SOME TESTS OF EQUALITY OF SEVERAL COVARIANCE MATRICES

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TABLE OF CONTENTS

LIST OF TABLES

ABSTRACT

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This dissertation deals with the distribution problems of certain criteria for testing hypotheses concerning covariance matrices from several p-variate normal populations where the random normal variates are real valued (Chapters I to IV) and complex valued (Chapter V).

The first three chapters deal with testing the hypothesis $H_0:\underset{\sim}{\Sigma}_1=\ldots=\underset{\sim}{\Sigma}_k$ where $\underset{\sim}{\Sigma}_i$ is the unknown covariance matrix of the i^{th} population, $i = 1,\ldots,k$. Chapter I considers the $U^{(p)}$ statistic (a constant times Hotelling's generalized T_0^2) and gives a general method of obtaining the distribution of $U^{(p)}$ by inversion of the Laplace transform of $U^{(p)}$. $U^{(p)}$ can be used to test H_0 for $k = 2$ as well as the general linear hypothesis and that of the independence of two sets of normal random variates. Approximations to $U^{(p)}$ are discussed and percentage points given. Chapter II introduces the max U-ratio (R_1) test of H_0 and provides exact $(k = 2, p = 2)$ and approximate $(k = 2)$ distributions of R_1 using results found in Chapter I. Chapter III considers the likelihood ratio (LR) criterion for testing H_0 for $k = 2$ and develops the distribution of the LR criterion for $p = 2$ and 4. Power comparisons of the LR and R_1 tests are made for selected alternatives and $p = 2$.

Chapter IV develops the max trace-ratio (R_2) for testing $H_0:\underset{\sim}{\Sigma}_1=\ldots=\underset{\sim}{\Sigma}_k=\lambda\ \underset{\sim}{\Sigma}_0$, where $\underset{\sim}{\Sigma}_0$ is known and λ is unspecified. The max trace-ratio has the same distribution as Hartley's F_{max} test of the equality of several variances from univariate normal populations. Distributions of R_2 are obtained for k = 2,3 and 4 and the power function studied for k = 2.

Chapter V deals with the approximation of the largest characteristic root of a complex Wishart matrix using a technique due to Pillai. Comparison of the approximate and exact distributions are made and percentage points tabulated.

Finally Chapter VI summarizes the studies of the first five chapters and presents suggestions for future research.

INTRODUCTION

Let $\pi_1,\ldots,\pi_k$ be k populations having the p-variate normal distribution $N(\underset{\sim}{\mu}_i, \underset{\sim}{\Sigma}_i)$, $i = 1,\ldots,k$, where $\underset{\sim}{\mu}_i(p \times 1)$ and $\underset{\sim}{\Sigma}_i(p \times p)$ are unknown. The problem of testing the hypothesis $\underset{\sim}{\mu}_1 = \ldots = \underset{\sim}{\mu}_k$, the simple multivariate analysis of variance (MANOVA) problem, has been considered by many authors (see Anderson [1] or Roy [40], for example). One of the basic assumptions underlying MANOVA is the equality of the covariance matrices $\underset{\sim}{\Sigma}_1,\ldots,\underset{\sim}{\Sigma}_k$. Thus the problem of testing the hypothesis H_0: $\underset{\sim}{\Sigma}_1 = \ldots = \underset{\sim}{\Sigma}_k$ of the equality of covariance matrices is of basic importance in multivariate analysis. This problem is the primary theme of this dissertation.

When $k = 2$ there are a number of test criteria available for testing $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$ against the so-called one-sided alternatives (namely each characteristic root of $\underset{\sim}{\Sigma}_1 \underset{\sim}{\Sigma}_2^{-1} \geq 1$ and $\mathrm{tr}\, \underset{\sim}{\Sigma}_1 \underset{\sim}{\Sigma}_2^{-1} > p$; reversal of the inequalities gives the other hypothesis). A discussion of the distributions and merits of four such criteria - Roy's largest root, Hotelling's trace T_0^2 (or $U^{(p)}$), Pillai's trace $V^{(p)}$ and Wilks' criterion - may be found in Pillai and Jayachandran [36] for the bivariate case. W. F. Mikhail [27] and Anderson and Das Gupta [2] have studied the monotonicity of the power functions of some of these tests. Giri [12], [13] has shown that $V^{(p)}$ is locally best invariant and unbiased. For $k > 2$ and two-sided alternatives, the only test which appears available is the likelihood ratio (LR) criterion first introduced by Wilks [42], later modified by Bartlett [3] and studied by Box [4], Korin [26] and Sugiura and Nagao [41]. For

$k = 2$ and two-sided alternatives, Roy's largest-smallest root test is available but it has not been explored in any detail.

In this study, first a new test proposed by Pillai [39] to test H_0 against two-sided alternatives is investigated at length (Chapters I - III). The test is based on the maximum of ratios of independent Hotelling traces. Later a second test also proposed by Pillai [39] is studied for testing the hypothesis $H_0^{'}$: $\underset{\sim}{\Sigma}_1 = \dots = \underset{\sim}{\Sigma}_k = \lambda \underset{\sim}{\Sigma}_0$ where $\underset{\sim}{\Sigma}_0$ is given and λ is unspecified (see Chapter IV). This is a generalization of the sphericity test. Both of the proposed tests are based on the union-intersection approach (Roy [40]).

The topic of discussion of Chapter I is the distribution (exact and approximate) of Hotelling's generalized T_0^2 statistic (or equivalently $U^{(p)}$) in connection with testing equality of two covariance matrices as well as two other tests which have the same distribution problems. Chapter II introduces the new test for H_0 based on the max U-ratio which will be denoted by R_1. The null distribution problem for R_1 is studied and for $k = 2$ the non-null problem is considered and power tabulations are obtained.

The LR criterion is considered in Chapter III. The distribution of the LR criterion is obtained for $p = 2$ and 4 and $k = 2$. The power of the LR test is determined and compared with that of the power of the test based on R_1.

Chapter IV considers the hypothesis $H_0^{'}$: $\underset{\sim}{\Sigma}_1 = \dots = \underset{\sim}{\Sigma}_k = \lambda \underset{\sim}{\Sigma}_0$, which as mentioned earlier is a multipopulation extension of the test of sphericity (see Anderson [1]). The test proposed involves the ratio of the maximum to the minimum of the traces of $\underset{\sim}{\Sigma}_0^{-1} \underset{\sim}{S}_i$, where the $\underset{\sim}{S}_i$ are the sample sums of product matrices. This criterion, max trace-ratio to be denoted by R_2, has the same distribution as the F_{max} statistic introduced by Hartley [16] as a short cut test of the equality of several variances. (It should be noted that the multivariate hypothesis considered here reduces to the hypothesis of the equality of variances when $p = 1$.) The null distribution of R_2 is considered for $k = 2$, 3 and 4, and the non-null distribution for $k = 2$.

Finally Chapter V deals with a problem in complex multivariate analysis - that of the distribution of the largest characteristic root of a complex Wishart matrix and an approximation to it. Some applications are discussed.

It should be noted that the classes of alternatives for which the max U-ratio and the max trace-ratio have reasonable power are at this time unknown. It seems possible to invent alternatives for which the tests proposed here are insensitive. These problems will require further study.

CHAPTER I

THE EXACT DISTRIBUTION OF HOTELLING'S GENERALIZED T_0^2

1. Introduction and Summary

Let $\underset{\sim}{S}_1$ and $\underset{\sim}{S}_2$ be two symmetric matrices of order p estimating the same covariance matrix, where $\underset{\sim}{S}_2$ is positive definite having a Wishart distribution with n_2 degrees of freedom, and $\underset{\sim}{S}_1$ is at least positive semi-definite having a non-central Wishart distribution with n_1 degrees of freedom. Then Hotelling's generalized T_0^2 statistic is defined by [17]:

$$T_0^2 = n_2 \text{ tr } \underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1} = n_2 U^{(s)} ,$$

where $s(= \min (n_1,p))$ is the number of non-zero characteristic roots of $\underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$. When $n_1 \geq p$, $U^{(s)} = U^{(p)}$. When $n_1 < p$ the density function of the characteristic roots of $\underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$ can be obtained from that for $n_1 \geq p$ if in the latter case the following changes are made:

$$(n_1,n_2,p) \rightarrow (p,n_1 + n_2-p,n_1).$$

Hence the density of $U^{(s)}$ can be easily derived from that of $U^{(p)}$ and therefore only the case of $U^{(p)}$ is considered here.

The exact null distribution of T_0^2 (i.e., when the non-centrality matrix is null) was obtained by Hotelling [17] for $p = 2$. Davis [10] has shown that the null density of T_0^2 satisfies an ordinary linear homogeneous differential equation of order p. The non-null distribution has been attempted by Constantine [8] using zonal polynomials and hypergeometric functions of matrix arguments. However, his results hold only for $|U^{(p)}| < 1$. Pillai and Jayachandran [35], [36] have obtained the non-null distribution of $U^{(2)}$ using zonal polynomials up to the sixth degree.

An approximation to the null distribution of $U^{(p)}$ has been suggested by Pillai [28], [29] and studied by Pillai and Samson [38]. Ito [18] has obtained an asymptotic expansion for the null distribution of T_0^2 which he later extended to the non-null case [19]. Davis [11] has further studied the asymptotic null distribution.

Grubbs [15] has provided some exact percentage points for $U^{(2)}$ for n_1 and n_2 less than 50. Using the exact moment quotients of $U^{(p)}$, Pillai [32] has provided extensive tables of approximate percentage points for $U^{(p)}$. Further, Pillai and Jayachandran [35] have obtained some exact percentage points of $U^{(2)}$ in connection with power function studies. Recently Davis [11] has tabulated exact percentage points of T_0^2/n_1 for $p = 3$ and 4 using the differential equation approach [10]. He also provides comparisons of the accuracy of several approximations.

It may be pointed out that the null distribution of the characteristic roots of $\underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$ (see Eq. (2.1)) is of the same form as those of the characteristic roots of matrices arising in each of the following tests of hypotheses except that the two parameters m and n involved there (see below) have to be defined differently in each case [28], [32]: (i) Independence between a p-set and a q-set in a $(p + q)$-variate normal population and (ii) Equality of covariance matrices in two p-variate normal populations. In view of this, the null distribution of $U^{(p)}$ for the three tests is also of the same form. Pillai[29] considered the use of $U^{(p)}$ for tests of (i) and (ii) as well, and Pillai and Jayachandran [35], [36] have shown that the power functions of the $U^{(p)}$ test against appropriate alternatives for tests of (i) and (ii) and the general linear hypothesis behave more or less in the same manner.

Still, however, there are no explicit expressions available for the exact null distribution of $U^{(p)}$ (or T_0^2) for $p > 2$ except one obtained for $U^{(3)}$ as an infinite series by Pillai and Chang through transformation of variables [34]. In this chapter there is presented a method for deriving the exact null distribution of $U^{(p)}$ employing inverse Laplace transforms. Density functions are given for $p = 2$, m a non-negative integer, $p = 3$, $m = 0, 1, 2, 3, 4$ and 5, and $p = 4$, $m = 0, 1$ and 2, where $m = (n_1-p-1)/2$. Exact upper percentage points are tabulated for $p = 2, 3$ and 4, various significance levels, and selected values of m and n $(= (n_2-p-1)/2)$. Also, two approximations similar to Pillai's [28], [29] are presented. Finally, the non-null density of $U^{(2)}$ is given using zonal polynominals up to the sixth degree.

The exact densities and the approximations derived in this chapter will be used to develop the distribution of R_1.

2. The Laplace Transform of $U^{(p)}$

The joint density function of $\lambda_1, \lambda_2, \ldots, \lambda_p$, the characteristic roots of $\underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$, has the form [40]:

$$(2.1)\quad f(\lambda_1,\ldots,\lambda_p) = C(p,m,n) \prod_{i=1}^{p} \lambda_i^{m}/(1+\lambda_i)^{m+n+p+1} \prod_{i>j} (\lambda_i-\lambda_j) ,$$

$$0 < \lambda_1 < \ldots < \lambda_p < \infty ,$$

where $C(p,m,n) = \pi^{p/2} \prod_{i=1}^{p} \Gamma(\frac{1}{2}(2m+2n+p+i+2))/\{\Gamma(\frac{1}{2}(2m+i+1))\Gamma(\frac{1}{2}(2n+i+1))\Gamma(\frac{1}{2}i)\}$.

and m and n are defined in section 1. Then $U^{(p)} = \sum_{i=1}^{p} \lambda_i = \text{tr}\, \underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$,

and the Laplace transform of $U^{(p)}$ with respect to (2.1) is:

$$(2.2) \qquad L(t;\, p,m,n) = E(\exp(-t \sum_{i=1}^{p} \lambda_i))$$

$$= C \int \underset{G}{\cdots} \int \exp(-t \sum_{i=1}^{p} \lambda_i) \prod_{i=1}^{p} \lambda_i^m/(1+\lambda_i)^{m+n+p+1} \prod_{i>j} (\lambda_i - \lambda_j) \prod_{i=1}^{p} d\lambda_i$$

where

$$G = \{ (\lambda_1, \ldots, \lambda_p) \mid 0 < \lambda_1 < \ldots < \lambda_p < \infty \} ,$$

$C = C\,(p,m,n)$ and $t \geq 0$.

Upon making the transformation

$$x_i = 1/(1 + \lambda_{p-i+1}),\ i = 1,\ldots,p,$$

we may write (2.2) as:

$$(2.3) \quad L(t;\, p,m,n) = e^{pt}\, C \int \underset{R}{\cdots} \int \exp(-t \sum_{i=1}^{p} x_i^{-1}) \prod_{i=1}^{p} x_i^{\,n}(1-x_i)^m \prod_{i>j} (x_i - x_j) \prod_{i=1}^{p} dx_i$$

where

$$\mathcal{B} = \{ (x_1,\ldots,x_p) \mid 0 \le x_1 < x_2 < \ldots < x_p \le 1 \} .$$

Next we note that $\prod_{i>j} (x_i - x_j)$ may be written as the Vandermonde determinant

$$\begin{vmatrix} x_p^{p-1} & x_p^{p-2} & \cdots & x_p & 1 \\ x_{p-1}^{p-1} & x_{p-1}^{p-2} & \cdots & x_{p-1} & 1 \\ \vdots & & & \vdots & \\ x_1^{p-1} & x_1^{p-2} & \cdots & x_1 & 1 \end{vmatrix} ,$$

and that the elementary properties of determinants allows (2.3) to be written:

(2.4) $L(t;p,m,n)$

$$= e^{pt}\, C \int \underset{\mathcal{B}}{\cdots} \int \exp\left(-t \sum_{i=1}^{p} x_i^{-1}\right) \begin{vmatrix} (1-x_p)^m\, x_p^{n+p-1} & \cdots & (1-x_p)^m\, x_p^{n} \\ \vdots & & \vdots \\ (1-x_1)^m\, x_1^{n+p-1} & \cdots & (1-x_1)^m\, x_1^{n} \end{vmatrix} \prod_{i=1}^{p} dx_i .$$

If we take m to be a non-negative integer and expand $(1-x_i)^m$ as a binomial series, the determinant in (2.4) is:

$$(2.5)\quad \begin{vmatrix} \sum_{i_p=0}^{m} \binom{m}{i_p} (-1)^{i_p} x_p^{n+q_p+i_p} & \cdots & \sum_{i_1=0}^{m} \binom{m}{i_1}(-1)^{i_1} x_p^{n+q_1+i_1} \\ \vdots & & \vdots \\ \sum_{i_p=0}^{m} \binom{m}{i_p}(-1)^{i_p} x_1^{n+q_p+i_p} & \cdots & \sum_{i_1=0}^{m} \binom{m}{i_1}(-1)^{i_1} x_1^{n+q_1+i_1} \end{vmatrix},$$

where $q_j = j-1$. (2.5) can be further reduced to the form

$$(2.6)\quad \sum_{i_p=0}^{m} \cdots \sum_{i_1=0}^{m} \{ \prod_{j=1}^{p} \binom{m}{i_j} \} (-1)^{\sum_{j=1}^{p} i_j} \begin{vmatrix} x_p^{n+q_p+i_p} & \cdots & x_p^{n+q_1+i_1} \\ \vdots & & \vdots \\ x_1^{n+q_p+i_p} & \cdots & x_1^{n+q_1+i_1} \end{vmatrix}.$$

The expansion (2.6) allows us to throw (2.4) into the form [30]:

$$(2.7)\quad L(t;p,m,n) = e^{pt}\, C \sum_{i_p=0}^{m} \cdots \sum_{i_1=0}^{m} \{ \prod_{j=1}^{p} \binom{m}{i_j} \}(-1)^{\sum_{j=1}^{p} i_j} R(n;q_p+i_p,\ldots,q_1+i_1;t)$$

where

(2.8) $R(n;a_p,a_{p-1},\ldots,a_1;t)$

$$= \begin{vmatrix} \int_0^1 x_p^{n+a_p} e^{-t/x_p} dx_p & \cdots & \int_0^1 x_p^{n+a_1} e^{-t/x_p} dx_p \\ \vdots & & \vdots \\ \int_0^{x_2} x_1^{n+a_p} e^{-t/x_1} dx_1 & \cdots & \int_0^{x_2} x_1^{n+a_1} e^{-t/x_1} dx_1 \end{vmatrix} .$$

Now permuting the columns of the determinants so that the indices form a decreasing sequence, dropping all determinants which are zero and combining like terms in (2.7) gives

$$\text{(2.9)} \qquad L(t;p,m,n) = e^{pt} \, C \sum_{\mathcal{S}} k_{i_p \, i_{p-1} \ldots i_1} R(n;\, i_p,\, i_{p-1},\ldots,i_1;t)$$

where

$$\mathcal{S} = \{ \, (i_1,\ldots,i_p) \mid 0 \le i_1 < i_2 < \ldots < i_p \le m + p-1 \, \}$$

and the $k_{i_p \ldots i_1}$ depend on p and m. The constants $k_{i_p \ldots i_1}$ have been tabulated in Table 1.1 for $p = 3$, $m = 0\ (1)\ 5$ and $p = 4$, $m = 0, 1, 2$.

Thus we have expressed the Laplace transform of $U^{(p)}$ as a linear combination of the determinants $R(n; i_p,\ldots,i_1; t)$.

3. A Reduction Formula for $R(n;a_p,\ldots,a_1; t)$

With the expression (2.9) for the Laplace transform of $U^{(p)}$ we need to evaluate the determinants $R(n; a_p,\ldots,a_1; t)$. This will be done by means of a reduction formula similar to the one developed by Pillai [30].

We will state here the notation and lemmas that are needed and give only an outline of the approach as the results are analogous to those of Pillai [28], [30]. Let

$$(3.1)\quad V(x; q_k, q_{k-1},\ldots,q_1; t) = \begin{vmatrix} \int_0^x x_k^{q_k} e^{-t/x_k} dx_k & \cdots & \int_0^x x_k^{q_1} e^{-t/x_k} dx_k \\ \vdots & & \vdots \\ \int_0^{x_2} x_1^{q_k} e^{-t/x_1} dx_1 & \cdots & \int_0^{x_2} x_1^{q_1} e^{-t/x_1} dx_1 \end{vmatrix}$$

(Note that $R(n; a_p,\ldots,a_1; t) = V(1; n + a_p,\ldots,n + a_1; t)$.)

Now (3.1) will involve integrals of the type

$$I(x';q,F;t) = \int_0^{x'} y^q F(y) e^{-ty} dy,$$

where $F(y)$ is a function of y such that the integral exists and in our context could be of the form

$$(3.3)\quad \int_0^y x_{k-1}^{q_{k-1}} e^{-t/x_{k-1}} dx_{k-1} \cdots \int_0^{x_2} x_1^{q_1} e^{-t/x_1} dx_1 .$$

When $F(y)$ has the form (3.3) we will denote (3.2) by $I(x';q,q_{k-1},\ldots q_1;t)$.

The following lemma involving (3.2) is obtained by integration by parts.

Lemma 3.1: The integral

(3.4) $$I(x'; q, F; t) = [1/(q+1)] \{ I_0(x'; q+1, F; t) - I(x'; q+1, F'; t) - t\, I(x'; q-1, F; t) \}$$

where

$$I_0(x'; q+1, F; t) = y^{q+1} F(y)\, e^{-t/y} \Big|_0^{x'}$$

and $$F'(y) = \frac{d}{dy} F(y).$$

Lemma 3.2: If σ is any permutation of $(1, 2, \ldots, k)$ then

$$\sum_{\sigma} I(x; q_{\sigma(k)}, \ldots, q_{\sigma(1)}; t) = \prod_{j=1}^{k} I(x; q_j; t)$$

where the summation is over all possible permutations.

Let $V(x; q'_k, \ldots, q'_1; t')^{(i)}$ denote the determinant (3.1) when the indices of the ith row alone are different from those of the other rows, where the indices of the ith row are $q'_k, \ldots, q'_1, t'$. Then we have the following lemma.

Lemma 3.3:

$$\sum_{i=1}^{k} (-1)^{i-1} V(x; q_k',\ldots,q_1'; t')^{(i)}$$

$$=\sum_{j=1}^{k} (-1)^{k+j} I(x; q_j'; t') V(x; q_k,\ldots,q_{j+1}, q_{j-1},\ldots,q_1; t).$$

We now state the reduction formula for the determinant (3.1).

Theorem 3.1:

$$(3.5)\quad V(x; q_k,q_{k-1},\ldots,q_1; t) = [1/(q_k+1)] (A^{(k)} + B^{(k)} - tC^{(k)}) ,$$

where

$$A^{(k)} = x^{q_k+1} e^{-t/x} V(x;q_{k-1},\ldots,q_1; t) ,$$

$$B^{(k)} = 2 \sum_{j=1}^{k-1} (-1)^{k+j} I(x; q_j+q_k+1; 2t) V(x; q_{k-1},\ldots,q_{j+1},q_{j-1},\ldots,q_1;t)$$

and

$$C^{(k)} = V(x; q_k -1 ,q_{k-1},\ldots,q_1; t).$$

Proof: Expand the determinant by the first column. (Recall that the order of integrations must not be changed). Now using Lemma 1 we integrate by parts the term involving the element from the ith row and first column with respect to x_{k-i+1}. Next add the expressions obtained corresponding to each of the three terms on the right side of (3.4) and apply the above lemmas. The result follows.

The formula we require to evaluate the Laplace transform (2.9) follows as a corollary.

Corollary 3.1:

$$R(n;\ a_p,\ldots,a_1;\ t) = [1/(a_p+1)]\ (D^{(p)} + E^{(p)} - tF^{(p)}) , \tag{3.6}$$

where

$$D^{(p)} = e^{-t}\ R(n;\ a_{p-1},\ldots,a_1;\ t) ,$$

$$E^{(p)} = e^{-2t} \sum_{j=1}^{p-1} (-1)^{p+j} g(n;\ a_j+a_p+3,2;\ t)\ R(n:\ a_{p-1},\ldots,a_{j+1},a_{j-1},\ldots,a_1;t)$$

where

$$g(n;\ a,b;\ t) = \int_0^\infty e^{-tz}/(1 + z/b)^{bn+a}\ dz$$

and

$$F^{(p)} = R(n;\ a_p-1,\ a_{p-1},\ldots,a_1;\ t).$$

Proof: In (3.5) let $x = 1$, $q_j = n+a_j$ and make the change of variable $z = 2(1-y)/y$ in $I(1;\ 2n+a_j+a_p+1;\ 2t)$ to get

$$I(1;\ 2n+a_j+a_p+1;\ 2t) = \tfrac{1}{2} e^{-2t}\ g(n;\ a_j+a_p+3,2;\ t).$$

4. Use of Reduction Formula.

Now let us illustrate the use of (3.6) in deriving the expression for the determinant $R(n;\ 3,\ 1,\ 0;\ t)$. (3.6) yields:

$$(4.1)\quad R(n;\ 3,1,0;\ t) = [1/(n+4)]\ \{e^{-t}\ R(n;\ 1,0;\ t)$$

$$+ e^{-2t}\ g(n;\ 6,2;\ t)\ R(n;\ 1;\ t) - e^{-2t}\ g(n;\ 7,2;t)\ R(n;\ 0;\ t) -$$

$$- t\ R(n;\ 2,1,0;\ t)\ \}.$$

But $R(n;\ i;\ t) = I(1;\ n+i;\ t) = e^{-t}\ g(n;\ i+2,1;\ t)$ and a second use of the reduction formula yields

$$(4.2)\quad R(n;\ 2,1,0;\ t) = [1/(n+3)]\ \{\ e^{-t}\ R(n;\ 1,0;\ t)$$

$$+ e^{-2t}\ g(n;\ 5,2;\ t)\ R(n;\ 1;\ t)$$

$$- e^{-2t}\ g(n;\ 6,2;\ t)\ R(n;\ 0;\ t)\ \}$$

and

$$(4.3)\quad R(n;\ 1,0;\ t) = [e^{-2t}/(n+2)]\ \{\ g(n;\ 2,1;\ t) - g(n;\ 4,2;\ t)\ \}\ .$$

There are no terms corresponding to $t\,F^{(p)}$ of the Corollary in (4.2) and (4.3) since any determinant having two columns (indices) the same is zero.

We now integrate by parts $R(n;\,2,1,0;\,t)$, integrating one g-factor in each of the terms in (4.2), and in this connection we use the following result:

$$g(n;\,a,b;\,t) = (1/t)\,\{\,1-((bn+a)/b)\,g(n;\,a+1,b;\,t)\,\}\,.$$

This is done in order to bring the terms to a more suitable form for inversion as shown in the next section.

We thus obtain:

$$R(n;\,3,1,0;\,t) = [e^{-3t}/(n+4)]\,\{\,(g(n;\,2,1;\,t) - g(n;\,4,2;\,t))/(n+2)$$

$$+\,g(n;\,6,2;\,t)\,g(n;\,3,1;\,t) - g(n;\,7,2;\,t)\,g(n;\,2,1;\,t)$$

$$-\,[t/(n+3)]\,[(\,(1/t) - ((n+2)/t)\,g(n;\,3,1;\,t)$$

$$-\,((1/t) - ((n+2)/t)\,g(n;\,5,2;t))/(n+2)$$

$$+\,((1/t) - ((2n+5)/(2t))\,g(n;\,6,2;\,t))\,g(n;\,3,1;\,t)$$

$$-\,((1/t) - ((n+3)/t)\,g(n;\,7,2;\,t))\,g(n;\,2,1;\,t)]\,\}\,.$$

All terms involving t as a factor and the constant terms not involving integrals can be seen to vanish. This holds true in the general case. Upon simplification we get:

$$(4.4)\quad R(n;\ 3,1,0;\ t) = [e^{-3t}/(n+4)]\ \{\ [(2n+5)/((n+2)(n+3))]g(n;\ 2,1;t)$$

$$-\ [1/(n+2)]\ g(n;\ 4,2;\ t) - [1/(n+3)]\ g(n;\ 5,2;\ t)$$

$$+\ [(4n+11)/(2(n+3))]\ g(n;\ 6,2;\ t)\ g(n;\ 3,1;\ t)$$

$$-\ 2g(n;\ 7,2;\ t)\ g(n;\ 2,1;\ t)\ \}.$$

5. The Density Function of $U^{(p)}$

The uniqueness property of the Laplace transform will allow us now to obtain the density of $U^{(p)}$ using (2.9). The density may be written:

$$(5.1)\qquad f(u) = C \sum_{\mathcal{S}} k_{i_p\, i_{p-1} \cdots i_1}\ R^*(n;\ i_p,\ldots,i_1;\ u)$$

where $R^*(n;\ i_p,\ldots,\ i_1;\ u)$ is the inverse Laplace transform of $R(n;\ i_p,\ldots,i_1;\ t)$.

We will illustrate the method of obtaining the R^* functions with the help of $R(n;\ 3,1,0;\ t)$ in (4.4).

If we denote the inverse Laplace transform of $g(n;\ a,b;\ t)$ by $g^*(n;\ a,b;\ t)$ we see that

$$g^*(n;\ a,b;\ u) = 1/(1+u/b)^{bn+a}\ .$$

Also the function whose transform is $g(n;\ a,b;\ t)\ g(n';\ a',b';t)$ is given by the convolution

$$(5.2)\qquad g^*(n;\ a,b;\ u) * g^*(n';\ a',\ b';\ u)$$

where * denotes the convolution operator. We may write (5.2) as:

$$(5.3)\quad h(n,n';\ a,a';\ b,b';\ u) = \int_0^u \frac{dx}{(1+x/b)^{bn+a}\ (1+(u-x)/b')^{b'n'+a'}}\ .$$

Then from (4.4) we find:

$$(5.4)\ R^*(n;\ 3,1,0;\ t) = [1/(n+4)]\ \{\ [\ (2n+5)/((n+2)(n+3))]\ g^*(n;\ 2,1;\ u)$$

$$-\ [1/(n+2)]\ g^*(n;\ 4,2;\ u) - [1/(n+3)]\ g^*(n;\ 5,2;\ u)$$

$$+\ [(4n+11)/(2(n+3))]\ h(n,n;\ 6,3;\ 2,1;\ u)$$

$$-\ 2\ h(n,n;\ 7,2;\ 2,1;\ u)\ \}\ .$$

(5.4) may be further simplified by using the expression below which is obtained by integration by parts.

$$h(n,n';\ a,a';\ b,b';\ u) = [b/(bn+a)]\ (g(n';\ a',b';\ u) - g(n;\ a,b;\ u))$$

$$+\ [b(b'n'+a')/(b'(bn+a))]\ h(n,n';\ a-1,a'+1;\ b,b';u).$$

Finally upon simplification we have

$$R^*(n;\ 3,1,0;\ t) = [1/(n+4)]\ \{\ [1/((n+2)(n+3))]\ g^*(n;\ 2,1;\ u)$$

$$-\ [1/(n+2)]g^*(n;\ 4,2;\ u)\ -\ [1/(n+3)]\ g^*(n;\ 5,2;\ u)$$

$$+\ [2/(n+3)]\ g^*(n;\ 6,2;\ u)\ +\ [3/(2(n+3))]\ h(n,n;\ 6,3;\ 2,1;\ u)\ \}\ .$$

In calculating the R^* functions it should be noted that

$$R^*(n;\ a_p,a_{p-1},\ldots,a_1;\ u) = R^*(n+a_1;\ a_p-a_1,\ldots,0;\ u)\ ,$$

where we may take $0 \leq a_1 < a_2 < \ldots < a_p \leq m+p-1$, so that the only R^*'s which need be determined are those with $a_1 = 0$.

R^* for $p = 2$ can be written as:

$$R^*(i,0) = [1/(n+i+1)]\left\{\ \sum_{\ell=1}^{i} \alpha_{1\ell}(n)\ g(n;\ell+1,\ 1;u)\right.$$

$$\left. -\left(\sum_{\ell=1}^{i} \alpha_{1\ell}(n)\right) g(n;\ i+3,2;\ u)\ \right\}$$

where $\alpha_{i\ell}(n) = \prod_{k=1}^{\ell-1} \{(n+k+1)/(n+i+1-k)\}$ if $\ell > 1$ and $\alpha_{i1}(n) = 1$.

R^* for $p = 3$ can be written in the form:

$$(5.5)\quad R^*(n;\ i,j,0;\ u) = [1/(n+i+1)] \Big\{ \sum_{\ell=1}^{i-2} \alpha_{ij\ell}(n)\, g^*(n;\ \ell+2,1;\ u)$$

$$+ \sum_{\ell=1}^{i} \beta_{ij\ell}(n)\, g^*(n;\ \ell + j + 2,2;\ u)$$

$$+ \gamma_{ij}(n)\, h(n,n;\ i+j+2,3;\ 2,1;\ u) \Big\} \text{ for } i > 2.$$

If $i = 2$, the last two terms are obtained by substituting 2 for i, but the first term becomes $\alpha_{211}(n)\, g^*(n;\ 2,1;\ u)$. The coefficients $\alpha_{ij\ell}(n)$, $\beta_{ij\ell}(n)$ and $\gamma_{ij}(n)$ for $1 \leq j < i \leq 7$ are presented in Table 1.2. These provide the density function of $U^{(3)}$ for $m = 0(1)5$.
R^* for $p = 4$ can be expressed as:

$$(5.6)\quad R^*(n;\ i,j,k,0;\ u) = [1/(n+i+1)] \{ \alpha_{ijk}(n)\, g^*(n;\ 2,1;\ u)$$

$$+ \sum_{\ell=1}^{i+j-2} \beta_{ijk\ell}(n)\, g^*(n;\ k+\ell+2,2;\ u) + \sum_{\ell=1}^{i} \gamma_{ijk\ell}(n) h(n,n;\ \ell+4,3;\ 2,1;\ u)$$

$$+ \delta_{ijk}(n)\, h(n,n;\ i+3,j+k+3;\ 2,2;\ u) \} .$$

The coefficients involved in (5.6) are given in Table 1.3 for $1 \leq k < j < i \leq 5$. These terms provide the density function of $U^{(4)}$ for $m = 0,1$ and 2.

6. The Distribution of $U^{(3)}$ and $U^{(4)}$

The distribution function of $U^{(p)}$, say, $G(z;\ p,m,n) = P(U^{(p)} \leq z)$ may be obtained from the density function (5.1) upon integration. We have:

$$(6.1)\quad G(z;\ p,m,n) = C \sum_{\mathcal{S}} k_{i_p \cdots i_1} \int_0^z R^*(n;\ i_p,\ldots,i_1;\ u)du.$$

The distribution functions of $U^{(p)}$ for $p = 3$ and 4 are thus seen to be obtained by the integration of $g^*(n;\ a,b;\ u)$ and $h(n,n';\ a,a';\ b,b';\ u)$ with respect to u. Now

$$(6.2)\quad \int_0^z h(n,n';\ a,a';\ b,b';\ u)du$$

$$= [b'/(b'n'+a'-1)]\{[b/(bn+a-1)](1-g^*(n;\ a-1,b;\ z))$$

$$- h(n,n';\ a,a'-1;\ b,b';\ z)\}.$$

(6.3) is obtained by interchange of the order of integration. Finally, evaluation of $h(n,n';\ a,a';\ b,b';\ z)$ makes use of the following method. If

$$P(z;\ p,q,c,d) = \int_0^z \frac{dx}{(c+x)^p\ (d-x)^q}$$

where p and q are non-negative integers, $c > 0$ and $d > z$ then

(6.4) $P(z;\ p,q,c,d) = A_1 \{ \ln (1+z/c) \} - B_1 \{ \ln (1-z/d) \}$

$$-\sum_{i=2}^{p} \frac{A_i}{i-1} \left\{ \frac{1}{(c+z)^{i-1}} - \frac{1}{c^{i-1}} \right\} + \sum_{j=2}^{q} \frac{B_j}{j-1} \left\{ \frac{1}{(d-z)^{j-1}} - \frac{1}{d^{j-1}} \right\}$$

where

$$A_{p-i} = [\prod_{\ell=1}^{i} (q+\ell-1)]/[i!\ (c+d)^{q+i}] \quad \text{and} \quad B_{q-j} = [\prod_{\ell=1}^{j} (p+\ell-1)]/[j!(c+d)^{p+j}].$$

(6.4) is obtained by using partial fraction expansions. We can then write:

$$h(n,n';\ a,a';\ b,b';\ z) = b^{bn+a}\, b'^{\,b'n'+a'}\, P(z;\ bn+a, b'n'+a', b, b'+u),$$

where we take $bn+a$ and $b'n'+a'$ to be non-negative integers.

The integrals of the R^* functions involved in the distribution of $U^{(p)}$ for $p = 3$, $m = 0(1)5$ and $p = 4$, $m = 0$, 1 and 2 are provided in Appendix A in simplified form.

7. Computation of Percentage Points of $U^{(2)}$, $U^{(3)}$ and $U^{(4)}$

Tables of percentage points have been prepared for $U^{(p)}$ for $p = 3$, $m = 0\ (1)\ 5$ and $p = 4$, $m = 0$, 1 and 2, for $\alpha = .10$, .025 and .005, and $n = 5\ (5)\ 80\ (10)\ 100$ using the exact expressions discussed in the previous sections. Further, the percentage points of $U^{(2)}$ using the formula for the distribution found in [17] or [38] are presented for $m = -.5$ (.5) 5 (5) 50 (10) 100, 130, 160, 200, for $\alpha = .10$, .05, .025, .01 and

.005, and for n = 5 (5) 50 (10) 100, 130, 160, 200. These computations (as well as those described throughout this dissertation) were carried out on the CDC 6500 computer at the Purdue University Computing Center using double precision arithmetic. The percentage points are given to five significant digits in Tables 1.4 and 1.5.

8. Approximation to the Distribution of $U^{(p)}$

Pillai [28], [29] has suggested an approximation to the distribution of $U^{(p)}$ which involves an F-type (Type II Beta) distribution with the first moment of the approximate distribution being the same as that of $U^{(p)}$. Here we propose two similar approximations by fitting the first two moments and the first three moments of $U^{(p)}$ respectively to an F-type distribution.

The density function to be used in the approximation has the form:

$$(8.1) \qquad f(x) = x^a/\{\beta(a+1,b-a-1)\ K^{a+1}(1+x/K)^b\},\ 0 < x < \infty .$$

The distribution can be expressed as the incomplete beta integral $I_w(a+1, b-a-1)$, where $w = x/(x+K)$. (8.1) has the first three central moments:

$$\mu_{F1} = K\ (a+1)/(b-a-2),$$

$$\mu_{F2} = [K^2(a+1)(b-1)]/[(b-a-2)^2(b-a-3)] ,$$

and

$$\mu_{F3} = [2K^3(a+1)(b-1)(a+b)]/[(b-a-2)^3(b-a-3)(b-a-4)] .$$

The first three central moments of $U^{(p)}$ are given in [32], [38] and are:

$$\mu_1 = p(2m+p+1)/(2n) ,$$

$$\mu_2 = [p\ (2m+p+1)(2m+2n+p+1)(2n+p)]/[4n^2(n-1)(2n+1)] ,$$

and

$$\mu_3 = [p(2m+n+p+1)(2m+p+1)(2m+2n+p+1)(n+p)(2n+p)]/[2n^3(n-1)(n-2)(n+1)(2n+1)].$$

Pillai's approximation (A_1) with one moment fitted yields

$$a = \tfrac{1}{2}\, p(2m+p+1)-1,\ b = \tfrac{1}{2}\, p(2m+2n+p+1) + 1, \text{ and } K = p.$$

By setting $\mu_{F1} = \mu_1$ and $\mu_{F2} = \mu_2$ and taking $K = p$, we find the parameters for approximation A_2:

$$a = [\mu_2(\mu_1-p) + \mu_1{}^2(\mu_1+p)]/(p\ \mu_2) ,$$

and

$$b = [\mu_1(\mu_1 + p)^2 + \mu_1\, \mu_2 + 2p\ \mu_2]/(p\ \mu_2) .$$

Finally equating the first three moments yields the parameters for approximation A_3:

$$a = (2\mu_1{}^3\mu_2 + 3\mu_1{}^2\mu_3 - 6\mu_1\mu_2{}^2 - \mu_2\mu_3)/(\mu_2\mu_3 + 4\mu_1\mu_2{}^2 - \mu_1{}^2\mu_3) ,$$

$$b = [(a+1)(a+3) - \mu_1{}^2/\mu_2] \;/\; [(a+1) - \mu_1{}^2/\mu_2] ,$$

and

$$K = \mu_1(b-a-2)/(a+1).$$

Tables 1.6 and 1.7 indicate the accuracy of the three approximations A_1, A_2 and A_3. The percentage points for $p = 3$, $m = 0$ and 3, and $p = 4$, $m = 0$ and 2, and for $\alpha = .05$ and .01 were calculated for various values of n using the exact and approximate distributions. It can be seen that the approximations A_2 and A_3 are considerable improvements over Pillai's original approximation A_1, as is to be expected, with A_3 generally better than A_2. A_3 provides about three significant digits accuracy in the percentage points for $n \geq 10$. In some cases $n \geq 5$ is sufficient for this accuracy. A_2 provides the same accuracy for n slightly larger, usually around 10 to 15. A_1 does not provide this degree of accuracy until n is at least 40, and often n needs to be much larger. It has also been observed that the distributions associated with A_1, A_2 and A_3 closely approximate the distribution of $U^{(p)}$ not only in the upper tail but throughout the entire range of $U^{(p)}$ to the same degree of accuracy mentioned above for the percentage points. This inference has been based both on the study of percentiles as well as probability comparisons with the agreement in both cases being about three places or more. Thus the distribution function for A_3 provides a good approximation to the exact distribution of $U^{(p)}$ for $n \geq 10$ and for the whole range of $U^{(p)}$.

9. The Non-Null Density of $U^{(2)}$

For $p = 2$ the non-null density function of the characteristic roots $(\theta_i = \lambda_i/(1+\lambda_i)$, $i=1,2)$ of the matrix involved in the three different tests mentioned in Section 1 can be expressed in the following form using zonal polynomials up to the sixth degree [20], [21], [35], [36]:

$$(9.1)\quad K\{1 + A_{11}\, a_1 + \sum_{i=2}^{3} \sum_{j=1}^{2} A_{ij}\, a_{ij} + \sum_{i=4}^{5} \sum_{j=1}^{3} A_{ij}\, a_{ij}$$

$$+ \sum_{j=1}^{4} A_{6j}\, a_{6j} + \ldots\}\, a_2^{m}\, [(1-\theta_1)(1-\theta_2)]^{n}\, (\theta_2 - \theta_1)\,,$$

$$0 < \theta_1 < \theta_2 < 1,$$

where $a_1 = \theta_1 + \theta_2$ and $a_2 = \theta_1\, \theta_2$, the a_{ij}'s are functions of a_1 and a_2, and K and the A_{ij}'s are constants which depend on the non-null parameters of the hypothesis being tested. Expressions for K, a_{ij} and A_{ij} are available in [21].

Now $U^{(2)} = \sum_{i=1}^{2} \theta_i/(1-\theta_i)$, and we proceed as before to take the Laplace transform of $U^{(2)}$ with respect to (9.1). Thus

$$E(\exp(-t\, U^{(2)})) = K \int\int_R \exp(-t \sum_{i=1}^{2} \theta_i/(1-\theta_i)) F_1(\theta_1,\theta_2)$$

$$a_2^{m}[(1-\theta_1)(1-\theta_2)]^{n}\, (\theta_2-\theta_1)\, d\theta_1\, d\theta_2,$$

where $\mathcal{R} = \{(\theta_1,\theta_2) \mid 0 < \theta_1 < \theta_2 < 1\}$ and $F_1(\theta_1,\theta_2)$ is the summation in the braces in equation (9.1). Upon making the change of variable

$$X_1 = 1 - \theta_2 \ , \ X_2 = 1 - \theta_1$$

we have the transform of $U^{(2)}$ as

$$(9.2) \quad e^{2t} \ K \int\int_{\mathcal{R}'} \exp(-t \sum_{i=1}^{2} x_i^{-1}) F_2(x_1,x_2)[(1-x_1)(1-x_2)]^m (x_1 x_2)^n (x_2-x_1) \ dx_1 \, dx_2$$

where $\mathcal{R}' = \{(x_1,x_2) \mid 0 < x_1 < x_2 < 1\}$,

$$F_2(x_1,x_2) = 1 + A_{11}\, c_1 + \sum_{i=2}^{3} \sum_{j=1}^{2} A_{ij}\, c_{ij} + \sum_{i=4}^{5} \sum_{j=1}^{3} A_{ij}\, c_{ij} + \sum_{j=1}^{4} A_{6j}\, c_{6j} + \ldots ,$$

where $c_1 = 2 - x_1 - x_2$, $c_2 = (1-x_1)(1-x_2)$ and the c_{ij}'s are the same functions as the a_{ij}'s if we replace a_1 and a_2 by c_1 and c_2 respectively. (The c_{ij}'s may be found in Appendix B in terms of $d_1 = x_1 + x_2$ and $d_2 = x_1 x_2$.)

As in Section 2, we write $[(1-x_1)(1-x_2)]^m (x_1 x_2)^n (x_2-x_1)$ as a Vandermonde determinant and expand $(1-x_i)^m$, for m a non-negative integer, in a binomial series and (9.2) becomes:

$$(9.3)\quad e^{2t}\, K \sum_{r=0}^{m} \sum_{s=0}^{m} \binom{m}{r}\binom{m}{s}(-1)^{r+s} \iint_{R'} \exp\left(-t \sum_{i=1}^{2} x_i^{-1}\right) F_2(x_1,x_2)$$

$$D(n+r+1,\ n+s)\,dx_1\, dx_2,$$

where

$$D(q_2,q_1) = \begin{vmatrix} x_2^{q_2} & x_2^{q_1} \\ x_1^{q_2} & x_1^{q_1} \end{vmatrix} .$$

(9.3) can then be put in the form

$$e^{2t}\, K \sum_{r=0}^{m} \sum_{s=0}^{m} \binom{m}{r}\binom{m}{s}(-1)^{r+s} L(r,s;\, t)$$

where $L(r,s;\, t)$ is obtained by replacing 1 with $R(n;\, r+1,s;\, t)$, c_{ij} with $\Psi_{ij}(t)$ and c_1 with $\Psi_1(t)$ in $F_2(x_1,x_2)$ where

$$\Psi_{ij}(t) = \iint_{R'} \exp\left(-t \sum_{i=1}^{2} x_i^{-1}\right) c_{ij}(x_1,x_2)\, D(n+r+1,n+s)\,dx_1\, dx_2 .$$

To determine the $\Psi_{ij}(t)$'s, we make use of the following lemma due to Pillai [33] on the multiplication of elementary symmetric functions and Vandermonde determinants.

