Sequential Estimation of a Poisson Integer Mean

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SUMMARY

A sequential method is investigated for estimating the mean of a Poisson distribution when the mean is assumed to be a nonnegative integer.

1. INTRODUCTION

One observes a sequence of random variables $X_1, X_2, \ldots$ which are identically and independently distributed Poisson variables with mean λ, i.e. $P_\lambda(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = \prod_{j=1}^{n} e^{-\lambda} \lambda^{x_j}/x_j!$ for $x_j = 0,1,\ldots$.
It is assumed that λ is an unknown nonnegative integer which one would like to estimate with an arbitrarily small uniform (for all λ) bound on the probabilities of error.

The problem of estimating restricted parameters was first considered by Hammersley (1950) from a fixed sample size point of view. The present work is based on the work of Robbins (1970) in which he proposes a general sequential approach and solves the problem of estimating a normal integer mean. In contrast to the normal case, there is no fixed sample size procedure which will insure an arbitrarily small uniform bound on the error probabilities for the Poisson case.

2. Fixed Sample Size Approach

2.1. Fixed Sample Size Rules.

Note that $EX_i = \lambda$ and $\text{var}(X_i) = \lambda$. Also, for a sample of size n, $\bar{X}_n = (X_1 + \ldots + X_n)/n$ is unbiased and sufficient for λ, and for the unrestricted parameter space $[0, \infty)$, it is a maximum likelihood estimator. In addition, for large n, the quantity $(\bar{X}_n - \lambda)/\sqrt{\lambda/n}$ is approximately normal with mean zero and variance one.

A class of reasonable procedures can be characterized as follows: For $i = 0, 1, \ldots$, choose i_- such that $(i-1) < i_- < i$ and set $i_+ = (i+1)_-$. Then, given a sample of size n, estimate that $\lambda = i$ if $i_- \leq \bar{X}_n < i_+$. A typical rule in this class is that with $i_+ = i + \frac{1}{2}$. The maximum likelihood estimates for this problem are discussed by Hammersley (1950). In this case, $i_+ = 1/\log((i+1)/i)$ for $i > 0$ and $i_+ = 0$ for $i = 0$. Also, $i_+ \to i + \frac{1}{2}$ as $i \to \infty$.

2.2. Fixed Sample Size Error Probabilities

Let $P_i^* =$ the probability of error when i is the true value of the parameter λ. Now,

$$P_i^* = P_i (\bar{X}_n < i_-) + P_i (\bar{X}_n \geq i_+).$$

Using the results of Blackwell and Hodges (1959) for large deviation probabilities and assuming that $i_+ = i + \frac{1}{2}$ for all i, it follows that

$$\log P_i^* \sim -n ((i + \tfrac{1}{2}) \log ((i + \tfrac{1}{2})/i) - \tfrac{1}{2}) \quad \text{as } n \to \infty$$

In addition, $\log ((i + \frac{1}{2})/i) \to 1/(2 i + \frac{1}{2})$ as $i \to \infty$. Hence,

$$\log P_i^* \sim -n/(8 i + 1) \quad \text{as } i, n \to \infty. \tag{1}$$

Clearly, for a preassigned value of n, it is not possible to insure a small uniform bound on the error probabilities. This is seen to be true for any fixed sample size rule by considering the standard test of the hypothesis $\lambda = i$ vs $\lambda = i + 1$ for large i.

Hence, with the aim of devising a decision procedure that will insure a small uniform bound on the error probabilities, one is led to consider sequential procedures.

3. Sequential Approach

3.1. A Sequential Procedure

For $\lambda > 0$, let

$$f_\lambda^n = f_\lambda(X_1, \ldots, X_n) = \prod_{j=1}^{n} e^{-\lambda} \lambda^{X_j}/X_j! \quad (X_j = 0, 1, \ldots)$$

$$= e^{-n\lambda} \lambda^{S_n}/X_1! \ldots X_n! ,$$

where $S_n = X_1 + \ldots + X_n$.

