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Asymptotic Expansions for the Distribution of the Characteristic Roots of $\underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$ When Population Roots Are Not All Distinct*

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1. Introduction and Summary. The asymptotic expansions of the first two terms for the distribution of the characteristic (ch.) roots of $S_1 S_2^{-1}$ have been obtained by the authors [4] when all population roots are distinct, both in the real and complex cases, where S_1 and S_2 are independent sample covariance matrices of degrees n_1 and n_2 respectively from normal populations. However, if the population roots are not all distinct, the formulae derived break down and the situations become more complicated. In the case of q smallest roots equal, the first term of the asymptotic expansion in the real case has also been obtained by the authors [4]. In the present paper, we extend this result to the second term (Section 2) and derive the corresponding formulae (Section 3) for the complex case, because not all results in the complex case are counterparts of the real. Finally, we show in Section 4 that the results obtained by other authors [1], [2] and [3] are all special cases of the formulae obtained here.

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2. Two-sample real case. We follow the notations in Section 2 of [4]. Let $\underset{\sim}{S}_j$ $(j = 1, 2)$ be independently distributed as Wishart $(n_j, p, \underset{\sim}{\Sigma}_j)$, and let the ch. roots of $\underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$ and $(\underset{\sim}{\Sigma}_1 \underset{\sim}{\Sigma}_2^{-1})^{-1}$ be b_i and a_i $(i = 1,\dots,p)$ respectively such that $b_1 > b_2 > \dots > b_p > 0$ and $0 < a_1 < \dots < a_k < a_{k+1} = \dots = a_p = a$, $(1 \leq k \leq p-1)$. Further, let us denote

$$\underset{\sim}{A} = \text{diag}\,(a_1, \dots, a_k, a, \dots, a)$$
$$\underset{\sim}{B} = \text{diag}\,(b_1, b_2, \dots, b_p)$$

and $n = n_1 + n_2$. Then from (3.7) of [4], the joint distribution of $b_1, b_2, \dots, b_p$ depends on the definite integral

$$\mathcal{J} = 2^p \int_{N(\underset{\sim}{I})} |\underset{\sim}{I} + \underset{\sim}{A}\,\underset{\sim}{Q}\,\underset{\sim}{B}\,\underset{\sim}{Q}'|^{-\frac{1}{2}n} \, (\underset{\sim}{Q}'d\,\underset{\sim}{Q}) \quad ,$$

where $N(\underset{\sim}{I})$ and $(\underset{\sim}{Q}'d\,\underset{\sim}{Q})$ are defined in [4].

Under the condition $a_j = a$ $(j = k+1, \dots, p)$ then from the definitions of r_j, r_{jt} and c_{jt} in [4], we have

$$r_j = \frac{a}{1 + ab_j} \qquad \text{if} \quad j = k+1, \dots, p$$

$$r_{tj} = r_t - r_j = \frac{a^2\, b_{jt}}{(1 + ab_j)\,(1 + ab_t)} \qquad \text{if} \quad j, t = k+1, \dots, p$$

and

$$c_{jt} = r_{tj}\, b_{jt} - r_j\, r_t\, b_{jt}^2 = c_{tj} = 0, \quad \text{if} \quad j, t = k+1, \dots, p. \tag{2.1}$$

where $b_{jt} = b_j - b_t$.

From (4.4) of [4], we also have

(2.2) $$s_{jt} = 0 \qquad \text{if } j, t = k+1, \ldots, p.$$

Since we wish to compute up to the second term in the asymptotic expansion of $\mathcal{J}$, we need to investigate the groups of terms up to the fourth order of $\underset{\sim}{S}$. Under conditions (2.1) and (2.2), (3.16) of [4] reduces to

