the $\overline{X}$ and R tests for N = 10 given by (3.2.8) and (3.2.9). Table 3.2 gives the results comparing "Fine Grid" vs. "Regular". We note that all "regular" probabilities are within .003 of the "fine" values, with most within .001. In every case the "fine" probability is slightly less than the "regular" value. For our purposes, it was concluded that the "regular" approximation to (3.2.1) was sufficient.

Table 3.2

P_a, the Average Probability of Acceptance

μ	σ	Regular	Fine Grid
1	1	.985	.982
5	1	.694	.693
7	1	.463	.463
1	4	.867	.866
5	4	.617	.617
7	4	.413	.413
1	7	.539	.538
5	7	.387	.387
7	7	.261	.261

3.4 Distribution Moments

At this point it is now appropriate to introduce the use of moments as a means of describing a posteriori process distributions. Having indicated briefly above the need for considering sample sizes greater than 10, future judgments in choosing an $\bar{X}$ and R test combination will be based both on P_a, the probability of overall acceptance, and how acceptable are the first four a posteriori moments. Two separate classifications of a posteriori process distributions were used:

(a) in lot-by-lot testing, only those lots accepted were retained; or

(b) those lots that were rejected were sorted 100 percent to fixed limits, and then combined with the lots that were previously accepted.

Class (a) would include those processes where 100 percent sorting of rejected material is for some reason not feasible. Class (b) would include most other processes.

Consider class (a), where the a posteriori distribution contains only lots which were accepted by the $\bar{X}$ and R test combination. We wish to derive the first four moments of this special distribution. Assume that X has a normal distribution F(X) in order to facilitate an illustrative solution. We will later examine deviations from normality with the aid of the Burr distribution.

Let μ = mean of F(X)

σ = standard deviation of F(X)

$E[(X - \mu_o)^r]_{\mu,\sigma}$ = r^{th} moment of X about μ_o, given by μ, σ.

Then:

$$E[(X - \mu_o)^r]_{\mu,\sigma} = \int_{-\infty}^{\infty} \frac{(X - \mu_o)^r \, e^{-\frac{(X-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\;\sigma} \, dX \qquad (3.4.1)$$

Let

$$y = \frac{X - \mu}{\sigma}$$

Substituting, we have:

$$E[(X - \mu_o)^r]_{\mu,\sigma} = \int_{-\infty}^{\infty} \frac{(\mu + \sigma y - \mu_o)^r \, e^{-y^2/2}}{\sqrt{2\pi}} \, dy \qquad (3.4.2)$$

$$= \int_{-\infty}^{\infty} \sigma^r \; (y + \delta)^r \; \phi(y) \, dy,$$

where

$$\delta = \frac{\mu - \mu_o}{\sigma}$$

$\phi(y)$ = standard normal density function

Equation (3.4.2) may be evaluated for r = 1, 2, 3, 4:

$$E[(X - \mu_o)]_{\mu,\sigma} = \delta \; \sigma \qquad (3.4.3)$$

$$E[(X - \mu_o)^2]_{\mu,\sigma} = \sigma^2(1 + \delta^2) \qquad (3.4.4)$$

$$E[(X - \mu_o)^3]_{\mu,\sigma} = \sigma^3(3\delta + \delta^3) \qquad (3.4.5)$$

$$E[(X - \mu_o)^4]_{\mu,\sigma} = \sigma^4(3 + 6\delta^2 + \delta^4) \qquad (3.4.6)$$

Now, let

$$E_a[(X - \mu_o)^r] = r^{th} \text{ moment of the accepted product}$$

Then:

$$E_a[(X - \mu_o)^r] = \frac{\int_\mu \int_\sigma \left[P(a \mid \mu, \sigma)\right] f_1(\mu)\ f_2(\sigma) \left\{E[(X - \mu_o)^r]_{\mu, \sigma}\right\} d\sigma\, d\mu}{\int_\mu \int_\sigma [P(a \mid \mu, \sigma)]\ f_1(\mu)\ f_2(\sigma)\ d\mu\, d\sigma} \qquad (3.4.7)$$

where:

$P(a \mid \mu, \sigma)$ = probability of acceptance, given μ and σ

$f_1(\mu)$ = density function of μ

$f_2(\sigma)$ = density function of σ

Again we consider the discrete case, so (3.4.7) becomes

$$E_a[(X - \mu_o)^r] = \frac{\sum_{all_i} \sum_{all_j} [P(a \mid \mu_i, \sigma_j)]\ p_i\ q_j\ \{E[(X - \mu_o)^r]_{\mu_i, \sigma_j}\}}{\sum_{all_i} \sum_{all_j} [P(a \mid \mu_i, \sigma_j)]\ p_i\ q_j} \qquad (3.4.8)$$

where $P(a \mid \mu_i, \sigma_j)$, p_i and q_j are defined as in (3.2.2). Equation (3.4.8) can then be evaluated to give the a posteriori moments of the accepted product distribution.

We now wish to consider class (b) of the a posteriori process distributions: an amalgam of accepted product combined with the retained product from rejected lots which have been 100 percent sorted. We will consider two limits for the 100 percent sorting: either discard parts which lie outside $\mu_o \pm \frac{3T}{8}$, or outside $\mu_o \pm \frac{T}{2}$. The purpose of considering the tighter limits, as discussed by Burr (1967a), is to reduce the possibility that lots which are badly off center could give excessive values of

$$\sum_{i=1}^{N_1 - N_2} (X_i - \mu_o)^2/(N_1 - N_2) \quad ,$$

after sorting. (Here N_1 = lot size, N_2 = the number of parts rejected in the sorting.)

Let $\mu_o \pm K\,T$ be the 100 percent sorting limits. We can assume $T = 1$ without loss of generality. Again, assuming μ and σ to be the mean and standard deviation of our normal process distribution $F(X)$, the truncated proportion of the material retained after sorting would be:

$$A_X(\mu,\ \sigma,\ K) = \int_{\mu_o - K}^{\mu_o + K} \frac{e^{-\frac{(X - \mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\ \sigma}\ dX \tag{3.4.9}$$

Define

$$E[(X - \mu_o)^r] = \tag{3.4.10}$$

$$\int_{\mu_o - K}^{\mu_o + K} \frac{(X - \mu_o)^r\ e^{-\frac{(X - \mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\ \sigma}\ dX \ ,$$

for given μ, σ, K.

Then letting $Z = \frac{X - \mu}{\sigma}$, (3.4.10) reduces to

$$E[(X - \mu_o)^r]_{\mu,\sigma,K} = \tag{3.4.11}$$

$$\sigma^r \int_{-\frac{K}{\sigma} - \delta}^{\frac{K}{\sigma} - \delta} (Z + \delta)^r \, \phi(Z) \, dZ \qquad (3.4.7)$$

where $\delta = \dfrac{\mu - \mu_o}{\sigma}$

$\phi(Z)$ = standard normal density function

Equation (3.4.11) may be evaluated for r = 1, 2, 3, 4. For ease of notation, let

$$U = \frac{K}{\sigma} - \delta$$

$$L = -\frac{K}{\sigma} - \delta$$

$P = A_X(\mu, \sigma, K)$, given in (3.4.9)

$\phi(t)$ = standard normal density function

$\Phi(t)$ = standard normal distribution function

Noting $P = \Phi(U) - \Phi(L)$, we have:

$$E[(X - \mu_o)]_{\mu,\sigma,K} = \sigma[\phi(L) - \phi(U) + \delta P] \tag{3.4.12}$$

$$E[(X - \mu_o)^2]_{\mu,\sigma,K} = \sigma^2[(1 + \delta^2) P - U\phi(L) + L\phi(U)] \tag{3.4.13}$$

$$E[(X - \mu_o)^3]_{\mu,\sigma,K} = \sigma^3[(\frac{K^2}{\sigma^2} - \frac{K\delta}{\sigma} + \delta^2 + 2)\phi(L)$$

$$- (\frac{K^2}{\sigma^2} + \frac{K\delta}{\sigma} + \delta^2 + 2)\phi(U) + (3\delta + \delta^3) P] \tag{3.4.14}$$

$$E[(X - \mu_o)^4]_{\mu,\sigma,K} = \sigma^4[(- \frac{K^3}{\sigma^3} + \frac{K^2\delta}{\sigma^2} - \frac{K\delta^2}{\sigma} + \delta^3 \tag{3.4.15}$$

$$- \frac{3K}{\sigma} + 5\delta)\phi(L) - (\frac{K^3}{\sigma^3} + \frac{K^2\delta}{\sigma^2} + \frac{K\delta^2}{\sigma} + \delta^3$$

$$+ \frac{3K}{\sigma} + 5\delta) \phi(U) + (3 + 6\delta^2 + \delta^4) P]$$

Now, let:

$E[(X - \mu_o)^r]$ = r^{th} moment of our combined product: accepted plus sorted-rejected.

Then:

$$E[(X - \mu_o)^r] = \tag{3.4.16}$$

$$\frac{\int_\mu \int_\sigma f_1(\mu) f_2(\sigma) \left\{P(a|\mu,\sigma)E[(X-\mu_o)^r]_{\mu,\sigma} + [1-P(a|\mu,\sigma)]E[(X-\mu_o)^r]_{\mu,\sigma,K}\right\} d\sigma\, d\mu}{\int_\mu \int_\sigma f_1(\mu)\ f_2(\sigma) \left\{P(a|\mu,\sigma) + [1 - P(a|\mu,\sigma)]\ A_X(\mu,\sigma,K)\right\} d\sigma\, d\mu}$$

In the discrete case (3.4.16) becomes

$$E[(X - \mu_o)^r] = \tag{3.4.17}$$

$$\frac{\sum_{all_i} \sum_{all_j} p_i q_j \left\{P(a|\mu_i,\sigma_j)E[(X-\mu_o)^r]_{\mu_i,\sigma_j} + [1-P(a|\mu_i,\sigma_j)]E[(X-\mu_o)^r]_{\mu_i,\sigma_j,K}\right\}}{\sum_{all_i} \sum_{all_j} p_i q_j \left\{P(a|\mu_i,\sigma_j) + [1 - P(a|\mu_i,\sigma_j)]\ A_X(\mu_i,\ \sigma_j,\ K)\right\}}$$

3.5 Evaluation of the Moments

We now would like to evaluate numerically (3.4.8) and (3.4.17).

Let

$$\mu_i = .02i - .01,\ i = 1, \ldots, 18$$
$$= .02i + .01,\ i = -1, \ldots, -18 \qquad (3.4.11)$$
$$\sigma_j = .055 + .01j,\ j = 1, \ldots, 20$$

as before, and

$$\delta_{ij} = \frac{\mu_i - \mu_o}{\sigma_j} = \frac{\mu_i}{\sigma_j} \quad ,$$

assuming $\mu_o = 0$.

In addition, $\{p_i\}$ and $\{q_j\}$ can be found in Tables 2.6 and 2.7, respectively. $E[(X - \mu_o)^r]_{\mu_i,\sigma_j}$ and $E[(X - \mu_o)^r]_{\mu_i,\sigma_j,K}$ are given by (3.4.3) through (3.4.6) and (3.4.12) through (3.4.15), respectively, for r = 1, 2, 3, 4, and

$$A_X(\mu_i, \sigma_j, K) = \Phi\left(\frac{K}{\sigma_j} - \delta_{ij}\right) - \Phi\left(-\frac{K}{\sigma_j} - \delta_{ij}\right) .$$

Thus we need only to solve for $P(a \mid \mu_i, \sigma_j)$ in order to evaluate (3.4.8) and (3.4.17). Now $P(a \mid \mu_i, \sigma_j)$ is the probability of overall acceptance, given μ_i and σ_j. It is equal to the product of the respective probabilities of acceptance on the $\overline{X}$ and R tests, as given in (3.2.3). The form of the range test changes as the sample size N increases, however, as discussed in Chapter II. Since the recommended range test is on

$$S_{nm} = \sum_{i=1}^{m} R_{ni}, \quad \text{for } N = nm > 10,$$

we use the sum-range probabilities given in Table 2.1 in computing $P(a \mid \mu_i, \sigma_j)$. The $\overline{X}$ test does not change except for increased sample size. As has already been suggested in Chapter II, we will consider sample sizes where n = 8, 10, m = 2, 3. We can now evaluate (3.4.8) and (3.4.17) for r = 1, 2, 3, 4. Combining these moments with P_a as the criteria, we can select the test constants c_1 and c_2 which provide the best compromise among those considered, for a given sample size N. It should be noted that in the above case, when normality is assumed, the first and third moments are zero because of symmetry. This will not be true in general, however, as we shall see in later sections of this chapter. If we denote the a posteriori moments as M_r, r = 1, 2, 3, 4, we have:

$M_1 = M_3 = 0.$

$SD = \sqrt{M_2}$ = a posteriori standard deviation for accepted product,

and $\alpha_4 = M_4/M_2^2$

Let the tests be of the form:

$$\text{a)}\quad \mu_o - c_1 T \leq \bar{X}_N \leq \mu_o + c_1 T \qquad (3.5.1)$$

$$\text{b)}\quad S_{nm} \leq c_2 T$$

Then Table 3.3 gives the values for (c_1, c_2) for various sample sizes N = nm, using the above discussed criteria.

Table 3.3

Test Constants (c_1, c_2).

N	Normal	Relaxed
20	(.13, .933)	(.14, .933)
24	(.13, 1.30)	(.14, 1.30)
30	(.13, 1.45)	(.14, 1.45)

Note more than one set of (c_1, c_2) is given for a particular N. The first set of values is recommended for "normal" usage. However, in the case of a "long history" of acceptances without a rejection, the set to the right, or "relaxed" set of constants can be used. For the present, "long history" will remain undefined, with the judgment of the test user called upon in a particular situation.

3.6 Criteria Evaluation Summary

Table 3.4 provides a representative sample of P_a, SD, and α_4 used as criteria to select the above test constants. Values for both class (a) and class (b) a posteriori process distributions are given, where class (a) contains only accepted material, and class (b) contains both accepted and rejected-sorted material. Since SD is the a posteriori standard deviation, a reasonable requirement would be that SD $\leq T/6 = .1667$. We note that for class (a), all SD values fall well below this limit. However, under class (b) a posteriori distributions, an undesirable a priori distribution can give an SD value for $K = \frac{1}{2}$ as high as .20, in some cases not listed in these tables. If we let K = 3/8, we can reduce this to .18, a value only slightly larger than T/6. Evidently Burr's conjecture (1967a) as given in his paper was correct, and sorting of rejected lots should be to $\mu_o \pm \frac{3T}{8}$ instead of $\mu_o \pm \frac{T}{2}$. It should be noted here that the higher sample sizes were necessary in order to achieve a reduction in SD for poor a priori process distributions, as compared to the SD obtained for a sample size of ten. We will use this fact later in providing a general testing procedure

Table 3.4. Summary of Criteria Used to Select Test Constants (c_1, c_2) Given in Table 3.3, for Sample Sizes of 30, 24, and 20.

	Dist. for		(.13, c_2)					(.14, c_2)				
				class (a)		class(b) K=.375			class (a)		class(b), K=.375	
N	u	σ	P_a	SD	α_4	SD	α_4	P_a	SD	α_4	SD	α_4
30	1	1	.998	.104	3.04	.104	3.04	.999	.104	3.04	.104	3.04
	1	3	.929	.114	3.41	.117	3.36	.931	.114	3.41	.117	3.36
	1	6	.699	.125	3.59	.139	3.13	.701	.125	3.59	.139	3.14
	2	1	.948	.106	3.06	.109	3.01	.970	.107	3.05	.109	3.02
	2	3	.881	.116	3.40	.122	3.25	.902	.117	3.38	.122	3.26
	2	6	.663	.127	3.58	.142	3.05	.678	.128	3.55	.142	3.06
	6	1	.520	.114	2.92	.172	2.25	.554	.117	2.88	.172	2.25
	6	3	.485	.124	3.24	.175	2.25	.516	.127	3.18	.175	2.26
	6	6	.365	.134	3.42	.181	2.23	.389	.137	3.36	.181	2.24
24	1	1	.996	.104	3.04	.104	3.03	.998	.104	3.04	.104	3.04
	1	3	.910	.113	3.39	.116	3.31	.913	.113	3.40	.118	3.35
	1	6	.673	.124	3.60	.137	3.09	.676	.124	3.60	.139	3.11
	2	1	.945	.106	3.07	.109	3.01	.966	.107	3.05	.109	3.02
	2	3	.863	.115	3.39	.121	3.20	.883	.116	3.37	.122	3.25
	2	6	.638	.126	3.58	.140	3.01	.653	.127	3.56	.142	3.03
	6	1	.519	.114	2.92	.172	2.25	.553	.117	2.88	.172	2.25
	6	3	.475	.123	3.23	.175	2.25	.507	.126	3.17	.176	2.25
	6	6	.353	.133	3.43	.180	2.23	.376	.136	3.37	.181	2.23
20	1	1	.993	.103	3.04	.104	3.03	.996	.103	3.04	.104	3.03
	1	3	.902	.113	3.40	.117	3.34	.907	.113	3.40	.117	3.35
	1	6	.668	.124	3.64	.139	3.15	.673	.124	3.64	.139	3.16
	2	1	.941	.105	3.06	.109	3.01	.963	.107	3.05	.109	3.02
	2	3	.856	.115	3.40	.122	3.23	.876	.116	3.37	.122	3.25
	2	6	.634	.126	3.62	.142	3.06	.649	.127	3.60	.142	3.08
	6	1	.518	.114	2.93	.172	2.25	.552	.117	2.88	.172	2.25
	6	3	.472	.123	3.24	.175	2.25	.503	.126	3.18	.175	2.26
	6	6	.351	.134	3.47	.181	2.24	.374	.137	3.40	.181	2.25

to be followed. Note also that α_4 is well-controlled in every case. As 100 percent sorting increases, we note that α_4 drops in value as low as 2.2, thus indicating the a posteriori distribution has a "square-shouldered" shape when compared to the normal distribution. This is to be expected since we are truncating rather heavily on undesirable distributions. Tables of N = 16 are not given. A complete summary of recommendations derived from the above will be given at the end of this chapter, after taking into account some further generalizations of the above work.

3.7 Bias Introduction

In Lemma 1 proved in Chapter I it was demonstrated the potential effect that a so-called bias term could have on the total variability. That is, we proved that $E(X - \mu_o)^2 = \sigma_X^2 + \sigma_\mu^2 + (\mu_\mu - \mu_o)^2$, where $(\mu_\mu - \mu_o)^2$ is the bias term. We would like the statistical tests on $\bar{X}$ and S_{nm} to reject process distributions F(X) in which the bias term is excessive. In this section we will speak of two types of bias. The first is an a priori bias, where the a priori distribution of process mean μ is not centered about the nominal mean μ_o. The second, and perhaps the more important, is an a posteriori bias, equal to the first moment of the outgoing distribution (i.e., what we have referred to as class (a) or class (b) distributions) about μ_o. In order to reduce confusion, we will refer to the a priori bias as "a priori" or "off-centered", and the a posteriori bias as simply "bias".

As discussed in Chapter I, Burr has indicated an "acceptable" a priori bias term might be one such that $| \mu_\mu - \mu_o | \leq .05T$, where T is the tolerance range for component part X. In the general case given by (1.6.9) in Chapter I, if c = .05, then $a^2 + b^2 \leq .025$ gives the upper bound for the variability of σ_X and σ_μ, assuming T = 1. That is, approximately 91 percent of the total permissible variability remains. If c = .10, however, only 64 percent of the permissible variability remains. Since σ_X and σ_μ should normally be the major sources of process variability, it is clear that an a priori bias of .10T is undesirable.

In order to see how the test combination which we have just found for a given sample size N will reduce a priori bias, we shall consider values of $| \mu_\mu - \mu_o |$ up to, say, .10T. It should be noted, of course, that although we are placing particular emphasis on both types of bias in this section, the overall variability of the process is that with which we are primarily concerned. We will consider a bias level less than .05T as acceptable, and unacceptable if larger than .05T. This of course assumes that the other sources of variability are near the maximum permissible. If σ_X and σ_μ are quite low, then naturally a higher level of bias is permissible. We cannot expect an immediate drop in acceptance probability as the bias increases beyond the .05T level, but we would want a rather steep slope in the acceptance probability OC curve to occur at that point.

3.8 Off-Center Distributions of μ

Since it is necessary to consider only a positive a priori bias when dealing with symmetric distributions of μ, we will assume $\mu_o = 0$, and consider: $E(\mu_i) = .02b$, where $b = 1, 2, \ldots, 5$. The first seven $\{\mu_i\}$ distributions given in Table 2.6 were used, but were centered symmetrically about .02b instead of $\mu_o = 0$. Thus the scale of the distributions remains invariant, but the location has been shifted to .02b. Denote such a shifted density function $f_1'(\mu)$, or in the discrete case, by p_i'. We will again use the seven a priori $\{\sigma_j\}$ distributions given in Table 2.7. Since we are allowing μ_o to remain zero and simply shifting the $\{\mu_i\}$ distributions in location. The actual formulation for the $\bar{X}$ and S_{nm} tests remains identical to our earlier work. As a result, we may immediately use (3.2.1), (3.4.7), and (3.4.16) by simply substituting the new μ distribution $f_1'(\mu)$ for $f_1(\mu)$ in all three equations. Similarly, (3.2.2), (3.4.8), and (3.4.17) also follow directly by substituting p_i' and p_i. Here

$$p_i' = p_{i-b}\ , \quad \begin{array}{l} i = b+1,\ldots,18+b \\ \ \ = -1,\ldots,-18+b \end{array} \tag{3.8.1}$$

$$p_i' = p_{i-b-1},\ i = 1,\ldots,b \tag{3.8.2}$$

for a fixed b. Note that since p_o was defined earlier as a point of zero probability, so also is p_o'.

In order to indicate the increase in total a priori variability due to the μ distribution being off-centered, Table 3.5 gives a representative sample of the total variation of the a priori process distribution with no a priori bias, given by TO, compared to various levels of positive a priori bias, given by TB. Table values are given as:

$$TO = 6(\sigma_X^2 + \sigma_\mu^2)^{\frac{1}{2}}$$

$$TB = 6[\sigma_X^2 + \sigma_\mu^2 + (.02b)^2]^{\frac{1}{2}} ,$$

and can be directly compared to component part tolerance T = 1. If the table value is less than one, the total variability is acceptable, if greater than one, the variability is excessive.