For $\lambda = 0$ let $f_0^n = \lim_{\lambda \to 0} f_\lambda^n$. Thus, f_0^n equals 1 or 0 according as S_n is zero or positive. Now, for i and j positive, $f_i^n/f_j^n = e^{-n(i-j)}(i/j)^{S_n}$, and f_i^n/f_0^n equals f_i^n or ∞ according as S_n is zero or positive. This is consistent with the above if it is agreed that $(i/j)^{S_n} = 1$ when $j = 0$ and $S_n = 0$.

Lemma. Let j and n be fixed positive integers. Then for $0 < k \leq j$,

$$(f_{k-1}^n/f_k^n) \leq (f_{j-1}^n/f_j^n). \tag{2}$$

Proof. $k \leq j$ implies $((k-1)/k) \leq ((j-1)/j)$. Hence, $(f_{k-1}^n/f_k^n) = e^n((k-1)/k)^{S_n} \leq e^n((j-1)/j)^{S_n} = (f_{j-1}^n/f_j^n)$ since S_n is nonnegative. Q.E.D.

Now, let

$$L_i^n = \begin{cases} \min\,(f_i^n/f_{i+1}^n,\ f_i^n/f_{i-1}^n) & \text{for } i > 0 \\ f_0^n/\ f_1^n & \text{for } i = 0. \end{cases}$$

Consider the following rule:

Fix $\alpha > 1$. Stop at $N = n$ and guess $\lambda = i$ as soon as $L_i^n \geq \alpha$ for some $i = 0,1,\ldots.$ First notice that there is no ambiguity in the guess since $L_i^n \geq \alpha$ for some i implies that $L_j^n < 1$ for all $j \neq i$.

The form of the rule can be considerably simplified as follows: Suppose $i > 0$. Then, $L_i^n \geq \alpha$ implies that $f_i^n/f_{i+1}^n = e^n(i/(i+1))^{S_n} \geq \alpha$ or $\bar{X}_n \leq i_+ - i_+ (\log \alpha)/n$, where $\bar{X}_n = S_n/n$ and $i_+ = 1/\log((i+1)/i)$. Similarly, for $i > 1$, $L_i^n \geq \alpha$ implies $\bar{X}_n \geq i_- + i_- (\log \alpha)/n$, where $i_- = (i-1)_+ = 1/\log(i/(i-1))$. Also, $f_1^n/f_0^n = f_1^n = e^{-n}$ or ∞ according as S_n is zero or positive. Thus, $L_0^n \geq \alpha$ implies $n \geq \log \alpha$ and $S_n = 0$, and $L_1^n \geq \alpha$ implies $f_1^n/f_0^n \geq \alpha$ or simply $S_n > 0$.

Thus, the rule can be rewritten as follows:

(3) Stop at $N = n$ as soon as one of the following is true:

(a) for some $i > 0$,

$$i_- + i_- (\log \alpha)/n \leq \bar{X}_n \leq i_+ - i_+ (\log \alpha)/n, \text{ and guess that } \lambda = i\ ;$$

(b) $n \geq \log \alpha$ and $S_n = 0$, guess $\lambda = 0$.

Note that as $n \to \infty$, $i_- (\log \alpha)/n \to 0$ and $i_+(\log \alpha)/n \to 0$ and $\bar{X}_n$ converges almost surely to λ, an integer. Thus, if $i_- < i < i_+$, the procedure will terminate with probability one. This is seen to be true from the inequality $(n+1)^{-1} < \log((n+1)/n) < n^{-1}$. It is interesting to note that as $i \to \infty$, $i_+ \to i + \frac{1}{2}$.