Lemma 9.1: Let $D(g_p, g_{p-1},\ldots,g_1)$, $g_j \geq 0$, $j = 1, \ldots, p$, denote the determinant

$$D(g_p, g_{p-1}, \ldots, g_1) = \begin{vmatrix} x_p^{g_p} & x_p^{g_{p-1}} & \ldots & x_p^{g_1} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ x_1^{g_p} & x_1^{g_{p-1}} & \ldots & x_1^{g_1} \end{vmatrix} . .$$

If d_r $(r \leq p)$ denotes the $r^{\underline{th}}$ elementary symmetric function (e s f) in p x's, then

$$(9.4) \qquad d_r\, D(g_p, g_{p-1}, \ldots, g_1) = \Sigma' D\,(g_p', g_{p-1}', \ldots, g_1')$$

where $g_j' = g_j + \delta$, $j = 1,2,\ldots,p$, $\delta = 0,1$ and Σ' denotes the sum over the $\binom{p}{r}$ combinations of p g's taken r at a time for which r indices $g_j' = g_j$ such that $\delta = 0$.

(ii)

$$d_r\, d_k\, D(g_p, g_{p-1}, \ldots, g_1) = \Sigma''\, D(g_p'', g_{p-1}'', \ldots, g_1''),$$

where $h \leq p$, $g_j'' = g_j' + \delta$, $j = 1, \ldots, p$, $\delta = 0,1$ and Σ'' denotes summation over the $\binom{p}{r}\binom{p}{h}$ terms obtained by taking h at a time of the p g's in each D in Σ' in (9.4) for which h indices $g_j'' = g_j' + 1$ while for the other indices $g_j'' = g_j'$.

(iii) $(d_r)^k (d_k)^\ell D(g_p, g_{p-1},\dots,g_1), (k, \ell \geq 0)$ can be expressed as a sum of $\binom{p}{r}^k \binom{p}{h}^\ell$ determinants obtained by performing on $D(g_p, g_{p-1}, \dots, g_1)$ in any order (i) k times and (i) ℓ times with $r = h$.

However, if at least two of the indices in any determinant are equal, the corresponding term in the summation vanishes.

As a simple consequence of Lemma 9.1, we see that

$$d_1^k D(g_2,g_1) = \sum_{j=0}^{k} \binom{k}{j} D(g_2 + j, g_1 + k-j) \tag{9.5}$$

and

$$d_2^\ell D(g_2,g_1) = D(g_2 + \ell, g_1 + \ell). \tag{9.6}$$

To illustrate the procedure for determining the $\Psi_{ij}(t)$ functions we consider $\Psi_{21}(t)$. First we note that

$$c_{21} = 8 - 8d_1 + 3d_1^2 - 4d_2.$$

Now using (9.5) and (9.6) we have

$$c_{21} D(g_2,g_1) = 8 D(g_2,g_1) - 8 D(g_2+1,g_1) - 8 D(g_2,g_1+1)$$

$$+ 3 D(g_2+2,g_1) + 2 D(g_2+1,g_1+1) + 3 D(g_2,g_1+2) \quad .$$

By taking $g_2 = n+r+1$ and $g_1 = n+s$, we can write $\Psi_{21}(t)$ as:

$$\Psi_{21}(t) = 8\ R(n;\ r+1,s;\ t) - 8\ R(n;\ r+2,s;\ t)$$

$$- 8\ R(n;\ r+1,\ s+1;\ t) + 3\ R(n;\ r+3,s;\ t)$$

$$+ 2\ R(n;\ r+2,s+1;\ t) + 3\ R(n;\ r+1,s+2;\ t)\ .$$

The Ψ functions are given in terms of the R functions in Appendix C.

Finally the density function of $U^{(2)}$ is obtained by inverting the R functions as done in Section 5 to obtain the R^* functions. The density function of $U^{(2)}$ is:

$$(9.7)\quad f(u) = K \sum_{r=0}^{m} \sum_{s=0}^{m} \binom{m}{r}\binom{m}{s}(-1)^{r+s} \{R^*(n;\ r+1,s;\ u)$$

$$+ A_{11}\ \Psi_1^*\ (u) + \sum_{i=2}^{3} \sum_{j=1}^{2} A_{ij}\ \Psi_{ij}^*\ (u)$$

$$+ \sum_{i=4}^{5} \sum_{j=1}^{3} A_{ij}\ \Psi_{ij}^*\ (u) + \sum_{j=1}^{4} A_{6j}\ \Psi_{6j}^*\ (u) + \ldots\}$$

where $\Psi_{ij}^*\ (u)$ is obtained from $\Psi_{ij}(t)$ by replacing R by R^*.

Table 1.1a. $k_{i_3 i_2 i_1}$ Coefficients for m = 0, 1, 2, 3, 4, 5.

	m=0	m=1		m=2			m=3				m=4					m=5					
(i_3,i_2) \ i_1	0	0	1	0	1	2	0	1	2	3	0	1	2	3	4	0	1	2	3	4	5
(2,1)	1	1		1			1				1					1					
(3,1)		-1		-2			-3				-4					-5					
(3,2)		1	-1	3	-4		6	-10			10	-20				15	-35				
(4,1)				1			3				6					10					
(4,2)				-2	3		-8	15			-20	45				-40	105				
(4,3)				1	-2	1	6	-15	10		20	-60	50			50	-175	175			
(5,1)							-1				-4					-10					
(5,2)							3	-6			15	-36				45	-126				
(5,3)							-3	8	-6		-20	64	-60			-75	280	-315			
(5,4)							1	-3	3	-1	10	-36	45	-20		50	-210	315	-175		
(6,1)											1					5					
(6,2)											-4	10				-24	70				
(6,3)											6	-20	20			45	-175	210			
(6,4)											-4	15	-20	10		-40	175	-280	175		
(6,5)											1	-4	6	-4	1	15	-70	126	-105	35	
(7,1)																-1					
(7,2)																5	-15				
(7,3)																-10	40	-50			
(7,4)																10	-45	75	-50		
(7,5)																-5	24	-45	40	-15	
(7,6)																1	-5	10	-10	5	-1

Table 1.1b. $k_{i_4 i_3 i_2 i_1}$ Coefficients for m = 0, 1, 2.

	m=0	m=1		m=2		
(i_4,i_3,i_2) \ i_1	0	0	1	0	1	2
(3,2,1)	1	1		1		
(4,2,1)		-1		-2		
(4,3,1)		1		3		
(4,3,2)		-1	1	-4	5	
(5,2,1)				1		
(5,3,1)				-2		
(5,3,2)				3	-4	
(5,4,1)				1		
(5,4,2)				-2	3	
(5,4,3)				1	-2	1

Table 1.2. Coefficients for $R^*(n;\, i,j,0)$ (Note: $r_i = n+i$, $s_i = 2n+i$, and $r_3/r_1r_4 = r_3(r_1r_4)^{-1}$)

(i,j) \ ℓ	$\alpha_{ij\ell}(n)$: 1	2	3	4	5	$\gamma_{ij}(n)$
(2,1)	$1/r_2s_5$					$1/s_5$
(3,1)	$1/r_2r_3$					$3/2r_3$
(3,2)	$1/r_3s_7$					$3/s_7$
(4,1)	$3/r_2s_7$	$-1/2r_3r_4$				$3(4n+15)/2r_4s_7$
(4,2)	$1/r_3r_4$	$-1/r_3r_4$				$(8n+25)/2r_3r_4$
(4,3)	$1/r_4s_9$	$-3/2r_3r_4$				$3(4n+13)/2r_3s_9$
(5,1)	$2/r_2r_4$	$-3/2r_4r_5$	$-1/2r_4r_5$			$5(4n+17)/4r_4r_5$
(5,2)	$3/r_3s_9$	$-3s_7/2r_3r_4r_5$	$-1/r_4r_5$			$c_{52}/2r_3r_5s_9$
(5,3)	$1/r_4r_5$	$-9/2r_3r_5$	$-3/2r_4r_5$			$15s_7/4r_3r_5$
(5,4)	$1/r_5s_{11}$	$-(7n+24)/2r_3r_4r_5$	$-2/r_4r_5$			$5(4n+15)/2r_3s_{11}$
(6,1)	$5/r_2s_9$	$-3(4n+15)/4r_4r_5r_6$	$-3r_3/2r_4r_5r_6$	$-1/2r_5r_6$		$c_{61}/4r_5r_6s_9$
(6,2)	$2/r_3r_5$	$-3(4n+15)/2r_3r_5r_6$	$-3s_7/2r_4r_5r_6$	$-1/r_5r_6$		$3(8n+41)s_7/4r_3r_5r_6$
(6,3)	$3/r_4s_{11}$	$-3a_{632}/4r_3r_4r_5r_6$	$-9/2r_5r_6$	$-3/2r_5r_6$		$c_{63}/4r_3r_4r_6s_{11}$
(6,4)	$1/r_5r_6$	$-3(3n+11)/r_3r_4r_6$	$-3s_9/r_4r_5r_6$	$-2/r_5r_6$		$c_{64}/2r_3r_4r_6$
(6,5)	$1/r_6s_{13}$	$-3a_{652}/4r_3r_4r_5r_6$	$-3(3n+11)/2r_4r_5r_6$	$-5/2r_5r_6$		$c_{65}/4r_3r_4s_{13}$
(7,1)	$3/r_2r_5$	$-5(4n+17)/4r_5r_6r_7$	$-3(4n+15)r_3/4r_4r_5r_6r_7$	$-3r_3/2r_5r_6r_7$	$-1/2r_6r_7$	$c_{71}/8r_5r_6r_7$
(7,2)	$5/r_3s_{11}$	$-5(4n+17)s_9/4r_3r_5r_6r_7$	$-3(4n+15)/2r_5r_6r_7$	$-3s_7/2r_5r_6r_7$	$-1/r_6r_7$	$c_{72}/4r_3r_6r_7s_{11}$
(7,3)	$2/r_4r_6$	$-15a_{732}/4r_3r_4r_6r_7$	$-3a_{733}/4r_4r_5r_6r_7$	$-9r_4/2r_5r_6r_7$	$-3/2r_6r_7$	$c_{73}/8r_3r_4r_6r_7$
(7,4)	$3/r_5s_{13}$	$-3a_{742}/2r_3r_4r_5r_6r_7$	$-3a_{743}/r_4r_5r_6r_7$	$-3s_9/r_5r_6r_7$	$-2/r_6r_7$	$c_{74}/2r_3r_4r_7s_{13}$
(7,5)	$1/r_6r_7$	$-a_{752}/4r_3r_4r_5r_7$	$-(23n+91)s_{11}/4r_4r_5r_6r_7$	$-15/2r_6r_7$	$-5/2r_6r_7$	$c_{75}/8r_3r_4r_7$
(7,6)	$1/r_7s_{15}$	$-3a_{762}/4r_3r_4r_6r_7$	$-3a_{763}/2r_4r_5r_6r_7$	$-(11n+43)/2r_5r_6r_7$	$-3/r_6r_7$	$c_{76}/4r_3r_4s_{15}$

$a_{632} = 12n^2+105n+232$, $a_{652} = 8n^2+65n+133$, $a_{732} = 4n^2+37n+88$, $a_{733} = a_{632}$, $a_{742} = 11n^3+161n^2+781n+1251$,
$a_{743} = 4n^2+39n+97$, $a_{752} = 58n^2+495n+1073$, $a_{762} = 12n^2+109n+254$, $a_{763} = 5n^2+43n+94$, $c_{52} = 15(2n^2+16n+31)$
$c_{61} = 15(4n^2+38n+91)$, $c_{63} = 3(36n^3+486n^2+2151n+3136)$, $c_{64} = 3(8n^2+65n+133)$, $c_{65} = 15(4n^2+34n+73)$
$c_{71} = 21(4n^2+42n+113)$, $c_{72} = 35(4n^3+60n^2+296n+477)$, $c_{73} = 168n^3+2394n^2+11193n+17304$, $c_{74} = 21(4n^3+61n^2+304n+501)$
$c_{75} = 35(4n^2+36n+83)$, $c_{76} = 21(4n^2+38n+93)$

Table 1.2. (Continued)

$\beta_{ij\ell}(n)$

(i,j) \ ℓ	1	2	3	4	5	6	7
(2,1)	$-1/r_2$	$2/s_5$					
(3,1)	$-1/r_2$	$-1/r_3$	$2/r_3$				
(3,2)	$-2/r_3$	$1/r_3$	$2/s_7$				
(4,1)	$-1/r_2$	$-1/r_4$	$-s_5/2r_3r_4$	$2(3n+11)/r_4s_7$			
(4,2)	$-2/r_3$	$-s_5/r_3r_4$	s_7/r_3r_4	$2/r_4$			
(4,3)	$-(3n+8)/r_3r_4$	$1/r_4$	$s_7/2r_3r_4$	$2/s_9$			
(5,1)	$-1/r_2$	$-1/r_5$	$-s_5/2r_4r_5$	$-s_5/2r_4r_5$	$4/r_5$		
(5,2)	$-2/r_3$	$-s_5/r_3r_5$	$-s_5/r_4r_5$	$b_{524}/2r_3r_4r_5$	$2(3n+14)/r_5s_9$		
(5,3)	$-(3n+8)/r_3r_4$	$-(3n+8)/r_4r_5$	$(4n+17)/2r_4r_5$	$(4n+17)/2r_3r_5$	$2/r_5$		
(5,4)	$4r_3/r_4r_5$	$1/r_5$	$1/r_5$	$s_9/2r_3r_5$	$2/s_{11}$		
(6,1)	$-1/r_2$	$-1/r_6$	$-s_5/2r_5r_6$	$-r_3s_5/2r_4r_5r_6$	$-s_5s_7/4r_4r_5r_6$	$b_{616}/r_5r_6s_9$	
(6,2)	$-2/r_3$	$-s_5/r_3r_6$	$-s_5/r_5r_6$	$-s_5s_7/2r_4r_5r_6$	$b_{625}/2r_3r_5r_6$	$4/r_6$	
(6,3)	$-(3n+8)/r_3r_4$	$-(3n+8)/r_4r_6$	$-(3n+8)s_7/2r_4r_5r_6$	$b_{634}/2r_4r_5r_6$	$b_{635}s_9/4r_3r_4r_5r_6$	$2(3n+17)/r_6s_{11}$	
(6,4)	$-4r_3/r_4r_5$	$-2r_3s_7/r_4r_5r_6$	$2/r_6$	s_9/r_4r_6	$r_5s_9/r_3r_4r_6$	$2/r_6$	
(6,5)	$-b_{651}/r_4r_5r_6$	$1/r_6$	$s_9/2r_5r_6$	$s_9/2r_4r_6$	$s_9s_{11}/4r_3r_4r_6$	$2/s_{13}$	
(7,1)	$-1/r_2$	$-1/r_7$	$-s_5/2r_6r_7$	$-r_2s_5/2r_5r_6r_7$	$-r_3s_5s_7/4r_4r_5r_6r_7$	$-s_5s_7/4r_5r_6r_7$	$b_{717}/r_5r_6r_7$
(7,2)	$-2/r_3$	$-s_5/r_3r_7$	$-s_5/r_6r_7$	$-s_5s_7/2r_5r_6r_7$	$-s_5s_7/2r_5r_6r_7$	$b_{726}/4r_3r_5r_6r_7$	$b_{727}/r_6r_7s_{11}$
(7,3)	$-(3n+8)/r_3r_4$	$-(3n+8)/r_4r_7$	$-(3n+8)s_7/2r_4r_6r_7$	$-(3n+8)s_7/2r_5r_6r_7$	$b_{735}/4r_4r_5r_6r_7$	$b_{736}/4r_3r_4r_6r_7$	$4/r_7$
(7,4)	$-4r_3/r_4r_5$	$-2r_3s_7/r_4r_5r_7$	$-2r_3s_7/r_5r_6r_7$	$b_{744}/r_5r_6r_7$	$b_{745}/r_4r_6r_7$	$b_{746}s_{11}/2r_3r_4r_6r_7$	$2(3n+20)/r_7s_{13}$
(7,5)	$-b_{751}/r_4r_5r_6$	$-b_{752}/r_5r_6r_7$	$(4n+23)/2r_6r_7$	$(4n+23)/2r_6r_7$	$(4n+23)s_{11}/4r_4r_6r_7$	$(4n+23)s_{11}/4r_3r_4r_7$	$2/r_7$
(7,6)	$-b_{761}/r_5r_6r_7$	$1/r_7$	r_5/r_6r_7	$s_{11}/2r_6r_7$	$s_{11}/2r_4r_7$	$s_{11}s_{13}/4r_3r_4r_7$	$2/s_{15}$

$b_{524} = 6n^2+46n+89$, $b_{616} = 2(5n^2+45n+102)$, $b_{625} = 8n^2+68n+149$, $b_{634} = 6n^2+55n+128$, $b_{635} = b_{634}$.

$b_{651} = 5n^2+35n+62$, $b_{717} = 6n^2+58n+144$, $b_{726} = 20n^3+280n^2+1338n+2181$, $b_{727} = 2(5n^2+55n+152)$

$b_{735} = 16n^3+236n^2+1179n+1992$, $b_{736} = b_{735}$, $b_{744} = 3n^2+32n+87$, $b_{745} = b_{744}$, $b_{746} = b_{744}$, $b_{751} = 5n^2+35n+62$

$b_{752} = b_{751}$, $b_{761} = 6n^2+46n+92$.

Table 1.3. Coefficients for $R^*(n;\ i,j,k,0)$.

(i,j,k) \ ℓ	$\gamma_{ijk\ell}(n)$ 1	2	3	4	5	$\alpha_{ijk}(n)$	$\delta_{ijk}(n)$
(3,2,1)	$1/r_3s_5$	$-1/r_3r_3$	$1/r_3s_7$			$1/r_2r_3r_3s_5s_7$	$1/r_3r_3s_7$
(4,2,1)	$1/r_3s_5$	$1/2r_3r_4$	$-(5n+19)/r_3r_4s_7$	$3/2r_3r_4$		$3/r_2r_3r_4s_5s_7$	$4/r_3r_4s_7$
(4,3,1)	$3/2r_3r_4$	$-3/r_4s_7$	$-3/2r_3r_4$	$3/r_3s_9$		$3/r_2r_3r_4s_7s_9$	$12/r_4s_7s_9$
(4,3,2)	$3/r_4s_7$	$-(5n+16)/2r_3r_4r_4$	$r_2/r_3r_4s_9$	$1/2r_3r_4$		$1/r_3r_4r_4s_7s_9$	$4/r_4r_4s_9$
(5,2,1)	$1/r_3s_5$	$1/2r_3r_5$	$1/2r_4r_5$	$-3(3n+13)/2r_3r_4r_5$	$3(4n+19)/2r_3r_5s_9$	$6/r_2r_3r_4s_5s_9$	$2(5n+24)/r_3r_4r_5s_9$
(5,3,1)	$3/2r_3r_4$	$3/2r_4r_5$	$-(7n+32)/2r_4r_4r_5$	$-(7n+32)/2r_3r_4r_5$	$(8n+33)/2r_3r_4r_5$	$2/r_2r_3r_4r_4r_5$	$4(5n+21)/r_4r_4r_5s_9$
(5,3,2)	$3/r_4s_7$	$3/2r_4r_5$	$-c_{5323}/2r_3r_4r_5s_9$	$3r_2/2r_3r_4r_5$	$3/2r_3r_4$	$3/r_3r_4r_5s_7s_9$	$3(5n+18)/r_4r_4r_5s_9$
(5,4,1)	$3(4n+15)/2r_4r_5s_7$	$-2/r_4r_5$	$-2/r_4r_5$	$-s_9/r_3r_4r_5$	$3(4n+17)/2r_3r_4s_{11}$	$6/r_2r_4r_5s_7s_{11}$	$d_{542}/r_4r_4r_5s_9s_{11}$
(5,4,2)	$(8n+25)/2r_3r_4r_5$	$-(7n+24)/r_3r_5s_9$	$-(7n+24)/2r_3r_4r_5$	$3r_2/r_3r_4s_{11}$	$3/2r_3r_4$	$3/r_3r_4r_5s_9s_{11}$	$4(10n+37)/r_4r_5s_9s_{11}$
(5,4,3)	$3(4n+13)/r_3r_5s_9$	$-3(3n+11)/2r_3r_5r_5$	$r_2/r_3r_5s_{11}$	$r_2/2r_3r_4r_5$	$r_6/2r_3r_4r_5$	$1/r_4r_5r_5s_9s_{11}$	$2(5n+19)/r_4r_5r_5s_{11}$

$c_{5323} = 3(8n^2+65n+128)$ $d_{542} = 2(20n^2+158n+309)$

(i,j,k) \ ℓ	$\beta_{ijk\ell}(n)$ 1	2	3	4	5	6	7
(3,2,1)	$-1/r_2r_3s_7$	$2/r_3r_3s_5s_7$	$1/r_3r_3s_7$				
(4,2,1)	$-1/r_2r_3r_4$	$-(2n-1)/r_3r_4s_5s_7$	0	$3/r_3r_4s_7$			
(4,3,1)	$-1/r_2r_4s_9$	$-3/r_3r_4s_9$	$12/r_3r_4s_7s_9$	$(2n-3)/r_3r_4s_7s_9$	$3/r_3r_4s_9$		
(4,3,2)	$-2/r_3r_4s_9$	$4/r_3r_4r_4s_9$	$-1/r_3r_4s_7$	$s_5/r_3r_4r_4s_9$	$1/r_3r_4s_9$		
(5,2,1)	$-3/r_2r_3s_9$	$-b_{5212}/r_3r_4r_5s_5s_9$	$-1/2r_3r_4r_5$	0	$3(4n+19)/2r_3r_4r_5s_9$		
(5,3,1)	$-1/r_2r_4r_5$	$-6/r_3r_5s_9$	$-b_{5313}/2r_3r_4r_4r_5s_9$	0	$b_{5315}/2r_3r_4r_4r_5s_9$	$(8n+33)/r_3r_4r_5s_9$	
(5,3,2)	$-2/r_3r_4r_5$	$-3n/r_3r_4r_5s_9$	$6(5n+18)/r_4r_4r_5s_7s_9$	$-b_{5324}/2r_3r_4r_4r_5s_9$	$3s_5/r_3r_4r_5s_9$	$3/2r_3r_4r_5$	
(5,4,1)	$-1/r_2r_5s_{11}$	$-3/r_4r_5s_{11}$	$-3(4n+17)s_5/2r_3r_4r_5s_9s_{11}$	$b_{5414}/r_4r_4s_5s_7s_9s_{11}$	$b_{5415}/r_4r_4r_5s_9s_{11}$	$b_{5416}/r_3r_4r_5s_9s_{11}$	$3(4n+17)/2r_3r_4r_5s_{11}$
(5,4,2)	$-2/r_3r_5s_{11}$	$-3s_5/r_3r_4r_5s_{11}$	$3(14n+47)/r_3r_4r_5s_9s_{11}$	$-1/r_3r_4r_5$	$b_{5425}/r_3r_4r_5s_9s_{11}$	$3s_5/r_3r_4r_5s_{11}$	$3/r_3r_4s_{11}$
(5,4,3)	$-(3n+8)/r_3r_4r_5s_{11}$	$-(n-3)/r_4r_5r_5s_{11}$	$1/2r_3r_4r_5$	$-3/r_3r_5s_9$	$(3n+8)/r_4r_5r_5s_{11}$	$s_5/r_3r_4r_5s_{11}$	$1/2r_3r_4r_5$

$b_{5212} = 4n^2+16n-15$, $b_{5313} = 6n^2+11n-60$, $b_{5315} = 3(2n^2+5n-14)$, $b_{5324} = 4n^2+67n+180$, $b_{5414} = 4(20n^2+158n+309)$

$b_{5415} = 2n^2-19n-120$, $b_{5416} = 3(2n^2+7n-7)$, $b_{5425} = 2n^2-25n-108$

Table 1.4. Upper α percentage points of $U^{(2)}$.

α = .10

m \ n	5	10	15	20	25	30	35	40	45	50	60	70	80	90	100	130	160	200
-.5	.81989	.40130	.26509	.19782	.15776	.13118	.11226	.098110	.087126	.078354	.065219	.055855	.048842	.043394	.039039	.030005	.024367	.019484
0	1.1454	.55537	.36557	.27230	.21690	.18022	.15414	.13465	.11954	.10747	.089423	.076562	.066935	.059458	.053484	.041096	.033367	.026677
.5	1.4593	.70291	.46151	.34329	.27323	.22689	.19398	.16940	.15035	.13515	.11242	.096228	.084115	.074710	.067196	.051621	.041907	.033502
1	1.7667	.84667	.55477	.41222	.32786	.27213	.23259	.20307	.18019	.16195	.13467	.11526	.10074	.089465	.080461	.061801	.050166	.040100
1.5	2.0702	.98795	.64625	.47975	.38136	.31641	.27035	.23599	.20937	.18815	.15643	.13386	.11698	.10388	.093417	.071741	.058230	.046542
2	2.3709	1.1275	.73646	.54628	.43402	.35998	.30750	.26836	.23806	.21390	.17781	.15213	.13293	.11804	.10614	.081505	.066149	.052868
2.5	2.6696	1.2657	.82570	.61205	.48606	.40302	.34418	.30032	.26637	.23931	.19890	.17016	.14867	.13200	.11869	.091130	.073955	.059102
3	2.9669	1.4030	.91420	.67722	.53759	.44562	.38049	.33195	.29438	.26445	.21976	.18798	.16423	.14581	.13110	.10064	.081670	.065264
3.5	3.2631	1.5395	1.0021	.74191	.58872	.48787	.41648	.36330	.32215	.28937	.24043	.20564	.17964	.15948	.14339	.11007	.089309	.071364
4	3.5584	1.6754	1.0895	.80620	.63952	.52984	.45223	.39443	.34971	.31410	.26094	.22316	.19494	.17305	.15558	.11941	.096885	.077413
4.5	3.8529	1.8107	1.1765	.87015	.69003	.57156	.48776	.42536	.37710	.33867	.28131	.24056	.21012	.18652	.16768	.12869	.10441	.083418
5	4.1469	1.9456	1.2632	.93383	.74030	.61307	.52310	.45612	.40433	.36310	.30157	.25786	.22522	.19991	.17971	.13791	.11188	.089385
10	7.0696	3.2810	2.1187	1.5611	1.2348	1.0210	.87010	.75798	.67141	.60256	.49997	.42720	.37292	.33086	.29733	.22800	.18488	.14765
15	9.9787	4.6048	2.9643	2.1798	1.7218	1.4221	1.2109	1.0542	.93331	.83723	.69421	.59287	.51732	.45883	.41222	.31592	.25608	.20444
20	12.883	5.9240	3.8057	2.7946	2.2052	1.8200	1.5488	1.3477	1.1927	1.0695	.88635	.75665	.66002	.58526	.52569	.40269	.32632	.26044
25	15.785	7.2409	4.6448	3.4073	2.6867	2.2161	1.8850	1.6396	1.4506	1.3004	1.0772	.91931	.80169	.71072	.63827	.48874	.39595	.31595
30	18.685	8.5564	5.4827	4.0188	3.1670	2.6111	2.2202	1.9306	1.7075	1.5304	1.2673	1.0812	.94266	.83555	.75026	.57430	.46516	.37109
35	21.585	9.8711	6.3197	4.6295	3.6465	3.0053	2.5546	2.2208	1.9637	1.7598	1.4568	1.2426	1.0831	.95990	.86180	.65949	.53405	.42598
40	24.485	11.185	7.1561	5.2396	4.1254	3.3989	2.8885	2.5105	2.2195	1.9887	1.6458	1.4035	1.2232	1.0839	.97302	.74440	.60270	.48066
45	27.384	12.499	7.9921	5.8493	4.6040	3.7922	3.2220	2.7998	2.4749	2.2172	1.8345	1.5642	1.3630	1.2076	1.0840	.82910	.67117	.53518
50	30.282	13.812	8.8278	6.4586	5.0822	4.1852	3.5552	3.0889	2.7300	2.4455	2.0230	1.7246	1.5026	1.3312	1.1947	.91362	.73948	.58957
60	36.079	16.439	10.499	7.6767	6.0380	4.9704	4.2209	3.6663	3.2397	2.9014	2.3993	2.0448	1.7812	1.5777	1.4158	1.0822	.87573	.69803
70	41.875	19.064	12.169	8.8942	6.9932	5.7550	4.8860	4.2432	3.7487	3.3567	2.7751	2.3645	2.0594	1.8237	1.6364	1.2505	1.0116	.80618
80	47.671	21.690	13.838	10.111	7.9479	6.5393	5.5507	4.8196	4.2573	3.8116	3.1505	2.6839	2.3371	2.0694	1.8566	1.4184	1.1472	.91408
90	53.467	24.315	15.508	11.328	8.9024	7.3232	6.2151	5.3957	4.7656	4.2663	3.5256	3.0029	2.6146	2.3149	2.0766	1.5860	1.2826	1.0218
100	59.263	26.939	17.177	12.545	9.8566	8.1069	6.8793	5.9716	5.2737	4.7207	3.9004	3.3217	2.8919	2.5601	2.2963	1.7535	1.4179	1.1294
130	76.649	34.813	22.184	16.193	12.719	10.457	8.8710	7.6984	6.7971	6.0830	5.0241	4.2774	3.7228	3.2949	2.9548	2.2552	1.8229	1.4514
160	94.035	42.687	27.191	19.842	15.580	12.807	10.862	9.4244	8.3196	7.4444	6.1470	5.2322	4.5529	4.0289	3.6125	2.7562	2.2272	1.7728
200	117.22	53.184	33.865	24.705	19.394	15.939	13.516	11.725	10.349	9.2590	7.6434	6.5045	5.6590	5.0068	4.4887	3.4234	2.7656	2.2007

Table 1.4. (Continued)

$\alpha = .05$

m^E	5	10	15	20	25	30	35	40	45	50	60	70	80	90	100	130	160	200
-.5	1.0676	.50646	.33087	.24551	.19512	.16187	.13830	.12071	.10710	.096237	.080011	.068466	.059832	.053132	.047781	.036694	.029783	.023805
0	1.4508	.68072	.44292	.32797	.26031	.21577	.18424	.16074	.14255	.12806	.10642	.091038	.079539	.070618	.063497	.048749	.039560	.031614
.5	1.8163	.84638	.54905	.40592	.32188	.26663	.22755	.19846	.17596	.15803	.13129	.11228	.098082	.087070	.78280	.060084	.048751	.038955
1	2.1771	1.0071	.65171	.48120	.38127	.31566	.26929	.23479	.20812	.18689	.15521	.13272	.11591	.10289	.092493	.070979	.057584	.046008
1.5	2.5304	1.1645	.75207	.55469	.43919	.36344	.30995	.27017	.23943	.21497	.17849	.15259	.13326	.11827	.10631	.081570	.066169	.052861
2	2.8802	1.3197	.85078	.62688	.49605	.41032	.34982	.30485	.27012	.24249	.20130	.17206	.15024	.13333	.11984	.091934	.074569	.059567
2.5	3.2273	1.4732	.94824	.69809	.55209	.45650	.38909	.33900	.30033	.26957	.22373	.19121	.16694	.14814	.13314	.10212	.082825	.066156
3	3.5724	1.6255	1.0447	.76853	.60750	.50214	.42787	.37272	.33015	.29630	.24587	.21010	.18342	.16274	.14626	.11217	.090964	.072652
3.5	3.9161	1.7767	1.1405	.83835	.66238	.54733	.46627	.40609	.35966	.32274	.26776	.22878	.19970	.17718	.15922	.12210	.099008	.079070
4	4.2585	1.9271	1.2356	.90766	.71683	.59214	.50434	.43917	.38890	.34895	.28946	.24729	.21584	.19148	.17206	.13192	.10697	.085423
4.5	4.6001	2.0768	1.3302	.97654	.77092	.63665	.54213	.47200	.41792	.37495	.31097	.26564	.23183	.20566	.18479	.14167	.11486	.091719
5	4.9409	2.2260	1.4243	1.0451	.82470	.68088	.57968	.50462	.44675	.40077	.33234	.28386	.24771	.21973	.19743	.15134	.12269	.097966
10	8.3254	3.6997	2.3512	1.7180	1.3520	1.1140	.94696	.82337	.72824	.65278	.54066	.46138	.40235	.35671	.32036	.24534	.19878	.15864
15	11.692	5.1584	3.2652	2.3798	1.8694	1.5383	1.3063	1.1349	1.0031	.89865	.74365	.63418	.55277	.48986	.43979	.33656	.27257	.21743
20	15.052	6.6111	4.1738	3.0367	2.3824	1.9585	1.6619	1.4429	1.2747	1.1415	.94395	.80458	.70101	.62102	.55740	.42632	.34512	.27522
25	18.409	8.0608	5.0796	3.6909	2.8929	2.3764	2.0153	1.7489	1.5444	1.3825	1.1426	.97352	.84792	.75096	.67388	.51515	.41690	.33236
30	21.764	9.5087	5.9837	4.3435	3.4019	2.7928	2.3673	2.0536	1.8128	1.6224	1.3403	1.1415	.99393	.88008	.78958	.60334	.48813	.38904
35	25.119	10.956	6.8866	4.9950	3.9098	3.2083	2.7184	2.3574	2.0804	1.8614	1.5372	1.3088	1.1393	1.0086	.90473	.69106	.55896	.44539
40	28.472	12.402	7.7888	5.6458	4.4170	3.6230	3.0688	2.6605	2.3474	2.0999	1.7335	1.4755	1.2842	1.1367	1.0194	.77843	.62948	.50148
45	31.825	13.847	8.6905	6.2961	4.9237	4.0373	3.4187	2.9632	2.6139	2.3378	1.9294	1.6419	1.4287	1.2644	1.1338	.86552	.69976	.55736
50	35.178	15.292	9.5917	6.9459	5.4300	4.4511	3.7683	3.2655	2.8801	2.5755	2.1249	1.8079	1.5729	1.3918	1.2480	.95238	.76984	.61307
60	41.883	18.182	11.393	8.2448	6.4417	5.2780	4.4665	3.8692	3.4116	3.0500	2.5153	2.1393	1.8607	1.6460	1.4756	1.1256	.90954	.72409
70	48.587	21.071	13.194	9.5428	7.4526	6.1040	5.1639	4.4722	3.9423	3.5237	2.9050	2.4700	2.1478	1.8996	1.7027	1.2982	1.0488	.83470
80	55.291	23.959	14.995	10.840	8.4629	6.9295	5.8608	5.0746	4.4725	3.9969	3.2941	2.8003	2.4345	2.1528	1.9293	1.4705	1.1876	.94500
90	61.994	26.847	16.795	12.137	9.4729	7.7546	6.5574	5.6766	5.0023	4.4697	3.6829	3.1301	2.7208	2.4056	2.1556	1.6425	1.3262	1.0551
100	68.697	29.735	18.595	13.434	10.483	8.5795	7.2536	6.2784	5.5318	4.9423	4.0714	3.4597	3.0068	2.6581	2.3816	1.8142	1.4646	1.1649
130	88.806	38.397	23.993	17.324	13.511	11.053	9.3411	8.0825	7.1192	6.3587	5.2358	4.4473	3.8637	3.4147	3.0586	2.3285	1.8789	1.4937
160	108.91	47.059	29.391	21.213	16.538	13.525	11.428	9.8856	8.7055	7.7741	6.3990	5.4338	4.7196	4.1702	3.7346	2.8417	2.2922	1.8216
200	135.72	58.608	36.587	26.397	20.573	16.821	14.209	12.289	10.820	9.6604	7.9491	6.7482	5.8599	5.1766	4.6350	3.5252	2.8424	2.2581

Table 1.4. (Continued)

$\alpha = .025$

m \ n	5	10	15	20	25	30	35	40	45	50	60	70	80	90	100	130	160	200
-.5	1.3390	.61533	.39755	.29333	.23233	.19230	.16403	.14300	.12675	.11381	.094513	.080810	.070576	.062642	.056312	.043211	.035056	.028007
0	1.7821	.80882	.52024	.38299	.30293	.25051	.21354	.18607	.16486	.14798	.12284	.10499	.091673	.081352	.073120	.056091	.045495	.036342
.5	2.2056	.99189	.63584	.46728	.36920	.30510	.25994	.22641	.20054	.17997	.14933	.12761	.11140	.098842	.088829	.068125	.055247	.044125
1	2.6183	1.1690	.74728	.54838	.43289	.35752	.30446	.26511	.23476	.21063	.17472	.14927	.13029	.11559	.10387	.079639	.064576	.051570
1.5	3.0243	1.3421	.85596	.62734	.49484	.40846	.34771	.30268	.26796	.24039	.19934	.17027	.14859	.13181	.11844	.090793	.073612	.058780
2	3.4257	1.5126	.96266	.70475	.55551	.45832	.39003	.33942	.30043	.26947	.22340	.19078	.16647	.14766	.13266	.10168	.082429	.065814
2.5	3.8239	1.6810	1.0679	.78100	.61522	.50736	.43162	.37553	.33232	.29803	.24703	.21092	.18402	.16321	.14662	.11236	.091076	.072711
3	4.2197	1.8478	1.1719	.85632	.67417	.55575	.47265	.41113	.36376	.32617	.27030	.23075	.20130	.17851	.16036	.12287	.099585	.079498
3.5	4.6136	2.0135	1.2751	.93091	.73249	.60360	.51320	.44631	.39483	.35398	.29328	.25034	.21836	.19362	.17392	.13324	.10798	.086192
4	5.0061	2.1781	1.3775	1.0049	.79030	.65101	.55337	.48115	.42558	.38150	.31602	.26971	.23523	.20857	.18733	.14349	.11628	.092809
4.5	5.3974	2.3419	1.4792	1.0783	.84768	.69805	.59321	.51569	.45607	.40878	.33855	.28890	.25195	.22337	.20061	.15364	.12449	.099358
5	5.7878	2.5051	1.5805	1.1514	.90469	.74477	.63277	.54998	.48632	.43585	.36091	.30794	.26852	.23804	.21378	.16371	.13264	.10585
10	9.6628	4.1150	2.5754	1.8670	1.4622	1.2009	1.0185	.88402	.78082	.69914	.57810	.49274	.42932	.38035	.34140	.26155	.21144	.16863
15	13.516	5.7067	3.5549	2.5694	2.0080	1.6466	1.3948	1.2094	1.0674	.95511	.78894	.67192	.58509	.51810	.46486	.35529	.28750	.22918
20	17.360	7.2912	4.5278	3.2658	2.5485	2.0874	1.7665	1.5307	1.3501	1.2075	.99658	.84825	.73827	.65349	.58615	.44768	.36209	.28853
25	21.201	8.8721	5.4974	3.9591	3.0860	2.5254	2.1357	1.8495	1.6305	1.4577	1.2023	1.0228	.88986	.78742	.70609	.53897	.43576	.34711
30	25.040	10.451	6.4649	4.6504	3.6216	2.9617	2.5033	2.1668	1.9095	1.7065	1.4068	1.1963	1.0404	.92036	.82511	.62949	.50878	.40514
35	28.878	12.028	7.4311	5.3404	4.1561	3.3968	2.8697	2.4830	2.1874	1.9543	1.6103	1.3689	1.1902	1.0526	.94346	.71947	.58132	.46278
40	32.714	13.605	8.3964	6.0296	4.6896	3.8310	3.2353	2.7984	2.4646	2.2014	1.8132	1.5408	1.3393	1.1843	1.0613	.80901	.65350	.52011
45	36.550	15.181	9.3610	6.7181	5.2225	4.2647	3.6003	3.1133	2.7413	2.4480	2.0156	1.7124	1.4881	1.3156	1.1788	.89822	.72539	.57719
50	40.386	16.756	10.325	7.4061	5.7550	4.6978	3.9649	3.4277	3.0175	2.6942	2.2176	1.8835	1.6365	1.4465	1.2959	.98717	.79703	.63407
60	48.056	19.906	12.252	8.7811	6.8188	5.5632	4.6930	4.0555	3.5689	3.1855	2.6207	2.2249	1.9325	1.7076	1.5295	1.1644	.93978	.74734
70	55.726	23.055	14.179	10.155	7.8817	6.4276	5.4202	4.6824	4.1194	3.6760	3.0229	2.5656	2.2277	1.9680	1.7623	1.3410	1.0820	.86013
80	63.395	26.203	16.105	11.529	8.9440	7.2913	6.1468	5.3087	4.6693	4.1659	3.4246	2.9056	2.5223	2.2279	1.9946	1.5172	1.2237	.97255
90	71.063	29.351	18.030	12.902	10.006	8.1546	6.8728	5.9345	5.2188	4.6553	3.8258	3.2452	2.8165	2.4873	2.2266	1.6930	1.3652	1.0847
100	78.732	32.499	19.955	14.274	11.067	9.0175	7.5986	6.5600	5.7678	5.1443	4.2266	3.5844	3.1104	2.7464	2.4582	1.8685	1.5063	1.1966
130	101.74	41.941	25.729	18.391	14.250	11.605	9.7744	8.4349	7.4137	6.6101	5.4278	4.6009	3.9908	3.5225	3.1518	2.3939	1.9288	1.5313
160	124.74	51.382	31.502	22.507	17.432	14.191	11.949	10.309	9.0583	8.0746	6.6277	5.6161	4.8699	4.2973	3.8442	2.9182	2.3502	1.8651
200	155.41	63.970	39.198	27.993	21.674	17.639	14.847	12.806	11.250	10.026	8.2264	6.9685	6.0409	5.3292	4.7663	3.6161	2.9110	2.3091

Table 1.4. (Continued)

α = .01

m \ n	5	10	15	20	25	30	35	40	45	50	60	70	80	90	100	130	160	200
-.5	1.7419	.76665	.48801	.35740	.28181	.23256	.19795	.17229	.15252	.13682	.11345	.096896	.084558	.075006	.067393	.051661	.041884	.033444
0	2.2704	.98500	.62391	.45581	.35887	.29587	.25166	.21893	.19373	.17373	.14399	.12294	.10726	.095124	.085455	.065486	.053081	.042378
.5	2.7740	1.1908	.75136	.54786	.43084	.35493	.30173	.26238	.23210	.20808	.17240	.14715	.12836	.11382	.10224	.078325	.063478	.050670
1	3.2642	1.3893	.87386	.63613	.49976	.41143	.34959	.30389	.26875	.24088	.19950	.17025	.14847	.13164	.11823	.090556	.073379	.058567
1.5	3.7459	1.5831	.99306	.72186	.56662	.46619	.39595	.34408	.30421	.27261	.22571	.19257	.16791	.14885	.13368	.10237	.082938	.066188
2	4.2220	1.7736	1.1099	.80577	.63197	.51968	.44121	.38329	.33879	.30355	.25125	.21432	.18685	.16562	.14872	.11386	.092240	.073604
2.5	4.6939	1.9617	1.2250	.88828	.69618	.57220	.48561	.42175	.37271	.33388	.27628	.23562	.20539	.18204	.16345	.12511	.10134	.080858
3	5.1629	2.1479	1.3387	.96970	.75949	.62394	.52934	.45961	.40609	.36371	.30090	.25657	.22362	.19817	.17792	.13617	.11028	.087983
3.5	5.6295	2.3326	1.4513	1.0502	.82205	.67505	.57252	.49698	.43902	.39315	.32517	.27722	.24159	.21407	.19218	.14705	.11909	.094999
4	6.0943	2.5161	1.5630	1.1301	.88401	.72563	.61524	.53394	.47159	.42225	.34916	.29763	.25934	.22978	.20626	.15780	.12778	.10192
4.5	6.5576	2.6987	1.6739	1.2093	.94546	.77578	.65757	.57055	.50384	.45106	.37291	.31782	.27690	.24531	.22019	.16843	.13637	.10877
5	7.0198	2.8804	1.7842	1.2879	1.0065	.82554	.69957	.60687	.53581	.47963	.39644	.33782	.29430	.26070	.23399	.17896	.14488	.11555
10	11.606	4.6716	2.8665	2.0574	1.6016	1.3101	1.1079	.95949	.84604	.75651	.62427	.53131	.46242	.40932	.36716	.28045	.22686	.18079
15	16.163	6.4407	3.9303	2.8110	2.1828	1.7822	1.5049	1.3019	1.1469	1.0247	.84457	.71815	.62459	.55256	.49541	.37804	.30561	.24341
20	20.711	8.2012	4.9863	3.5575	2.7576	2.2484	1.8966	1.6393	1.4431	1.2886	1.0611	.90160	.78369	.69300	.62108	.47355	.38261	.30460
25	25.254	9.9571	6.0382	4.3002	3.3289	2.7114	2.2853	1.9739	1.7367	1.5500	1.2753	1.0830	.94091	.83170	.74516	.56776	.45852	.36487
30	29.795	11.711	7.0877	5.0406	3.8979	3.1723	2.6720	2.3066	2.0285	1.8097	1.4880	1.2630	1.0969	.96924	.86814	.66107	.53366	.42451
35	34.333	13.462	8.1355	5.7795	4.4655	3.6318	3.0573	2.6381	2.3190	2.0683	1.6997	1.4420	1.2519	1.1059	.99033	.75372	.68023	.48367
40	38.871	15.213	9.1823	6.5173	5.0321	4.0903	3.4417	2.9686	2.6087	2.3260	1.9105	1.6203	1.4063	1.2420	1.1119	.84586	.68237	.54246
45	43.408	16.963	10.228	7.2543	5.5979	4.5481	3.8254	3.2985	2.8978	2.5830	2.1208	1.7980	1.5601	1.3775	1.2331	.93761	.75616	.60096
50	47.944	18.712	11.274	7.9907	6.1632	5.0053	4.2085	3.6278	3.1863	2.8396	2.3306	1.9753	1.7135	1.5127	1.3538	1.0290	.82968	.65922
60	57.016	22.210	13.363	9.4624	7.2924	5.9185	4.9736	4.2852	3.7621	3.3516	2.7491	2.3289	2.0194	1.7821	1.5945	1.2111	.97604	.77518
70	66.086	25.707	15.452	10.933	8.4205	6.8306	5.7375	4.9416	4.3369	3.8625	3.1667	2.6815	2.3244	2.0506	1.8343	1.3925	1.1217	.89055
80	75.156	29.203	17.540	12.403	9.5479	7.7419	6.5007	5.5972	4.9110	4.3727	3.5834	3.0334	2.6287	2.3185	2.0735	1.5733	1.2669	1.0055
90	84.226	32.698	19.627	13.872	10.675	8.6526	7.2633	6.2522	5.4845	4.8823	3.9997	3.3848	2.9324	2.5859	2.3122	1.7537	1.4118	1.1201
100	93.295	36.193	21.714	15.341	11.801	9.5630	8.0255	6.9068	6.0575	5.3915	4.4156	3.7358	3.2358	2.8530	2.5505	1.9337	1.5563	1.2344
130	120.50	46.677	27.973	19.746	15.179	12.292	10.311	8.8690	7.7751	6.9176	5.6615	4.7872	4.1445	3.6525	3.2641	2.4725	1.9885	1.5762
160	147.71	57.160	34.231	24.150	18.555	15.020	12.594	10.830	9.4912	8.4421	6.9059	5.8371	5.0516	4.4506	3.9762	3.0098	2.4195	1.9168
200	183.98	71.136	42.575	30.020	23.056	18.657	15.638	13.443	11.778	10.474	8.5639	7.2356	6.2598	5.5133	4.9243	3.7251	2.9929	2.3698

Table 1.4. (Continued)

$a = .005$

m \ n	5	10	15	20	25	30	35	40	45	50	60	70	80	90	100	130	160	200
-.5	2.0863	.88778	.55868	.40683	.31969	.26322	.22368	.19446	.17198	.15416	.12769	.10897	.095037	.084262	.075682	.057970	.046976	.037495
0	2.6855	1.1249	.70418	.51145	.40129	.33009	.28031	.24356	.21532	.19294	.15973	.13627	.11882	.10533	.094588	.072430	.058682	.046830
.5	3.2561	1.3480	.84031	.60906	.47729	.39228	.33292	.28915	.25554	.22892	.18945	.16158	.14085	.12484	.11210	.085812	.069512	.055465
1	3.8110	1.5629	.97094	.70250	.54992	.45165	.38312	.33262	.29387	.26320	.21773	.18565	.16181	.14339	.12874	.098530	.079802	.063666
1.5	4.3561	1.7726	1.0979	.79314	.62028	.50911	.43166	.37463	.33090	.29629	.24503	.20888	.18202	.16128	.14479	.11079	.089715	.071566
2	4.8947	1.9785	1.2223	.88174	.68898	.56517	.47899	.41557	.36696	.32852	.27160	.23148	.20169	.17868	.16039	.12270	.099348	.079241
2.5	5.4285	2.1817	1.3447	.96881	.75642	.62015	.52538	.45569	.40229	.36008	.29761	.25359	.22092	.19570	.17564	.13434	.10876	.086739
3	5.9588	2.3828	1.4655	1.0547	.82286	.67428	.57103	.49514	.43703	.39110	.32316	.27531	.23980	.21240	.19062	.14576	.11799	.094093
3.5	6.4865	2.5823	1.5851	1.1396	.88848	.72771	.61607	.53405	.47127	.42168	.34833	.29670	.25840	.22885	.20536	.15701	.12708	.10133
4	7.0120	2.7804	1.7038	1.2236	.95343	.78057	.66060	.57251	.50511	.45188	.37319	.31782	.27675	.24508	.21990	.16810	.13604	.10846
4.5	7.5359	2.9773	1.8216	1.3070	1.0178	.83293	.70470	.61059	.53860	.48177	.39778	.33871	.29490	.26112	.23428	.17905	.14490	.11551
5	8.0584	3.1734	1.9386	1.3899	1.0817	.88487	.74842	.64833	.57179	.51139	.42214	.35939	.31287	.27700	.24851	.18990	.15365	.12248
10	13.242	5.1049	3.0863	2.1988	1.7041	1.3898	1.1728	1.0142	.89315	.79785	.65742	.55894	.48608	.43001	.38552	.29419	.23782	.18942
15	18.393	7.0117	4.2134	2.9900	2.3110	1.8809	1.5848	1.3687	1.2041	1.0747	.88437	.75114	.65272	.57707	.51710	.39416	.31841	.25345
20	23.532	8.9087	5.3317	3.7735	2.9108	2.3656	1.9908	1.7177	1.5100	1.3468	1.1071	.93959	.81597	.72102	.64583	.49184	.39709	.31591
25	28.666	10.801	6.4454	4.5527	3.5067	2.8466	2.3934	2.0635	1.8129	1.6161	1.3274	1.1258	.97713	.86307	.77279	.58807	.47454	.37736
30	33.797	12.690	7.5565	5.3293	4.1002	3.3254	2.7939	2.4073	2.1138	1.8836	1.5459	1.3104	1.1369	1.0038	.89854	.68331	.55115	.43810
35	38.925	14.577	8.6657	6.1042	4.6920	3.8025	3.1928	2.7496	2.4134	2.1498	1.7633	1.4940	1.2956	1.1436	1.0234	.77782	.62713	.49832
40	44.053	16.463	9.7737	6.8779	5.2826	4.2786	3.5907	3.0910	2.7120	2.4150	1.9798	1.6767	1.4536	1.2827	1.1476	.87177	.70263	.55812
45	49.180	18.348	10.881	7.6507	5.8725	4.7538	3.9878	3.4316	3.0099	2.6795	2.1956	1.8588	1.6111	1.4213	1.2713	.96528	.77775	.61761
50	54.306	20.233	11.987	8.4230	6.4617	5.2284	4.3843	3.7716	3.3072	2.9434	2.4110	2.0404	1.7680	1.5594	1.3946	1.0584	.85256	.67682
60	64.557	24.001	14.199	9.9660	7.6387	6.1763	5.1760	4.4502	3.9005	3.4701	2.8404	2.4026	2.0809	1.8346	1.6402	1.2439	1.0014	.79463
70	74.807	27.767	16.410	11.508	8.8144	7.1229	5.9664	5.1277	4.4926	3.9956	3.2688	2.7637	2.3927	2.1089	1.8849	1.4286	1.1496	.91179
80	85.056	31.533	18.619	13.049	9.9893	8.0687	6.7560	5.8043	5.0839	4.5203	3.6963	3.1240	2.7038	2.3824	2.1289	1.6126	1.2972	1.0285
90	95.304	35.298	20.828	14.589	11.164	9.0139	7.5449	6.4803	5.6746	5.0443	4.1233	3.4837	3.0143	2.6554	2.3724	1.7962	1.4444	1.1448
100	105.55	39.063	23.037	16.129	12.338	9.9586	8.3335	7.1558	6.2648	5.5679	4.5498	3.8430	3.3244	2.9280	2.6155	1.9794	1.5912	1.2608
130	136.30	50.357	29.661	20.747	15.857	12.791	10.697	9.1807	8.0336	7.1369	5.8275	4.9191	4.2529	3.7441	3.3430	2.5275	2.0303	1.6075
160	167.04	61.649	36.284	25.364	19.376	15.622	13.059	11.204	9.8008	8.7042	7.1035	5.9935	5.1798	4.5585	4.0689	3.0740	2.4679	1.9529
200	208.03	76.704	45.114	31.518	24.066	19.395	16.207	13.900	12.156	10.793	8.8034	7.4245	6.4142	5.6429	5.0354	3.8013	3.0501	2.4121

Table 1.5. Upper α Percentage Points of $U^{(3)}$ and $U^{(4)}$.

