On the exact distribution of Hotelling's generalized T_0^2.*

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1. <u>Introduction and summary</u>. Let $\underset{\sim}{S}_1$ and $\underset{\sim}{S}_2$ be two symmetric matrices of order p estimating the same covariance matrix, where $\underset{\sim}{S}_2$ is positive definite having a Wishart distribution with n_2 degrees of freedom, and $\underset{\sim}{S}_1$ is at least positive semi-definite having a non-central Wishart distribution with n_1 degrees of freedom. Then Hotelling's generalized T_0^2 statistic is defined by [5]:

$$T_0^2 = n_2 \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1} = n_2 U^{(s)} ,$$

where $s(= \min (n_1,p))$ is the number of non-zero characteristic roots of $\underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$. When $n_1 \geq p$, $U^{(s)} = U^{(p)}$. When $n_1 < p$ the density function of the characteristic roots of $\underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$ can be obtained from that for $n_1 \geq p$ if in the latter case the following changes are made:

$$(n_1,n_2,p) \to (p,n_1 + n_2-p,n_1).$$

Hence the density of $U^{(s)}$ can be easily derived from that of $U^{(p)}$ and therefore only the case of $U^{(p)}$ is considered here.

The exact null distribution of T_0^2 (i.e., when the non-centrality matrix is null) was obtained by Hotelling [5] for p = 2. Davis [2] has shown that the null density of T_0^2 satisfies an ordinary linear homogeneous differential

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equation of order p. The non-null distribution has been attempted by Constantine [1] using zonal polynomials and hypergeometric functions of matrix arguments. However, his results hold only for $|U^{(p)}| < 1$. Pillai and Jayachandran [13] have obtained the non-null distribution of $U^{(2)}$ using zonal polynomials up to the sixth degree.

An approximation to the null distribution of $U^{(p)}$ has been suggested by Pillai [8], [9] and studied by Pillai and Samson [15]. Ito [6] has obtained an asymptotic expansion for the null distribution of T_0^2 which he later extended to the non-null case [7]. Davis [3] has further studied the asymptotic null distribution.

Grubbs [4] has provided some exact percentage points for $U^{(2)}$ for n_1 and n_2 less than 50. Using the exact moment quotients of $U^{(p)}$, Pillai [11] has provided extensive tables of approximate percentage points for $U^{(p)}$. Further, Pillai and Jayachandran [13] have obtained some exact percentage points of $U^{(2)}$ in connection with power function studies. Recently Davis [3] has tabulated exact percentage points of T_0^2/n_1 for p = 3 and 4 using the differential equation approach [2]. He also provides compairsons of the accuracy of several approximations.

It may be pointed out that the null distribution of the characteristic roots of $\underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$ (see Eq. (2.1))is of the same form as those of the characteristic roots of matrices arising in each of the following tests of hypotheses except that the two parameters m and n involved there (see below) have to be defined differently in each case [9], [11]: (i) Independence between a p-set and a q-set in a (p + q)-variate normal population and (ii) Equality of covariance matrices in two p-variate normal populations. In view of this, the null distribution of $U^{(p)}$ for the three tests is also of the same form. Pillai [9] considered the use of $U^{(p)}$ for tests of (i) and (ii) as well, and

Pillai and Jayachandran [13], [14] have shown that the power functions of the $U^{(p)}$ test against appropriate alternatives for tests of (i) and (ii) and the general linear hypothesis behave more or less in the same manner.

Still, however, there are no explicit expressions available for the exact null distribution of $U^{(p)}$ (or T_0^2) for $p > 2$ except one obtained for $U^{(3)}$ as an infinite series by Pillai and Chang through transformation of variables [12]. In this paper there is presented a method for deriving the exact null distribution of $U^{(p)}$ employing inverse Laplace transforms. Density functions are given for $p = 3$, $m = 0, 1, 2, 3, 4$ and 5, and $p = 4$, $m = 0, 1$ and 2, where $m = (n_1-p-1)/2$. In addition, exact upper percentage points are tabulated for $p = 2, 3$ and 4, various significance levels, and selected values of m and n $(= (n_2-p-1)/2)$. Also, two approximations similar to Pillai's [8], [9] are presented.

The exact densities and the approximations derived in the paper are further being used to develop the distributions of some test criteria involving the maximum of ratios of independent $n_i U^{(p)}/n_j$ for the tests of equality of several covariance matrices [16]. The results of this study will be reported later.

2. <u>The Laplace transform of $U^{(p)}$</u>. The joint density function of $\lambda_1, \lambda_2, \ldots, \lambda_p$, the characteristic roots of $\underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$, has the form [17]:

$$(2.1) \qquad f(\lambda_1,\ldots,\lambda_p) = C(p,m,n) \prod_{i=1}^{p} \lambda_i^m/(1+\lambda_i)^{m+n+p+1} \prod_{i>j} (\lambda_i-\lambda_j) \;,$$

$$0 < \lambda_1 < \ldots < \lambda_p < \infty \;,$$

where $C(p,m,n) = \pi^{p/2} \prod_{i=1}^{p} \Gamma(\frac{1}{2}(2m+2n+p+i+2))/\{\Gamma(\frac{1}{2}(2m+i+1))\Gamma(\frac{1}{2}(2n+i+1))\Gamma(\frac{1}{2}i)\}$.

and m and n are defined in section 1. Then $U^{(p)} = \sum_{i=1}^{p} \lambda_i = \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$, and the Laplace transform of $U^{(p)}$ with respect to (2.1) is:

$$(2.2) \qquad L(t;\, p,m,n) = E(\exp\,(-t \sum_{i=1}^{p} \lambda_i))$$

$$= C \int \underset{A}{\cdots} \int \exp(-t \sum_{i=1}^{p} \lambda_i) \prod_{i=1}^{p} \lambda_i^{m}/(1+\lambda_i)^{m+n+p+1} \prod_{i>j}^{p} (\lambda_i-\lambda_j) \prod_{i=1}^{p} d\lambda_i$$

where

$$A = \{\, (\lambda_1,\ldots,\lambda_p) \mid 0 < \lambda_1 < \ldots < \lambda_p < \infty \,\},$$

$$C = C\,(p,m,n) \text{ and } t \geq 0.$$

Upon making the transformation

$$x_i = 1/(1 + \lambda_{p-i+1}),\ i = 1,\ldots,p\,,$$

we may write (2.2) as:

$$(2.3) \quad L(t;\, p,m,n) = e^{pt}\, C \int \underset{B}{\cdots} \int \exp\,(-t \sum_{i=1}^{p} x_i^{-1}) \prod_{i=1}^{m} x_i^{\,n}(1-x_i)^{m} \prod_{i>j} (x_i-x_j) \prod_{i=1}^{p} dx_i$$

where

$$B = \{\, (x_1,\ldots,x_p) \mid 0 \leq x_1 < x_2 < \ldots < x_p \leq 1 \,\}.$$

Next we note that $\prod_{i>j} (x_i-x_j)$ may be written as the Vandermonde determinant

$$\begin{vmatrix} x_p^{\,p-1} & x_p^{\,p-2} & \cdots & x_p & 1 \\ x_{p-1}^{\,p-1} & x_{p-1}^{\,p-2} & \cdots & x_{p-1} & 1 \\ \vdots & & & \vdots & \\ x_1^{\,p-1} & x_1^{\,p-2} & \cdots & x_1 & 1 \end{vmatrix},$$

and that the elementary properties of determinants allows (2.3) to be written:

$$(2.4)\quad L(t;p,m,n) = e^{pt} C \int_{B}\cdots\int \exp\left(-t\sum_{i=1}^{p} x_i^{-1}\right) \begin{vmatrix} (1-x_p)^m\, x_p^{n+p-1} & \cdots & (1-x_p)^m\, x_p^{\,n} \\ \vdots & & \vdots \\ (1-x_1)^m\, x_1^{n+p-1} & \cdots & (1-x_1)^m\, x_1^{\,n} \end{vmatrix} \prod_{i=1}^{p} dx_i \;.$$

If we take m to be a non-negative integer and expand $(1-x_i)^m$ as a binomial series, the determinant in (2.4) is:

$$(2.5)\quad \begin{vmatrix} \sum_{i_p=0}^{m} \binom{m}{i_p}(-1)^{i_p}\, x_p^{n+q_p+i_p} & \cdots & \sum_{i_1=0}^{m} \binom{m}{i_1}(-1)^{i_1}\, x_p^{n+q_1+i_1} \\ \vdots & & \vdots \\ \sum_{i_p=0}^{m} \binom{m}{i_p}(-1)^{i_p}\, x_1^{n+q_p+i_p} & \cdots & \sum_{i_1=0}^{m} \binom{m}{i_1}(-1)^{i_1}\, x_1^{n+q_1+i_1} \end{vmatrix} ,$$

where $q_j = j-1$. (2.5) can be further reduced to the form

$$(2.6)\quad \sum_{i_p=0}^{m} \cdots \sum_{i_1=0}^{m} \left\{ \prod_{j=1}^{p} \binom{m}{i_j} \right\} (-1)^{\sum_{j=1}^{p} i_j} \begin{vmatrix} x_p^{n+q_p+i_p} & \cdots & x_p^{n+q_1+i_1} \\ \vdots & & \vdots \\ x_1^{n+q_p+i_p} & \cdots & x_1^{n+q_1+i_1} \end{vmatrix} .$$

The expansion (2.6) allows us to throw (2.4) into the form [10]:

$$(2.7)\quad L(t;p,m,n) = e^{pt}\, C \sum_{i_p=0}^{m} \cdots \sum_{i_1=0}^{m} \Big\{ \prod_{j=1}^{p} \binom{m}{i_j} \Big\} (-1)^{\sum_{j=1}^{p} i_j} R(n; q_p+i_p, \ldots, q_1+i_1; t)$$

where

$$(2.8)\quad R(n;a_p,a_{p-1},\ldots,a_1;t) = \begin{vmatrix} \int_0^1 x_p^{n+a_p} e^{-t/x_p}\, dx_p & \cdots & \int_0^1 x_p^{n+a_1} e^{-t/x_p}\, dx_p \\ \vdots & & \vdots \\ \int_0^{x_2} x_1^{n+a_p} e^{-t/x_1}\, dx_1 & \cdots & \int_0^{x_2} x_1^{n+a_1} e^{-t/x_1}\, dx_1 \end{vmatrix} .$$

Now permuting the columns of the determinants so that the indices form a decreasing sequence, dropping all determinants which are zero and combining like terms in (2.7) gives

$$(2.9)\quad L(t;p,m,n) = e^{pt}\, C \sum_{\mathcal{D}} k_{i_p\, i_{p-1} \cdots i_1} R(n;\, i_p,\, i_{p-1},\ldots,i_1;t)$$

where

$$\mathcal{D} = \{\, (i_1,\ldots,i_p) \mid 0 \le i_1 < i_2 < \cdots < i_p \le m + p-1 \,\}$$

and the $k_{i_p \cdots i_1}$ depend on p and m. The constants $k_{i_p \cdots i_1}$ have been tabulated in Table 1 for p = 3, m = 0 (1) 5 and p = 4, m = 0, 1, 2.

Thus we have expressed the Laplace transform of $U^{(p)}$ as a linear combination of the determinants $R(n;\, i_p,\ldots,i_1;\, t)$.

3. <u>A reduction formula for $R(n;\, a_p,\ldots,a_1;\, t)$</u>. With the expression (2.9) for the Laplace transform of $U^{(p)}$ we need to evaluate the determinants $R(n;\, a_p,\ldots,a_1;\, t)$. This will be done by means of a reduction formula similar to the one developed by Pillai [10].