3.2 Minimum Sample Size

Recall that a guess of $\lambda = 0$ implies $n \geq \log \alpha$. Also, note that for large α and small n, $i_- + i_- (\log \alpha)/n > i_+ - i_+ (\log \alpha)/n$. But,

$i_- + i_-(\log\alpha)/n$ decreases to i_- and $i_+ - i_+(\log\alpha)/n$ increases to i_+ as $n \to \infty$. Thus, for each i, there is a minimum sample size, call it m_i, which is the smallest sample size which will admit a guess of $\lambda = i$. For conciseness, m_i will be identified with any number less than m_i and greater than $m_i - 1$. To find m_i, set $i_- + i_-(\log\alpha)/n = i_+ - i_+(\log\alpha)/n$. Solving for n gives

$$(4) \qquad m_i = \begin{cases} \log\alpha & \text{for } i = 0,1 \\ (\log\alpha)(\log((i+1)/(i-1)))/\log(i^2/(i^2-1)) & \text{for } i > 1. \end{cases}$$

Note that $n \geq m_i$ does not imply that $i_- + i_-(\log\alpha)/n \leq i \leq i_+ - i_+(\log\alpha)/n$. It will be necessary to use the minimum value of n, call it n_i, such that this expression is valid. Clearly, $n_0 = \log\alpha$. For $i > 1$, $i_- + i_-(\log\alpha)/n = i$ implies $n = (\log\alpha)/(i\log(i/(i-1))-1)$, and for $i = 1$, the inequality $i_- + i_-(\log\alpha)/n < i$ is valid for all $n \geq 1$. Similarly, for $i \geq 1$, $i = i_+ - i_+(\log\alpha)/n$ implies $n = (\log\alpha)/(1-i\log((i+1)/i))$. Hence,

$n_i = (\log\alpha)/\min(i\log(i/(i-1))-1,\ 1-i\log((i+1)/i))$ for $i > 1$ and $n_1 = (\log\alpha)/(1-\log 2)$.

Integrating the Taylor expansion for x^{-1} about the point $(x+1)/2$, gives $\log x = 2(z + z^3/3 + \ldots)$, where $z = (x-1)/(x+1)$, $x > 0$. It follows that for $x > 1$, $\log x > 2(x-1)/(x+1)$. (This inequality could have been obtained in a more statistical manner by an application of Jensen's inequality). Setting $x = (i+1)/(i-1)$ yields $\log((i+1)/(i-1)) > 2/i$, which upon rearrangement gives $1-i\log((i+1)/i) < i\log(i/(i-1))-1$. Hence,

(5) $$n_i = \begin{cases} (\log \alpha)/(1-i \log ((i+1)/i)) & \text{for } i \geq 1 \\ \log \alpha & \text{for } i = 0 \end{cases}$$

Note that as $i \to \infty$,

(6) $$n_i \sim (2i+1) \log \alpha.$$

3.3 Bounds on Error Probabilities

Let P_i = the probability of error when $\lambda = i$, and let $A_{n,j} = \{N=n, \text{ guess } \lambda = j\}$. Then,

$$P_i = \sum_{j \neq i} \sum_{n \geq m_j} \int_{A_{n,j}} f_i^n.$$

For brevity the differential term $d\mu_n$ will be omitted. Note $P_0 = 0$. Now, let

$$a_i = \sum_{j<i} \sum_{n \geq m_j} \int_{A_{n,j}} f_i^n \quad \text{and} \quad b_i = \sum_{j>i} \sum_{n \geq m_j} \int_{A_{n,j}} f_i^n.$$

Thus, $P_i = a_i + b_i$. Now,

$$a_i = \sum_{j<i} \sum_{n \geq m_j} \int_{A_{n,j}} (f_j^n)(f_{j+1}^n/f_j^n)\ldots(f_i^n/f_{i-1}^n).$$

Recall that by (2), $(f_{k-1}^{n}/ f_k^{n}) \geq (f_{m-1}^{n}/ f_m^{n})$ for $m \leq k$, or equivalently, $(f_k^{n}/f_{k-1}^{n}) \leq (f_m^{n}/f_{m-1}^{n})$. Also, on $A_{n,j}$, $(f_j^{n}/f_{j+1}^{n}) \geq \alpha$, so

$$\max ((f_{j+1}^{n}/f_j^{n}),\ldots,(f_i^{n}/f_{i-1}^{n})) = (f_{j+1}^{n}/f_j^{n}) \leq \alpha^{-1}.$$