$\alpha = .10$

	p = 3						p = 4		
n \ m	0	1	2	3	4	5	0	1	2
5	2.0526	2.9240	3.7808	4.6300	5.4747	6.3164	3.1949	4.3113	5.1454
10	.98227	1.3888	1.7865	2.1791	2.5686	2.9558	1.5170	2.0366	2.5485
15	.64348	.90724	1.1645	1.4181	1.6692	1.9186	.99104	1.3280	1.6593
20	.47812	.67309	.86300	1.0499	1.2349	1.4184	.73534	.98436	1.2290
25	.38028	.53487	.68529	.83323	.97950	1.1246	.58438	.78180	.97558
30	.31565	.44369	.56819	.69057	.81152	.93144	.48479	.64831	.80872
35	.26979	.37904	.48523	.58957	.69265	.79482	.41418	.55372	.69056
40	.23555	.33083	.42340	.51432	.60412	.69312	.36152	.48320	.60250
45	.20902	.29349	.37554	.45610	.53565	.61446	.32073	.42861	.53435
50	.18786	.26373	.33739	.40971	.48111	.55183	.28821	.38510	.48005
55	.17059	.23944	.30628	.37188	.43664	.50078	.26167	.34960	.43576
60	.15623	.21925	.28041	.34045	.39970	.45838	.23961	.32010	.39895
65	.14409	.20219	.25858	.31391	.36851	.42258	.22098	.29519	.36787
70	.13371	.18760	.23990	.29121	.34184	.39198	.20504	.27387	.34129
75	.12472	.17497	.22373	.27157	.31877	.36550	.19124	.25542	.31828
80	.11686	.16394	.20961	.25441	.29861	.34238	.17918	.23930	.29819
90	.10379	.14558	.18611	.22587	.26509	.30391	.15911	.21249	.26475
100	.093344	.13091	.16735	.20308	.23833	.27322	.14309	.19107	.23805

$\alpha = .025$

	p = 3						p = 4		
n \ m	0	1	2	3	4	5	0	1	2
5	2.9692	4.0964	5.1996	6.2904	7.3737	8.4521	4.4223	5.8326	7.2228
10	1.3259	1.8108	2.2817	2.7448	3.2031	3.6577	1.9559	2.5623	3.1567
15	.84829	1.1544	1.4505	1.7408	2.0274	2.3113	1.2477	1.6308	2.0052
20	.62289	.84610	1.0615	1.2724	1.4802	1.6859	.91499	1.1945	1.4673
25	.49194	.66749	.83668	1.0021	1.1649	1.3260	.72209	.94205	1.1564
30	.40642	.55105	.69029	.82629	.96011	1.0924	.59628	.77756	.95411
35	.34621	.46916	.58744	.70290	.81643	.92858	.50776	.66192	.81198
40	.30152	.40844	.51124	.61153	.71014	.80746	.44212	.57621	.70667
45	.26704	.36162	.45252	.54117	.62827	.71427	.39149	.51014	.62553
50	.23963	.32443	.40589	.48531	.56332	.64031	.35127	.45765	.56109
55	.21733	.29418	.36797	.43990	.51053	.58023	.31853	.41496	.50868
60	.19882	.26908	.33653	.40226	.46679	.53047	.29138	.37955	.46523
65	.18322	.24793	.31004	.37055	.42995	.48855	.26849	.34970	.42861
70	.16988	.22986	.28741	.34347	.39849	.45276	.24894	.32421	.39733
75	.15836	.21424	.26786	.32007	.37132	.42186	.23203	.30218	.37031
80	.14830	.20061	.25080	.29966	.34762	.39491	.21728	.28295	.34673
90	.13158	.17797	.22246	.26577	.30826	.35016	.19277	.25101	.30756
100	.11825	.15991	.19987	.23876	.27691	.31451	.17323	.22554	.27634

Table 1.5. (Continued)

$\alpha = .005$

n \ m	p = 3: 0	1	2	3	4	5	p = 4: 0	1	2
5	4.2391	5.7062	7.1391	8.5541	9.9592	11.354	6.0986	7.8971	9.6669
10	1.7401	2.3122	2.8655	3.4083	3.9445	4.4761	2.4737	3.1758	3.8619
15	1.0830	1.4329	1.7696	2.0988	2.4231	2.7438	1.5349	1.9654	2.3846
20	.78457	1.0359	1.2771	1.5123	1.7436	1.9720	1.1107	1.4205	1.7215
25	.61474	.81071	.99841	1.1811	1.3604	1.5376	.86980	1.1116	1.3462
30	.50524	.66578	.81933	.96866	1.1151	1.2596	.71463	.91292	1.1051
35	.42880	.56475	.69463	.82084	.94452	1.0665	.60640	.77442	.93710
40	.37243	.49031	.60284	.71212	.81910	.92470	.52662	.67238	.81342
45	.32915	.43320	.53246	.62879	.72308	.81594	.46538	.59409	.71857
50	.29488	.38799	.47678	.56291	.64722	.73023	.41689	.53212	.64352
55	.26707	.35133	.43164	.50952	.58572	.66063	.37755	.48185	.58265
60	.24404	.32099	.39430	.46537	.53493	.60322	.34499	.44026	.53231
65	.22468	.29547	.36291	.42826	.49210	.55498	.31761	.40528	.48996
70	.20816	.27371	.33614	.39663	.45576	.51388	.29424	.37545	.45386
75	.19390	.25494	.31305	.36934	.42437	.47845	.27408	.34970	.42271
80	.18147	.23857	.29293	.34557	.39699	.44758	.25651	.32726	.39556
90	.16084	.21143	.25956	.30616	.35169	.39643	.22735	.29004	.35054
100	.14443	.18983	.23301	.27481	.31566	.35576	.20415	.26042	.31471

Table 1.6. Comparison of Three Approximations to the Upper Percentage Points of $U^{(p)}$, p = 3 and 4.

p = 3, m = 0

n	5% Points: A_1	A_2	A_3	Exact	1% Points: A_1	A_2	A_3	Exact
5	2.3284	2.5311	2.5064	2.4959	3.1473	3.5804	3.6951	3.6581
10	1.1102	1.1564	1.1562	1.1540	1.4432	1.5321	1.5623	1.5581
15	.72741	.74723	.74777	.74702	.93288	.96959	.98252	.98145
20	.54069	.55162	.55207	.55174	.68865	.70852	.71559	.71518
30	.35717	.36192	.36218	.36208	.45174	.46023	.46326	.46316
40	.26663	.26927	.26943	.26939	.33604	.34072	.34239	.34235
50	.21270	.21438	.21449	.21447	.26750	.27046	.27151	.27150
60	.17691	.17808	.17815	.17814	.22218	.22421	.22494	.22493
80	.13237	.13302	.13306	.13306	.16594	.16707	.16748	.16747
100	.10574	.10616	.10619	.10618	.13242	.13314	.13340	.13340

Table 1.6. (Continued)

p = 3, m = 3

	5% Points				1% Points			
n	A_1	A_2	A_3	Exact	A_1	A_2	A_3	Exact
5	5.1093	5.4901	5.4723	5.4373	6.5800	7.3703	7.6114	7.5217
10	2.3850	2.4651	2.4700	2.4640	2.9285	3.0762	3.1258	3.1187
15	1.5488	1.5818	1.5845	1.5827	1.8691	1.9275	1.9465	1.9452
20	1.1455	1.1633	1.1649	1.1642	1.3701	1.4009	1.4107	1.4103
30	.75264	.76019	.76090	.76070	.89200	.90477	.90866	.90860
40	.56025	.56440	.56480	.56471	.66088	.66781	.66987	.66986
50	.44615	.44876	.44901	.44897	.52479	.52912	.53039	.53039
60	.37064	.37244	.37261	.37259	.43514	.43810	.43896	.43896
80	.27689	.27789	.27799	.27798	.32430	.32593	.32640	.32640
100	.22099	.22162	.22168	.22168	.25845	.25948	.25977	.25977

p = 4, m = 0

	5% Points				1% Points			
n	A_1	A_2	A_3	Exact	A_1	A_2	A_3	Exact
5	3.4776	3.8348	3.8146	3.7913	4.4646	5.1839	5.3980	5.3339
10	1.6584	1.7383	1.7419	1.7377	2.0579	2.2039	2.2532	2.2474
15	1.0867	1.1207	1.1230	1.1217	1.3324	1.3925	1.4127	1.4114
20	.80777	.82644	.82786	.82732	.98442	1.0169	1.0277	1.0272
30	.53359	.54169	.54237	.54221	.64628	.66012	.66464	.66455
40	.39833	.40282	.40321	.40315	.48094	.48857	.49103	.49100
50	.31776	.32062	.32087	.32083	.38294	.38776	.38930	.38929
60	.26430	.26627	.26645	.26643	.31811	.32143	.32248	.32248
80	.19775	.19885	.19895	.19894	.23764	.23948	.24007	.24006
100	.15797	.15867	.15874	.15873	.18965	.19083	.19120	.19120

p = 4, m = 2

	5% Points				1% Points			
n	A_1	A_2	A_3	Exact	A_1	A_2	A_3	Exact
5	5.8164	6.3526	6.3433	6.2964	7.2646	8.3276	8.6636	8.5540
10	2.7410	2.8548	2.8631	2.8558	3.2927	3.4960	3.5626	3.5550
15	1.7870	1.8342	1.8384	1.8363	2.1168	2.1982	2.2236	2.2222
20	1.3246	1.3501	1.3525	1.3517	1.5577	1.6010	1.6140	1.6136
30	.87232	.88323	.88428	.88405	1.0183	1.0364	1.0416	1.0416
40	.65014	.65615	.65673	.65663	.75606	.76593	.76869	.76869
50	.51812	.52192	.52228	.52224	.60115	.60735	.60905	.60905
60	.43066	.43327	.43352	.43349	.49890	.50315	.50430	.50430
80	.32194	.32339	.32353	.32352	.37223	.37459	.37521	.37521
100	.25704	.25797	.25806	.25805	.29685	.29834	.29874	.29874

CHAPTER II

THE MAX U-RATIO TEST OF EQUALITY OF COVARIANCE MATRICES

1. Introduction and Summary

Let $\underset{\sim}{x}_{ij}$ (px1) be a random sample of size n_i+1 from $N(\underset{\sim}{\mu}_i, \underset{\sim}{\Sigma}_i)$, $i = 1,\ldots,k$. Let $\underset{\sim}{S}_i$, $i = 1,\ldots,k$, be the sample sum of products (S.P.) matrix defined by

$$\underset{\sim}{S}_i = \sum_{j=1}^{n_i+1} (\underset{\sim}{x}_{ij} - \bar{\underset{\sim}{x}}_i)(\underset{\sim}{x}_{ij} - \bar{\underset{\sim}{x}}_i)' ,$$

where $\bar{\underset{\sim}{x}}_i = \sum_{j=1}^{n_i+1} \underset{\sim}{x}_{ij}/(n_i+1)$. To test the hypothesis $H_0: \underset{\sim}{\Sigma}_1 = \ldots = \underset{\sim}{\Sigma}_k$ we proceed in the following manner. First divide the sample from the i^{th} population into two independent subsamples of size $n_{1,i}+1$ and $n_{2,i}+1$, such that $n_{1,i}+1 + n_{2,i}+1 = n_i+1$, and calculate $\underset{\sim}{S}_{1,i}$ and $\underset{\sim}{S}_{2,i}$ from

$$\underset{\sim}{S}_{\ell,i} = \sum_{j=1}^{n_{\ell,i}+1} (\underset{\sim}{x}_{ij}^{(\ell)} - \bar{\underset{\sim}{x}}_i^{(\ell)})(\underset{\sim}{x}_{ij}^{(\ell)} - \bar{\underset{\sim}{x}}_i^{(\ell)})', \quad \ell = 1,2,$$

where $\underset{\sim}{x}_{ij}^{(\ell)}$ is the j^{th} observation from the ℓ^{th} subsample from the i^{th} population and $\bar{\underset{\sim}{x}}_i^{(\ell)}$ is the mean of the ℓ^{th} subsample from the i^{th} population. Now let

$$U_i = n_{1,i+1} \operatorname{tr} \underset{\sim}{S}_{2,i} \underset{\sim}{S}_{1,i+1}^{-1} / n_{2,i} , \quad i=1,\ldots,k-1, \tag{1.1}$$

and

$$U_k = \begin{cases} n_{2,k} \text{ tr } S_{1,1} S_{2,k}^{-1} / n_{1,1} \text{ , if } k > 2 \text{ ,} \\ n_{1,1} \text{ tr } S_{2,2} S_{1,1}^{-1} / n_{2,2} \text{ , if } k = 2. \end{cases}$$

(Note that the definition of U_i does not agree with the previous Chapter. Omitting the upper suffix is of course a notational change.) Next compute the statistic

$$R_1 = \{\max_{1 \le i \le k} U_i\} / \{\min_{1 \le j \le k} U_j\},$$

the max U-ratio statistic, and reject H_0 if $R_1 > c_\alpha$, where c_α depends upon the subsample sizes, the original sample sizes, the number of variates and the level (α) of the test.

If we have $2k$ populations and have prior knowledge of the equality of pairs of covariance matrices, i.e., $\Sigma_{2i-1} = \Sigma_{2i}$, $i=1,\ldots,k$, then to test the hypothesis H_0: $\Sigma_1 = \ldots = \Sigma_{2k}$, we consider the statistic based on 2k independent samples:

$$R_1 = \{\max_{1 \le i \le k} U_i\} / \{\min_{1 \le j \le k} U_j\},$$

(1.2) where $$U_i = n_{2i+1} \text{ tr } S_{2i} S_{2i+1}^{-1} / n_{2i} \text{ , } i = 1,\ldots,k-1,$$

and

$$U_k = \begin{cases} n_{2k} \text{ tr } S_1 S_{2k}^{-1} / n_1 \text{ , if } k > 2 \text{ ,} \\ n_1 \text{ tr } S_4 S_1^{-1} / n_4 \text{ , if } k = 2, \end{cases}$$

and S_j, $j = 1,\ldots,2k$, is the S.P. matrix of the j^{th} population. Again H_0 is rejected if $R_1 > c'_\alpha$.

Here we will consider the exact distribution of the ratio of the maximum to the minimum of two U statistics for $p = 2$ when the U statistics arise from samples of the same size. Also to be considered is an approximation to the distribution of the max U-ratio employing the F-type approximation to $U^{(p)}$ introduces in Chapter I. A comparison is made between the exact and approximate percentage points. Percentage points have been tabulated for selected parameter values. The non-null distribution of R_1 for $k = 2$, $p = 2$ in the identically distributed case is also considered by using the non-null density function of $U^{(p)}$ obtained in Chapter I.

Before we proceed with investigation of the distribution problems involved here, a short discussion of the rationale behind the max U-ratio test will be undertaken.

2. Preliminary Remarks

The max U-ratio test introduced here is based on the union-intersection approach to hypothesis testing due to Roy (see [40]). First let us consider the subsampling scheme and define the U_i's as in (1.1). We see that each U_i may be used to test the equality of a pair of covariance matrices, namely $\Sigma_i = \Sigma_{i+1}$, if $i \neq k$, and $\Sigma_1 = \Sigma_k$, if $i = k$. Now if $i \neq j$, the ratio U_i/U_j will simultaneously test $\Sigma_i = a_{ij}\Sigma_{i+1}$ ($\Sigma_i\Sigma_{i+1}^{-1} = a_{ij} I$) and $\Sigma_j = a_{ij}\Sigma_{j+1}$ ($\Sigma_j\Sigma_{j+1}^{-1} = a_{ij} I$) for $i,j \neq k$, and if i or j is k, the appropriate equality is replaced by $\Sigma_1 = a_{ij}\Sigma_k$($\Sigma_1\Sigma_k^{-1} = a_{ij} I$). The introduction of the constant a_{ij} becomes necessary when the ratio of the two U statistics is taken, and the a_{ij} disappears in consideration of the distribution problem. Now the pair of equalities given above yields $\underset{\sim}{\Sigma}_i \underset{\sim}{\Sigma}_{i+1}^{-1} = \underset{\sim}{\Sigma}_j \underset{\sim}{\Sigma}_{j+1}^{-1}$.

Now if $Q_{ij} = U_i/U_j$, $i \neq j$, we see that

$$U_{(1)}/U_{(k)} = 1/\max_{i,j} Q_{ij} = \min_{i,j} Q_{ij} \leq Q_{ij} \leq \max_{i,k} Q_{ij} = U_{(k)}/U_{(1)},$$

where $U_{(1)} \leq U_{(2)} \leq \cdots \leq U_{(k)}$ are the ordered U_i. Thus all $\binom{k}{2}$ possible ratios of pairs of U_i's lie between $1/R_1$ and $R_1(=U_{(k)}/U_{(1)})$, and we can use R_1 to simultaneously test the equality (up to a multiplicative constant) of the pairs of covariance matrices $(\underset{\sim}{\Sigma}_1, \underset{\sim}{\Sigma}_2)$, $(\underset{\sim}{\Sigma}_2, \underset{\sim}{\Sigma}_3), \ldots, (\underset{\sim}{\Sigma}_1, \underset{\sim}{\Sigma}_k)$ and equivalently determine if there are any differences between all possible pairs of $\underset{\sim}{\Sigma}_1\underset{\sim}{\Sigma}_2^{-1}$, $\underset{\sim}{\Sigma}_2\underset{\sim}{\Sigma}_3^{-1}, \ldots, \underset{\sim}{\Sigma}_1\underset{\sim}{\Sigma}_k^{-1}$.

To illustrate how these tests lead to the equality of all the covariances matrices, we consider the case of $k = 3$. If $R_1 < c_\alpha$, we see that by taking all possible ratios of U_1, U_2 and U_3 the following equalities hold: $\underset{\sim}{\Sigma}_1 = a_{12} \underset{\sim}{\Sigma}_2$ and $\underset{\sim}{\Sigma}_2 = a_{12} \underset{\sim}{\Sigma}_3$, $\underset{\sim}{\Sigma}_2 = a_{23} \underset{\sim}{\Sigma}_3$ and $\underset{\sim}{\Sigma}_1 = a_{23} \underset{\sim}{\Sigma}_3$, $\underset{\sim}{\Sigma}_1 = a_{13} \underset{\sim}{\Sigma}_2$ and $\underset{\sim}{\Sigma}_1 = a_{13} \underset{\sim}{\Sigma}_3$, and a similar set of equations with a_{ji} replacing a_{ij} which we do not write since $a_{ji} = a_{ij}$. By using these three equalities it is easily seen that: (1) $a_{12} = a_{23} = a_{13} = 1$ and (2) $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2 = \underset{\sim}{\Sigma}_3$. The same conclusion is also reached if we consider the equivalent equalities obtained from above, namely $\underset{\sim}{\Sigma}_1\underset{\sim}{\Sigma}_2^{-1} = \underset{\sim}{\Sigma}_2 \underset{\sim}{\Sigma}_3^{-1}$, $\underset{\sim}{\Sigma}_2 \underset{\sim}{\Sigma}_3^{-1} = \underset{\sim}{\Sigma}_1 \underset{\sim}{\Sigma}_3^{-1}$ and $\underset{\sim}{\Sigma}_1 \underset{\sim}{\Sigma}_2^{-1} = \underset{\sim}{\Sigma}_1 \underset{\sim}{\Sigma}_3^{-1}$. So if R_1 is not significant, we can conclude that the covariance matrices involved are all equal.

Although the above discussion centered on the subsampling scheme and $k = 3$, similar arguments will hold for the case when we have prior knowledge of the equality of pairs of covariances matrices and also for arbitrary k.

3. An Alternate Expression for the Density of $U^{(2)}$

In order to consider the exact distribution of the max U-ratio for $p = 2$ and $k = 2$, we will derive an expression for the density function of $U^{(2)} = \text{tr}\, S_1 S_2^{-1}$, where S_i is an S.P. matrix with n_i degrees of freedom, for $m = (n_1-3)/2$ an integer. The joint density function of the characteristic roots of $S_1 S_2^{-1}$ for $p = 2$ has the form (see (2.1) of Chapter I.):

$$f(\lambda_1,\lambda_2) = C(2,m,n)\, \lambda_1^m \lambda_2^m/\{(1+\lambda_1)(1+\lambda_2)\}^{m+n+3}(\lambda_2-\lambda_1)$$

where $0 < \lambda_1 < \lambda_2 < \infty$ and $n = (n_2-3)/2$.

Now let $U = \lambda_1+\lambda_2$ and $G = \lambda_1\lambda_2$ to obtain:

$$f(u,g) = C(2,m,n)\, g^m/(1+u+g)^{m+n+3}, \quad 0 < u < \infty, \quad 0 < g < u^2/4 .$$

Next assuming that m is a non-negative integer, in order to integrate out g we use the following result which is obtained by integration by parts:

$$\int_a^b x^{q-1}/(d+x)^p\, dx = (p-1)^{-1}\left\{ \sum_{i=0}^{q-1} c(p,q,i)\left[a^{q-i-1}/(d+a)^{p-i-1} - b^{q-i-1}/(d+b)^{p-i-1}\right] \right\}$$

where q is a positive integer, $d > 0$, $0 \leq a < b < \infty$, and $c(p,q,0) = 1$, $c(p,q,i) = \prod_{j=1}^{i} (q-j)/(p-j-1),\ i > 0.$

Thus we find

$$(3.1)\ f(u) = \frac{C(2,m,n)}{m+n+2} \left(c_m/(1+u)^{n+2} - \sum_{i=0}^{m} c_i \frac{(u^2/4)^{m-i}}{(1+u/2)^{2(m+n+2-i)}} \right),$$

where $c_i = c(q,m,i)$, $q = m+n+3$.

4. The Exact Distribution of R_1 for $p = 2$ and $k = 2$.

Let $U_1^{(2)}$ and $U_2^{(2)}$ be independent identically distributed random variables having the density (3.1). Let $U_{(1)}$ and $U_{(2)}$ denote the ordered $U_1^{(2)}$ and $U_2^{(2)}$. Then the joint density of $U_{(1)}$ and $U_{(2)}$ is:

$$f(u_{(1)}, u_{(2)}) = 2\, f(u_{(1)})\, f(u_{(2)})$$

Now let $Z = U_{(1)}/U_{(2)}$ and $T = U_{(2)}$. (Note $R_1 = 1/Z$. We will consider $Z = 1/R_1$ for convenience throughout.) Then

$$f(z,t) = 2 \left(\frac{C(2,m,n)}{m+n+2} \right)^2 \left\{ c_m^2 \frac{t}{(1+zt)^{n+2}(1+t)^{n+2}} \right.$$

$$- c_m \sum_{i=0}^{m} c_i/4^{m-i} \frac{t^{2m-2i+1}}{(1+zt)^{n+2}(1+t/2)^{2(m+n+2-i)}}$$

$$- c_m \sum_{i=0}^{m} c_i/4^{m-i} \frac{z^{2m-2i}\, t^{2m-2i+1}}{(1+t)^{n+2}(1+zt/2)^{2(m+n+2-i)}}$$

$$+\sum_{i=0}^{m}\sum_{j=0}^{m} c_i c_j/4^{2m-i-j} \frac{z^{2m-2i}\, t^{4m-2i-2j+1}}{(1+zt/2)^{2(m+n+2-i)}(1+t/2)^{2(m+n+2-j)}} \} \quad .$$

In order to integrate out t for $0 < t < \infty$, we make use of the following lemma.

Lemma 4.1: Let $a > 0$ and $q+r > p+1$. Then

$$(4.1) \quad \int_0^\infty \frac{t^p\, dt}{(1+at)^q(1+zt)^r} = \frac{1}{a^{p+1}} \sum_{i=0}^{\infty} \binom{-r}{i}(-1)^i \beta(p+i+1, q+r-p-1)(1-z/a)^i \, ,$$

for $0 < z < 2a$.

Proof:

$$\int_0^\infty \frac{t^p}{(1+at)^q} \frac{dt}{(1+zt)^r} = \int_0^\infty \frac{t^p dt}{(1+at)^{q+r}(1-(a-zt)/(1+at))^r}$$

$$= \int_0^\infty \frac{t^p}{(1+at)^{q+r}} \sum_{i=0}^{\infty} \binom{-r}{i}(-1)^i \left(\frac{(a-z)t}{1+at}\right)^i dt \quad .$$

Now let $x = at/(1+at)$ and interchange the order of integration and summation to obtain

$$\frac{1}{a^{p+1}} \sum_{i=0}^{\infty} \binom{-r}{i}(-1)^i(1-z/a)^i \int_0^1 x^{p+i}(1-x)^{q+r-p-2}\, dx$$

Thus (4.1) becomes

$$\frac{1}{a^{p+1}} \sum_{i=0}^{\infty} \binom{-r}{i}(-1)^i \beta(p+i+1\, , q+r-p-1)(1-z/a)^i.$$

From the condition for convergence

$$-1 < \frac{(a-z)t}{1+at} < 1 \quad , \quad \text{for all } 0 \le t < \infty \; ,$$

we have: $0 < z < 2a.$

Then using Lemma 4.1 we obtain

$$f(z) = 2\left(\frac{C(2,m,n)}{m+n+2}\right)^2 \{ c_m^{\;2} \sum_{k=0}^{\infty} \binom{-(n+2)}{k}(-1)^k \beta(k+2,2n+2)(1-z)^k$$

$$- 4c_m \sum_{i=0}^{m} c_i \sum_{k=0}^{\infty} \binom{-(n+2)}{k}(-1)^k \beta(2(m-i+1)+k,3n+4)(1-2z)^k$$

$$- c_m \sum_{i=0}^{m} c_i/4^{m-i} \; z^{2m-2i} \sum_{k=0}^{\infty} \binom{-2(m+n+2-i)}{k}(-1)^k \beta(2(m-i+1)+k,3n+4) (1-z/2)^k$$

$$+ 4\sum_{i=0}^{m} \sum_{j=0}^{m} c_i \, c_j \, z^{2m-2i} \sum_{k=0}^{\infty} \binom{-2(m+n+2-i)}{k}(-1)^k \beta(2(2m-i-j+1)+k,4n+6) (1-z)^k \},$$

$$0 < z < 1.$$

Then

(4.2) $$P(Z \le x) = \int_0^x f(z)dz$$

$$= 2\left(\frac{C(2,m,n)}{m+n+2}\right)^2 \left\{ c_m^{\;2} \sum_{k=0}^{\infty} \binom{-(n+2)}{k}(-1)^k \beta(k+2,2n+2)[1-(1-x)^{k+1}]/(k+1)\right.$$

$$-2c_m \sum_{i=0}^{m} c_i \sum_{k=0}^{\infty} \binom{-(n+2)}{k}(-1)^k \beta(2(m-i+1)+k,3n+4)[1-(1-2x)]^{k+1}/(k+1)$$

$$-2c_m \sum_{i=0}^{m} c_i \sum_{k=0}^{\infty} \binom{-2(m+n+2-i)}{k}(-1)^k \beta(2(m-i+1)+k, 3n+4) \int_0^{x/2} z^{2m-2i}(1-z)^k dz$$

$$+\sum_{i=0}^{m} \sum_{j=0}^{m} c_i c_j \sum_{k=0}^{\infty} \binom{-2(m+n+2-i)}{k}(-1)^k \beta(2(2m-i-j+1)+k, 4n+6)$$

$$\int_0^{x/2} z^{2m-2i}(1-z)^k \, dz.$$

(4.2) is then the distribution function of Z from which the distribution of R_1 is found by using

$$P(R_1 \leq x) = 1 - P(Z \leq 1/x).$$

5. An Approximation to the Distribution of R_1

In Chapter I approximations to the distribution of $U^{(p)}$ were considered. They were obtained by fitting the exact moments of $U^{(p)}$ to the moments of an F-type distribution. The form for the approximating density function is:

$$f(u) = u^a / \{\beta(a+1, b-a-1) \, K^{a+1}(1+u/k)^b\} ,$$

$$0 < u < \infty.$$

By assuming that the U statistics have this approximate distribution, we will proceed to derive the distribution of $1/R_1$.

In general let $U_1,\ldots,U_k$ be independent having density functions

$$f(u_i) = u_i^{a_i} / \{\beta(a_i+1, b_i-a_i-1)\, K_i^{a_i+1} (1+u_i/k_i)^{b_i}\},$$
$$0 < u_i < \infty \;, \quad i = 1,\ldots,k.$$

Let $Y_i = U_i/h_i$, where $h_i > 0$. Then

$$f(y_i) = [\beta(a_i+1, b_i-a_i-1)]^{-1} (h_i/K_i)^{a_i+1}\, y_i^{a_i} / (1+h_i\, y_i/k_i)^{b_i},$$
$$0 < y_i < \infty \;, \qquad i = 1,\ldots,k\;.$$

Now let $Y_{(1)} < \ldots < Y_{(k)}$ be the ordered Y_i's and let $M_i = Y_{(i)}/Y_{(k)}$, $i = 1,\ldots,k-1$ and $t = Y_{(k)}$.

Then with the Jacobian of transformation being t^{k-1}, simplification yields:

$$f(M_1,\ldots,M_{k-1},t) = C_k \sum_\sigma \frac{t^{a+k-1} \prod_{i=1}^{k-1} M_i^{a_{\sigma(i)}}}{\{\prod_{i=1}^{k-1} (1+ h_{\sigma(i)}/K_{\sigma(i)} t M_i)^{b_{\sigma(i)}}\} (1+ \frac{h_{\sigma(k)}}{K_{\sigma(k)}} t)^{b_{\sigma(k)}}} \;,$$

where $C_k = \prod_{i=1}^{k} \{(h_i/K_i)^{a_i+1} / \beta(a_i+1, b_i-a_i-1)\}$,

$$0 < M_1 < \ldots < M_{k-1} < 1 \;, \quad 0 < t < \infty, \; a = \Sigma a_i, \; b = \Sigma b_i$$

and the summation is over all possible permutations σ of $(1,\ldots,k)$.

If $h_1/K_1 = h_2/K_2 = \ldots = h_k/K_k = L$, we have

$$f(M_1,\ldots,M_{k-1},t) = C_k \sum_{\sigma} \left\{ \prod_{i=1}^{k-1} \frac{M_i^{a_{\sigma(i)}}}{(1+LM_i t)^{b_{\sigma(i)}}} \right\} \frac{t^{a+k-1}}{(1+Lt)^{b_{\sigma(k)}}} .$$

Let $z = Lt/(1+Lt)$, $0 < z < 1$, and simplify to obtain

$$f(M_1,\ldots,M_{k-1},z) = C_k' \sum_{\sigma} \left\{ \prod_{i=1}^{k-1} \frac{M_i^{a_{\sigma(i)}}}{(1-(1-M_i)z)^{b_{\sigma(i)}}} \right\} z^{a+k-1}(1-z)^{b-a-k-1},$$

where $C_k' = 1/\{\prod_{i=1}^{k} \beta(a_i+1,b_i-a_i-1)\}$.

Now expand $(1-(1-M_i)z)^{-b_{\sigma(i)}}$ in a binominal series to get

$$C_k' \sum_{\sigma} \sum_{i_1=0}^{\infty} \cdots \sum_{i_{k-1}=0}^{\infty} \left\{ \prod_{j=1}^{k-1} \binom{-b_{\sigma(j)}}{i_j} (-1)^{i_j} (1-M_j)^{i_j} M_j^{a_{\sigma(j)}} \right\}$$

$$z^{a+\sum_{j=1}^{k-1} i_j+k-1} (1-z)^{b-a-k-1}$$

Integrating out z yields

$$f(M_1,\ldots,M_{k-1}) = C_k' \sum_{\sigma} \sum_{i_1=0}^{\infty} \cdots \sum_{i_{k-1}=0}^{\infty} \left\{ \prod_{j=1}^{k-1} \binom{-b_{\sigma(j)}}{i_j} (-1)^{i_j} (1-M_j)^{i_j} M_j^{a_{\sigma(j)}} \right\}$$

$$\beta\left(a+\sum_{j=1}^{k-1} i_j +k, b-a-k\right)$$

To integrate out $M_2,\ldots,M_{k-1}$, where $M_1<M_2<\ldots<M_{k-1}<1$, we need to evaluate

$$\int_{M_1}^{1}\cdots\int_{M_{k-2}}^{1}\{\prod_{j=2}^{k-1} M_j^{a_{\sigma(j)}}(1-M_j)^{i_j}\}dM_2\ldots dM_{k-1}.$$

The first integration yields:

$$\int_{M_{k-2}}^{1} M_{k-1}^{a_{\sigma(k-1)}}(1-M_{k-1})^{i_{k-1}}\, dM_{k-1}$$

$$= \int_0^{1-M_{k-2}} y^{i_{k-1}}\,(1-y)^{a_{\sigma(k-1)}}\,dy$$

$$= \sum_{\ell_{k-1}=0}^{\infty}\binom{a_{\sigma(k-1)}}{\ell_{k-1}}(-1)^{\ell_{k-1}}\int_0^{1-M_{k-2}} y^{i_{k-1}+\ell_{k-1}}\,dy$$

$$= \sum_{\ell_{k-1}=0}^{\infty}\binom{a_{\sigma(k-1)}}{\ell_{k-1}}(-1)^{\ell_{k-1}}\frac{(1-M_{k-2})^{i_{k-1}+\ell_{k-1}+1}}{i_{k-1}+\ell_{k-1}+1}.$$

The series is finite if $a_{\sigma(k-1)}$ is a non negative integer. We can continue in this manner to obtain

$$(5.1)\quad f(M_1) = C_k'\sum_{\sigma}\sum_{i_1=0}^{\infty}\cdots\sum_{i_{k-1}=0}^{\infty}\;\sum_{\ell_2=0}^{\infty}\cdots\sum_{\ell_{k-1}=0}^{\infty}\Psi(\underset{\sim}{i},\underset{\sim}{\ell},\underset{\sim}{a},\underset{\sim}{b},\sigma,k)\; M_1^{a_{\sigma(1)}}(1-M_1)^{\sum_{j=1}^{k-1}i_j+\sum_{j=2}^{k-1}\ell_j+k-2},$$

where $\Psi(\underset{\sim}{i}, \underset{\sim}{\ell}, \underset{\sim}{a}, \underset{\sim}{b}, \sigma, k) = \left\{ \prod_{j=1}^{k-1} \binom{-b_{\sigma(j)}}{i_j} (-1)^{i_j} \right\} \beta(a+\sum_{j=1}^{k-1} i_j + k, b-a-k)$

$$\prod_{j=2}^{k-1} \left\{ \binom{a_{\sigma(j)}}{\ell_j} (-1)^{\ell_j} / [\sum_{m=0}^{j-2} (i_{k-m-1}+\ell_{k-j-1}) + j - 1] \right\}$$

Then

$$P(M_1 \leq x) = C_k' \sum_{\sigma} \sum_{i_1=0}^{\infty} \cdots \sum_{i_{k-1}=0}^{\infty} \sum_{\ell_2=0}^{\infty} \cdots \sum_{\ell_{k-1}=0}^{\infty} \Psi(\underset{\sim}{i}, \underset{\sim}{\ell}, \underset{\sim}{a}, \underset{\sim}{b}, \sigma, k)$$

$$B_x(a_{\sigma(1)}+1, \sum_{j=1}^{k-1} i_j + \sum_{j=2}^{k-1} \ell_j + k-1) \; ,$$

where $B_x(c,d) = \int_0^x t^{c-1}(1-t)^{d-1}\, dt.$

If $k = 2$, but the h's and K's are different so that $h_1/K_1 \neq h_2/K_2$, we can write the density of M_1 as

$$(5.2) \quad C_2 \sum_{\sigma} \int_0^{\infty} \frac{t^{a+1} \qquad dt}{(1+h_{\sigma(1)}/K_{\sigma(1)} t M_1)^{b_{\sigma(1)}} (1+h_{\sigma(2)}/K_{\sigma(2)} t)^{b_{\sigma(2)}}}$$

Using Lemma 4.1, (5.2) yields

$$(5.3) \quad f(M_1) = C_2 \sum_{\sigma} (K_{\sigma(2)}/h_{\sigma(2)})^{a+2} \sum_{i=0}^{\infty} \binom{-b_{\sigma(1)}}{i} (-1)^i \; \beta(a+i+2, b-a-2)$$
$$M_1^{a_{\sigma(1)}} (1 - \frac{h_{\sigma(1)} K_{\sigma(2)}}{h_{\sigma(2)\;\sigma(1)}} M_1)^i$$

where $a = a_1 + a_2$, $b = b_1 + b_2$ and

$$0 < h_{\sigma(2)} K_{\sigma(2)} / (h_{\sigma(1)} K_{\sigma(2)}) < 2 \quad , \text{for all } \sigma,$$

$$\Rightarrow \tfrac{1}{2} < h_1 K_2 / (h_2 K_1) < 2.$$

If $K_1 = K_2$ and $h_1 = h_2$ we have

$$(5.4) \quad f(M_1) = C_2' \sum_{\sigma} \sum_{i=0}^{\infty} \binom{-b_{\sigma(1)}}{i} (-1)^i \beta(a+i+2, b-a-2) M_1^{a_{\sigma(1)}} (1-M_1)^i ;$$

and if $a_1 = a_2$, $b_1 = b_2$, $h_1 = h_2$, $K_1 = K_2$ we get

$$(5.5) \quad f(M_1) = 2/\{\beta(a_1+1, b_1-a_1-1)\}^2 \sum_{i=0}^{\infty} \binom{-b_1}{i} (-1)^i \beta(a+i+2, b-a-2) M_1^{a_1} (1-M_1)^i .$$

Then taking (5.1), (5.3), (5.4) or (5.5) for the density of M_1 under various conditions we can approximate the distribution of $1/R_1$ using the distribution of M_1.

6. Comparison of Exact and Approximate Percentage Points of R_1 for $k = 2$ and $p = 2$

By using approximation A_3 of Chapter I for the distribution of $U^{(2)}$, percentage points for R_1 have been computed for $n_1 = 3, 5$ and 7 and various n_2 in the identically distributed case by using the density (5.5). Exact percentage points were also calculated using (4.2). Comparison of the approximate and exact percentage points may be found in

Table 2.1. As can be noted from this table, as n_2 (or n) increases, the approximate percentage points become more accurate. This is to be expected since it is the nature of the F-type approximation to $U^{(p)}$ to improve as n_2(or n) increases. Also as n_1 increases the accuracy of the approximate percentage points increase. This would seem to indicate that for n_1 moderately large (around 10 or 15) the approximate percentage points should agree with the exact ones to three significant digits for n_2 as low as 10 or 15.

It should also be noted that the approximation to the distribution of R_1 holds for $p > 2$ and thus the R_1 test can be used for more than the bivariate case by using the given approximation.

Table 2.1. Comparison of Exact and Approximate Percentage Points of R_1 for $k=2$, $p=2$ and $\alpha = .05$

	$n_1=3$		$n_1=5$		$n_1=7$	
n_2	Exact	Approximate	Exact	Approximate	Exact	Approximate
10	8.566	8.013	5.870	5.518	4.963	4.658
15	7.406	7.229	4.951	4.850	4.099	4.027
20	6.932	6.848	4.580	4.534	3.762	3.731
25	6.676	6.627	4.380	4.354	3.581	3.563
30	6.515	6.483	4.256	4.239	3.468	3.456
35	6.405	6.383	4.170	4.159	3.390	3.382
40	6.326	6.309	4.108	4.100	3.334	3.328
50	6.217	6.207	4.024	4.019	3.257	3.254
60	6.147	6.140	3.970	3.966	3.208	3.206
80	6.061	6.058	3.903	3.902	3.148	3.147
100	6.011	6.009	3.865	3.864	3.113	3.112

7. Computation of Percentage Points

Using (5.5), approximate percentage points have been calculated for R_1 for $k = 2$, and $p = 2$, 3 and 4. They are found in Table 2.2. Due to the poor rate of convergence of the infinite series involved for certain variations in the degrees of freedom, some percentage points were unobtainable. The percentage points are given to three significant digits. For p = 2 adjustments have been made in percentage points where exact percentage points were available. The error in the approximate percentage points is believed to be no more than several places in the last digit given.