(2.3) $$K = \prod_{j<t}^{k} \left(\frac{2\pi}{n\, c_{jt}} \right)^{\frac{1}{2}} \prod_{j=1}^{k} \prod_{t=k+1}^{p} \left(\frac{2\pi}{n\, c_{jt}^{0}} \right)^{\frac{1}{2}},$$

where

$$c_{jt}^{0} = r_{tj}\, b_{jt} - r_j\, r_t\, b_{jt}^{2} = c_{tj}^{0} \qquad j = 1, \ldots, k; \quad t = k+1, \ldots, p.$$

Now let $S' = \sum_{j<t}^{p} c_{jt}^{-1}$ and

$$S'' = \sum_{j<s<t}^{p} \left(\frac{c_{st}}{c_{js}\, c_{jt}} + \frac{c_{jt}}{c_{js}\, c_{st}} + \frac{c_{js}}{c_{jt}\, c_{st}} \right).$$

Under conditions (2.1) and (2.2), S' turns out to be

$$\sum_{j<t}^{k} c_{jt}^{-1} + \sum_{j=1}^{k} \sum_{t=k+1}^{p} c_{jt}^{0^{-1}}.$$

In S", if both j and s, or j and t, or s and t are greater than k, the corresponding term containing c_{js} or c_{jt} or c_{st} as a factor vanishes.

Now $\frac{3}{4n} \sum_{j<t}^{p} \frac{c_{jt}}{c_{jt}}$ which originally contributed $\frac{3}{4n} \binom{p}{2}$,

now under conditions.(2.1) and (2.2), gives

$$\frac{3}{4n} \left\{ \binom{k}{2} + kq \right\}, \quad \text{and} \quad \frac{1}{2n} \sum_{j<s<t}^{p} \left(\frac{c_{js}\, c_{jt}}{c_{js}\, c_{jt}} + \frac{c_{js}\, c_{st}}{c_{js}\, c_{st}} + \frac{c_{jt}\, c_{st}}{c_{jt}\, c_{st}} \right)$$

which originally gave $\frac{3}{2n} \binom{p}{3}$, now becomes

$$\frac{1}{2n} \left\{ \binom{k}{3} + \binom{k}{2} q + \frac{k}{2} q\,(q-1) \right\} + \frac{1}{2n} \left\{ \binom{k}{3} + \binom{k}{2} q \right\} + \frac{1}{2n} \left\{ \binom{k}{3} + \binom{k}{2} q \right\}$$

$$\text{i.e. } \frac{3}{2n} \left\{ \binom{k}{3} + \binom{k}{2} q \right\} + \frac{k}{4n}\, q\,(q-1) \quad ,$$

where $q = p - k$. After simplification, (3.17) of [4] turns out to be

$$(2.4) \qquad K \left\{ \frac{1}{2n}\, S' + \frac{3}{4n} \left[\binom{k}{2} + kq \right] + \frac{p-2}{3n}\, S' - \frac{1}{8n}\, S'' \right.$$

$$\left. + \frac{3}{2n} \left[\binom{k}{3} + \binom{k}{2} q \right] + \frac{k}{4n}\, q\,(q-1) \right\} .$$

Similarly, (3.18) of [4] becomes

$$(2.5) \qquad K\left\{ \frac{1}{8n} S'' - \frac{p-2}{4n} S' - \frac{1}{2n}\left[\binom{k}{3} + \binom{k}{2} q \right]\right\},$$

Finally, since

$$\operatorname{tr} \underset{\sim}{S}^2 = -2\left\{ \sum_{j<t}^{k} s_{jt}^2 + \sum_{j=1}^{k} \sum_{t=k+1}^{p} s_{jt}^2 \right\},$$

it is easy to see that $(p-2)\operatorname{tr} \underset{\sim}{S}^2 / 4!$ gives

$$(2.6) \qquad -\frac{p-2}{12n} K S' .$$

Adding (2.3) - (2.6) and factoring K out, we have the following theorem:

<u>Theorem 2.1</u>. Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be diagonal matrices with $0 < a_1 < \ldots < a_k < a_{k+1} = \ldots = a_p = a$, $(1 \leq k \leq p-1)$ and $b_1 > b_2 > \ldots > b_p > 0$, then for large degrees of freedom $n = n_1 + n_2$, the first two terms in the expansion for $\mathcal{J}$ are given by

$$(2.7) \qquad \mathcal{J} = 2^p \frac{\pi^{\frac{1}{2}q^2}}{\Gamma_q(\frac{1}{2}q)} \prod_{j=1}^{k} (1 + a_j b_j)^{-\frac{n}{2}} \prod_{j=k+1}^{p} (1 + ab_j)^{-\frac{n}{2}} \prod_{j<t}^{k} \left(\frac{2\pi}{n c_{jt}}\right)^{\frac{1}{2}}$$

$$\cdot \prod_{j=1}^{k} \prod_{t=k+1}^{p} \left(\frac{2\pi}{n c_{jt}^0}\right)^{\frac{1}{2}} \left\{ 1 + \frac{1}{2n}\left[\sum_{j<t}^{k} c_{jt}^{-1} + \sum_{j=1}^{k} \sum_{t=k+1}^{p} c_{jt}^{0^{-1}} + \alpha(p,k) \right] + \ldots \right\},$$

where

$$(2.8) \qquad \alpha(p,k) = \frac{k}{12} \{(k-1)(4k+1) + 6(p^2-k^2)\}$$

and $q = p - k$.

<u>Theorem 2.2</u>. The asymptotic distribution of the ch. roots, $b_1 > b_2 > \ldots > b_p > 0$ of $\underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$, for large degrees of freedom $n = n_1 + n_2$, when ch. roots of $(\underset{\sim}{\Sigma}_1 \underset{\sim}{\Sigma}_2^{-1})^{-1}$ are $0 < a_1 < \ldots < a_k < a_{k+1} = \ldots = a_p = a$ $(1 \le k \le p-1)$ is given by

$$(2.9) \qquad C_0\, a^{\frac{1}{2} q n_1} \prod_{j=1}^{k} a_j^{\frac{1}{2} n_1} \prod_{j=1}^{k} (1+a_j b_j)^{-\frac{1}{2} n} \prod_{j=k+1}^{p} (1+a\, b_j)^{-\frac{1}{2} n}$$

$$\cdot \prod_{j<t}^{p} (b_j - b_t) \prod_{j<t}^{k} \left(\frac{2\pi}{n\, c_{jt}}\right)^{\frac{1}{2}} \prod_{j=1}^{k} \prod_{t=k+1}^{p} \left(\frac{2\pi}{n\, c_{jt}^{0}}\right)^{\frac{1}{2}} \prod_{j=1}^{p} b_j^{\frac{1}{2}(n_1-p-1)} ab_j$$

$$\cdot \left\{ 1 + \frac{1}{2n} \left[\sum_{j<t}^{k} c_{jt}^{-1} + \sum_{j=1}^{k} \sum_{t=k+1}^{p} c_{jt}^{0^{-1}} + \alpha\,(p,k) \right] + \ldots \right\},$$

where

$$C_0 = \pi^{\frac{1}{2} q^2}\, \Gamma p\,(\tfrac{1}{2}\, n) \left\{ \Gamma\, q\,(\tfrac{1}{2}\, q)\, \Gamma p\,(\tfrac{1}{2}\, n_1)\, \Gamma\, p\,(\tfrac{1}{2}\, n_2) \right\}^{-1}$$

and $\alpha\,(p,k)$ is defined by (2.8).