Therefore, $((f_{j+1}^{n}/f_j^{n})\ldots(f_i^{n}/f_{i-1}^{n})) \leq \alpha^{-(i-j)}$. Thus,

$$a_i \leq \sum_{j<i} \sum_{n \geq m_j} \int_{A_{n,j}} f_j^{n} \ \alpha^{-(i-j)}$$

$$= \sum_{j<i} (\alpha^{-(i-j)} \sum_{n \geq m_j} \int_{A_{n,j}} f_j^{n}).$$

Since

$$\sum_{n \geq m_j} \int_{A_{n,j}} f_j^{n} = 1 - P_j \leq 1 ,$$

it follows that

$$a_i \leq \sum_{j=0}^{i-1} \alpha^{-(i-j)} \quad \text{or}$$

(7) $$a_i \leq (1 - \alpha^{-i})/(\alpha-1) \leq 1/(\alpha-1) \quad \text{for all } i.$$

In an entirely analogous fashion, it can be shown that

(8) $$b_i \leq 1/(\alpha-1).$$

Thus, adding (7) and (8) gives

(9) $$P_i \leq (2-\alpha^{-i})/(\alpha-1) \leq 2/(\alpha-1) \quad \text{for all } i.$$

Hence, by using a sequential procedure, one can obtain an arbitrarily small uniform bound on the error probabilities for all i.

3.4. Asymptotic Sample Size

As in section 3.2, when considering sample sizes, no distinction will be made between n, an integer, and any real number less than n but greater than $n - 1$. Recall that $S_n = 0$ for every n when $\lambda = 0$, so $N = \log \alpha$ and $E_0 N = \log \alpha$.

Now, for $i \geq 1$, let $k_i = 1/(1-i \log((i+1)/i))$, and let $k_0 = 1$. Then, $n_i = k_i \log \alpha$. Recall that n_i is the smallest sample size such that $i_- + i_- (\log \alpha)/n \leq i \leq i_+ - i_+ (\log \alpha)/n$.

Let $i \geq 1$ and $k > k_i$ be fixed. Let $n = k \log \alpha$. Thus, $n > n_i$ and

$$\begin{aligned} P_i(N > n) &\leq P_i(i_- + i_- (\log \alpha)/n > \bar{X}_n) \\ &\quad + P_i(i_+ - i_+ (\log \alpha)/n < \bar{X}_n) \\ &= P_i(a > z_n) + P_i(b < z_n), \end{aligned}$$

where $a = i_- + i_-/k-i$, $b = i_+ - i_+/k-i$, and $z_n = \bar{X}_n - i$. Now, since $k > k_i$, it follows that $a < 0$ and $b > 0$. Also, $b < -a$ by the same arguement used to find n_i in section 3.2. Therefore, $P_i(N > n) \leq P_i(|z_n| > b)$. The letter probability can be bounded by the Markov inequality with $r = 3$ (see, for example Loeve (1963)). This gives $P_i(|z_n| > b) \leq E|z_n|^3/b^3$. Now, $E|z_n|^3 \leq n^{-2} E|X-i|^3$, where X is a Poisson random variable with mean i. Let $K = b^{-3} E|X-i|^3$. Clearly, K is a finite positive constant for i and k fixed since all moments exist for the Poisson distribution. Hence,

(10) $$P_i\,(N > n) \leq K\,n^{-2} = K(k \log \alpha)^{-2}.$$

By letting $\alpha \to \infty$ in the above expression, it is seen that $P_i(N > n) \to 0$ as $\alpha \to \infty$. Since k was arbitrary subject only to the condition $k > k_i$, it follows that N is asymptotically less than or equal to n_i as $\alpha \to \infty$, i.e.

(11) $$N \lesssim n_i = k_i \log \alpha \quad \text{as} \quad \alpha \to \infty \quad \text{for all} \quad i.$$

3.5 Asymptotic Expected Sample Size

This section deals with the study of the behavior of E_iN as $\alpha \to \infty$.