8. The Non-Null Distribution of R_1 for $p=2$, $k = 2$ and $m = 0$

The density function of $U^{(2)}$ under the alternative hypothesis for $k = 2$ ($\Sigma_1 \neq \Sigma_2$) was considered in Chapter I using zonal polynominals up to the sixth degree. When $m = 0$, this density may be written:

$$f(m,n,\gamma_1,\gamma_2,u) = K\sum_{i=1}^{7}\sum_{j=0}^{q_i} g_{ij}\, R^*(n;\, i+j,\, j;\, u)$$

where $q_1 = 3$, $q_2 = q_3 = 2$, $q_4 = q_5 = 1$, $q_6 = q_7 = 0$, the γ_i's are the characteristic roots of $\underset{\sim}{\Sigma}_1 \underset{\sim}{\Sigma}_2^{-1}$ and the g_{ij} are functions of the A_{ij}'s and may be found in Appendix D, and $K = (\gamma_1 \gamma_2)^{-3/2}$ $C(2,0,n)$.

Now let U_1 have the density $f_2(m,n,\gamma_1,\gamma_2,u_1)$ and U_2 have the density $f(m,n, 1/\gamma_1, 1/\gamma_2, u_2)$. Then the joint density of $U_{(1)}$ and $U_{(2)}$ is given by

$$[C(2,0,n)]^2 \Big\{ \sum_{(i,j)} \sum_{(i',j')} g_{ij}\, g'_{i'j'}\, [R^*(n;i+j,j;u_{(1)})R^*(n;i'+j',j';u_{(2)})$$

$$+ R^*(n;i+j,j;u_{(2)})\, R^*(n;i'+j',j';u_{(1)})]\Big\}$$

Making the transformation $M_1 = u_{(1)}/u_{(2)}$ and $t = u_{(2)}$ and integrating out t , $0 < t < \infty$ we get:

$$f(M_1) = [C(2,0,n)]^2 \left\{ \sum_{(i,j)} \sum_{(i',j')} g_{ij}\, g'_{i'j'} \right.$$

$$\left[\int_0^\infty R^*(n;i+j,j;M_1 t)\, R^*(n,i'+j',j';t)t\, dt \right.$$

$$\left.\left. + \int_0^\infty R^*(n;i+j,j;t)\, R^*(n;i'+j',j';M_1 t)t\, dt \right] \right\}$$

A typical term in the integral

$$T^*(n;i,j;i',j';M_1) = \int_0^\infty R(n;i+j,j;t)\, R^*(n;i'+j',j';M_1 t)t\, dt$$

is of the form

$$(8.1) \qquad \int_0^\infty \frac{t\, dt}{(1+M_1/d)^b(1+tM_1/c)^a} \text{ , where c,d are 1 and 2.}$$

But (8.1) is

$$c^2 \int_0^\infty \frac{t\, dt}{(1+c/dt)^b(1+M_1 t)^a} \quad ,$$

and we can use Lemma 4.1 to perform the integration. Thus

$$(8.2) \quad f(M_1) = [C(2,0,n)]^2 \left\{ \sum_{(i,j)} \sum_{(i',j')} g_{ij}\, g'_{i'j'} \, [T^*(n;i,j;i',j';M_1) + T^*(n;i',j';i,j;M_1)] \right\}$$

The distribution of M_1 is found upon integration of $f(M_1)$ from 0 to x.

The power of R_1 $(=1/M_1)$ has been computed using (8.2) for various γ_i's and different sample sizes. The power will be compared with that of the power of the likelihood ratio criterion for testing $H_0:\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$ in Chapter III (see Table 3.2).

Table 2.2. Upper .05 Points of R_1 for $k = 2$

$p = 2$

n_2/n_1	3	5	7	10	12	14	16	18	20	25	30	35	40	45	50	60	70	80	100
10	8.57	5.87	4.96	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----
12	8.02	5.47	4.54	3.90	3.73	----	----	----	----	----	----	----	----	----	----	----	----	----	----
14	7.64	5.14	4.26	3.60	3.40	3.20	3.13	----	----	----	----	----	----	----	----	----	----	----	----
16	7.33	4.92	4.05	3.42	3.24	3.04	2.97	2.88	2.79	----	----	----	----	----	----	----	----	----	----
18	7.12	4.69	3.88	3.30	3.09	2.93	2.82	2.75	2.67	2.55	----	----	----	----	----	----	----	----	----
20	6.93	4.58	3.76	3.18	2.99	2.83	2.73	2.65	2.58	2.46	2.39	----	----	----	----	----	----	----	----
25	6.68	4.38	3.58	3.02	2.82	2.68	2.57	2.49	2.43	2.31	2.23	2.17	2.13	2.10	2.07	----	----	----	----
30	6.52	4.26	3.47	2.92	2.72	2.58	2.47	2.39	2.33	2.21	2.13	2.07	2.03	2.00	1.97	1.93	----	----	----
35	6.41	4.17	3.39	2.85	2.65	2.51	2.40	2.32	2.26	2.14	2.06	2.00	1.96	1.93	1.90	1.86	1.83	1.81	----
40	6.33	4.11	3.33	2.80	2.60	2.46	2.35	2.27	2.21	2.09	2.01	1.95	1.91	1.87	1.85	1.81	1.78	1.75	----
45	6.27	4.06	3.29	2.76	2.56	2.42	2.31	2.23	2.17	2.05	1.97	1.91	1.87	1.83	1.81	1.77	1.74	1.71	1.68
50	6.22	4.02	3.26	2.73	2.53	2.39	2.28	2.20	2.14	2.02	1.94	1.88	1.84	1.80	1.77	1.73	1.70	1.68	1.65
60	6.15	3.97	3.21	2.68	2.48	2.34	2.24	2.16	2.09	1.97	1.89	1.83	1.79	1.75	1.73	1.68	1.65	1.63	1.59
70	6.10	3.93	3.17	2.65	2.45	2.31	2.21	2.12	2.06	1.94	1.86	1.80	1.76	1.72	1.69	1.65	1.62	1.59	1.56
80	6.06	3.90	3.15	2.62	2.43	2.29	2.18	2.10	2.03	1.92	1.83	1.78	1.73	1.69	1.67	1.62	1.59	1.56	1.53
100	6.01	3.86	3.11	2.59	2.39	2.26	2.15	2.07	2.00	1.88	1.80	1.74	1.70	1.65	1.63	1.58	1.55	1.53	1.49

Table 2.2. (Continued)

p = 3

n_2/n_1	3	5	7	10	12	14	16	18	20	25	30	35	40	45	50	60	70	80	100
12	5.8	4.3	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----
14	5.5	4.0	3.4	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----
16	5.3	3.9	3.3	2.87	2.72	----	----	----	----	----	----	----	----	----	----	----	----	----	----
18	5.1	3.7	3.2	2.76	2.62	2.52	2.44	----	----	----	----	----	----	----	----	----	----	----	----
20	5.0	3.6	3.0	2.68	2.54	2.44	2.36	2.30	2.26	----	----	----	----	----	----	----	----	----	----
25	4.7	3.38	2.89	2.53	2.40	2.30	2.22	2.17	2.12	2.04	1.98	----	----	----	----	----	----	----	----
30	4.55	3.28	2.80	2.44	2.31	2.21	2.14	2.08	2.03	1.95	1.89	1.85	1.82	----	----	----	----	----	----
35	4.47	3.22	2.73	2.38	2.25	2.15	2.08	2.02	1.97	1.89	1.83	1.79	1.76	1.74	1.72	----	----	----	----
40	4.41	3.17	2.69	2.34	2.21	2.11	2.04	1.98	1.93	1.85	1.79	1.75	1.72	1.69	1.67	1.64	1.61	----	----
45	4.36	3.13	2.65	2.31	2.17	2.08	2.00	1.95	1.90	1.82	1.76	1.72	1.68	1.66	1.64	1.61	1.58	1.57	----
50	4.33	3.10	2.63	2.28	2.15	2.05	1.98	1.92	1.88	1.79	1.73	1.69	1.66	1.63	1.61	1.58	1.56	1.54	----
60	4.27	3.06	2.59	2.24	2.11	2.02	1.94	1.88	1.84	1.75	1.69	1.65	1.62	1.59	1.57	1.54	1.52	1.50	1.47
70	4.24	3.03	2.56	2.22	2.09	1.99	1.92	1.86	1.81	1.73	1.67	1.62	1.59	1.57	1.54	1.51	1.49	1.47	1.44
80	4.21	3.01	2.54	2.20	2.07	1.97	1.90	1.84	1.79	1.71	1.65	1.60	1.57	1.55	1.52	1.49	1.47	1.45	1.42
100	4.17	2.98	2.51	2.17	2.04	1.94	1.87	1.81	1.77	1.68	1.62	1.58	1.54	1.52	1.49	1.46	1.44	1.42	1.39

Table 2.2. (Continued)

p = 4

n_2/n_1	5	7	10	12	14	16	18	20	25	30	35	40	45	50	60	70	80	100
14	3.6	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----
16	3.4	3.0	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----
18	3.3	2.8	2.52	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----
20	3.1	2.7	2.44	2.33	2.24	----	----	----	----	----	----	----	----	----	----	----	----	----
25	2.94	2.57	2.29	2.19	2.11	2.05	2.00	1.97	----	----	----	----	----	----	----	----	----	----
30	2.85	2.48	2.21	2.10	2.03	1.97	1.92	1.88	1.82	1.77	----	----	----	----	----	----	----	----
35	2.78	2.42	2.15	2.05	1.97	1.91	1.87	1.83	1.76	1.71	1.68	1.65	----	----	----	----	----	----
40	2.74	2.38	2.11	2.01	1.93	1.87	1.83	1.79	1.72	1.68	1.64	1.62	1.59	1.58	----	----	----	----
45	2.70	2.35	2.08	1.98	1.90	1.84	1.80	1.76	1.69	1.65	1.61	1.59	1.56	1.55	1.52	----	----	----
50	2.68	2.32	2.06	1.95	1.88	1.82	1.77	1.74	1.67	1.62	1.59	1.56	1.54	1.52	1.50	1.49	----	----
60	2.64	2.29	2.03	1.92	1.85	1.79	1.74	1.70	1.64	1.59	1.55	1.53	1.51	1.49	1.46	1.44	1.43	----
70	2.61	2.26	2.00	1.90	1.82	1.76	1.72	1.68	1.61	1.56	1.53	1.50	1.48	1.46	1.44	1.42	1.40	1.38
80	2.59	2.25	1.99	1.88	1.81	1.75	1.70	1.66	1.60	1.55	1.51	1.48	1.46	1.45	1.42	1.40	1.38	1.36
100	2.56	2.22	1.96	1.86	1.78	1.72	1.68	1.64	1.57	1.52	1.49	1.46	1.44	1.42	1.39	1.37	1.36	1.33

CHAPTER III

THE LIKELIHOOD RATIO TEST OF THE HYPOTHESIS $\Sigma_1 = \Sigma_2$

1. Introduction and Summary

Let π_1 and π_2 be two populations having the p-variate normal distributions $N(\mu_i, \Sigma_i)$, $i = 1,2$, respectively where μ_i and Σ_i (positive definite), $i = 1,2$, are unknown. Let S_1 and S_2 be the independent sum of products matrices from samples of size N_1 and N_2 from π_1 and π_2 respectively. Then the likelihood ratio criterion for testing the hypothesis H_0: $\Sigma_1 = \Sigma_2$ against the alternative $\Sigma_1 \neq \Sigma_2$ is given by [42], [1]:

$$\Lambda = |S_1|^{N_1/2} |S_2|^{N_2/2} |S_1+S_2|^{-(N_1+N_2)/2} (N_1+N_2)^{p(N_1+N_2)/2} N_1^{-pN_1/2} N_2^{-pN_2/2} .$$

This criterion has been modified by Bartlett [3] to

$$(1.1) \qquad V = |S_1|^{n_1/2} |S_2|^{n_2/2} |S_1+S_2|^{-(n_1+n_2)/2}$$

where $\qquad n_i = N_i - 1,\ i = 1,2.$

The distribution of V for $p = 1$ and 2 and $n_1 = n_2$ and the moments of V for general p, n_1 and n_2 may be found in Anderson [1]. Box [4] has given a review of work done on V until 1949 and has given approximate and asymptotic distributions for log V. Korin [26] has used a series of central chi-square distributions for approximating the distribution of log V and has undertaken some computer simulations of the power of the test employing log V. Sugiura and Nagao [41] have shown that the test using V is unbiased.

Using the method employed by Anderson [1] to obtain the distribution of V for $p = 2$ and $n_1 = n_2$, namely, by identifying the moments of V with their appropriate distributions, we find here the density of V for $p = 4$ and $n_1 = n_2$. Also an alternate derivation of the density of V for $p = 2$ and $n_1 = n_2$ is given by reverting to the distribution of the characteristic roots of $\underset{\sim}{S}_1(\underset{\sim}{S}_1+\underset{\sim}{S}_2)^{-1}$. Lastly the non-null distribution of V for $p = 2$ and $n_1 = n_2$ is determined by using zonal polynomials up to the sixth degree. Tabulations of the power of V are given for various degrees of freedom and various alternatives. Comparison is made with the power of the R_1 test introduced in Chapter II. Percentage points for a function of V for $p = 2$ and $n_1 = n_2$ are provided.

2. The Distribution of V for p = 2

(1.1) may be rewritten as

$$V = |\underset{\sim}{S}_1(\underset{\sim}{S}_1+\underset{\sim}{S}_2)^{-1}|^{n_1/2}\,|\underset{\sim}{I}-\underset{\sim}{S}_1(\underset{\sim}{S}_1+\underset{\sim}{S}_2)^{-1}|^{n_2/2}.$$

Thus we can express V as the product

$$V = \prod_{i=1}^{p} \theta_i^{n_1/2}(1-\theta_i)^{n_2/2}, \tag{2.1}$$

where $0 < \theta_1 < \ldots < \theta_p < 1$ are the characteristic roots of $\underset{\sim}{S}_1(\underset{\sim}{S}_1+\underset{\sim}{S}_2)^{-1}$. Under the null hypothesis the distribution of the θ_i's has the form [40]:

$$(2.2)\quad f(\theta_1,\ldots,\theta_p) = C(p,m,n) \prod_{i=1}^{p} \theta_i^{m} (1-\theta_i)^{n} \prod_{i>j} (\theta_i-\theta_j),$$

where $m = (n_1-p-1)/2$, $n = (n_2-p-1)/2$ and $C(p,m,n)$ is given in (2.1) of Chapter I.

The moments of V follow readily from (2.1) and (2.2) since

$$E(V^h) = E\left(\prod_{i=1}^{p} \theta_i^{hn_1/2} (1-\theta_i)^{hn_2/2}\right)$$

$$= C(p,m,n) \int \underset{R}{\cdots} \int \prod_{i=1}^{p} \theta_i^{m+hn_1/2} (1-\theta_i)^{n+hn_2/2} \prod_{i>j} (\theta_i-\theta_j) \prod_{i=1}^{p} d\theta_i ,$$

where $\quad R = \{(\theta_1,\ldots,\theta_p) | 0 < \theta_1 < \ldots < \theta_p < 1\}.$

Thus

$$(2.3)\quad E(V^h) = c(p,m,n)/c(p,m+hn_1/2 , n+hn_2 /2).$$

When $p = 1$, V has the beta distribution with parameters $m+1$ and $n+1$. For $p = 2$ and $n_1 = n_2 = n_0$, the distribution of V can be found by transformation. In (2.2) set $m=n$, $p = 2$ and let $g = \theta_1\theta_2$ and $w = (1 - \theta_1)(1 - \theta_2)$ to get

(2.4) $$f(g,w) = C(2,m,m)\, g^m w^m,$$

$$0 \le w^{\frac{1}{2}} + g^{\frac{1}{2}} \le 1.$$

Now if we let $z = gw$ and $t = w$, (2.4) becomes

$$f(z,t) = C(2,m,m)\, z^m/t,$$

where $[\frac{1}{2} - (\frac{1}{4} - z^{\frac{1}{2}})^{\frac{1}{2}}]^2 \le t \le [\frac{1}{2} + (\frac{1}{4} - z^{\frac{1}{2}})^{\frac{1}{2}}]^2$ and $0 \le z \le 1/16$.

Now integrate out t to obtain

(2.5) $$f(z) = 2C(2,m,m)z^m \ln\{[1+(1-4z^{\frac{1}{2}})^{\frac{1}{2}}]\ [1-(1-4z^{\frac{1}{2}})^{\frac{1}{2}}]\}$$

Finally (2.5) can be integrated by parts and a change of variable can be made to get the distribution of $z = \theta_1\theta_2(1-\theta_1)(1-\theta_2)$ as

(2.6) $$P(Z \le z) = [2/\beta(n_0-1,n_0-1)]\ z^{(n_0-1)/2} \ln\{(1+(1-4z^{\frac{1}{2}})^{\frac{1}{2}})/(1-(1-4z^{\frac{1}{2}})^{\frac{1}{2}})\}$$

$$+ 2\, I_{z_0}(n_0-1,n_0-1),$$

where

$$I_x(a,b) = [1/\beta(a,b)] \int_0^x t^{a-1}(1-t)^{b-1}\, dt$$

and $$z_0 = (1-(1-4z^{\frac{1}{2}})^{\frac{1}{2}})/2.$$

We need only note that $V = Z^{n_0/2}$ to obtain the distribution of V. This result agrees with Anderson [1].

3. The Distribution of V for $p = 4$

By using the expression for the moments of V given in Anderson [1] or (2.3) above, we have

$$E(V^h) = \prod_{i=1}^{p} \left\{ \prod_{j=1}^{2} \frac{\Gamma((n_j+hn_j+1-i)/2)}{\Gamma((n_j+1-i)/2)} \right\} \frac{\Gamma((\nu+1-i)/2)}{\Gamma((\nu+h\nu+1-i)/2)}$$

where $\nu = n_1+n_2$. Since $0 \leq V \leq 1$, the moments of V determine the distribution uniquely.

Now by using the duplication formula for the gamma function, namely,

$$\Gamma(\alpha+\tfrac{1}{2})\ \Gamma(\alpha+1) = \pi^{\frac{1}{2}}\ \Gamma(2\alpha+1)\ 2^{-2\alpha} ,$$

we can write the moments of V for $p = 2r$ as

$$E(V^h) = \prod_{j=1}^{r} \left\{ \prod_{i=1}^{2} \frac{\Gamma(n_i+hn_i+1-2j)}{\Gamma(n_i+1-2j)} \right\} \frac{\Gamma(\nu+1-2j)}{\Gamma(\nu+h\nu+1-2j)}$$

$$= \prod_{j=1}^{r} \frac{\beta(n_1+hn_1+1-2j, n_2+hn_2+1-2j)}{\beta(n_1+1-2j, n_2+1-2j)} \cdot \frac{\beta(\nu+h\nu+2-4j, 2j-1)}{\beta(\nu+2-4j, 2j-1)}$$

$$= \prod_{j=1}^{r} E(\{X_j^{n_1}(1-X_j)^{n_2}\}^h)\ E(\{Y_j^{n_1+n_2}\}^h)$$

where

$$(3.1) \qquad X_j \sim \beta(n_1+1-2j, n_2+1-2j),$$

$$Y_j \sim \beta(n_1+n_2+2-4j, 2j-1),$$

and all X_j's and Y_j's are independent. Thus if $p = 2r$, the distribution of V is the same as the distribution of the product

$$(3.2) \qquad \prod_{j=1}^{r} X_j^{n_1} (1-X_j)^{n_2} Y_j^{n_1+n_2}$$

where X_j's and Y_j's are defined in (3.1).

Here we will attempt to obtain the distribution of V for $r = 2$ ($p = 4$) and $n_1=n_2$. The following lemmas are needed in this connection.

Lemma 3.1. If X is distributed $\beta(a,a)$, then $4X(1-X)$ is distributed $\beta(a,1/2)$.

Proof: Let

$$f(x) = [\beta(a,a)]^{-1} x^{a-1}(1-x)^{a-1}$$

and let $y = 4x(1-x)$. The Jacobian is $1/(2(1-y)^{\frac{1}{2}})$ and $f(y) = [2\beta(a,a)]^{-1} (y/4)^{a-1} (1-y)^{-\frac{1}{2}} = [\beta(a,\frac{1}{2})]^{-1} y^{a-1}(1-y)^{-\frac{1}{2}}$.

Lemma 3.2. Let $X \sim \beta(a,b)$ and $Y \sim \beta(c,d)$, where X and Y are independent. Then the density of $Z = XY$ is

$$(3.3) \quad f(z) = K z^{c-1}(1-z)^{b+d-1} \sum_{i=0}^{f} \binom{f}{i}(-1)^i \beta(d,b+i)(1-z)^i,$$

where $0 < z < 1$, $K = 1/\{\beta(a,b)\beta(c,d)\}$, $f = a-c-d$, and the sum is finite if f is a non-negative integer and infinite otherwise.

<u>Proof</u>: Let $z = xy$ and $v = x$ with the Jacobian being $1/v$ and $0 < z < v < 1$. Then if the density of X and Y is

$$f(x,y) = [\beta(a,b)\ \beta(c_1 d)]\ x^{a-1}(1-x)^{b-1}\ y^{c-1}(1-y)^{d-1}$$

we get

$$f(z,v) = K\ z^{c-1}v^{a-c-d}(1-v)^{b-1}(v-z)^{d-1},$$

where $\qquad K = [\beta(a,b)\ \beta(c,d)]^{-1}.$

Writing $v^{a-c-d} = (1-(1-v))^{a-c-d} = \sum_{i=0}^{f} \binom{f}{i} (-1)^i (1-v)^i,$

where $f = a-c-d$, and integrating v from z to 1 we get

$$\int_z^1 f(z, v)dv = K\ z^{c-1} \sum_{i=0}^{f} \binom{f}{i}(-1)^i \int_z^1 (1-v/z)^{b+i-1}(v-z)^{d-1}\ dv.$$

But

$$\int_z^1 (1-v)^{b+i-1}(v-z)^{d-1}dv = (1-z)^{b+d+i+1} \int_0^1 (1-x)^{b+i-1}x^{d-1}\ dx$$

(Let $v = z+(1-z)x$.).

Thus

$$f(z) = K\, z^{c-1}(1-z)^{b+d-1} \sum_{i=0}^{f} \binom{f}{i}(-1)^i\, \beta(d,b+i)(1-z)^i.$$

Lemma 3.3. Let Z have the density (3.3) and let $X \sim \beta(g,h)$ where Z and X are independent. Then the density of $W = XZ$ is given by:

$$(3.4)\quad f(w) = K_1 w^{g-1}(1-w)^{b+d+h-1} \sum_{i=0}^{f_1} \sum_{j=0}^{f_2} \binom{f_1}{i}\binom{f_2}{j}(-1)^{i+j}\, \beta(d,b+i)\, \beta(h,b+d+i+j)(1-w)^{i+j},$$

where

$$K_1 = 1/\{\beta(a,b)\beta(c,d)\beta(g,h)\},\quad f_1 = a-c-d \quad\text{and}\quad f_2 = c-g-h.$$

Proof: The proof is similar to that of Lemma 3.2 and is omitted.

We now proceed to the density function of V for $p = 4$ and $n_1 = n_2 = n_0$. Using (3.2), we know that V has the same distribution as

$$X_1^{n_0}\,(1-X_1)^{n_0}\,Y_1^{2n_0}\,(1-X_2)^{n_0}\,Y_2^{2n_0},$$

where X_1, X_2, Y_1 and Y_2 are independent and have the following distributions:

$$X_1 \sim \beta(n_0-1,n_0-1)$$
$$X_2 \sim \beta(n_0-3,n_0-3)$$

$$Y_1 \sim \beta(2n_0-2,1)$$
$$Y_2 \sim \beta(2n_0-6,3).$$

We will consider the distribution of $W = 16\, V^{1/n_0}$, i.e.,
$W = 16X_1(1-X_1)\, Y_1^2\, X_2(1-X_2)\, Y_2^2$.

By using Lemma 3.1 we have:

$$4X_1(1-X_1) \sim \beta(n_0-1,\tfrac{1}{2})$$

and

$$4X_2(1-X_2) \sim \beta(n_0-3,\tfrac{1}{2}).$$

Then using Lemma 3.2, the density of $X = 16X_1(1-X_1)X_2(1-X_2)$ is

$$(3.5)\quad f(x) = K'\, x^{n_0-4} \sum_{i=0}^{\infty} \binom{3/2}{i}(-1)^i\, \beta(\tfrac{1}{2},i+\tfrac{1}{2})(1-x)^i,$$

where $0 < x < 1$ and $K' = 1/\{\beta(n_0-1,\tfrac{1}{2})\beta(n_0-3,\tfrac{1}{2})\}$.

Lemma 3.2 also gives the density function of $Y_3 = Y_1Y_2$ as:

$$f(y_3) = K''\, y_3^{2n_0-7}\, (1-y_3)^3 \sum_{i=0}^{1} \binom{1}{i}(-1)^i\, \beta(3,i+1)(1-y_3)^i,$$

where $0 < y_3 < 1$ and $K'' = 1/\{\beta(2n_0-2,1)\beta(2n_0-6,3)\}$. The density of $Y = Y_3^2 = (Y_1Y_2)^2$ is

$$(3.6)\quad f(y) = (K''/2)\, y^{n_0-4}(\tfrac{1}{4}-2/3\, y^{\frac{1}{2}} + y/2 - y^2/12),$$
$$0 < y < 1.$$

Now since X and Y are independent the joint density of X and Y is the product of (3.5) and (3.6). Then make the transformation $W = XY$ and $Z = X$, where $0 < W < Z < 1$. Thus

$$f(w,z) = (K'K''/2)z^{n_0-4}(w/z)^{n_0-4}[\tfrac{1}{4} - 2/3\,(w/z)^{\frac{1}{2}} - w/(2z)$$

$$- 1/12\,(w/z)^2] \sum_{i=0}^{\infty} \alpha_i (1-z)^i,$$

where $\alpha_i = \binom{3/2}{i}(-1)^i\, \beta(\tfrac{1}{2}, i+\tfrac{1}{2})$. Finally we integrate z from w to 1 using the method illustrated in Lemma 3.2 and interchange the order of summation to find:

$$f(w) = K\, w^{n_0-4}\{\tfrac{1}{4}\sum_{i=0}^{\infty}\sum_{j=0}^{i} \alpha_j (1-w)^{i+1}/(i+1)$$

$$- \tfrac{2}{3}\, w^{\frac{1}{2}} \sum_{i=0}^{\infty}\sum_{j=0}^{i} \binom{-3/2}{j}(-1)^j\, \alpha_{i-j}(1-w)^{i+1}/(i+1)$$

$$+\tfrac{1}{2}\, w \sum_{i=0}^{\infty}\sum_{j=0}^{i} (j+1)\, \alpha_{i-j}(1-w)^{i+1}/(i+1)$$

$$- \tfrac{1}{24}\, w^2 \sum_{i=0}^{\infty}\sum_{j=0}^{i} (j+1)(j+2)\, \alpha_{i-j}\, (1-w)^{i+1}/(i+1)\},$$

where $K = K'\, K''/2$

4. The Non-null Distribution of V for p = 2

Khatri [25] has shown that the characteristic roots of $S_1 S_2^{-1}$, $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_p < \infty$, where $\lambda_i = \theta_i/(1-\theta_i)$, for testing $\Sigma_1 = \Sigma_2$ has the non-null distribution

$$(4.1)\quad C(p,m,n)\ |G|^{-(2m+p-1)/2}\ |\Lambda|^{m}\ |I + \Lambda|^{-\nu/2}$$

$$ {}_1F_0(\nu/2,\ I - G^{-1},\ \Lambda(I + \Lambda)^{-1}) \prod_{i>j} (\lambda_i - \lambda_i)$$

where $G = \text{diag}(\gamma_i)$, γ_i's being the characteristic roots of $\Sigma_1\Sigma_2^{-1}$, $\Lambda = \text{diag}(\lambda_i)$, $\nu = n_1+n_2$, m, n and $C(p,m,n)$ are as previously defined, and the hypergeometric function of a matrix argument is defined by James [20]:

$$ {}_sF_t(a_1,\ldots,a_s;\ b_1,\ldots,b_t;\ S,\ T)$$

$$= \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_\kappa \cdots (a_s)_\kappa}{(b_1)_\kappa \cdots (b_t)_\kappa} \frac{C_\kappa(S)\, C_\kappa(T)}{C_\kappa(I)\ k!}$$

where $a_1,\ldots a_s, b_1,\ldots,b_t$ are real or complex constants and the coefficient $(a)_\kappa$ is given by

$$(a)_\kappa = \prod_{i=1}^{p} (a - (i-1)/2)_{k_i}\ ,$$

where

$$(a)_k = a(a+1)\ \ldots\ (a+k-1)\ ,$$

and κ of k is a partition of k,

$$\kappa = (k_1,\ldots,k_p) \text{ , } k_1 \geq k_2 \geq \cdots \geq k_p \geq 0,$$

such that $k_1 + \ldots + k_p = k$, and the zonal polynomials, $C_\kappa(\underset{\sim}{S})$, are expressible in terms of elementary symmetric functions of the characteristic roots of $\underset{\sim}{S}$.

For $p = 2$, (4.1) becomes

$$C(2,m,n)\ (\gamma_1\gamma_2)^{-(2m+3)/2}(\lambda_1\lambda_2)^m\ [(1+\lambda_1)(1+\lambda_2)]^{-\nu/2}$$

$$\text{(4.2)} \qquad {}_1F_0(\nu/2,\ \underset{\sim}{I} - \underset{\sim}{G}^{-1},\ \underset{\sim}{\Lambda}(\underset{\sim}{I} + \underset{\sim}{\Lambda})^{-1})\ (\lambda_2-\lambda_1).$$

Pillai and Jayachandran [36] have shown that using zonal polynomials up to the sixth degree the joint distribution of $g = \theta_1\theta_2$ and $w = (1-\theta_1)(1-\theta_2)$ by making transformations in (4.2) can be written as

$$\text{(4.3)} \qquad f(g,w) = K'' \sum_{i+2j=k=0}^{6} C''_{ij}\ g^{j+m}(1-w+g)^i\ w^n + \ldots ,$$

$$0 \leq g^{\frac{1}{2}} + w^{\frac{1}{2}} \leq 1,$$

where the C''_{ij}'s are functions of γ_1, γ_2, n_1 and n_2 as given in [21] in terms of constants A''_{ij} also available in [21], and $K'' = (\gamma_1\gamma_2)^{-(2m+3)/2}\ C(2,m,n)$. The A''_{ij}'s and C_{ij}'''s are given in Appendix E.

In order to determine the distribution of V we set $n_1=n_2=n_0 (m=n)$ and seek the density of $Z = V^{2/n_0}$ $(= \theta_1\theta_2(1-\theta_1)(1-\theta_2))$. Thus in (4.3) the following change of variable is made: $z = g\,w$, $t = w$ with Jacobian $1/t$; and we obtain

$$f(t,z) = K'' \sum_{i+2j=k=0}^{6} C''_{ij}\, z^{j+m}\, t^{-j-1}\, (1-t+z/t)^{i} + \ldots ,$$

$$[\tfrac{1}{2} - (\tfrac{1}{4} - z^{\frac{1}{2}})^{\frac{1}{2}}]^{2} \leq t \leq [\tfrac{1}{2} + (\tfrac{1}{4} - z^{\frac{1}{2}})]^{2}$$

and $0 < z < 1/16$.

Then

$$(4.4)\quad f(z) = K'' \sum_{i+2j=k=0}^{6} C''_{ij}\, z^{j+m} \int_{a}^{b} t^{-j-1}\, (1-t+z/t)^{i}\, dt,$$

where $a = [\tfrac{1}{2} - (\tfrac{1}{4} - z^{\frac{1}{2}})^{\frac{1}{2}}]^{2}$ and $b = [\tfrac{1}{2} + (\tfrac{1}{4} - z^{\frac{1}{2}})^{\frac{1}{2}}]^{2}$.

If

$$(4.5)\qquad h_{ij}(z) = \int_{a}^{b} t^{-j-1}(1-t+z/t)^{i}\, dt ,$$

(4.4) can be written

$$f(z) = K'' \sum_{i+2j=k=0}^{6} C''_{ij}\, z^{j+m}\, h_{ij}(z).$$

Simplified expressions for $h_{ij}(z)$ may be found in Appendix F. The method for determining the $h_{ij}(z)$'s will be illustrated by considering $h_{10}(z)$. Now

$$h_{10}(z) = \int_a^b t^{-1}(1-t+z/t)dt$$

$$= \ln b/a - (b-a) - z\,(1/b - 1/a)$$

$$= \ln b/a,$$

where we have made use of the relation

$$b^n - a^n + z^n(1/b^n - 1/a^n) = 0$$

for n a positive integer.

The formulae

$$b^n - a^n - z^n(1/b^n - 1/a^n) = 2(b^n - a^n),$$

and

$$b^n - a^n = 2^{-2n+1} \sum_{i=1}^{n} \binom{2n}{2i-1} (1-4z^{\frac{1}{2}})^{i-\frac{1}{2}}$$

are also useful in determining the remaining $h_{ij}(z)$ functions.

Then the distribution of Z, say $F(z) = P(Z \le z)$, is written as

$$(4.6) \quad F(z) = K'' \sum_{i+2j=k=0}^{6} C''_{ij} \int_0^z x^{j+m}\, h_{ij}(x)dx.$$

Now integration of $x^{j+m} h_{ij}(x)$ involves integrals of the following types:

(4.7) $$\int_0^z x^q \ln (a/b)\, dx \quad \text{, where } q = m+j,$$

and

(4.8) $$\int_0^z x^r(1-4x^{\frac{1}{2}})^{s/2}dx, \text{ where } r > 0 \text{ and } s = 1,3,5,\dots .$$

For (4.7) integration by parts yields

(4.9) $$2z^{q+1}/(q+1) \ln \{[1+(1-4x^{\frac{1}{2}})^{\frac{1}{2}}]/[1-(1-4x^{\frac{1}{2}})^{\frac{1}{2}}]\} +2/(q+1)\, B_{z_0}(2q+2,2q+2),$$

where $z_0 = (1-(1-4z^{\frac{1}{2}})^{\frac{1}{2}})/2$ and $B_x(c,d) = \int_0^x t^{c-1}(1-t)^{d-1}\, dt.$

For (4.8) the change of variable $t = 4x^{\frac{1}{2}}$ gives

(4.10) $$2^{-2r-1} B_{z_1}(2r+2,s/2+1) ,$$

where $z_1 = 4z^{\frac{1}{2}}.$

Now since $Z = V^{2/n_0}$, we have the distribution of V under the alternative hypothesis $\Sigma_1 \neq \Sigma_2$.

5. Percentage Points for V and Power Function Tabulations for V and R_1

The lower tail percentage points of $Z = V^{2/n_0}$ have been determined for $p = 2$ using (2.6) for $n_0 = 3(1)20(2)30(5)50(10)100$ and $\alpha = .10$, .05, .025, .01 and .005. They are found in Table 3.1. The percentage points for $\alpha = .05$ were then used to determine the power of the LR test for various values of γ_1, γ_2 the characteristic roots of $\underset{\sim}{\Sigma}_1\underset{\sim}{\Sigma}_2^{-1}$ using the non-null distribution (4.5). Power tabulations have also been obtained for the R_1 test using the distribution given in Chapter II for the same non-null parameters and degrees of freedom as used for the LR criterion.

6. Power Comparisons of the LR and R_1 Tests

Table 3.2 provides a comparison of the power of the LR test and the R_1 test of the hypothesis $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$. In this connection if the LR test is based on samples of size n_0+1 from each population, the R_1 test, using the subsampling procedure, divides each sample into subsamples of size 4 and n_0-3. (The subsample of size $4(m=0)$ is necessitated by the complexity and slow convergence of the non-null distribution.) On this basis power comparisons of the two tests are found in Table 3.2 for $n_0=$ 11, 14 and 19 and for $\alpha = .05$.

As the table indicates, neither test is uniformly better than the other. The LR test shows better power than R_1 when the difference $(\gamma_2-\gamma_1)$ is larger. With both γ's greater than one (or less than one) and their sum constant, the R_1 test seems to provide more power than the LR test as the non-null parameters tend to be equal and as the subsample sizes come closer together. It should also be noted that as the subsample sizes move farther apart, as in the case of $n_0 = 19$, the R_1 test provides less power than the LR test for all deviations from the null hypothesis considered here. This leads one to conjecture that the R_1 test will show maximum power for a given sample size when the subsamples are of equal sizes.

One remaining point which can be brought forth is that if the sum or the product of the γ_i's remains constant, both tests will provide the greatest power when the difference $(\gamma_2 - \gamma_1)$ is largest.

One remaining point which can be brought forth is that if the sum or the product of the γ_i's remains constant, both tests will provide the greatest power when the difference $(\gamma_2 - \gamma_1)$ is largest.

Table 3.1. Lower Tail Percentage Points of Z.

n_0/α	.10	.05	.025	.01	.005
3	2.4209 (-3)	1.0762 (-3)	4.8628 (-4)	1.7315 (-4)	8.0062(-5)
4	7.3490 (-3)	4.3052 (-3)	2.5485 (-3)	1.2883 (-3)	7.7369(-4)
5	1.2681 (-2)	8.5119 (-3)	5.7573 (-3)	3.4612 (-3)	2.3659(-3)
6	1.7535 (-2)	1.2762 (-2)	9.3448 (-3)	6.2288 (-3)	4.5990(-3)
7	2.1734 (-2)	1.6690 (-2)	1.2881 (-2)	9.1938 (-3)	7.1445(-3)
8	2.5318 (-2)	2.0198 (-2)	1.6183 (-2)	1.2127 (-2)	9.7734(-3)
9	2.8378 (-2)	2.3294 (-2)	1.9193 (-2)	1.4916 (-2)	1.2353(-2)
10	3.1005 (-2)	2.6020 (-2)	2.1909 (-2)	1.7515 (-2)	1.4815(-2)
11	3.3276 (-2)	2.8423 (-2)	2.4351 (-2)	1.9912 (-2)	1.7129(-2)
12	3.5253 (-2)	3.0550 (-2)	2.6547 (-2)	2.2111 (-2)	1.9284(-2)
13	3.6988 (-2)	3.2441 (-2)	2.8524 (-2)	2.4125 (-2)	2.1284(-2)
14	3.8521 (-2)	3.4130 (-2)	3.0310 (-2)	2.5969 (-2)	2.3135(-2)
15	3.9884 (-2)	3.5647 (-2)	3.1927 (-2)	2.7661 (-2)	2.4847(-2)
16	4.1104 (-2)	3.7014 (-2)	3.3398 (-2)	2.9214 (-2)	2.6432(-2)
17	4.2200 (-2)	3.8252 (-2)	3.4739 (-2)	3.0644 (-2)	2.7900(-2)
18	4.3191 (-2)	3.9379 (-2)	3.5966 (-2)	3.1963 (-2)	2.9262(-2)
19	4.4091 (-2)	4.0407 (-2)	3.7092 (-2)	3.3181 (-2)	3.0528(-2)
20	4.4912 (-2)	4.1350 (-2)	3.8130 (-2)	3.4311 (-2)	3.1707(-2)
22	4.6354 (-2)	4.3016 (-2)	3.9975 (-2)	3.6336 (-2)	3.3832(-2)
24	4.7579 (-2)	4.4441 (-2)	4.1564 (-2)	3.8097 (-2)	3.5694(-2)
26	4.8632 (-2)	4.5674 (-2)	4.2948 (-2)	3.9641 (-2)	3.7335(-2)
28	4.9547 (-2)	4.6751 (-2)	4.4162 (-2)	4.1005 (-2)	3.8792(-2)
30	5.0349 (-2)	4.7699 (-2)	4.5236 (-2)	4.2218 (-2)	4.0093(-2)
35	5.1980 (-2)	4.9638 (-2)	4.7444 (-2)	4.4731 (-2)	4.2805(-2)
40	5.3225 (-2)	5.1129 (-2)	4.9154 (-2)	4.6696 (-2)	4.4938(-2)
45	5.4208 (-2)	5.2312 (-2)	5.0517 (-2)	4.8271 (-2)	4.6658(-2)
50	5.5002 (-2)	5.3272 (-2)	5.1628 (-2)	4.9563 (-2)	4.8073(-2)
60	5.6208 (-2)	5.4736 (-2)	5.3330 (-2)	5.1553 (-2)	5.0263(-2)
70	5.7080 (-2)	5.5800 (-2)	5.4572 (-2)	5.3014 (-2)	5.1877(-2)
80	5.7740 (-2)	5.6608 (-2)	5.5518 (-2)	5.4131 (-2)	5.3116(-2)
90	5.8257 (-2)	5.7241 (-2)	5.6263 (-2)	5.5013 (-2)	5.4097(-2)
100	5.8672 (-2)	5.7752 (-2)	5.6864 (-2)	5.5728 (-2)	5.4893(-2)
120	5.9299 (-2)	5.8525 (-2)	5.7775 (-2)	5.6813 (-2)	5.6104(-2)
140	5.9750 (-2)	5.9081 (-2)	5.8432 (-2)	5.7599 (-2)	5.6983(-2)
160	6.0089 (-2)	5.9501 (-2)	5.8929 (-2)	5.8194 (-2)	5.7649(-2)
180	6.0354 (-2)	5.9829 (-2)	5.9318 (-2)	5.8660 (-2)	5.8172(-2)
200	6.0566 (-2)	6.0092 (-2)	5.9631 (-2)	5.9035 (-2)	5.8594(-2)

The numbers in parentheses indicate the power of 10 by which the tabulated values are to be multiplied.

Table 3.2. Comparison of the LR and R_1 Criteria for Testing $H_0:\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$ for p = 2 and α = .05.

		n_0 = 11		n_0 = 14		n_0 = 19	
γ_1	γ_2	LR	R_1	LR	R_1	LR	R_1
1.0	1.01	.05001	.050008	.05002	.05001	.05002	.05001
1.0	1.05	.05028	.05020	.05038	.05024	.05056	.05027
1.01	1.04	.05019	.05018	.05026	.05022	.05037	.05026
1.02	1.03	.05014	.05018	.05019	.05022	.05028	.05025
1.025	1.025	.05013	.05018	.05018	.05022	.05027	.05025
1.0	1.1	.05108	.05075	.05147	.05091	.05213	.05104
1.05	1.05	.05053	.05069	.05072	.05084	.05106	.05096
1.0	1.1025	.0513	.0508	.0517	.0510	.0524	.0511
.98	.98	.05009	.05012	.05012	.05014	.05018	.05017
.97	..99	.05012	.05012	.05016	.05015	.05023	.05017
.96	1.0	.05020	.05014	.05027	.05018	.05039	.05019
.95	1.05	.0507	.0501	.0509	.0501	.0513	.0501
.96	1.04	.0505	.05006	.0506	.05007	.0508	.05008
.98	1.02	.0501	.05002	.0501	.05002	.0502	.05002
1.0	1.2	.0540	.0528	.0554	.0534	.0580	.0538
1.1	1.1	.0520	.0527	.0528	.0533	.0541	.0537
1.2	1.2	.0584	.0595	.0613	.0615	.0652	.0631
1.18	1.18	.0569	.0580	.0593	.0596	.0633	.0610
1.15	1.15	.0549	.0557	.0566	.0570	.0595	.0580

CHAPTER IV

THE MAX TRACE-RATIO TEST OF THE HYPOTHESIS

H_0': $\Sigma_1 = \cdots = \Sigma_k = \lambda \Sigma_0$

1. Introduction and Summary

Let x_{ij} (p x 1), $i = 1,\ldots,k$ and $j = 1,\ldots,n_i+1$, be a random sample from $N(\mu_i, \Sigma_i)$ where μ_i and Σ_i are unknown. Let S_i (p x p) be the sample sum of products matrix from the i^{th} population, i.e.,

$$S_i = \sum_{j=1}^{n_i+1} (x_{ij} - \bar{x}_i)(x_{ij} - \bar{x}_i)',$$

where
$$\bar{x}_i = \sum_{j=1}^{n_i+1} x_{ij}/(n_i+1).$$

S_i has the Wishart distribution $W(p,n_i,\Sigma_i)$. To test the hypothesis

$$H_0': \Sigma_1 = \cdots = \Sigma_k = \lambda \Sigma_0 ,$$

where Σ_0 is given and λ is unspecified, we consider the following test statistic:

$$R_2 = \{ \max_{i \leq i \leq k} T_i\}/\{ \min_{i \leq j \leq k} T_j\},$$

where $T_i = \text{tr}\ \Sigma_0^{-1} S_i/n_i$. The hypothesis H_0' is a multipopulation version of the sphericity test $\Sigma = \lambda\ \Sigma_0$. The critical region of size α to reject H_0' is $\{S_i,\ i=1,\ldots,k | R_2 > K_\alpha\}$.

It is known that if $\Sigma_i = \Sigma_0$ then $n_i T_i/\lambda = \text{tr}\ \Sigma_0^{-1} S_i$ has the $\chi^2_{pn_i}$ distribution (the chi-square distribution with pn_i degrees of freedom). Thus the distribution of R_2 is the same as that of the F_{max} statistic introduced by Hartley [16] as a shortcut test of the equality of variances from k univariate normal populations. Hartley [16] has given approximate distributions and percentage points for F_{max} when the sample sizes are the same and for $k = 2(1)12$. Further approximate tabulations of percentage points have been carried out by H. A. David [9] for equal sample sizes.