Table 3.5

Total variability of selected a priori (μ,σ) distributions assuming no a priori bias (given by TO), compared with positive a priori bias up to .02b (given by TB). TO and TB may be compared directly to tolerance T=1

Distribution for		TO	TB				
μ	σ	No bias	b = 1	b = 2	b = 3	b = 4	b = 5
1	1	.711	.721	.750	.797	.857	.930
1	3	.793	.803	.829	.871	.927	.995
1	6	.979	.986	1.008	1.043	1.090	1.148
2	1	.767	.777	.804	.848	.905	.974
2	3	.845	.853	.878	.918	.972	1.036
2	6	1.021	1.028	1.049	1.083	1.128	1.184
6	1	1.506	1.510	1.525	1.548	1.580	1.621
6	3	1.546	1.551	1.565	1.589	1.619	1.659
6	6	1.649	1.654	1.667	1.688	1.718	1.755

3.9 Criteria Evaluation Summary Including Bias

Table 3.6 provides a representative sample of the numerical solution to P_a, the overall probability of acceptance, after including the a priori bias term. That is, (3.8.1) and (3.8.2) have been used in place of $\{p_i\}$ in computing P_a for b = 1,...,5. Note that the most pronounced drop in P_a occurs for b > 2; that is, when $| \mu_\mu - \mu_o | > .04$, the probability of overall acceptance of such an off-center process distribution is significantly reduced. This occurs to some extent even for process distributions which areotherwise in good control. We conclude that the $\bar{X}$ and S_{nm} tests will not accept a process distribution as easily when its mean is on the average off-center by more than .05T. This is as we had hoped.

Tables 3.7, 3.8, and 3.9 give a representative sample of the first four moments of the outgoing distributions for sample sizes 20, 24, and 30. Both class (a) and class (b) outgoing distributions are considered, with the a priori μ distribution off-center as much as .10T. Not all cases considered in order to find the best compromise $\bar{X}$ and S_{nm} test combination can be given in the tables. As a result, only those test combinations which seemed to provide the best compromise among those considered are given here. It was found that for a given sample size N, the $\bar{X}$ and S_{nm} test combinations selected were those given earlier in this chapter, and listed in Table 3.3.

The characteristics of the outgoing distribution (either class (a) or class (b)) were derived from the first four a posteriori moments. Let

Table 3.6. P_a, the Overall Probability of Acceptance, Assuming First No a Prior Bias and Then a Priori Bias of .02b. Given for X and R Test Constants (c_1, c_2) from Table 3.3, for Sample Sizes of 30, 24, and 20.

	Dist. for		(.13, c_2)				(.14, c_2)			
N	μ	σ	no bias	b=2	b=3	b=5	no bias	b=2	b=3	b=5
30	1	1	.998	.939	.849	.650	.999	.971	.897	.700
	1	3	.929	.872	.791	.606	.931	.902	.834	.652
	1	6	.699	.656	.595	.457	.701	.677	.627	.492
	2	1	.948	.893	.831	.662	.970	.919	.864	.710
	2	3	.881	.830	.774	.616	.902	.855	.805	.660
	2	6	.663	.625	.582	.464	.678	.644	.606	.497
	6	1	.520	.511	.500	.467	.554	.545	.534	.499
	6	3	.485	.476	.466	.435	.516	.508	.497	.465
	6	6	.365	.359	.351	.328	.389	.383	.375	.350
24	1	1	.996	.935	.847	.649	.998	.966	.894	.699
	1	3	.910	.854	.775	.595	.913	.882	.817	.641
	1	6	.673	.631	.574	.442	.676	.651	.604	.475
	2	1	.945	.890	.829	.660	.966	.917	.862	.708
	2	3	.863	.814	.758	.604	.883	.838	.789	.647
	2	6	.638	.602	.561	.447	.653	.621	.584	.479
	6	1	.519	.510	.499	.466	.553	.544	.533	.498
	6	3	.475	.467	.457	.427	.507	.498	.488	.456
	6	6	.353	.346	.339	.317	.376	.369	.362	.338
20	1	1	.993	.931	.845	.648	.996	.961	.891	.698
	1	3	.902	.846	.770	.591	.907	.873	.810	.637
	1	6	.668	.626	.571	.440	.673	.646	.600	.473
	2	1	.941	.887	.827	.658	.963	.914	.860	.705
	2	3	.856	.807	.752	.599	.876	.832	.783	.642
	2	6	.634	.598	.558	.444	.649	.617	.581	.476
	6	1	.518	.509	.498	.465	.552	.543	.532	.497
	6	3	.472	.464	.454	.424	.503	.495	.485	.453
	6	6	.351	.345	.338	.315	.374	.368	.360	.337

Table 3.7. Moments of Class (a) Outgoing Distributions, Where Only Accepted Product is Retained, Assuming a Priori Bias is .02b.

N = 30 Test Constants (.13,1.45)

Dist. for μ	σ	b = 2 M	SD	α_3	α_4	b = 5 M	SD	α_3	α_4
1	1	.034	.102	.005	3.06	.066	.095	.013	3.20
1	3	.034	.113	.000	3.45	.066	.106	.019	3.66
1	6	.034	.124	-.003	3.63	.067	.118	.021	3.83
2	1	.027	.104	-.039	3.09	.063	.098	-.043	3.17
2	3	.027	.114	-.027	3.45	.063	.109	-.026	3.59
2	6	.027	.125	-.019	3.62	.063	.121	-.015	3.76
6	1	.006	.114	-.021	2.92	.014	.113	-.052	2.94
6	3	.006	.124	-.015	3.24	.014	.123	-.038	3.26
6	6	.006	.134	-.011	3.43	.014	.134	-.027	3.44

Test Constants (.14,1.45)

Dist. for μ	σ	b = 2 M	SD	α_3	α_4	b = 5 M	SD	α_3	α_4
1	1	.037	.103	.001	3.05	.071	.096	.014	3.18
1	3	.037	.113	-.005	3.42	.071	.107	.018	3.63
1	6	.037	.124	-.007	3.61	.072	.119	.019	3.80
2	1	.030	.105	-.035	3.08	.068	.099	-.046	3.15
2	3	.030	.115	-.026	3.43	.068	.110	-.029	3.56
2	6	.030	.126	-.019	3.60	.068	.122	-.018	3.74
6	1	.006	.117	-.025	2.88	.016	.117	-.064	2.90
6	3	.006	.127	-.019	3.18	.016	.126	-.047	3.20
6	6	.006	.137	-.014	3.36	.016	.137	-.035	3.38

Table 3.7. (cont.)

Moments of class (b) outgoing distributions, where both accepted product and rejected product sorted to $\mu_o \pm 3T/8$ is retained, assuming a priori bias is .02b.

N = 30 Test Constants (.13,1.45)

Dist. For		b = 2				b = 5			
μ	σ	M	SD	α_3	α_4	M	SD	α_3	α_4
1	1	.040	.103	-.007	3.02	.099	.102	-.072	2.90
1	3	.039	.117	-.044	3.33	.097	.114	-.180	3.22
1	6	.037	.138	-.108	3.14	.089	.135	-.318	3.21
2	1	.040	.109	-.023	2.98	.098	.107	-.094	2.88
2	3	.039	.121	-.064	3.22	.096	.118	-.193	3.15
2	6	.036	.141	-.118	3.05	.088	.137	-.317	3.13
6	1	.030	.171	-.099	2.26	.073	.164	-.239	2.33
6	3	.029	.174	-.105	2.27	.071	.168	-.256	2.36
6	6	.026	.180	-.109	2.25	.064	.175	-.269	2.35

Test Constants (.14,1.45)

1	1	.040	.103	-.004	3.03	.099	.102	-.067	2.91
1	3	.039	.117	-.038	3.34	.097	.114	-.169	3.24
1	6	.037	.138	-.102	3.15	.090	.135	-.308	3.22
2	1	.040	.109	-.020	2.99	.098	.107	-.090	2.88
2	3	.039	.121	-.058	3.23	.096	.118	-.184	3.16
2	6	.036	.141	-.113	3.06	.089	.138	-.309	3.14
6	1	.030	.171	-.100	2.26	.073	.164	-.239	2.33
6	3	.029	.174	-.105	2.28	.071	.168	-.256	2.37
6	6	.026	.180	-.109	2.26	.065	.175	-.269	2.36

Table 3.8. Moments of Selected Outgoing Distributions, Assuming an a Priori Bias of .02b, and a Sample Size of N = 24. Test Constants Used are (.13, 1.30).

Class (a) Outgoing Distributions

Dist. for		b = 2				b = 5			
μ	σ	M	SD	α_3	α_4	M	SD	α_3	α_4
1	1	.034	.102	.006	3.06	.066	.095	.016	3.20
1	3	.034	.112	.000	3.43	.067	.106	.022	3.64
1	6	.034	.123	-.003	3.64	.067	.117	.025	3.85
2	1	.027	.104	-.037	3.09	.063	.098	-.041	3.17
2	3	.027	.113	-.026	3.44	.063	.108	-.024	3.57
2	6	.027	.124	-.018	3.63	.064	.120	-.013	3.77
6	1	.006	.114	-.021	2.93	.014	.113	-.051	2.94
6	3	.006	.123	-.015	3.23	.014	.123	-.038	3.24
6	6	.006	.133	-.011	3.43	.014	.133	-.027	3.44

Class (b) Outgoing Distributions (K=.375)

1	1	.040	.103	-.007	3.02	.099	.102	-.071	2.90
1	3	.039	.117	-.048	3.31	.097	.114	-.182	3.21
1	6	.036	.138	-.111	3.12	.089	.134	-.320	3.20
2	1	.040	.109	-.023	2.98	.098	.107	-.094	2.88
2	3	.039	.121	-.065	3.20	.096	.118	-.194	3.14
2	6	.036	.141	-.120	3.04	.088	.137	-.319	3.12
6	1	.030	.171	-.099	2.26	.073	.164	-.239	2.33
6	3	.029	.174	-.105	2.27	.071	.168	-.255	2.36
6	6	.026	.179	-.108	2.25	.064	.175	-.268	2.35

Table 3.9. Moments of Selected Outgoing Distributions, Assuming an a Priori Bias of .02b, and a Sample Size of N=20. Test Constants Used are (.13,.933).

Dist. for μ	Dist. for σ	b = 2 M	b = 2 SD	b = 2 α_3	b = 2 α_4	b = 5 M	b = 5 SD	b = 5 α_3	b = 5 α_4
Class (a) Outgoing Distributions									
1	1	.034	.102	.006	3.06	.067	.095	.019	3.20
1	3	.034	.112	.000	3.44	.067	.106	.026	3.65
1	6	.033	.123	-.003	3.68	.067	.118	.030	3.90
2	1	.027	.104	-.035	3.09	.063	.098	-.039	3.17
2	3	.027	.113	-.024	3.44	.064	.108	-.022	3.58
2	6	.027	.124	-.016	3.67	.064	.120	-.009	3.82
6	1	.006	.114	-.021	2.93	.014	.114	-.051	2.94
6	3	.006	.123	-.015	3.24	.014	.123	-.036	3.25
6	6	.006	.134	-.010	3.47	.014	.134	-.024	3.48
Class (b) Outgoing Distributions (K=.375)									
1	1	.040	.103	-.008	3.02	.099	.102	-.071	2.90
1	3	.039	.117	-.049	3.31	.097	.114	-.181	3.22
1	6	.036	.138	-.111	3.14	.089	.135	-.318	3.21
2	1	.040	.109	-.023	2.98	.098	.107	-.094	2.88
2	3	.039	.121	-.065	3.20	.096	.118	-.194	3.14
2	6	.036	.141	-.119	3.05	.088	.137	-.318	3.13
6	1	.030	.171	-.099	2.26	.073	.164	-.239	2.33
6	3	.029	.174	-.105	2.27	.071	.168	-.255	2.36
6	6	.026	.180	-.108	2.25	.064	.175	-.269	2.36

$M_i = i^{th}$ a posteriori moment,

$i = 1, 2, 3, 4$

Then we define:

$$M = M_1 \text{, the a posteriori bias} \tag{3.9.1}$$

$$SD = [M_2 - (M_1)^2]^{\frac{1}{2}}$$

$$\alpha_3 = \frac{M_3 - 3 M_2 M_1 + 2 M_1^3}{(SD)^3}$$

$$\alpha_4 = \frac{M_4 - 4 M_3 M_1 + 6 M_2 M_1^2 - 3 M_1^4}{(SD)^4}$$

In class (b) distributions where 100 percent sorting was used, it was again found that sorting to the limits $\mu_o \pm \frac{3T}{8}$ was better than $\mu_o \pm \frac{T}{2}$.

In summary, using both the moments of the outgoing distribution and the overall probability of acceptance as the criteria, we can conclude that, in general, distribution biases can be controlled by the $\bar{X}$ and S_{nm} test combinations given in Table 3.3.

3.10 Non-Normality

Up to this point we have always assumed the underlying process distribution to be normal. In Chapter I it was mentioned that one of the goals of this research was to initiate the investigation of the effects of non-normality on the $\bar{X}$ and S_{nm} test combination for a given sample size. That is, will the tests continue to discriminate between desirable and undesirable process distributions when these underlying distributions are not necessarily normal? Although not quite as significant, another question is: what is the effect of non-normal distributions of the process mean? Until now we have considered process mean distributions to be uniform, triangular, or normal. We now propose to add moderately skewed Burr distributions to this list.

Rather than considering a general process distribution assumed to be non-normal, it was decided it would be more advantageous to consider several specific distributions with varying departures from normality. Of particular interest is the effect of skewed distributions; i.e., $\alpha_3 \neq 0$. By using a Burr distribution to approximate any specific non-normal distribution we choose, we are able to develop a procedure by which any non-normal distribution (within the range of a Burr distribution) may be considered as the underlying process distribution. The same applies to the process mean distribution. Two such non-normal process distributions were considered in this research, as well as two non-normal process mean distributions.

We will first consider the two process distributions. For a part dimension X with distribution F(X), let $\alpha_{3:X}$ and $\alpha_{4:X}$ represent the skewness and kurtosis of X, respectively. A normal distribution will have $\alpha_{3:X} = 0.0$, $\alpha_{4:X} = 3.0$. A positive skewness of $\alpha_{3:X} = 1.0$ is quite large, and will not occur too often in a practical situation. A more likely value for $\alpha_{3:X}$ would be, say, around 0.5 or less. We will assume two values: $\alpha_{3:X} = .3$ and .5, approximately. Now, for a sample of size N from F(X), $(X_1,\ldots,X_N)$, the skewness and kurtosis of the distribution of $\overline{X} = \sum_{i=1}^{N} X_i/N$ can be expressed in terms of $\alpha_{3:X}$ and $\alpha_{4:X}$.

That is,

$$\alpha_{3:\overline{X}} = \frac{\alpha_{3:X}}{\sqrt{N}} \tag{3.10.1}$$

$$\alpha_{4:\overline{X}} = \frac{\alpha_{4:X} - 3.0}{N} + 3.0 \ , \tag{3.10.2}$$

for the skewness and kurtosis, respectively. For the general Burr distribution

$$\begin{aligned} F(X) &= 1 - (1 + X^c)^{-k} \ , \quad X \geq 0,\ c,\ k \geq 1 \\ &= 0 \qquad\qquad\qquad\quad , \quad X < 0 \end{aligned} \tag{3.10.3}$$

one can show that if $c = 2.86$, $k = 20.0$, then $F(X)$ is a non-normal distribution with $\alpha_{3:X} = .305$, $\alpha_{4:X} = 2.90$. Let $N = 30$. Then

$$\alpha_{3:\overline{X}} = \frac{.305}{\sqrt{30}} = .056$$

$$\alpha_{4:\overline{X}} = \frac{2.90 - 3.00}{30} + 3.00 = 2.997$$

We may then approximate the distribution of $\overline{X}$ with $\alpha_{3:\overline{X}}$, $\alpha_{4:\overline{X}}$ given above by taking a Burr distribution with $c = 4.44$, $k = 6.67$. We therefore have two Burr distributions: one which approximates a non-normal process distribution, say $F_{X_1}(X)$, and the other which approximates the distribution of the sample mean, say $F_{\overline{X}_1}(X)$.

As a second non-normal process distribution we take $c = 2.50$, $k = 13.33$, which gives $\alpha_{3:X'} = .508$, $\alpha_{4:X} = 3.20$. For $N = 30$, we have:

$$\alpha_{3:\overline{X}} = .093$$

$$\alpha_{4:\overline{X}} = 3.01$$

We may approximate the distribution of $\bar{X}_2$ by letting $c = 4.44$, $k = 5.71$ in a Burr distribution. Table 3.10 gives the mean M and standard deviation S of the four Burr distributions discussed above, in addition to a summary of c and k values. In order to standardize, with $\mu = 0$, $\sigma = 1$, for each F(t) take:

$$F(t) = 1 - [1 - (M + St)^c]^{-k} \quad , \text{ if } M + St \geq 0 \qquad (3.10.4)$$
$$= 0 \quad , \text{ otherwise.}$$

Table 3.10.

Burr Distribution Parameters

Distribution	M	S	c	k
$F_{X_1}(X)$	.316	.123	2.86	20.00
$F_{\bar{X}_1}(X)$	.608	.166	4.44	6.67
$F_{X_2}(X)$	.322	.143	2.50	13.33
$F_{\bar{X}_2}(X)$	.632	.174	4.44	5.71

$F_{X_1}(X)$ and $F_{X_2}(X)$ can then be used as process distributions replacing a normality assumption, and $F_{\bar{X}_1}(X)$ and $F_{\bar{X}_2}(X)$ can similarly be used as sample mean distributions.

A similar procedure to the above could be followed for N = 24, 20, in order to find $F_{\bar{X}_1}(X)$, $F_{\bar{X}_2}(X)$ in those cases. The resulting distributions for $F_{\bar{X}_1}(X)$, $F_{\bar{X}_2}(X)$ were so close to those already derived that it was decided to simply use $F_{\bar{X}_1}(X)$ and $F_{\bar{X}_2}(X)$ as given above, for $N = 24, 20$. What is important here is that the $\bar{X}$ and S_{nm} tests are applied to non-normal distributions, and this procedure is not affected by using $F_{\bar{X}_1}(X)$ and $F_{\bar{X}_2}(X)$ as given.

Consider now the distribution of the process means $\{\mu_i\}$. We have already used rather extensively seven discrete distributions of $\{\mu_i\}$ given in Table 2.6 based on uniform, triangular, or normal distributions. In Chapter II, two additional non-normal process mean distributions were considered, with $\alpha_3 = .401$, $\alpha_4 = 3.11$. Therefore, for the derivation of the $\bar{X}$ and S_{nm} combination tests considering non-normal cases, all nine a priori $\{\mu_i\}$ distributions were considered, along with the usual seven a priori $\{\sigma_j\}$ distributions, giving a total of 63 possible (μ, σ) distribution combinations.

3.11 Test Criteria Evaluation Assuming Non-Normality

We now have indicated the procedure for considering non-normal process distributions, sample mean distributions, and process mean distributions. We have chosen several cases of each, but have indicated that the procedure is not limited to these few cases. We now would like to proceed as before in determining the best compromise among the various $\bar{X}$ and S_{nm} test combinations considered for a given sample size N. We will evaluate P_a, the overall probability of acceptance, and the first four a posteriori moments, for each of our 63 (μ, σ) distribution combinations. We again will consider class (a) and class (b) a posteriori distributions as previously defined, as well as whether we should sort rejected material to $\mu_o \pm \frac{3T}{8}$ limits or $\mu_o \pm \frac{T}{2}$. The emphasis will be on

Table 3.11

Summary of $\alpha_{3:S_{nm}}$, $\alpha_{4:S_{nm}}$

S_{nm}	$\alpha_{3:S_{nm}}$	$\alpha_{4:S_{nm}}$
$S_{10,2}$	.23 to .33	3.07 to 3.23
$S_{8,3}$	.20 to .26	3.04 to 3.13
$S_{10,3}$	.19 to .27	3.05 to 3.15
(Normal)	.28	3.10
	.24	2.06
	.23	3.07

Included for comparison on the bottom half of Table 3.11 are values of $\alpha_{3:S_{nm}}$ and $\alpha_{4:S_{nm}}$ when the underlying distribution is assumed to be normal. Note that in every case the value assuming normality falls within the limits computed for the non-normal case. In addition, the limits given for the non-normal case are rather narrow for both $\alpha_{3:S_{nm}}$ and $\alpha_{4:S_{nm}}$. One can therefore conclude that the distribution of S_{nm} assuming normality is a good approximation to the distribution of S_{nm} in the non-normal case. These results correlate well with earlier work by Burr (1955) and Burr (1968) in which the robustness of the range for moderately skewed populations and small sample sizes is noted. We will, therefore, use the tables for S_{nm} derived in Chapter II when computing $P_{S_{nm}}(a \mid \sigma_j)$. The above suggests a rather extensive research area as yet basically untouched. One possible technique that this author may employ at some future time would be to first, extend Burr's tables (1968) to $\alpha_{3:S_{nm}}$ and $\alpha_{4:S_{nm}}$, and then to approximate the resulting sum-range distributions by appropriate Burr distributions. From the resulting approximated distribution functions, one could select percentage points useful in extending the author's work to a wide range of non-normal process distributions. For now, however, we will limit our interests to $F_{X_1}(X)$ and $F_{X_2}(X)$, for which $P_{S_{nm}}(a \mid \sigma_j)$ may be approximated by the tables of the numerical distribution of S_{nm} derived in Chapter II.

3.12 Probability of Acceptance Assuming Non-Normality

Based on this approximation, we can now calculate P_a for various $\bar{X}$ and S_{nm} test combinations for each of the 63 a priori (μ, σ) distribution combinations. Let

$$\mu_i = .02i - .01 \quad , \quad i=1,\ldots,18$$

$$= .02i + .01 \quad , \quad i=-1,\ldots,-18$$

$$\sigma_j = .055 + .01 \quad , \quad j=1,\ldots,20$$

Assume $F_{X_1}(X)$ as given in Table 3.10 is the underlying process distribution, with the corresponding sample mean distribution $F_{\bar{X}_1}(X)$. Then

$$P_{\bar{X}}(a \mid \mu_i, \sigma_j)$$

$$= P_{F_{\bar{X}_1}} (\text{acceptance on } \bar{X} \text{ test} \mid \mu_i, \sigma_j)$$

$$= P_{F_{\bar{X}_1}} (\mu_o - c_1 T < \bar{X} \leq \mu_o + c_1 T \mid \mu_i, \sigma_j)$$

$$= F_{\bar{X}_1} \left(\frac{\mu_o + c_1 T - \mu_i}{\sigma_j / \sqrt{N}} \right) - F_{\bar{X}_1} \left(\frac{\mu_o - c_1 T - \mu_i}{\sigma_j / \sqrt{N}} \right)$$

Let

$$\mu_o = 0, \ T = 1.$$

Then:

$$P_{\bar{X}}(a \mid \mu_i, \sigma_j) = \tag{3.12.1}$$

$$F_{X_1} \left[\left(\frac{c_1 - \mu_i}{\sigma_j} \right) \sqrt{N} \right] - F_{\bar{X}_1} \left[\left(\frac{- c_1 - \mu_i}{\sigma_j} \right) \sqrt{N} \right] ,$$

which can easily be evaluated from the Burr distribution $F_{\bar{X}_1}(X)$ when μ_i, σ_j, and N are known and c_1 is specified.