We will state here the notation and lemmas that are needed and give only an outline of the approach as the results are analogous to those of Pillai [8], [10]. Let

$$(3.1)\quad V(x;\, q_k,\, q_{k-1},\dots,q_1;\, t) = \begin{vmatrix} \int_0^x x_k^{q_k} e^{-t/x_k}\, dx_k & \cdots & \int_0^x x_k^{q_1} e^{-t/x_k}\, dx_k \\ \vdots & & \vdots \\ \int_0^{x_2} x_1^{q_k} e^{-t/x_1}\, dx_1 & \cdots & \int_0^{x_2} x_1^{q_1} e^{-t/x_1}\, dx_1 \end{vmatrix}$$

(Note that $R(n;\, a_p,\dots,a_1;\, t) = V(1;\, n + a_p,\dots,n + a_1;\, t)$.)

Now (3.1) will involve integrals of the type

$$(3.2)\qquad I(x';\, q,\, F;\, t) = \int_0^{x'} y^q\, F(y)\, e^{-t/y}\, dy$$

where $F(y)$ is a function of y such that the integral exists and in our context could be of the form

$$(3.3)\qquad \int_0^y x_{k-1}^{q_{k-1}} e^{-t/x_{k-1}}\, dx_{k-1} \cdots \int_0^{x_2} x_1^{q_1} e^{-t/x_1}\, dx_1 \,.$$

When $F(y)$ has the form (3.3) we will denote (3.2) by

$$I(x';\, q,\, q_{k-1},\dots,q_1;\, t)\ .$$

The following lemma involving (3.2) is obtained by integration by parts.

<u>Lemma 1</u>: The integral

$$(3.4)\qquad I(x';\, q,\, F;\, t) = [1/(q+1)]\, \{\, I_0(x';\, q+1,\, F;\, t) - I(x';\, q+1,\, F';\, t) - t\, I(x';\, q-1,\, F;\, t)\, \}$$

where

$$I_0(x';\, q+1,\, F;\, t) = y^{q+1}\, F(y)\, e^{-t/y}\Big|_0^{x'}$$

and $$F'(y) = \frac{d}{dy}\, F(y).$$

Lemma 2: If σ is any permutation of (1, 2,...,k) then

$$\sum_{\sigma} I(x;\ q_{\sigma(k)},\dots,q_{\sigma(1)};\ t) = \prod_{j=1}^{k} I(x;\ q_j;\ t)$$

where the summation is over all possible permutations.

Let $V(x;\ q_k',\dots,q_1';\ t')^{(i)}$ denote the determinant (3.1) when the indices of the ith row alone are different from those of the other rows, where the indices of the ith row are $q_k',\dots,q_1',\ t'$. Then we have the following lemma.

Lemma 3:

$$\sum_{i=1}^{k} (-1)^{i-1}\ V(x;\ q_k',\dots,q_1';\ t')^{(i)}$$

$$= \sum_{j=1}^{k} (-1)^{k+j}\ I(x;\ q_j';\ t')\ V(x;\ q_k,\dots,q_{j+1},\ q_{j-1},\dots,q_1;\ t)\ .$$

We now state the reduction formula for the determinant (3.1).

Theorem 1:

$$(3.5)\quad V(x;\ q_k,q_{k-1},\dots,q_1;\ t) = [1/(q_k+1)]\ (A^{(k)} + B^{(k)} - tC^{(k)})$$

where

$$A^{(k)} = x^{q_k+1}\ e^{-t/x}\ V(x;q_{k-1},\dots,q_1;\ t)$$

$$B^{(k)} = 2\sum_{j=1}^{k-1} (-1)^{k+j}\ I(x;\ q_j+q_k+1;\ 2t)\ V(x;\ q_{k-1},\dots,q_{j+1},q_{j-1},\dots,q_1;\ t)$$

and

$$C^{(k)} = V(x;\ q_k-1,\ q_{k-1},\dots,q_1;\ t).$$

Proof: Expand the determinant by the first column. (Recall that the order of integrations must not be changed). Now using Lemma 1 we integrate by parts the term involving the element from the ith row and first column with respect to x_{k-i+1}. Next add the expressions obtained corresponding to each of the three terms on the right side of (3.4) and apply the above lemmas. The result follows.

The formula we require to evaluate the Laplace transform (2.9) follows as a corollary.

Corollary 1:

$$(3.6) \quad R(n;\, a_p,\ldots,a_1;\, t) = [1/(a_p+1)]\, (D^{(p)} + E^{(p)} - tF^{(p)}) ,$$

where

$$D^{(p)} = e^{-t}\, R(n;\, a_{p-1},\ldots,a_1;\, t) ,$$

$$E^{(p)} = e^{-2t} \sum_{j=1}^{p-1} (-1)^{p+j}\, g(n;\, a_j+a_p+3,2;\, t)\, R(n:\, a_{p-1},\ldots,a_{j+1},\, a_{j-1},\ldots,a_1;t$$

where

$$g(n;\, a,b;\, t) = \int_0^\infty e^{-tz}/(1 + z/b)^{bn+a}\, dz$$

and

$$F^{(p)} = R(n;\, a_p-1,\, a_{p-1},\ldots,a_1;\, t) .$$

Proof: In (3.5) let $x = 1$, $q_j = n+a_j$ and make the change of variable $z = 2(1-y)/y$ in $I(1;\, 2n+a_j+a_p+1;\, 2t)$ to get

$$I(1;\, 2n+a_j+a_p+1;\, 2t) = \tfrac{1}{2}e^{-2t}\, g(n;\, a_j+a_p+3,2;\, t).$$

4. Use of the reduction formula. Now let us illustrate the use of (3.6) in deriving the expression for the determinant $R(n;\, 3,\, 1,\, 0;\, t)$. (3.6) yields:

(4.1) $R(n;\ 3,1,0;\ t) = [1/(n+4)]\ \{\ e^{-t}\ R(n;\ 1,0;t) + e^{-2t}\ g(n;\ 6,2;\ t)\ R(n;\ 1;\ t)$

$- e^{-2t}\ g(n;\ 7,2;\ t)\ R(n;\ 0;\ t) - t\ R(n;\ 2,1,0;\ t)\ \}$.

But $R(n;\ i;\ t) = I(1;\ n+i;\ t) = e^{-t}\ g(n;\ i+2,1;\ t)$ and a second use of the reduction formula yields

(4.2) $R(n;\ 2,1,0;\ t) = [1/(n+3)]\ \{\ e^{-t}\ R(n;\ 1,0;\ t)$

$+ e^{-2t}\ g(n;\ 5,2;\ t)\ R(n;\ 1;\ t) - e^{-2t}\ g(n;\ 6,2;\ t)\ R(n;\ 0;\ t)\ \}$

and

(4.3) $R(n;\ 1,0;\ t) = [e^{-2t}/(n+2)]\ \{\ g(n;\ 2,1;\ t) - g(n;\ 4,2;\ t)\ \}$.

There are no terms corresponding to $t\ F^{(p)}$ of the Corollary in (4.2) and (4.3) since any determinant having two columns (indices) the same is zero.

We now integrate by parts $R(n;\ 2,1,0;\ t)$, integrating one factor in each of the terms in (4.2), and in this connection we use the following result:

$$g(n;\ a,b;\ t) = (1/t)\ \{\ 1-((bn+a)/b)\ g(n;\ a+1,b;\ t)\ \}\ .$$

This is done in order to bring the terms to a more suitable form for inversion as shown in the next section.

We thus obtain:

$$R(n;\ 3,1,0;\ t) = [e^{-3t}/(n+4)]\ \{\ (g(n;\ 2,1;\ t) - g(n;\ 4,2;\ t))/(n+2)$$

$$+ g(n;\ 6,2;\ t)\ g(n;\ 3,1;\ t) - g(n;\ 7,2;\ t)\ g(n;\ 2,1;\ t)$$

$$- [t/(n+3)]\ [(\ (1/t) - ((n+2)/t)\ g(n;\ 3,1;\ t) - ((1/t) - ((n+2)/t)\ g(n;\ 5,2;t)))/(n+2)$$

$$+ ((1/t) - ((2n+5)/(2t))\ g(n;\ 6,2;\ t))\ g(n;\ 3,1;\ t)$$

$$- ((1/t) - ((n+3)/t)\ g(n;\ 7,2;\ t))\ g(n;\ 2,1;\ t)]\ \}\ .$$

All terms involving t as a factor and the constant terms not involving integrals can be seen to vanish. This holds true in the general case. Upon simplification we get:

$$(4.4)\quad R(n;\ 3,1,0;\ t) = [e^{-3t}/(n+4)]\ \{\ [(2n+5)/(n+2)(n+3)]\ g(n;\ 2,1;\ t)$$

$$- [1/(n+2)]\ g(n;\ 4,2;\ t) - [1/(n+3)]\ g(n;\ 5,2;\ t)$$

$$+ [(4n+11)/2(n+3)]\ g(n;\ 6,2;\ t)\ g(n;\ 3,1;\ t)$$

$$- 2g(n;\ 7,2;\ t)\ g(n;\ 2,1;\ t)\ \}\ .$$

5. <u>The density function of $U^{(p)}$.</u> The uniqueness property of the Laplace transform will allow us now to obtain the density of $U^{(p)}$ using (2.9). The density may be written:

$$(5.1)\qquad f(u) = C \sum_{\mathcal{D}} k_{i_p\, i_{p-1}\, \cdots\, i_1}\ R^*(n;\ i_p,\ldots,i_1;\ u)$$

where $R^*(n;\ i_p,\ldots,\ i_1;\ u)$ is the inverse Laplace transform of $R(n;\ i_p,\ldots,i_1;\ t)$.

We will illustrate the method of obtaining the R^* functions with the help of $R(n;\ 3,1,0;\ t)$ in (4.4).

If we denote the inverse Laplace transform of $g(n;\ a,b;\ t)$ by $g^*(n;\ a,b;\ t)$ we see that

$$g^*(n;\ a,b;\ u) = 1/(1+u/b)^{bn+a}\ .$$

Also the function whose transform is $g(n;\ a,b;\ t)\ g(n';\ a',b',t)$ is given by the convolution

$$(5.2)\qquad g^*(n;\ a,b;\ u) * g^*(n';\ a',b';\ u)$$

where * denotes the convolution operator. We may write (5.2) as:

$$(5.3)\quad h(n,n';\ a,a';\ b,b';\ u) = \int_0^u \frac{dx}{(1+x/b)^{bn+a}\,(1+(u-x)/b')^{b'n'+a'}} \ .$$

Then from (4.4) we find:

$$(5.4)\quad R^*(n;\ 3,1,0;\ t) = [1/(n+4)]\ \{\ [(2n+5)/(n+2)(n+3)]\ g^*(n;\ 2,1;\ u)$$

$$-\ [1/(n+2)]\ g^*(n;\ 4,2;\ u) - [1/(n+3)]\ g^*(n;\ 5,2;\ u)$$

$$+\ [(4n+11)/2(n+3)]\ h(n,n;\ 6,3;\ 2,1;\ u)$$

$$-\ 2\ h(n,n;\ 7,2;\ 2,1;\ u)\ \}\ .$$

(5.4) may be further simplified by using the expression below which is obtained by integration by parts.

$$h(n,n';\ a,a';\ b,b';\ u) = [b/(bn+a)]\ (g(n';\ a',b';\ u) - g(n;\ a,b;\ u))$$

$$+\ [b(b'n'+a')/b'(bn+a)]\ h(n,n';\ a-1,a'+1;\ b,b';\ u)\ .$$

Finally upon simplification we have

$$R^*(n;\ 3,1,0;\ t) = [1/(n+4)]\ \{\ [1/(n+2)(n+3)]\ g^*(n;\ 2,1;\ u)$$

$$-\ [1/(n+2)]g^*(n;\ 4,2;\ u) - [1/(n+3)]\ g^*(n;\ 5,2;\ u)$$

$$+\ [2/(n+3)]\ g^*(n;\ 6,2;\ u) + [3/2(n+3)]\ h(n,n;\ 6,3;\ 2,1;\ u)\ \}\ .$$

In calculating the R^* functions it should be noted that

$$R^*(n;\ a_p,a_{p-1},\ldots,a_1;\ u) = R^*(n+a_1;\ a_p-a_1,\ldots,0;\ u)\ ,$$

where we may take $0 \le a_1 < a_2 < \ldots < a_p \le m+p-1$, so that the only R^*'s which need be determined are those with $a_1 = 0$.