In this Chapter the exact distribution of R_2(or F_{max}) is obtained for k = 2,3 and 4 and unequal degrees of freedom. Using these exact distributions, percentage points have been obtained for k = 2 and 3 and various degrees of freedom. The distribution of R_2 under the alternative hypothesis is also considered for k = 2. The power of the test has been calculated for various alternatives.

2. Preliminary Remarks

In the max trace-ratio test described above, which is based on the union-intersection approach to testing hypotheses as is the max U-ratio test, we see that each T_i can be used to test $\Sigma_i = \Sigma_0$, where Σ_0 is known. When the ratio T_i/T_j is formed, we can use it to test $\Sigma_i = a_{ij}\Sigma_0$ and $\Sigma_j = a_{ij}\ \Sigma_0$, where a_{ij} is an unknown constant and disappears in the null distribution of the ratio. These two equalities are equivalent to:

$\underset{\sim}{\Sigma}_i = \underset{\sim}{\Sigma}_j = a_{ij}\,\underset{\sim}{\Sigma}_0$.

Now if $Q_{ij} = T_i/T_j$, $i \neq j$, we have:

$$T_{(1)}/T_{(k)} = 1/\max_{i,j} Q_{ij} = \min_{i,j} Q_{ij} \leq Q_{ij} \leq \max Q_{ij} = T_{(k)}/T_{(1)},$$

where $T_{(1)} \leq T_{(2)} \leq \cdots \leq T_{(k)}$ are the ordered T_i's. Thus all Q_{ij} lie between $1/R_2$ and R_2 $(=T_{(k)}/T_{(1)})$, and if $R_2 < K_\alpha$, we have that $\underset{\sim}{\Sigma}_i = a_{ij}\,\underset{\sim}{\Sigma}_0$ and $\underset{\sim}{\Sigma}_j = a_{ij}\,\underset{\sim}{\Sigma}_0$ simultaneously for $i,j = 1,\ldots,k$, $i \neq j$. This is equivalent to $\underset{\sim}{\Sigma}_i = \underset{\sim}{\Sigma}_j = a_{ij}\,\underset{\sim}{\Sigma}_0$ for $i,j = 1,\ldots,k$. Now however $a_{ij} = a_{ji}$, and a little algebra shows that (1) all the a_{ij}'s are equal (say to λ, unknown) and (2) $\underset{\sim}{\Sigma}_1 = \cdots = \underset{\sim}{\Sigma}_k = \lambda\,\underset{\sim}{\Sigma}_0$. Thus R_2 can be used to test H_0'.

It should also be noted that if $p = 1$ H_0' reduces to the hypothesis of testing the equality of variances from k univariate populations, which is the hypothesis considered by Hartley [16].

3. The Joint Distribution of $T_{(1)}/T_{(k)},\ldots,T_{(k-1)}/T_{(k)}$

Since n_iT_i/λ is distributed as $\chi^2_{pn_i}$ R_2 is the ratio of the largest to the smallest of k independent $\chi^2_{\nu_i}$ random variables divided by ν_i, where $\nu_i = pn_i$. Thus let us consider $X_1,X_2,\ldots,X_k$ to be k independent χ^2 random variables with $\nu_1,\ldots,\nu_k$ degrees of freedom respectively. The density of X_i is:

$$f_i(x_i) = [\Gamma(\nu_i/2)2^{\nu_i/2}]^{-1}\, x_i^{\nu_i/2 - 1}\, e^{-x_i/2}, \quad 0 < x_i < \infty.$$

Let $Y_i = X_i/\nu_i$, $i=1,\ldots,k$, then Y_i has the density function

$$f_i(y_i) = [r_i^{r_i}/\Gamma(r_i)]\, y_i^{r_i-1}\, e^{-r_i y_i},\ 0 < y_i < \infty,$$

where $\quad r_i = \nu_i/2.$

If $Y_{(1)} < Y_{(2)} < \ldots < Y_{(k)}$ denote the ordered Y_i's, the joint density of $Y_{(1)},\ldots,Y_{(k)}$ becomes

$$f(y_{(1)},\ldots,y_{(k)}) = \{\prod_{i=1}^{k} r_i^{r_i}/\Gamma(r_i)\} \sum_{\sigma} \exp(-\sum_{i=1}^{k} r_{\sigma(i)}\, y_{(i)}) \prod_{i=1}^{k} y_{(i)}^{r_{\sigma(i)}-1},$$

$$0 < y_{(1)} < \ldots < y_{(k)} < \infty,$$

where the summation is over all permutations $\sigma = (\sigma_1,\ldots,\sigma_k)$ of $(1,2,\ldots,k)$. For the sake of convenience we will consider the density function of $Y_{(1)}/Y_{(k)} = 1/R_2$. Now make the transformation

$$M_i = y_{(i)}/y_{(k)}\ ,\ i=1,\ldots,k-1\ ,$$

$$t = y_{(k)}.$$

The Jacobian of transformation is t^{k-1}. Then

$$f(M_1,\ldots,M_{k-1},t) = \{\prod_{i=1}^{k} r_i^{r_i}/\Gamma(r_i)\} \sum_{\sigma} \{\exp[-t(\sum_{i=1}^{k-1} r_{\sigma(i)}M_i + r_{\sigma(k)})]$$

$$\prod_{i=1}^{k-1} M_i^{r_{\sigma(i)}-1}\}\ t^{r-1}$$

where $r = \sum_{i=1}^{k} r_i$ and $0 < M_1 < \ldots < M_{k-1} < 1$, $0 < t < \infty$.

To integrate out t from 0 to ∞ we use the fact that

$$\int_0^\infty e^{-bt}\, t^{a-1}\, dt = \Gamma(a)/b^a.$$

Thus

$$(3.1)\quad f(M_1,\ldots,M_{k-1}) = C_k \sum_{\sigma} \{\prod_{i=1}^{k-1} M_i^{r_{\sigma(i)}-1}\}/(r_{\sigma(k)}+\sum_{i=1}^{k-1} r_{\sigma(i)} M_i)^r ,$$

where $$C_k = \Gamma(r) \prod_{i=1}^{k} \{r_i^{r_i}/\Gamma(r_i)\}.$$

To obtain the density function of M_1 we need to integrate out the remaining variables $M_2,\ldots,M_{k-1}$, where $0 < M_1 < M_2 < \ldots < M_{k-1} < 1$. From the distribution of M_1 we obtain the distribution of R_2 through the relation

$$P(R_2 \le x) = 1 - P(M_1 \le 1/x).$$

The distribution of M_1 will be treated in the following sections for $k = 2, 3$ and 4.

4. The Distribution of M_1 for k=2

Let us denote the density function of M_1 by $f_k(M_1)$ and the distribution function by $F_k(x) = P(M_1 \leq x)$. Then for k=2, (3.1) yields:

$$f_2(M_1) = C_2 \sum_{\sigma} M_1^{r_{\sigma(1)}-1} / [r^r_{\sigma(2)} (1+r_{\sigma(1)}M_1/r_{\sigma(2)})^r] ,$$

$$0 < M_1 < 1.$$

Then

$$(4.1) \quad F_2(x) = C_2 \sum_{\sigma} \int_0^x \frac{M_1^{r_{\sigma(1)}-1} \, dM_1}{r^r_{\sigma(2)}(1+r_{\sigma(1)}M_1/r_{\sigma(2)})^r} .$$

Upon making the transformation

$$t = \{r_{\sigma(1)}M_1/r_{\sigma(2)}\}/\{1+r_{\sigma(1)}M_1/r_{\sigma(2)}\},$$

we have

$$(4.2) \quad F_2(x) = C_2 \sum_{\sigma} r_{\sigma(1)}^{-r_{\sigma(1)}} r_{\sigma(2)}^{-r_{\sigma(2)}} \int_0^{x/(r_{\sigma(2)}/r_{\sigma(1)}+x)} t^{r_{\sigma(1)}-1} (1-t)^{r_{\sigma(2)}-1} dt.$$

$$= \sum_{\sigma} I_{x/(r_{\sigma(2)}/r_{\sigma(1)}+x)} (r_{\sigma(1)}, r_{\sigma(2)}),$$

where $I_y(a,b) = [\beta(a,b)]^{-1} \int_0^y t^{a-1} (1-t)^{b-1} dt.$

We will also provide an alternate expression for $F_2(x)$ by way of the following expression which will be needed in the next section and which has been used in Chapter II, Section 2.

Let q be a positive integer and let $0 \leq a < b < \infty$ and $d > 0$. Then

$$(4.3)\quad \int_a^b \frac{x^{q-1}}{(d+x)^p}\, dx = \frac{1}{p-1}\left\{\sum_{i=0}^{q-1} c(p,q,i)\left[\frac{a^{q-i-1}}{(d+a)^{p-i-1}} - \frac{b^{q-i-1}}{(d+b)^{p-i-1}}\right]\right\},$$

where $c(p,q,0) = 1$ and $c(p,q,i) = \prod_{j=1}^{i} (q-j)/(p-j-1)$.

Now if r_1 and r_2 are positive integers, i.e., the degrees of freedom are even, we can write (4.1) as

$$C_2(r-1)^{-1} \sum_{\sigma} r_{\sigma(1)}^{-r} \{c(r,r_{\sigma(1)},r_{\sigma(1)}-1)/(r_{\sigma(2)}/r_{\sigma(1)})^{r_{\sigma(2)}}$$

$$-\sum_{i=0}^{r_{\sigma(1)}-1} c(r,r_{\sigma(1)},i)\, x^{r_{\sigma(1)}-i-1}/(r_{\sigma(2)}/r_{\sigma(1)}+x)^{r-i-1}\}.$$

But $c(r_1 r_{\sigma(1)},r_{\sigma(1)}-1)/(r-1) = \Gamma(r_{\sigma(1)})\, \Gamma(r_{\sigma(2)})/\Gamma(r)$,

and further simplification yields

$$F_2(x) = 2 - C_2(r-1)^{-1} \sum_{\sigma} \sum_{i=0}^{r_{\sigma(1)}-1} c(r,r_{\sigma(1)},i) x^{r_{\sigma(1)}-i-1} /$$

$$\{r_{\sigma(1)}^{i+1}(r_{\sigma(2)}+r_{\sigma(1)}x)^{r-i-1}\}.$$

5. The Distribution of M_1 for $k=3$

When $k=3$ integration of M_2 from M_1 to 1 yields

$$(5.1)\quad f_3(M_1) = C_3 \sum_\sigma r_{\sigma(2)}^{-r} M_1^{r_{\sigma(1)}-1} \int_{M_1}^{1} \frac{M_2^{r_{\sigma(2)}-1}\, dM_2}{[(r_{\sigma(3)}+r_{\sigma(1)}M_1)/r_{\sigma(2)}+M_2]^r}.$$

Using (4.3) above and assuming the r_i are integers, we write (5.1) as

$$(5.2)\quad \frac{C_3}{r-1} \sum_\sigma r_{\sigma(2)}^{-r} M_1^{r_{\sigma(1)}-1} \Big\{ \sum_{i=0}^{r_{\sigma(2)}-1} c(r, r_{\sigma(2)}, i)\, M_1^{r_{\sigma(1)}-i-1} \Big/ [(r_{\sigma(3)}+r_{\sigma(1)}M_1)/r_{\sigma(2)}+M_1]^{r-i-1}$$

$$- \sum_{i=0}^{r_{\sigma(2)}-1} c(r, r_{\sigma(2)}, i) / [(r_{\sigma(3)}+r_{\sigma(1)}M_1)/r_{\sigma(2)}+1]^{r-i-1} \Big\}.$$

Upon rewriting (5.2) we have

$$f_3(M_1) = \frac{C_3}{r-1} \sum_\sigma r_{\sigma(2)}^{-r} \Big\{ \sum_{i=0}^{r_{\sigma(2)}-1} c(r, r_{\sigma(2)}, i) \left(\frac{r_{\sigma(2)}}{r_{\sigma(1)}+r_{\sigma(2)}}\right)^{r-i-1} \frac{M_1^{r_{\sigma(1)}+r_{\sigma(2)}-i-1}}{(Q_1+M_1)^{r-i-1}}$$

$$- \sum_{i=0}^{r_{\sigma(2)}-1} c(r, r_{\sigma(2)}, i) \left(\frac{r_{\sigma(2)}}{r_{\sigma(1)}}\right)^{r-i-1} \frac{M_1^{r_{\sigma(1)}-1}}{(Q_2+M_1)^{r-i-1}} \Big\}$$

where $Q_1 = r_{\sigma(3)}/(r_{\sigma(1)}+r_{\sigma(2)})$ and $Q_2 = (r_{\sigma(2)}+r_{\sigma(3)})/r_{\sigma(1)}$.

Now integrating M_1 from 0 to x and making a change of variable we find

$$F_3(x) = \frac{C_3}{r-1} \sum_\sigma r_{\sigma(2)}^{-r} \Big\{ \sum_{i=0}^{r_{\sigma(2)}-1} c(r, r_{\sigma(2)}, i) \left(\frac{r_{\sigma(2)}}{r_{\sigma(1)}+r_{\sigma(2)}} \right)^{r-i-1} Q_1^{-r_{\sigma(3)}}$$

$$\int_0^{\frac{x}{Q_1+x}} y^{r_{\sigma(1)}+r_{\sigma(2)}-i-2} (1-y)^{r_{\sigma(3)}-1} \, dy$$

$$- \sum_{i=0}^{r_{\sigma(2)}-1} c(r, r_{\sigma(2)}, i) \left(\frac{r_{\sigma(2)}}{r_{\sigma(1)}} \right)^{r-i-1} Q_2^{-(r_{\sigma(2)}+r_{\sigma(3)})-i-1}$$

$$\int_0^{\frac{x}{Q_2+x}} y^{r_{\sigma(1)}-1} (1-y)^{r_{\sigma(2)}+r_{\sigma(3)}-i-2} dy \Big\} .$$

Further simplification gives

$$F_3(x) = \frac{C_3}{r-1} \sum_\sigma \Big\{ \sum_{i=0}^{r_{\sigma(2)}-1} c(r, r_{\sigma(2)}, i) / \{ r_{\sigma(2)}^{i+1} r_{\sigma(3)}^{r_{\sigma(3)}} (r_{\sigma(1)}+r_{\sigma(2)})^{r_{\sigma(1)}+r_{\sigma(2)}-i} \}$$

$$B_{x/(Q_1+x)}(r_{\sigma(1)}+r_{\sigma(2)}-i-1, r_{\sigma(3)})$$

$$- \sum_{i=0}^{r_{\sigma(2)}-1} c(r, r_{\sigma(2)}, i) / \{ r_{\sigma(2)}^{i+1} \, r_{\sigma(1)}^{r_{\sigma(1)}} \, (r_{\sigma(2)}+r_{\sigma(3)})^{r_{\sigma(2)}+r_{\sigma(3)}-i-1} \}$$

$$B_{x/(Q_2+x)}(r_{\sigma(1)}, r_{\sigma(2)}+r_{\sigma(3)}-i-1) \Big\},$$

where $B_y(a,b) = \int_0^y t^{a-1} (1-t)^{b-1} \, dt.$

6. The Distribution of M_1 for $k=4$

By assuming that $r_i, i=1,..,4$ are integers and starting with (2.1) and integrating out M_3 and then M_2 using (4.3), we can write the distribution of M_1 for $k=4$ as

$$F_4(x)=\frac{C_4}{r-1}\sum_{\sigma} r_{\sigma(3)}^{-r}\sum_{i=0}^{r_{\sigma(3)}-1}\Big\{\sum_{j=0}^{r_{\sigma(2)}+r_{\sigma(3)}-i-1} D_{ij}^{(1)}(\underset{\sim}{r},\sigma)(K_{ij}^{(1)}(\underset{\sim}{r},\sigma)B_{ij}^{(1)}(\underset{\sim}{r},\sigma,x)$$

$$- K_{ij}^{(2)}(\underset{\sim}{r},\sigma)B_{ij}^{(2)}(\underset{\sim}{r},\sigma,x))$$

$$-\sum_{j=1}^{r_{\sigma(2)}-1} D_{ij}^{(2)}(\underset{\sim}{r},\sigma)(K_{ij}^{(3)}(\underset{\sim}{r},\sigma)B_{ij}^{(3)}(\underset{\sim}{r},\sigma,x)-K_{ij}^{(4)}(\underset{\sim}{r},\sigma)B_{ij}^{(4)}(\underset{\sim}{r},\sigma,x))\Big\},$$

where $\underset{\sim}{r} = (r_1,\ r_2,\ r_3,\ r_4)$,

$$D_{ij}^{(1)}(\underset{\sim}{r},\sigma) = c(r,r_{\sigma(3)},i)\ c(r-i-1,r_{\sigma(2)}+r_{\sigma(3)}-i-1,j),$$

$$D_{ij}^{(2)}(\underset{\sim}{r},\sigma) = c(r,r_{\sigma(3)},i)\ c(r-i-1,\ r_{\sigma(2)},j\ ),$$

$$K_{ij}^{(1)}(\underset{\sim}{r},\sigma) = 1/\{r_{\sigma(4)}^{r_{\sigma(4)}}(r_{\sigma(2)}+r_{\sigma(3)})^{j+1}$$

$$(r_{\sigma(1)}+r_{\sigma(2)}+r_{\sigma(3)})^{r_{\sigma(1)}+r_{\sigma(2)}+r_{\sigma(3)}-i-j-2}\},$$

$$K_{ij}^{(2)}(\underset{\sim}{r},\sigma) = 1/\{r_{\sigma(1)}^{r_{\sigma(1)}}\ (r_{\sigma(2)}+r_{\sigma(3)})^{j+1}$$

$$(r_{\sigma(2)}+r_{\sigma(3)}+r_{\sigma(4)})^{r_{\sigma(2)}+r_{\sigma(3)}+r_{\sigma(4)}-i-j-2}\},$$

$$K_{ij}^{(3)}(\underset{\sim}{r},\sigma) = 1/\{r_{\sigma(2)}^{j+1}(r_{\sigma(1)}+r_{\sigma(2)})^{r_{\sigma(1)}+r_{\sigma(2)}-j-1}$$

$$(r_{\sigma(3)}+r_{\sigma(4)})^{r_{\sigma(3)}+r_{\sigma(4)}-i-1}\},$$

$$K_{ij}^{(4)}(\underset{\sim}{r},\sigma) = 1/\{r_{\sigma(1)}^{r_{\sigma(1)}}\ r_{\sigma(2)}^{j+1}(r_{\sigma(2)}+r_{\sigma(3)}+r_{\sigma(4)})^{r_{\sigma(2)}+r_{\sigma(3)}+r_{\sigma(4)}-i-j-2}\},$$

$$B_{ij}^{(1)}(\underset{\sim}{r},\sigma,x) = B_{x/(Q_1+x)}(r_{\sigma(1)}+r_{\sigma(2)}+r_{\sigma(3)}-i-j-1, r_{\sigma(4)}-1),$$

where $Q_1 = r_{\sigma(4)}/(r_{\sigma(1)}+r_{\sigma(2)}+r_{\sigma(3)})$,

$$B_{ij}^{(2)}(\underset{\sim}{r},\sigma,x) = B_{x/(Q_2+x)}(r_{\sigma(1)}, r_{\sigma(2)}+r_{\sigma(3)}+r_{\sigma(4)}-i-j-2),$$

where $Q_2 = (r_{\sigma(2)}+r_{\sigma(3)}+r_{\sigma(4)})/r_{\sigma(1)}$,

$$B_{ij}^{(3)}(\underset{\sim}{r},\sigma,x) = B_{x/(Q_3+x)}(r_{\sigma(1)}+r_{\sigma(2)}-j-1, r_{\sigma(3)}+r_{\sigma(4)}-i-1),$$

where $Q_3 = (r_{\sigma(3)}+r_{\sigma(4)})/(r_{\sigma(1)}+r_{\sigma(2)})$,

and

$$B_{ij}^{(4)}(\underset{\sim}{r},\sigma,x) = B_{x/(Q_4+x)}(r_{\sigma(1)}, r_{\sigma(2)}+r_{\sigma(3)}+r_{\sigma(4)}-i-j-2),$$

where $Q_4 = Q_2$.

7. The Non-Null Distribution of R_2 for $k=2$

In this section we first consider the distribution of $nT/\lambda = \text{tr}\, \Sigma_0^{-1} S$, where S (p x p) has the distribution $W(p,n,\Sigma)$, and then obtain the non-null distribution of R_2 when $k=2$. We will use two results due to Box [5] which are stated as Theorems 7.1 and 7.2.

Theorem 7.1. If z (p x 1) has a p-variate normal distribution with mean 0 and covariance matrix Σ, and if $Q = z' M z$ is any real quadratic form of rank $r \leq p$, then Q is distributed like the quantity

$$X = \sum_{j=1}^{r} \gamma_j \chi_1^2$$

where each χ_1^2 variate is distributed independently of every other and the γ_j's are the r nonzero characteristic roots of ΣM.

Theorem 7.2. The exact distribution of $X = \sum_{j=1}^{r} \gamma_j \chi_{\nu_j}^2$, where $\nu_j=2g_j, j=1, \ldots, r$ are even integers, is a weighted finite sum of χ^2 distributions.

$$(7.1) \quad P(X > x_0) = \sum_{j=1}^{r} \sum_{s=1}^{g_j} \alpha_{js}\, P\{\chi_{2s}^2 > x_0/\gamma_j\} ,$$

where the α_{js}'s are constants involving only the γ_j's and are given by

$$\alpha_{js} = f_j^{(g_j-s)}(0)/(g_j-s)! ,$$

where $f_j^{(h)}(0)$ is obtained by differentiating $f_j(y)$ h times with respect to y and then putting y = 0 and

$$f_j(y) = \prod_{i \neq j}^{r} [(\gamma_j - \gamma_i)/\gamma_j + y\, \gamma_i/\gamma_j]^{-\nu_i/2} .$$

It can be shown (see Roy [40]) that $\underset{\sim}{S}$ can be written as $\underset{\sim}{Y}\,\underset{\sim}{Y}'$ where $\underset{\sim}{Y}(p \times n) = (\underset{\sim}{y}_1,\ldots,\underset{\sim}{y}_n)$, and the $\underset{\sim}{y}_i$'s are independently and identically distributed as $N(\underset{\sim}{0},\underset{\sim}{\Sigma})$. Thus the distributions of $\underset{\sim}{S}$ and $\underset{\sim}{Y}\,\underset{\sim}{Y}'$ are the same and

$$\begin{aligned} \operatorname{tr} \underset{\sim}{\Sigma}_0^{-1} \underset{\sim}{S} &= \operatorname{tr} \underset{\sim}{\Sigma}_0^{-1} \underset{\sim}{Y}\,\underset{\sim}{Y}' \\ &= \operatorname{tr} \underset{\sim}{\Sigma}_0^{-1} \sum_{i=1}^{n} \underset{\sim}{y}_i\, \underset{\sim}{y}_i' \\ &= \sum_{i=1}^{n} \operatorname{tr} \underset{\sim}{\Sigma}_0^{-1} \underset{\sim}{y}_i\, \underset{\sim}{y}_i' \\ &= \sum_{i=1}^{n} \underset{\sim}{y}_i' \underset{\sim}{\Sigma}_0^{-1} \underset{\sim}{y}_i . \end{aligned}$$

But Theorem 7.1 states that $\underset{\sim}{y}_i' \underset{\sim}{\Sigma}_0^{-1} \underset{\sim}{y}_i$ is distributed the same as $\sum_{j=1}^{p} \gamma_j \chi_1^2$ for each i where the γ_j's are the characteristic roots of $\underset{\sim}{\Sigma}\, \underset{\sim}{\Sigma}_0^{-1}$. Thus $\operatorname{tr} \underset{\sim}{\Sigma}_0^{-1} \underset{\sim}{S}$ has the same distribution as $\sum_{j=1}^{p} \gamma_j\, \chi_n^2$ since the sum of independent χ^2 random variables is a χ^2 random variable with degrees of freedom the sum of the degrees of freedom of the summands. (Note: if $\underset{\sim}{\Sigma}\, \underset{\sim}{\Sigma}_0^{-1} = \underset{\sim}{I}$, $\operatorname{tr} \underset{\sim}{\Sigma}_0^{-1} \underset{\sim}{S}$ is distributed χ_{pn}^2.)

In order to determine the distribution of $1/R_2$ we need the density function of $X = \sum_{j=1}^{r} \gamma_j \chi^2_{\nu_j}$ since we have shown above that $\operatorname{tr} \Sigma_0^{-1} S$ has the same distribution as X if $r \leq p$ and $\nu_i = n_i$. Thus we write (7.1) as

$$P(X \leq x) = \sum_{j=1}^{r} \sum_{s=1}^{g_j} \alpha_{js} P\{\chi^2_{2s} \leq x/\gamma_j\}, \quad \text{noting that} \sum_{j=1}^{r} \sum_{s=1}^{g_j} \alpha_{js} = 1.$$

(Let $x \to 0$ in (6.1).).

The density function is then

$$h(x) = \frac{d}{dx} P(X \leq x) = \sum_{j=1}^{r} \sum_{s=1}^{g_j} \alpha_{js}/\gamma_j \; f_{2s}(x/\gamma_j),$$

where $f_{2s}(t) = [\Gamma(s)\, 2^s]^{-1}\, t^{s-1}\, e^{-t/2}$, $0 < t < \infty$.

Now let $X_1 = \operatorname{tr} \Sigma_0^{-1} S_1/n_1$ and $X_2 = \operatorname{tr} \Sigma_0^{-1} S_2/n_2$ be independent with S_i having n_i degrees of freedom, $i = 1,2$. Let $\Sigma_1 \neq \Sigma_0$ and $\Sigma_2 \neq \Sigma_0$ be the covariance matrices associated with S_1 and S_2 respectively and let $\gamma_1,\ldots,\gamma_r$ $(r \leq p)$ and $\tau_1,\ldots,\tau_{r'}$ $(r' \leq p)$ be the characteristic roots of $\Sigma_1 \Sigma_0^{-1}/n_1$ and $\Sigma_2 \Sigma_0^{-1}/n_2$ respectively. Then the distribution of $Z = X_{(1)}/X_{(2)}$, where $X_{(1)} \leq X_{(2)}$ are the ordered X_i's, will enable us to determine the power of the test using R_2.

Thus let X_1 and X_2 have respectively the density functions

$$h_1(x_1) = \sum_{j=1}^{r} \sum_{s=1}^{g_j} \alpha_{js}/\gamma_j \; f_{2s}(x_1/\gamma_j)$$

and

$$h_2(x_2) = \sum_{j'=1}^{r'} \sum_{s'=1}^{g_{j'}} \beta_{j's'}/\tau_{j'}\, f_{2s'}(x_2/\tau_{j'}),$$

where the α_{js}'s and $\beta_{j's'}$'s are determined as in Theorem 7.2 and g_j = multiplicity of the j^{th} roots of $\Sigma_1 \Sigma_0^{-1}$ times $n_1/2$ and $g_{j'}$ is defined similarly. Let $0 < X_{(1)} \leq X_{(2)} < \infty$ be the ordered X_i's and let $Z = X_{(1)}/X_{(2)}$ and $W = X_{(2)}$. Then the joint density function of Z and W is

$$h(z,w) = \sum_{j=1}^{r} \sum_{j'=1}^{r'} \sum_{s=1}^{g_j} \sum_{s'=1}^{g_{j'}} \frac{\alpha_{js}\beta_{j's'}}{\gamma_j \tau_{j'}} w\, \{f_{2s}(zw/\gamma_j) f_{2s'}(w/\tau_{j'})$$

$$+ f_{2s}(w/\gamma_j) f_{2s'}(zw/\tau_{j'})\}.$$

Integrating w from 0 to ∞ yields

$$h(z) = \sum_{j=1}^{r} \sum_{j'=1}^{r'} \sum_{s=1}^{g_j} \sum_{s'=1}^{g_{j'}} \frac{\alpha_{js}\beta_{j's'}}{\beta(s,s')} \left\{ \left(\frac{\gamma_j}{\tau_{j'}}\right)^{s'} \frac{z^{s-1}}{(\gamma_j/\tau_{j'}+z)^{s+s'}}\right.$$

$$\left. + \left(\frac{\tau_{j'}}{\gamma_j}\right)^{s} \frac{z^{s'-1}}{(\tau_{j'}/\gamma_j+z)^{s+s'}} \right\}.$$

Thus

$$H(z) = \int_0^z h(x)dx$$

$$= \sum_{j=1}^{r} \sum_{j'=1}^{r'} \sum_{s=1}^{g_j} \sum_{s'=1}^{g_{j'}} \alpha_{js}\beta_{j's'} \{I_{x/(z+\gamma_j/\tau_{j'})}(s,s') + I_{z/(z+\tau_{j'}/\gamma_j)}(s,s')\}.$$

The distribution of $1/R_2$ under the alternative hypothesis when the n_i are even and $k = 2$ can be expressed as a finite linear combination of incomplete beta functions.

8. Computation of Percentage Points

Using the expressions derived in Sections 3 and 4 for the distribution of $1/R_2$, percentage points have been computed for R_2 for $k = 2$, $\nu_i = 2(1)20(2)30(5)50(10)100$, $i = 1, 2$ and $\alpha = .10, .05, .025, .01, .005$, and for $k = 3$, $\nu_i = 2(2)12(4)30(10)60(20)140$, $i = 1,2,3$ and $\alpha = .05$. These percentage points may be found in Tables 4.1 and 4.2 respectively.

9. Remarks on the Power of R_2 for $k = 2$ and $p = 2$

Table 4.3 provides some tabulations of the power of R_2 for testing H_0' for $k = 2$, $p = 2$ and some selected alternatives. These tabulations indicate several things. First we note that when the characteristic roots of $\Sigma_1 \Sigma_0^{-1}$ and $\Sigma_2 \Sigma_0^{-1}$ are equal, i.e., $\gamma_1' = \tau_1'$ and $\gamma_2' = \tau_2'$ ($\gamma_1' \neq \gamma_2'$ and $\tau_1' \neq \tau_2'$), the power of R_2 is rather low even for larger differences in the γ_i''s and τ_i''s. Also for (n_1+n_2) a constant, the power appears to be better when $n_1=n_2$. In fact for a given pair of γ_i''s and τ_i''s, the power of R_2 is often greater with equal degrees of freedom than unequal degrees of freedom. The test also appears to be unbiased at least for equal degrees of freedom.

Table 4.1. Upper Percentage Points of R_2 for k = 2.
$\alpha = .10$

ν_2/ν_1	2	3	4	5	6	7	8	9	10	11	12
2	19.00										
3	13.67	9.2766									
4	11.87	7.772	6.3882								
5	11.02	7.033	5.7050	5.0503							
6	10.55	6.601	5.3013	4.6625	4.2839						
7	10.26	6.320	5.0362	4.4068	4.0337	3.7870					
8	10.07	6.125	4.8495	4.2259	3.8564	3.6119	3.4381				
9	9.94	5.982	4.7115	4.0914	3.7241	3.4811	3.3082	3.1789			
10	9.85	5.874	4.6055	3.9876	3.6219	3.3798	3.2075	3.0785	2.9782		
11	9.78	5.790	4.5218	3.9052	3.5404	3.2989	3.1270	2.9982	2.8981	2.8179	
12	9.73	5.723	4.4540	3.8383	3.4741	3.2330	3.0613	2.9326	2.8325	2.7523	2.6866
13	9.69	5.668	4.3982	3.7829	3.4191	3.1782	3.0066	2.8780	2.7778	2.6976	2.6318
14	9.66	5.623	4.3515	3.7363	3.3727	3.1320	2.9604	2.8317	2.7315	2.6512	2.5854
15	9.63	5.584	4.3119	3.6966	3.3331	3.0924	2.9208	2.7921	2.6919	2.6115	2.5455
16	9.61	5.552	4.2778	3.6624	3.2989	3.0582	2.8866	2.7578	2.6575	2.5770	2.5109
17	9.60	5.524	4.2483	3.6326	3.2691	3.0283	2.8566	2.7278	2.6273	2.5468	2.4806
18	9.58	5.500	4.2225	3.6064	3.2428	3.0020	2.8303	2.7013	2.6008	2.5201	2.4538
19	9.57	5.479	4.1997	3.5833	3.2195	2.9786	2.8068	2.6778	2.5771	2.4963	2.4300
20	9.56	5.460	4.1795	3.5627	3.1988	2.9578	2.7859	2.6567	2.5560	2.4751	2.4086
22	9.55	5.429	4.1452	3.5276	3.1634	2.9221	2.7500	2.6207	2.5197	2.4386	2.3719
24	9.54	5.404	4.1172	3.4988	3.1342	2.8928	2.7204	2.5909	2.4897	2.4084	2.3416
26	9.53	5.383	4.0940	3.4748	3.1099	2.8682	2.6956	2.5659	2.4645	2.3831	2.3160
28	9.53	5.366	4.0744	3.4545	3.0892	2.8473	2.6745	2.5446	2.4431	2.3614	2.2942
30	9.52	5.352	4.0577	3.4371	3.0715	2.8293	2.6563	2.5263	2.4246	2.3428	2.2754
35	9.51	5.324	4.0251	3.4029	3.0365	2.7937	2.6204	2.4899	2.3878	2.3057	2.2380
40	9.51	5.304	4.0013	3.3777	3.0106	2.7674	2.5937	2.4629	2.3605	2.2781	2.2101
45	9.50	5.289	3.9831	3.3585	2.9908	2.7472	2.5731	2.4420	2.3394	2.2567	2.1885
50	9.50	5.277	3.9689	3.3433	2.9751	2.7311	2.5568	2.4254	2.3226	2.2397	2.1713
60	9.50	5.260	3.9480	3.3208	2.9518	2.7073	2.5325	2.4007	2.2976	2.2143	2.1456
70	9.50	5.249	3.9334	3.3050	2.9354	2.6904	2.5153	2.3832	2.2798	2.1963	2.1274
80	9.50	5.240	3.9227	3.2933	2.9232	2.6779	2.5025	2.3702	2.2665	2.1828	2.1137
90	9.50	5.234	3.9144	3.2843	2.9138	2.6682	2.4926	2.3601	2.2563	2.1724	2.1031
100	9.50	5.229	3.9079	3.2772	2.9063	2.6605	2.4847	2.3520	2.2480	2.1640	2.0946

Table 4.1. Continued

$\alpha = .10$

ν_2/ν_1	13	14	15	16	17	18	19	20	22	24	26
13	2.5769										
14	2.5304	2.4837									
15	2.4904	2.4436	2.4034								
16	2.4557	2.4088	2.3685	2.3335							
17	2.4253	2.3783	2.3379	2.3027	2.2719						
18	2.3984	2.3513	2.3108	2.2755	2.2446	2.2172					
19	2.3744	2.3272	2.2866	2.2513	2.2203	2.1928	2.1683				
20	2.3530	2.3057	2.2650	2.2295	2.1984	2.1708	2.1462	2.1242			
22	2.3161	2.2686	2.2277	2.1921	2.1608	2.1331	2.1083	2.0861	2.0478		
24	2.2856	2.2379	2.1968	2.1611	2.1296	2.1017	2.0768	2.0545	2.0159	1.9838	
26	2.2598	2.2120	2.1708	2.1349	2.1033	2.0753	2.0503	2.0278	1.9889	1.9566	1.9292
28	2.2379	2.1899	2.1486	2.1125	2.0808	2.0527	2.0275	2.0049	1.9658	1.9333	1.9057
30	2.2189	2.1708	2.1294	2.0932	2.0613	2.0331	2.0078	1.9851	1.9458	1.9131	1.8854
35	2.1812	2.1328	2.0910	2.0546	2.0225	1.9939	1.9684	1.9455	1.9058	1.8726	1.8445
40	2.1531	2.1044	2.0624	2.0257	1.9934	1.9646	1.9389	1.9158	1.8757	1.8422	1.8138
45	2.1313	2.0824	2.0402	2.0033	1.9708	1.9418	1.9160	1.8926	1.8523	1.8185	1.7898
50	2.1139	2.0648	2.0224	1.9854	1.9527	1.9236	1.8976	1.8741	1.8335	1.7995	1.7706
60	2.0879	2.0385	1.9958	1.9585	1.9256	1.8963	1.8700	1.8463	1.8053	1.7708	1.7415
70	2.0693	2.0198	1.9769	1.9394	1.9063	1.8768	1.8503	1.8264	1.7850	1.7503	1.7207
80	2.0555	2.0058	1.9627	1.9250	1.8918	1.8621	1.8355	1.8115	1.7698	1.7348	1.7050
90	2.0448	1.9949	1.9517	1.9139	1.8805	1.8507	1.8240	1.7999	1.7580	1.7228	1.6928
100	2.0362	1.9862	1.9429	1.9050	1.8714	1.8416	1.8148	1.7906	1.7485	1.7131	1.6829

ν_2/ν_1	28	30	35	40	45	50	60	70	80	90	100
28	1.8821										
30	1.8615	1.8409									
35	1.8204	1.7994	1.7571								
40	1.7893	1.7681	1.7252	1.6928							
45	1.7651	1.7436	1.7002	1.6674	1.6415						
50	1.7456	1.7239	1.6801	1.6468	1.6207	1.5995					
60	1.7163	1.6942	1.6497	1.6157	1.5890	1.5673	1.5343				
70	1.6952	1.6729	1.6277	1.5933	1.5661	1.5440	1.5103	1.4857			
80	1.6792	1.6567	1.6111	1.5762	1.5487	1.5262	1.4920	1.4669	1.4477		
90	1.6668	1.6441	1.5981	1.5629	1.5350	1.5123	1.4776	1.4521	1.4326	1.4171	
100	1.6568	1.6340	1.5876	1.5521	1.5240	1.5011	1.4659	1.4401	1.4202	1.4045	1.3917

Table 4.1. (Continued) $\alpha = .05$

ν_2/ν_1	2	3	4	5	6	7	8	9	10	11	12
2	39.00										
3	26.00	15.439									
4	22.36	12.40	9.6045								
5	20.91	11.04	8.3490	7.1464							
6	20.23	10.30	7.6510	6.4752	5.8198						
7	19.89	9.853	7.2136	6.0511	5.4045	4.9949					
8	19.70	9.560	6.9173	5.7608	5.1193	4.7130	4.4333				
9	19.60	9.358	6.7056	5.5509	4.9119	4.5076	4.2292	4.0260			
10	19.5	9.213	6.5480	5.3927	4.7548	4.3516	4.0740	3.8713	3.7168		
11	19.5	9.105	6.4270	5.2697	4.6320	4.2293	3.9520	3.7496	3.5952	3.4737	
12	19.5	9.023	6.3316	5.1717	4.5336	4.1309	3.8538	3.6514	3.4971	3.3755	3.2773
13	19.5	8.959	6.2550	5.0919	4.4531	4.0502	3.7730	3.5706	3.4162	3.2946	3.1962
14	19.5	8.908	6.1922	5.0259	4.3860	3.9828	3.7055	3.5030	3.3485	3.2267	3.1282
15	19.5	8.867	6.1401	4.9705	4.3295	3.9258	3.6483	3.4455	3.2909	3.1690	3.0703
16	19.5	8.833	6.0962	4.9233	4.2812	3.8770	3.5991	3.3962	3.2414	3.1193	3.0205
17	19.5	8.805	6.0588	4.8828	4.2394	3.8347	3.5565	3.3533	3.1983	3.0760	2.9771
18	19.5	8.781	6.0267	4.8477	4.2031	3.7977	3.5192	3.3158	3.1605	3.0381	2.9390
19	19.5	8.761	5.9989	4.8170	4.1712	3.7652	3.4863	3.2826	3.1272	3.0046	2.9053
20	19.4	8.744	5.9745	4.7899	4.1429	3.7363	3.4570	3.2531	3.0975	2.9747	2.8752
22	19.4	8.716	5.9341	4.7444	4.0952	3.6874	3.4074	3.2029	3.0469	2.9237	2.8240
24	19.4	8.694	5.9019	4.7078	4.0565	3.6476	3.3669	3.1619	3.0054	2.8820	2.7819
26	19.4	8.677	5.8758	4.6778	4.0245	3.6146	3.3332	3.1277	2.9709	2.8471	2.7467
28	19.4	8.663	5.8542	4.6527	3.9977	3.5868	3.3047	3.0988	2.9416	2.1875	2.7169
30	19.4	8.651	5.8361	4.6315	3.9749	3.5631	3.2804	3.0741	2.9166	2.7922	2.6913
35	19.4	8.629	5.8016	4.5906	3.9306	3.5168	3.2328	3.0255	2.8672	2.7422	2.6408
40	19.4	8.614	5.7772	4.5612	3.8985	3.4830	3.1980	2.9899	2.8310	2.7054	2.6035
45	19.4	8.603	5.7590	4.5391	3.8742	3.4574	3.1714	2.9626	2.8032	2.6772	2.5749
50	19.4	8.594	5.7450	4.5219	3.8552	3.4373	3.1505	2.9412	2.7812	2.6548	2.5523
60	19.4	8.582	5.7249	4.4970	3.8275	3.4078	3.1198	2.9095	2.7488	2.6218	2.5187
70	19.4	8.573	5.7111	4.4799	3.8083	3.3872	3.0982	2.8872	2.7260	2.5985	2.4950
80	19.4	8.567	5.7011	4.4673	3.7941	3.3720	3.0824	2.8708	2.7091	2.5812	2.4774
90	19.4	8.563	5.6935	4.4578	3.7833	3.3604	3.0702	2.8581	2.6961	2.5679	2.4638
100	19.4	8.559	5.6875	4.4502	3.7748	3.3512	3.0605	2.8481	2.6858	2.5573	2.4530

Table 4.1. (Continued) $\alpha = .05$

ν_2/ν_1	13	14	15	16	17	18	19	20	22	24	26
13	3.1150										
14	3.0469	2.9786									
15	2.9889	2.9204	2.8621								
16	2.9389	2.8703	2.8118	2.7614							
17	2.8953	2.8266	2.7680	2.7174	2.6733						
18	2.8571	2.7882	2.7294	2.6787	2.6345	2.5956					
19	2.8232	2.7542	2.6953	2.6445	2.6001	2.5611	2.5265				
20	2.7930	2.7238	2.6648	2.6139	2.5694	2.5303	2.4955	2.4645			
22	2.7415	2.6720	2.6128	2.5616	2.5169	2.4775	2.4426	2.4114	2.3579		
24	2.6991	2.6294	2.5699	2.5185	2.4736	2.4341	2.3990	2.3675	2.3137	2.2693	
26	2.6637	2.5938	2.5341	2.4825	2.4374	2.3976	2.3623	2.3308	2.2767	2.2319	2.1943
28	2.6336	2.5635	2.5036	2.4518	2.4065	2.3666	2.3312	2.2995	2.2451	2.2001	2.1622
30	2.6078	2.5375	2.4774	2.4254	2.3800	2.3399	2.3043	2.2725	2.2178	2.1726	2.1345
35	2.5568	2.4861	2.4255	2.3732	2.3274	2.2870	2.2511	2.2189	2.1637	2.1179	2.0794
40	2.5192	2.4480	2.3871	2.3345	2.2884	2.2477	2.2115	2.1791	2.1234	2.0772	2.0382
45	2.4902	2.4187	2.3576	2.3046	2.2583	2.2173	2.1809	2.1483	2.0922	2.0456	2.0063
50	2.4672	2.3955	2.3341	2.2809	2.2343	2.1932	2.1566	2.1237	2.0673	2.0204	1.9809
60	2.4331	2.3610	2.2992	2.2456	2.1987	2.1572	2.1203	2.0872	2.0301	1.9828	1.9427
70	2.4091	2.3366	2.2744	2.2206	2.1734	2.1317	2.0945	2.0611	2.0037	1.9559	1.9155
80	2.3912	2.3184	2.2560	2.2019	2.1545	2.1126	2.0752	2.0417	1.9839	1.9358	1.8951
90	2.3774	2.3044	2.2418	2.1875	2.1399	2.0979	2.0603	2.0266	1.9685	1.9202	1.8792
100	2.3664	2.2932	2.2305	2.1760	2.1283	2.0861	2.0484	2.0146	1.9563	1.9077	1.8665

	28	30	35	40	45	50	60	70	80	90	100
28	2.1299										
30	2.1020	2.0739									
35	2.0465	2.0180	1.9611								
40	2.0049	1.9761	1.9185	1.8752							
45	1.9727	1.9436	1.8853	1.8415	1.8073						
50	1.9470	1.9176	1.8587	1.8144	1.7798	1.7520					
60	1.9084	1.8786	1.8188	1.7737	1.7383	1.7099	1.6668				
70	1.8808	1.8507	1.7902	1.7444	1.7085	1.6796	1.6356	1.6038			
80	1.8601	1.8297	1.7686	1.7224	1.6860	1.6567	1.6120	1.5796	1.5549		
90	1.8440	1.8134	1.7518	1.7051	1.6684	1.6387	1.5935	1.5605	1.5354	1.5156	
100	1.8311	1.8004	1.7383	1.6913	1.6542	1.6242	1.5785	1.5452	1.5197	1.4996	1.4833