<u>3. Two-sample complex case.</u> Let $\underset{\sim}{S}_j$ $(j = 1,2)$ be independently distributed as complex Wishart $(n_j,\, p,\, \underset{\sim}{\Sigma}_j)$ and let $0 < a_1 < \ldots < a_k < a_{k+1} = \ldots = a_p = a$ $(1 \le k \le p-1)$ and $b_1 > b_2 > \ldots > b_p > 0$ be the ch. roots of $(\underset{\sim}{\Sigma}_1 \underset{\sim}{\Sigma}_2^{-1})^{-1}$ and $\underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$ respectively. We still denote

$$\underset{\sim}{A} = \operatorname{diag}\,(a_1, \ldots, a_k, a, \ldots, a)$$

and

$$\underset{\sim}{B} = \operatorname{diag}\,(b_1, b_2, \ldots, b_p) \quad ,$$

then from (5.6) of [4], we have

$$(3.1) \qquad \mathcal{J}_1 = \int_{N(\underset{\sim}{I})} |\underset{\sim}{I} + \underset{\sim}{A}\,\underset{\sim}{U}\,\underset{\sim}{B}\,\underset{\sim}{U}^*|^{-n} (\underset{\sim}{U}^*\, d\, \underset{\sim}{U})$$

where $n = n_1 + n_2$, and $N(\underset{\sim}{I})$ and $(\underset{\sim}{U}^*\, d\, \underset{\sim}{U})$ defined as in [4].

Partition the unitary matrix $\underset{\sim}{U}$ into the submatrices $\underset{\sim}{U}_1$ consisting of its first k, and $\underset{\sim}{U}_2$, the remaining q rows. If the integrand of (3.1) does not depend on $\underset{\sim}{U}_2$ then we can integrate over $\underset{\sim}{U}_2$ for fixed $\underset{\sim}{U}_1$ by the formula

$$(3.2) \qquad \int_{\underset{\sim}{U}_2} C_1\, (d\, \underset{\sim}{U}) = C_2\, (d\, \underset{\sim}{U}_1) \quad ,$$

where $C_1 = \pi^{p(p-1)} \left\{ \tilde{\Gamma}_p\, (p) \right\}^{-1}, \quad C_2 = \pi^{k(p-1)} \left\{ \tilde{\Gamma}_k\, (p) \right\}^{-1}$

and $\tilde{\Gamma}x(y)$ and $(d\, \underset{\sim}{U}_1)$ defined similarly as in [4].

Apply the transformation $\underset{\sim}{U} = \exp\, (i\, \underset{\sim}{H})$ (See (5.7) of [4]) the parametrization of $\underset{\sim}{U}$ may be obtained by writing

$$(3.3) \qquad \underset{\sim}{U} = \begin{pmatrix} \underset{\sim}{U}_1 \\ \underset{\sim}{U}_2 \end{pmatrix} = \exp \left\{ i \begin{pmatrix} \underset{\sim}{H}_{11} & \underset{\sim}{H}_{12} \\ \underset{\sim}{H}_{12}^* & \underset{\sim}{0} \end{pmatrix} \right\} ,$$

where $\underset{\sim}{H}_{11}$ is a $k \times k$ Hermitian matrix and $\underset{\sim}{H}_{12}$ is a $k \times q$ rectangular complex matrix.

From (5.8) of [4], it can be shown that

$$(3.4) \qquad \frac{\pi^{k(p-1)}}{\tilde{\Gamma}_k(p)} \; (d\, \underset{\sim}{U}_1) = (d\, \underset{\sim}{H}_{11})\; (d\, \underset{\sim}{H}_{12})\; \{ 1 + 0\, (\text{squares of } |h_{jt}|\text{'s} \} \; .$$

where the symbols $(d\, \underset{\sim}{H}_{11})$ and $(d\, \underset{\sim}{H}_{12})$ stand for $\prod_{j<t}^{k} dh_{jt\,R}\, dh_{jt\,I}$ and $\prod_{j=1}^{k} \prod_{t=k+1}^{p} dh_{jt\,R}\, dh_{jt\,I}$.