R^* for $p = 3$ can be written in the form:

$$(5.5)\quad R^*(n;\ i,j,0;\ u) = [1/(n+i+1)]\ \{ \sum_{\ell=1}^{i-2} \alpha_{ij\ell}(n)\ g^*(n;\ \ell+2,1;\ u)$$

$$+ \sum_{\ell=1}^{i} \beta_{ij\ell}(n)\ g^*(n;\ \ell + j + 2,2;\ u)$$

$$+ \gamma_{ij}(n)\ h(n,n;\ i+j+2,3;\ 2,1;\ u)\ \}\ \text{for } i > 2.$$

If $i = 2$, the last two terms are obtained by substituting 2 for i, but the first term becomes $\alpha_{211}(n)\ g^*(n;\ 2,1;\ u)$. The coefficients $\alpha_{ij\ell}(n)$, $\beta_{ij\ell}(n)$ and γ_{ij} for $1 \leq j < i \leq 7$ are presented in Table 2. These provide the density function of $U^{(3)}$ for $m = 0(1)5$.

R^* for $p = 4$ can be expressed as:

$$(5.6)\quad R^*(n;\ i,j,k,0;\ u) = [1/(n+i+1)]\ \{ \alpha_{ijk}(n)\ g^*(n;\ 2,1;\ u)$$

$$\sum_{\ell=1}^{i+j-2} \beta_{ijk\ell}(n)\ g^*(n;\ k+\ell+2,2;\ u) + \sum_{\ell=1}^{i} \gamma_{ijk\ell}(n) h(n,n;\ \ell+4,3;\ 2,1;\ u)$$

$$+ \delta_{ijk}(n)\ h(n,n;\ i+3,j+k+3;\ 2,2;\ u)\ \}\ .$$

The coefficients involved in (5.6) are given in Table 3 for $1 \leq k < j < i \leq 5$. These terms provide the density function of $U^{(4)}$ for $m = 0,1$ and 2.

6. The distribution of $U^{(3)}$ and $U^{(4)}$. The distribution function of $U^{(p)}$, say, $G(z;\ p,m,n) = P\ (U^{(p)} \leq z)$ may be obtained from the density function (5.1) upon integration. We have:

$$(6.1)\quad G(z;\ p,m,n) = C \sum_{\mathcal{D}} k_{i_p} \ \cdots\ {}_{i_1} \int_0^z R^*(n;\ i_p,\ldots,i_1;\ u)du.$$

The distribution functions of $U^{(p)}$ for $p = 3$ and 4 are thus seen to be obtained by the integration of $g^*(n;\ a,b;\ u)$ and $h(n,n';\ a,a';\ b,b';\ u)$ with respect to u. Now.

$$(6.2)\qquad \int_0^z g^*(n;\ a,b;\ u)du = [b/(bn+a-1)]\ (1 - g^*(n;\ a-1,b;\ z))$$

and

$$(6.3)\qquad \int_0^z h(n,n';\ a,a';\ b,b';\ u)du = [b'/(b'n'+a'-1)]\{[b/(bn+a-1)](1-g^*(n;\ a-1,b;\ z)) - h(n,n';\ a,a'-1;\ b,b';\ z)\}.$$

(6.3) is obtained by interchange of the order of integration. Finally, evaluation of $h(n,n';\ a,a';\ b,b';\ z)$ makes use of the following method. If

$$P(z;\ p,q,c,d) = \int_0^z \frac{dx}{(c+x)^p\ (d-x)^q}$$

where p and q are non-negative integers, c and d are non-negative, then

$$(6.4)\qquad P(z;\ p,q,c,d) = A_1\ \{\ \ln\ (1+z/c)\ \} - B_1\ \{\ \ln\ (1+z/d)\ \}$$
$$- \sum_{i=2}^{p} \frac{A_i}{i-1}\left\{ \frac{1}{(c+z)^{i-1}} - \frac{1}{c^{i-1}} \right\} + \sum_{j=2}^{q} \frac{B_j}{j-1}\left\{ \frac{1}{(d-z)^{j-1}} - \frac{1}{d^{j-1}} \right\}$$

where

$$A_{p-i} = \left[\ \prod_{\ell=1}^{i}\ (q+\ell-1)\right]/[i!\ (c+d)^{q+i}] \text{ and } B_{q-j} = \left[\ \prod_{\ell=1}^{j}\ (p+\ell-1)\right]/[j!(c+d)^{p+j}]\ .$$

(6.4) is obtained by using partial fraction expansions. We can then write:

$$h(n,n';\ a,a';\ b,b';\ z) = b^{bn+a}\ b'^{\,b'n'+a'}\ P(z;\ bn+a,b'n'+a',b,b'+u)\ .$$

where we take $bn+a$ and $b'n'+a'$ to be non-negative integers.

7. Computation of percentage points of $U^{(2)}$, $U^{(3)}$ and $U^{(4)}$. Tables of percentage points have been prepared for $U^{(p)}$ for $p = 3$, $m = 0\ (1)\ 5$ and $p = 4$, $m = 0$, 1 and 2, for $\alpha = .10$, .025 and .005, and $n = 5\ (5)\ 80\ (10)\ 100$ using the exact expressions discussed in the previous sections. Further, the percentage points of $U^{(2)}$ using the formula for the distribution found in [5] or [15] are presented for $m = -.5\ (.5)\ 5\ (5)\ 50\ (10)\ 100$, 130, 160, 200, for $\alpha = .10$, .05, .025, .01 and .005, and for $n = 5\ (5)\ 50\ (10)\ 100$, 130, 160, 200. These computations (as well as those described in the next section) were carried out on the CDC 6500 computer at the Purdue University Computing Center using double precision arithmetic. The percentage points are given to five significant digits in Tables 4 and 5.

8. Approximation to the distribution of $U^{(p)}$. Pillai [8], [9] has suggested an approximation to the distribution of $U^{(p)}$ which involves an F-type (Type II Beta) distribution with the first moment of the approximate distribution being the same as that of $U^{(p)}$. Here we propose two similar approximations by fitting the first two moments and the first three moments of $U^{(p)}$ respectively to an F-type distribution.

The density function to be used in the approximation has the form:

$$(8.1)\qquad f(x) = x^a/\{\beta(a+1,b-a-1)\ K^{a+1}\ (1+x/K)^b\},\ 0 < x < \infty\ .$$

The distribution can be expressed as the incomplete beta integral $I_w\ (a+1, b-a-1)$, where $w = x/(x+K)$. (8.1) has the first three central moments:

$$\mu_{F1} = K\ (a+1)/(b-a-2)\ ,$$

$$\mu_{F2} = [K^2(a+1)(b-1)]/[(b-a-2)^2(b-a-3)]\ ,$$

and

$$\mu_{F3} = [2K^3(a+1)(b-1)(a+b)]/[(b-a-2)^3(b-a-3)(b-a-4)]\ .$$

The first three central moments of $U^{(p)}$ are given in [11], [15] and are:

$$\mu_1 = p(2m+p+1)/(2n) ,$$

$$\mu_2 = [p\ (2m+p+1)(2m+2n+p+1)(2n+p)]/[4n^2(n-1)(2n+1)] ,$$

and

$$\mu_3 = [p(2m+n+p+1)(2m+p+1)(2m+2n+p+1)(n+p)(2n+p)]/[2n^3(n-1)(n-2)(n+1)(2n+1)] .$$

Pillai's approximation (A_1) with one moment fitted yields

$$a = \frac{1}{2}\, p(2m+p+1)-1,\ b = \frac{1}{2}\, p(2m+2n+p+1) + 1, \text{ and } K = p.$$

By setting $\mu_{F1} = \mu_1$ and $\mu_{F2} = \mu_2$ and taking $K = p$, we find the parameters for approximation A_2:

$$a = [\mu_2(\mu_1-p) + \mu_1^{\,2}(\mu_1+p)]/(p\ \mu_2) ,$$

and

$$b = [\mu_1(\mu_1 + p)^2 + \mu_1\ \mu_2 + 2p\ \mu_2]/(p\ \mu_2) .$$

Finally equating the first three moments yields the parameters for approximation A_3:

$$a = (2\mu_1^{\,3}\mu_2+3\mu_1^{\,2}\mu_3 - 6\mu_1\mu_2^{\,2} - \mu_2\mu_3)/(\mu_2\mu_3+4\mu_1\mu_2^{\,2} - \mu_1^{\,2}\mu_3) ,$$

$$b = [(a+1)(a+3) - \mu_1^{\,2}/\mu_2] \,/\, [(a+1) - \mu_1^{\,2}/\mu_2] ,$$

and

$$K = \mu_1(b-a-2)/(a+1) .$$

Tables 6 and 7 indicate the accuracy of the three approximations A_1, A_2 and A_3. The percentage points for $p = 3$, $m = 0$ and 3, and $p = 4$, $m = 0$ and 2, and for

$\alpha = .05$ and .01 were calculated for various values of n using the exact and approximate distributions. It can be seen that the approximations A_2 and A_3 are considerable improvements over Pillai's original approximation A_1, as is to be expected, with A_3 generally better than A_2. A_3 provides about three significant digits accuracy in the percentage points for $n \geq 10$. In some cases $n \geq 5$ is sufficient for this accuracy. A_2 provides the same accuracy for n slightly larger, usually around 10 to 15. A_1 does not provide this degree of accuracy until n is at least 40, and often n needs to be much larger. It has also been observed that the distributions associated with A_1, A_2 and A_3 closely approximate the distribution of $U^{(p)}$ not only in the upper tail but throughout the entire range of $U^{(p)}$ to the same degree of accuracy mentioned above for the percentage points. This inference has been based both on the study of percentiles as well as probability comparisons with the agreement in both cases being about three places or more. Thus the distribution function for A_3 provides a good approximation to the exact distribution of $U^{(p)}$ for $n \geq 10$ and for the whole range of $U^{(p)}$.