Lemma. Let $i \geq 1$ and $k' \geq k > k_i$. Then there exists a positive number K which may depend on i and k but not on k' or α, such that

(12) $$P_i(n > k' \log \alpha) \leq K(k' \log \alpha)^{-2}.$$

Proof. Let $n = k \log \alpha$ and $n' = k' \log \alpha$. In the previous section it was shown that

$P_i\ (N > n') \leq K'\ (n')^{-2}$ where $K' = (i_+ - i_+/k' - i)^{-3}\ E|X - i|^3$. Since $k' \geq k$, it is clear that $K \geq K'$. Hence, $P_i(N > n') \leq K(n')^{-2}$. Q.E.D.

Theorem. For $i \geq 0$,

$$E_i N \lesssim n_i = k_i \log \alpha \quad \text{as } \alpha \to \infty. \tag{13}$$

Proof. The case where $i = 0$ has already been considered. Let $i \geq 1$ and $k > k_i$ be fixed. Set $n = k \log \alpha$. For convenience it will be assumed that $k \log \alpha$ is an integer. Now,

$$E_i N = \sum_{j \geq 1} j\ P_i(n = j)$$

$$\leq n + \sum_{j > n} j\ P_i(N = j)$$

$$= n + (n+1)\ P_i\ (N > n) + \sum_{j > n} P_i(N > j).$$

By the previous lemma,

$$(n+1)\ P_i\ (N > n) \leq K\ (k \log \alpha + 1)\ (k \log \alpha)^{-2}.$$

Clearly, as $\alpha \to \infty$, this term goes to zero. Applying the previous lemma to each term of the last summation above gives

$$\sum_{j > n} P_i(N > j) \leq K \sum_{j > k \log \alpha} j^{-2}.$$

This series is clearly convergent. Thus, as $\alpha \to \infty$, this term also approaches zero. Hence, $E_i N \underset{\sim}{<} n = k \log \alpha$ as $\alpha \to \infty$. Since k was arbitrary, subject only to the condition that $k > k_i$, it follows that $E_i N \underset{\sim}{<} n_i = k_i \log \alpha$ as $\alpha \to \infty$. Q.E.D.

Note that $k_i = (1 - i \log ((i+1)/i))^{-1} \to 2i + 1$ as $i \to \infty$. Therefore,

$$(14) \qquad E_i N \underset{\sim}{<} (2i + 1) \log \alpha \quad \text{as} \quad i, \alpha \to \infty.$$

3.6 Asymptotic Optimality

The following two lemmas will be useful in proving the main result of this section. Let F_n be the σ- algebra generated by $(X_1, \ldots, X_n)$.

Lemma 1. Let N be any stopping rule with $P_i(N < \infty) = 1$ and let A be any set such that $A \cap \{N=n\}$ is in F_n for all n. If $P_i(A) > 0$ and $P_{i+1}(A) > 0$, then

$$(15) \qquad E_i(\log(f_i^N / f_{i+1}^N) | A) \geq \log (P_i(A)/P_{i+1}(A)).$$

Proof. $E_i(\log (f_i^N/f_{i+1}^N)|A) = - E_i(\log (f_{i+1}^N/f_i^N)|A)$

$$\geq - \log E_i((f_{i+1}^N/f_i^N)|A)$$

by Jensen's inequality. Let $A_n = \{N=n\} \cap A$. Then,

$$E_i((f_{i+1}^N/f_i^N)|A) = (P_i(A))^{-1} \sum_n \int_{A_n} (f_{i+1}^n/f_i^n)f_i^n$$

$$= (P_i(A))^{-1} \sum_n \int_{A_n} f_{i+1}^n$$

$$= P_{i+1}(A)/P_i(A)$$

Substituting this expression into the above inequality gives the desired result. Q.E.D.