As discussed briefly earlier in this chapter, to compute the eighth and ninth sets of p_i, say $\{p_i^{(8)}\}$ and $\{p_i^{(9)}\}$, let M = .408, S = .163, c = 2.86, k = 10.0 in (3.10.4) and call the resulting distribution $\Phi^*(t)$. Then define:

$$p_i^{(8)} = \Phi^* \left(\frac{\mu_i + .01}{.10} \right) - \Phi^* \left(\frac{\mu_i - .01}{.10} \right) \tag{3.12.2}$$

$$i = \pm 1, \ldots, \pm 18$$

$$p_i^{(9)} = \Phi^* \left(\frac{\mu_i + .01}{.20} \right) - \Phi^* \left(\frac{\mu_i - .01}{.20} \right) \tag{3.12.3}$$

$$i = \pm 1, \ldots, \pm 18$$

Here we are letting $\sigma = .10 = T/10$, $\sigma = .20 = T/5$, respectively, to give the eighth $\{\mu_i\}$ distribution an acceptable variability level with respect to $T = 1$, and the ninth $\{\mu_i\}$ distribution a clearly unacceptable variance. $\{p_i^{(8)}\}$ and $\{p_i^{(9)}\}$ were then corrected to sum to one by dividing each p_i, $\forall_i$, by the sum of the $\{p_i\}$.

We now can compute the 63 (μ, σ) a priori probability distributions; (3.12.1) gives $P_{\overline{X}}(a \mid \mu_i, \sigma_j)$, while $P_{S_{nm}}(a \mid \sigma_j)$ can be obtained from the sum-range tables given in Chapter II. Hence we can compute P_a:

$$P_a = \sum_{\text{all}_i} \sum_{\text{all}_j} P_{\overline{X}}(a \mid \mu_i, \sigma_j) \, P_{S_{nm}}(a \mid \sigma_j) \, p_i \, q_j \qquad (3.12.4)$$

for a given set of a priori (μ, σ) distributions.

3.13 Evaluation of Outgoing Moments Assuming Non-Normality

We now turn to the problem of computing the a posteriori process distribution moments, assuming the underlying process distribution is non-normal. We will, of course, use $F_{X_1}(X)$ and $F_{X_2}(X)$ as previously defined. Two classifications of a posteriori process distributions will again be used, with class (a) denoting only those lots accepted, and class (b) including those rejected lots which have been sorted 100 percent to certain limits, in addition to the accepted lots. A close investigation of the derivations leading up to (3.4.8) and (3.4.17), under the assumption of normality, indicates that all that is necessary to extend these equations to the non-normal case is a technique for finding $E(X - \mu_o)^r]_{\mu_i, \sigma_j}$, the r^{th} moment of X about μ_o given μ_i, σ_j for accepted lots, and $E[(X - \mu_o)^r]_{\mu_i,\sigma_j,K}$, defined for lots truncated by 100 percent sorting to limits $\mu_o \pm KT$. All other functions in (3.4.8) and (3.4.17) have been discussed earlier when considering P_a except $A_X(\mu_i, \sigma_j, K)$, and it can easily be shown to be:

$$A_X(\mu_i, \sigma_j, K) = F_{X_1}\left(\frac{K}{\sigma_j} - \delta_{ij}\right) - F_{X_1}\left(- \frac{K}{\sigma_j} - \delta_{ij}\right) , \qquad (3.13.1)$$

where $F_{X_1}(X)$ is the non-normal process distribution. The immediate problem is then to evaluate $E[(X - \mu_o)^r]_{\mu_i,\sigma_j}$ and $E[(X - \mu_o)^r]_{\mu_i,\sigma_j,K}$ for r = 1, 2, 3, 4.

Two possible methods of attacking this problem were considered. First, since we are expressing the distribution of X in terms of a Burr distribution, one possible choice might be to evaluate the first four moments of our Burr distributions $F_{X_1}(X)$ and $F_{X_2}(X)$ as briefly summarized in the first part of Chapter II. For $E[(X - \mu_o)^r]_{\mu_i,\sigma_j,K}$, a method of truncating the moments would be obtained. The moments could then be expressed as functions of beta functions and evaluated from the appropriate tables. The second possible solution to the problem involves further approximations using the Burr distribution, and is based on the rather intuitive assumption that as the process distribution deviates from normality, so will the process moments tend to deviate from normal moments. Taking (3.4.2) and (3.4.11), or to be more precise the particular equations for r = 1, 2, 3, 4 following (3.4.2) and (3.4.11), we replace all normal distribution functions Φ and density functions ϕ by $F_{X_1}(X)$ and $f_{X_1}(X)$, the derivative with respect to X of $F_{X_1}(X)$, respectively. That is, take the functional form of the first four moments as given by (3.4.3) through (3.4.6) and (3.4.12) through (3.4.15) and replace Φ and ϕ by $F_{X_1}(X)$ and $f_{X_1}(X)$, respectively. The resulting moments should be close approximations to the exact moments for moderately skewed distributions. The density function of the Burr distribution given by (3.10.4) is:

$$f(t) = \frac{c\ k\ S\ (M + St)^{c-1}}{[1 + (M + St)^{c}]^{k+1}} \quad , \text{ if } M + St \geq 0 \qquad (3.13.2)$$

$$= 0 \qquad\qquad , \text{ otherwise}$$

A distinct advantage of the second method over the first is the ease in which method two can be implmented on a computer. This fact is quite important, since we will be considering 63 a priori (μ, σ) distributions with up to 360 discrete (μ_i, σ_j) combinations for each a priori distribution, plus several $\bar{X}$ and S_{nm} test combinations and two K values. The computer can easily be programmed to evaluate a Burr distribution (or density) function for any computed value, so the second method is preferable to the first because the first method involves evaluating beta functions, which involve an integral. Moreover, the simplicity of the second method is appealing in cases where the process distribution is moderately skewed, so that the approximation is likely to be a good one. Finally, since it is not the purpose of this research to investigate all possible choices of process distributions, but rather to give an indication of how the derived $\bar{X}$ and S_{nm} test combinations will react when the process distribution deviates somewhat from normality, the second method of finding the a posteriori distributions seemed to the author preferable to the first method. Again, a broad area of additional research is available here for future study. The problem of deriving moments over broad regions of non-normal process distributions is an important one, and perhaps could best be attacked by using both techniques described here.

Adopting method two for now, $E[(X - \mu_o)^r]_{\mu_i, \sigma_j}$ and $E[(X - \mu_o)^r]_{\mu_i, \sigma_j, K}$ can be evaluated explicitly. We then can evaluate (3.4.8) for $E_a[(X - \mu_o)^r]$, and (3.4.17) for $E[(X - \mu_o)^r]$, r = 1, 2, 3, 4, using first $F_{X_1}(X)$ and $F_{\overline{X}_1}(X)$, and then $F_{X_2}(X)$ and $F_{\overline{X}_2}(X)$. Combining these a posteriori moments with P_a as our criteria, we again will select the best compromise among the various constants c_1 and c_2 considered for a given sample size N.

3.14 Summary of Criteria Evaluation Assuming Non-Normality

Tables 3.12 through 3.15 provide a summary of the results for the non-normal cases considered. For ease of notation, we refer to the non-normal process distribution $F_{X_1}(X)$ and its corresponding sample mean distribution $F_{\overline{X}_1}(X)$ as "Skew 1", and to the $F_{X_2}(X)$, $F_{\overline{X}_2}(X)$ combination as "Skew 2". All other notations should be familiar to the reader. It should be noted that an expanded set of a priori (μ, σ) distribution combinations is given. Sets eight and nine of $\{\mu, \sigma\}$ were added, of course, to study the effects of non-normal process mean distributions. Sets three and six of $\{\mu_i\}$ were included in order to give a direct comparison as to the effects of the non-normal process mean distributions. That is, sets three and eight were constructed identically except for the difference in the distribution assumed, as were sets six and nine. Set one of the $\{\mu_i\}$ was included in order to study the effects of non-normal process distributions when compared to earlier work in this chapter. In order to keep the tables to a reasonable size, only 10 (μ, σ) combinations are given for each skewed distribution, although a total of 63 combinations were considered in the research.

The following general conclusions can be drawn from the summary tables or from untabulated results, for N = 30:

a) Both the a posteriori mean M and skewness α_3 were negligibly small for class (a) distributions, over all 63 (μ, σ) combinations. All M values were within a range of $\mu_o \pm .00002$, except for $\{\mu_i\}$ sets eight and nine, which had negative M values down to -.006. Likewise, all α_3 values were within $0 \pm .0002$ except for $\{\mu_i\}$ sets eight and nine, which had positive α_3 values up to .025. This held true for both Skew 1 and Skew 2, as well as for the sample sizes N considered, and hence these values are omitted from the table summaries.

b) Both Skew 1 andSkew 2 were handled well under class (a) distributions: SD < T/6, and $2.87 \leq \alpha_3 \leq 3.41$ in all cases.

c) P_a is virtually identical to the values obtained earlier assuming normality. The difference would increase if the process distribution deviated further from normality than presently assumed, but using $\bar{\bar{X}}$ and S_{nm} as our tests, since they are both averages, will always cause a tendency toward normality to occur.

d) Very little differences can be detected when comparing $\{\mu_i\}$ sets: three vs. eight, and six vs. nine, although α_3's for sets eight and nine are greater than α_3's for sets three and six, respectively. We can conclude that non-normal process mean distributions are handled about as well as normal ones.

e) Under class (b) distributions, K = .375 provides slightly better results for M and SD than does K = .500. However, K = .500 does a much better job of "centering" α_3 about zero than does K = .375. This holds true for Skew 1 and 2, as well as test combinations (.13, 1.45) and (.14, 1.45). α_4 is well controlled in either case.

f) Under class (b) distributions, there is little difference between Skew 1 and Skew 2 <u>except</u> in α_3, which has wider scattering of values about zero for Skew 2 than for Skew 1. This we would expect.

g) For well-controlled, or "good", a priori (μ, σ) distributions (i.e., $\mu = 1, 2$; $\sigma = 1, 2$), all four moments remain in good control, including α_3.

h) As the a priori (μ, σ) distributions deviate more from these so-called "good" distributions in (g), α_3 remains less and less in control. In nearly all cases, α_3 for test combinations (.14, 1.45) is less than α_3 for (.13, 1.45). But the big reduction in α_3 is brought about by using K = .500 instead of K = .375. The significance of this fact will be discussed later in the summary.

i) Under class (b) distributions, SD $\dot{\leq}$ T/6 even for K = .500.

Tables 3.14 and 3.15 reflect similar conclusions drawn for sample sizes N = 24, 20. They have been further condensed so as not to repeat information. For example, P_a is approximately the same for Skew 1 and Skew 2, so was only given for Skew 2. SD and α_4 for class (a) distributions are approximately the same for Skew 1 and 2, so are given only for Skew 2. All results for (.13, 1.30) vs. (.14, 1.30) were approximately the same except for α_3, and so are given only for (.14, 1.30). This is <u>not</u> to say that the various $\bar{X}$ and S_{nm} test combinations considered had little effect; these were again reduced to only two choices, for which distinctions were apparent only in α_3.

Table 3.12. Summary of Test Criteria Assuming Non-normality for (.13, 1.45), N = 30.

Dist. for		Class (a)			Class (b)							
					K = .375				K = .500			
μ	σ	P_a	SD	α_4	M	SD	α_3	α_4	M	SD	α_3	α_4
Skew 1												
1	1	.998	.104	3.04	-.000	.104	-.001	3.04	-.000	.104	-.001	3.04
1	3	.929	.114	3.41	-.003	.117	-.034	3.34	-.004	.119	-.087	3.56
3	1	.800	.109	3.00	-.033	.125	-.064	2.73	-.035	.127	-.137	2.89
3	3	.745	.119	3.33	-.034	.134	.019	2.82	-.037	.139	-.110	3.08
6	1	.520	.114	2.92	-.093	.144	.228	2.34	-.104	.155	.055	2.48
6	3	.485	.124	3.24	-.090	.149	.299	2.46	-.105	.161	.103	2.58
8	1	.801	.110	2.99	-.037	.124	-.026	2.76	-.039	.126	-.097	2.91
8	3	.745	.120	3.32	-.038	.133	.047	2.85	-.041	.137	-.080	3.11
9	1	.508	.114	2.92	-.098	.143	.258	2.37	-.110	.153	.079	2.50
9	3	.473	.124	3.24	-.095	.148	.326	2.49	-.110	.160	.124	2.60
Skew 2												
1	1	.998	.104	3.04	-.000	.104	-.001	3.04	-.000	.104	-.001	3.04
1	3	.929	.114	3.41	-.003	.118	-.048	3.35	-.004	.119	-.097	3.57
3	1	.800	.109	3.00	-.034	.127	-.054	2.70	-.035	.128	-.129	2.87
3	3	.745	.119	3.33	-.035	.135	.029	2.76	-.038	.140	-.111	3.07
6	1	.520	.114	2.92	-.098	.146	.320	2.34	-.108	.156	.108	2.48
6	3	.485	.124	3.24	-.094	.150	.390	2.43	-.109	.163	.151	2.58
8	1	.801	.110	2.99	-.038	.125	-.014	2.72	-.039	.126	-.088	2.88
8	3	.745	.120	3.32	-.039	.134	.059	2.80	-.042	.138	-.080	3.10
9	1	.508	.114	2.92	-.102	.145	.347	2.38	-.113	.155	.129	2.52
9	3	.473	.124	3.24	-.099	.149	.415	2.48	-.113	.162	.170	2.61

Table 3.13. Summary of Test Criteria Assuming Non-normality for (.14, 1.45), N = 30.

Dist. for			Skew 1									
			Class (a)		Class (b)							
					K = .375				K = .500			
μ	σ	P_a	SD	α_4	M	SD	α_3	α_4	M	SD	α_3	α_4
1	1	.999	.104	3.04	-.000	.104	-.000	3.04	-.000	.104	-.000	3.04
1	3	.931	.114	3.41	-.003	.117	-.033	3.34	-.004	.119	-.086	3.56
3	1	.833	.112	2.98	-.029	.126	-.072	2.74	-.030	.128	-.145	2.90
3	3	.775	.121	3.29	-.030	.135	.006	2.82	-.033	.140	-.119	3.08
6	1	.554	.117	2.88	-.088	.147	.217	2.31	-.100	.158	.051	2.45
6	3	.516	.127	3.18	-.086	.152	.286	2.43	-.100	.164	.098	2.55
8	1	.835	.112	2.97	-.033	.125	-.031	2.76	-.035	.127	-.101	2.91
8	3	.777	.122	3.28	-.034	.134	.037	2.85	-.037	.138	-.086	3.10
9	1	.542	.117	2.87	-.094	.146	.251	2.35	-.105	.156	.079	2.47
9	3	.505	.127	3.18	-.091	.150	.316	2.47	-.106	.163	.122	2.57
			Skew 2									
1	1	.999	.104	3.04	-.000	.104	-.000	3.04	-.000	.104	-.000	3.04
1	3	.931	.114	3.41	-.003	.118	-.047	3.36	-.004	.119	-.095	3.58
3	1	.833	.112	2.98	-.030	.128	-.066	2.71	-.031	.129	-.139	2.88
3	3	.775	.121	3.29	-.031	.136	.014	2.77	-.034	.140	-.122	3.07
6	1	.554	.117	2.88	-.092	.149	.299	2.30	-.103	.159	.099	2.44
6	3	.516	.127	3.18	-.089	.153	.368	2.39	-.104	.166	.142	2.54
8	1	.835	.112	2.97	-.034	.126	-.022	2.73	-.035	.127	-.094	2.89
8	3	.777	.122	3.28	-.035	.135	.047	2.80	-.038	.139	-.087	3.09
9	1	.542	.117	2.87	-.098	.148	.330	2.34	-.109	.158	.125	2.48
9	3	.505	.127	3.18	-.094	.151	.397	2.44	-.109	.164	.164	2.57

Table 3.14. Summary of Test Criteria Assuming Non-normality (Skew 2 Only) for Test Constants (c_1, c_2), N = 24, 20.

N = 24, Test Constant c_2 = 1.30

Dist. for		(.13, c_2)	(.14, c_2)	(.14, c_2), Skew 2							
				Class (a)		Class (b)					
						K = .375			K = .500		
μ	σ	P_a	P_a	SD	α_4	M	SD	α_4	M	SD	α_4
1	1	.996	.998	.104	3.04	-.000	.104	3.04	-.000	.104	3.04
1	3	.910	.913	.113	3.40	-.004	.118	3.33	-.005	.119	3.57
3	1	.798	.831	.112	2.98	-.030	.128	2.71	-.031	.129	2.88
3	3	.730	.760	.121	3.28	-.032	.136	2.76	-.035	.140	3.06
6	1	.519	.553	.117	2.88	-.092	.149	2.30	-.103	.159	2.44
6	3	.475	.507	.126	3.17	-.090	.153	2.39	-.105	.165	2.54
8	1	.798	.833	.112	2.97	-.034	.126	2.73	-.035	.127	2.89
8	3	.730	.761	.121	3.27	-.036	.135	2.79	-.039	.139	3.09
9	1	.507	.541	.117	2.88	-.098	.148	2.35	-.109	.158	2.48
9	3	.464	.496	.126	3.17	-.095	.151	2.44	-.110	.164	2.57

N = 20, Test Constant c_2 = .933

1	1	.993	.996	.103	3.04	-.000	.104	3.03	-.000	.104	3.04
1	3	.902	.907	.113	3.40	-.004	.118	3.33	-.005	.119	3.57
3	1	.796	.828	.112	2.98	-.030	.127	2.71	-.031	.129	2.88
3	3	.724	.754	.121	3.29	-.032	.136	2.76	-.035	.140	3.07
6	1	.518	.552	.117	2.88	-.092	.149	2.30	-.103	.159	2.45
6	3	.472	.503	.126	3.18	-.090	.153	2.40	-.105	.165	2.54
8	1	.796	.830	.112	2.97	-.034	.126	2.73	-.035	.127	2.89
8	3	.724	.755	.121	3.28	-.037	.135	2.79	-.039	.139	3.09
9	1	.506	.540	.118	2.88	-.098	.148	2.35	-.109	.158	2.48
9	3	.461	.492	.127	3.17	-.095	.151	2.44	.110	.164	2.57

Table 3.15. Summary of α_3 Assuming Non-normality, for Test Constant (c_1, c_2), N = 24, 20.

Dist. for		Class (b)							
		K = .375				K = .500			
		$(.13, c_2)$		$(.14, c_2)$		$(.13, c_2)$		$(.14, c_2)$	
μ	σ	Skew 1	Skew 2	Skew 1	Skew 2	Skew 1	Skew 2	Skew 1	Skew 2
N = 24, Test Constant c_2 = 1.30									
1	1	-.001	-.001	-.001	-.001	-.001	-.001	-.001	-.001
1	3	-.042	-.058	-.041	-.057	-.098	-.108	-.096	-.106
3	1	-.062	-.052	-.071	-.064	-.135	-.127	-.143	-.137
3	3	.017	.027	.005	.013	-.112	-.114	-.121	-.124
6	1	.230	.322	.218	.300	.056	.109	.052	.101
6	3	.299	.391	.286	.369	.101	.150	.097	.141
8	1	-.024	-.012	-.029	-.021	-.096	-.086	-.100	-.092
8	3	.046	.057	.036	.045	-.083	-.083	-.089	-.090
9	1	.259	.348	.252	.331	.080	.130	.080	.126
9	3	.325	.415	.315	.397	.123	.169	.121	.164
N = 20, Test Constant c_2 = .933									
1	1	-.002	-.002	-.001	-.001	-.002	-.002	-.001	-.001
1	3	-.043	-.060	-.043	-.059	-.099	-.109	-.098	-.107
3	1	-.060	-.051	-.069	-.062	-.134	-.125	-.141	-.135
3	3	.018	.029	.007	.015	-.111	-.112	-.120	-.122
6	1	.231	.323	.219	.301	.057	.110	.053	.102
6	3	.300	.393	.287	.371	.102	.151	.098	.143
8	1	-.023	-.011	-.028	-.019	-.094	-.084	-.098	-.090
8	3	.047	.058	.037	.047	-.082	-.081	-.087	-.088
9	1	.260	.349	.253	.333	.081	.131	.081	.127
9	3	.326	.417	.317	.399	.123	.170	.122	.165

Hence, separate tables for α_3 are given for N = 20, 24. Again we note the slight superiority in most cases of $c_1 = .14$ over $c_1 = .13$, and the significant superiority of K = .500 over K = .375.

We may summarize our conclusions for the non-normal cases considered as follows:

a) If the a priori (μ, σ) distribution is clearly an acceptable one, we will have little problems regardless of whether the underlying process distribution is normal or not. An $\bar{X}$ test constant of $c_1 = .13$ seems the best compromise, as does sorting any rejected material to $\mu_o \pm \frac{3T}{8}$.

b) If, however, the results of the S_{nm} test point to excessive variation, and moreover if a frequency curve or some other technique indicates a definite positive skewness away from normality, the above conclusions suggest using $c_1 = .14$ for future tests and possibly sorting rejected material to $\mu_o \pm \frac{T}{2}$. The same should apply to negative skewness as well, since α_3 would then have a tendency to be off-center in the negative direction.

Sorting rejected material to $\mu_o \pm \frac{T}{2}$, however, still creates possible excessive values of $\sum_{i=1}^{N_1-N_2} (X_i - \mu_o)^2/(N_1-N_2)$, as discussed earlier in in this chapter. It would therefore probably be more beneficial to try to reduce the skewness in the lot or process distribution, rather than increasing the sorting limits from $\mu_o \pm \frac{3T}{8}$ to $\mu_o \pm \frac{T}{2}$. The recommended procedures which follow reflect this thought.

Having briefly considered a few non-normal process distributions and determined the best compromise $\bar{X}$ and S_{nm} test combinations for various sample sizes N, we now will combine these conclusions with earlier results from the normal and bias cases in order to recommend a general procedure to be followed for acceptance sampling.

3.15 Acceptance Sampling Tests

The outline for the general procedure given here is similar to one proposed by Burr (1967a). Significant changes have been made, however, in the sample size, the form of the range test, and the generality of the proposed production model. We will assume for the moment that no prior information is available on either the process distribution or the past history of the producer. The next section will consider the case when more information is available.