Table 1a. $k_{i_3 i_2 i_1}$ Coefficients for m = 0, 1, 2, 3, 4, 5.

	m=0	m=1		m=2			m=3				m=4					m=5					
(i_3,i_2) \ i_1	0	0	1	0	1	2	0	1	2	3	0	1	2	3	4	0	1	2	3	4	5
(2,1)	1	1		1			1				1					1					
(3,1)		-1		-2			-3				-4					-5					
(3,2)		1	-1	3	-4		6	-10			10	-20				15	-35				
(4,1)				1			3				6					10					
(4,2)				-2	3		-8	15			-20	45				-40	105				
(4,3)				1	-2	1	6	-15	10		20	-60	50			50	-175	175			
(5,1)							-1				-4					-10					
(5,2)							3	-6			15	-36				45	-126				
(5,3)							-3	8	-6		-20	64	-60			-75	280	-315			
(5,4)							1	-3	3	-1	10	-36	45	-20		50	-210	315	-175		
(6,1)											1					5					
(6,2)											-4	10				-24	70				
(6,3)											6	-20	20			45	-175	210			
(6,4)											-4	15	-20	10		-40	175	-280	175		
(6,5)											1	-4	6	-4	1	15	-70	126	-105	35	
(7,1)																-1					
(7,2)																5	-15				
(7,3)																-10	40	-50			
(7,4)																10	-45	75	-50		
(7,5)																-5	24	-45	40	-15	
(7,6)																1	-5	10	-10	5	-1

Table 1b. $k_{i_4 i_3 i_2 i_1}$ Coefficients for m = 0, 1, 2.

	m=0	m=1		m=2		
(i_4,i_3,i_2) \ i_1	0	0	1	0	1	2
(3,2,1)	1	1		1		
(4,2,1)		-1		-2		
(4,3,1)		1		3		
(4,3,2)		-1	1	-4	5	
(5,2,1)				1		
(5,3,1)				-2		
(5,3,2)				3	-4	
(5,4,1)				1		
(5,4,2)				-2	3	
(5,4,3)				1	-2	1

Table 2. Coefficients for $R^*(n; i,j,0)$ (Note: $r_i = n+i$, $s_i = 2n+i$, and $r_3/r_1r_4 = r_3(r_1r_4)^{-1}$)

(i,j) \ ℓ	$\alpha_{ij\ell}(n)$: 1	2	3	4	5	$\gamma_{ij}(n)$
(2,1)	$1/r_2s_5$					$1/s_5$
(3,1)	$1/r_2r_3$					$3/2r_3$
(3,2)	$1/r_3s_7$					$3/s_7$
(4,1)	$3/r_2s_7$	$-1/2r_3r_4$				$3(4n+15)/2r_4s_7$
(4,2)	$1/r_3r_4$	$-1/r_3r_4$				$(8n+25)/2r_3r_4$
(4,3)	$1/r_4s_9$	$-3/2r_3r_4$				$3(4n+13)/2r_3s_9$
(5,1)	$2/r_2r_4$	$-3/2r_4r_5$	$-1/2r_4r_5$			$5(4n+17)/4r_4r_5$
(5,2)	$3/r_3s_9$	$-3s_7/2r_3r_4r_5$	$-1/r_4r_5$			$c_{52}/2r_3r_5s_9$
(5,3)	$1/r_4r_5$	$-9/2r_3r_5$	$-3/2r_4r_5$			$15s_7/4r_3r_5$
(5,4)	$1/r_5s_{11}$	$-(7n+24)/2r_3r_4r_5$	$-2/r_4r_5$			$5(4n+15)/2r_3s_{11}$
(6,1)	$5/r_2s_9$	$-3(4n+15)/4r_4r_5r_6$	$-3r_3/2r_4r_5r_6$	$-1/2r_5r_6$		$c_{61}/4r_5r_6s_9$
(6,2)	$2/r_3r_5$	$-3(4n+15)/2r_3r_5r_6$	$-3s_7/2r_4r_5r_6$	$-1/r_5r_6$		$3(8n+41)s_7/4r_3r_5r_6$
(6,3)	$3/r_4s_{11}$	$-3a_{632}/4r_3r_4r_5r_6$	$-9/2r_5r_6$	$-3/2r_5r_6$		$c_{63}/4r_3r_4r_6s_{11}$
(6,4)	$1/r_5r_6$	$-3(3n+11)/r_3r_4r_6$	$-3s_9/r_4r_5r_6$	$-2/r_5r_6$		$c_{64}/2r_3r_4r_6$
(6,5)	$1/r_6s_{13}$	$-3a_{652}/4r_3r_4r_5r_6$	$-3(3n+11)/2r_4r_5r_6$	$-5/2r_5r_6$		$c_{65}/4r_3r_4s_{13}$
(7,1)	$3/r_2r_5$	$-5(4n+17)/4r_5r_6r_7$	$-3(4n+15)r_3/4r_4r_5r_6r_7$	$-3r_3/2r_5r_6r_7$	$-1/2r_6r_7$	$c_{71}/8r_5r_6r_7$
(7,2)	$5/r_3s_{11}$	$-5(4n+17)s_9/4r_3r_5r_6r_7$	$-3(4n+15)/2r_5r_6r_7$	$-3s_7/2r_5r_6r_7$	$-1/r_6r_7$	$c_{72}/4r_3r_6r_7s_{11}$
(7,3)	$2/r_4r_6$	$-15a_{732}/4r_3r_4r_6r_7$	$-3a_{733}/4r_4r_5r_6r_7$	$-9r_4/2r_5r_6r_7$	$-3/2r_6r_7$	$c_{73}/8r_3r_4r_6r_7$
(7,4)	$3/r_5s_{13}$	$-3a_{742}/2r_3r_4r_5r_6r_7$	$-3a_{743}/r_4r_5r_6r_7$	$-3s_9/r_5r_6r_7$	$-2/r_6r_7$	$c_{74}/2r_3r_4r_7s_{13}$
(7,5)	$1/r_6r_7$	$-a_{752}/4r_3r_4r_5r_7$	$-(23n+91)s_{11}/4r_4r_5r_6r_7$	$-15/2r_6r_7$	$-5/2r_6r_7$	$c_{75}/8r_3r_4r_7$
(7,6)	$1/r_7s_{15}$	$-3a_{762}/4r_3r_4r_6r_7$	$-3a_{763}/2r_4r_5r_6r_7$	$-(11n+43)/2r_5r_6r_7$	$-3/r_6r_7$	$c_{76}/4r_3r_4s_{15}$

$a_{632} = 12n^2+105n+232$, $a_{652} = 8n^2+65n+133$, $a_{732} = 4n^2+37n+88$, $a_{733} = a_{632}$, $a_{742} = 11n^3+161n^2+781n+1251$,
$a_{743} = 4n^2+39n+97$, $a_{752} = 58n^2+495n+1073$, $a_{762} = 12n^2+109n+254$, $a_{763} = 5n^2+43n+94$, $c_{52} = 15(2n^2+16n+31)$
$c_{61} = 15(4n^2+38n+91)$, $c_{63} = 3(36n^3+486n^2+2151n+3136)$, $c_{64} = 3(8n^2+65n+133)$, $c_{65} = 15(4n^2+34n+73)$
$c_{71} = 21(4n^2+42n+113)$, $c_{72} = 35(4n^3+60n^2+296n+477)$, $c_{73} = 168n^3+2394n^2+11193n+17304$, $c_{74} = 21(4n^3+61n^2+304n+501)$
$c_{75} = 35(4n^2+36n+83)$, $c_{76} = 21(4n^2+38n+93)$

Table 2. (Continued)

$\beta_{ij\ell}(n)$

(i,j) \ ℓ	1	2	3	4	5	6	7
(2,1)	$-1/r_2$	$2/s_5$					
(3,1)	$-1/r_2$	$-1/r_3$	$2/r_3$				
(3,2)	$-2/r_3$	$1/r_3$	$2/s_7$				
(4,1)	$-1/r_2$	$-1/r_4$	$-s_5/2r_3r_4$	$2(3n+11)/r_4s_7$			
(4,2)	$-2/r_3$	$-s_5/r_3r_4$	s_7/r_3r_4	$2/r_4$			
(4,3)	$-(3n+8)/r_3r_4$	$1/r_4$	$s_7/2r_3r_4$	$2/s_9$			
(5,1)	$-1/r_2$	$-1/r_5$	$-s_5/2r_4r_5$	$-s_5/2r_4r_5$	$4/r_5$		
(5,2)	$-2/r_3$	$-s_5/r_3r_5$	$-s_5/r_4r_5$	$b_{524}/2r_3r_4r_5$	$2(3n+14)/r_5s_9$		
(5,3)	$-(3n+8)/r_3r_4$	$-(3n+8)/r_4r_5$	$(4n+17)/2r_4r_5$	$(4n+17)/2r_3r_5$	$2/r_5$		
(5,4)	$4r_3/r_4r_5$	$1/r_5$	$1/r_5$	$s_9/2r_3r_5$	$2/s_{11}$		
(6,1)	$-1/r_2$	$-1/r_6$	$-s_5/2r_5r_6$	$-r_3s_5/2r_4r_5r_6$	$-s_5s_7/4r_4r_5r_6$	$b_{616}/r_5r_6s_9$	
(6,2)	$-2/r_3$	$-s_5/r_3r_6$	$-s_5/r_5r_6$	$-s_5s_7/2r_4r_5r_6$	$b_{625}/2r_3r_5r_6$	$4/r_6$	
(6,3)	$-(3n+8)/r_3r_4$	$-(3n+8)/r_4r_6$	$-(3n+8)s_7/2r_4r_5r_6$	$b_{634}/2r_4r_5r_6$	$b_{635}s_9/4r_3r_4r_5r_6$	$2(3n+17)/r_6s_{11}$	
(6,4)	$-4r_3/r_4r_5$	$-2r_3s_7/r_4r_5r_6$	$2/r_6$	s_9/r_4r_6	$r_5s_9/r_3r_4r_6$	$2/r_6$	
(6,5)	$-b_{651}/r_4r_5r_6$	$1/r_6$	$s_9/2r_5r_6$	$s_9/2r_4r_6$	$s_9s_{11}/4r_3r_4r_6$	$2/s_{13}$	
(7,1)	$-1/r_2$	$-1/r_7$	$-s_5/2r_6r_7$	$-r_2s_5/2r_5r_6r_7$	$-r_3s_5s_7/4r_4r_5r_6r_7$	$-s_5s_7/4r_5r_6r_7$	$b_{717}/r_5r_6r_7$
(7,2)	$-2/r_3$	$-s_5/r_3r_7$	$-s_5/r_6r_7$	$-s_5s_7/2r_5r_6r_7$	$-s_5s_7/2r_5r_6r_7$	$b_{726}/4r_3r_5r_6r_7$	$b_{727}/r_6r_7s_{11}$
(7,3)	$-(3n+8)/r_3r_4$	$-(3n+8)/r_4r_7$	$-(3n+8)s_7/2r_4r_6r_7$	$-(3n+8)s_7/2r_5r_6r_7$	$b_{735}/4r_4r_5r_6r_7$	$b_{736}/4r_3r_4r_6r_7$	$4/r_7$
(7,4)	$-4r_3/r_4r_5$	$-2r_3s_7/r_4r_5r_7$	$-2r_3s_7/r_5r_6r_7$	$b_{744}/r_5r_6r_7$	$b_{745}/r_4r_6r_7$	$b_{746}s_{11}/2r_3r_4r_6r_7$	$2(3n+20)/r_7s_{13}$
(7,5)	$-b_{751}/r_4r_5r_6$	$-b_{752}/r_5r_6r_7$	$(4n+23)/2r_6r_7$	$(4n+23)/2r_6r_7$	$(4n+23)s_{11}/4r_4r_6r_7$	$(4n+23)s_{11}/4r_3r_4r_7$	$2/r_7$
(7,6)	$-b_{761}/r_5r_6r_7$	$1/r_7$	r_5/r_6r_7	$s_{11}/2r_6r_7$	$s_{11}/2r_4r_7$	$s_{11}s_{13}/4r_3r_4r_7$	$2/s_{15}$

$b_{524} = 6n^2+46n+89$, $b_{616} = 2(5n^2+45n+102)$, $b_{625} = 8n^2+68n+149$, $b_{634} = 6n^2+55n+128$, $b_{635} = b_{634}$,

$b_{651} = 5n^2+35n+62$, $b_{717} = 6n^2+58n+144$, $b_{726} = 20n^3+280n^2+1338n+2181$, $b_{727} = 2(5n^2+55n+152)$

$b_{735} = 16n^3+236n^2+1179n+1992$, $b_{736} = b_{735}$, $b_{744} = 3n^2+32n+87$, $b_{745} = b_{744}$, $b_{746} = b_{744}$, $b_{751} = 5n^2+35n+62$

$b_{752} = b_{751}$, $b_{761} = 6n^2+46n+92$.

Table 3. Coefficients for $R^*(n;\, i,j,k,0)$.