Table 4.1. (Continued) $\alpha = .025$

ν_2/ν_1	2	3	4	5	6	7	8	9	10	11	12
2	79.00										
3	49.34	25.218									
4	42.73	19.47	14.147								
5	40.64	17.18	11.989	9.9046							
6	39.9	16.05	10.872	8.8166	7.7463						
7	39.6	15.42	10.211	8.1621	7.0998	6.4581					
8	39.5	15.05	9.7855	7.7317	6.6714	6.0320	5.6071				
9	39.4	14.81	9.4954	7.4311	6.3693	5.7302	5.3057	5.0043			
10	39.4	14.64	9.2885	7.2114	6.1463	5.5062	5.0815	4.7799	4.5552		
11	39.4	14.53	9.1357	7.0454	5.9758	5.3342	4.9087	4.6066	4.3815	4.2075	
12	39.4	14.45	9.0195	6.9164	5.8420	5.1983	4.7718	4.4691	4.2435	4.0690	3.9301
13	39.4	14.39	8.9291	6.8139	5.7345	5.0885	4.6608	4.3574	4.1313	3.9563	3.8171
14	39.4	14.34	8.8570	6.7309	5.6466	4.9983	4.5693	4.2651	4.0384	3.8629	3.7233
15	39.4	14.30	8.7987	6.6626	5.5736	4.9229	4.4926	4.1875	3.9602	3.7843	3.6442
16	39.4	14.27	8.7505	6.6056	5.5121	4.8591	4.4275	4.1215	3.8936	3.7172	3.5767
17	39.4	14.24	8.7103	6.5574	5.4598	4.8045	4.3715	4.0648	3.8362	3.6594	3.5185
18	39.4	14.22	8.6762	6.5163	5.4147	4.7573	4.3231	4.0154	3.7863	3.6090	3.4677
19	39.3	14.20	8.6470	6.4807	5.3756	4.7162	4.2807	3.9722	3.7425	3.5647	3.4231
20	39.3	14.18	8.6217	6.4498	5.3414	4.6800	4.2433	2.9340	3.7037	3.5255	3.3835
22	39.3	14.15	8.5802	6.3987	5.2844	4.6195	4.1805	3.8698	3.6383	3.4593	3.3166
24	39.3	14.13	8.5475	6.3583	5.2391	4.5710	4.1300	3.8178	3.5854	3.4055	3.2622
26	39.3	14.11	8.5211	6.3257	5.2021	4.5313	4.0885	3.7750	3.5416	3.3611	3.2172
28	39.3	14.10	8.4993	6.2987	5.1715	4.4983	4.0538	3.7392	3.5050	3.3237	3.1793
30	39.3	14.08	8.4809	6.2762	5.1458	4.4704	4.0245	3.7088	3.4738	3.2919	3.1469
35	39.3	14.06	8.4457	6.2331	5.0966	4.4168	3.9677	3.6498	3.4131	3.2299	3.0838
40	39.3	14.04	8.4204	6.2024	5.0615	4.3784	3.9269	3.6072	3.3691	3.1847	3.0378
45	39.3	14.02	8.4012	6.1795	5.0353	4.3496	3.8962	3.5750	3.3358	3.1505	3.0028
50	39.3	14.01	8.3862	6.1617	5.0149	4.3273	3.8723	3.5499	3.3097	3.1237	2.9753
60	39.3	13.99	8.3640	6.1357	4.9855	4.2948	3.8375	3.5133	3.2716	3.0844	2.9350
70	39.3	13.98	8.3485	6.1178	4.9651	4.2724	3.8135	3.4879	3.2451	3.0570	2.9069
80	39.3	13.97	8.3369	6.1046	4.9502	4.2561	3.7959	3.4693	3.2257	3.0369	2.8862
90	39.3	13.96	8.3280	6.0945	4.9389	4.2436	3.7825	3.4551	3.2108	3.0215	2.8703
100	39.3	13.96	8.3209	6.0864	4.9299	4.2338	3.7719	3.4439	3.1991	3.0093	2.8577

Table 4.1. Continued $\alpha = .025$

ν_2/ν_1	13	14	15	16	17	18	19	20	22	24	26
13	3.7036										
14	3.6094	3.5149									
15	3.5300	3.4352	3.3552								
16	3.4622	3.3670	3.2867	3.2180							
17	3.4036	3.3082	3.2276	3.1586	3.0990						
18	3.3525	3.2568	3.1759	3.1067	3.0468	2.9945					
19	3.3076	3.2115	3.1304	3.0610	3.0009	2.9484	2.9021				
20	3.2677	3.1714	3.0901	3.0204	2.9601	2.9074	2.8609	2.8196			
22	3.2002	3.1034	3.0216	2.9516	2.8909	2.8379	2.7911	2.7495	2.6787		
24	3.1453	3.0480	2.9658	2.8954	2.8344	2.7810	2.7340	2.6921	2.6209	2.5625	
26	3.0997	3.0020	2.9195	2.8487	2.7874	2.7338	2.6864	2.6443	2.5726	2.5139	2.4649
28	3.0614	2.9633	2.8803	2.8093	2.7477	2.6938	2.6462	2.6038	2.5318	2.4727	2.4233
30	3.0286	2.9302	2.8469	2.7756	2.7137	2.6596	2.6117	2.5692	2.4967	2.4373	2.3876
35	2.9646	2.8653	2.7813	2.7094	2.6469	2.5923	2.5439	2.5009	2.4277	2.3675	2.3172
40	2.9178	2.8178	2.7333	2.6608	2.5979	2.5428	2.4941	2.4507	2.3767	2.3160	2.2651
45	2.8822	2.7817	2.6966	2.6237	2.5604	2.5049	2.4558	2.4121	2.3376	2.2763	2.2250
50	2.8541	2.7532	2.6677	2.5944	2.5308	2.4750	2.4256	2.3817	2.3066	2.2449	2.1932
60	2.8130	2.7113	2.6251	2.5512	2.4870	2.4307	2.3809	2.3365	2.2607	2.1982	2.1459
70	2.7842	2.6819	2.5953	2.5209	2.4563	2.3996	2.3494	2.3047	2.2282	2.1652	2.1124
80	2.7630	2.6602	2.5732	2.4984	2.4335	2.3765	2.3260	2.2810	2.2041	2.1407	2.0874
90	2.7467	2.6436	2.5562	2.4812	2.4160	2.3587	2.3080	2.2628	2.1855	2.1217	2.0681
100	2.7338	2.6304	2.5428	2.4675	2.4020	2.3446	2.2937	2.2483	2.1706	2.1066	2.0527

ν_2/ν_1	28	30	35	40	45	50	60	70	80	90	100
28	2.3815										
30	2.3455	2.3093									
35	2.2745	2.2377	2.1649								
40	2.2219	2.1847	2.1109	2.0560							
45	2.1814	2.1438	2.0691	2.0136	1.9705						
50	2.1492	2.1113	2.0359	1.9797	1.9361	1.9013					
60	2.1013	2.0628	1.9863	1.9291	1.8846	1.8490	1.7954				
70	2.0673	2.0284	1.9509	1.8929	1.8478	1.8115	1.7569	1.7176			
80	2.0420	2.0028	1.9245	1.8659	1.8201	1.7834	1.7279	1.6879	1.6577		
90	2.0224	1.9829	1.9040	1.8448	1.7986	1.7614	1.7053	1.6647	1.6339	1.6097	
100	2.0067	1.9670	1.8876	1.8279	1.7813	1.7438	1.6871	1.6460	1.6147	1.5902	1.5703

Table 4.1. Continued $\alpha = .01$

ν_2/ν_1	2	3	4	5	6	7	8	9	10	11	12
2	199.0										
3	116.0	47.467									
4	103.0	34.99	23.155								
5	100.2	30.82	19.022	14.940							
6	99.6	29.06	17.102	13.008	11.073						
7	99.4	28.21	16.068	11.931	9.9832	8.8854					
8	99.4	27.76	15.457	11.266	9.3004	8.1939	7.4959				
9	99.4	27.48	15.070	10.828	8.8411	7.7248	7.0208	6.5411			
10	99.4	27.30	14.810	10.523	8.5161	7.3894	6.6795	6.1956	5.8467		
11	99.4	27.17	14.626	10.304	8.2768	7.1400	6.4242	5.9363	5.5844	5.3197	
12	99.3	27.08	14.491	10.139	8.0952	6.9487	6.2271	5.7355	5.3808	5.1139	4.9062
13	99.3	27.00	14.388	10.013	7.9538	6.7983	6.0712	5.5759	5.2187	4.9498	4.7405
14	99.3	26.93	14.306	9.9134	7.8412	6.6775	5.9452	5.4466	5.0869	4.8161	4.6053
15	99.3	26.88	14.239	9.8331	7.7498	6.5788	5.8417	5.3399	4.9779	4.7054	4.4932
16	99.3	26.83	14.184	9.7670	7.6745	6.4969	5.7554	5.2505	4.8864	4.6122	4.3988
17	99.3	26.79	14.137	9.7118	7.6114	6.4280	5.6825	5.1748	4.8087	4.5330	4.3183
18	99.3	26.75	14.096	9.6648	7.5580	6.3694	5.6203	5.1099	4.7419	4.4647	4.2489
19	99.3	26.72	14.061	9.6245	7.5121	6.3190	5.5665	5.0538	4.6840	4.4055	4.1886
20	99.3	26.69	14.029	9.5893	7.4724	6.2753	5.5198	5.0048	4.6333	4.3535	4.1356
22	99.3	26.64	13.976	9.5310	7.4069	6.2032	5.4425	4.9236	4.5491	4.2670	4.0473
24	99.2	26.60	13.933	9.4844	7.3552	6.1464	5.3815	4.8592	4.4821	4.1979	3.9765
26	99.2	26.56	13.897	9.4462	7.3133	6.1005	5.3321	4.8069	4.4276	4.1416	3.9188
28	99.2	26.53	13.866	9.4141	7.2786	6.0627	5.2913	4.7638	4.3824	4.0949	3.8708
30	99.2	26.51	13.839	9.3868	7.2493	6.0309	5.2572	4.7276	4.3445	4.0556	3.8303
35	99.2	26.45	13.786	9.3332	7.1927	5.9701	5.1919	4.6583	4.2719	3.9801	3.7524
40	99.2	26.41	13.746	9.2937	7.1517	5.9265	5.1455	4.6091	4.2202	3.9263	3.6968
45	99.2	26.38	13.715	9.2632	7.1205	5.8937	5.1107	4.5723	4.1815	3.8860	3.6550
50	99.2	26.35	13.690	9.2389	7.0959	5.8680	5.0836	4.5437	4.1516	3.8547	3.6227
60	99.2	26.32	13.652	9.2026	7.0594	5.8303	5.0441	4.5022	4.1081	3.8094	3.5757
70	99.2	26.29	13.625	9.1767	7.0336	5.8039	5.0166	4.4735	4.0781	3.7781	3.5432
80	99.2	26.27	13.605	9.1572	7.0144	5.7843	4.9963	4.4524	4.0561	3.7553	3.5195
90	99.2	26.25	13.590	9.1422	6.9994	5.7692	4.9807	4.4362	4.0392	3.7378	3.5014
100	99.2	26.24	13.577	9.1300	6.9875	5.7571	4.9684	4.4234	4.0259	3.7240	3.4871

Table 4.1. (Continued) $\alpha = .01$

ν_2/ν_1	13	14	15	16	17	18	19	20	22	24	26
13	4.5733										
14	4.4368	4.2993									
15	4.3235	4.1850	4.0698								
16	4.2281	4.0886	3.9726	3.8747							
17	4.1466	4.0062	3.8895	3.7909	3.7066						
18	4.0763	3.9352	3.8178	3.7186	3.6337	3.5603					
19	4.0151	3.8732	3.7552	3.6554	3.5701	3.4962	3.4318				
20	3.9613	3.8188	3.7001	3.5999	3.5141	3.4398	3.3749	3.3178			
22	3.8714	3.7276	3.6079	3.5067	3.4201	3.3451	3.2795	3.2217	3.1246		
24	3.7994	3.6545	3.5338	3.4318	3.3444	3.2687	3.2025	3.1442	3.0461	2.9667	
26	3.7404	3.5945	3.4730	3.3702	3.2821	3.2059	3.1392	3.0804	2.9814	2.9013	2.8352
28	3.6913	3.5445	3.4222	3.3188	3.2301	3.1533	3.0861	3.0269	2.9271	2.8464	2.7797
30	3.6499	3.5023	3.3792	3.2752	3.1860	3.1087	3.0411	2.9815	2.8810	2.7996	2.7324
35	3.5700	3.4207	3.2962	3.1908	3.1005	3.0222	2.9537	2.8932	2.7912	2.7086	2.6402
40	3.5128	3.3621	3.2364	3.1300	3.0387	2.9596	2.8903	2.8291	2.7260	2.6423	2.5730
45	3.4698	3.3180	3.1914	3.0841	2.9921	2.9123	2.8424	2.7806	2.6765	2.5919	2.5219
50	3.4364	3.2837	3.1563	3.0483	2.9557	2.8753	2.8049	2.7427	2.6376	2.5524	2.4817
60	3.3879	3.2339	3.1052	2.9962	2.9026	2.8213	2.7501	2.6871	2.5807	2.4943	2.4226
70	3.3544	3.1994	3.0699	2.9601	2.8657	2.7838	2.7119	2.6484	2.5410	2.4537	2.3812
80	3.3299	3.1742	3.0440	2.9336	2.8387	2.7562	2.6839	2.6200	2.5118	2.4238	2.3507
90	3.3112	3.1549	3.0243	2.9134	2.8180	2.7352	2.6625	2.5982	2.4894	2.4008	2.3272
100	3.2965	3.1398	3.0087	2.8974	2.8017	2.7185	2.6455	2.5809	2.4716	2.3826	2.3086
ν_2/ν_1	28	30	35	40	45	50	60	70	80	90	100
28	2.7236										
30	2.6759	2.6278									
35	2.5827	2.5337	2.4377								
40	2.5147	2.4649	2.3674	2.2958							
45	2.4629	2.4125	2.3137	2.2411	2.1854						
50	2.4222	2.3713	2.2713	2.1978	2.1414	2.0967					
60	2.3621	2.3104	2.2087	2.1337	2.0761	2.0303	1.9622				
70	2.3201	2.2677	2.1646	2.0885	2.0299	1.9833	1.9138	1.8642			
80	2.2890	2.2361	2.1319	2.0549	1.9955	1.9482	1.8776	1.8271	1.7892		
90	2.2650	2.2118	2.1067	2.0289	1.9689	1.9210	1.8494	1.7982	1.7597	1.7296	
100	2.2461	2.1924	2.0866	2.0082	1.9476	1.8993	1.8269	1.7751	1.7360	1.7055	1.6809

Table 4.1. (Continued)

$\alpha = .005$

ν_2/ν_1	2	3	4	5	6	7	8	9	10	11	12
2	399.0										
3	223.6	76.056									
4	203.1	54.40	33.303								
5	199.5	48.13	26.774	20.178							
6	199.0	45.83	24.011	17.296	14.354						
7	199.0	44.81	22.640	15.789	12.789	11.188					
8	199.0	44.28	21.883	14.911	11.855	10.223	9.2358				
9	199.0	43.95	21.427	14.360	11.252	9.5919	8.5868	7.9243			
10	199.0	43.71	21.128	13.993	10.841	9.1553	8.1342	7.4603	6.9875		
11	199.0	43.54	20.917	13.736	10.548	8.8398	7.8043	7.1205	6.6402	6.2869	
12	199.0	43.39	20.760	13.548	10.332	8.6041	7.5556	6.8629	6.3760	6.0176	5.7440
13	199.0	43.27	20.636	13.404	10.167	8.4228	7.3629	6.6622	6.1695	5.8065	5.5292
14	199.0	43.17	20.536	13.292	10.038	8.2802	7.2102	6.5023	6.0042	5.6372	5.3566
15	199.0	43.09	20.451	13.201	9.9351	8.1655	7.0867	6.3724	5.8696	5.4988	5.2154
16	199.0	43.01	20.380	13.125	9.8505	8.0717	6.9853	6.2652	5.7581	5.3840	5.0979
17	199.0	42.94	20.317	13.061	9.7799	7.9936	6.9008	6.1756	5.6645	5.2874	4.9988
18	199.0	42.88	20.262	13.006	9.7201	7.9277	6.8293	6.0996	5.5850	5.2051	4.9143
19	199.0	42.83	20.213	12.958	9.6687	7.8714	6.7683	6.0346	5.5168	5.1343	4.8415
20	199.0	42.78	20.169	12.915	9.6239	7.8226	6.7155	5.9783	5.4577	5.0729	4.7782
22	199.0	42.69	20.093	12.844	9.5494	7.7425	6.6291	5.8861	5.3605	4.9717	4.6737
24	199.0	42.62	20.030	12.785	9.4897	7.6793	6.5612	5.8137	5.2842	4.8920	4.5912
26	199.0	42.56	19.977	12.735	9.4406	7.6279	6.5065	5.7555	5.2228	4.8278	4.5246
28	199.0	42.51	19.931	12.693	9.3993	7.5852	6.4614	5.7077	5.1723	4.7750	4.4698
30	199.0	42.47	19.892	12.657	9.3639	7.5490	6.4235	5.6677	5.1302	4.7309	4.4239
35	199.0	42.38	19.812	12.584	9.2942	7.4788	6.3506	5.5913	5.0500	4.6470	4.3367
40	199.0	42.31	19.752	12.530	9.2424	7.4274	6.2981	5.5367	4.9931	4.5877	4.2750
45	199.0	42.26	19.705	12.488	9.2023	7.3879	6.2582	5.4957	4.9504	4.5434	4.2290
50	199.0	42.21	19.667	12.454	9.1703	7.3566	6.2267	5.4635	4.9173	4.5090	4.1934
60	199.0	42.15	19.611	12.402	9.1222	7.3099	6.1802	5.4163	4.8688	4.4591	4.1418
70	199.0	42.10	19.570	12.366	9.0878	7.2767	6.1473	5.3831	4.8350	4.4244	4.1062
80	199.0	42.07	19.540	12.338	9.0620	7.2517	6.1227	5.3584	4.8100	4.3988	4.0800
90	199.0	42.04	19.516	12.317	9.0419	7.2323	6.1036	5.3393	4.7907	4.3792	4.0599
100	199.0	42.02	19.497	12.300	9.0257	7.2168	6.0883	5.3241	4.7753	4.3636	4.0440

Table 4.1. (Continued) $\alpha = .005$

ν_2/ν_1	13	14	15	16	17	18	19	20	22	24	26
13	5.3113										
14	5.1360	4.9584									
15	4.9923	4.8127	4.6651								
16	4.8726	4.6911	4.5420	4.4175							
17	4.7716	4.5884	4.4379	4.3121	4.2056						
18	4.6853	4.5006	4.3487	4.2218	4.1143	4.0221					
19	4.6108	4.4247	4.2716	4.1437	4.0352	3.9422	3.8616				
20	4.5459	4.3585	4.2044	4.0754	3.9661	3.8724	3.7911	3.7200			
22	4.4387	4.2490	4.0928	3.9622	3.8514	3.7563	3.6738	3.6016	3.4813		
24	4.3538	4.1621	4.0043	3.8722	3.7601	3.6638	3.5803	3.5071	3.3852	3.2877	
26	4.2852	4.0917	3.9324	3.7990	3.6858	3.5885	3.5041	3.4301	3.3067	3.2080	3.1273
28	4.2286	4.0337	3.8730	3.7385	3.6242	3.5261	3.4408	3.3661	3.2415	3.1417	3.0601
30	4.1812	3.9850	3.8232	3.6876	3.5725	3.4735	3.3876	3.3122	3.1865	3.0857	3.0033
35	4.0910	3.8921	3.7279	3.5903	3.4733	3.3727	3.2853	3.2086	3.0805	2.9777	2.8935
40	4.0271	3.8262	3.6603	3.5211	3.4027	3.3008	3.2122	3.1344	3.0045	2.9001	2.8145
45	3.9796	3.7772	3.6099	3.4695	3.3499	3.2470	3.1575	3.0789	2.9474	2.8418	2.7551
50	3.9428	3.7393	3.5709	3.4295	3.3091	3.2053	3.1150	3.0357	2.9030	2.7963	2.7087
60	3.8896	3.6845	3.5146	3.3717	3.2500	3.1450	3.0535	2.9732	2.8386	2.7302	2.6411
70	3.8528	3.6467	3.4758	3.3320	3.2093	3.1035	3.0112	2.9301	2.7941	2.6845	2.5943
80	3.8259	3.6191	3.4475	3.3030	3.1796	3.0731	2.9803	2.8986	2.7615	2.6510	2.5600
90	3.8054	3.5980	3.4259	3.2808	3.1570	3.0500	2.9567	2.8745	2.7367	2.6254	2.5337
100	3.7891	3.5813	3.4088	3.2634	3.1391	3.0318	2.9381	2.8556	2.7171	2.6053	2.5130
ν_2/ν_1	28	30	35	40	45	50	60	70	80	90	100
28	2.9920										
30	2.9345	2.8763									
35	2.8232	2.7637	2.6483								
40	2.7430	2.6824	2.5648	2.4795							
45	2.6826	2.6211	2.5016	2.4148	2.3489						
50	2.6354	2.5731	2.4521	2.3641	2.2972	2.2445					
60	2.5665	2.5031	2.3796	2.2896	2.2210	2.1670	2.0872				
70	2.5187	2.4544	2.3291	2.2375	2.1677	2.1126	2.0310	1.9734			
80	2.4836	2.4187	2.2918	2.1991	2.1283	2.0723	1.9893	1.9305	1.8867		
90	2.4568	2.3913	2.2633	2.1696	2.0979	2.0412	1.9570	1.8973	1.8527	1.8181	
100	2.4356	2.3697	2.2407	2.1462	2.0738	2.0165	1.9313	1.8708	1.8256	1.7903	1.7621

Table 4.2. The Upper 5% Points of R_2 for k = 3

$\nu_1 = 2$

$\nu_3 \backslash \nu_2$	2	4	6	8	10	12	14	18	22
2	87.49								
4	58.38	33.02							
6	55.67	30.03	26.88						
8	55.01	29.25	26.03	25.14					
10	54.69	28.92	25.68	24.76	24.37				
12	54.49	28.74	25.48	24.55	24.14	23.90			
14	54.35	28.61	25.34	24.40	23.98	23.73	23.56		
18	54.16	28.45	25.17	24.21	23.77	23.51	23.33	23.08	
22	54.03	28.34	25.06	24.09	23.64	23.37	23.18	22.92	22.75
26	53.94	28.27	24.98	24.00	23.54	23.26	23.07	22.80	22.62
30	53.88	28.22	24.92	23.94	23.47	23.19	22.99	22.71	22.53
40	53.77	28.13	24.82	23.83	23.35	23.06	22.85	22.56	22.37
50	53.70	28.07	24.76	23.76	23.28	22.98	22.76	22.47	22.26
60	53.66	28.04	24.72	23.72	23.23	22.92	22.71	22.40	22.19
80	53.61	27.99	24.67	23.66	23.16	22.85	22.63	22.32	22.10
100	53.57	27.96	24.64	23.62	23.13	22.81	22.59	22.27	22.05
120	53.55	27.94	24.62	23.60	23.10	22.78	22.55	22.23	22.01
140	53.53	27.93	24.61	23.58	23.08	22.76	22.53	22.21	21.98

$\nu_3 \backslash \nu_2$	26	30	40	50	60	80	100	120	140
26	22.49								
30	22.39	22.29							
40	22.22	22.11	21.92						
50	22.12	22.00	21.80	21.67					
60	22.04	21.92	21.71	21.58	21.48				
80	21.94	21.82	21.60	21.46	21.36	21.22			
100	21.88	21.76	21.53	21.38	21.28	21.13	21.04		
120	21.84	21.71	21.48	21.33	21.22	21.07	20.98	20.91	
140	21.81	21.68	21.45	21.29	21.18	21.03	20.93	20.86	20.81

Table 4.2. (Continued)

$\nu_1 = 4$

ν_2/ν_3	4	6	8	10	12	14	18	22	26
4	15.46								
6	12.98	10.58							
8	12.17	9.754	8.908						
10	11.81	9.374	8.513	8.107					
12	11.62	9.169	8.297	7.884	7.655				
14	11.51	9.046	8.166	7.747	7.514	7.371			
18	11.38	8.908	8.020	7.594	7.355	7.207	7.035		
22	11.31	8.833	7.941	7.511	7.268	7.117	6.940	6.840	
26	11.26	8.786	7.891	7.456	7.214	7.060	6.879	6.776	6.710
30	11.23	8.753	7.857	7.423	7.176	7.021	6.837	6.731	6.663
40	11.17	8.701	7.804	7.368	7.119	6.960	6.771	6.661	6.589
50	11.14	8.671	7.774	7.336	7.086	6.925	6.733	6.620	6.545
60	11.12	8.651	7.754	7.316	7.064	6.902	6.708	6.593	6.517
80	11.09	8.627	7.729	7.290	7.037	6.874	6.676	6.559	6.481
100	11.07	8.612	7.715	7.275	7.021	6.857	6.658	6.539	6.459
120	11.06	8.602	7.705	7.265	7.011	6.846	6.645	6.523	6.445
140	11.06	8.595	7.698	7.258	7.003	6.838	6.637	6.516	6.434

ν_2/ν_3	30	40	50	60	80	100	120	140
30	6.614							
40	6.537	6.455						
50	6.492	6.405	6.354					
60	6.461	6.373	6.319	6.283				
80	6.424	6.331	6.275	6.236	6.187			
100	6.401	6.306	6.248	6.208	6.156	6.124		
120	6.386	6.289	6.229	6.188	6.135	6.101	6.078	
140	6.375	6.277	6.216	6.174	6.119	6.085	6.061	6.043

Table 4.2. (Continued)

$\nu_1 = 6$

ν_3/ν_2	6	8	10	12	14	18	22	26
6	8.363							
8	7.567	6.776						
10	7.183	6.388	5.995					
12	6.968	6.167	5.769	5.539				
14	6.835	6.028	5.626	5.393	5.244			
18	6.685	5.869	5.461	5.223	5.071	4.893		
22	6.605	5.784	5.372	5.131	4.977	4.794	4.693	
26	6.556	5.732	5.318	5.075	4.918	4.733	4.630	4.565
30	6.523	5.698	5.281	5.037	4.879	4.692	4.587	4.520
40	6.474	5.646	5.228	4.981	4.822	4.631	4.522	4.453
50	6.446	5.618	5.199	4.951	4.790	4.597	4.487	4.415
60	6.428	5.600	5.180	4.932	4.770	4.576	4.464	4.391
80	6.406	5.578	5.158	4.909	4.747	4.550	4.437	4.362
100	6.393	5.566	5.145	4.896	4.733	4.535	4.421	4.345
120	6.384	5.557	5.137	4.887	4.724	4.526	4.410	4.334
140	6.378	5.551	5.131	4.881	4.718	4.519	4.403	4.326

ν_3/ν_2	30	40	50	60	80	100	120	140
30	4.474							
40	4.405	4.330						
50	4.365	4.288	4.243					
60	4.340	4.260	4.214	4.183				
80	4.309	4.226	4.177	4.145	4.104			
100	4.291	4.206	4.156	4.122	4.079	4.054		
120	4.279	4.193	4.141	4.107	4.063	4.036	4.018	
140	4.271	4.183	4.131	4.096	4.051	4.023	4.005	3.991

Table 4.2. (Continued)

$\nu_1 = 8$

ν_3/ν_2	8	10	12	14	18	22	26	30
8	6.002							
10	5.616	5.230						
12	5.393	5.004	4.775					
14	5.250	4.858	4.627	4.477				
18	5.084	4.687	4.452	4.299	4.116			
22	4.994	4.593	4.355	4.200	4.014	3.909		
26	4.938	4.535	4.295	4.138	3.950	3.843	3.775	
30	4.902	4.496	4.255	4.097	3.906	3.798	3.730	3.683
40	4.848	4.440	4.197	4.037	3.843	3.732	3.662	3.613
50	4.819	4.410	4.165	4.005	3.809	3.697	3.625	3.575
60	4.801	4.392	4.146	3.985	3.788	3.675	3.602	3.551
80	4.779	4.370	4.124	3.961	3.763	3.649	3.574	3.522
100	4.767	4.357	4.111	3.948	3.749	3.634	3.559	3.506
120	4.759	4.349	4.103	3.940	3.740	3.624	3.548	3.495
140	4.753	4.343	4.097	3.934	3.734	3.617	3.541	3.487

ν_3/ν_2	40	50	60	80	100	120	140
40	3.540						
50	3.499	3.456					
60	3.473	3.428	3.399				
80	3.441	3.395	3.364	3.326			
100	3.423	3.375	3.343	3.304	3.280		
120	3.411	3.362	3.329	3.289	3.264	3.248	
140	3.403	3.353	3.320	3.278	3.253	3.236	3.223

Table 4.2. (Continued)

$\nu_1 = 10$

ν_3/ν_2	10	12	14	18	22	26	30	40	50	60	80	100	120	140
10	4.845													
12	4.618	4.389												
14	4.471	4.240	4.090											
18	4.295	4.062	3.908	3.723										
22	4.197	3.961	3.806	3.617	3.509									
26	4.137	3.898	3.742	3.550	3.441	3.371								
30	4.096	3.856	3.698	3.505	3.394	3.323	3.275							
40	4.037	3.795	3.634	3.439	3.326	3.253	3.203	3.128						
50	4.006	3.762	3.601	3.403	3.289	3.215	3.164	3.087	3.044					
60	3.987	3.742	3.581	3.382	3.266	3.192	3.140	3.061	3.017	2.989				
80	3.964	3.719	3.557	3.357	3.240	3.164	3.111	3.031	2.985	2.955	2.918			
100	3.952	3.706	3.544	3.343	3.225	3.149	3.095	3.013	2.966	2.935	2.897	2.874		
120	3.943	3.698	3.535	3.334	3.216	3.139	3.085	3.002	2.953	2.922	2.883	2.859	2.843	
140	3.938	3.692	3.529	3.328	3.209	3.132	3.078	2.994	2.945	2.913	2.873	2.848	2.832	2.821

$\nu_1 = 12$

ν_3/ν_2	12	14	18	22	26	30	40	50	60	80	100	120	140
12	4.160												
14	4.010	3.858											
18	3.828	3.675	3.487										
22	3.725	3.570	3.379	3.269									
26	3.661	3.503	3.310	3.199	3.127								
30	3.617	3.458	3.264	3.150	3.078	3.028							
40	3.553	3.392	3.194	3.079	3.005	2.953	2.876						
50	3.519	3.357	3.157	3.041	2.965	2.913	2.834	2.790					
60	3.498	3.336	3.135	3.017	2.941	2.888	2.807	2.763	2.735				
80	3.475	3.311	3.109	2.990	2.913	2.859	2.777	2.730	2.701	2.665			
100	3.461	3.298	3.095	2.975	2.897	2.843	2.759	2.712	2.681	2.644	2.622		
120	3.453	3.289	3.086	2.966	2.888	2.833	2.748	2.700	2.669	2.630	2.607	2.592	
140	3.447	3.283	3.080	2.959	2.881	2.826	2.740	2.692	2.660	2.620	2.597	2.581	2.570

Table 4.2. (Continued)

$\nu_1 = 14$

ν_3/ν_2	14	18	22	26	30	40	50	60	80	100	120	140
14	3.706											
18	3.520	3.331										
22	3.413	3.221	3.110									
26	3.345	3.151	3.038	2.964								
30	3.299	3.103	2.988	2.914	2.862							
40	3.231	3.031	2.914	2.838	2.785	2.706						
50	3.195	2.993	2.874	2.797	2.744	2.662	2.618					
60	3.172	2.970	2.850	2.772	2.718	2.635	2.590	2.561				
80	3.147	2.943	2.822	2.743	2.688	2.604	2.557	2.527	2.491			
100	3.133	2.928	2.807	2.727	2.672	2.586	2.538	2.508	2.470	2.449		
120	3.124	2.919	2.797	2.717	2.661	2.575	2.526	2.495	2.457	2.434	2.419	
140	3.118	2.913	2.791	2.711	2.654	2.568	2.518	2.486	2.447	2.424	2.409	2.398

$\nu_1 = 18$

ν_3/ν_2	18	22	26	30	40	50	60	80	100	120	140
18	3.139										
22	3.027	2.912									
26	2.954	2.838	2.762								
30	2.904	2.786	2.710	2.656							
40	2.829	2.709	2.630	2.575	2.492						
50	2.788	2.667	2.587	2.531	2.446	2.398					
60	2.763	2.641	2.560	2.504	2.417	2.369	2.339				
80	2.735	2.611	2.529	2.472	2.384	2.335	2.303	2.267			
100	2.719	2.595	2.513	2.455	2.366	2.315	2.284	2.246	2.224		
120	2.710	2.585	2.502	2.444	2.354	2.303	2.271	2.232	2.210	2.195	
140	2.703	2.578	2.495	2.437	2.346	2.295	2.262	2.223	2.200	2.184	2.174

Table 4.2. (Continued)

$\nu_1 = 22$

ν_3/ν_2	22	26	30	40	50	60	80	100	120	140
22	2.796									
26	2.720	2.643								
30	2.667	2.589	2.534							
40	2.587	2.506	2.450	2.363						
50	2.543	2.461	2.404	2.315	2.266					
60	2.516	2.433	2.375	2.285	2.234	2.203				
80	2.484	2.401	2.341	2.250	2.198	2.166	2.127			
100	2.467	2.383	2.323	2.231	2.178	2.145	2.106	2.083		
120	2.457	2.372	2.312	2.218	2.165	2.132	2.092	2.069	2.054	
140	2.450	2.364	2.304	2.210	2.157	2.123	2.082	2.059	2.044	2.033

$\nu_1 = 26$

ν_3/ν_2	26	30	40	50	60	80	100	120	140
26	2.564								
30	2.509	2.453							
40	2.425	2.367	2.278						
50	2.378	2.319	2.228	2.177					
60	2.349	2.289	2.197	2.144	2.111				
80	2.315	2.254	2.160	2.106	2.072	2.032			
100	2.297	2.235	2.140	2.085	2.051	2.010	1.987		
120	2.285	2.223	2.127	2.072	2.037	1.996	1.972	1.957	
140	2.277	2.215	2.119	2.063	2.028	1.986	1.962	1.946	1.935

Table 4.2. (Continued)

$\nu_1 = 30$

ν_3/ν_2	30	40	50	60	80	100	120	140
30	2.396							
40	2.308	2.217						
50	2.259	2.166	2.113					
60	2.229	2.134	2.080	2.046				
80	2.192	2.096	2.041	2.005	1.964			
100	2.173	2.075	2.019	1.983	1.940	1.916		
120	2.160	2.061	2.005	1.969	1.925	1.901	1.885	
140	2.152	2.053	1.996	1.959	1.915	1.890	1.874	1.863

$\nu = 40$

ν_3/ν_2	40	50	60	80	100	120	140
40	2.123						
50	2.069	2.014					
60	2.035	1.978	1.942				
80	1.995	1.936	1.899	1.854			
100	1.972	1.913	1.874	1.828	1.802		
120	1.956	1.898	1.859	1.812	1.786	1.768	
140	1.948	1.888	1.848	1.801	1.774	1.757	1.745

$\nu_1 = 50$

ν_3/ν_2	50	60	80	100	120	140
50	1.957					
60	1.920	1.882				
80	1.876	1.837	1.789			
100	1.851	1.811	1.762	1.734		
120	1.835	1.794	1.745	1.717	1.698	
140	1.824	1.783	1.733	1.704	1.686	1.673

$\nu_1 = 60$

ν_3/ν_2	60	80	100	120	140
60	1.843				
80	1.796	1.747			
100	1.769	1.719	1.690		
120	1.752	1.701	1.671	1.652	
140	1.741	1.688	1.658	1.639	1.625

$\nu_1 = 80$

ν_3/ν_2	80	100	120	140
80	1.696			
100	1.666	1.635		
120	1.647	1.615	1.594	
140	1.633	1.601	1.580	1.565

$\nu_1 = 100$

ν_3/ν_2	100	120	140
100	1.603		
120	1.582	1.560	
140	1.567	1.545	1.529

$\nu_1 = 120$

ν_3/ν_2	120	140
120	1.537	
140	1.522	1.505

$\nu_1 = 140$

ν_3/ν_2	140
140	1.489

Table 4.3. The Power of R_2 for Testing H_0' for $k = 2$ and $p = 2$ and for Selected Alternatives

γ_1'	γ_2'	τ_1'	τ_2'	$n_1=6$ $n_2=6$	$n_1=6$ $n_2=14$	$n_1=6$ $n_2=20$	$n_1=10$ $n_2=10$	$n_1=10$ $n_2=20$	$n_1=14$ $n_2=14$	$n_1=14$ $n_2=20$	$n_1=20$ $n_2=20$
2.0	3.0	4.0	6.0	.208	.247	.260	.326	.397	.436	.489	-----
.5	3.5	2.0	3.0	.377	.461	.484	.567	.659	.707	.759	.844
.5	3.5	2.0	4.0	.441	.532	.557	.645	.735	.782	.828	.900
.5	3.5	2.0	5.0	.485	.578	.603	.692	.779	.823	.865	.927
2.0	3.0	.5	3.5	.377	.522	.567	.567	.711	.707	.786	.844
2.0	4.0	.5	3.5	.441	.601	.648	.645	.787	.782	.853	.900
2.0	5.0	.5	3.5	.485	.650	.697	.692	.828	.823	.886	.927
1.0	2.0	1.0	2.0	.0598	.0602	.0602	.0610	.0613	.0615	.0617	.0619
1.0	5.0	1.0	5.0	.0956	.0967	.0967	.0988	.0994	.100	.100	.101
1.0	10.0	1.0	10.0	.125	.126	.126	.127	.128	.128	.128	.129

CHAPTER V

AN APPROXIMATION TO THE DISTRIBUTION OF THE LARGEST ROOT OF A COMPLEX WISHART MATRIX

1. Introduction and Summary

Let $\underset{\sim}{X}(q\text{x}r)$ $(r \geq q)$ be a complex valued random matrix whose columns are independent and have the q-variate complex normal distribution $N_c(\underset{\sim}{M},\underset{\sim}{\Sigma})$ (Wooding [43], Goodman [14]). The distribution of $\underset{\sim}{X}\,\underset{\sim}{\bar{X}}'$ is then complex Wishart $W_c(q,r,\underset{\sim}{\Sigma})$ (Goodman [14], Khatri [23]). If $\underset{\sim}{M} = \underset{\sim}{0}$ and $\underset{\sim}{\Sigma} = \underset{\sim}{I}_q$ (the qxq identity matrix), the distribution of $0 \leq f_1 \leq f_2 \leq \ldots \leq f_q < \infty$, the characteristic roots of $\underset{\sim}{X}\,\underset{\sim}{\bar{X}}'$, is given by (Khatri [23], James [20]):

$$C_1 \left(\prod_{j=1}^{q} f_j^m \right) \exp\left(-\sum_{j=1}^{q} f_j\right) \prod_{j>k} (f_j - f_k)^2, \tag{1.1}$$

where

$$C_1 = 1 \Big/ \left(\prod_{j=1}^{q} \Gamma(m+j)\,\Gamma(j) \right) \quad \text{and} \quad m = r-q. \tag{1.2}$$

This distributional form also arises in another manner. Suppose that $\underset{\sim}{X}(q\text{x}r)$ is distributed $N_c(\underset{\sim}{0}, \underset{\sim}{I}_q)$ and that $\underset{\sim}{S}(q\text{x}q)$ is independent of $\underset{\sim}{X}$ with distribution $W_c(q,s,\underset{\sim}{I}_q)$. Then the distribution of the characteristic roots of $[\underset{\sim}{X}\,\underset{\sim}{\bar{X}}'(\underset{\sim}{S}+\underset{\sim}{X}\,\underset{\sim}{\bar{X}}')^{-1}]$, say $0 \leq w_1 \leq w_2 \leq \ldots \leq w_q \leq 1$, has the form:

(1.3) $$C_2 \prod_{j=1}^{q} [w_j^m(1-w_j)^n] \prod_{j>k} (w_j - w_k)^2,$$

where

(1.4) $$C_2 = \prod_{j=1}^{q} \Gamma(m+n+q+j)/(\Gamma(m+j)\Gamma(n+j)\Gamma(j)),$$

$m=r-q$ and $n=s-q$.

By making the transformation $f_j = nw_j$, $j=1,\ldots,q$, and allowing $n\to\infty$, the distribution of $0 \le f_1 \le f_2 \le \ldots \le f_q < \infty$ is that given by (1.1) (Khatri [23]).

Because of the similarity of handling the classical problem of point estimation and hypothesis testing for normal populations in the complex case with that in the real case, the largest (or smallest) characteristic root has been proposed as a test criterion by Khatri [23], [24].

The distribution of the largest characteristic root $(f_q$ or $w_q)$ has been given by Khatri [22] as follows:

(1.5) $$P\{f_q \le x;\ m\} = C_1|(\gamma_{i+j-2})| \quad ,$$

where C_1 is defined in (1.2),

(1.6) $$\gamma_{i+j-2} = \int_0^x z^{m+i+j-2}\, e^{-z}\, dz,\ i,j=1,\ldots,q,$$

and (γ_{i+j-2}) is a $q\text{x}q$ matrix; and

(1.7) $$P\{w_q \le x;\ m\} = C_2|(\beta_{i+j-2})|,$$

where C_2 is defined in (1.4),

$$\beta_{i+j-2} = \int_0^x w^{m+i+j-2}(1-w)^n dw, \quad i,j=1,\ldots,q, \tag{1.8}$$

and (β_{i+j-2}) is a qxq matrix.

Pillai and Jouris [37], using an approach due to Pillai [31], have suggested an approximation to the distribution (1.7) and have obtained various percentage points. The purpose here is to suggest an approximation to the distribution of f_q (1.5) using a similar approach and to tabulate upper tail percentage points using this approximation.

2. Approximation to the C.D.F. of f_q

By using integration by parts for integral values of m, (1.6) can be written as:

$$\gamma_k = \int_0^x z^{m+k} e^{-z} dz = (m+k)! - T_k \tag{2.1}$$

where

$$T_k = (m+k)!\, e^{-x} \sum_{j=0}^{m+k} x^j/j! \quad .$$

By definition

$$|(\gamma_{i+j-2})| = \sum_{\underset{\sim}{j}} \text{sign } (\underset{\sim}{j}) \prod_{k=1}^{q} (\gamma_{k+j_k-2}) \ , \tag{2.2}$$

where $\sum\limits_{\underset{\sim}{j}}$ denotes the summation over the permutation $\underset{\sim}{j} = (j_1, j_2, \ldots, j_q)$ of $(1,2,\ldots,q)$. Using the expansion of γ_k in (2.1) and neglecting terms of the type $T_\ell T_k$ (that is, all terms involving e^{-bx} for $b \geq 2$) we find that:

$$\gamma_{j_1-1}\,\gamma_{j_2} \doteq (m+j_1-1)!\,\gamma_{j_2} + (m+j_2)!\,\gamma_{j_1-1} - (m+j_1-1)!\,(m+j_2)!,$$

$$\gamma_{j_1-1}\,\gamma_{j_2}\,\gamma_{j_3+1} \doteq (j_1-1)!\,j_2!\,\gamma_{j_3+1} + (j_1-1)!(j_3+1)!\,\gamma_{j_2}$$

$$+\, j_2!(j_3+1)!\,\gamma_{j_1-1} - 2(j_1-1)!\,j_2!(j_3+1)!$$

and in general

$$\prod_{k=1}^{q} \gamma_{k+j_k-2} \doteq \sum_{\alpha=1}^{q} \Big(\prod_{\substack{k=1\\ k\neq\alpha}}^{q} (m+k+j_k-2)! \Big)\, \gamma_{\alpha+j_\alpha-2} - (q-1) \prod_{k=1}^{q} (m+k+j_k-2)!. \tag{2.3}$$

Upon using (2.3) in the definition of $|(\gamma_{i+j-2})|$ given in (2.2), we can approximate (1.5) by

$$C_1|(\gamma_{i+j-2})| \doteq C_1 \sum_{\underset{\sim}{j}} \sum_{\alpha=1}^{q} \operatorname{sign}(\underset{\sim}{j}) \Big(\prod_{\substack{k=1\\ k\neq\alpha}}^{q} (m+k+j_k-2)! \Big) \gamma_{\alpha+j_\alpha-2} - C_1(q-1)\,|((m+i+j-2)!)|$$

$$= C_1 \sum_{k=0}^{2q-2} G_k'\gamma_k - (q-1),$$

since $|((m+i+j-2)!)| = C_1^{-1}$ and where G_k' is the sum of the cofactors of $(m+k)!$ in the $q \times q$ matrix

$$G = \begin{bmatrix} m! & (m+1)! & \cdots & (m+q-1)! \\ (m+1)! & (m+2)! & \cdots & (m+q)! \\ \vdots & & & \\ \vdots & & & \\ (m+q-1)! & (m+q)! & \cdots & (m+2q-2)! \end{bmatrix} .$$

Thus for $q \geq 2$ we have

$$(2.4) \qquad P\{f_q \leq x;\, m\} \doteq C_1 \sum_{k=0}^{2q-2} G_k' \gamma_k - (q-1).$$

Explicit simplified expressions for (2.4) when $q=2,3,4$ and 5 are given below in (2.5), (2.6), (2.7) and (2.8) respectively.

$$(2.5) \qquad P\{f_2 \leq x;m\} \doteq \frac{1}{(m+1)!}\,[(m+1)_2\gamma_0-2(m+1)\gamma_1+\gamma_2]\ -1,$$

where $(a)_k = a(a+1)\ \ldots\ (a+k-1)$.

$$(2.6) \quad P\{f_3 \leq x;m\} = \frac{1}{2(m+2)!}\,[(m+2)(m+1)_3\gamma_0-4(m+1)_3\gamma_1+6(m+1)_2\gamma_2-4(m+2)\gamma_3 + \gamma_4]\ -2.$$

$$(2.7)\quad P\{f_4 \le x;m\} \doteq \frac{1}{6(m+3)!}\,[(m+1)_2(m+2)_2(m+3)_2\gamma_0 - 6(m+1)(m+2)_2(m+3)_2\gamma_1$$

$$+ 15(m+2)(m+3)_2(m+9/5)\gamma_2 - 20(m+2)(m+3)(m+17/5)\,\gamma_3$$

$$+ 15(m+3)(m+14/5)\,\gamma_4 - 6(m+3)\,\gamma_5 + \gamma_6] - 3.$$

$$(2.8)\quad P\{f_5 \le x;m\} \doteq \frac{1}{24(m+4)!}\,[(m+1)_2(m+2)_2(m+3)_2(m+4)_2\,\gamma_0$$

$$- 8(m+1)(m+2)_2(m+3)_2(m+4)_2\gamma_1$$
$$+28(m+2)(m+3)_2(m+4)_2(m+12/7)\gamma_2$$
$$-56(m+2)(m+3)(m+4)_2(m+22/7)\gamma_3$$
$$+70(m+3)(m+4)(m^2+51/7m+86/7)\gamma_4$$
$$-56(m+3)(m+4)(m+29/7)\gamma_5 + 28(m+4)(m+26/7)\gamma_6$$
$$- 8(m+4)\gamma_7 + \gamma_8] - 4.$$

The approximation is thus a linear combination of incomplete gamma functions and is simpler than the exact C.D.F. which involves products of q incomplete gamma functions.