<u>Lemma 2</u>. For any $\alpha > 1$, let N be any stopping rule such that $P_i(N < \infty)=1$ for all i
and let there be an associated terminal decision rule such that such that $P_i(\text{error}) \leq 2/(\alpha-1)$ for all i. Then, for every i,

$$(16) \qquad E_i(\log (f_i^N/f_{i+1}^N)) \gtrsim \log \alpha \text{ as } \alpha \to \infty.$$

<u>Proof</u>. Let $C_i = \{\text{guess } i\}$. Then $P_i(\text{error}) = P_i(C_i^c)$. Without loss of generality, assume that $P_i(C_j) > 0$ for all i and j. Any decision rule can be modified on a set of arbitrarily small probability to meet this condition. Now, let i be fixed. Clearly,

$$E_i \log (f_i^N/f_{i+1}^N) = P_i(C_i)\, E_i\, (\log (f_i^N/f_{i+1}^N)|C_i)$$

$$+ P_i(C_i^c)E_i(\log (f_i^N/f_{i+1}^N)\, /C_i^c).$$

Applying the previous lemma, first with $A = C_i$ and then with $A = C_i^c$, yields

(17) $$E_i \log (f_i^N/f_{i+1}^N) \geq P_i (C_i) \log (P_i (C_i)/P_{i+1} (C_i))$$

$$+ P_i (C_i^c) \log (P_i (C_i^c)/P_{i+1} (C_i^c)).$$

Now, $P_i (C_i^c) \leq 2/(\alpha-1)$, so $P_i (C_i) \geq (\alpha-3)/(\alpha-1)$

and $P_{i+1} (C_i) < P_{i+1} (C_{i+1}^c) \leq 2/(\alpha-1)$. Thus,

$$P_i (C_i) \log (P_i (C_i)/P_{i+1} (C_i)) \geq ((\alpha-3)/(\alpha-1)) \log ((\alpha-3)/2)$$

$$\sim \log \alpha \quad \text{as} \quad \alpha \to \infty.$$

Also,

$$P_i(C_i^c) \log (P_i (C_i^c)/P_{i+1} (C_i^c)) > P_i (C_i^c) \log P_i (C_i^c)$$

$$\to 0 \quad \text{as} \quad \alpha \to \infty$$

since $P_i(C_i^c) \leq 2/(\alpha-1)$. Now, combining the above with (17) gives the desired result. Q.E.D.

<u>Theorem</u>. For any $\alpha > 1$, let (N^*, d^*) be the stopping rule and terminal decision function described in section 3.1, and let (N, d) be any stopping rule and associated terminal decision function such that $E_i N < \infty$ and P_i (error) $\leq 2/(\alpha-1)$ for all i, Then, for every i,

$$E_i N^* \lesssim E_i N \text{ as } \alpha \to \infty. \tag{18}$$

<u>Proof</u>. Since the Poisson variables $X_1, X_2, \ldots$ are identically and independently distributed and $E_i N < \infty$ for all i, the following well-known equality is valid for all i: $E_i \log (f_i^N/f_{i+1}^N) = (E_i N)\ (E_i\ (f_i(X)/f_{i+1}(X)))$. Recall that $f_i(X)/f_{i+1}(X) = e\ (i/(i+1))^X$ for $i \geq 1$. So,

$$\begin{aligned} E_i \log (f_i(X)/f_{i+1}(X)) &= E_i(1 + X \log (i/(i+1))) \\ &= 1 - i \log ((i+1)/i) \\ &= k_i^{-1}. \end{aligned}$$

This is also valid for $i = 0$. Hence,

$$E_i N = k_i E_i \log (f_i^N/f_{i+1}^N) \text{ for all } i.$$

Now, by Lemma 2, $\log \alpha \lesssim E_i \log (f_i^N/f_{i+1}^N)$, so $E_i N \gtrsim k_i \log \alpha$. But by the theorem of section 3.5, $E_i N^* \lesssim k_i \log \alpha$ for all i as $\alpha \to \infty$. Therefore $E_i N^* \lesssim E_i N$ for all i as $\alpha \to \infty$. Q.E.D.

4. Comparison of Fixed and Sequential Plans

The sequential procedure is obviously for superior to any fixed sample size plan since it is only with a sequential plan that one can obtain a small uniform bound on the error probabilities for the whole parameter space.