(i,j,k) ℓ	$\gamma_{ijk\ell}(n)$ 1	2	3	4	5	$\alpha_{ijk}(n)$	$\delta_{ijk}(n)$
(3,2,1)	$1/r_3s_5$	$-1/r_3r_3$	$1/r_3s_7$			$1/r_2r_3r_3s_5s_7$	$1/r_3r_3s_7$
(4,2,1)	$1/r_3s_5$	$1/2r_3r_4$	$-(5n+19)/r_3r_4s_7$	$3/2r_3r_4$		$3/r_2r_3r_4s_5s_7$	$4/r_3r_4s_7$
(4,3,1)	$3/2r_3r_4$	$-3/r_4s_7$	$-3/2r_3r_4$	$3/r_3s_9$		$3/r_2r_3r_4s_7s_9$	$12/r_4s_7s_9$
(4,3,2)	$3/r_4s_7$	$-(5n+16)/2r_3r_4r_4$	$r_2/r_3r_4s_9$	$1/2r_3r_4$		$1/r_3r_4r_4s_7s_9$	$4/r_4r_4s_9$
(5,2,1)	$1/r_3s_5$	$1/2r_3r_5$	$1/2r_4r_5$	$-3(3n+13)/2r_3r_4r_5$	$3(4n+19)/2r_3r_5s_9$	$6/r_2r_3r_4s_5s_9$	$2(5n+24)/r_3r_4r_5s_9$
(5,3,1)	$3/2r_3r_4$	$3/2r_4r_5$	$-(7n+32)/2r_4r_4r_5$	$-(7n+32)/2r_3r_4r_5$	$(8n+33)/2r_3r_4r_5$	$2/r_2r_3r_4r_4r_5$	$4(5n+21)/r_4r_4r_5s_9$
(5,3,2)	$3/r_4s_7$	$3/2r_4r_5$	$-c_{5323}/2r_3r_4r_5s_9$	$3r_2/2r_3r_4r_5$	$3/2r_3r_4$	$3/r_3r_4r_5s_7s_9$	$3(5n+18)/r_4r_4r_5s_9$
(5,4,1)	$3(4n+15)/2r_4r_5s_7$	$-2/r_4r_5$	$-2/r_4r_5$	$-s_9/r_3r_4r_5$	$3(4n+17)/2r_3r_4s_{11}$	$6/r_2r_4r_5s_7s_{11}$	$d_{542}/r_4r_4r_5s_9s_{11}$
(5,4,2)	$(8n+25)/2r_3r_4r_5$	$-(7n+24)/r_3r_5s_9$	$-(7n+24)/2r_3r_4r_5$	$3r_2/r_3r_4s_{11}$	$3/2r_3r_4$	$3/r_3r_4r_5s_9s_{11}$	$4(10n+37)/r_4r_5s_9s_{11}$
(5,4,3)	$3(4n+13)/r_3r_5s_9$	$-3(3n+11)/2r_3r_5r_5$	$r_2/r_3r_5s_{11}$	$r_2/2r_3r_4r_5$	$r_6/2r_3r_4r_5$	$1/r_4r_5r_5s_9s_{11}$	$2(5n+19)/r_4r_5r_5s_{11}$

$c_{5323} = 3(8n^2+65n+128)$ $d_{542} = 2(20n^2+158n+309)$

(i,j,k) ℓ	$\beta_{ijk\ell}(n)$ 1	2	3	4	5	6	7
(3,2,1)	$-1/r_2r_3s_7$	$2/r_3r_3s_5s_7$	$1/r_3r_3s_7$				
(4,2,1)	$-1/r_2r_3r_4$	$-(2n-1)/r_3r_4s_5s_7$	0	$3/r_3r_4s_7$			
(4,3,1)	$-1/r_2r_4s_9$	$-3/r_3r_4s_9$	$12/r_3r_4s_7s_9$	$(2n-3)/r_3r_4s_7s_9$	$3/r_3r_4s_9$		
(4,3,2)	$-2/r_3r_4s_9$	$4/r_3r_4r_4s_9$	$-1/r_3r_4s_7$	$s_5/r_3r_4r_4s_9$	$1/r_3r_4s_9$		
(5,2,1)	$-3/r_2r_3s_9$	$-b_{5212}/r_3r_4r_5s_5s_9$	$-1/2r_3r_4r_5$	0	$3(4n+19)/2r_3r_4r_5s_9$		
(5,3,1)	$-1/r_2r_4r_5$	$-6/r_3r_5s_9$	$-b_{5313}/2r_3r_4r_4r_5s_9$	0	$b_{5315}/2r_3r_4r_4r_5s_9$	$(8n+33)/r_3r_4r_5s_9$	
(5,3,2)	$-2/r_3r_4r_5$	$-3n/r_3r_4r_5s_9$	$6(5n+18)/r_4r_4r_5s_7s_9$	$-b_{5324}/2r_3r_4r_4r_5s_9$	$3s_5/r_3r_4r_5s_9$	$3/2r_3r_4r_5$	
(5,4,1)	$-1/r_2r_5s_{11}$	$-3/r_4r_5s_{11}$	$-3(4n+17)s_5/2r_3r_4r_5s_9s_{11}$	$b_{5414}/r_4r_4s_5s_7s_9s_{11}$	$b_{5415}/r_4r_4r_5s_9s_{11}$	$b_{5416}/r_3r_4r_5s_9s_{11}$	$3(4n+17)/2r_3r_4r_5s_{11}$
(5,4,2)	$-2/r_3r_5s_{11}$	$-3s_5/r_3r_4r_5s_{11}$	$3(14n+47)/r_3r_4r_5s_9s_{11}$	$-1/r_3r_4r_5$	$b_{5425}/r_3r_4r_5s_9s_{11}$	$3s_5/r_3r_4r_5s_{11}$	$3/r_3r_4s_{11}$
(5,4,3)	$-(3n+8)/r_3r_4r_5s_{11}$	$-(n-3)/r_4r_5r_5s_{11}$	$1/2r_3r_4r_5$	$-3/r_3r_5s_9$	$(3n+8)/r_4r_5r_5s_{11}$	$s_5/r_3r_4r_5s_{11}$	$1/2r_3r_4r_5$

$b_{5212} = 4n^2+16n-15$, $b_{5313} = 6n^2+11n-60$, $b_{5315} = 3(2n^2+5n-14)$, $b_{5324} = 4n^2+67n+180$, $b_{5414} = 4(20n^2+158n+309)$

$b_{5415} = 2n^2-19n-120$, $b_{5416} = 3(2n^2+7n-7)$, $b_{5425} = 2n^2-25n-108$

Table 4. Upper α percentage points of $U^{(2)}$.

α = .10

m \ n	5	10	15	20	25	30	35	40	45	50	60	70	80	90	100	130	160	200
-.5	.81989	.40130	.26509	.19782	.15776	.13118	.11226	.098110	.087126	.078354	.065219	.055855	.048842	.043394	.039039	.030005	.024367	.019484
0	1.1454	.55537	.36557	.27230	.21690	.18022	.15414	.13465	.11954	.10747	.089423	.076562	.066935	.059458	.053484	.041096	.033367	.026677
.5	1.4593	.70291	.46151	.34329	.27323	.22689	.19398	.16940	.15035	.13515	.11242	.096228	.084115	.074710	.067196	.051621	.041907	.033502
1	1.7667	.84667	.55477	.41222	.32786	.27213	.23259	.20307	.18019	.16195	.13467	.11526	.10074	.089465	.080461	.061801	.050166	.040100
1.5	2.0702	.98795	.64625	.47975	.38136	.31641	.27035	.23599	.20937	.18815	.15643	.13386	.11698	.10388	.093417	.071741	.058230	.046542
2	2.3709	1.1275	.73646	.54628	.43402	.35998	.30750	.26836	.23806	.21390	.17781	.15213	.13293	.11804	.10614	.081505	.066149	.052868
2.5	2.6696	1.2657	.82570	.61205	.48606	.40302	.34418	.30032	.26637	.23931	.19890	.17016	.14867	.13200	.11869	.091130	.073955	.059102
3	2.9669	1.4030	.91420	.67722	.53759	.44562	.38049	.33195	.29438	.26445	.21976	.18798	.16423	.14581	.13110	.10064	.081670	.065264
3.5	3.2631	1.5395	1.0021	.74191	.58872	.48787	.41648	.36330	.32215	.28937	.24043	.20564	.17964	.15948	.14339	.11007	.089309	.071364
4	3.5584	1.6754	1.0895	.80620	.63952	.52984	.45223	.39443	.34971	.31410	.26094	.22316	.19494	.17305	.15558	.11941	.096885	.077413
4.5	3.8529	1.8107	1.1765	.87015	.69003	.57156	.48776	.42536	.37710	.33867	.28131	.24056	.21012	.18652	.16768	.12869	.10441	.083418
5	4.1469	1.9456	1.2632	.93383	.74030	.61307	.52310	.45612	.40433	.36310	.30157	.25786	.22522	.19991	.17971	.13791	.11188	.089385
10	7.0696	3.2810	2.1187	1.5611	1.2348	1.0210	.87010	.75798	.67141	.60256	.49997	.42720	.37292	.33086	.29733	.22800	.18488	.14765
15	9.9787	4.6048	2.9643	2.1798	1.7218	1.4221	1.2109	1.0542	.93331	.83723	.69421	.59287	.51732	.45883	.41222	.31592	.25608	.20444
20	12.883	5.9240	3.8057	2.7946	2.2052	1.8200	1.5488	1.3477	1.1927	1.0695	.88635	.75665	.66002	.58526	.52569	.40269	.32632	.26044
25	15.785	7.2409	4.6448	3.4073	2.6867	2.2161	1.8850	1.6396	1.4506	1.3004	1.0772	.91931	.80169	.71072	.63827	.48874	.39595	.31595
30	18.685	8.5564	5.4827	4.0188	3.1670	2.6111	2.2202	1.9306	1.7075	1.5304	1.2673	1.0812	.94266	.83555	.75026	.57430	.46516	.37109
35	21.585	9.8711	6.3197	4.6295	3.6465	3.0053	2.5546	2.2208	1.9637	1.7598	1.4568	1.2426	1.0831	.95990	.86180	.65949	.53405	.42598
40	24.485	11.185	7.1561	5.2396	4.1254	3.3989	2.8885	2.5105	2.2195	1.9887	1.6458	1.4035	1.2232	1.0839	.97302	.74440	.60270	.48066
45	27.384	12.499	7.9921	5.8493	4.6040	3.7922	3.2220	2.7998	2.4749	2.2172	1.8345	1.5642	1.3630	1.2076	1.0840	.82910	.67117	.53518
50	30.282	13.812	8.8278	6.4586	5.0822	4.1852	3.5552	3.0889	2.7300	2.4455	2.0230	1.7246	1.5026	1.3312	1.1947	.91362	.73948	.58957
60	36.079	16.439	10.499	7.6767	6.0380	4.9704	4.2209	3.6663	3.2397	2.9014	2.3993	2.0448	1.7812	1.5777	1.4158	1.0822	.87573	.69803
70	41.875	19.064	12.169	8.8942	6.9932	5.7550	4.8860	4.2432	3.7487	3.3567	2.7751	2.3645	2.0594	1.8237	1.6364	1.2505	1.0116	.80618
80	47.671	21.690	13.838	10.111	7.9479	6.5393	5.5507	4.8196	4.2573	3.8116	3.1505	2.6839	2.3371	2.0694	1.8566	1.4184	1.1472	.91408
90	53.467	24.315	15.508	11.328	8.9024	7.3232	6.2151	5.3957	4.7656	4.2663	3.5256	3.0029	2.6146	2.3149	2.0766	1.5860	1.2826	1.0218
100	59.263	26.939	17.177	12.545	9.8566	8.1069	6.8793	5.9716	5.2737	4.7207	3.9004	3.3217	2.8919	2.5601	2.2963	1.7535	1.4179	1.1294
130	76.649	34.813	22.184	16.193	12.719	10.457	8.8710	7.6984	6.7971	6.0830	5.0241	4.2774	3.7228	3.2949	2.9548	2.2552	1.8229	1.4514
160	94.035	42.687	27.191	19.842	15.580	12.807	10.862	9.4244	8.3196	7.4444	6.1470	5.2322	4.5529	4.0289	3.6125	2.7562	2.2272	1.7728
200	117.22	53.184	33.865	24.705	19.394	15.939	13.516	11.725	10.349	9.2590	7.6434	6.5045	5.6590	5.0068	4.4887	3.4234	2.7656	2.2007