3. Computation of Percentage Points

By using the approximation obtained in the previous section, upper 10%, 5%, 2.5%, 1% and .5% points were obtained for the C.D.F. of f_q for $q = 2\ (1)\ 11$. The percentage points are given to five significant digits for $m = 0(1)20(2)30(5)50(10)100$ in Table 5.2.

Some exact percentage points were also tabulated for comparisons with the approximate ones. Table 5.1 below displays some representative values of both the exact and approximate percentage points. As can be seen from this table, the approximate and exact percentage points usually agree through five significant digits. This same degree of accuracy has been found in the approximation suggested in [37] to the distribution (1.3).

Table 5.1

Comparison of the Approximate and Exact Percentage Points for the C.D.F. of the Largest Root f_q.

		1%		5%	
q	m	Approximate	Exact	Approximate	Exact
2	15	32.6968	32.6968	28.9562	28.9561
3	30	58.6083	58.6083	53.8994	53.8992
4	60	103.5274	103.5274	97.5795	97.5791
5	100	160.1230	160.1230	153.0122	153.0118
6	20	59.6795	59.6795	55.2224	55.2221
7	10	48.2632	48.2632	44.2295	44.2293
8	70	139.8957	139.8956	133.5577	133.5572
9	30	88.6527	88.8526	83.5302	83.5298
10	5	52.0095	52.0094	47.9146	47.9143
11	18	78.7225	78.7205	73.9153	73.9145

4. Applications

The complex multivariate normal and related distribution have been found useful in such areas as physics and time series analysis. Under certain basic assumptions Bronk [7] has found that the distribution (1.1) is that of the energy levels of atomic nuclei. Goodman [14] has noted several applications of complex multivariate theory to time series analysis. Brillinger [6] has shown that the asymptotic distributions of the matrix of second-order periodograms and the matrix of spectral densities of a strictly stationary time series are complex Wishart (the distribution of whose characteristic roots is given by (1.1)). It has been noted in Section 1 that many hypothesis testing problems in the complex case can be handled as in the real case. It is hoped that the findings here will be useful in the areas mentioned above as well as in other fields.

Table 5.2

Upper α Points of the Largest Root

	q=2					q=3				
m \ α	.10	.05	.025	.01	.005	.10	.05	.025	.01	.005
0	5.912	6.900	7.839	9.031	9.904	9.649	10.829	11.931	13.308	14.305
1	7.664	8.753	9.778	11.068	12.007	11.465	12.722	13.890	15.324	16.391
2	9.287	10.461	11.156	12.933	13.928	13.177	14.502	15.729	17.250	18.344
3	10.830	12.081	13.243	14.691	15.736	14.817	16.204	17.485	19.067	20.203
4	12.319	13.637	14.858	16.374	17.464	16.403	17.848	19.178	20.818	21.992
5	13.766	15.147	16.422	18.000	19.133	17.947	19.445	20.822	22.515	23.727
6	15.181	16.620	17.946	19.582	20.755	19.456	21.005	22.425	24.170	25.416
7	16.570	18.064	19.438	21.129	22.339	20.937	22.534	23.996	25.788	27.067
8	17.937	19.483	20.902	22.645	23.891	22.394	24.036	25.538	27.376	28.687
9	19.286	20.881	22.343	24.136	25.415	23.830	25.515	27.055	28.938	30.278
10	20.619	22.261	23.764	25.605	26.916	25.248	26.975	28.551	30.477	31.846
11	21.938	23.626	25.168	27.054	28.397	26.649	28.417	30.028	31.995	33.392
12	23.244	24.976	26.556	28.486	29.859	28.036	29.843	31.488	33.494	34.919
13	24.540	26.313	27.930	29.903	31.305	29.411	31.255	32.933	34.978	36.428
14	25.826	27.640	29.292	31.306	32.736	30.773	32.654	34.364	36.446	37.922
15	27.102	28.956	30.642	32.697	34.154	32.125	34.041	35.782	37.900	39.401
16	28.371	30.263	31.983	34.076	35.560	33.467	35.418	37.189	39.342	40.867
17	29.323	31.562	33.314	35.445	36.954	34.800	36.785	38.585	40.773	42.321
18	30.886	32.852	34.636	36.804	38.338	36.124	38.142	39.972	42.193	43.764
19	32.134	34.136	35.950	38.154	39.713	37.441	39.491	41.349	43.603	45.196
20	33.376	35.412	37.257	39.496	41.079	38.751	40.832	42.717	45.003	46.619
22	35.844	37.947	39.850	42.157	43.787	41.351	43.493	45.431	47.779	49.438
24	38.292	40.460	42.419	44.791	46.465	43.926	46.127	48.116	50.525	52.224
26	40.724	42.953	44.966	47.401	49.118	46.481	48.738	50.777	53.243	54.912
28	43.140	45.428	47.494	49.990	51.748	49.016	51.328	53.415	55.937	57.714
30	45.542	47.888	50.004	52.559	54.357	51.535	53.899	56.032	58.608	60.422
35	51.495	53.978	56.214	58.908	60.802	57.765	60.256	62.498	65.203	67.106
40	57.385	59.997	62.343	65.168	67.152	63.917	66.524	68.870	71.696	73.681
45	63.224	65.956	68.408	71.356	73.423	70.002	72.721	75.164	78.104	80.168
50	69.018	71.865	74.416	77.481	79.628	76.032	78.857	81.392	84.441	86.578
60	80.499	83.559	86.297	89.580	91.876	87.955	90.977	93.685	96.936	99.213
70	91.865	95.121	98.032	101.52	103.95	99.730	102.93	105.80	109.24	111.64
80	103.14	106.58	109.65	113.32	115.88	111.39	114.76	117.78	121.39	123.92
90	114.34	117.95	121.17	125.02	127.70	122.94	126.48	129.64	133.42	136.05
100	125.47	129.24	132.61	136.62	139.42	134.42	138.11	141.40	145.33	148.08

Table 5.2 (continued)

	q=4					q=5				
m \ α	.10	.05	.025	.01	.005	.10	.05	.025	.01	.005
0	13.441	14.768	15.997	17.520	18.615	17.265	18.715	20.050	21.694	22.871
1	15.293	16.683	17.967	19.552	20.689	19.139	20.644	22.025	23.724	24.938
2	17.058	18.506	19.839	21.482	22.659	20.940	22.494	23.920	25.669	26.918
3	18.758	20.260	21.640	23.337	24.550	22.682	24.284	25.751	27.548	28.830
4	20.408	21.960	23.383	25.131	26.378	24.378	26.025	27.531	29.374	30.686
5	22.017	23.616	25.081	26.876	28.156	26.035	27.725	29.267	31.153	32.495
6	23.592	25.236	26.740	28.581	29.892	27.659	29.389	30.968	32.895	34.265
7	25.138	26.825	28.366	30.250	31.591	29.254	31.024	32.636	34.603	36.001
8	26.659	28.387	29.964	31.890	33.259	30.824	32.632	34.277	36.282	37.706
9	28.158	29.926	31.537	33.503	34.890	32.373	34.216	35.893	37.936	39.384
10	29.638	31.444	33.088	35.093	36.516	33.901	35.780	37.487	39.566	41.039
11	31.101	32.944	34.620	36.662	38.111	35.412	37.325	39.062	41.175	42.672
12	32.548	34.427	36.134	38.213	39.686	36.907	38.853	40.619	42.766	44.286
13	33.982	35.894	37.632	39.746	41.244	38.388	40.365	42.159	44.339	45.882
14	35.402	37.349	39.116	41.264	42.785	39.855	41.864	43.685	45.897	47.462
15	36.811	38.790	40.586	42.767	44.312	41.309	43.349	45.197	47.440	49.026
16	38.209	40.220	42.044	44.258	45.824	42.752	44.822	46.696	48.970	50.577
17	39.597	41.639	43.490	45.736	47.325	44.185	46.284	48.183	50.487	52.115
18	40.976	43.048	44.926	47.203	48.813	45.608	47.735	49.660	51.992	53.640
19	42.346	44.448	46.352	48.660	50.290	47.022	49.176	51.126	53.487	55.154
20	43.708	45.840	47.768	50.106	51.757	48.426	50.609	52.582	54.971	56.658
22	46.410	48.598	50.576	52.972	54.663	51.212	53.448	55.468	57.912	59.636
24	49.085	51.328	53.354	55.806	57.535	53.970	56.256	58.321	60.818	62.579
26	51.736	54.031	56.104	58.610	60.376	56.701	59.037	61.145	63.694	65.489
28	54.366	56.712	58.829	61.387	63.189	59.409	61.793	63.943	66.541	68.371
30	56.976	59.371	61.532	64.141	65.978	62.095	64.526	66.718	69.364	71.226
35	63.426	65.939	68.203	70.933	72.853	68.729	71.271	73.561	76.322	78.263
40	69.785	72.409	74.769	77.613	79.611	75.263	77.910	80.292	83.161	85.177
45	76.069	78.797	81.248	84.200	86.272	81.713	84.460	86.928	89.900	90.967
50	82.288	85.115	87.654	90.708	92.851	88.092	90.933	93.485	96.555	98.708
60	94.566	97.579	100.28	103.53	105.80	100.67	103.69	106.40	109.65	111.93
70	106.67	109.86	112.71	116.13	118.53	113.06	116.24	119.10	122.52	124.91
80	118.64	121.98	124.98	128.57	131.08	125.29	128.63	131.62	135.20	137.70
90	130.49	133.99	137.12	140.86	143.48	137.40	140.88	143.99	147.72	150.34
100	142.25	145.89	149.14	153.04	155.76	149.39	153.01	156.25	160.12	162.83

Table 5.2 (continued)

m \ α	q=6 .10	q=6 .05	q=6 .025	q=6 .01	q=6 .005	q=7 .10	q=7 .05	q=7 .025	q=7 .01	q=7 .005
0	21.111	22.667	24.093	25.843	27.093	24.973	26.623	28.130	29.975	31.289
1	23.001	24.605	26.073	27.871	29.154	26.875	28.568	30.113	32.002	33.345
2	24.827	26.476	27.983	29.827	31.141	28.720	30.454	32.635	33.965	35.337
3	26.601	28.292	29.837	31.724	33.067	30.518	32.291	33.906	35.876	37.275
4	28.331	30.064	31.643	33.572	34.944	32.275	34.086	35.734	37.742	39.167
5	30.025	31.796	33.410	35.379	36.777	33.998	35.845	37.524	39.569	41.019
6	31.686	33.496	35.142	37.149	38.573	35.691	37.572	39.282	41.362	42.837
7	33.321	35.166	36.844	38.888	40.337	37.356	39.271	41.010	43.125	44.623
8	34.930	36.810	38.519	40.598	42.072	39.998	40.946	42.713	44.861	46.382
9	36.518	38.432	40.170	42.283	43.781	40.619	42.598	44.393	46.573	48.116
10	38.086	40.033	41.799	43.946	45.467	42.220	44.229	46.051	48.263	49.828
11	39.637	41.615	43.409	45.589	47.131	43.803	45.843	47.691	49.933	51.519
12	41.171	43.180	45.001	47.212	48.777	45.370	47.439	49.312	51.585	53.191
13	42.690	44.729	46.577	48.819	50.404	46,922	49.019	50.917	53.219	54.845
14	44.195	46.264	48.137	50.410	52.016	48.460	50.585	52.508	54.838	56.484
15	45.688	47.785	49.684	51.986	53.612	49.985	52.137	54.084	56.442	58.107
16	47.169	49.294	51.218	53.548	55.195	51.499	53.677	55.647	58.033	59.716
17	48.639	50.792	52.739	55.098	56.764	53.001	55.206	57.198	59.611	61.313
18	50.099	52.279	54.249	56.636	58.321	54.493	56.723	58.738	61.177	62.897
19	51.549	53.755	55.749	58.163	59.866	55.974	58.230	60.267	62.737	64.469
20	52.991	55.222	57.239	59.680	61.401	57.447	59.727	61.786	64.276	66.031
22	55.848	58.130	60.191	62.683	64.441	60.366	62.694	64.795	67.334	69.124
24	58.675	60.006	63.093	65.652	67.444	63.254	65.628	67.770	70.357	72.180
26	61.474	63.852	65.997	68.589	70.414	66.113	68.532	70.714	73.348	75.202
28	64.249	66.673	68.858	71.497	73.354	68.946	71.410	73.629	76.309	78.194
30	67.001	69.470	71.694	74.378	76.267	71.756	74.262	76.519	79.242	81.158
35	73.794	76.368	78.686	81.480	83.445	78.688	81.296	83.643	86.472	88.460
40	80.479	83.154	85.560	88.458	90.494	85.508	88.212	90.643	93.572	95.629
45	87.075	89.845	92.334	95.330	97.434	92.232	95.028	97.540	100.56	102.68
50	93.594	96.454	99.023	102.11	104.28	98.875	101.76	104.35	107.46	109.64
60	106.44	109.47	112.19	115.12	117.74	111.96	115.00	117.74	121.02	123.32
70	119.07	122.26	125.12	128.55	130.95	124.81	128.01	130.88	134.32	136.74
80	131.54	134.88	137.86	141.45	143.96	137.49	140.83	143.83	147.42	149.93
90	143.87	147.34	150.45	154.18	156.79	150.02	153.50	156.61	160.34	162.95
100	156.07	159.68	162.91	166.78	169.48	162.42	166.02	169.25	173.11	175.82

Table 5.2 (continued)

m	q=8 .10	q=8 .05	q=8 .025	q=8 .01	q=8 .005	q=9 .10	q=9 .05	q=9 .025	q=9 .01	q=9 .005
0	28.847	30.582	32.163	34.094	35.466	32.731	34.544	36.192	38.202	39.628
1	30.758	32.532	34.148	36.119	37.518	34.650	36.499	38.197	40.226	41.677
2	32.619	34.430	36.079	38.088	39.514	36.522	38.406	40.117	42.199	43.675
3	34.435	36.283	37.963	40.009	41.460	38.355	40.272	42.013	44.129	45.628
4	36.215	38.097	39.808	41.884	43.365	40.152	42.102	43.871	46.021	47.543
5	37.961	39.878	41.618	43.733	45.232	41.919	43.900	45.697	47.879	49.423
6	39.679	41.628	43.396	45.546	47.067	43.658	45.670	47.493	49.707	51.272
7	41.371	43.352	45.148	47.329	48.872	45.372	47.414	49.263	51.507	53.094
8	43.040	45.051	46.874	49.087	50.652	47.064	49.135	51.009	53.283	54.890
9	44.688	46.729	48.578	50.821	52.407	48.736	50.834	52.734	55.036	56.663
10	46.317	48.387	50.261	52.534	54.140	50.388	52.515	54.438	56.769	58.415
11	47.928	50.026	51.925	54.227	55.853	52.024	54.177	56.124	58.482	60.148
12	49.523	51.649	53.572	55.902	57.548	53.644	55.823	57.793	60.178	61.862
13	51.104	53.256	55.203	57.561	59.226	55.248	57.453	59.446	61.858	63.560
14	52.670	54.849	56.819	59.204	60.887	56.839	59.069	61.084	63.522	65.242
15	54.224	56.428	58.420	60.832	62.534	58.418	60.672	62.709	65.172	66.909
16	55.765	57.995	60.009	62.447	64.167	59.984	62.262	64.320	66.809	68.563
17	57.295	59.550	61.586	64.050	65.787	61.538	63.841	65.919	68.432	70.204
18	58.815	61.094	63.151	65.640	67.395	63.082	65.408	67.507	70.044	71.832
19	60.324	62.627	64.706	67.219	68.991	64.616	66.965	69.084	71.645	73.449
20	61.824	64.150	66.250	68.788	70.576	66.141	68.512	70.651	73.235	75.055
22	64.798	67.170	69.310	71.895	73.716	69.163	71.578	73.756	76.386	78.237
24	67.739	70.156	72.335	74.966	76.819	72.152	74.610	76.826	79.500	81.382
26	70.651	73.111	75.328	78.004	79.887	75.111	77.611	79.863	82.580	84.492
28	73.536	76.038	78.292	81.102	82.925	78.043	80.584	82.871	85.630	87.571
30	76.397	78.940	81.230	83.992	85.935	80.950	83.530	85.852	88.653	90.621
35	83.454	86.095	88.472	91.335	93.348	88.120	90.795	93.200	96.099	98.135
40	90.394	93.128	95.586	98.546	100.63	95.169	97.933	100.42	103.41	105.51
45	97.234	100.06	102.59	105.64	107.79	102.11	104.96	107.52	110.61	112.77
50	103.99	106.90	109.51	112.64	114.85	108.97	119.90	114.54	117.70	119.92
60	117.28	120.35	123.10	126.40	128.72	122.44	125.55	128.32	131.64	133.97
70	130.34	133.56	136.44	139.90	142.32	135.68	138.93	141.83	145.31	147.74
80	143.21	146.56	149.57	153.17	155.69	148.77	152.12	155.13	158.75	161.28
90	155.92	159.41	162.53	166.27	168.89	161.64	165.12	168.24	172.01	174.62
100	168.49	172.09	175.32	179.19	181.91	174.38	177.98	181.21	185.09	187.80

Table 5.2 (continued)

m	q=10, α=.10	q=10, .05	q=10, .025	q=10, .01	q=10, .005	q=11, .10	q=11, .05	q=11, .025	q=11, .01	q=11, .005
0	36.624	38.507	40.219	42.301	43.777	40.523	42.473	44.243	46.394	47.916
1	38.548	40.466	42.207	44.324	45.824	42.452	44.435	46.233	48.416	49.960
2	40.431	42.381	44.151	46.301	47.824	44.343	46.357	48.181	50.396	51.962
3	42.276	44.258	46.055	48.238	49.783	46.200	48.243	50.093	52.339	53.926
4	44.089	46.101	47.925	50.140	51.706	48.025	50.097	51.973	54.248	55.856
5	45.872	47.915	49.765	52.009	53.596	49.823	51.923	53.824	56.128	57.756
6	47.629	49.701	51.576	53.851	55.458	51.596	53.724	55.648	57.981	59.628
7	49.363	51.462	53.362	55.666	57.293	53.346	55.501	57.449	59.808	61.474
8	51.075	53.202	55.126	57.457	59.104	55.075	57.256	59.227	61.614	63.298
9	52.767	54.920	56.868	59.227	60.892	56.785	58.991	60.985	63.398	65.100
10	54.440	56.620	58.591	60.977	62.661	58.477	60.708	62.724	65.163	66.883
11	56.097	58.302	60.296	62.708	64.410	60.153	62.408	64.445	66.909	68.647
12	57.738	59.969	61.984	64.422	66.142	61.813	64.092	66.151	68.639	70.394
13	59.365	61.620	63.656	66.120	67.857	63.458	65.762	67.841	70.354	72.125
14	60.977	63.257	65.314	67.803	69.557	65.091	67.417	69.516	72.053	73.841
15	62.577	64.880	66.959	69.472	71.243	66.710	69.059	71.178	73.739	75.542
16	64.165	66.491	68.590	71.127	72.915	68.318	70.689	72.827	75.410	77.231
17	65.742	68.091	70.210	72.770	74.574	69.914	72.308	74.466	77.073	78.907
18	67.308	69.679	71.818	74.402	76.221	71.500	73.915	76.093	78.722	80.571
19	68.864	71.257	73.415	76.022	77.857	73.075	75.511	77.707	80.358	82.224
20	70.410	72.825	75.002	77.637	79.482	74.641	77.097	79.311	81.986	83.867
22	73.475	75.933	78.147	80.820	82.701	77.747	80.245	82.500	85.210	87.121
24	76.508	79.006	81.257	83.973	85.885	80.818	83.355	85.641	88.398	90.336
26	79.510	82.048	84.335	87.092	89.032	83.859	86.336	88.756	91.552	93.519
28	82.484	85.062	87.382	90.179	92.149	86.872	89.487	91.840	94.676	96.669
30	85.433	88.050	90.405	93.242	95.246	89.859	92.510	94.897	97.771	99.791
35	92.701	95.414	97.848	100.78	102.84	97.226	99.966	102.45	105.40	107.48
40	99.853	102.65	105.16	108.18	110.31	104.46	107.29	109.83	112.88	115.03
45	106.87	10[illegible].77	112.36	115.47	117.65	111.60	114.50	117.11	120.25	122.46
50	113.85	116.80	119.46	122.65	124.90	118.63	121.62	124.30	127.52	129.77
60	127.52	130.63	133.42	136.76	139.11	132.47	135.60	138.41	141.78	144.15
70	140.94	144.18	147.10	150.59	153.03	146.05	149.32	152.24	155.76	158.22
80	154.14	157.52	160.55	164.19	166.72	159.41	162.80	165.85	169.49	172.05
90	167.18	170.68	173.82	177.58	180.21	172.59	176.10	179.26	183.03	185.68
100	180.06	183.68	186.93	190.82	193.54	185.61	189.25	192.50	196.40	199.12

CHAPTER VI

SUMMARY AND FURTHER STUDIES

1. Summary

This dissertation has been concerned with the null and non-null distribution problems of certain criteria for testing hypotheses about covariance matrices from several multivariate normal populations. Chapters I to IV considered problems dealing with real valued normal random variates, while Chapter V dealt with complex valued normal random variates.

In Chapter I a general method employing Laplace transformations was developed in order to obtain the distribution of $U^{(p)}$ (a constant times Hotelling's T_0^2 statistic) which can be used to test $H_0:\underset{\sim}{\Sigma}_1=\underset{\sim}{\Sigma}_2$ as well as the general linear hypothesis and the hypothesis of independence between two sets of multivariate normal variates. The exact null distribution of $U^{(p)}$ was obtained for $p = 3$, $m = 0(1)5$ and $p = 4$, $m = 0, 1$ and 2. The exact non-null density function of $U^{(2)}$ was developed by using zonal polynomials up to the sixth degree. Several approximations to the distribution of $U^{(p)}$ were given. Percentage points were calculated using the exact distributions.

Chapter II introduced the max U-ratio (R_1) criterion for testing $H_0:\underset{\sim}{\Sigma}_1=\ldots=\underset{\sim}{\Sigma}_k$. Exact ($p = 2$, $k = 2$) and approximate ($k = 2$) distributions for R_1 were considered. The approximate distributions rely on the F-type approximation to $U^{(p)}$ given in Chapter I. The non-null distribution of R_1 for $k = 2$, $p = 2$ and $m = 0$ was found by using the non-null density of $U^{(2)}$ developed in Chapter I.

Chapter III undertook the discussion of the LR criterion for testing $\underset{\sim}{\Sigma}_1=\underset{\sim}{\Sigma}_2$. The exact distribution of the LR criterion under the null hypothesis was given for $p = 2$ by considering transformations of the characteristic roots of $\underset{\sim}{S}_1 (\underset{\sim}{S}_1+\underset{\sim}{S}_2)^{-1}$, and for $p = 4$ by identifing the moments of the LR criterion with those of the product of certain independent beta variates. The non-null distribution of the LR criterion was found by employing zonal polynomials up to the sixth degree. Percentage points of the LR criterion for $p = 2$ were computed and the power of the R_1 and LR tests compared for selected alternatives.

Consideration of the hypothesis $H_0': \underset{\sim}{\Sigma}_1=\ldots=\underset{\sim}{\Sigma}_k=\lambda\underset{\sim}{\Sigma}_0$, where $\underset{\sim}{\Sigma}_0$ is given and λ unknown, was undertaken in Chapter IV where the max trace-ratio (R_2) test for H_0' was introduced. The distribution of R_2 was found for $k = 2, 3$ and 4 and the power function was obtained for $k = 2$. The distribution of R_2 is the same as Hartley's F_{max} test of the equality of several variances from normal populations. Selected percentage points of R_2 were obtained for $k = 2$ and 3.

The approximate distribution of the largest root of a complex Wishart matrix is found in Chapter V by using a technique due to Pillai. Selected percentage points were tabulated, and comparisons made between the exact and approximate percentage points.

2. Suggestions for Further Work

Although this dissertation has considered and solved a number of problems, several more need further study. Listed below are problems for future research.

(i) The distribution of $U^{(p)}$ for $p = 3$ and 4 and larger m

needs to be considered, as well as $p > 4$. Development of a general formula may be possible.

(ii) To make the max U-ratio test more applicable the exact and approximate distributions of R_1 for $k > 2$ need to be found.

(iii) The distribution of the LR criterion for larger p needs to be determined.

(iv) Further work needs to be done on the R_2 test of H_0' using the techniques developed in Chapter IV.

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APPENDICES

APPENDIX A

INTEGRALS OF R^* FUNCTIONS FOR THE DISTRIBUTIONS OF $U^{(3)}$ and $U^{(4)}$.

In order to obtain the distribution of $U^{(3)}$ and $U^{(4)}$ the integrals of $R^*(n;i,j,0;u)$ and $R^*(n;i,j,k,0;u)$ are required. These integrals are given below for $p = 3$ and $1 \leq j < i \leq 7$ and for $p = 4$ and $1 \leq k < j < i \leq 5$. The following notation is used: $r_i = n + i$, $s_i = 2n + i$ and $g^*(n;a,b;u)$ and $h(n,n';a,a';b,b';z)$ are as defined in Chapter I.

$$\int_0^z R^*(n;2,1,0;u)du = B_{21}\{1 - r_2 s_3 g^*(n;1,1;z) + 2r_1r_2s_5 g^*(n;3,2;z)$$

$$- r_1s_3s_5 g^*(n;4,2;z) - r_1r_2s_3 h(n,n;5,2;2,1;z)\},$$

where $$B_{21} = [r_1r_2^2r_3s_3s_5]^{-1}.$$

$$\int_0^z R^*(n;3,1,0;u)du = B_{31}\{3 - s_3s_5 g^*(n;1,1;z) + 2r_1r_3s_5 g^*(n;3,2;z)$$

$$+ r_1s_3s_5 g^*(n;4,2;z) - (4n + 11)r_1s_3 g^*(n;5,2,z)$$

$$- 3/2\, r_1s_3s_5 h(n,n;6,2;2,1;z)\},$$

where $B_{31} = [r_1r_2r_3r_4s_3s_5]^{-1}.$

$$\int_0^z R^*(n;\ 3,2,0;\ u)du = B_{32}\{3 - r_2s_5\ g^*(n;\ 1,1;\ z) + 2r_1s_5s_7\ g^*(n;\ 4,2;\ z)$$

$$- 2r_1r_2s_7\ g^*(n;\ 5,2;\ z) - r_1s_5s_7\ g^*(n;\ 6,2;\ z)$$

$$- 3r_1r_3s_5\ h(n,n;\ 7,2;\ 2,1;\ z)\},$$

where $$B_{32} = [r_1r_2r_3r_4s_5s_7]^{-1}.$$

$$\int_0^z R^*(n;\ 4,1,0;\ u)du = B_{41}\{6r_4 - 3r_3r_4s_3\ g^*(n;\ 1,1;\ z) + \tfrac{1}{2}r_1s_3s_7g^*(n;2,1;z)$$

$$+ 2r_1r_3r_4s_7\ g^*(n;\ 3,2;\ z) + r_1r_3s_3s_7\ g^*(n;\ 4,2;\ z) + r_1r_2s_3s_7g^*(n;5,2;z)$$

$$- \tfrac{1}{2}(6n + 19)r_1s_3s_7\ g^*(n;\ 6,2;\ z) - \frac{3}{2}\ (4n+15)r_1r_3s_3\ h(n,n;7,2;2,1;z)\},$$

where $B_{41} = [r_1r_2r_3r_4r_5s_3s_7]^{-1}.$

$$\int_0^z R^*(n;\ 4,2,0;\ u)du = B_{42}\{2 - r_2r_3\ g^*(n;1,1;\ z) + r_1r_3\ g^*(n;\ 2,1;\ z)$$

$$+ 2r_1\ r_3r_4\ g^*(n;\ 4,2;\ z) + 2r_1r_2r_3\ g^*(n;\ 5,2;\ z) - r_1r_2s_7g^*(n;6,2;z)$$

$$- r_1r_3s_7\ g^*(n;\ 7,2;\ z) - \tfrac{1}{2}(8n + 25)r_1r_3\ h(n,n;\ 8,2;\ 2,1;\ z)\},$$

where $$B_{42} = [r_1r_2r_3^2r_4r_5]^{-1}.$$

$$\int_0^z R^*(n;\ 4,3,0;\ u)du = D_{43}\{6r_2 - r_2r_3s_5\ g^*(n;\ 1,1;\ z)$$

$$+ \tfrac{3}{2}\ r_1s_5s_9\ g^*(n;\ 2,1;\ z) + 2(3n + 8)r_1r_2s_9\ g^*(n;\ 5,2;\ z)$$

$$- r_1r_2s_5s_9\ g^*(n;\ 6,2;\ z) - r_1r_2s_5s_9\ g^*(n;\ 7,2;\ z)$$

$$- \tfrac{1}{2}\ r_1s_5s_7s_9\ g^*(n;\ 8,2;\ z) - \tfrac{3}{2}(4n+13)r_1r_4s_5\ h(n,n;\ 9,2;\ 2,1;\ z)\},$$

where $D_{43} = [r_1r_2r_3r_4r_5s_5s_9]^{-1}$.

$$\int_0^z R^*(n;\ 5,1,0;\ u)du = D_{51}\{10r_3r_5 - 2r_3r_5s_3s_7\ g^*(n;\ 1,1;\ z)$$

$$+ \tfrac{3}{2}\ r_1r_3s_3s_7\ g^*(n;\ 2,1;\ z) + \tfrac{1}{2}\ r_1r_2s_3s_7\ g^*(n;\ 3,1;\ z)$$

$$+ 2r_1r_3r_4r_5s_7\ g^*(n;\ 3,2;\ z) + r_1r_3r_4s_3s_7\ g^*(n;\ 4,2;\ z)$$

$$+ r_1r_2r_3s_3s_7\ g^*(n;\ 5,2;\ z) + \tfrac{1}{2}r_1r_2s_3s_5s_7\ g^*(n;\ 6,2;\ z)$$

$$- \tfrac{1}{2}(16n^2 + 116n + 213)\ r_1r_3s_3\ g^*(n;\ 7,2;\ z)$$

$$- \tfrac{5}{4}\ (4n + 17)\ r_1r_3\ s_3s_7\ h(n,n;\ 8,2;\ 2,1;\ z)\}$$

where $D_{51} = [r_1r_2r_3\ r_4r_5r_6\ s_3s_7]^{-1}$

$$\int_0^z R^*(n;\ 5,2,0;\ u)du = D_{52}\{15r_4r_5 - 3r_2r_4r_5s_7\ g^*(n;\ 1,1;\ z)$$

$$+ \frac{3}{2}\ r_1s_7{}^2s_9\ g^*(n;\ 2,1;\ z) + r_1r_2s_7s_9\ g^*(n;\ 3,1;\ z)$$

$$+ 2r_1r_4r_5s_7s_9\ g^*(n;\ 4,2;\ z) + 2r_1r_2r_4s_7s_9\ g^*(n;\ 5,2;\ z)$$

$$+ r_1r_2s_5s_7s_9\ g^*(n;\ 6,2;\ z) - (6n^2 + 46n + 89)r_1r_2s_9g^*(n;\ 7,2;\ z)$$

$$- \tfrac{1}{2}(6n^2 + 46n + 89)r_1s_7s_9\ g^*(n;\ 8,2;\ z)$$

$$- \frac{15}{2}(2n^2 + 16n + 31)r_1r_4s_7\ h(n,n;\ 9,2;\ 2,1;\ z)\},$$

where $D_{52} = [r_1r_2r_3r_4r_5r_6s_7s_9]^{-1}$.

$$\int_0^z R^*(n;\ 5,3,0;\ u)du = D_{53}\{15r_2r_3 - r_2r_3s_5s_7\ g^*(n;\ 1,1;\ z)$$

$$+ \frac{9}{2}\ r_1r_4s_5s_7\ g^*(n;\ 2,1;\ z) + \frac{3}{2}\ r_1r_2s_5s_7\ g^*(n;\ 3,1;\ z)$$

$$+ 2(3n + 8)r_1r_2r_5s_7\ g^*(n;\ 5,2;\ z) + (3n + 8)r_1r_2s_5s_7\ g^*(n;\ 6,2;\ z)$$

$$- (4n + 17)r_1r_2r_3s_5\ g^*(n;\ 7,2;\ z) - \tfrac{1}{2}(4n + 17)r_1r_2s_5s_7\ g^*(n;\ 8,2;\ z)$$

$$- \tfrac{1}{2}(4n + 17)r_1r_4s_5s_7\ g^*(n;\ 9,2;\ z) - \frac{15}{4}\ r_1r_4s_5s_7{}^2\ h(n,n;\ 10,2;\ 2,1;\ z)\},$$

where $D_{53} = [r_1r_2r_3r_4r_5r_6s_5s_7]^{-1}$.

$$\int_0^z R^*(n;\,5,4,0;\,u)du = D_{54}\{10r_2r_4 - r_2r_3r_4s_7\, g^*(n;\,1,1;\,z)$$

$$- \tfrac{1}{2}(7n+24)r_1s_7s_{11}\, g^*(n;\,2,1;\,z) + 2r_1r_2s_7s_{11}\, g^*(n;\,3,1;\,z)$$

$$+ 4r_1r_2r_3s_7s_{11}\, g^*(n;\,6,2;\,z) - 2r_1r_2r_3r_4s_{11}\, g^*(n;\,7,2;\,z)$$

$$- r_1r_2r_3s_7s_{11}\, g^*(n;\,8,2;\,z) - r_1r_2r_4s_7s_{11}\, g^*(n;\,9,2;\,z)$$

$$-\tfrac{1}{2}r_1r_4s_7s_9s_{11}\, g^*(n;\,10,2;\,z) - \tfrac{5}{2}(4n+15)r_1r_4r_5s_7\, h(n,n;\,11,2;\,2,1;\,z)\},$$

where $\qquad D_{54} = [r_1r_2r_3r_4r_5r_6\, s_7s_{11}]^{-1}.$

$$\int_0^z R^*(n;\,6,1,0;\,u)du = D_{61}\{15r_5r_6 - 5r_4r_5r_6s_3\, g^*(n;\,1,1;\,z)$$

$$+ \tfrac{3}{4}(4n+15)r_1s_3s_9\, g^*(n;\,2,1;\,z) + \tfrac{3}{2}\, r_1r_2s_3s_9\, g^*(n;\,3,1;\,z)$$

$$+ \tfrac{1}{2}\, r_1r_2s_3s_9\, g^*(n;\,4,1;\,z) + 2\, r_1r_4r_5r_6s_9\, g^*(n;\,3,2;\,z)$$

$$+ r_1r_4r_5s_3s_9\, g^*(n;\,4,2;\,z) + r_1r_2r_4s_3s_9\, g^*(n;\,5,2;\,z)$$

$$+ \tfrac{1}{2}\, r_1r_2s_3s_5s_9\, g^*(n;\,6,2;\,z) + \tfrac{1}{2}r_1r_2s_3s_5s_9\, g^*(n;7,2;\,z)$$

$$- \tfrac{1}{4}(20n^2 + 160n + 333)r_1s_3s_9\, g^*(n;\,8,2;\,z)$$

$$- \tfrac{15}{4}(4n^2 + 38n + 91)\, r_1r_4s_3\, h(n,n;\,9,2;\,2,1;\,z)\},$$

where $\qquad D_{61} = [r_1r_2r_4r_5r_6r_7s_3s_9]^{-1}.$

$$\int_0^z R^*(n;\ 6,2,0;\ u)du = D_{62}\{6r_6 - 2r_2r_4r_6\ g^*(n;\ 1,1;\ z)$$

$$+ \tfrac{3}{2}(4n + 15)r_1r_4\ g^*(n;\ 2,1;\ z) + \tfrac{3}{2}\ r_1r_2s_7\ g^*(n;\ 3,1;\ z)$$

$$+ r_1r_2r_3\ g^*(n;\ 4,1;\ z) + 2r_1r_4r_5r_6\ g^*(n;\ 4,2;\ z)$$

$$+ 2r_1r_2r_4r_5\ g^*(n;\ 5,2;\ z) + r_1r_2r_4s_5\ g^*(n;\ 6,2;\ z)$$

$$+ r_1r_2r_3s_5\ g^*(n;\ 7,2;\ z) - \tfrac{1}{2}(8n^2+68n+149)r_1r_2\ g^*(n;\ 8,2;\ z)$$

$$- \tfrac{1}{2}(8n^2+68n+149)r_1r_4\ g^*(n;\ 9,2;\ z) - \tfrac{3}{4}(8n+41)r_1r_4s_7\ h(n;n;\ 10,2;\ 2,1;z)\},$$

where $D_{62} = [r_1r_2r_3r_4r_5r_6r_7]^{-1}$.

$$\int_0^z R^*(n;\ 6,3,0;\ u)du = D_{63}\{27r_2r_3r_5r_6 - 3r_2r_3r_4r_5r_6s_5\ g^*(n;\ 1,1;\ z)$$

$$+ \tfrac{3}{4}(12n^2 + 105n+232)r_1r_4s_5s_{11}\ g^*(n;\ 2,1;\ z) + \tfrac{9}{2}r_1r_2r_4{}^2s_5s_{11}\ g^*(n;3,1;z)$$

$$+ \tfrac{3}{2}\ r_1r_2r_3r_4s_5s_{11}\ g^*(n;\ 4,1,\ z)+2(3n+8)r_1r_2r_4r_5r_6s_{11}\ g^*(n;\ 5,2;\ z)$$

$$+(3n+8)r_1r_2r_4r_5s_5s_{11}\ g^*(n;\ 6,2;\ z)+(3n+8)r_1r_2r_3r_4s_5s_{11}g^*(n;\ 7,2;\ z)$$

$$-\tfrac{1}{2}(6n^2+55n+128)r_1r_2r_3s_5s_{11}\ g^*(n;\ 8,2;\ z)$$

$$-\tfrac{1}{2}(6n^2+55n+128)r_1r_2r_4s_5s_{11}\ g^*(n;\ 9,2;\ z)$$

$$-\tfrac{1}{4}(6n^2+55n+128)r_1r_4s_5s_9s_{11}\ g^*(n;\ 10,2;\ z)$$

$$- \tfrac{3}{4}(36n^3+486n^2+2151n+3136)r_1r_4r_5s_5\ h(n,n;\ 11,2;\ 2,1;\ z)\},$$

where $D_{63} = [r_1r_2r_3r_4^{\ 2}r_5r_6r_7s_5s_{11}]^{-1}$.

$$\int_0^z R^*(n;6,4,0;u)du = D_{64}\{6r_2 - r_2r_3r_4\ g^*(n;\ 1,\ 1;\ z)$$

$$+ 3(3n + 11)r_1r_5\ g^*(n;\ 2,\ 1;\ z) + 3r_1r_2s_9\ g^*(n;\ 3,1;\ z)$$

$$+ 2r_1r_2r_3\ g^*(n;\ 4,1;\ z) + 4r_1r_2r_3r_6\ g^*(n;\ 6,2;\ z)$$

$$+4r_1r_2r_3^{\ 2}\ g^*(n;\ 7,2;\ z) - 2r_1r_2r_3r_5\ g^*(n;\ 8,2;\ z)$$

$$- 2r_1r_2r_3r_5\ g^*(n;\ 9,2;\ z) - r_1r_2r_5s_9\ g^*(n;\ 10,2;\ z)$$

$$- r_1r_5^{\ 2}s_9\ g^*(n;\ 11,2;\ z) - \tfrac{3}{2}(8n^2+65n+133)r_1r_5\ h(n,n;\ 12,2;\ 2,1;\ z)\},$$

where $D_{64} = [r_1r_2r_3r_4r_5r_6r_7]^{-1}$.

$$\int_0^z R^*(n;\ 6,5,0;u)du = D_{65}\{15r_2r_3r_5 - r_2r_3r_4r_5s_7\ g^*(n;\ 1,1;\ z)$$

$$+ \tfrac{3}{4}(8n^2+65n+133)r_1s_7s_{13}\ g^*(n;\ 2,1;\ z)+\tfrac{3}{2}(3n+11)r_1r_2s_7s_{13}g^*(n;3,1;z)$$

$$+\tfrac{5}{2}\ r_1r_2r_3s_7s_{13}\ g^*(n;\ 4,1;\ z)+2(5n^2+35n+62)r_1r_2r_3s_{13}g^*(n;\ 7,2;\ z)$$

$$-r_1r_2r_3r_5s_7s_{13}\ g^*(n;\ 8,2;\ z)-r_1r_2r_3r_4s_7s_{13}\ g^*(n;\ 9\ ,2;\ z)$$

$$-\tfrac{1}{2}r_1r_2r_3s_7s_9s_{13}g^*(n;\ 10,2;\ z)-\tfrac{1}{2}r_1r_2r_5s_7s_9s_{13}\ g^*(n;\ 11,2;\ z)$$

$$-\tfrac{1}{4}r_1r_5s_7s_9s_{11}s_{13}g^*(n;12,2;z)-\tfrac{15}{4}(4n^2+34n+73)r_1r_5r_6s_7h(n;n;13,2;2,1;z)\},$$

where $D_{65} = [r_1r_2r_3r_4r_5r_6r_7s_7s_{13}]^{-1}$.

$$\int_0^z R^*(n;\ 7,1,0;\ u)du = D_{71}\{21r_4r_6r_7-3r_4r_6r_7s_3s_9g^*(n;\ 1,1;\ z)$$

$$+\tfrac{5}{4}(4n+17)r_1r_4s_3s_9\ g^*(n;\ 2,1;\ z)+\tfrac{3}{4}(4n+15)r_1r_2s_3s_9\ g^*(n;\ 3,1;\ z)$$

$$+\tfrac{3}{2}r_1r_2r_3s_3s_9\ g^*(n;\ 4,1;\ z)+\tfrac{1}{2}r_1r_2r_4s_3s_9\ g^*(n;\ 5,\ 1;\ z)$$

$$+2r_1r_4r_5r_6r_7s_9\ g^*(n;\ 3,2;\ z)+r_1r_4r_5r_6s_3s_9\ g^*(n;\ 4,2;\ z)$$

$$+r_1r_2r_4r_5s_3s_9\ g^*(n;\ 5,2;\ z)+\tfrac{1}{2}r_1r_2r_4s_3s_5s_9\ g^*(n;\ 6,2;\ z)$$

$$+\tfrac{1}{2}r_1r_2r_3s_3s_5s_9\ g^*(n;\ 7,2;\ z)+\tfrac{1}{4}r_1r_2s_3s_5s_7s_9\ g^*(n;\ 8,2;\ z)$$

$$-\tfrac{1}{4}(48n^3+644n^2+2962n+4677)r_1r_4s_3\ g^*(n;9,2;z)$$

$$-\tfrac{21}{8}(4n^2+42n+113)r_1r_4s_3s_9\ h(n,n;\ 10,2;\ 2,1;\ z)\},$$

where $D_{71} = [r_1r_2r_4r_5r_6r_7r_8s_3s_9]^{-1}$.

$$\int_0^z R^*(n;\ 7,2,0;\ u)du = D_{72}\{35r_4r_5r_6r_7-5r_2r_4r_5r_6r_7s_9g^*(n;1,1;z)$$

$$+\tfrac{5}{4}(4n+17)r_1r_4s_9{}^2s_{11}\ g^*(n;\ 2,1;\ z)+\tfrac{3}{2}(4n+15)r_1r_2r_4s_9s_{11}g^*(n;3,1;z)$$

$$+\tfrac{3}{2}\ r_1r_2r_3s_7s_9s_{11}\ g^*(n;\ 4,1;\ z)+r_1r_2r_3r_4s_9s_{11}g^*(n;\ 5,1;\ z)$$

$$+ \ 2r_1r_4r_5r_6r_7s_9s_{11}\ g^*(n;\ 4,2;\ z)+2r_1r_2r_4r_5r_6s_9s_{11}g^*(n;\ 5,2;\ z)$$

$$+ \ r_1r_2r_4r_5s_5s_9s_{11}\ g^*(n;\ 6,2;\ z)+r_1r_2r_3r_4s_5s_9s_{11}\ g^*(n;\ 7,2;\ z)$$

$$+\tfrac{1}{2}r_1r_2r_3s_5s_7s_9s_{11}\ g^*(n;\ 8,2;\ z)$$

$$-\tfrac{1}{2}(20n^3+280n^2+1338n+2181)r_1r_2r_4s_{11}\ g^*(n;\ 9,2;\ z)$$

$$-\tfrac{1}{4}(40n^4+780n^3+5756n^2+19080n+23991)r_1r_4s_9\ g^*(n;\ 10,2;\ z)$$

$$-\ \tfrac{35}{4}(4n^3+60n^2+296n+477)r_1r_4r_5s_9\ g^*(n;\ n;\ 11,2;\ 2,1;\ z)\},$$

where $\quad D_{72} = [r_1r_2r_3r_4r_5r_6r_7r_8s_9s_{11}]^{-1}.$

$$\int_0^z R^*(n;\ 7,3,0;\ u)du = D_{73}\{42r_2r_3r_5r_7-2r_2r_3r_5r_7s_5s_9\ g^*(n;\ 1,1;\ z)$$

$$+\ \tfrac{15}{4}\ (4n^2+37n+88)r_1r_5s_5s_9\ g^*(n;\ 2,1;\ z)$$

$$+\ \tfrac{3}{4}\ (12n^2+105n+232)r_1r_2s_5s_9\ g^*(n;\ 3,1;\ z)$$

$$+\ \tfrac{9}{2}\ r_1r_2r_3r_4s_5s_9\ g^*(n;\ 4,1;\ z)+\tfrac{3}{2}\ r_1r_2r_3r_4s_5s_9\ g^*(n;\ 5,1;\ z)$$

$$+2\ (3n+8)r_1r_2r_5r_6r_7s_9\ g^*(n;\ 5,2;\ z)+(3n+8)r_1r_2r_5r_6s_5s_9g^*(n;6,2;z)$$

$$+(3n+8)r_1r_2r_3r_5s_5s_9\ g^*(n;\ 7,2;\ z)$$

$$+\tfrac{1}{2}(3n+8)r_1r_2r_3s_5s_7s_9\ g^*(n;\ 8,2;\ z)$$

$$- \tfrac{1}{2}(16n^3+236n^2+1179n+1992)r_1r_2r_3s_5\ g^*(n;\ 9,2;\ z)$$

$$- \tfrac{1}{4}(16n^3+236n^2+1179n+1992)r_1r_2s_5s_9\ g^*(n;\ 10,2;\ z)$$

$$- \tfrac{1}{4}(16n^3+236n^2+1179n+1992)r_1r_5s_5s_9\ g^*(n;\ 11,2;\ z)$$

$$- \frac{1}{8}(168n^3+2394n^2+11193n+17304)r_1r_5s_5s_9\ h(n,n;\ 12,2;\ 2,1;\ z),$$

where $D_{73} = [r_1r_2r_3r_4r_5r_6r_7r_8s_5s_{11}]^{-1}$.