Let i be fixed and suppose that one could somehow (perhaps by a two-stage sampling procedure) pick a sample size which would give a reasonable bound on the error probability for the true parameter, i.e. by (1), pick n such that $\log P_i^* = -n/(8i+1)$. Suppose further that i is large enough for this expression to validly approximate the fixed sample size error probability and for (14) to be approximately valid as $\alpha \to \infty$. Obviously, a knowledge of i is being assumed, but this fact will temporarily be neglected. Now let $\log(2/(\alpha-1)) = -n/(8i+1)$. Then, $\log(\alpha-1) = n/(8i+1) + \log 2$. Recall that by (13) and (14), $E_i N \lesssim k_i \log \alpha$ as $\alpha \to \infty$, and that this letter expression is asymptotic to $(2i+1) \log \alpha$ as $i \to \infty$. Hence, as $\alpha \to \infty$, $E_i N$ will be asymptotically less than or equal to $n/4$. Thus, even if it were possible to slect a suitable n for a fixed sample size procedure, the sequential procedure required on the average only $\frac{1}{4}$ as many observations to attain the same bound on the error probability as this error probability goes to zero for large i.

5. Monte Carlo Results

To investigate the properties of the procedure described in 3.1 for various values of α and λ, a Fortran program for an IBM-360-90 was written. Sequences of Poisson variables with a given mean were generated, the stopping and terminal decision rules were applied, and the results were tabulated. For each value of α and λ, 1000 sequences were generated.

For convenience, an arbitrary upper bound of 1000 was set on the length of the sequences. At the point of truncation, the decision function was taken to be the maximum likelihood estimate of λ. For the data presented in Table I, this truncation point was reached for only one sequence.

For each value of the pair (α, λ), $(\alpha = 3, 5, 21, 41, 81;\ \lambda = 1, 3, 5, 10, 20)$ the following quantities are tabulated:

(a) mean = i = the true value of λ;

(b) P (err) = average number of incorrect decisions;

(c) TP(err) = theoretical bound on the error probability = $2/(\alpha-1)$;

(d) Av-N = the average sample size;

(e) Tav - N = $k_i \log \alpha$ = theoretical asymptotic bound for the expected sample size; and

(f) Fix - N = the sample size which would be required to distinguish the hypothesis $\lambda = i$ from $\lambda = i+1$ or $i-1$ with an error probability less than or equal to $(2 - \alpha^{-i})/(\alpha-1)$. (When $\lambda = i$, the sequential procedure has error probability less than or equal to this quantity by (9).). The normal approximation was used to calculate Fix - N.

These results point out that in many cases, the true error probability may be somewhat less than the theoretical bound. This is due mostly to the inequalities introduced in the derivation of (9). It is not surprising that the calculated average sample size is greater than Tav-N since the latter quantity is an asymptotic bound. The average sample sizes obtained do, however, compare favorably with the corresponding fixed sample size values for the moderate values of α used.

Table I.

Results of Monte Carlo Experiment

Mean	P(err)	TP(err)	Av-N	Tav-N	Fix-N
1	0.214	1.0	4.11	3.58	1
3	0.305	1.0	11.28	8.02	1
5	0.293	1.0	17.87	12.43	1
10	0.318	1.0	34.34	23.42	1
20	0.343	1.0	63.82	45.38	1
1	0.192	0.5	5.83	5.24	3
3	0.192	0.5	16.92	11.75	6
5	0.173	0.5	26.63	18.21	10
10	0.219	0.5	51.36	34.31	19
20	0.215	0.5	99.08	66.47	37
1	0.035	0.1	11.04	9.92	11
3	0.042	0.1	31.62	22.23	33
5	0.049	0.1	51.05	34.44	55
10	0.055	0.1	100.66	64.91	109
20	0.051	0.1	191.76	125.75	217
1	0.029	0.05	13.44	12.10	16
3	0.023	0.05	37.15	27.12	47
5	0.024	0.05	61.64	42.01	77
10	0.025	0.05	121.83	79.17	154
20	0.018	0.05	235.85	153.38	308
1	0.007	0.025	16.05	14.32	21
3	0.006	0.025	42.68	32.09	61
5	0.013	0.025	71.25	49.71	101
10	0.013	0.025	140.10	93.69	202
20*	0.010	0.025	270.85	181.50	403

*One sequence in this group was truncated at 1000.

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