Table 4. (Continued)

α = .05

m \ r	5	10	15	20	25	30	35	40	45
-.5	1.0676	.50646	.33087	.24551	.19512	.16187	.13830	.12071	.10710
0	1.4508	.66072	.44292	.32797	.26031	.21577	.18424	.16074	.14255
.5	1.8183	.84638	.54905	.40592	.32188	.26663	.22755	.19846	.17596
1	2.1771	1.0071	.65171	.48120	.38127	.31566	.26929	.23479	.20812
1.5	2.5304	1.1645	.75207	.55469	.43919	.36344	.30995	.27017	.23943
2	2.8802	1.3197	.85078	.62688	.49605	.41032	.34982	.30485	.27012
2.5	3.2273	1.4732	.94824	.69809	.55209	.45650	.38909	.33900	.30033
3	3.5724	1.6255	1.0447	.76853	.60750	.50214	.42787	.37272	.33015
3.5	3.9161	1.7767	1.1405	.83835	.66238	.54733	.46627	.40609	.35966
4	4.2585	1.9271	1.2356	.90766	.71683	.59214	.50434	.43917	.38890
4.5	4.6001	2.0768	1.3302	.97654	.77092	.63665	.54213	.47200	.41792
5	4.9409	2.2260	1.4243	1.0451	.82470	.68088	.57968	.50462	.44675
10	8.3254	3.6997	2.3512	1.7180	1.3520	1.1140	.94696	.82337	.72824
15	11.692	5.1584	3.2652	2.3798	1.8694	1.5383	1.3063	1.1349	1.0031
20	15.052	6.6111	4.1738	3.0367	2.3824	1.9585	1.6619	1.4429	1.2747
25	18.409	8.0608	5.0796	3.6909	2.8929	2.3764	2.0153	1.7489	1.5444
30	21.764	9.5087	5.9837	4.3435	3.4019	2.7928	2.3673	2.0536	1.8128
35	25.119	10.956	6.8866	4.9950	3.9098	3.2083	2.7184	2.3574	2.0804
40	28.472	12.402	7.7888	5.6458	4.4170	3.6230	3.0688	2.6605	2.3474
45	31.825	13.847	8.6905	6.2961	4.9237	4.0373	3.4187	2.9632	2.6139
50	35.178	15.292	9.5917	6.9459	5.4300	4.4511	3.7683	3.2655	2.8801
60	41.883	18.182	11.393	8.2448	6.4417	5.2780	4.4665	3.8692	3.4116
70	48.587	21.071	13.194	9.5428	7.4526	6.1040	5.1639	4.4722	3.9423
80	55.291	23.959	14.995	10.840	8.4629	6.9295	5.8608	5.0746	4.4725
90	61.994	26.847	16.795	12.137	9.4729	7.7546	6.5574	5.6766	5.0023
100	68.697	29.735	18.595	13.434	10.483	8.5795	7.2536	6.2784	5.5318
130	68.806	38.397	23.993	17.324	13.511	11.053	9.3411	8.0825	7.1192
160	108.91	47.059	29.391	21.213	16.538	13.525	11.428	9.8856	8.7055
200	135.72	58.608	36.587	26.397	20.573	16.821	14.209	12.289	10.820

m \ r	50	60	70	80	90	100	130	160	200
-.5	.096237	.080011	.068466	.059832	.053132	.047781	.036694	.029783	.023805
0	.12806	.10642	.091038	.079539	.070618	.063497	.048749	.039560	.031614
.5	.15803	.13129	.11228	.098082	.087070	.78280	.060084	.048751	.038955
1	.18689	.15521	.13272	.11591	.10289	.092493	.070979	.057584	.046008
1.5	.21497	.17849	.15259	.13326	.11827	.10631	.081570	.066169	.052861
2	.24249	.20130	.17206	.15024	.13333	.11984	.091934	.074569	.059567
2.5	.26957	.22373	.19121	.16694	.14814	.13314	.10212	.082825	.066156
3	.29630	.24587	.21010	.18342	.16274	.14626	.11217	.090964	.072652
3.5	.32274	.26776	.22878	.19970	.17718	.15922	.12210	.099008	.079070
4	.34895	.28946	.24729	.21584	.19148	.17206	.13192	.10697	.085423
4.5	.37495	.31097	.26564	.23183	.20566	.18479	.14167	.11486	.091719
5	.40077	.33234	.28386	.24771	.21973	.19743	.15134	.12269	.097966
10	.65278	.54066	.46138	.40235	.35671	.32036	.24534	.19878	.15864
15	.89865	.74365	.63418	.55277	.48986	.43979	.33656	.27257	.21743
20	1.1415	.94395	.80458	.70101	.62102	.55740	.42632	.34512	.27522
25	1.3825	1.1426	.97352	.84792	.75096	.67388	.51515	.41690	.33236
30	1.6224	1.3403	1.1415	.99393	.88008	.78958	.60334	.48813	.38904
35	1.8614	1.5372	1.3088	1.1393	1.0086	.90473	.69106	.55896	.44539
40	2.0999	1.7335	1.4755	1.2842	1.1367	1.0194	.77843	.62948	.50148
45	2.3378	1.9294	1.6419	1.4287	1.2644	1.1338	.86552	.69976	.55736
50	2.5755	2.1249	1.8079	1.5729	1.3918	1.2480	.95238	.76984	.61307
60	3.0500	2.5153	2.1393	1.8607	1.6460	1.4756	1.1256	.90954	.72409
70	3.5237	2.9050	2.4700	2.1478	1.8996	1.7027	1.2982	1.0488	.83470
80	3.9969	3.2941	2.8003	2.4345	2.1528	1.9293	1.4705	1.1876	.94500
90	4.4697	3.6829	3.1301	2.7208	2.4056	2.1556	1.6425	1.3262	1.0551
100	4.9423	4.0714	3.4597	3.0068	2.6581	2.3816	1.8142	1.4646	1.1649
130	6.3587	5.2358	4.4473	3.8637	3.4147	3.0586	2.3285	1.8789	1.4937
160	7.7741	6.3990	5.4338	4.7196	4.1702	3.7346	2.8417	2.2922	1.8216
200	9.6604	7.9491	6.7482	5.8599	5.1766	4.6350	3.5252	2.8424	2.2581

Table 4. (Continued)

α = .025

m \ n	5	10	15	20	25	30	35	40	45	50	60	70	80	90	100	130	160	200
-.5	1.3390	.61533	.39755	.29333	.23233	.19230	.16403	.14300	.12675	.11381	.094513	.080810	.070576	.062642	.056312	.043211	.035056	.028007
0	1.7821	.80882	.52024	.38299	.30293	.25051	.21354	.18607	.16486	.14798	.12284	.10499	.091673	.081352	.073120	.056091	.045495	.036342
.5	2.2056	.99189	.63584	.46728	.36920	.30510	.25994	.22641	.20054	.17997	.14933	.12761	.11140	.098842	.088829	.068125	.055247	.044125
1	2.6183	1.1690	.74728	.54838	.43289	.35752	.30446	.26511	.23476	.21063	.17472	.14927	.13029	.11559	.10387	.079639	.064576	.051570
1.5	3.0243	1.3421	.85596	.62734	.49484	.40846	.34771	.30268	.26796	.24039	.19934	.17027	.14859	.13181	.11844	.090793	.073612	.058780
2	3.4257	1.5126	.96266	.70475	.55551	.45832	.39003	.33942	.30043	.26947	.22340	.19078	.16647	.14766	.13266	.10168	.082429	.065814
2.5	3.8239	1.6810	1.0679	.78100	.61522	.50736	.43162	.37553	.33232	.29803	.24703	.21092	.18402	.16321	.14662	.11236	.091076	.072711
3	4.2197	1.8478	1.1719	.85632	.67417	.55575	.47265	.41113	.36376	.32617	.27030	.23075	.20130	.17851	.16036	.12287	.099585	.079498
3.5	4.6136	2.0135	1.2751	.93091	.73249	.60360	.51320	.44631	.39483	.35398	.29328	.25034	.21836	.19362	.17392	.13324	.10798	.086192
4	5.0061	2.1781	1.3775	1.0049	.79030	.65101	.55337	.48115	.42558	.38150	.31602	.26971	.23523	.20857	.18733	.14349	.11628	.092809
4.5	5.3974	2.3419	1.4792	1.0783	.84768	.69805	.59321	.51569	.45607	.40878	.33855	.28890	.25195	.22337	.20061	.15364	.12449	.099358
5	5.7878	2.5051	1.5805	1.1514	.90469	.74477	.63277	.54998	.48632	.43585	.36091	.30794	.26852	.23804	.21378	.16371	.13264	.10585
10	9.6628	4.1150	2.5754	1.8670	1.4622	1.2009	1.0185	.88402	.78082	.69914	.57810	.49274	.42932	.38035	.34140	.26155	.21144	.16863
15	13.516	5.7067	3.5549	2.5694	2.0080	1.6466	1.3948	1.2094	1.0674	.95511	.78894	.67192	.58509	.51810	.46486	.35529	.28750	.22918
20	17.360	7.2912	4.5278	3.2658	2.5485	2.0874	1.7665	1.5307	1.3501	1.2075	.99658	.84825	.73827	.65349	.58615	.44768	.36209	.28853
25	21.201	8.8721	5.4974	3.9591	3.0860	2.5254	2.1357	1.8495	1.6305	1.4577	1.2023	1.0228	.88986	.78742	.70609	.53897	.43576	.34711
30	25.040	10.451	6.4649	4.6504	3.6216	2.9617	2.5033	2.1668	1.9095	1.7065	1.4068	1.1963	1.0404	.92036	.82511	.62949	.50878	.40514
35	28.878	12.028	7.4311	5.3404	4.1561	3.3968	2.8697	2.4830	2.1874	1.9543	1.6103	1.3689	1.1902	1.0526	.94346	.71947	.58132	.46278
40	32.714	13.605	8.3964	6.0296	4.6896	3.8310	3.2353	2.7984	2.4646	2.2014	1.8132	1.5408	1.3393	1.1843	1.0613	.80901	.65350	.52011
45	36.550	15.181	9.3610	6.7181	5.2225	4.2647	3.6003	3.1133	2.7413	2.4480	2.0156	1.7124	1.4881	1.3156	1.1788	.89822	.72539	.57719
50	40.386	16.756	10.325	7.4061	5.7550	4.6978	3.9649	3.4277	3.0175	2.6942	2.2176	1.8835	1.6365	1.4465	1.2959	.98717	.79703	.63407
60	48.056	19.906	12.252	8.7811	6.8188	5.5632	4.6930	4.0555	3.5689	3.1855	2.6207	2.2249	1.9325	1.7076	1.5295	1.1644	.93978	.74734
70	55.726	23.055	14.179	10.155	7.8817	6.4276	5.4202	4.6824	4.1194	3.6760	3.0229	2.5656	2.2277	1.9680	1.7623	1.3410	1.0820	.86013
80	63.395	26.203	16.105	11.529	8.9440	7.2913	6.1468	5.3087	4.6693	4.1659	3.4246	2.9056	2.5223	2.2279	1.9946	1.5172	1.2237	.97255
90	71.063	29.351	18.030	12.902	10.006	8.1546	6.8728	5.9345	5.2188	4.6553	3.8258	3.2452	2.8165	2.4873	2.2266	1.6930	1.3652	1.0847
100	78.732	32.499	19.955	14.274	11.067	9.0175	7.5986	6.5600	5.7678	5.1443	4.2266	3.5844	3.1104	2.7464	2.4582	1.8685	1.5063	1.1966
130	101.74	41.941	25.729	18.391	14.250	11.605	9.7744	8.4349	7.4137	6.6101	5.4278	4.6009	3.9908	3.5225	3.1518	2.3939	1.9288	1.5313
160	124.74	51.382	31.502	22.507	17.432	14.191	11.949	10.309	9.0583	8.0746	6.6277	5.6161	4.8699	4.2973	3.8442	2.9182	2.3502	1.8651
200	155.41	63.970	39.198	27.993	21.674	17.639	14.847	12.806	11.250	10.026	8.2264	6.9685	6.0409	5.3292	4.7663	3.6161	2.9110	2.3091