Given a tolerance T and nominal mean value μ_0 for a part, the following plan will safely control the distribution of part dimensions for the accepted lots:

A. N = 30

1. Choose three random samples, each made up of 10 parts, from the lot. By random we mean that each one of the parts in the lot is given an equal chance of being chosen in a sample. Let: $(X_{11}, X_{12}, \ldots, X_{1,10})$, $(X_{21}, \ldots, X_{2,10})$, and $(X_{31}, \ldots, X_{3,10})$ be the three samples of ten each. Then compute:

$$\bar{X}_{30} = \sum_{i=1}^{3} \sum_{j=1}^{10} X_{ij}/30$$

$$R_{10,i} = \max(X_{i1},\ldots,X_{i,10}) - \min(X_{i1},\ldots,X_{i,10}),$$
$$i = 1, 2, 3$$
$$S_{10,3} = \sum_{i=1}^{3} R_{10,i}$$

2. Accept the lot if both of the following criteria are met:

   a) $\mu_o - .13T \leq \bar{X}_{30} \leq \mu_o + .13T$ (3.15.1)

   b) $S_{10,3} \leq 1.45T$

3. Reject the lot if either one or both of the requirements in step 2 is not met.

4. Report information on all rejected lots to the producer. Such lots should also be 100 percent sorted within the limits $\mu_o \pm \frac{3T}{8}$, if the rejected lots are to be used.

5. Two conditions may occur under which steps 2 and 4 could be altered:

   a) If the producer has some past history of providing material which is usually acceptable under step 2, one may then relax the $\bar{X}_{30}$ test a bit in order to reduce the probability of rejecting a lot that should be accepted. Part (a) of step 2 then becomes:

      (a′) $\mu_o - .14T \leq \bar{X}_{30} \leq \mu_o + .14T$

b) If when testing a particular lot, one suspects, either from the past history of the producer or from the data, that the distribution of the past dimensions is not well-controlled and possibly skewed, then modifications are needed. The $\overline{X}_{30}$ test should then be relaxed to (a′) given above, and rejected material sorted to limits $\mu_o \pm \frac{3T}{8}$. The information indicating a skewed distribution should be reported to the producer in an effort to reduce the skewness, if possible. The test user must be aware, however, that skewed process distributions that are not well-controlled will likely give skewed a posteriori distributions.

B. N = 24

1. Take three random samples of eight each, and compute $\overline{X}_{24}$ and $S_{8,3} = \sum_{i=1}^{3} R_{8,i}$

2. Accept if both of the following are met:

   a) $\mu_o - .13T \leq \overline{X}_{24} \leq \mu_o + .13T$ (3.15.2)

   b) $S_{8,3} \leq 1.30T$

3. Reject, otherwise.

4. Steps 4 and 5 remain the same for sorting, the relaxed test, and handling skewness, except test (a′) becomes.

   $\mu_o - .14T \leq \overline{X}_{24} \leq \mu_o + .14T$

C. $N = 20$

1. Take two random samples of ten each, and compute $\overline{X}_{20}$ and $S_{10,2} = \sum_{i=1}^{2} R_{10,i}$.

2. Accept if both of the following are met:

   a) $\mu_o = .13T \leq \overline{X}_{20} \leq \mu_o + .13T$ (3.15.3)

   b) $S_{10,2} \leq .933T$

3. Reject, otherwise.

4. Steps 4 and 5 remain the same, with test (a′) becoming:

   $$\mu_o - .14T \leq \overline{X}_{20} \leq \mu_o + .14T$$

The results of this chapter have indicated that for very good process distributions, a sample size of ten does about as well as the higher sample sizes. Therefore, if one is confident that a process distribution is likely to be well-controlled, the test procedure given below may be used. The danger in doing this must be understood, however, for such a test will reject with a high probability only very poor process distributions. A process distribution which has excessive variability, but not extremely so, is likely to be accepted. With this in mind, such a test procedure would be:

1. For a random sample of size ten, compute the average $\overline{X}_{10}$ and the range R_{10}.

2. Accept the lot if <u>both</u> of the following criteria are met:

   a) $\mu_o - .14T \leq \overline{X}_{10} \leq \mu_o + .14T$ (3.15.4)

   b) $R_{10} \leq .50T$

3. Reject, otherwise.

4. Sort rejected lots 100 percent to limits $\mu_o \pm \frac{3T}{8}$.

A further note on the control of the process distribution would be appropriate here. Some concern is expressed in step 5 of the acceptance sampling procedure concerning the shape of the process distribution itself. If a producer has a history of supplying material which is rejected by the $\overline{X}$ or S_{nm} tests, a further check of the process distribution should be made. To do this, take an additional sample up to a total of 100 to 150 parts and construct a frequency curve in order to check for deviations from normality. If excess variability or considerable skewness is evident from the frequency curve, one might suggest to the producer that tighter process controls are needed than he is now employing. Chapters IV and V of this thesis will develop such process control procedures for achieving desired distributions.

CHAPTER IV

PROCESS CONTROL: RANDOMLY ACTING ASSIGNABLE CAUSES

4.1 Introduction

In Chapter I the general field of process control was divided into two categories: a) tool wear, and b) randomly acting assignable causes, where each category is characterized by the changes occurring in its process mean. In this chapter we will investigate methods of controlling process distributions with characteristic (b) present, with Chapter V covering the tool wear problem. We will again assume the field of application to be measurable component parts which combine into assemblies in a known fashion. In addition, we are reasonably "close" to the actual production process and thus will have some control over the procedures following a test rejection. Our goal is to develop statistial tests, which, if used periodically, will safely control the distribution of the part dimensions, in commonly occurring cases.

As discussed by Burr (1967a), the ideal process for providing the desired distribution of part characteristics is one in which the process is in control with μ_X very close to μ_o and σ_X sufficiently small. In general, we are assuming that the process is not yet in control. Thus it is expected that efforts will be made to determine and correct the causes of this lack of control, in addition to employing the techniques given in this chapter for safe control of the part dimension distribution. Shewhart control charts for $\overline{X}$ and R are an effective means of seeking such an ideal process. Duncan (1955) also provides a criterion for control of a process, subject to random shifts in the process mean.

In order to develop the statistical tests we seek, it was necessary to set up and solve a probabilistic model representing a process with randomly acting assignable causes. Although the model presented here, as well as the derivation of the two statistical tests, is the work of the author, he is much in debt to the work of Burr, both published (1967a) and unpublished (1966), which provided direction and guidance throughout.

The two tests will have the following form, given a tolerance T and nominal mean value μ_o for a part:

1. For a periodic sample of size five, compute $\overline{X}$ and R.
2. The process will be considered satisfactory at this time if both of the following are met:

   (a) $\mu_o - c_1 T \leq \overline{X} \leq \mu_o + c_1 T$ (4.1.1.)

   (b) $R \leq c_2 T$
3. If step 2 is not satisfied, other measures must be taken, which will be discussed in detail later.

We use a sample size of five in keeping with past work in process control.

4.2. Range Test

Because the randomly acting assignable causes model is primarily concerned with the process mean, we will first consider the range test necessary to control the process variability. The range of σ_X values considered, from T/15 to T/4, can be used in constructing an OC curve for the probability of acceptance on the range test. Various c_2 values for the R test given in (4.1.1) were considered, beginning with $c_2 = .615$ as suggested by Burr (1967). The criteria for choosing a c_2 were as follows:

1) Recalling the three sets of σ_X defined in section (2.4) of Chapter II, where

   $A = \{ \sigma_X \mid \sigma_X \leq .125\ T\}$

   $B = \{ \sigma_X \mid \sigma_X \geq .185\ T\}$

   $C = \{ \sigma_X \mid .125\ T \leq \sigma_X \leq .185\ T\}$

   we want the probability of acceptance for $\sigma_X \in A$ to be high, and for $\sigma_X \in B$ to be low. However, for a sample of size five, the OC curve is not too steep, so emphasis has been placed on a high probability of acceptance for $\sigma_X \in A$.

2) In order to reduce somewhat the probability of acceptance for $\sigma_X \in B$, it was decided to reduce c_2 below .615. For $c_2 = .55$, the value selected for the R test, the reduction in the probability of acceptance for $\sigma_X \in B$ was .10 or more, while the reduction in the probability of acceptance for $\sigma_X \in A$ was .01 or less.

3) Even with the reduction in c_2 mentioned above, the probability of acceptance for $\sigma_X = .185$ is .781. In order to reduce the probability of accepting a process with excessive variation, the amount of time between testing procedures should be relatively short when the testing procedures are first being implemented. A rule-of-thumb for the amount of time between testing procedures for a randomly acting assignable causes process will be included in the summary given at the end of this chapter. When past history indicates the variability is in control, the time between testing may be gradually increased.

Based on the above criteria, $c_2 = .55$ was selected, as mentioned above. Table 4.1 gives the probability of acceptance given σ_X, say $P_R(a \mid \sigma_X)$, for $\sigma_j = .055 + .01j$, j=1,...,20. The distribution of X from which these calculations are made is assumed to be N(0,1). Thus

$$P_R(a \mid \sigma_X) = P(R_5 \leq .55T \mid \sigma_X = \sigma_j)$$

$$= P(R_5 \leq .55T/\sigma_j \mid \sigma = 1)$$

Again, let T = 1.0. Tables by Harter and Clemm (1959) were used to find $P_R(a \mid \sigma_X)$.

Table 4.1.

Probability of Acceptance on the Range Test

σ_j	$P_R(a \mid \sigma_j)$	σ_j	$P_R(a \mid \sigma_j)$
.065	1.000	.165	.873
.075	1.000	.175	.829
.085	1.000	.185	.781
.095	1.000	.195	.732
.105	.998	.205	.681
.115	.994	.215	.632
.125	.984	.225	.583
.135	.968	.235	.538
.145	.943	.245	.495
.155	.911	.255	.454

4.3 Randomly Acting Assignable Causes Model

We now will develop the randomly acting assignable causes model with which we will determine the $\overline{X}$ test which, along with the above R test, best controls the distribution of the part dimensions for a sample of size five. As indicated in the title of this model, we are assuming the process mean μ randomly takes on various values, about a nominal mean μ_o, some of which are unsatisfactory. We will assume these values to be discrete levels of μ. Each level of μ will remain for a random time length, at which point a new discrete level of μ will be chosen, with a new random time length. If at any time a rejection occurs in an $\bar{X}$ test, the process will in practice be immediately reset to μ_o, and a new random time length will be chosen by nature. We will make the further simplifying assumption that random time lengths can only end at the time an $\bar{X}$ and R test is made. This will allow us to solve the model as a stochastic process, rather than resorting to a Monte Carlo study of the model (Burr, 1966).

It is clear from the above that we need to:

(a) select discrete μ levels, some of which are unsatisfactory;

(b) assume an appropriate distribution for the random time length:

(c) assume an appropriate method for "nature" to select a new μ_j level once a random time length at a given μ_i level is over, and we have not rejected.

Without loss of generality, let tolerance T = 1. To satisfy (a), let us first determine what would be an undesirable μ level. Clearly a μ level more than two standard deviations away from μ_o would be undesirable, if maintained for very long. If we choose $\sigma_X = T/8 = .125$ (an upper bound on σ_X presented in Chapter I), then a μ level outside $\mu_o \pm 2\sigma_X$ or $\mu_o \pm .25$ could be considered undesirable. On the basis of the above, it was decided to assume: $\mu_i = .05i$, $i = 0, \pm 1, \ldots, \pm 6$. Rather than assuming the process standard deviation to be fixed, it was decided to consider various discrete levels of σ_X, from low to excessive values when compared to T=1. Thus we set $\sigma_j = .035 + .03j$, $j = 1, 2, \ldots, 6$. Values σ_1, σ_2, and σ_3 are clearly acceptable, while σ_5 and σ_6 are not acceptable.

To satisfy (b), we will consider as the random time length distribution a negative binomial distribution. For an arbitrary fixed real k, and $0 < p < 1$, the sequence $\{f(t;\, k,\, p)\}$, where

$$f(t;k,p) = \binom{t+k-2}{t-1} p^k (1-p)^{t-1} \qquad (4.3.1)$$

$$t = 1,2,3,\ldots$$

is called a negative binomial distribution. For positive integer k, it is often referred to as the probability distribution for the waiting time to the k^{th} failure. As such it is a natural choice for the random time length distribution. Note that for $k = 1$, the distribution reduces to the geometric distribution, which was used by Burr (1967a) in a similar model. We will consider a subclass of distributions (4.3.1) with $k = 1, 2, 3$. The parameter t will represent "time" in the stochastic process, with one time unit being the time between periodic testing of the process distribution. In order to provide some continuity among the three negative binomial distributions, we will select p, the probability of failure, in such a way that the expected time length E(t) will be the same in all three distributions. In the model E(t) corresponds to the expected number of periodic testing procedures before a change occurs in the μ level. E(t) will be quite important to us, as will be noted later. Now

$$E(t) = \sum_{t=1}^{\infty} t \binom{t+k-2}{t-1} p^k (1-p)^{t-1}$$

$$= p^k\, G(t,\, k,\, p),$$

$$G(t,k,p) = \sum_{t=1}^{\infty} t \binom{t + k - 2}{t - 1} (1-p)^{t-1}$$

In order to solve for G(t,k,p), let

$$Q = 1 - p,$$

$$T = C + Q\,[(1 - Q)^{-k}] \qquad (4.3.2)$$

where C is a constant.

Using the binomial expansion series,

$$T = C + Q \left[\sum_{i=1}^{\infty} \binom{t + k - 2}{t - 1} Q^{t-1} \right]$$

$$= C + \sum_{t=1}^{\infty} \binom{t + k - 2}{t - 1} Q^{t}$$

Then:

$$\frac{dT}{dQ} = \sum_{t=1}^{\infty} t \binom{t + k - 2}{t - 1} Q^{t-1}$$

$$= \sum_{t=1}^{\infty} t \binom{t + k - 2}{t - 1} (1-p)^{t-1}$$

$$= G(t, k, p)$$

Therefore

$$G(t,k,p) = \frac{dT}{dQ} = Qk(1-Q)^{-(k+1)} + (1-Q)^{-k}$$

$$= (1-p)k\,p^{-(k+1)} + p^{-k},$$

by taking the derivative w.r.t. Q of (4.3.2). Hence

$$E(t) = p^k G(t,k,p)$$

$$= \frac{(1-p)k}{p} + 1$$

$$= \frac{k}{p} - k + 1 \qquad (4.3.3)$$

We now wish to select p, for k = 1,2,3, such that $E(t) = \frac{k}{p} - k + 1$ is the same. Table 4.2 gives values for p when E(t) = 2,5,10 for k = 1,2,3.

Table 4.2.

Negative Binomial Parameter p Values

E(t)	k = 1	k = 2	k = 3
2	1/2	—	—
5	1/5	1/3	3/7
10	1/10	2/11	1/4

These values for (k,p) were used in equation (4.3.1) to compute the random time length distribution, with p omitted for k = 2, 3 under E(t) = 2.

We now need to find (c), a method of selecting a new μ_j level once a random time length at a given μ_i level is completed, and we have not rejected. We do this by invoking $\{\pi_i\}$, a discrete a priori μ distribution. We will consider four discrete $\{\pi_i\}$ distributions, discretized from a normal distribution with mean zero and variance σ_r, where $\sigma_r = .05\ (r + 1)$, r = 1,2,3,4. Thus σ_1 gives an a priori μ distribution which has an acceptable σ_μ, while σ_r, r = 2,3,4 give $\{\pi_{ir}\}$ which have increasing unacceptable σ_μ. This will allow us to judge how well the $\bar{X}$ test reduces excessive σ_μ. Table 4.3 gives the a priori σ_μ for the four $\{\pi_{ir}\}$, as well as π_{ir}, the a priori probability of μ_i occurring, $i = 0, \pm 1, \ldots, \pm 6$, for $r = 1, 2, 3, 4$. Note that the $\{\pi_{ir}\}$ have been corrected, so that $\sum_i \pi_{ir} = 1$, $r = 1,2,3,4$.

Table 4.3

Values of π_{ir}

i	μ_i	$r = 1$ $\sigma = .10$	$r = 2$ $\sigma = .15$	$r = 3$ $\sigma = .20$	$r = 4$ $\sigma = .25$
0	.00	.19765	.13649	.11105	.09878
±1	±.05	.17486	.12919	.10764	.09682
±2	±.10	.12112	.10952	.09806	.09121
±3	±.15	.06567	.08318	.08394	.08256
±4	±.20	.02787	.05657	.06752	.07180
±5	±.25	.00925	.03448	.05106	.06001
±6	±.30	.00240	.01881	.03625	.04820
a priori σ_μ =		.1003	.1371	.1568	.1671

We now may summarize the model procedure. Initially, let the μ level be μ_0. Equation (4.3.1), the negative binomial distribution, chooses a random time length for μ_0. As we test on $\bar{X}$ periodically, we will either accept or reject the process at each time interval. If we accept, the process remains at μ_0 until its random time length is over. At that point in time, $\{\pi_{ir}\}$ will be invoked to choose a new μ level, distribution (4.3.1) will choose another random time length, and we proceed as before, testing periodically. If, however, we reject the process at any time, we will automatically reset the process to μ_0, choose a new random time length via (4.3.1), and continue periodic testing. Thus, level μ_i, $i \neq 0$, can only be reached by all acceptances during a random time interval <u>and</u> the invoking of $\{\pi_{ir}\}$. Level μ_i, $i=0$, can be reached both by invoking $\{\pi_{ir}\}$ and by a test rejection at any μ_i level. Equation (4.3.1) is invoked whenever a new μ_i is reached, whether by invoking $\{\pi_{ir}\}$ or a test rejection.

As one can easily see, the model is a stochastic process, in which we are interested in several things:

(1) Over a long period of time, what is the <u>proportion</u> of that time spent at level μ_i? Knowing this, we can compute the a posteriori standard deviation, for example.

(2) Over a long time period, what are the moments of the a posteriori distribution considering

(a) only material accepted, or

(b) both accepted material and sorted-rejected material?

We will refer to these two classes of distributions as class (a) and class (b), respectively. In order to find answers to these questions, we will assume for the moment that the underlying process distribution is normal. Later sections of the chapter will deal with the problem of biasedness (i.e., the μ_i distribution not centered about the chosen nominal mean μ_o) and nonnormality.

We are interested in computing the probability of acceptance on the $\bar{X}$ test for a given mean μ_i and standard deviation σ_j. So

P (acceptance on $\bar{X}$ test $| \mu_i, \sigma_j$)

$$= P_{\bar{X}} (a \mid \mu_i, \sigma_j)$$

$$= P(\mu_o - c_1 T \leq \bar{X}_5 \leq \mu_o + c_1 T \mid \mu_i, \sigma_j)$$

$$= \int_{\mu_o - c_1 T}^{\mu_o + c_1 T} \frac{e^{-\frac{(\bar{X} - \mu_i)^2}{2\sigma_j^2/5}}}{\sqrt{2\pi}\ \sigma_j/\sqrt{5}} \, d\bar{x}$$

$$= \int_{\frac{\mu_o - c_1 T - \mu_i}{\sigma_j/\sqrt{5}}}^{\frac{\mu_o + c_1 T - \mu_i}{\sigma_j/\sqrt{5}}} \phi(t)\, dt,$$

where $\phi(t)$ = N(0,1) density function. Denoting $\Phi(t)$ as the N(0,1) distribution function, then

$$P_{\bar{X}}(a \mid \mu_i, \sigma_j) = \Phi\left[\left(\frac{\mu_o + c_1 T - \mu_i}{\sigma_j}\right)\sqrt{5}\right] - \Phi\left[\left(\frac{\mu_o - c_1 T - \mu_i}{\sigma_j}\right)\sqrt{5}\right] \quad (4.3.4)$$

Letting $\mu_o = 0$, $T = 1$, (4.3.4) becomes

$$P_{\bar{X}}(a \mid \mu_i, \sigma_j) = \Phi\left[\left(\frac{c_1 - \mu_i}{\sigma_j}\right)\sqrt{5}\right] - \Phi\left[\left(\frac{-c_1 - \mu_i}{\sigma_j}\right)\sqrt{5}\right] \quad (4.3.5)$$

As previously mentioned, we will assume

$$\mu_i = .05i, \; i = 0, \pm 1, \ldots, \pm 6$$

$$\sigma_j = .035 + .03j, \; j = 1, 2, \ldots, 6$$

An earlier similar model by Burr (1967a) selected a value for c_1 to be .168. Based on these preliminary results, we will consider values $c_1 = .09(.01).28$, and attempt to find the most the most desirable c_1 value under the generality of the above discussed model, including moderately skewed process distributions, for a sample size of five.

4.4. Model Evaluation: Time to Change

In solving any time series model, it is quite important to "get started right." The author is indebted to the unpublished notes by Burr (1966) for guidance in the "right" direction. The key is to answer the question: "What is the expected time spent at μ_i before a change occurs?" Here "change" is defined as either (a) completing a time length run with all acceptances and thus invoking $\{\pi_{ir}\}$, or (b) rejecting on the $\bar{X}$ test before the end of a run, thus resetting automatically to μ_o without invoking $\{\pi_{ir}\}$.

Let us find (a), the expected time spent at μ_i given that we have all acceptances on the $\overline{X}$ test. Now t = 1 is the smallest time length we can have, since the first periodic $\overline{X}$ test occurs at t = 1. For the sake of brevity, denote

$A_{ij} = P_{\overline{X}}(a \mid \mu_i, \sigma_j)$, for a fixed c_1,

$p_k(t) = f(t; k, p)$, for a fixed p,

EA(i) = expected time spent at μ_i in which all acceptances occur on the periodic $\overline{X}$ tests.

It should be noted here that, although not explicitly stated hereafter, all derivations are for fixed values of c_1, σ_j, k, p, and r. Then:

$$EA(i) = \sum_{t=1}^{\infty} t\, p_k(t)\, A_{ij}^t$$

$$= p^k A_{ij} \sum_{t=1}^{\infty} t \binom{t+k-2}{t-1} [(1-p) A_{ij}]^{t-1} \qquad (4.4.1)$$

Letting $Q = (1-p) A_{ij}$, and using a similar technique to the one given by (4.3.2) and following, one can reduce (4.4.1) to

$$EA(i) = p^k A_{ij} \left[Q\, k(1-Q)^{-(k+1)} + (1-Q)^{-k}\right]$$

$$= p^k A_{ij} \left[\frac{1 + (k-1)\, Q}{(1-Q)^{k+1}} \right]$$

$$= p^k A_{ij} \left\{ \frac{1 + (k-1)\,(1-p)\, A_{ij}}{[1 - (1-p)\, A_{ij}]^{k+1}} \right\} . \qquad (4.4.2)$$

Let PA(i) = probability of a run of all acceptances at level μ_i.

Then

$$PA(i) = \sum_{t=1}^{\infty} p_k(t)\, A_{ij}^{t}$$

$$= p^k A_{ij} \sum_{t=1}^{\infty} \binom{t + k - 2}{t - 1} \left[(1-p)\, A_{ij}\right]^{t-1}$$

$$= \frac{p^k A_{ij}}{\left[1 - (1-p)\, A_{ij}\right]^k}, \tag{4.4.3}$$

using the binomial expansion series.