Note that

$$(3.5) \qquad \begin{cases} c_{jt} = r_{tj}\, b_{jt} - r_j\, r_t\, b_{jt}^2 = c_{tj} = 0 \\ h_{jtR} = h_{jtI} = 0 \end{cases}$$

for $j, t = k + 1, \ldots , p$,

where $r_{jt} = r_j - r_t = \dfrac{a_j}{1 + a_j\, b_j} - \dfrac{a_t}{1 + a_t\, b_t}$

and $b_{jt} = b_j - b_t$.

Since we wish to compute up to the second term in the asymptotic expansion of ϑ_1 we need to investigate the groups of terms up to the fourth order of $\underset{\sim}{H}$. Under conditions (3.5), then (5.19), (5.20) and (5.21) of [4] become

$$(3.6)\quad \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\Big(-n\sum_{j<t}^{k} c_{jt}\, h_{jt}\,\overline{h_{jt}} - n\sum_{j=1}^{k}\sum_{t=k+1}^{p} c_{jt}^{0}\, h_{jt}\,\overline{h_{jt}}\Big)$$

$$\cdot \prod_{j<t}^{k} dh_{jtR}\, dh_{jtI} \prod_{j=1}^{k}\prod_{t=k+1}^{p} dh_{jtR}\, dh_{jtI}$$

$$= \prod_{j<t}^{k} \frac{\pi}{n\, c_{jt}} \prod_{j=1}^{k}\prod_{t=k+1}^{p} \frac{\pi}{n\, c_{jt}^{0}} = C\ ,$$

$$(3.7)\quad \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\Big(-n\sum_{j<t}^{k} c_{jt}\, h_{jt}\,\overline{h_{jt}} - n\sum_{j=1}^{k}\sum_{t=k+1}^{p} c_{jt}^{0}\, h_{jt}\,\overline{h_{jt}}\Big)$$

$$\cdot\, h_{uvc}^{2m} \prod_{j<t}^{k} dh_{jtR}\, dh_{jtI} \prod_{j=1}^{k}\prod_{t=k+1}^{p} dh_{jtR}\, dh_{jtI}$$

$$= \begin{cases} C \cdot 1 \cdot 3 \cdot 5 \cdots (2m-1)(2n\, c_{uv})^{-m} & \text{if } v = 1, \ldots, k \\ C \cdot 1 \cdot 3 \cdot 5 \cdots (2m-1)(2n\, c_{uv}^{0})^{-m} & \text{if } v = k+1, \ldots, p \end{cases}$$

and

$$(3.8)\quad \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\Big(-n\sum_{j<t}^{k} c_{jt}\, h_{jt}\,\overline{h_{jt}} - n\sum_{j=1}^{k}\sum_{t=k+1}^{p} c_{jt}^{0}\, h_{jt}\,\overline{h_{jt}}\Big)$$

$$\cdot\,(h_{uv}\,\overline{h_{uv}})^{m} \prod_{j<t}^{k} dh_{jtR}\, dh_{jtI} \prod_{j=1}^{k}\prod_{t=k+1}^{p} dh_{jtR}\, dh_{jtI}$$

$$= \begin{cases} \dfrac{C\,(m!)}{(n\, c_{uv})^{m}} & \text{if } v = 1, \ldots, k \\[2ex] \dfrac{C\,(m!)}{(n\, c_{uv}^{0})^{m}} & \text{if } v = k+1, \ldots, p \end{cases}$$

respectively, where h_{uvc} denotes either h_{uvR} or h_{uvI} and

$$c^0_{jt} = r_{tj}\, b_{jt} - r_j\, r_t\, b^2_{jt} = c^0_{tj} \qquad j = 1,\dots,k;\ t = k+1,\dots,p.$$

Therefore we have the following theorem:

<u>Theorem 3.1</u>. Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be diagonal matrices with $0 < a_1 < \dots < a_k < a_{k+1} = \dots = a_p = a$ $(1 \le k \le p-1)$ and $b_1 > b_2 > \dots > b_p > 0$. Then for large n, the first two terms in the expansion for ϑ_1 are given by

$$(3.9)\quad \vartheta_1 = \frac{\pi^{q(q-1)}}{\tilde{\Gamma}_q(q)} \prod_{j=1}^{k} (1 + a_j\, b_j)^{-n} \prod_{j=k+1}^{p} (1 + a\, b_j)^{-n} \prod_{j<t}^{k} \frac{\pi}{nc_{jt}}$$

$$\cdot \prod_{j=1}^{k} \prod_{t=k+1}^{p} \frac{\pi}{nc^0_{jt}} \left\{ 1 + \frac{1}{3n} \left[\sum_{j<t}^{k} c^{-1}_{jt} + \sum_{j=1}^{k} \sum_{t=k+1}^{p} c^0_{jt} + \beta(p,k) \right] + \dots \right\},$$

where

$$(3.10)\qquad \beta(p,k) = \frac{k}{2} \,\{ (k-1)(2k-1) + 3(p-k)(p+k-1) \}$$

and $q = p - k$.

The proof is analogous to that of Theorem 2.1.

<u>Theorem 3.2</u>. The asymptotic distribution of the ch. roots, $b_1 > b_2 > \dots > b_p > 0$, of $\underset{\sim}{S_1}\, \underset{\sim}{S_2}^{-1}$ for large degrees of freedom $n = n_1 + n_2$ when the roots of $(\underset{\sim}{\Sigma_1}\, \underset{\sim}{\Sigma_2}^{-1})^{-1}$ are $0 < a_1 < \dots < a_k < a_{k+1} = \dots = a_p = a$ $(1 \le k \le p-1)$, is given by

(3.11) $$C_3\, a^{qn_1} \prod_{j=1}^{k} a_j^{\,n_1} \prod_{j=1}^{k} (1 + a_j b_j)^{-n} \prod_{j=k+1}^{p} (1 + ab_j)^{-n}$$

$$\cdot \prod_{j<t}^{p} (b_j - b_t)^2 \prod_{j<t}^{k} \frac{\pi}{nc_{jt}} \prod_{j=1}^{k} \prod_{t=k+1}^{p} \frac{\pi}{nc^{0}_{jt}} \prod_{j=1}^{p} b_j^{\,n_1-p}\, db_j$$

$$\cdot \left\{ 1 + \frac{1}{3n} \left[\sum_{j<t}^{k} c_{jt}^{-1} + \sum_{j=1}^{k} \sum_{t=k+1}^{p} c_{jt}^{0^{-1}} + \beta(p,k) \right] + \ldots \right\} ,$$

where

$$C_3 = \pi^{q(q-1)}\, \tilde{\Gamma}_p(n) \left\{ \tilde{\Gamma}_q(q)\, \tilde{\Gamma}_p(n_1)\, \tilde{\Gamma}_p(n_2) \right\}^{-1}$$

and $\beta(p,k)$ is defined by (3.10).

4. Remark: Replace b_j by $n_1 b_j/n_2$ $(j = 1,\ldots,p)$, and let n_2 tend to infinity and rewrite n_1 as n, then (2.9) and (3.11) become the one-sample cases. i.e.

(4.1) $$\left(\frac{n}{2}\right)^{\frac{1}{2}pn} \pi^{\frac{1}{2}q^2} \left\{ \Gamma_q\left(\tfrac{1}{2}q\right) \Gamma_p\left(\tfrac{1}{2}n\right) \right\}^{-1} a^{\frac{1}{2}qn} \prod_{j=1}^{k} a_j^{\frac{1}{2}n}$$

$$\cdot \prod_{j<t}^{p} (b_j - b_t) \exp\left(-\tfrac{1}{2} n \sum_{j=1}^{k} a_j b_j - \tfrac{1}{2} na \sum_{j=k+1}^{p} b_j\right) \prod_{j<t}^{k} \left(\frac{2\pi}{n\gamma_{jt}}\right)^{\frac{1}{2}}$$