Table 4. (Continued)

α = .01

m \ n	5	10	15	20	25	30	35	40	45	50	60	70	80	90	100	130	160	200
-.5	1.7419	.76665	.48801	.35740	.28181	.23256	.19795	.17229	.15252	.13682	.11345	.096896	.084558	.075006	.067393	.051661	.041884	.033444
0	2.2704	.98500	.62391	.45581	.35887	.29587	.25166	.21893	.19373	.17373	.14399	.12294	.10726	.095124	.085455	.065486	.053081	.042378
.5	2.7740	1.1908	.75136	.54786	.43084	.35493	.30173	.26238	.23210	.20808	.17240	.14715	.12836	.11382	.10224	.078325	.063478	.050670
1	3.2642	1.3893	.87386	.63613	.49976	.41143	.34959	.30389	.26875	.24088	.19950	.17025	.14847	.13164	.11823	.090556	.073379	.058567
1.5	3.7459	1.5831	.99306	.72186	.56662	.46619	.39595	.34408	.30421	.27261	.22571	.19257	.16791	.14885	.13368	.10237	.082938	.066188
2	4.2220	1.7736	1.1099	.80577	.63197	.51968	.44121	.38329	.33879	.30355	.25125	.21432	.18685	.16562	.14872	.11386	.092240	.073604
2.5	4.6939	1.9617	1.2250	.88828	.69618	.57220	.48561	.42175	.37271	.33388	.27628	.23562	.20539	.18204	.16345	.12511	.10134	.080858
3	5.1629	2.1479	1.3387	.96970	.75949	.62394	.52934	.45961	.40609	.36371	.30090	.25657	.22362	.19817	.17792	.13617	.11028	.087983
3.5	5.6295	2.3326	1.4513	1.0502	.82205	.67505	.57252	.49698	.43902	.39315	.32517	.27722	.24159	.21407	.19218	.14705	.11909	.094999
4	6.0943	2.5161	1.5630	1.1301	.88401	.72563	.61524	.53394	.47159	.42225	.34916	.29763	.25934	.22978	.20626	.15780	.12778	.10192
4.5	6.5576	2.6987	1.6739	1.2093	.94546	.77578	.65757	.57055	.50384	.45106	.37291	.31782	.27690	.24531	.22019	.16843	.13637	.10877
5	7.0198	2.8804	1.7842	1.2879	1.0065	.82554	.69957	.60687	.53581	.47963	.39644	.33782	.29430	.26070	.23399	.17896	.14488	.11555
10	11.606	4.6716	2.8665	2.0574	1.6016	1.3101	1.1079	.95949	.84604	.75651	.62427	.53131	.46242	.40932	.36716	.28045	.22686	.18079
15	16.163	6.4407	3.9303	2.8110	2.1828	1.7822	1.5049	1.3019	1.1469	1.0247	.84457	.71815	.62459	.55256	.49541	.37804	.30561	.24341
20	20.711	8.2012	4.9863	3.5575	2.7576	2.2484	1.8966	1.6393	1.4431	1.2886	1.0611	.90160	.78369	.69300	.62108	.47355	.38261	.30460
25	25.254	9.9571	6.0382	4.3002	3.3289	2.7114	2.2853	1.9739	1.7367	1.5500	1.2753	1.0830	.94091	.83170	.74516	.56776	.45852	.36487
30	29.795	11.711	7.0877	5.0406	3.8979	3.1723	2.6720	2.3066	2.0285	1.8097	1.4880	1.2630	1.0969	.96924	.86814	.66107	.53366	.42451
35	34.333	13.462	8.1355	5.7795	4.4655	3.6318	3.0573	2.6381	2.3190	2.0683	1.6997	1.4420	1.2519	1.1059	.99033	.75372	.68023	.48367
40	38.871	15.213	9.1823	6.5173	5.0321	4.0903	3.4417	2.9686	2.6087	2.3260	1.9105	1.6203	1.4063	1.2420	1.1119	.84586	.68237	.54246
45	43.408	16.963	10.228	7.2543	5.5979	4.5481	3.8254	3.2985	2.8978	2.5830	2.1208	1.7980	1.5601	1.3775	1.2331	.93761	.75616	.60096
50	47.944	18.712	11.274	7.9907	6.1632	5.0053	4.2085	3.6278	3.1863	2.8396	2.3306	1.9753	1.7135	1.5127	1.3538	1.0290	.82968	.65922
60	57.016	22.210	13.363	9.4624	7.2924	5.9185	4.9736	4.2852	3.7621	3.3516	2.7491	2.3289	2.0194	1.7821	1.5945	1.2111	.97604	.77518
70	66.086	25.707	15.452	10.933	8.4205	6.8306	5.7375	4.9416	4.3369	3.8625	3.1667	2.6815	2.3244	2.0506	1.8343	1.3925	1.1217	.89055
80	75.156	29.203	17.540	12.403	9.5479	7.7419	6.5007	5.5972	4.9110	4.3727	3.5834	3.0334	2.6287	2.3185	2.0735	1.5733	1.2669	1.0055
90	84.226	32.698	19.627	13.872	10.675	8.6526	7.2633	6.2522	5.4845	4.8823	3.9997	3.3848	2.9324	2.5859	2.3122	1.7537	1.4118	1.1201
100	93.295	36.193	21.714	15.341	11.801	9.5630	8.0255	6.9068	6.0575	5.3915	4.4156	3.7358	3.2358	2.8530	2.5505	1.9337	1.5563	1.2344
130	120.50	46.677	27.973	19.746	15.179	12.292	10.311	8.8690	7.7751	6.9176	5.6615	4.7872	4.1445	3.6525	3.2641	2.4725	1.9885	1.5762
160	147.71	57.160	34.231	24.150	18.555	15.020	12.594	10.830	9.4912	8.4421	6.9059	5.8371	5.0516	4.4506	3.9762	3.0098	2.4195	1.9168
200	183.98	71.136	42.575	30.020	23.056	18.657	15.638	13.443	11.778	10.474	8.5639	7.2356	6.2598	5.5133	4.9243	3.7251	2.9929	2.3698

Table 4. (Continued)

$\alpha = .005$

m^n	5	10	15	20	25	30	35	40	45	50	60	70	80	90	100	130	160	200
-.5	2.0863	.88778	.55868	.40683	.31969	.26322	.22368	.19446	.17198	.15416	.12769	.10897	.095037	.084262	.075682	.057970	.046976	.037495
0	2.6855	1.1249	.70418	.51145	.40129	.33009	.28031	.24356	.21532	.19294	.15973	.13627	.11882	.10533	.094588	.072430	.058682	.046830
.5	3.2561	1.3480	.84031	.60906	.47729	.39228	.33292	.28915	.25554	.22892	.18945	.16158	.14085	.12484	.11210	.085812	.069512	.055465
1	3.8110	1.5629	.97094	.70250	.54992	.45165	.38312	.33262	.29387	.26320	.21773	.18565	.16181	.14339	.12874	.098530	.079802	.063666
1.5	4.3561	1.7726	1.0979	.79314	.62028	.50911	.43166	.37463	.33090	.29629	.24503	.20888	.18202	.16128	.14479	.11079	.089715	.071566
2	4.8947	1.9785	1.2223	.88174	.68898	.56517	.47899	.41557	.36696	.32852	.27160	.23148	.20169	.17868	.16039	.12270	.099348	.079241
2.5	5.4285	2.1817	1.3447	.96881	.75642	.62015	.52538	.45569	.40229	.36008	.29761	.25359	.22092	.19570	.17564	.13434	.10876	.086739
3	5.9588	2.3828	1.4655	1.0547	.82286	.67428	.57103	.49514	.43703	.39110	.32316	.27531	.23980	.21240	.19062	.14576	.11799	.094093
3.5	6.4865	2.5823	1.5851	1.1396	.88848	.72771	.61607	.53405	.47127	.42168	.34833	.29670	.25840	.22885	.20536	.15701	.12708	.10133
4	7.0120	2.7804	1.7038	1.2236	.95343	.78057	.66060	.57251	.50511	.45188	.37319	.31782	.27675	.24508	.21990	.16810	.13604	.10846
4.5	7.5359	2.9773	1.8216	1.3070	1.0178	.83293	.70470	.61059	.53860	.48177	.39778	.33871	.29490	.26112	.23428	.17905	.14490	.11551
5	8.0584	3.1734	1.9386	1.3899	1.0817	.88487	.74842	.64833	.57179	.51139	.42214	.35939	.31287	.27700	.24851	.18990	.15365	.12248
10	13.242	5.1049	3.0863	2.1988	1.7041	1.3898	1.1728	1.0142	.89315	.79785	.65742	.55894	.48608	.43001	.38552	.29419	.23782	.18942
15	18.393	7.0117	4.2134	2.9900	2.3110	1.8809	1.5848	1.3687	1.2041	1.0747	.88437	.75114	.65272	.57707	.51710	.39416	.31841	.25345
20	23.532	8.9087	5.3317	3.7735	2.9108	2.3656	1.9908	1.7177	1.5100	1.3468	1.1071	.93959	.81597	.72102	.64583	.49184	.39709	.31591
25	28.666	10.801	6.4454	4.5527	3.5067	2.8466	2.3934	2.0635	1.8129	1.6161	1.3274	1.1258	.97713	.86307	.77279	.58807	.47454	.37736
30	33.797	12.690	7.5565	5.3293	4.1002	3.3254	2.7939	2.4073	2.1138	1.8836	1.5459	1.3104	1.1369	1.0038	.89854	.68331	.55115	.43810
35	38.925	14.577	8.6657	6.1042	4.6920	3.8025	3.1928	2.7496	2.4134	2.1498	1.7633	1.4940	1.2956	1.1436	1.0234	.77782	.62713	.49832
40	44.053	16.463	9.7737	6.8779	5.2826	4.2786	3.5907	3.0910	2.7120	2.4150	1.9798	1.6767	1.4536	1.2827	1.1476	.87177	.70263	.55812
45	49.180	18.348	10.881	7.6507	5.8725	4.7538	3.9878	3.4316	3.0099	2.6795	2.1956	1.8588	1.6111	1.4213	1.2713	.96528	.77775	.61761
50	54.306	20.233	11.987	8.4230	6.4617	5.2284	4.3843	3.7716	3.3072	2.9434	2.4110	2.0404	1.7680	1.5594	1.3946	1.0584	.85256	.67682
60	64.557	24.001	14.199	9.9660	7.6387	6.1763	5.1760	4.4502	3.9005	3.4701	2.8404	2.4026	2.0809	1.8346	1.6402	1.2439	1.0014	.79463
70	74.807	27.767	16.410	11.508	8.8144	7.1229	5.9664	5.1277	4.4926	3.9956	3.2688	2.7637	2.3927	2.1089	1.8849	1.4286	1.1496	.91179
80	85.056	31.533	18.619	13.049	9.9893	8.0687	6.7560	5.8043	5.0839	4.5203	3.6963	3.1240	2.7038	2.3824	2.1289	1.6126	1.2972	1.0285
90	95.304	35.298	20.828	14.589	11.164	9.0139	7.5449	6.4803	5.6746	5.0443	4.1233	3.4837	3.0143	2.6554	2.3724	1.7962	1.4444	1.1448
100	105.55	39.063	23.037	16.129	12.338	9.9586	8.3335	7.1558	6.2648	5.5679	4.5498	3.8430	3.3244	2.9280	2.6155	1.9794	1.5912	1.2608
130	136.30	50.357	29.661	20.747	15.857	12.791	10.697	9.1807	8.0336	7.1369	5.8275	4.9191	4.2529	3.7441	3.3430	2.5275	2.0303	1.6075
160	167.04	61.649	36.284	25.364	19.376	15.622	13.059	11.204	9.8008	8.7042	7.1035	5.9935	5.1798	4.5585	4.0689	3.0740	2.4679	1.9529
200	208.03	76.704	45.114	31.518	24.066	19.395	16.207	13.900	12.156	10.793	8.8034	7.4245	6.4142	5.6429	5.0354	3.8013	3.0501	2.4121

Table 5. Upper α percentage points of $U^{(3)}$ and $U^{(4)}$.