Consider now the expected time until a rejection on the $\bar{X}$ test occurs. Let

ER(i) = expected time spent at level μ_i until a rejection occurs on the $\bar{X}$ test.

$R_{ij} = 1 - A_{ij}$

$$p_k(T \geq t) = \sum_{i=t}^{\infty} f(i;\, k,\, p) \tag{4.4.4}$$

Let PR(i) = probability of a rejection on $\bar{X}$ occurring at level μ_i.

Then:

$$PR(i) = 1 - PA(i) \tag{4.4.5}$$

as given in (4.4.3). Now

$$ER(i) = \sum_{t=1}^{\infty} t\, P(\text{rejecting on } \bar{X} \text{ test at time } t)$$

$$= \sum_{t=1}^{\infty} t\, p_k(T \geq t)\, A_{ij}^{t-1}\, R_{ij} \tag{4.4.6}$$

To evaluate (4.4.6), we must first evaluate (4.4.4):

$$p_k(T \geq t) = 1 - p_k(T \leq t - 1)$$

$$= 1 - \sum_{i=1}^{t-1} \binom{i + k - 2}{i - 1} p^k (1-p)^{i-1}$$

= 1 - [probability of (t + k - 2) trials with k or more failures]

$$= 1 - \left[\text{cumulative binomial of} \binom{t + k - 2}{k \text{ or more}}\right] \tag{4.4.7}$$

The bracket term cannot in general be expressed in any form simplier than a summation. It was therefore necessary to use cumulative binomial tables (Harvard, 1955) to evaluate $p_k(T \geq t)$ for $k = 2,3$. These values are given in Table 4.4 for the p given in Table 4.2. For $k = 1$, however, one can show that

$$p_1(T \geq t) = (1-p)^{t-1}, \quad t = 1,2,\ldots \tag{4.4.8}$$

Hence (4.4.6) can be evaluated directly for $k = 1$.

<u>k = 1</u>

$$ER(i) = \sum_{t=1}^{\infty} t(1-p)^{t-1} A_{ij}^{t-1} R_{ij}$$

$$= R_{ij} \sum_{t=1}^{\infty} t\,[(1-p)\,A_{ij}]^{t-1}$$

$$= \frac{R_{ij}}{[1 - (1-p)\,A_{ij}]^2} \tag{4.4.9}$$

Table 4.4. $P_k(T \geq t) = \sum_{i=t}^{\infty} f(i; k,p)$, where $f(i; k,p)$ is the Negative Binomial Distribution.

	k = 2		k = 3	
t	p = .18	p =1/3	p = .25	p = .43
1	1.00000	1.00000	1.00000	1.00000
2	.96760	.88889	.98437	.92049
3	.91446	.74074	.94922	.78454
4	.84911	.59259	.89648	.62955
5	.77765	.46091	.83057	.48230
6	.70441	.35117	.75641	.35641
7	.63233	.26337	.67854	.25595
8	.56339	.19509	.60068	.17960
9	.49877	.14307	.52559	.12365
10	.43916	.10405	.45520	.08378
11	.38485	.07515	.39068	.05600
12	.33587	.05395	.33260	.03711
13	.29205	.03854	.28113	.02421
14	.25312	.02740	.23609	.01570
15	.21874	.01941	.19711	.01010
16	.18854	.01370	.16370	.00646
17	.16213	.00964	.13531	.00410
18	.13911	.00677	.11134	.00259
19	.11913	.00474	.09126	.00163
20	.10183	.00331	.07452	.00102
21	.08690	.00231	.06065	.00063
22	.07405	.00160	.04920	.00039
23	.06301	.00111	.03980	.00024
24	.05354	.00077	.03211	.00015
25	.04544	.00053	.02584	.00009
26	.03852	.00037	.02074	.00006
27	.03262	.00026	.01661	.00003
28	.02760	.00018	.01328	.00002
29	.02333	.00012	.01060	.00001
30	.01970	.00008	.00844	.00001
31	.01662	.00006	.00671	.00000
32	.01401	.00004	.00532	↓
33	.01180	.00003	.00422	
34	.00994	.00002	.00334	
35	.00836	.00001	.00264	
36	.00703	.00001	.00208	
37	.00590	.00001	.00164	
38	.00496	.00000	.00129	
39	.00416	↓	.00102	
40	.00349		.00080	

Table 4.4. (cont.)

	$p_k(T \geq t)$			
	k = 2		k = 3	
t	p = .18	p = 1/3	p = .25	p = .43
41	.00293	0	.00063	0
42	.00245		.00049	
43	.00205		.00038	
44	.00172		.00030	
45	.00144		.00024	
46	.00120		.00018	
47	.00101		.00014	
48	.00084		.00011	
49	.00070		.00009	
50	.00059		.00007	
51	.00050		.00005	
52	.00041		.00004	
53	.00035		.00003	
54	.00029		.00002	
55	.00024		.00002	
56	.00020		.00001	
57	.00017		.00001	
58	.00014		.00001	
59	.00012		.00001	
60	.00010		.00001	
61	.00009		.00000	
62	.00007			
63	.00006			
64	.00005			
65	.00004			
66	.00003			
67	.00002			
68	.00002			
69	.00002			
70	.00002			
71	.00001			
72	.00001			
73	.00001			
74	.00001			
75	.00001			
76	.00000			

4.5 Model Evaluation: Time to Cycle Completion

The time series model can be considered to have completed a "cycle" whenever $\{\pi_{ir}\}$ is invoked. That is, we are choosing a μ_i level to begin the process over again. We are therefore interested in determining the total expected time spent at level μ_i until $\{\pi_{ir}\}$ is invoked. Let

$$E(i) = EA(i) + ER(i) \tag{4.5.1}$$

= total time at level μ_i until a "change", as earlier defined.

If a rejection occurs, however, $\{\pi_{ir}\}$ is not invoked, but the process level is simply reset to μ_o. Thus all levels of μ_i contribute time spent at $\mu_o = 0.0$ before $\{\pi_{ir}\}$ is invoked. If TO is the total expected time spent at level $\mu_o = 0.0$ until $\{\pi_{ir}\}$ is invoked, then $TO = E(0) + [PR(0)]\ (TO)$, which simplifies to:

$$TO = E(0)\ /\ PA(0) \tag{4.5.2}$$

In addition, each μ_i level, $i \neq 0$, contributes an expected time of $[PR(i)]\ (TO)$ to level $\mu_o = 0.0$.

We now can compute the total expected time spent at each μ_i level until $\{\pi_{ir}\}$ is invoked. Let

$ET(i)$ = total expected time spent at each μ_i level until $\{\pi_{ir}\}$ is invoked.

Then, for $i \neq 0$ and fixed r,

$$ET(i) = (\pi_{ir})\ E(i) \tag{4.5.3}$$

For $i = 0$,

$$ET(0) = (\pi_{or})\ (TO) + (TO) \sum_{\substack{i=-6 \\ i\neq 0}}^{+6} (\pi_{ir})\ PR(i)$$

$$= TO \left[\pi_{or} + \sum_{\substack{i=-6 \\ i\neq 0}}^{+6} (\pi_{ir})\ PR(i)\right] \qquad (4.5.4)$$

If the underlying process distribution is assumed symmetric (as, for example, in the normal case), then (4.5.4) may be written as:

$$ET(0) = TO \left[\pi_{or} + 2 \sum_{i=1}^{6} (\pi_{ir})\ PR(i)\right] \qquad (4.5.5)$$

Now, (4.5.3) and (4.5.4) give the expected time spent at each μ_i level, $\forall_i$, for any "cycle" of the time series process. As mentioned earlier, a new cycle begins whenever $\{\pi_{ir}\}$ is invoked to randomly choose a new μ_i level. Therefore, if we take the ratio of ET(i) to the total expected time over all μ_i levels, we will have the expected proportion of time during a cycle spent at each μ_i level. Since these expected proportions hold true for any cycle of time, we will have the expected proportion of time spent at each μ_i level over any long period of time. Thus, let

$$P_i = \frac{ET(i)}{\sum_{i=-6}^{+6} ET(i)} \quad , \ i = 0, \pm 1, \ldots, \pm 6 \qquad (4.5.6)$$

Then P_i gives the a posteriori, or long-term, probability of level μ_i occurring. The a posteriori standard deviation of the process mean, say SDM, can then be found:

$$SDM = \left(\sum_{i=-6}^{+6} (\mu_o - .05i)^2 \, P_i \right)^{1/2} \tag{4.5.7}$$

In addition, the overall a posteriori probability of acceptance is given by

$$P_a = \sum_{i=-6}^{+6} (P_i)\,(A_{ij}) \tag{4.5.8}$$

Recall that the P_i and corresponding SDM and Pa are computed for fixed c_1, σ_j, k, p, and r.

4.6 Reset Error

One of the practical problems in this type of model is reset error; that is, errors in resetting the process mean exactly to μ_o following a test rejection. Burr added this feature in his unpublished model (1966). Suppose we now assume that the probability is .6 of resetting exactly to μ_o, and a probability of .2 each of resetting to μ_1 or μ_{-1}. This will give us some indication as to the effects of reset error. Note that this will change ET(i), for i = -1, 0, 1, thus affecting the P_i. Define:

T_{ij} = expected future time spent at level μ_j before an all acceptance run invokes $\{\pi_{ir}\}$, given level μ_i has just been reached.

Then:

$$T_{11} = E(1) + PR(1)\,[.2T_{-1,1} + .6T_{01} + .2T_{11}] \tag{4.6.1}$$

$$T_{01} = PR(0)\,[.2T_{-1,1} + .6T_{01} + .2T_{11}] \tag{4.6.2}$$

$$T_{-1,1} = PR(-1)\,[\,.2T_{-1,1} + .6T_{01} + .2T_{11}] \qquad (4.6.3)$$

In general,

$$T_{ii} = E(i) + PR(i)[.2T_{-1,i} + .6T_{oi} + .2T_{1i}] \qquad (4.6.4)$$

$$T_{ij} = PR(i)[.2T_{-1,j} + .6T_{oj} + .2T_{1j}]\,, \qquad (4.6.5)$$

for i, j = -1, 0, 1; i ≠ j

Solving (4.6.1), (4.6.2), and (4.6.3) for T_{11}, T_{01}, and $T_{-1,1}$, we find:

$$T_{11} = E(1)\left[\frac{1 - .6\,PR(0) - .2\,PR(-1)}{D}\right] \qquad (4.6.6)$$

$$T_{01} = E(1)\left[\frac{.2\,PR(0)}{D}\right] \qquad (4.6.7)$$

$$T_{-1,1} = E(1)\left[\frac{.2\,PR(-1)}{D}\right], \qquad (4.6.8)$$

where D = 1 - .2PR(1) - .6PR(0) - .2PR(-1)

If we define q_i to be the probability of resetting to μ_i, i=-1,0,1 then in general:

$$T_{ii} = E(i)\left[\frac{D + q_i\,PR(i)}{D}\right] \qquad (4.6.9)$$

$$T_{ij} = E(j)\left[\frac{q_j \cdot PR(i)}{D}\right] \qquad (4.6.10)$$

i, j = -1, 0, +1; i ≠ j

Using (4.6.9) and (4.6.10), we solve for ET (i), i = -1, 0, + 1:

$$ET(i) = \sum_{k=-1}^{+1} (\pi_{kr}) T_{ki}$$

$$+ \left(\sum_{k=-1}^{+1} q_k T_{ki}\right)\left[\sum_{k=-6}^{-2} (\pi_{kr})PR(k) + \sum_{k=2}^{6} (\pi_{kr}) PR(k)\right], \quad (4.6.11)$$

$i = -1, 0, +1$

$ET(i)$ for $i = -6,\ldots,-2,+2,\ldots,+6$ remains as given in (4.5.3). The $ET(i)$ given in (4.6.11) can then be used to find the P_i given in (4.5.6).

4.7 Moments Evaluation

We now would like to consider the a posteriori moments for class (a) and (b) outgoing distributions as defined earlier. Let X have a normal distribution F(X). We will later consider F(X) to be non-normal. Let

μ = mean of F(X)

σ = standard deviation of F(X)

$E[(X-\mu_o)^m]_{\mu,\sigma}$ = m^{th} moment of X about μ_o, given μ, σ

$E_a[(X-\mu_o)^m]$ = m^{th} moment of X about μ_o of the accepted product (class (a))

$E[(X-\mu_o)^m]_{\mu,\sigma,K}$ = m^{th} moment of X about μ_o, given μ, σ, and K.

Here $\mu_o \pm KT$ represents the 100 percent sorting limits.

$E[(X-\mu_o)^m]$ = m^{th} moment of the combined class (b) product: accepted plus sorted-rejected material.

With the above definitions, and equating the m^{th} and r^{th} moments, (3.4.1), (3.4.2), (3.4.9), (3.4.10), and (3.4.11) of chapter III follow directly as presented there, and will not be repeated here. On the basis of (3.4.1) and (3.4.2), we have

$$E_a[(X-\mu_o)^m] = \frac{\sum_{i=-6}^{+6} (P_i)\,(A_{ij})\,\{E[(X-\mu_o)^m]_{\mu_i,\sigma_j}\}}{\sum_{i=-6}^{+6} (P_i)\,(A_{ij})}\,, \qquad (4.7.1)$$

$$m = 1,\ 2,\ 3,\ 4$$

for a fixed value of c_1, σ_j, k, p, and r. Equations (3.4.9), (3.4.10), and (3.4.11) give us:

$$E[(X - \mu_o)^m] =$$

$$\frac{\sum_{i=-6}^{+6}(P_i)\left\{(A_{ij})E[(X-\mu_o)^m]_{\mu_i,\sigma_j} + (R_{ij})E[(X-\mu_o)^m]_{\mu_i,\sigma_j,K}\right\}}{\sum_{i=-6}^{+6}(P_i)\left\{A_{ij} + (R_{ij})\,A_X(\mu_i,\ \sigma_j,\ K)\right\}}\,, \qquad (4.7.2)$$

$$m = 1,\ 2,\ 3,\ 4$$

(4.7.1) and (4.7.2) can be computed both with and without set-up error.

Denote the a posteriori moments as M_m, $m = 1,2,3,4$. Then

$$M = M_1$$

$$SD = (M_2 - M_1^2)^{\frac{1}{2}} \qquad (4.7.3)$$

$$\alpha_3 = \frac{M_3 - 3\,M_2\,M_1 + 2\,M_1^3}{(SD)^3}$$

$$\alpha_4 = \frac{M_4 - 4\,M_3\,M_1 + 6\,M_2\,M_1^2 - 3\,M_1^4}{(SD)^4}$$

When the underlying process distribution is assumed symmetric (as in the normal case),

$$M = \alpha_3 = 0.$$

4.8 Test Evaluation Criteria

It should be recalled that the randomly acting assignable causes model as set up is used to determine the best $\bar{X}$ test for a sample size of five. The range test was considered earlier in this chapter. Two criteria will be used to choose the constant c_1 in the $\bar{X}$ test. First, since $\bar{X}$ is an estimate of the process mean, we would like the $\bar{X}$ test procedure to reduce SDM, the a posteriori standard deviation of the process mean given by (4.5.7), to an acceptable level. Equation (1.6.9) gives the upper limits on the three sources of variability. We will assume various levels of σ_X and biasedness, and compute the upper bound on σ_μ. We then should select the largest c_1 value which will still reduce SDM below the upper bound on σ_μ. Note that we are incorporating the bias term into the model, so biasedness will not be considered as a separate section.

Let us compute the upper bounds on the a posteriori σ_μ:

A. 1) Let $\sigma_X = .065$, $| \mu_\mu - \mu_o | = .05$

Then: $\sigma_\mu^2 + (.065)^2 + (.05)^2 \leq .0278$, from (1.6.9).

Thus $\sigma_\mu^2 \leq .0211$

or $\sigma_\mu \leq .145$

Therefore when $\sigma_X = .065$, the upper bound on SDM (as defined by (4.5.7)) is .145, assuming small bias.

2) let $\sigma_X = .065$, $| \mu_\mu - \mu_o | = .10$

Then $\sigma_\mu \leq .116$, or the upper bound on SDM is .116, assuming "excessive" bias.

B. 1) Let $\sigma_X = .095$, $| \mu_\mu - \mu_o | = .05$

Then, proceeding similarly, the upper bound on SDM is .128. If $| \mu_\mu - \mu_o | = .10$, then SDM $\leq .094$

C. Let $\sigma_X = .125$, $| \mu_\mu - \mu_o | = .05$

Then SDM $\leq .100$.

For $\sigma_X > T/8$, SDM should be reduced as much as possible. In such a case the range test is likely to be rejected, leading to an investigation as to the cause of the excessive variation in the process.

The second criterion we will use in selecting the test is based on the a posteriori moments, both class (a) and class (b). We would hope that the a posteriori standard deviation, SD, is less than T/6, and that the a posteriori kurtosis, α_4, is close to three. It should be noted that SD will always be greater than the σ_X assumed for the process, thus forcing SD > T/6 whenever σ_X is assumed excessive. At this point, however, the R test should detect the excessive variability, and steps will be taken to alleviation to this problem.

Finally, we will note the comparison of the average probability of acceptance for the first $\bar{X}$ test (computed by assuming the distribution of the μ_i to be one of the $\{\pi_{ir}\}$ given in Table 4.3) with the a posteriori average probability of acceptance given by (4.5.8).

4.9 Summary of Results

Only a summary of the overall results are included in the following table. It was found that E(t), as given in (4.3.3), the expected number of periodic testing procedures before a change occurs in the μ level, is important in the selection of the proper $\bar{X}$ test. However, because there seems to be no practical way to estimate E(t) in a given situation, it would do little good to base the $\bar{X}$ test on the value of E(t). It should be noted, however, that $E(T) \leq 2$ can be dangerous, in that you are more likely to accept poor process distributions. If the test user discovers from feedback that he is accepting distributions that should be rejected, a decrease in the time between tests should be considered. With this in mind, the best compromise among the various $\bar{X}$ test considered was

$$\mu_o - .19T \leq \bar{X} \leq \mu_o + .19T \ . \qquad (4.9.1)$$

Table 4.5 provides a summary of the values of the above mentioned criteria for $c_1 = .19$. Six values of σ_X are considered. Parameters k and p are used to denote the value of E(t), as given in table 4.2. r denotes the a priori $\{\pi_{ir}\}$ distribution of μ, SDM denotes the a posteriori σ_μ, Pa is the average probability of acceptance (given for both the first $\bar{X}$ test and a posteriori), and SD and α_4 denote respectively the standard deviation and kurtosis of both class (a) and class (b) outgoing distributions. Values are given for both "with" and "without" set-up error. The following general conclusions may be drawn from Table 4.5:

(1) As indicated where $\sigma_X = .065$, little difference in the criteria due to assuming three random run length distributions ($k = 1, 2, 3$) was observed whenever $E(t)$ was constant. As a result, values for the table are given only for $k = 2$.

(2) The standard deviation SD for both class (a) and class (b) outgoing distributions is approximately less than or equal to T/6 for $\sigma_X \leq .155$. For $\sigma_X > .155$, the range test should detect the excessive variation, and correction action should be taken.

(3) The kurtosis α_4 is well-controlled at all times.

(4) All results for class (b) distributions are given with the limits for sorting rejected material set at $\mu_o \pm \frac{3T}{8}$. These limits seemed to give the best results.

(5) The $\bar{X}$ test given by (4.9.1) allows for some reset error.

A small bias term, say .05T or less, has been allowed for in the above $\bar{X}$ tests. If a larger bias term is present, say .10T or more, an effort must be made to reduce $\sigma_X < \frac{T}{10}$, or else the $\bar{X}$ test will usually reject. Corrective action should then be taken to eliminate the excessive bias.

4.10 Randomly Acting Assignable Causes: Non-normal Case

Suppose now that the underlying process distribution is non-normal, rather than normal as assumed above. What effect will this have on our $\bar{X}$ test? Again we will limit the discussion to moderately skewed distributions, such as the two discussed in Chapter III and denoted by

Table 4.5. Summary of the Criteria Used to Select the $\bar{X}$ Test for Randomly Acting Assignable Causes.

No Set-Up Error

							Class (a)		Class (b)	
σ_X	k	p	r	SDM	P_a a priori	P_a a posteriori	SD	α_4	SD	α_4
.065	2	.333	1	.080	.930	.977	.099	2.89	.103	2.97
			2	.090	.809	.950	.101	2.97	.110	3.17
			3	.094	.727	.934	.100	3.07	.113	3.31
			4	.096	.681	.926	.099	3.13	.114	3.40
.095	2	.333	1	.076	.918	.968	.118	2.99	.121	2.98
			2	.086	.799	.941	.119	3.06	.126	3.02
			3	.091	.721	.926	.119	3.11	.128	3.06
			4	.093	.677	.919	.118	3.13	.129	3.08
.125	2	.333	1	.072	.901	.958	.141	3.02	.143	2.96
			2	.083	.787	.932	.142	3.06	.147	2.95
			3	.089	.713	.918	.142	3.08	.148	2.94
			4	.091	.671	.911	.142	3.10	.149	2.94
.155	2	.333	1	.069	.880	.944	.167	3.02	.167	2.96
			2	.081	.771	.920	.168	3.04	.170	2.94
			3	.086	.702	.908	.168	3.06	.171	2.92
			4	.089	.663	.902	.168	3.07	.171	2.92
.185	2	.333	1	.065	.855	.926	.194	3.01	.193	2.97
			2	.078	.753	.904	.195	3.03	.195	2.96
			3	.084	.689	.893	.195	3.04	.195	2.95
			4	.088	.653	.888	.195	3.04	.195	2.94
.215	2	.333	1	.063	.828	.902	.222	3.01	.219	3.00
			2	.076	.733	.883	.223	3.02	.220	2.99
			3	.082	.675	.873	.223	3.03	.220	2.99
			4	.085	.642	.868	.224	3.03	.220	2.98

Table 4.5. (cont.)