$$\cdot \prod_{j=1}^{k} \prod_{t=k+1}^{p} \left(\frac{2\pi}{n\gamma^{0}_{jt}}\right)^{\frac{1}{2}} \prod_{j=1}^{p} b_j^{\frac{1}{2}(n-p-1)}\, db_j \left\{ 1 + \frac{1}{2n} \left[\sum_{j<t}^{k} \gamma_{jt}^{-1} + \sum_{j=1}^{k} \sum_{t=k+1}^{p} \gamma_{jt}^{0^{-1}} \right] + \ldots \right\}$$

and

$$(4.2)\qquad n^{pn}\ \pi^{q(q-1)}\left\{\tilde{\Gamma}_q\ (q)\ \tilde{\Gamma}_p\ (n)\right\}^{-1} a^{qn} \prod_{j=1}^{k} a_j^n \prod_{j<t}^{p} (b_j - b_t)^2$$

$$\exp\left(-n\sum_{j=1}^{k} a_j b_j - n a \sum_{j=k+1}^{p} b_j\right) \prod_{j<t}^{k} \frac{\pi}{n\ \gamma_{jt}} \prod_{j=1}^{k} \prod_{t=k+1}^{p} \frac{\pi}{n\ \gamma_{jt}^{0}}$$

$$\cdot \prod_{j=1}^{p} b_j^{n-p}.db_j \left\{1 + \frac{1}{3n}\left[\sum_{j<t}^{k} \gamma_{jt}^{-1} + \sum_{j=1}^{k}\sum_{t=k+1}^{p} \gamma_{jt}^{0^{-1}}\right] + \ldots\right\}$$

respectively, where

$$\gamma_{jt} = (a_t - a_j)(b_j - b_t) \qquad \text{for } j,\ t = 1,\ldots,k$$

$$\gamma_{jt}^{0} = (a - a_j)(b_j - b_t) \qquad \text{for } j = 1,\ldots,k;\ \ t = k+1,\ldots,p.$$

The first term in the asymptotic expansion of (4.1) gives the result in (3.11) of [3].

Using the same convention as in Section 4 of [4], then the restriction $1 \leq k \leq p-1$ can be written as $0 \leq k \leq p$.

(I) If $k = 0$, i.e. $q = p$, then $a_1 = a_2 = \ldots = a_p = a$, $\alpha(p,k)=0$, $\beta(p,k) = 0$, (i) (2.9) reduces to (4.9) of [4], and (ii) (3.11) becomes

$$\pi^{p(p-1)}\ \tilde{\Gamma}_p(n)\left\{\tilde{\Gamma}_p(p)\ \tilde{\Gamma}_p(n_1)\ \tilde{\Gamma}_p\ (n_2)\right\}^{-1} a^{pn_1} \prod_{j<t}^{p} (b_j - b_t)^2$$

$$\prod_{j=1}^{p} b_j^{n_1 - p}\ (1 + ab_j)^{-n}\, db_j \quad ,$$

both of which are the joint distribution of $b_1, b_2, \ldots, b_p$ under null hypothesis $\underset{\sim}{\Sigma}_1 = a^{-1} \underset{\sim}{\Sigma}_2$ (for the real case see [5]) and are exact forms where we assume no asymptotic condition.

Similarly, we can obtain the corresponding formulae from (4.1) and (4.2).

(II) If $k = p$, i.e. $q = 0$, then $0 < a_1 < a_2 < \ldots < a_p$, $\alpha(p,k) = \alpha(p)$, $\beta(p,k) = \beta(p)$, and (2.7), (2.8), (2.9), (3.9), (3.10) and (3.11) reduce to (3.14), (3.15), (3.20), (5.22), (5.23) and (5.27) of [4] respectively. Similarly, (4.1) reduces to (1.8) of [1]. If we take the first term in the asymptotic expansion of (3.20) in [4], it is Chang's result [2].

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