α = .10

	p = 3						p = 4		
n \ m	0	1	2	3	4	5	0	1	2
5	2.0526	2.9240	3.7808	4.6300	5.4747	6.3164	3.1949	4.3113	5.1454
10	.98227	1.3888	1.7865	2.1791	2.5686	2.9558	1.5170	2.0366	2.5485
15	.64348	.90724	1.1645	1.4181	1.6692	1.9186	.99104	1.3280	1.6593
20	.47812	.67309	.86300	1.0499	1.2349	1.4184	.73534	.98436	1.2290
25	.38028	.53487	.68529	.83323	.97950	1.1246	.58438	.78180	.97558
30	.31565	.44369	.56819	.69057	.81152	.93144	.48479	.64831	.80872
35	.26979	.37904	.48523	.58957	.69265	.79482	.41418	.55372	.69056
40	.23555	.33083	.42340	.51432	.60412	.69312	.36152	.48320	.60250
45	.20902	.29349	.37554	.45610	.53565	.61446	.32073	.42861	.53435
50	.18786	.26373	.33739	.40971	.48111	.55183	.28821	.38510	.48005
55	.17059	.23944	.30628	.37188	.43664	.50078	.26167	.34960	.43576
60	.15623	.21925	.28041	.34045	.39970	.45838	.23961	.32010	.39895
65	.14409	.20219	.25858	.31391	.36851	.42258	.22098	.29519	.36787
70	.13371	.18760	.23990	.29121	.34184	.39198	.20504	.27387	.34129
75	.12472	.17497	.22373	.27157	.31877	.36550	.19124	.25542	.31828
80	.11686	.16394	.20961	.25441	.29861	.34238	.17918	.23930	.29819
90	.10379	.14558	.18611	.22587	.26509	.30391	.15911	.21249	.26475
100	.093344	.13091	.16735	.20308	.23833	.27322	.14309	.19107	.23805

Table 5. (Continued)

α = .025

	p = 3						p = 4		
n \ m	0	1	2	3	4	5	0	1	2
5	2.9692	4.0964	5.1996	6.2904	7.3737	8.4521	4.4223	5.8326	7.2228
10	1.3259	1.8108	2.2817	2.7448	3.2031	3.6577	1.9559	2.5623	3.1567
15	.84829	1.1544	1.4505	1.7408	2.0274	2.3113	1.2477	1.6308	2.0052
20	.62289	.84610	1.0615	1.2724	1.4802	1.6859	.91499	1.1945	1.4673
25	.49194	.66749	.83668	1.0021	1.1649	1.3260	.72209	.94205	1.1564
30	.40642	.55105	.69029	.82629	.96011	1.0924	.59628	.77756	.95411
35	.34621	.46916	.58744	.70290	.81643	.92858	.50776	.66192	.81198
40	.30152	.40844	.51124	.61153	.71014	.80746	.44212	.57621	.70667
45	.26704	.36162	.45252	.54117	.62827	.71427	.39149	.51014	.62553
50	.23963	.32443	.40589	.48531	.56332	.64031	.35127	.45765	.56109
55	.21733	.29418	.36797	.43990	.51053	.58023	.31853	.41496	.50868
60	.19882	.26908	.33653	.40226	.46679	.53047	.29138	.37955	.46523
65	.18322	.24793	.31004	.37055	.42995	.48855	.26849	.34970	.42861
70	.16988	.22986	.28741	.34347	.39849	.45276	.24894	.32421	.39733
75	.15836	.21424	.26786	.32007	.37132	.42186	.23203	.30218	.37031
80	.14830	.20061	.25080	.29966	.34762	.39491	.21728	.28295	.34673
90	.13158	.17797	.22246	.26577	.30826	.35016	.19277	.25101	.30756
100	.11825	.15991	.19987	.23876	.27691	.31451	.17323	.22554	.27634

Table 5. (Continued)

$\alpha = .005$

	p = 3						p = 4		
n \ m	0	1	2	3	4	5	0	1	2
5	4.2391	5.7062	7.1391	8.5541	9.9592	11.354	6.0986	7.8971	9.6669
10	1.7401	2.3122	2.8655	3.4083	3.9445	4.4761	2.4737	3.1758	3.8619
15	1.0830	1.4329	1.7696	2.0988	2.4231	2.7438	1.5349	1.9654	2.3846
20	.78457	1.0359	1.2771	1.5123	1.7436	1.9720	1.1107	1.4205	1.7215
25	.61474	.81071	.99841	1.1811	1.3604	1.5376	.86980	1.1116	1.3462
30	.50524	.66578	.81933	.96866	1.1151	1.2596	.71463	.91292	1.1051
35	.42880	.56475	.69463	.82084	.94452	1.0665	.60640	.77442	.93710
40	.37243	.49031	.60284	.71212	.81910	.92470	.52662	.67238	.81342
45	.32915	.43320	.53246	.62879	.72308	.81594	.46538	.59409	.71857
50	.29488	.38799	.47678	.56291	.64722	.73023	.41689	.53212	.64352
55	.26707	.35133	.43164	.50952	.58572	.66063	.37755	.48185	.58265
60	.24404	.32099	.39430	.46537	.53493	.60322	.34499	.44026	.53231
65	.22468	.29547	.36291	.42826	.49210	.55498	.31761	.40528	.48996
70	.20816	.27371	.33614	.39663	.45576	.51388	.29424	.37545	.45386
75	.19390	.25494	.31305	.36934	.42437	.47845	.27408	.34970	.42271
80	.18147	.23857	.29293	.34557	.39699	.44758	.25651	.32726	.39556
90	.16084	.21143	.25956	.30616	.35169	.39643	.22735	.29004	.35054
100	.14443	.18983	.23301	.27481	.31566	.35576	.20415	.26042	.31471

Table 6. Comparison of three approximations to the upper percentage points of $U^{(p)}$, p = 3 and 4.

p = 3, m = 0

	5% Points				1% Points			
n	A_1	A_2	A_3	Exact	A_1	A_2	A_3	Exact
5	2.3284	2.5311	2.5064	2.4959	3.1473	3.5804	3.6951	3.6581
10	1.1102	1.1564	1.1562	1.1540	1.4432	1.5321	1.5623	1.5581
15	.72741	.74723	.74777	.74702	.93288	.96959	.98252	.98145
20	.54069	.55162	.55207	.55174	.68865	.70852	.71559	.71518
30	.35717	.36192	.36218	.36208	.45174	.46023	.46326	.46316
40	.26663	.26927	.26943	.26939	.33604	.34072	.34239	.34235
50	.21270	.21438	.21449	.21447	.26750	.27046	.27151	.27150
60	.17691	.17808	.17815	.17814	.22218	.22421	.22494	.22493
80	.13237	.13302	.13306	.13306	.16594	.16707	.16748	.16747
100	.10574	.10616	.10619	.10618	.13242	.13314	.13340	.13340

Table 6. (Continued)

p = 3, m = 3

n	5% Points A$_1$	A$_2$	A$_3$	Exact	1% Points A$_1$	A$_2$	A$_3$	Exact
5	5.1093	5.4901	5.4723	5.4373	6.5800	7.3703	7.6114	7.5217
10	2.3850	2.4651	2.4700	2.4640	2.9285	3.0762	3.1258	3.1187
15	1.5488	1.5818	1.5845	1.5827	1.8691	1.9275	1.9465	1.9452
20	1.1455	1.1633	1.1649	1.1642	1.3701	1.4009	1.4107	1.4103
30	.75264	.76019	.76090	.76070	.89200	.90477	.90866	.90860
40	.56025	.56440	.56480	.56471	.66088	.66781	.66987	.66986
50	.44615	.44876	.44901	.44897	.52479	.52912	.53039	.53039
60	.37064	.37244	.37261	.37259	.43514	.43810	.43896	.43896
80	.27689	.27789	.27799	.27798	.32430	.32593	.32640	.32640
100	.22099	.22162	.22168	.22168	.25845	.25948	.25977	.25977

p = 4, m = 0

n	5% Points A$_1$	A$_2$	A$_3$	Exact	1% Points A$_1$	A$_2$	A$_3$	Exact
5	3.4776	3.8348	3.8146	3.7913	4.4646	5.1839	5.3980	5.3339
10	1.6584	1.7383	1.7419	1.7377	2.0579	2.2039	2.2532	2.2474
15	1.0867	1.1207	1.1230	1.1217	1.3324	1.3925	1.4127	1.4114
20	.80777	.82644	.82786	.82732	.98442	1.0169	1.0277	1.0272
30	.53359	.54169	.54237	.54221	.64628	.66012	.66464	.66455
40	.39833	.40282	.40321	.40315	.48094	.48857	.49103	.49100
50	.31776	.32062	.32087	.32083	.38294	.38776	.38930	.38929
60	.26430	.26627	.26645	.26643	.31811	.32143	.32248	.32248
80	.19775	.19885	.19895	.19894	.23764	.23948	.24007	.24006
100	.15797	.15867	.15874	.15873	.18965	.19083	.19120	.19120

p = 4, m = 2

n	5% Points A$_1$	A$_2$	A$_3$	Exact	1% Points A$_1$	A$_2$	A$_3$	Exact
5	5.8164	6.3526	6.3433	6.2964	7.2646	8.3276	8.6636	8.5540
10	2.7410	2.8548	2.8631	2.8558	3.2927	3.4960	3.5626	3.5550
15	1.7870	1.8342	1.8384	1.8363	2.1168	2.1982	2.2236	2.2222
20	1.3246	1.3501	1.3525	1.3517	1.5577	1.6010	1.6140	1.6136
30	.87232	.88323	.88428	.88405	1.0183	1.0364	1.0416	1.0416
40	.65014	.65615	.65673	.65663	.75606	.76593	.76869	.76869
50	.51812	.52192	.52228	.52224	.60115	.60735	.60905	.60905
60	.43066	.43327	.43352	.43349	.49890	.50315	.50430	.50430
80	.32194	.32339	.32353	.32352	.37223	.37459	.37521	.37521
100	.25704	.25797	.25806	.25805	.29685	.29834	.29874	.29874

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