With Set-Up Error

σ_X	k	p	r	SDM	P_a a priori	P_a a posteriori	Class (a) SD	Class (a) α_4	Class (b) SD	Class (b) α_4
.065	2	.333	1	.086	.930	.971	.104	2.79	.108	2.85
			2	.098	.809	.939	.106	2.82	.116	2.97
			3	.103	.727	.921	.105	2.89	.120	3.07
			4	.105	.681	.912	.104	2.94	.121	3.13
.095	2	.333	1	.082	.918	.961	.121	2.95	.125	2.92
			2	.094	.799	.929	.123	2.99	.131	2.93
			3	.100	.721	.911	.123	3.03	.134	2.94
			4	.102	.677	.902	.122	3.06	.135	2.95
.125	2	.333	1	.079	.901	.948	.144	3.01	.146	2.94
			2	.091	.787	.917	.146	3.03	.151	2.90
			3	.097	.713	.901	.146	3.06	.152	2.88
			4	.100	.671	.893	.145	3.07	.153	2.87
.155	2	.333	1	.075	.880	.934	.169	3.01	.170	2.95
			2	.089	.771	.905	.171	3.03	.173	2.91
			3	.095	.702	.890	.171	3.05	.174	2.89
			4	.098	.663	.883	.171	3.06	.174	2.88
.185	2	.333	1	.072	.855	.915	.196	3.01	.195	2.97
			2	.086	.753	.889	.197	3.03	.197	2.94
			3	.093	.689	.876	.198	3.04	.197	2.93
			4	.096	.653	.869	.198	3.04	.198	2.93
.215	2	.333	1	.069	.828	.892	.223	3.01	.220	2.99
			2	.083	.733	.868	.225	3.02	.221	2.99
			3	.090	.675	.856	.226	3.03	.222	2.98
			4	.094	.642	.850	.226	3.03	.222	2.98

$F_{X_1}(X)$ and $F_{X_2}(X)$. The distributions of $\bar{X}_1$ and $\bar{X}_2$ will change however, since we are now considering a sample size of five.

Let $F_{X_1}(X)$, $F_{X_2}(X)$ be given by (3.10.4), Chapter III, with parameters as given in Table 3.10. Then

$$\alpha_{3:\bar{X}_1} = \frac{\alpha_{3:X_1}}{\sqrt{5}} = .136$$

$$\alpha_{4:\bar{X}_1} = \frac{\alpha_{4:X_1} - 3.0}{5} + 3.0 = 3.019$$

and, similarly,

$$\alpha_{3:\bar{X}_2} = .227$$

$$\alpha_{4:\bar{X}_2} = 3.040$$

Table 4.6 gives the parameters M, S, c, and k necessary to approximate $F_{\bar{X}_1}(X)$ and $F_{\bar{X}_2}(X)$ by two Burr distributions.

Table 4.6

Burr Distribution Parameters

Distribution	M	S	c	k
$F_{\bar{X}_1}(X)$	.603	.176	4.158	6.124
$F_{\bar{X}_2}(X)$	.553	.179	3.679	6.692

In computing the criterion used to select the $\bar{X}$ test under the non-normal assumption, the technique described in detail in Chapter III was employed. That is, $F_{X_1}(X)$, its derivative (density function) $f_{X_1}(X)$, and $F_{\bar{X}_1}(X)$ were used in place of the normality assumptions for X_1 and $\bar{X}_1$ in equations (4.3.4) through (4.7.2) of this chapter. This procedure was repeated for $F_{X_2}(X)$, $f_{X_2}(X)$, and $F_{\bar{X}_2}(X)$. The resulting equations were then evaluated giving approximate results for the two moderately skewed distributions considered. It should be noted that the results obtained were used only as an indication of the effects of non-normality on the $\bar{X}$ test. Extensive investigation would be necessary to complete a thorough study of the total effects of non-normality. The author hopes to complete this study at a later date.

The results indicated that the $\bar{X}$ test handled moderately skewed lot or process distributions very well. Without exception the criteria of SDM and the moments of the outgoing distribution were quite acceptable for the two non-normal cases considered. It proved unnecessary to reproduce extensive tables for these cases, since results were quite comparable to those given in Table 4.5. However, under the normality assumption, the outgoing distribution mean M and skewness α_3 were both zero. Table 4.7 gives a range of values for M_1 and α_3 for $F_{X_2}(X)$, the more skewed of the two cases considered. It is evident from the table that M_1 and α_3 are at quite acceptable levels. Again, sorting limits of $\mu_o \pm \frac{3T}{8}$ were used.

Table 4.7.

Range Values for M_1 and α_3, Assuming $F_{X_2}(X)$, $c_1 = .19$.

	With or Without Set-Up Error	
Class (a)	-.001 to +.001	.000 to .003
Class (b)	-.002 to +.001	-.144 to +.035

4.11 Test Procedure for Randomly Acting Assignable Causes

We now will summarize the above results for the randomly acting assignable causes model by recommending a general procedure to be followed. The outline given here contains some similarities with one proposed by Burr (1967). It is assumed that the testing procedure will occur at regular (equal spaced) time intervals. The user is encouraged to test at least five times at a given μ level, on the average.

Given a tolerance T and a nominal mean value μ_o for a part, the following test plans will safely control the distribution of the part dimensions:

1) For each regular periodic sample of five parts from the process, compute the mean $\overline{X}$ and the range R.

2) The process can be considered satisfactory at this time if both of the following are met:

a) $\mu_o - .19T \leq \overline{X} \leq \mu_o + .19T$ (4.11.1)

b) $R \leq .55T$

3) The process is considered unsatisfactory at this time if either one, or both, of the requirements in step 2 is not met.

4) Appropriate corrective action in the event of step 3 should be taken, and the production sorted back to the previous sampling time, to the limits $\mu_o \pm \frac{3T}{8}$. A rejection on the R test is a clear indication that σ_X is excessive, and steps should be taken to find out why. A rejection on the $\bar{X}$ test usually indicates a change in level μ away from μ_o. The process level μ should then be reset as closely as possible to μ_o. Some allowance has been made for both reset error and biasedness, although care should be exercised in avoiding both cases.

CHAPTER V

TOOL WEAR

5.1 Introduction

In Chapters I and IV we mentioned two areas of process control under consideration: (a) tool wear, and (b) randomly acting assignable causes. We will investigate the tool wear problem in this chapter. By tool wear, we mean a manufacturing process where typically the part dimension mean μ is gradually changing in one direction. We thus have more information than in a random manufacturing process as discussed in Chapter IV. Since such a process often occurs through the gradual deterioration or "wear" of a manufacturing tool or die, a process with this characteristic is sometimes referred to as a tool wear process.

Significant contributions have been made in this area. Bhattacharjee, Pandit, and Mohan (1963) derive distributions which arise in cases where tool wear or other systematic causes must be accommodated. They indicate the application to setting exact tolerances for component parts and their assemblies. Mohan, Bhattacharji, and Mishra (1964) present a probabilistic method for analyzing machine errors when automatic resetting of the machine is done. The total error is found as a function of the inherent variability of the process, the reset magnitude, and the rate of tool wear. Hall and Eilon (1963) provide procedures for determining optimum resetting of a process average, subject to linear tool wear. Smith and Vemuganti (1968) present a model which provides a rule for tool wear process revision, considering the costs of the available actions and sample information.

In the author's work, we again assumed the field of application to be measurable component parts which combine into assemblies in a known fashion. Our goal is to develop two simple statistical tests which, if used periodically, will safely control the distribution of the part dimensions. In order to develop these tests, it was necessary to set up and solve a probabilistic model representing a tool wear process. The author is much in debt to the work of Burr, both published (1967a) and unpublished (1966), which provided direction and guidance throughout the modeling procedure. The form of the statistical tests will be patterned after those in Chapter IV, with a periodic sample of five used to construct $\bar{X}$ and R tests. The tests will be outlined in length at the end of this chapter.

5.2 Tool Wear Range Test

Because we are primarily concerned in the tool wear model with the process mean, we will again consider separately the range test necessary to control the process variability. The discussion of the range test given in Chapter IV is appropriate in the tool wear model as well. Hence we will adopt the same test: $R_5 \leq .55T$, as derived there.

Table 4.1 gives the probability of acceptance given σ_X, $P_R(a|\sigma_X)$, for $\sigma_j = .055 + .01j$, $j = 1,2,\ldots,20$.

5.3 Tool Wear Model: Introduction

We now will develop the tool wear model, with which we will determine the $\bar{X}$ test which, along with the above R test, appears to best control the distribution of the part dimensions for a sample of size five. As indicated earlier, we will assume the process mean μ has a steady drift in one direction. We can assume without loss of generality that the drift is in the positive direction. A similar model can be developed for negative drift. Past experience in the tool wear field indicates that in a great many cases the drift is approximately linear, so we can reasonably assume the tool wear drift is linear in nature. That is, let

$$\text{tool wear / unit time period} = rT,$$

for tolerance T and $0 \leq r < 1$. Here a unit time period corresponds to the time period between testing, assuming regular (equal-spaced) test intervals. In order to keep the manufacturing process mean from drifting too far from the nominal mean μ_o, we want to test often enough to keep the tool wear rate relatively low. For this reason, we considered r values of .01, .04, and .07. That is, if $\sigma_X = T/8$, $r = .01$ indicates a tool wear rate of approximately $\sigma_X/12$; $r = .04$ gives a rate of approximately $\sigma_X/3$; and $r = .07$ gives a rate slightly over $\sigma_X / 2$. The first is a slow tool wear rate, whereas one would usually not want the rate too much above the latter.

Periodic tests will be made, each with a sample of five, and the process mean reset following a rejection on the $\bar{X}$ test. Two methods of resetting were briefly mentioned in Chapter I:

1) reset down from the point of rejection by a fixed amount, say a fraction of the tolerance T; or

2) reset to a fixed point, say μ_o - aT, for a constant a.

We will incorporate each method of reset into the model, beginning first with method (1). Earlier work by Burr (1967a) indicated that a reasonable amount to reset was $2(c_1 T)$, when the $\bar{X}$ test is defined as:

$$\mu_o - c_1T \leq \bar{X} \leq \mu_o + c_1T \ . \qquad (5.3.1)$$

Following a rejection, method (1) will reset the process mean near the acceptable lower limit for $\bar{X}$, as given by (5.3.1).

5.4 Reset by a Fixed Amount

Since with the assumed positive tool wear drift the process mean will increase by an amount rT for each unit time interval, it is convenient to measure the process mean μ in units of rT, or for T=1, in units of r. Thus at time t = i with $\mu = \mu_i$, $\mu = \mu_i + r$ at t = i + 1, unless a rejection on the $\bar{X}$ test occurred at time t=i. With this unit of μ measurement in mind, define

R_i = probability of having to reset at the i^{th} μ level (i.e., the probability that the $\bar{X}$ test gives a rejection at the i^{th} test time)

σ_X = standard deviation of part dimension X.

Then, if the distribution of X is assumed normal,

$$R_i = \int_{c_1}^{\infty} \frac{e^{-\frac{(\bar{X} - ir)^2}{2\sigma_X^2/5}}}{\sqrt{2\pi}\ \sigma_X/\sqrt{5}}\ d\bar{X}\ , \tag{5.4.1}$$

for T = 1 and fixed c_1. Note that with a positive tool wear drift, we are not allowing the $\bar{X}$ test to reject if $\bar{X} \leq \mu_o - c_1T$. The reason for this is two-fold: (1) the positive drift should quickly bring the process mean above $\mu_o - c_1T$, and (2) if we rejected and tried to reset, we would have to reset to a much worse (lower) μ level, or else change the method of reset. Neither solution would be satisfactory. Since we are assuming positive drift, the danger area for the process mean is on the high side, not the low side.

Now (5.4.1) may be written as

$$R_i = 1 - \Phi\left[\left(\frac{c_1 - ir}{\sigma_X}\right)\sqrt{5}\right] \tag{5.4.2}$$

where Φ is the standard normal distribution function. Let

A_i = probability of accepting on the $\bar{X}$ test at the i^{th} μ level.

Then, of course,

$$A_i = 1 - R_i \tag{5.4.3}$$

$$= \Phi\left[\left(\frac{c_1 - ir}{\sigma_X}\right)\sqrt{5}\right]$$

Define:

P_{ij} = probability of going from the i^{th} μ level to the j^{th} μ level in one time unit.

Now at level i, we have two possibilities:

1) If we accept on the $\overline{X}$ test, after one time unit we reach μ level $(i + 1)$.

2) If we reject on the $\overline{X}$ test, we reset immediately to μ level $i - \frac{2c_1}{r}$ (remember, μ units are expressed in terms of r). Thus after one time unit we have reached μ level $i - \frac{2c_1}{r} + 1$.

In summary, then:

$$P_{j-1,j} = A_{j-1} \tag{5.4.4}$$

$$P_{j + \frac{2c_1}{r} - 1, j} = R_{j + \frac{2c_1}{r} - 1} \tag{5.4.5}$$

$$P_{ij} = 0, \quad \text{for } i \neq j-1,\ j + \frac{2c_1}{r} - 1 \tag{5.4.6}$$

Define:

π_i = the steady state, or long term, probability of being at the i^{th} μ level, given c_1, r, and σ_X.

From markov chain theory, we note that (5.4.4), (5.4.5), and (5.4.6) define a positive recurrent, a periodic markov chain. Thus a unique solution to the π_i, $\forall_i$, is given by:

$$\pi_j = \sum_{\text{all}_i} \pi_i P_{ij} \tag{5.4.7}$$

where

$$\sum_{\text{all}_i} \pi_i = 1,$$

$$\pi_i \geq 0, \quad \forall_i .$$

Substituting (5.4.4), (5.4.5), and (5.4.6) into (5.4.7) gives, for the j^{th} μ level,

$$\pi_j = (\pi_{j-1})(P_{j-1,j}) + (\pi_{j + \frac{2c_1}{r} - 1})(P_{j + \frac{2c_1}{r} - 1, j}) \quad (5.4.8)$$

Equation (5.4.8) may be evaluated recursively by setting an "appropriate" $\pi_i = 1$, and then normalizing so that $\sum_{all_i} \pi_i = 1$. The normalized π_i, $\forall_i$, can then be used to compute the a posteriori σ_μ and probability of acceptance, for example.

5.5 Evaluation of Markov Chain

It is clear from the definition of (5.4.1) that the μ index i can take on both positive and negative values. An important question in the solution of (5.4.8) is the determination of the lower and upper limits on index i for a given set of c_1, r, and σ_X. For the purposes of this thesis, it was decided to consider a range on index i such that $.0001 \leq A_i \leq .9999$, for a given set of c_1, r, and σ_X. For a N(0,1) distribution F(X), this occurs for $-3.86 \leq X \leq +3.86$. Thus by setting

$$\left(\frac{c_1 - ir}{\sigma_X}\right)\sqrt{5} = -3.86$$

and

$$\left(\frac{c_1 - ir}{\sigma_X}\right)\sqrt{5} = +3.86$$

and solving for i, we find:

$$i_L = \frac{c_1 - \sigma_X(1.726)}{r} \quad (5.5.1)$$

$$i_U = \frac{c_1 + \sigma_X (1.726)}{r} \tag{5.5.2}$$

and

$$i_U - i_L = \frac{3.452\ \sigma_X}{r} \tag{5.5.3}$$

We will assume

$$\begin{aligned} A_i &= 0, \quad \text{for } i > i_U \\ &= 1, \quad \text{for } i < i_L \end{aligned} \tag{5.5.4}$$

For example, let $\sigma_X = .065$, $r = .01$. Then:

$$i_L = 100c_1 - 11.22$$

$$i_U = 100c_1 + 11.22$$

Since we want the index i to be integer-valued, we take $i_L = 100c_1 - 11$, $i_U = 100c_1 + 11$. Now, for $i_U + 1 = 100c_1 + 12$, we are sure to reset, because of (5.5.4). On the other hand, the lowest μ level for which we have a possibility of rejection is $i_L = 100c_1 - 11$, since the probability of rejection at $i_L - 1$ is zero, again from (5.5.4). If we reject at i_L, we reset down by an amount $\frac{2c_1}{r} = 200c_1$, so in one time unit we arrive at μ level

$$(100c_1 - 11) - 200c_1 + 1 = -100c_1 - 10.$$

Thus the highest μ level we can obtain is $100c_1 + 12$, the lowest $-100c_1 - 10$, giving a total number of μ levels in this particular case as

$$(100c_1 + 12) - (-100c_1 - 10) + 1 = 200c_1 + 23 .$$

Thus in this case one can evaluate (5.4.8) by setting

$$\pi_{(-100c_1 + 13)} = 1.0 ,$$

and evaluating recursively

$$\pi_{(-100c_1 + 13 + j + 1)} = A_{(-100c_1 + 12 + j + 1)}\ \pi_{(-100c_1 + 12 + j + 1)}$$

for $j = 0,1,\ldots, 200c_1 - 2$

Using these values, find:

$$\pi_{(-100c_1 + 12-j)} = \frac{\pi_{(-100c_1 + 13-j)} - R_{(100c_1 + 12-j)}\ \pi_{(100c_1 + 12-j)}}{A_{(-100c_1 + 12-j)}} ,$$

for $j = 0,1,\ldots,22$.

A check in evaluation of the above is provided by:

$$\pi_{(-100c_1 -10)} = R_{(100c_1 - 11)}\ \pi_{(100c_1 - 11)}$$

The π_i, for $i = -100c_1 - 10$ to $100c_1 + 12$, are then normalized by dividing by $\sum_i \pi_i$.

The above evaluation procedure can be generalized as follows: Consider six σ_X values:

$$\sigma_j = .035 + .03j \ , \ j = 1,2,\ldots,6$$

For a fixed set of r and σ_j, select from Table 5.1 the appropriate constant L.

Table 5.1

L Constants Used in Evaluating Equation (5.4.8).

r \ σ_j	.065	.095	.125	.155	.185	.215
.01	11	16	21	26	31	37
.04	2	4	5	6	7	9

$\underline{r}$ = .07 was not considered in this model. Values in Table 5.1 may be found by evaluating $\left[\frac{1.726\ \sigma_j}{r}\right]$, where $[x]$ denotes the largest integer less than or equal to x. Then, for test constant c_1, set

$$\pi_{(-\frac{c_1}{r} + L + 2)} = 1.0 \tag{5.5.5}$$

Evaluate recursively:

$$\pi_{(-\frac{c_1}{r} + L + 3 + j)} = A_{(-\frac{c_1}{r} + L + 2 + j)}\ \pi_{(-\frac{c_1}{r} + L + 2 + j)}$$

$$j = 0,1,\ldots,\frac{2c_1}{r} - 2 \tag{5.5.6}$$

Using these values, find:

$$\pi_{(-\frac{c_1}{r} + L + 1 - j)} = \frac{\pi_{(-\frac{c_1}{r} + L+2-j)} - R_{(\frac{c_1}{r} + L+1-j)}\ \pi_{(\frac{c_1}{r} + L+1-j)}}{A_{(-\frac{c_1}{r} + L + 1 - j)}},$$

$$j = 0,1,\ldots,2L, \tag{5.5.7}$$

with a check of the above given by:

$$\pi_{(-\frac{c_1}{r} - L + 1)} = R_{(\frac{c_1}{r} - L)}\ L_{(\frac{c_1}{r} - L)} \tag{5.5.8}$$

Then normalize, setting

$$p_i = \frac{\pi_i}{\sum_i \pi_i} \tag{5.5.9}$$

for $i = -\frac{c_1}{r} - L + 1$ to $\frac{c_1}{r} + L + 1$. Here p_i is the long term, or steady state, probability of the i^{th} μ level occurring.

Thus far we have assumed that μ is discrete-valued, "jumping" from level i to i + 1 during one time unit. A more reasonable assumption is that μ has a uniform distribution during any time unit (i.e., this assumes the tool wear is linear, which we earlier indicated was a reasonable assumption). In order to adjust the model for this, we note:

$$\int_a^b \frac{x\,dx}{b-a} = \frac{b+a}{2}$$

$$\int_a^b \frac{x^2\,dx}{b-a} = \frac{a^2 + ab + b^2}{3}$$

Let a = i - 1, b = i,

$$M_1 = \sum_i i\,p_i$$

$$M_2 = \sum_i i^2\,p_i$$

Then the long term, or steady state, mean and variance of the process mean are given by:

$$E(\mu) = r\left[\sum_i p_i \int_{i-1}^{i} x\,dx\right] \tag{5.5.10}$$

$$= r\,(M_1 - .5)$$

$$\sigma_\mu^2 = r^2 \left[\sum_i p_i \int_{i-1}^{i} x^2 \, dx \right] - [E(\mu)]^2 \qquad (5.5.11)$$

$$= r^2(M_2 - M_1 + 1/3) - [E(\mu)]^2 ,$$

where we multiply by r and r^2 respectively to change the unit of measurement back in terms of T = 1 instead of in terms of r.

5.6 Derivation of Moments

As in the previous models, we want to evaluate the first four moments of the outgoing distribution. We again will consider two classes of outgoing distributions:

a) all material, whether accepted or rejected by the periodic testing, is retained; or

b) the material rejected is sorted 100 percent to fixed limits, and then combined with the accepted material.

Class (a) then makes no allowance for sorting of rejected material whatsoever. A comparison of the moments between class (a) and class (b) outgoing distributions will indicate whether 100 percent sorting of rejected material makes a significant contribution to the outgoing distribution. This, of course, is of interest, since sorting can be both time consuming and expensive. Recall that sorting in this case means sorting only the material since the previous testing procedure.

Consider first a class (a) distribution, where no sorting occurs. Let $E_a[(X - \mu_o)^m] = m^{th}$ moment of a class (a) distribution, where the a denotes class (a). Then:

$$E_a[(X - \mu_o)^m] = \tag{5.6.1}$$

$$\sum_{i=-\frac{c_1}{r}-L+1}^{\frac{c_1}{r}+L+1} p_i \int_{(i-1)r}^{ir} \frac{d\mu}{r} \int_{-\infty}^{\infty} \frac{(X - \mu_o)^m e^{-\frac{(X-\mu)^2}{2\sigma_X^2}}}{\sqrt{2\pi}\,\sigma_X} dX$$

let $t = \dfrac{X - \mu}{\sigma_X}$

Then (5.6.1) simplifies to:

$$E_a[(X - \mu_o)^m] =$$

$$\sum_{i=-\frac{c_1}{r}-L+1}^{\frac{c_1}{r}+1+1} p_i \int_{(i-1)r}^{ir} \frac{d\mu}{r} \int_{-\infty}^{\infty} \sigma_X^m (t + \delta)^m \phi(t)\, dt\ , \tag{5.6.2}$$

where $\delta = \dfrac{\mu - \mu_o}{\sigma_X}$

$\phi(t)$ = standardized normal density function

Define

$$I_m = \int_{-\infty}^{\infty} (t + \delta)^m \phi(t)\, dt \quad . \tag{5.6.3}$$

Then, $I_1 = \delta$

$I_2 = 1 + \delta^2$

$I_3 = 3\delta + \delta^3$

$I_4 = 3 + 6\delta^2 + \delta^4$

Define $y_i = ir$, for $i = -\frac{c_1}{r} - L$ to $\frac{c_1}{r} + L + 1$. Then, letting $\mu_o = 0$, substituting (5.6.3) into (5.6.2), and evaluating the integral over μ, we find:

$$E_a[(X - \mu_o)^m] = \sum_{i=-\frac{c_1}{r} - L + 1}^{\frac{c_1}{r} + L + 1} p_i \, W_{im} \quad , \tag{5.6.4}$$

where

$$W_{i1} = \frac{y_i^2 - y_{i-1}^2}{2r} \tag{5.6.5}$$

$$W_{i2} = \sigma_X^2 + \frac{y_i^3 - y_{i-1}^3}{3r} \tag{5.6.6}$$

$$W_{i3} = \frac{3\sigma_X^2}{2r} \left(y_i^2 - y_{i-1}^2\right) + \frac{y_i^4 - y_{i-1}^4}{4r} \tag{5.6.7}$$

$$W_{i4} = 3\sigma_X^4 + \frac{2\sigma_X^2}{r} \left(y_i^3 - y_{i-1}^3\right) + \frac{y_i^5 - y_{i-1}^5}{5r} \tag{5.6.8}$$

If we define $M_m = E_a[(X - \mu_o)^m]$, $m = 1,2,3,4$, then the outgoing distribution mean, standard deviation, skewness, and kurtosis are given respectively by:

$$M_1 \tag{5.6.9}$$

$$SD = (M_2 - M_1^2)^{\frac{1}{2}}$$

$$\alpha_3 = \frac{M_3 - 3\,M_2\,M_1 + 2\,M_1^3}{(SD)^3}$$

$$\alpha_4 = \frac{M_4 - 4\,M_3\,M_1 + 6\,M_2\,M_1^2 - 3\,M_1^4}{(SD)^4}$$

Consider now class (b) outgoing distributions, where rejected material is sorted 100 percent to $\mu_o \pm KT$, and then combined with the accepted material. We will again consider $K = 3/8$ and $1/2$, and assume $T = 1.0$. Define

$E_{b,K,R}[(X - \mu_o)^m] = m^{th}$ moment of the class (b) outgoing distribution, considering only the rejected material.

Then:

$$E_{b,K,R}[(X - \mu_o)^m] = \qquad (5.6.10)$$

$$\frac{\displaystyle\sum_{i=-\frac{c_1}{r}-L+1}^{\frac{c_1}{r}+L+1} p_i R_i \int_{(i-1)r}^{ir} \frac{d\mu}{r} \int_{\mu_o-K}^{\mu_o+K} \frac{(X - \mu_o)^m \, e^{-\frac{(X-\mu)^2}{2\sigma_X^2}}}{\sqrt{2\pi}\,\sigma_X} \, dX}{\displaystyle\sum_{i=-\frac{c_1}{r}-L+1}^{\frac{c_1}{r}+L+1} p_i \, R_i \int_{\mu_o-K}^{\mu_o+K} \frac{e^{-\frac{(X-\mu)^2}{2\,\sigma_X^2}}}{\sqrt{2\pi}\,\sigma_X} \, dX}$$

where

$$\delta = \frac{\mu - \mu_o}{\sigma_X}$$

$$U = \frac{K}{\sigma_X} - \delta$$

$$L = -\frac{K}{\sigma_X} - \delta$$

Then (5.6.10) may be written as:

$$E_{b,K,R}[(X - \mu_o)^m] = \tag{5.6.11}$$

$$\frac{\displaystyle\sum_{i=-\frac{c_1}{r}-L+1}^{\frac{c_1}{r}+L+1} p_i\, R_i \int_{(i-1)r}^{ir} \frac{d\mu}{r} \int_L^U \sigma_X^m\, (t+\delta)^m\, \phi(t)\, dt}{\displaystyle\sum_{i=-\frac{c_1}{r}-L+1}^{\frac{c_1}{r}+L+1} p_i\, R_i \int_L^U \phi(t)\, dt}$$

Define

$$I_{m,K}(\mu) = \int_L^U \sigma_X^m (t+\delta)^m\, \phi(t)\, dt \tag{5.6.12}$$

$$P = \Phi(U) - \Phi(L), \text{ for}$$

Φ = standard normal distribution function

Then:

$$I_{1,K}(\mu) = \sigma_X[\phi(L) - \phi(U) + \delta P]$$

$$I_{2,K}(\mu) = \sigma_X^2\, [P(1+\delta^2) + L\phi(U) - U\,\phi(L)]$$

$$I_{3,K}(\mu) = \sigma_X^3 \left[\left(\frac{K^2}{\sigma_X^2} - \frac{K\delta}{\sigma_X} + \delta^2 + 2\right)\phi(L) - \left(\frac{K^2}{\sigma_X^2} + \frac{K\delta}{\sigma_X} + \delta^2 + 2\right)\phi(U) + P(3\delta + \delta^3)\right]$$

$$I_{4,K}(\mu) = \sigma_X^4 \Big[\Big(- \frac{K^3}{\sigma_X^3} + \frac{K^2\delta}{\sigma_X^2} - \frac{K\delta^2}{\sigma_X} + \delta^3 - \frac{3K}{\sigma_X} + 5\delta\Big) \phi(L)$$

$$- \Big(\frac{K^3}{\sigma_X^3} + \frac{K^2\delta}{\sigma_X^2} + \frac{K\delta^2}{\sigma_X} + \delta^3 + \frac{3K}{\sigma_X} + 5\delta\Big) \phi(U)$$

$$+ P(3 + 6\delta^2 + \delta^4)\Big]$$

Equation (5.6.11) may now be written as:

$$E_{b,K,R}[(X - \mu_o)^m] = \frac{\displaystyle\sum_{i=-\frac{c_1}{r} - L + 1}^{\frac{c_1}{r} + L + 1} p_i \; R_i \int_{(i-1)r}^{ir} \frac{[I_{m,K}(\mu)]}{r} \, d\mu}{\displaystyle\sum_{i=-\frac{c_1}{r} - L + 1}^{\frac{c_1}{r} + L + 1} p_i \; R_i \; P_i} \tag{5.6.13}$$

Since $I_{m,K}(\mu)$ is a function of μ through δ, U, and L, we will approximate the integral over μ by using Simpson's Rule with n = 2. Let

$$Z_{i,m,K} = \int_{(i-1)r}^{ir} \frac{[I_{m,K}(\mu)]}{r} \, d\mu \tag{5.6.14}$$

$$\Delta\mu = \frac{ir - (i - 1)r}{n} = \frac{r}{2}$$

Then:

$$Z_{i,m,K} = \frac{1}{6}\left[I_{m,K}(y_{i-1}) + 4\, I_{m,K}(y_{i-1} + \Delta\mu) + I_{m,K}(y_i)\right], \qquad (5.6.15)$$

recalling the definition of y_i. Here note that the δ, P, U, and L defined following (5.6.10) and (5.6.12) should now be written as functions of i: δ_i, P_i, U_i, and L_i, since we are using the y_i as discrete values of μ. Thus

$$E_{b,K,R}[(X - \mu_o)^m] = \qquad (5.6.16)$$

$$\frac{\displaystyle\sum_{i=-\frac{c_1}{r}-L+1}^{\frac{c_1}{r}+L+1} p_i\, R_i\, Z_{i,m,K}}{\displaystyle\sum_{i=-\frac{c_1}{r}-L+1}^{\frac{c_1}{r}+L+1} p_i\, R_i\, P_i}$$

Define

$E_b[(X - \mu_o)^m]$ = m^{th} moment of the class (b) outgoing distribution (i.e., combining accepted plus sorted-rejected material)

Then for m = 1, 2, 3, 4:

$$E_b[(X - \mu_o)^m] = \frac{\displaystyle\sum_{i=-\frac{c_1}{r} - L + 1}^{\frac{c_1}{r} + L + 1} p_i \left[A_i W_{im} + R_i Z_{i,m,K}\right]}{\displaystyle\sum_{i=-\frac{c_1}{r} - L + 1}^{\frac{c_1}{r} + L + 1} p_i \left[A_i + R_i P_i\right]} \tag{5.6.17}$$

If we define $MM_m = E_b[(X - \mu_o)^m]$, $m = 1,2,3,4$ then the class (b) outgoing distribution mean, standard deviation, skewness, and kurtosis are given respectively by:

$$MM_1$$

$$SD = [MM_2 - (MM_1)^2]^{\frac{1}{2}} \tag{5.6.18}$$

$$\alpha_3 = \frac{MM_3 - 3\, MM_2\, MM_1 + 2(MM_1)^3}{(SD)^3}$$

$$\alpha_4 = \frac{MM_4 - 4\, MM_3\, MM_1 + 6\, MM_2(MM_1)^2 - 3(MM_1)^4}{(SD)^4}$$

5.7 Evaluation of Test Criteria

In order to find the most desirable c_1 value for the $\overline{X}$ test with a sample size of five, (5.5.10), (5.5.11), (5.6.9), and (5.6.18) were evaluated, for:

$c_1 = .14(.02)\ .20$

$r = .01, .04$ (for $r = .07$, see second half of this chapter)

$\sigma_X = \sigma_j = .035 + .03j,\ j = 1,2,\ldots,6$

$K = .375, .500$

One can easily show that $E(\mu)$ given by (5.5.10) is the same as M_1 given by (5.6.9). These values will be listed under M_1. Let SDM represent the a posteriori standard deviation of the process mean, given by (5.5.11). The remainder of the notation used is defined by (5.6.9) and (5.6.18). Table 5.2 gives the values for the above defined criteria under the normality assumption. The table includes both class (a) and class (b) outgoing distributions, as previously defined. For a tool wear drift of $r = .04$, it was found that $c_1 = .14$ gives the most desirable results. If $r = .01$, $c_1 = .16$ is recommended. If $.01 < r < .04$, a reasonable constant value for the $\bar{X}$ test would be $c_1 = .15$.

The following conclusions may be drawn from Table 5.2:

1) SDM remains in good control, even for very excessive a priori σ_X.

2) The outgoing distribution skewness and kurtosis are quite satisfactory in all cases.

3) The outgoing distribution mean and standard deviation are reasonably in control for $\sigma_X \leq .155$, beyond which the mean is a bit too negative and the standard deviation is too large (note that SD > σ_X, always.) However, the

Table 5.2. Summary of Test Criteria for Tool Wear Model, Assuming Reset by a Fixed Amount (Normal Case).

				Class (a) NO SORTING			
c_1	r	σ_i	SDM	M_1	SD	α_3	α_4
.14	.04	.065	.086	.014	.095	.035	2.84
		.095	.089	.008	.120	.007	2.95
		.125	.093	.000	.147	-.007	2.99
		.155	.097	-.009	.175	-.012	3.00
		.185	.102	-.019	.204	-.015	3.00
		.215	.106	-.030	.234	-.015	3.00
.16	.01	.065	.095	-.014	.115	-.002	2.50
		.095	.097	-.030	.135	-.004	2.74
		.125	.099	-.048	.159	-.005	2.87
		.155	.102	-.068	.185	-.005	2.93
		.185	.105	-.088	.212	-.006	2.96
		.215	.108	-.110	.240	-.006	2.97

			Class (b) WITH SORTING							
			K = .375				K = .500			
c_1	r	σ_i	MM_1	SD	α_3	α_4	MM_1	SD	α_3	α_4
.14	.04	.065	.014	.095	.034	2.84	.014	.095	.035	2.84
		.095	.008	.119	-.012	2.91	.008	.120	.006	2.95
		.125	-.001	.145	-.054	2.90	.000	.147	-.014	2.97
		.155	-.011	.172	-.076	2.92	-.009	.174	-.034	2.95
		.185	-.023	.199	-.082	2.94	-.020	.202	-.049	2.94
		.215	-.035	.227	-.081	2.98	-.032	.230	-.057	2.94
.16	.01	.065	-.014	.115	-.002	2.50	-.014	.115	-.002	2.50
		.095	-.030	.135	-.008	2.73	-.030	.135	-.004	2.74
		.125	-.049	.159	-.015	2.85	-.048	.159	-.006	2.86
		.155	-.068	.185	-.020	2.90	-.068	.185	-.010	2.92
		.185	-.089	.211	-.023	2.94	-.089	.212	-.013	2.94
		.215	-.111	.239	-.024	2.96	-.110	.240	-.016	2.96

range test will tend to reject such excessive variation, leading to an investigation as to the causes of such excessive variation. Thus the combination of the $\overline{X}$ and R tests will provide good control.

4) Little difference is gained in 100 percent sorting of rejected material, nor is there much difference in sorting to $\mu_o \pm \frac{3T}{8}$ or $\mu_o \pm \frac{T}{2}$ when one does sort. It is concluded that in most cases, sorting is not necessary. A summary of test results and conclusions will be given following a brief discussion of the effects of non-normal process distributions on the $\overline{X}$ test.

5.8 Reset by a Fixed Amount: Non-normal Case

In order to consider the effects of moderately skewed process distributions on the $\overline{X}$ test, it is necessary only to recall the derivation and procedures used in Chapters III and IV. Since the sample size considered is five, the procedure used is identical to that given in Chapter IV; in place of the normality assumption used in equations (5.4.1) through (5.6.17) of this chapter, use the two sets (i.e., $F_{X_1}(X)$, $f_{X_1}(X)$, $F_{\overline{X}_1}(X)$ or $F_{X_2}(X)$, $f_{X_2}(X)$, $F_{\overline{X}_2}(X)$) of moderately skewed Burr distributions given by (3.10.4), Chapter III, with parameters as given in Tables 3.10 and 4.8. By using first one set and then the other, we get some indication of how well the $\overline{X}$ test handles moderately skewed distributions. Table 5.3 gives the values of the various criteria used to find the most desirable c_1 for the $\overline{X}$ test, with a sample size

Table 5.3. Summary of Test Criteria for Tool Wear Model, Assuming Reset by a Fixed Amount (Non-normal Case).

				Skew 1			
				Class (a) (NO SORTING)			
c_1	r	σ_i	SDM	M_1	SD	α_3	α_4
.14	.04	.065	.086	.014	.095	.033	2.84
		.095	.089	.008	.120	.004	2.95
		.125	.094	.000	.147	-.010	2.99
		.155	.098	-.009	.176	-.015	3.00
		.185	.103	-.019	.205	-.017	3.00
		.215	.108	-.030	.234	-.017	3.00
.16	.01	.065	.95	-.015	.115	-.003	2.50
		.095	.97	-.031	.136	-.006	2.74
		.125	.100	-.050	.160	-.007	2.87
		.155	.103	-.070	.186	-.009	2.93
		.185	.106	-.092	.213	-.009	2.96
		.215	.110	-.114	.242	-.010	2.98
				Skew 2			
.14	.04	.065	.086	.014	.095	.032	2.84
		.095	.090	.008	.120	.002	2.96
		.125	.094	-.000	.147	-.011	2.99
		.155	.099	-.009	.176	-.017	3.00
		.185	.104	-.020	.205	-.018	3.00
		.215	.108	-.031	.234	-.018	3.00
.16	.01	.065	.095	-.015	.115	-.004	2.50
		.095	.097	-.032	.136	-.006	2.75
		.125	.100	-.051	.160	-.008	2.87
		.155	.103	-.072	.186	-.010	2.93
		.185	.107	-.094	.214	-.010	2.96
		.215	.111	-.117	.242	-.011	2.98

Table 5.3. (cont.)

Skew 1

			Class (b) WITH SORTING							
			K = .375				K = .500			
c_1	r	σ_1	MM_1	SD	α_3	α_4	MM_1	SD	α_3	α_4
.14	.04	.065	.014	.095	.031	2.84	.014	.095	.033	2.84
		.095	.008	.119	-.020	2.90	.008	.120	.002	2.95
		.125	-.001	.145	-.062	2.90	-.000	.147	-.021	2.96
		.155	-.012	.172	-.082	2.92	-.010	.175	-.043	2.94
		.185	-.024	.200	-.085	2.95	-.021	.203	-.058	2.93
		.215	-.035	.227	-.080	2.99	-.033	.231	-.065	2.94
.16	.01	.065	-.015	.115	-.004	2.50	-.015	.155	-.003	2.50
		.095	-.031	.136	-.011	2.73	-.031	.136	-.006	2.74
		.125	-.050	.159	-.019	2.85	-.050	.160	-.010	2.86
		.155	-.071	.185	-.024	2.91	-.070	.186	-.014	2.92
		.185	-.093	.212	-.026	3.94	-.092	.213	-.018	2.94
		.215	-.115	.240	-.028	2.97	-.114	.241	-.020	2.96

Skew 2

.14	.04	.065	.014	.095	.028	2.83	.014	.095	.032	2.84
		.095	.007	.119	-.025	2.89	.008	.120	-.001	2.94
		.125	-.002	.146	-.064	2.90	-.000	.147	-.027	2.95
		.155	-.012	.173	-.082	2.92	-.010	.175	-.047	2.94
		.185	-.024	.201	-.083	2.95	-.022	.203	-.061	2.93
		.215	-.036	.228	-.078	2.99	-.034	.231	-.067	2.94
.16	.01	.065	-.015	.115	-.005	2.50	-.015	.115	-.004	2.50
		.095	-.032	.136	-.013	2.73	-.032	.136	-.007	2.74
		.125	-.051	.160	-.020	2.85	-.051	.160	-.012	2.86
		.155	-.073	.186	-.025	2.91	-.072	.186	-.016	2.91
		.185	-.095	.213	-.026	2.94	-.094	.213	-.020	2.94
		.215	-.118	.241	-.027	2.97	-.117	.241	-.022	2.96

of five. The two sets of Burr distributions are denoted in the tables by "Skew 1" and "Skew 2", respectively. Conclusions similar to those drawn for the normal case are also applicable in the non-normal cases considered. SDM, the a posteriori σ_μ, is slightly larger, with SDM $\doteq$ T/9 when σ_j = .215T. However, variation in σ_X this large will quickly be detected by the range test, so this is not likely to cause problems.

5.9 Test Procedure When Resetting by a Fixed Amount

We will summarize the results by giving an outlined procedure to be followed when testing a tool wear process. Let T be the tolerance and μ_o the nominal mean value for a part. Then the following procedure will safely control the distribution of the part dimensions:

1) For each regular (equal spaced) periodic sample of five parts from the process, find the average $\bar{X}$ and the range R.

2) Samples should be taken often enough such that the tool wear rate r between testing procedures is $.01T \leq r \leq .04T$. If the tool wear rate r is unknown, assume r = .04T until enough past history has been observed to estimate r.

3) The process is considered satisfactory at this time if both of the following are met:

a) $\mu_o - c_1 T \leq \bar{X} \leq \mu_o + c_1 T$ (5.9.1)

b) $R \leq .55T$

where if $r \doteq .01T$, $c_1 = .16$

$\doteq .02T$, $c_1 = .15$

$\doteq .04T$, $c_1 = .14$.

4) If either or both of the requirements in step 3 is not met, action is required. If R is excessive, sort the recent product looking for the source of the excess variability. If $\bar{X}$ is beyond the upper limit if μ increases, or the lower limit if μ decreases, then reset by an amount $2c_1T$. Reset down if μ increases, up if μ decreases.

5) 100 percent sorting of rejected material is not necessary, if rejected by the $\bar{X}$ test.

5.10 Resetting to a Fixed Point: Introduction

At the beginning of this chapter, in introducing the tool wear model, two methods of resetting were mentioned. The first, resetting by a fixed amount, has been used in the model presented in the first half of this chapter. We now will consider the second method of reset: resetting to a fixed point, say μ_o - aT, for some constant a (assuming positive tool wear drift). In an earlier model by Burr (1967a), the constant a was given as .1. We will consider a values of .050 (.025) .150. The tool wear rate per unit time interval will again be assumed to be rT, with r = .01, .04, .07. Initially, let $\mu = \mu_o - aT$ at time t = 0. Assume T = 1.0. Since μ increases by an amount r during every unit time interval, the μ level at time t is $\mu_o - a + tr$. We are interested in the probability of acceptance on the $\bar{X}$ test at time t, say $P_{\bar{X}}(a \mid t)$. Assuming normality,

$$P_{\bar{X}}(a|t) = \int_{\mu_o - c_1}^{\mu_o + c_1} \frac{e^{-\frac{(\bar{X} - \mu_o + a - tr)^2}{2\sigma_X^2/\sqrt{5}}}}{\sqrt{2\pi}\ \sigma_X/\sqrt{5}} \, dX$$

$$= \Phi\left[\left(\frac{a + c_1 - tr}{\sigma_X}\right)\sqrt{5}\right] - \Phi\left[\left(\frac{a - c_1 - tr}{\sigma_X}\right)\sqrt{5}\right] \qquad (5.10.1)$$

for fixed a, c_1, r, and σ_X, and where

Φ = standard normal distribution function.

Note that (5.10.1) is independent of the nominal mean μ_o.

An important question now is: "What is the probability of a time interval (run) of length h occurring"? A time interval has length h if the $\bar{X}$ test is accepted for the first (h-1) periodic tests, and rejected on the h^{th} $\bar{X}$ test. Let

$p_a(h)$ = probability of a time interval of length h

It should be pointed out that beginning here and throughout the model, the definitions and derivations are for fixed values of c_1, r, σ_X, and a. However, we will continue to use a as a subscript throughout the notation. The reason for this will become clear later. Using (5.10.1),

$$p_a(h) = \left[\prod_{t=1}^{h-1} P_{\bar{X}}(a|t)\right]\left[1 - P_{\bar{X}}(a|h)\right] \qquad (5.10.2)$$

$$= \left\{\prod_{t=1}^{h-1}\left[\Phi\left(\frac{a + c_1 - tr}{\sigma_X/\sqrt{5}}\right) - \Phi\left(\frac{a - c_1 - tr}{\sigma_X/\sqrt{5}}\right)\right]\right\}$$

$$\left\{1 - \Phi\left(\frac{a + c_1 - hr}{\sigma_X/\sqrt{5}}\right) + \Phi\left(\frac{a - c_1 - hr}{\sigma_X/\sqrt{5}}\right)\right\}$$

Note that

$$\sum_{h=1}^{\infty} p_a(h) = 1 \ . \tag{5.10.3}$$

We now would like to solve for the a posteriori $E(\mu \mid a)$ and $\sigma^2_{\mu|a}$.

$$E(\mu|a) = \sum_{h=1}^{\infty} p_a(h) \text{ (average } \mu \text{ value up to time h)}$$

$$= \sum_{h=1}^{\infty} p_a(h) \left[\int_{\mu_o-a}^{\mu_o-a+hr} \frac{\mu \, d\mu}{hr} \right]$$

$$= \sum_{h=1}^{\infty} p_a(h) \left(\mu_o - a + \frac{rh}{2}\right)$$

$$= \mu_o - a + \frac{r}{2} \sum_{h=1}^{\infty} h \, p_a(h)$$

$$= \mu_o - a + \frac{r}{2} E(h), \tag{5.10.4}$$

for $E(h) = \sum_{h=1}^{\infty} h \, p_a(h)$.

Similarly,

$$E(\mu^2 \mid a) = \sum_{h=1}^{\infty} p_a(h) \left[\int_{\mu_o-a}^{\mu_o-a+hr} \frac{\mu^2}{hr} \, d\mu \right]$$

Recalling that

$$\int_c^d \frac{x^2 \, dx}{d - c} = \frac{d^2 + d\,c + c^2}{3}$$

then for $z = \mu_o - a$,

$$E(\mu^2|a) = \sum_{h=1}^{\infty} p_a(h) \left[\frac{(z+hr)^2 + z(z+hr) + z^2}{3} \right]$$

$$= \sum_{h=1}^{\infty} p_a(h) \left[z^2 + z\,h\,r + \frac{h^2 r^2}{3} \right]$$

$$= z^2 + zr \sum_{h=1}^{\infty} h\, p_a(h) + \frac{r^2}{3} \sum_{h=1}^{\infty} h^2\, p_a(h) \qquad (5.10.5)$$

Then for $E(h^2) = \sum_{h=1}^{\infty} h^2\, p_a(h)$,

$$\sigma^2_{\mu|a} = E(\mu^2 \mid a) - \left[E(\mu \mid a)\right]^2$$

$$= z^2 + zr\,E(h) + \frac{r^2}{3} E(h^2)$$

$$- \left[z^2 + zr\,E(h) + \frac{r^2}{4} (E(h))^2 \right]$$

So

$$\sigma^2_{\mu|a} = \frac{r^2}{12} \left\{ 4\,E(h^2) - 3 \left[E(h)\right]^2 \right\} . \qquad (5.10.6)$$

Thus by knowing the point to which we reset each time a rejection occurs we can directly compute the a posteriori $E(\mu|a)$ and $\sigma^2_{\mu|a}$.

5.11 Resetting to a Fixed Point: Reset Error

Whenever we reset by aiming for a particular value of μ, we always have the possibility of reset error. Suppose we reset to μ_o - a with probability .6, and to $(\mu_o - a) \pm .025$ with probability .2 each. We may then recompute $E(\mu|a)$ and $\sigma^2_{\mu|a}$ for a = .075, .100, .125. Let

$E(\mu^m, a)$ = a posteriori m^{th} moment of the process mean, assuming the reset error given above, $m = 1,2$

$V(\mu, a)$ = a posteriori variance of the process mean, assuming reset error

Then:

$$E(\mu,a) = .2\left[E(\mu \mid a - .025) + E(\mu \mid a + .025)\right] + .6\left[E(\mu \mid a)\right] \tag{5.11.1}$$

for a = .075, .100, .125.

Now

$$E(\mu^2 \mid a) = \sigma^2_{\mu \mid a} + [E(\mu \mid a)]^2$$

Hence

$$E(\mu^2, a) = .6\left\{\sigma^2_{\mu \mid a} + [E(\mu \mid a)]^2\right\} + .2\left\{\sigma^2_{\mu|a-.025} + \left[E(\mu|a-.025)\right]^2 + \sigma^2_{\mu|a+.025} + \left[E(\mu|a+.025)\right]^2\right\} \tag{5.11.2}$$

Thus

$$V(\mu, a) = E(\mu^2, a) - [E(\mu, a)]^2 \tag{5.11.3}$$

for a = .075, .100, .125.

It should now be clear why it was necessary to explicitly denote the constant a throughout the notation.

5.12 Derivation of Moments when Resetting to a Fixed Point

We now wish to evaluate the first four moments of the outgoing distribution. We again will consider two classes of outgoing distributions, as given earlier in this chapter:

a) all material, whether accepted or rejected by the periodic testing, is retained; or

b) the rejected material is sorted 100 percent to fixed limits, and then combined with the accepted material.

By comparing class (a) with class (b), we can determine whether 100 percent sorting of rejected material is worthwhile.

Consider first a class (a) distribution, where no sorting occurs. To avoid notational confusion between the (a) in class (a) distributions and the constant a used in the model, we will use A to denote class (a) whenever necessary.

Define:

$E_A[(X - \mu_o)_a^m] = m^{th}$ moment of the class (a) distribution, given c_1, r, σ_X, and a.

Now the $p_a(h)$ give the probability of a time interval of length h occurring. If we can find the m^{th} moment for each time length, from (j-1) to j, $\forall_j$, then we can sum over all $j = 1,\ldots,\infty$ and thus find the overall m^{th} moment $E_A[(X - \mu_o)_a^m]$. Define

$q_a(i)$ = probability of the process mean μ being

$$\mu_o - a + (i-1)r \leq \mu \leq \mu_o - a + ir, \; i=1,\ldots,\infty$$

The $q_a(i)$ may be computed at least two (equivalent) ways. For example, one can easily see that:

$$q_a(1) = 1$$
$$q_a(2) = 1 - p_a(1)$$
$$\vdots$$
$$q_a(i) = 1 - \sum_{j=1}^{i-1} p_a(j) \qquad (5.12.1)$$
$$\vdots$$

The i^{th} term may also be expressed as:

$$q_a(i) = q_a(i-1)\left\{\Phi\left[\left(\frac{a+c_1 - (i-1)r}{\sigma_X}\right)\sqrt{5}\right] - \Phi\left[\left(\frac{a-c_1 - (i-1)r}{\sigma_X}\right)\sqrt{5}\right]\right\}$$

(5.12.2)

Then:

$$E_A\left[(X - \mu_o)_a^m\right] =$$

$$\frac{\sum_{i=1}^{\infty} q_a(i)\int_{\mu_o-a(i-1)r}^{\mu_o-a+ir} \frac{d\mu}{r}\int_{-\infty}^{\infty} \frac{(X-\mu_o)^m e^{-\frac{(X-\mu)^2}{2\sigma_X^2}}}{\sqrt{2\pi}\ \sigma_X} dX}{\sum_{i=1}^{\infty} q_a(i)}$$

$$= \frac{\sum_{i=1}^{\infty} q_a(i)\int_{\mu_o-a+(i-1)r}^{\mu_o-a+ir} \frac{d\mu}{r}\int_{-\infty}^{\infty} \sigma_X^m\ (t+\delta)^m\ \phi(t)\ dt}{\sum_{i=1}^{\infty} q_a(i)} \qquad (5.12.3)$$

for

$$\delta = \frac{\mu - \mu_o}{\sigma_X}$$

$\phi(t)$ = standard normal density function

Now, letting

$$I_m = \int_{-\infty}^{\infty} (t + \delta)^m \phi(t)\, dt$$

$$y_{ai} = -a + ir, \; i = 0,1,\ldots$$

and setting $\mu_o = 0$, we can reduce (5.12.3) to:

$$E_A\left[(X - \mu_o)_a^m\right] = \frac{\sum_{i=1}^{\infty} q_a(i)\, W_{ma}(i)}{\sum_{i=1}^{\infty} q_a(i)}, \tag{5.12.4}$$

where

$$W_{ia}(i) = \frac{y_{ai}^2 - y_{a,i-1}^2}{2r}$$

$$W_{2a}(i) = \sigma_X^2 + \frac{y_{ai}^3 - y_{a,i-1}^3}{3r} \tag{5.12.5}$$

$$W_{3a}(i) = \frac{3\sigma_X^2}{2r}\left(y_{ai}^2 - y_{a,i-1}^2\right) + \frac{y_{ai}^4 - y_{a,i-1}^4}{4r}$$

$$W_{4a}(i) = 3\sigma_X^4 + \frac{2\sigma_X^2}{r}\left(y_{ai}^3 - y_{a,i-1}^3\right) + \frac{y_{ai}^5 - y_{a,i-1}^5}{5r}$$

If we define $M_m = E_A\left[(X - \mu_o)_a^m\right]$, $m = 1,2,3,4$, then the outgoing distribution mean, standard deviation, skewness, and kurtosis are given by equations (5.6.9) listed earlier in this chapter.

If we now consider the case of set-up error, then

$$E_A\left[(X-\mu_o)_a^m\right] = \tag{5.12.6}$$

$$\frac{.2\left[\sum_{i=1}^{\infty} q_{(a-.025)}(i)\,W_{m(a-.025)}(i) + \sum_{i=1}^{\infty} q_{(a+.025)}(i)\,W_{m(a+.025)}(i)\right] + .6\left[\sum_{i=1}^{\infty} q_a(i)W_{ma}(i)\right]}{.2\left[\sum_{i=1}^{\infty} q_{(a-.025)}(i) + \sum_{i=1}^{\infty} q_{(a+.025)}(i)\right] + .6\left[\sum_{i=1}^{\infty} q_a(i)\right]}$$

The outgoing distribution mean, standard deviation, skewness, and kurtosis may also be found in the case of set-up error.

Consider now class (b) distributions, where the rejected material is sorted 100 percent to limits $\mu_o \pm KT$, and then combined with the accepted material. Define

$PA_a(i)$ = probability of accepting on the $\bar{X}$ test at time t=i

$PR_a(i)$ = probability of rejecting on the $\bar{X}$ test at time t=i.

Implicit in the above definitions is the assumption that the process has been accepted on all $\bar{X}$ tests through time t = i - 1.

$E_{b,K,R}\left[(X-\mu_o)_a^m\right]$ = m^{th} moment of the class (b) outgoing distribution, considering only the rejected material.

Then:

$$PA_a(1) = \Phi\left[\left(\frac{a + c_1 - r}{\sigma_X}\right)\sqrt{5}\right] - \Phi\left[\left(\frac{a - c_1 - r}{\sigma_X}\right)\sqrt{5}\right]$$

$$= q_a(2)$$

In general,

$$PA_a(i) = q_a(i+1), \; i=1,2,\ldots \tag{5.12.7}$$

Now

$$PR_a(i) = p_a(i) \; , \; i=1,2,\ldots \tag{5.12.8}$$

for $p_a(i)$ given in (5.10.2).

Then

$$E_{b,K,R}\left[(X - \mu_o)_a^m\right] =$$

$$\frac{\sum_{i=1}^{\infty} p_a(i) \int_{\mu_o-a+(i-1)r}^{\mu_o-a+ir} \frac{d\mu}{r} \int_{\mu_o-K}^{\mu_o+K} \frac{(X - \mu_o)^m \, e^{-\frac{(X-\mu)^2}{2\sigma_X^2}}}{\sqrt{2\pi}\,\sigma_X} \, dX}{\sum_{i=1}^{\infty} p_a(i) \int_{\mu_o-a+(i-1)r}^{\mu_o-a+ir} \frac{d\mu}{r} \int_{\mu_o-K}^{\mu_o+K} \frac{e^{-\frac{(X-\mu)^2}{2\sigma_X^2}}}{\sqrt{2\pi}\,\sigma_X} \, dX}$$

$$= \frac{\sum_{i=1}^{\infty} p_a(i) \int_{\mu_o-a+(i-1)r}^{\mu_o-a+ir} \frac{d\mu}{r} \left[I_{m,K}(\mu)\right]}{\sum_{i=1}^{\infty} p_a(i) \int_{\mu_o-a+(i-1)r}^{\mu_o-a+ir} \frac{d\mu}{r} \int_L^U \phi(t) \, dt} \; , \tag{5.12.9}$$

where

$$I_{m,K}(\mu) = \int_L^U \sigma_X^m \, (t + \delta)^m \, \phi(t) \, dt \tag{5.12.10}$$

for $$t = \frac{X - \mu}{\sigma_X}$$

$$\delta = \frac{\mu - \mu_o}{\sigma_X}$$

$$U = \frac{K}{\sigma_X} - \delta$$

$$L = -\frac{K}{\sigma_X} - \delta$$

$\phi(t)$ = standard normal density function

Let $P = \Phi(U) - \Phi(L)$, and evaluate (5.12.10) using the procedure given following equation (5.6.12) earlier in this chapter. Let

$$Z_{a,m,K}(i) = \int_{\mu_o - a + (i-1)r}^{\mu_o - a + ir} \left[I_{m,K}(\mu)\right] \frac{d\mu}{r}$$

for $i = 1,2,\ldots$

$m = 1,2,3,4$

and evaluate using Simpson's Rule with $n = 2$.

$$\Delta\mu = \frac{(\mu_o - a + ir) - [\mu_o - a + (i-1)r]}{n}$$

$$= \frac{r}{2}$$

Then, for y_{ai} previously defined,

$$Z_{a,m,K}(i) = \frac{1}{6}\left[I_{m,K}(y_{a,i-1}) + 4I_{m,K}(y_{a,i-1} + \Delta\mu) + I_{m,K}(y_{ai})\right]$$

$i = 1,2,\ldots$

$m = 1,2,3,4$

Here note that the δ, P. U, and L defined following (5.12.10) should now be written as: $\delta_a(i)$, $P_a(i)$, $U_a(i)$, and $L_a(i)$, since we are using y_{ai} as discrete values of μ. Thus

$$E_{b,K,R}\left[(X - \mu_o)_a^m\right] = \frac{\sum_{i=1}^{\infty} p_a(i)\, Z_{a,m,K}(i)}{\sum_{i=1}^{\infty} p_a(i)\, P_a(i)} \qquad (5.12.13)$$

Define

$E_b\left[(X - \mu_o)_a^m\right] = m^{th}$ moment of the class (b) outgoing distribution

Then for m = 1,2,3,4,

$$E_b\left[(X - \mu_o)_a^m\right] = \frac{\sum_{i=1}^{\infty}\left[q_a(i+1)\, W_{ma}(i) + p_a(i)\, Z_{a,m,K}(i)\right]}{\sum_{i=1}^{\infty}\left[q_a(i+1) + p_a(i)\, P_a(i)\right]} \qquad (5.12.14)$$

For the case of set-up error,

$$E_b\left[(X - \mu_o)_a^m\right] = \tag{5.12.15}$$

$$\left\{.2\left\{\sum_{i=1}^{\infty}\left[q_{(a-.025)}(i+1)\, W_{m(a-.025)}(i) + p_{(a-.025)}(i)\; Z_{(a-.025),m,K}(i)\right]\right.\right.$$

$$\left.+ \sum_{i=1}^{\infty}\left[q_{(a+.025)}(i+1) W_{m(a+.025)}(i) + p_{(a+.025)}(i)\, Z_{(a+.025),m,K}(i)\right]\right\}$$

$$\left. + .6 \sum_{i=1}^{\infty}\left[q_a(i+1)W_{ma}(i)+p_a(i)Z_{a,m,K}(i)\right]\right\}$$

$$\div \left\{ .2 \left\{ \sum_{i=1}^{\infty}\left[q_{(a-.025)}(i+1) + p_{(a-.025)}(i)\; P_{(a-.025)}(i)\right] + \sum_{i=1}^{\infty}\left[q_{(a+.025)}(i+1) + p_{(a+.025)}(i)\; P_{(a+.025)}(i)\right]\right\}\right.$$

$$\left. + .6 \sum_{i=1}^{\infty}\left[q_a(i+1) + p_a(i)\; P_a(i)\right]\right\}$$

If we define $MM_m = E_b\left[(X - \mu_o)_a^m\right]$, $m = 1,2,3,4$, then the class (b) outgoing distribution mean, stándard deviation, skewness, and kurtosis are given respectively by equations (5.6.18) listed earlier in this chapter. This may be done for both with and without set-up error, using (5.12.14) and (5.12.15).

5.13 Test Criteria Evaluation when Resetting to a Fixed Point

In order to find the most desirable c_1 value for the $\overline{X}$ test with a sample size of five, equations (5.10.4), (5.10.6), (5.11.1), (5.11.3), (5.12.4), (5.12.6), (5.12.14), and (5.12.15) were evaluated, for:

$c_1 = .14(.02).20$

$r = .01,\ .04,\ .07$

$\sigma_X = \sigma_j = .035 + .03j,\ j = 1,\ldots,6$

$K = .375, .500$

$a = .050\ (.025) .150$, except when set-up error is assumed, and then $a = .075$, $.100$, $.125$.

Let SDM represent the a posteriori standard deviation of the process mean, σ_{μ}. The remainder of the notation should be quite familiar by now. Table 5.4 gives a summary of the criteria used to select the most appropriate c_1 values. For a tool wear rate of $r = .01$, $c_1 = .18$ seemed to be the best compromise choice. For $r = .04$, $c_1 = .16$ was chosen; and for $r = .07$, $c_1 = .14$ was chosen. Interpolation for $.01 \leq r \leq .07$ is possible in order to select the best compromise c_1 value. The above results are given for $a = .125$; that is, we will reset to $\mu_o - T/8$ following a rejection on the $\overline{X}$ test, with positive drift. This seemed to work out best among the alternatives considered. Values in Table 5.4 are given for class (a) distributions only; that, no sorting of rejected material is considered. When 100 percent sorting of rejected material was assumed, very little effect was noticed, so these values were omitted from the table. Finally, all results are given assuming reset error. Following a brief discussion of the effects of moderately skewed process distributions, we will summarize these results in an outlined test procedure.

5.14 Reset to a Fixed Point: Non-normal Case

In order to consider the effects of moderately skewed process distributions on the $\overline{X}$ test, a procedure analogous to that used earlier

Table 5.4. Summary of Selection Criteria for Positive Tool Wear with Reset to μ_o - T/8 (Normal Case). $c_1 = \bar{X}$ Test Constant, r = Tool Wear Rate.

c_1	r	σ_X	SDM	M_1	SD	α_3	α_4
.18	.01	.065	.088	.021	.108	.008	2.56
		.095	.091	.012	.126	.017	2.80
		.125	.087	-.001	.149	.024	2.91
		.155	.076	-.018	.174	.029	2.96
		.185	.062	-.036	.199	.029	2.99
		.215	.050	-.054	.225	.023	3.00
.16	.04	.065	.089	.027	.111	.029	2.55
		.095	.091	.023	.131	.037	2.78
		.125	.092	.019	.154	.037	2.89
		.155	.090	.012	.179	.035	2.95
		.185	.087	.005	.205	.031	2.97
		.215	.083	-.003	.232	.027	2.99
.14	.07	.065	.089	.026	.111	.054	2.57
		.095	.091	.026	.132	.055	2.79
		.125	.092	.023	.156	.051	2.89
		.155	.092	.020	.181	.045	2.95
		.185	.091	.015	.208	.039	2.97
		.215	.089	.011	.235	.034	2.99

in this chapter was again employed. The reader may wish to review section 5.8 before continuing. Table 5.5 gives the values of the various criteria used to find the most desirable c_1 and the $\bar{X}$ test, under the assumption of non-normal process distributions. The results were quite close to those obtained in the normal case. Therefore, values are given only for the moderately skewed Burr distribution denoted earlier by "Skew 2".

5.15 Test Procedure When Resetting to a Fixed Point

The following procedure should be used whenever the tool wear process is reset to a fixed point following a rejection on the $\bar{X}$ test. Let T be the tolerance and μ_o the nominal mean value for a part. The following procedure will safely control the distribution of part dimensions:

1) For each regular (equal spaced) periodic sample of five parts from the process, find the average $\bar{X}$ and the range R.

2) Samples should be taken often enough such that the tool wear rate r between testing procedures is $.01T \leq r \leq .07T$. If the tool wear rate r is unknown, assure r = .04T until enough past history has been observed to estimate r.

3) The process is considered satisfactory at this time if both of the following are met:

a) $\mu_o - c_1 T \leq \bar{X} \leq \mu_o + c_1 T$ (5.15.1)

b) $R \leq .55T$,

Table 5.5. Summary of Selection Criteria for Positive Tool Wear with Reset to μ_o - T/8 (Non-normal Case). c_1 = $\bar{X}$ Test Constant, r = Tool Wear Rate, Skew 2 Only.

c_1	r	σ_X	SDM	M_1	SD	α_3	α_4
.18	.01	.065	.087	.021	.108	.011	2.56
		.095	.090	.011	.126	.019	2.81
		.125	.088	-.001	.149	.024	2.91
		.155	.077	-.017	.174	.028	2.96
		.185	.062	-.036	.200	.029	2.99
		.215	.050	-.055	.226	.024	3.00
.16	.04	.065	.089	.027	.111	.030	2.55
		.095	.091	.024	.131	.037	2.78
		.125	.092	.019	.154	.038	2.89
		.155	.091	.013	.179	.035	2.95
		.185	.088	.006	.206	.032	2.97
		.215	.084	-.003	.233	.028	2.99
.14	.07	.065	.089	.027	.111	.054	2.57
		.095	.091	.026	.132	.054	2.78
		.125	.092	.024	.156	.050	2.89
		.155	.092	.020	.181	.045	2.94
		.185	.092	.016	.208	.040	2.97
		.215	.090	.011	.235	.034	2.99

where if $r \doteq .01T$, $c_1 = .18$

$\doteq .04T$, $c_1 = .16$

$\doteq .07T$, $c_1 = .14$.

4) If either or both of the requirements in step 3 is not met, action is required. If R is excessive, sort the recent product looking for the source of the excess variability. If $\bar{X}$ is beyond the upper limit if μ increases, reset to $\mu_o - T/8$; if $\bar{X}$ is beyond the lower limit if μ decreases, reset to $\mu_o + T/8$.

5) 100 percent sorting of rejected material is not necessary, if rejected by the $\bar{X}$ test.

By comparing the above procedure to that given in section 5.9, one notes that the above is a more relaxed test on $\bar{X}$, for a given tool wear rate r. This is due to the method of reset following a rejection, and is a natural recommendation for resetting to a fixed point rather than by a fixed amount.

CHAPTER VI

SUMMARY

The current application of specification limits, with no other control over the distribution of part dimensions, has made it difficult for design engineers to safely take much advantage of the square-root tolerance formula given by (1.3.2). What is needed is emphasis instead upon controlling the distribution of the parts, specifically by maintaining the average close to the desired nominal mean μ_o, while preventing the variability from becoming excessive. It is then possible and safe for the design engineer to use (1.3.2). We maintain this part distributional control through the use of statistical tests based on the sample mean and range. Such tests have been developed in the areas of acceptance sampling and process control, with both randomly acting assignable causes and tool wear models considered under process control. It is hoped that these tests can be implemented with ease, and yet be powerful enough to produce assemblies meeting all design engineering requirements.

Several areas of future research were mentioned in the body of the thesis. These include the effects of non-normal process distributions on (a) the overall probability of acceptance for the $\bar{X}$ and R tests, (b) the distribution of the sum-range, and (c) the derivation of the moments of the outgoing distribution. Item (c) has been investigated in this research to a certain extent, but further investigation might prove worthwhile. The author hopes to consider these problems at a later date, and suggests them as well to the interested reader.

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Richard Eugene James, a United States citizen, was born August 14, 1944 in Glendale, California. He graduated from Norman High School, Norman, Oklahoma, in June, 1962. He received his Bachelor of Science degree in mathematics from Oklahoma Baptist University in June, 1966. Beginning in September, 1966, he attended Purdue University, and received his Master of Science degree in Mathematical Statistics in January, 1968. He attended Purdue on a NASA fellowship until February, 1969, and then served as a graduate research instructor.