$$\int_0^z R^*(n;\ 7,4,0;\ u)du = D_{74}\{42r_2r_4r_6r_7-3r_2r_3r_4r_6r_7s_9\ g^*(n;\ 1,1;\ z)$$

$$+ \frac{3}{2}(11n^3+161n^2+781n+1251)r_1s_9s_{13}\ g^*(n;\ 2,1;\ z)$$

$$+3(4n^2+39n+97)r_1r_2s_9s_{13}\ g^*(n;\ 3,1;\ z)+3r_1r_2r_3s_9^2s_{13}\ g^*(n;\ 4,1;\ z)$$

$$+2r_1r_2r_3r_4s_9s_{13}\ g^*(n;\ 5,1;\ z)+4r_1r_2r_3r_6r_7s_9s_{13}\ g^*(n;\ 6,2;\ z)$$

$$+4r_1r_2r_3^2r_6s_9s_{13}\ g^*(n;\ 7,2;\ z)+2r_1r_2r_3^2s_7s_9s_{13}\ g^*(n;\ 8,2;\ z)$$

$$-2(3n^2+32n+87)r_1r_2r_3r_4s_{13}\ g^*(n;\ 9,2;\ z)$$

$$-(3n^2+32n+87)r_1r_2r_3s_9s_{13}\ g^*(n;\ 10,2;\ z)$$

$$-(3n^2+32n+87)r_1r_2r_5s_9s_{13}\ g^*(n;\ 11,2;\ z)$$

$$-\tfrac{1}{2}(12n^4+272n^3+2313n^2+8752n+12441)r_1r_5s_9\ g^*(n;\ 12,2;\ z)$$

$$- \frac{21}{2}(4n^3+61n^2+304n+501)r_1r_5r_6s_9\ h(n,n;\ 13,2;\ 2,1;\ z)\},$$

where $\qquad D_{74} = [r_1r_2r_3r_4r_5r_6r_7r_8s_9s_{13}]^{-1}$

$$\int_0^z R^*(n;\ 7,5,0;\ u)du = D_{75}\{35r_2r_3r_4r_5-r_2r_3r_4r_5s_7s_9\ g^*(n;\ 1,1;\ z)$$

$$+\tfrac{1}{4}(58n^2+495n+1073)r_1r_6s_7s_9\ g^*(n;\ 2,1;\ z)$$

$$+\tfrac{1}{4}(23n+91)r_1r_2s_7s_9s_{11}\ g^*(n;\ 3,1;\ z)+\frac{15}{2}r_1r_2r_3r_5s_7s_9g^*(n;4,1;z)$$

$$+\frac{5}{2}\ r_1r_2r_3r_4s_7s_9\ g^*(n;\ 5,1;\ z)+2(5n^2+35n+62)r_1r_2r_3r_7s_9g^*(n;7,2;z)$$

$$+(5n^2+35n+62)r_1r_2r_3s_7s_9\ g^*(n;\ 8,2;\ z)-(4n+23)r_1r_2r_3r_4r_5s_7g^*(n;9,2;z)$$

$$-\tfrac{1}{2}(4n+23)r_1r_2r_3r_4s_7s_9\ g^*(n;\ 10,2;\ z)-\tfrac{1}{2}(4n+23)r_1r_2r_3r_5s_7s_9g^*(n;11,2;z)$$

$$-\tfrac{1}{4}(4n+23)r_1r_2r_5s_7s_9s_{11}\ g^*(n;\ 12,2;\ z)-\tfrac{1}{4}(4n+23)r_1r_5r_6s_7s_9s_{11}g^*(n;13,2;z)$$

$$- \frac{35}{8}\ (4n^2+36n+83)r_1r_5r_6s_7s_9\ h(n,n;\ 14,2;\ 2,1;\ z)\},$$

where $\qquad D_{75} = [r_1r_2r_3r_4r_5r_6r_7r_8s_7s_9]^{-1}.$

$$\int_0^z R^*(n;\ 7,6,0;\ u)du = D_{76}\{21r_2r_3r_5r_6-r_2r_3r_4r_5r_6s_9\ g^*(n;\ 1,1;\ z)$$

$$+\ \frac{3}{4}(12n^2+109n+254)r_1r_5s_9s_{15}\ g^*(n;\ 2,1;\ z)$$

$$+\ \frac{3}{2}(5n^2+43n+94)r_1r_2s_9s_{15}\ g^*(n;\ 3,1;\ z)$$

$$+\tfrac{1}{2}(11n+43)r_1r_2r_3s_9s_{15}\ g^*(n;\ 4,1;\ z)+3r_1r_2r_3r_4s_9s_{15}g^*(n;5,1;z)$$

$$+(6n^2+46n+92)r_1r_2r_3s_9s_{15}\ g^*(n;\ 8,2;\ z)$$

$$-2r_1r_2r_3r_4r_5r_6s_{15}\ g^*(n;\ 9,2;\ z)-r_1r_2r_3r_4r_5s_9s_{15}\ g^*(n;\ 10,2;\ z)$$

$$-r_1r_2r_3r_4r_5s_9s_{15}\ g^*(n;\ 11,2;\ z)-\tfrac{1}{2}r_1r_2r_3r_5s_9s_{11}s_{15}g^*(n;\ 12,2;\ z)$$

$$-\tfrac{1}{2}r_1r_2r_5r_6s_9s_{13}s_{15}\ g^*(n;\ 13,2;\ z)-\tfrac{1}{4}r_1r_5r_6s_9s_{11}s_{13}s_{15}g^*(n;\ 14,2;\ z)$$

$$-\ \tfrac{21}{4}(4n^2+38n+93)r_1r_5r_6r_7s_9\ h(n,n;\ 15,2;\ 2,1;\ z)\},$$

where $\qquad D_{73} = [r_1r_2r_3r_4r_5r_6r_7r_8s_9s_{15}]^{-1}.$

$$\int_0^z R^*(n;\ 3,2,1,0;\ u)du = D_{321}\{3-r_2s_3s_5\ g^*(n;\ 1,1;\ z)$$

$$+2r_1r_2r_3s_5^{\ 2}\ g^*(n;\ 3,2;\ z)\ -\ r_1r_5s_3s_5^{\ 2}\ g^*(n;\ 4,2;\ z)$$

$$+2r_1r_2r_3s_3s_7g^*(\ n;\ 5,2;\ z)\ -\ r_1r_2s_3s_5^{\ 2}\ g^*(n;\ 6,2;\ z)$$

$$-r_1r_2r_3s_3s_5s_7\ h(n;\ n;\ 5,2;\ 2,1;\ z)+r_1r_2s_3s_5^{\ 2}s_7h(n,n;\ 6,2;\ 2,1;\ z)$$

$$-r_1r_2r_3s_3s_5^{\ 2}\ h(n,n;\ 7,2;\ 2,1;\ z)-2r_1r_2^{\ 2}s_3s_5\ h(n,n;\ 6,5;\ 2,2;\ z)\},$$

where $\qquad D_{321} = [r_1r_2^{\ 2}r_3^{\ 2}r_4s_3s_5^{\ 2}s_7]^{-1}.$

$$\int_0^z R^*(n;\ 4,2,1,0;\ u)du = D_{421}\{6 - 3r_2r_3s_3\ g^*(n;\ 1,1;\ z)$$

$$+2r_1r_2r_3s_5s_7\ g^*(n;\ 3,2;\ z) - 6r_1r_3s_3s_5\ g^*(n;\ 4,2;\ z)$$

$$-r_1r_2r_3s_3s_7\ g^*(n;\ 5,2;\ z) + r_1r_2s_3s_7^{\ 2}\ g^*(n;\ 6,2;\ z)$$

$$-3r_1r_2r_3s_3s_5\ g^*(n;\ 7,2;\ z) - r_1r_2r_3r_4s_3s_7\ h(n,n;\ 5,2;\ 2,1;\ z)$$

$$-\tfrac{1}{2}r_1r_2r_3s_3s_5s_7\ h(n,n;\ 6,2;\ 2,1;\ z) + (5n+19)r_1r_2r_3s_3s_5\ h(n,n;\ 7,2;2,1;z)$$

$$-\frac{3}{2}\ r_1r_2r_3s_3s_5s_7\ h(n,n;\ 8,2;\ 2,1;\ z) - 8r_1r_2^{\ 2}r_3s_3\ h(n,n;7,5;2,2;z)\},$$

where $\qquad D_{421} = [r_1r_2^{\ 2}r_3^{\ 2}\ r_4r_5s_3s_5s_7]^{-1}.$

$$\int_0^z R^*(n;\ 4,3,1,0;\ u)du = D_{431}\{18 - 3r_3s_3s_5\ g^*(n;\ 1,1;\ z)$$

$$+2r_1r_3^{\ 2}s_5s_7\ g^*(n;\ 3,2;\ z) + 3r_1r_3s_3s_5s_7\ g^*(n;\ 4,2;\ z)$$

$$-3(4n^2+40n+79)r_1r_3s_3\ g^*(n;\ 5,2;\ z) + r_1s_3s_5s_7s_9\ g^*(n;\ 6,2;\ z)$$

$$+15r_1r_3s_3s_5\ g^*(n;\ 7,2;\ z) - 3r_1r_3s_3s_5s_7\ g^*(n;\ 8,2;\ z)$$

$$-\frac{3}{2}r_1r_3s_3s_5s_7s_9\ \ h(n,n;\ 6,2;\ 2,1;\ z) + 3r_1r_3^{\ 2}s_3s_5s_9\ h(n,n;\ 7,2;\ 2,1;\ z)$$

$$+\frac{3}{2}\ r_1r_3s_3s_5s_7s_9\ h(n,n;\ 8,2;\ 2,1;\ z) - 3r_1r_3r_4s_3s_5s_7h(n,n;9,2;2,1;z)$$

$$-12r_1r_2r_3s_3s_5\ h(n,n;\ 7,6;\ 2,2;\ z)\},$$

where $D_{431} = [r_1 r_2 r_3^2 r_4 r_5 s_3 s_5 s_7 s_9]^{-1}$.

$$\int_0^z R^*(n;\ 4,3,2,0;\ u)du = D_{432}\{6 - r_2 r_3 s_5\ g^*(n;\ 1,1;\ z)$$

$$+2r_1 r_3 r_4 s_5 s_7\ g^*(n;\ 4,2;\ z) - 8r_1 r_2 r_3 s_7\ g^*(n;\ 5,2;\ z)$$

$$-r_1 r_4 s_5 s_7 s_9\ g^*(n;\ 6,2;\ z) + 3r_1 r_3 r_4 s_5 s_9\ g^*(n;\ 7,2;\ z)$$

$$-2r_1 r_2 r_3 s_5 s_7\ g^*(n;\ 8,2;\ z) - r_1 r_3 r_4 s_5 s_7\ g^*(n;\ 9,2;\ z)$$

$$-3r_1 r_3^2 r_4 s_5 s_9\ h(n,n;\ 7,2;\ 2,1;\ z) + \tfrac{1}{2}(5n+16) r_1 r_3 s_5 s_7 s_9 h(n,n;8,2;2,1;z)$$

$$-r_1 r_2 r_3 r_4 s_5 s_7\ h(n,n;\ 9,2;\ 2,1;\ z) - \tfrac{1}{2} r_1 r_3 r_4 s_5 s_7 s_9\ h(n,n;\ 10,2;\ 2,1;\ z)$$

$$-4r_1 r_2 r_3 s_5 s_7\ h(n,n;\ 8,6;\ 2,2;\ z)\},$$

where $D_{432} = [r_1 r_2 r_3^2 r_4^2 r_5 s_5 s_7 s_9]^{-1}$.

$$\int_0^z R^*(n;\ 5,2,1,0;\ u)du = D_{521}\{30r_5 - 6r_2 r_5 s_3 s_7\ g^*(n;\ 1,1;\ z)$$

$$+6r_1 r_2 r_4 r_5 s_5 s_7\ g^*(n;\ 3,2;\ z) + (2n^3-3n^2-104n-210) r_1 s_3 s_7\ g^*(n;\ 4,2;\ z)$$

$$-2r_1 r_2 s_3 s_7 s_9\ g^*(n;\ 5,2;\ z) - \tfrac{1}{2} r_1 r_2 s_3 s_5 s_7 s_9\ g^*(n;\ 6,2;\ z)$$

$$+(6n^2+46n+89) r_1 r_2 s_3 s_9\ g^*(n;\ 7,2;\ z) - \tfrac{3}{2}(4n+19) r_1 r_2 s_3 s_5 s_7 g^*(n;\ 8,2;\ z)$$

$$-r_1 r_2 r_4 r_5 s_3 s_7 s_9\ h(n,n;\ 5,2;\ 2,1;\ z) - \tfrac{1}{2} r_1 r_2 r_4 s_3 s_5 s_7 s_9 h(n,n;6,2;2,1;z)$$

$-\frac{1}{2}r_1r_2r_3s_3s_5s_7s_9\ h(n,n;\ 7,2;\ 2,1;\ z)$

$+\frac{3}{2}(3n+13)r_1r_2s_3s_5s_7s_9\ h(n,n;\ 8,2;\ 2,1;\ z)$

$-\frac{3}{2}(4n+19)r_1r_2r_4s_3s_5s_7\ h(n,n;\ 9,2;\ 2,1;\ z)$

$-4(5n+24)r_1r_2^2\ s_3s_7\ h(n,n;\ 8,5;\ 2,2;\ z)\},$

where $D_{521} = [r_1r_2^2r_3r_4r_5r_6s_3s_5s_7s_9]^{-1}.$

$$\int_0^z R^*(n;\ 5,3,1,0;\ u)\ du = D_{531}\{30s_9 - 2s_3s_5s_7s_9\ g^*(n;\ 1,1;\ z)$$

$+ 2r_1r_3r_4s_5s_7s_9\ g^*(n;\ 3,2;\ z) + 6r_1r_4^2s_3s_5s_7\ g^*(n;\ 4,2;\ z)$

$-(58n^2+401n+660)r_1s_3s_7\ g^*(n;\ 5,2;\ z) - \frac{3}{2}r_1r_4s_3s_5s_7s_9g^*(n;\ 6,2;\ z)$

$+(4n+17)r_1r_4s_3s_5s_9\ g^*(n;\ 7,2;\ z) - \frac{1}{2}(2n^2-29n-156)r_1s_3s_5s_7g^*(n;8,2;z)$

$-(8n+33)r_1r_4s_3s_5s_7\ g^*(n;\ 9,2;\ z) - \frac{3}{2}r_1r_4r_5s_3s_5s_7s_9h(n,n;6,2;2,1;z)$

$-\frac{3}{2}r_1r_3r_4s_3s_5s_7s_9\ h(n,n;\ 7,2;\ 2,1;\ z)$

$+\frac{1}{2}(7n+32)r_1r_3s_3s_5s_7s_9\ h(n,n;\ 8,2;\ 2,1;\ z)$

$+\frac{1}{2}(7n+32)r_1r_4s_3s_5s_7s_9\ h(n,n;\ 9,2;\ 2,1;\ z)$

$-\frac{1}{2}(8n+33)r_1r_4s_3s_5s_7s_9\ h(n,n;\ 10,2;\ 2,1;\ z)$

$$- 4(5n+21)r_1 r_2 s_3 s_5 s_7 \; h(n,n;\, 8,6;\, 2,2;\, z)\},$$

where $$B_{531} = [r_1 r_2 r_3 r_4^{\,2} r_5 r_6 s_3 s_5 s_7 s_9]^{-1}.$$

$$\int_0^z R^*(n;\, 5,3,2,0;\, u)du = D_{532}\{45r_4 - 3r_2 r_4 s_5 s_7 \; g^*(n;\, 1,1;\, z)$$

$$+2r_1 r_4 s_5 s_7^{\,2} s_9 \; g^*(n;\, 4,2;\, z) = 6nr_1 r_2 r_4 s_7^{\,2} \; g^*(n;\, 5,2;\, z)$$

$$-3(n+12)r_1 r_3 s_5 s_7^{\,2} \; g^*(n;\, 6,2;\, z)$$

$$-(4n^4+62n^3+368n^2+1997n+1044)r_1 s_5 \; g^*(n;\, 7,2;\, z)$$

$$+ \frac{3}{2}(4n^2+47n+108)r_1 s_5 s_7^{\,2} \; g^*(n;\, 8,2;\, z) - 6r_1 r_2 r_4 s_5 s_7^{\,2} \; g^*(n;\, 9,2;\, z)$$

$$- \frac{3}{2} r_1 r_4 s_5 s_7^{\,2} s_9 \; g^*(n;\, 10,2;\, z) \; -3r_1 r_3 r_4 r_5 s_5 s_7 s_9 h(n,n;7,2;2,1;z)$$

$$- \frac{3}{2} r_1 r_3 r_4 s_5 s_7^{\,2} s_9 \; h(n,n;\, 8,2;\, 2,1;\, z)$$

$$+ \frac{3}{2} (8n^2+65n+128)r_1 r_4 s_5 s_7^{\,2} \; h(n,n;\, 9,2;\, 2,1;\, z)$$

$$- \frac{3}{2} r_1 r_2 r_4 s_5 s_7^{\,2} s_9 \; h(n,n;\, 10,2;\, 2,1;\, z)$$

$$- \frac{3}{2} r_1 r_4 r_5 s_5 s_7^{\,2} s_9 \; h(n,n;\, 11,2;\, 2,1;\, z)$$

$$- 6(5n+18)r_1 r_2 r_3 s_5 s_7 \; h(n,n;\, 8,7;\, 2,2;\, z)\},$$

where $$D_{532} = [r_1 r_2 r_3 r_4^{\,2} r_5 r_6 s_5 s_7^{\,2} s_9]^{-1}.$$

$$\int_0^z R^*(n;\ 5,4,1,0;\ u)du = D_{541}\{60r_4s_9 - 6r_3r_4s_3s_7s_9\ g^*(n;\ 1,1;\ z)$$

$$+2r_1r_3r_4^2s_7^2s_9\ g^*(n;\ 3,2;\ z) + 3r_1r_3r_4s_3s_7^2s_9\ g^*(n;\ 4,2;\ z)$$

$$+3(4n+17)r_1r_2r_4s_3s_7^2\ g^*(n;\ 5,2;\ z)$$

$$-\tfrac{1}{2}(48n^4+1012n^3+7212n^2+21407n+22764)r_1s_3s_7\ g^*(n;\ 6,2;\ z)$$

$$+6r_1r_3r_4^2s_3s_9s_{11}g^*(n;\ 7,2;\ z)+(2n^3+71n^2+417n+636)r_1s_3s_7^2g^*(n;\ 8,2;\ z)$$

$$-(4n^2-5n-96)r_1r_4s_3s_7^2\ g^*(n;\ 9,2;\ z)-\tfrac{3}{2}(4n+17)r_1r_4s_3s_7^2s_9g^*(n;10,2;z)$$

$$-\tfrac{3}{2}(4n+15)r_1r_3r_4s_3s_7s_9s_{11}\ h(n,n;\ 7,2;\ 2,1;\ z)$$

$$+2r_1r_3r_4s_3s_7^2s_9s_{11}\ h(n,n;\ 8,2;\ 2,1;\ z)+2r_1r_3r_4s_3s_7^2s_9s_{11}h(n,n;9,2;2,1;z)$$

$$+\ r_1r_4s_3s_7^2s_9^2s_{11}\ h(n,n;\ 10,2;\ 2,1;\ z)$$

$$+\ \tfrac{3}{2}(4n+17)r_1r_4r_5s_3s_7^2s_9\ h(n,n;\ 11,2;\ 2,1;\ z)$$

$$-\ 4(20n^2+158n+309)r_1r_2r_3s_3s_7\ h(n,n;\ 8,7;\ 2,2;\ z)\},$$

where $$D_{541} = [r_1r_2r_3r_4^2r_5r_6s_3s_7^2s_9s_{11}]^{-1}.$$

$$\int_0^z R^*(n;\, 5,4,2,0;\, u)du = D_{542}\{30r_4 - 3r_2r_3r_4s_7\, g^*(n;\, 1,1;\, z)$$

$$+2r_1r_3r_4^2s_7s_9\, g^*(n;\, 4,2;\, z) + 6r_1r_2r_3r_4s_7s_9\, g^*(n;\, 5,2;\, z)$$

$$- 3(14n+47)r_1r_2r_4s_7\, g^*(n;\, 6,2;\, z)-3r_1r_3r_4s_7s_9s_{11}\, g^*(n;\, 7,2;\, z)$$

$$+(24n^4+408n^3+2562n^2+7038n+7128)r_1r_3\, g^*(n;\, 8,2;\, z)$$

$$+(2n^2+71n+204)r_1r_3r_4s_7\, g^*(n;\, 9,2;\, z)-6r_1r_2r_3r_4s_7s_9g^*(n;\, 10,2;\, z)$$

$$-3r_1r_3r_4r_5s_7s_9\, g^*(n;\, 11,2;\, z)-\tfrac{1}{2}(8n+25)r_1r_3r_4s_7s_9s_{11}h(n,n;8,2;2,1;z)$$

$$+(7n+24)r_1r_3r_4^2s_7s_{11}\, h(n,n;\, 9,2;\, 2,1;\, z)$$

$$+\tfrac{1}{2}(7n+24)r_1r_3r_4s_7s_9s_{11}\, h(n,n;\, 10,2;\, 2,1;\, z)$$

$$-3r_1r_2r_3r_4r_5s_7s_9\, h(n,n;\, 11,2;\, 2,1;\, z)-\tfrac{3}{2}r_1r_3r_4r_5s_7s_9s_{11}h(n,n;12,2;2,1;z)$$

$$-8(10n+37)r_1r_2r_3^2r_4\, h(n,n;\, 9,7;\, 2,2;\, z)\},$$

where $$D_{542} = [r_1r_2r_3^2r_5r_6s_7s_9s_{11}r_4^2]^{-1}.$$

$$\int_0^z R^*(n;\ 5,4,3,0;\ u)du = D_{543}\{30r_2 - r_2r_3s_5s_7\ g^*(n;\ 1,1;\ z)$$

$$+2(3n+8)r_1r_2r_5s_7s_9\ g^*(n;\ 5,2;\ z) + (n-3)r_1r_2s_5s_7s_9\ g^*(n;\ 6,2;\ z)$$

$$-r_1r_2r_5s_5s_9s_{11}\ g^*(n;\ 7,2;\ z) - \frac{3}{2}\ r_1r_5s_5s_7s_9s_{11}\ g^*(n;\ 8,2;\ z)$$

$$+(2n^4+426n^3+2799n^2+8061n+8580)r_1s_5\ g^*(n;\ 9,2;\ z)$$

$$-(3n+8)r_1r_2s_5s_7s_9\ g^*(n;\ 10,2;\ z) - 2r_1r_2r_5s_5s_7s_9\ g^*(n;11,2;z)$$

$$-\tfrac{1}{2}r_1r_5s_5s_7s_9s_{11}\ g^*(n;\ 12,2;\ z)$$

$$-\frac{3}{2}(4n+13)r_1r_4r_5s_5s_7s_{11}\ h(n,n;\ 9,2;\ 2,1;\ z)$$

$$+\frac{3}{2}(3n+11)r_1r_4s_5s_7s_9s_{11}\ h(n,n;\ 10,2;\ 2,1;\ z)$$

$$-r_1r_2r_3r_5s_5s_7s_9\ h(n,n;\ 11,2;\ 2,1;\ z) - \tfrac{1}{2}r_1r_2r_5s_5s_7s_9s_{11}h(n,n;12,2;2,1;z)$$

$$-\tfrac{1}{2}r_1r_5r_6s_5s_7s_9s_{11}\ h(n,n;\ 13,2;\ 2,1;\ z)$$

$$-\ 4(5n+19)r_1r_2r_3s_5s_9\ h(n,n;\ 10,7;\ 2,2;\ z)\},$$

where $$D_{543} = [r_1r_2r_3r_4r_5^2r_6s_5s_7s_9s_{11}]^{-1}.$$

APPENDIX B

EXPRESSIONS FOR c_{ij} FOR THE NON-NULL DENSITY FUNCTION OF $U^{(2)}$.

The expressions for the c_{ij} in terms of the elementary symmetric functions $d_1 = x_1 + x_2$ and $d_2 = x_1 x_2$ for the non-null density function of $U^{(2)}$ are given by:

$$c_{11} = 2 - d_1$$

$$c_{21} = 8 - 8d_1 + 3d_1^2 - 4d_2$$

$$c_{22} = 1 - d_1 + d_2$$

$$c_{31} = 16 - 24d_1 + 18d_1^2 - 5d_1^3 - 24d_2 + 12d_1d_2$$

$$c_{32} = 2 - 3d_1 + d_1^2 + 2d_2 - d_1d_2$$

$$c_{41} = 128 - 256d_1 + 288d_1^2 - 160d_1^3 + 35d_1^4 - 384d_2 + 384\,d_1d_2 - 120d_1^2d_2 + 48d_2^2$$

$$c_{42} = 8 - 16\,d_1 + 11d_1^2 - 3d_1^3 + 4d_2 - 4d_1d_2 + 3d_1^2\,d_2 - 4d_2^2$$

$$c_{43} = 1 - 2d_1 + d_1^2 + 2d_2 - 2d_1d_2 + d_2^2$$

$$c_{51} = 256 - 640\,d_1 + 960\,d_1^2 - 800d_1^3 + 350d_1^4$$
$$- 63\,d_1^5 - 1280d_2 + 1920\,d_1d_2 - 1200d_1^2d_2$$
$$+ 280\,d_1^3d_2 + 480d_2^2 - 240d_1d_2^2$$

$$c_{52} = 16 - 40\,d_1 + 42d_1^2 - 23d_1^3 + 5d_1^4 - 8d_2$$
$$+ 12d_1d_2 + 6d_1^2d_2 - 5d_1^3d_2 - 24d_2^2 + 12d_1d_2^2$$

$$c_{53} = 2 - 5d_1 + 4d_1^2 - d_1^3 + 4d_2 - 6d_1d_2 + 2d_1^2d_2$$
$$+ 2d_2^2 - d_1d_2^2$$

$$c_{61} = 1024 - 3072d_1 + 5760d_1^2 - 6400d_1^3 + 4200d_1^4$$
$$- 1512d_1^5 + 231d_1^6 - 7680d_2 + 15360d_1d_2$$
$$- 14400d_1^2d_2 + 6720d_1^3d_2 - 1260d_1^4d_2$$
$$+ 5760d_2^2 - 5760d_1d_2^2 + 1680a_1^2a_2^2 - 320d_2^3$$

$$c_{62} = 128 - 384d_1 + 544d_1^2 - 448d_1^3 + 195d_1^4$$
$$- 35d_1^5 - 256d_2 + 512d_1d_2 - 216d_1^2d_2 - 40d_1^3d_2$$
$$+ 35d_1^4d_2 - 336d_2^2 + 336d_1d_2^2 - 120d_1^2d_2^2$$
$$+ 48d_2^3$$

$$c_{63} = 8 - 24d_1 + 27d_1^2 - 14d_1^3 + 3d_1^4 + 12d_2$$
$$- 24d_1d_2 + 18d_1^2d_2 - 6d_1^3d_2 + 3d_1^2d_2^2 - 4d_2^3$$

$$c_{64} = 1 - 3d_1 + 3d_1^2 - d_1^3 + 3d_2 - 6d_1d_2 + 3d_1^2d_2$$
$$+ 3d_2^2 - 3d_1d_2^2 + d_2^3$$

APPENDIX C

Ψ_{ij} FUNCTIONS FOR THE NON-NULL DENSITY OF $U^{(2)}$

The $\Psi_{ij}(t)$ functions in the Laplace transform of $U^{(2)}$ with respect to its non-null density function are provided below in terms of the R functions introduced in Section 9 of Chapter I. For ease in writing we will use the simplified notation $R(a_2,a_1)$ for $R(n;a_2,a_1;t)$. Note: to obtain $\Psi_{ij}^*(u)$ merely replace R by R^* in the following expressions.

$$\Psi_{11}(t) = 2\,R(r+1,s) - R(r+2,s) - R(r+1,s+1)$$

$$\Psi_{21}(t) = 8\,R(r+1,s) - 8\,R(r+2,s) - 8\,R(r+1,s+1) + 3\,R(r+3,s) + 2\,R(r+2,s+1) + 3\,R(r+1,s+2)$$

$$\Psi_{22}(t) = R(r+1,s) - R(r+2,s) - R(r+1,s+1) + R(r+2,s+1)$$

$$\Psi_{31}(t) = 16\,R(r+1,s) - 24\,R(r+2,s) - 24\,R(r+1,s+1) + 18\,R(r+3,s) + 12\,R(r+2,s+1) + 18\,R(r+1,s+2) - 5\,R(r+4,s) - 3\,R(r+3,s+1) - 3\,R(r+2,s+2) - 5\,R(r+1,s+3)$$

$$\Psi_{32}(t) = 2\ R(r+1,s) - 3\ R(r+2,s) - 3\ R(r+1,s+1) + R(r+3,s) + 4(r+2,s+1) + R(r+1,s+2) - R(r+3,s+1) - R(r+2,s+2)$$

$$\Psi_{41}(t) = 128\ R(r+1,s) - 256\ R(r+2,s) - 256\ R(r+1,s+1) + 288\ R(r+3,s) + 192\ R(r+2,s+1) + 288\ R(r+1,s+2) - 160\ R(r+4,s) - 96\ R(r+3,s+1) - 96\ R(r+2,s+2) - 160\ R(r+1,s+3) \pm 35\ R(r+5,s) + 20\ R(r+4,s+1) + 18\ R(r+3,s+2) + 20\ R(s+2,s+3) + 35\ R(r+1,s+4)$$

$$\Psi_{42}(t) = 8\ R(r+1,s) - 16\ R(r+2,s) - 16\ R(r+1,s+1) + 11\ R(r+3,s) + 26\ R(r+2,s+1) + 11\ R(r+1,s+2) - 3\ R(r+4,s) - 13\ R(r+3,s+1) - 13\ R(r+2,s+2) - 3\ R(r+1,s+4) + 3\ R(r+4,s+1) + 2\ R(r+3,s+2) + 3\ R(r+2,s+3)$$

$$\Psi_{43}(t) = R(r+1,s) - 2\ R(r+2,s) - 2\ R(r+1,s+1) + R(r+3,s) + 4\ R(r+2,s+1) + R(r+1,s+2) - 2\ R(r+3,s+1) - 2\ R(r+2,s+2) + R(r+3,s+2)$$

$$\Psi_{51}(t) = 256(r+1,s) - 640R(r+2,s) - 640R(r+1,s+1) + 960R(r+3,s) + 640R(r+2,s+1) + 960R(r+1,s+2) - 800\ R(r+4,s) - 480R(r+3,s+1) - 480R(r+2,s+2) - 800R(r+1,s+3) + 350R(r+5,s) + 200R(r+4,s+1) + 180R(r+3,s+2) + 200R(r+2,s+3) + 350R(r+1,s+4) - 63R(r+6,s) - 35R(r+5,s+1) - 30R(r+4,s+2) - 30R(r+3,s+3) - 35R(r+2,s+4) - 63R(r+1,s+5)$$

$$\Psi_{52}(t) = 16\,R(r+1,s) - 40\,R(r+2,s) - 40\,R(r+1,s+1)$$
$$+ 42\,R(r+3,s) + 76\,R(r+2,s+1) + 42\,R(r+1,s+2)$$
$$- 23\,R(r+4,s) - 57\,R(r+3,s+1) - 57\,R(r+2,s+2)$$
$$- 23\,R(r+1,s+3) + 5\,R(r+5,s) + 26\,R(r+4,s+1)$$
$$+ 18\,R(r+3,s+2) + 26\,R(r+2,s+3) + 5\,R(r+1,s+4)$$
$$- 5\,R(r+5,s+1) - 3\,R(r + 4,s+2) - 3\,R(r+3,s+3)$$
$$- 5\,R(r+2,s+4)$$

$$\Psi_{53}(t) = 2\,R(r+1,s) - 5\,R(r+2,s) - 5\,R(r+1,s+1)$$
$$+ 4\,R(r+3,s) + 12\,R(r+2,s+1) + 4\,R(r+1,s+2)$$
$$- R(r+4,s) - 9\,R(r+3,s+1) - 9R(r+2,s+2)$$
$$- R(r+1,s+3) + 2\,R(r+4,s+1) + 6\,R(r+3,s+2)$$
$$+ 2\,R(r+2,s+3) - R(r+4,s+2) - R(r+3,s+3)$$

$$\Psi_{61}(t) = 1024\,R(r+1,s) - 3072\,R(r+2,s) - 3072\,R(r+1,s+1)$$
$$+ 5760\,R(r+3,s) + 3840\,R(r+2,s+1) + 5760\,R(r+1,s+2)$$
$$- 6400\,R(r+4,s) - 3840\,R(r+3,s+1) - 3840\,R(r+2,s+2)$$
$$- 6400\,R(r+1,s+3) + 4200\,R(r+5,s) + 2400\,R(r+4,s+1)$$
$$+ 2160\,R(r+3,s+2) + 2400\,R(r+2,s+3) + 4200\,R(r+1,s+4)$$
$$- 1512\,R(r+6,s) - 840R(r+5,s+1) - 720\,R(r+4,s+2)$$
$$- 720\,R(r+3,s+3) - 840\,R(r+2,s+4) - 1512\,R(r+1,s+5)$$
$$+ 231\,R(r+7,s) + 126\,R(r+6,s+1) + 105\,R\,(r+5,\ s+2)$$
$$+ 100\,R(r+4,s+3) + 105\,R(r+3,s+4) + 126\,R(r+2,s+5)$$
$$+ 231\,R(r+1,s+6)$$

$$\Psi_{62}(t) = 128\ R(r+1,s) - 384\ R(r+2,s) - 384\ R(r+1,s+1)$$
$$+ 544\ R(r+3,s) + 832\ R(r+2,s+1) + 544\ R(r+1,s+2)$$
$$- 448\ R(r+4,s) - 832\ R(r+3,s+1) - 832\ R(r+2,s+2)$$
$$- 448\ R(r+1,s+3) + 195\ R(r+5,s) + 564\ R(r+4,s+1)$$
$$+ 402\ R(r+3,\ s+2) + 564\ R(r+2,s+3) + 195\ R(r+1,s+4)$$
$$- 35\ R(r+6,s) - 215\ R(r+5,s+1) - 134\ R(r+4,s+2)$$
$$- 134\ R(r+3,s+3) - 215\ R(r+2,s+4) - 35\ R(r+1,s+5)$$
$$+ 35\ R(r+6,s+1) + 20\ R(r+5,s+2) + 18\ R(r+4,s+3)$$
$$+ 20\ R(r+3,s+4) +35\ R(r+2,s+5)$$

$$\Psi_{63}(t) = 8\ R(r+1,s) - 24\ R(r+2,s) - 24\ R(r+1,s+1)$$
$$+ 27\ R(r+3,s) + 66\ R(r+2,s+1) + 27\ R(r+1,s+2)$$
$$- 14\ R(r+4,s) - 66\ R\ (r+3,s+1) - 66\ R(r+2,s+2)$$
$$-14\ R(r+1,s+3) + 3\ R(r+5,s) + 30\ R(r+4,s+1)$$
$$+ 54\ R(r+3,s+2) + 30\ R(r+2,s+3) + 3\ R(r+1,s+4)$$
$$- 6\ R(r+5,s+1) - 18\ R(r+4,s+2) - 18\ R(r+3,s+3)$$
$$- 6\ R(r+2,s+4) + 3\ R(r+5,s+2) + 2\ R(r+4,s+3)$$
$$+ 3\ R(r+3,s+4)$$

$$\Psi_{64}(t) = R(r+1,s) - 3\ R(r+2,s) - 3\ R(r+1,s+1)$$
$$+ 3\ R(r+3,s) + 9\ R(r+2,s+1) + 3\ R(r+1,s+2)$$
$$- R(r+4,s) - 9\ R(r+3,s+1) - 9\ R(r+2,s+2)$$
$$- R(r+1,s+3) + 3\ R(r+4,s+1) + 9\ R(r+3,s+2)$$
$$+ 3\ R(r+2,s+3) - 3\ R(r+4,s+2) - 3\ R(r+3,s+3)$$
$$+ R(r+4,s+3)$$

APPENDIX D

g_{ij} COEFFICIENTS FOR THE NON-NULL DENSITY OF $U^{(2)}$ WHEN $m = 0$

The coefficients g_{ij} for the non-null density function of $U^{(2)}$ for $m = 0$ are given below in terms of the A_{ij} which may be found in [21].

$$\begin{aligned} g_{10} = 1 &+ 2A_{11} + 8A_{21} + A_{22} + 16A_{31} + 2A_{32} \\ &+ 128A_{41} + 8A_{42} + A_{43} + 256A_{51} + 16A_{52} \\ &+ 2A_{53} + 1024A_{61} + 128A_{62} + 8A_{63} + A_{64} \end{aligned}$$

$$\begin{aligned} g_{11} = &-A_{21} + A_{22} - 6A_{31} + 3A_{32} - 96A_{41} + 15A_{42} \\ &+ 3A_{43} - 320A_{51} + 34A_{52} + 8A_{53} - 1920A_{61} \\ &+ 288a_{62} + 39A_{63} + 6A_{64} \end{aligned}$$

$$\begin{aligned} g_{12} = &-2A_{41} - A_{42} + A_{43} - 20A_{51} - 8A_{52} + 4A_{53} \\ &- 240A_{61} - 162A_{62} + 24A_{63} + 6A_{64} \end{aligned}$$

$$g_{13} = -5A_{61} - 2A_{62} - A_{63} + A_{64}$$

$$\begin{aligned} g_{20} = &-A_{11} - 8A_{21} - A_{22} - 24A_{31} - 3A_{32} - 256A_{41} \\ &- 16A_{42} - 2A_{43} - 640A_{51} - 40A_{52} - 5A_{53} \\ &- 3072A_{61} - 384A_{62} - 24A_{63} - 3A_{64} \end{aligned}$$

$$g_{21} = 2A_{31} - A_{32} + 64A_{41} - 10A_{42} - 2A_{43} + 320A_{51} - 34A_{52} - 8A_{53} + 2560A_{61} - 384A_{62} - 52A_{63} - 8A_{64}$$

$$g_{22} = 5A_{51} + 2A_{52} - A_{53} + 120A_{61} + 81A_{62} - 12A_{63} - 3A_{64}$$

$$g_{30} = 3A_{21} + 18A_{31} + A_{32} + 288A_{41} + 11A_{42} + A_{43} + 960A_{51} + 42A_{52} + 4A_{53} + 5760A_{61} + 544A_{62} + 27A_{63} + 3A_{64}$$

$$g_{31} = -15A_{41} + 3A_{42} - 150A_{51} + 21A_{52} + 2A_{53} - 1800A_{61} + 369A_{62} + 27A_{63} + 3A_{64}$$

$$g_{32} = -21A_{61} - 15A_{62} + 3A_{63}$$

$$g_{40} = -5A_{31} - 160A_{41} - 3A_{42} - 800A_{51} - 23A_{52} - A_{53} - 6400A_{61} - 448A_{62} - 14A_{63} - A_{64}$$

$$g_{41} = 28A_{51} - 5A_{52} + 672A_{61} - 180A_{62} - 6A_{63}$$

$$g_{50} = 35A_{41} + 350A_{51} + 5A_{52} + 4200A_{61} + 195A_{62} + 3A_{63}$$

$$g_{51} = -105A_{61} + 35A_{62}$$

$$g_{60} = -63A_{51} - 1512A_{61} - 35A_{62}$$

$$g_{70} = 231\ A_{61}\ .$$

APPENDIX E

THE COEFFICIENTS C''_{ij} FOR THE NON-NULL DISTRIBUTION OF THE LR CRITERION FOR $k = 2$ AND $p = 2$

The coefficients C''_{ij} for the non-null distribution of the LR criterion for $k = 2$ and $p = 2$ are given below in terms of the constants A''_{ij} which may be found in [21] and which are also provided here.

$C''_{00} = 1$, $C''_{10} = A''_{11}$, $C''_{20} = 3A''_{21}$, $C''_{01} = A''_{22} - 4\,A''_{21}$,

$C''_{30} = 5\,A''_{31}$, $C''_{11} = A''_{32} - 12\,A''_{31}$, $C''_{40} = 35\,A''_{41}$,

$C''_{21} = 3A''_{43} - 120\,A''_{41}$, $C''_{02} = A''_{43} - 4\,A''_{42} + 48\,A''_{41}$, $C''_{50} = 63\,A''_{51}$,

$C''_{31} = 5\,A''_{52} - 280\,A''_{51}$, $C''_{12} = A''_{53} - 12\,A''_{52} + 240\,A''_{51}$, $C''_{60} = 231A''_{61}$,

$C''_{41} = 35\,A''_{62} - 1260\,A''_{61}$, $C''_{22} = 3\,A''_{63} - 120\,A''_{62} + 1680\,A''_{61}$,

$C''_{03} = A''_{64} - 4\,A''_{63} + 48\,A''_{62} - 320\,A''_{61}$,

where

$A''_{11} = \nu\, b_1/4$

$$A''_{21} = \nu(\nu+2)\ b_{21}/(8.4!) \ ,$$

$$A''_{22} = \nu(\nu-1)\ b_2/6 \ ,$$

$$A''_{31} = \nu(\nu+2)(\nu+4)\ b_{31}/(2^5.5!) \ ,$$

$$A''_{32} = \nu(\nu+2)(\nu-1)\ b_1 b_2/40 \ ,$$

$$A''_{41} = 3[\nu(\nu+2)\ldots(\nu+6)]\ b_{41}/(2^7.8!) \ ,$$

$$A''_{42} = \nu(\nu+2)(\nu+4)(\nu-1)\ b_{42}/(7.2^4.4!) \ ,$$

$$A''_{43} = \nu(\nu+1)(\nu+2)(\nu-1)\ b_2{}^2/120 \ ,$$

$$A''_{51} = [\nu(\nu+2)\ldots(\nu+8)]\ b_{51}/(3.2^9.8!) \ ,$$

$$A''_{52} = [\nu(\nu+2)\ldots(\nu+6)(\nu-1)]\ b_{52}/(3.2^5.6!) \ ,$$

$$A''_{53} = \nu(\nu+1)(\nu+2)(\nu+4)(\nu-1)\ b_1 b_2{}^2/(5.7.2^5) \ ,$$

$$A''_{61} = [\nu(\nu+2)\ldots(\nu+10)]\ b_{61}/(33.2^{12}.8!) \ ,$$

$$A''_{62} = 3[\nu(\nu+2)\ldots(\nu+8)(\nu-1)]\ b_{62}/(11.2^8.8!) \ ,$$

$$A''_{63} = 5[\nu(\nu+1)(\nu+2)\ldots(\nu+6)(\nu-1)]\ b_{63}/(3.2^5.7!) \ ,$$

$$A''_{64} = \nu(\nu+1)(\nu+2)(\nu+3)(\nu+4)(\nu-1)\ b_2{}^3/5.7.9.2^4) \ ,$$

where $\nu = n_1+n_2$, $b_1 = 2 - (1/\gamma_1+1/\gamma_2)$,

$b_2 = [1-1/\gamma_1][1-1/\gamma_2]$, $b_{21} = 3b_1^2 - 4b_2$, $b_{22} = b_2$,

$b_{31} = 5b_1^3 - 12b_1b_2$, $b_{32} = b_1b_2$, $b_{41}=35b_1^4-120b_1^2b_2+48b_2^2$,

$b_{42} = b_2b_{21}$, $b_{43} = b_2^2$, $b_{51} = 63b_1^5 - 280\ b_1^3b_2+240b_1b_2^2$,

$b_{52} = b_2b_{31}$, $b_{53} = b_1b_2^2$, $b_{61}=231b_1^6-1260b_1^4b_2+1680b_1^2b_2^2-320b_2^3$,

$b_{62} = b_2\ b_{41}$, $b_{63} = b_2^2\ b_{21}$, $b_{64} = b_2^3$.

APPENDIX F

$h_{ij}(z)$ FUNCTIONS FOR THE NON-NULL DENSITY FUNCTION OF THE LR CRITERIA.

Tabulated below are the functions $h_{ij}(z)$ appearing in the non-null density function of the LR criterion for testing $\Sigma_1 = \Sigma_2$ (see (4.5) of Chapter III. The following notation is used:

$$H = (1-4z^{\frac{1}{2}})^{\frac{1}{2}}$$

$$a = (1-H)^2/4$$

and

$$b = (1+H)^2/4 \quad .$$

$$h_{00}(z) = \ln(b/a)$$

$$h_{10}(z) = \ln(b/a)$$

$$h_{20}(z) = (H+H^3)/2 + (1-2z)\ln(b/a)$$

$$h_{01}(z) = H$$

$$h_{30}(z) = 3(H+H^3)/2 + (1-6z)\ln(b/a)$$

$$h_{11}(z) = (5\,H+H^3)/4 - z\ln(b/a)$$

$$h_{40}(z) = (97\,H + 103H^3 + 7H^5 + H^7)/32 - 2z\,(H+H^3) + (1-12z + 6z^2)\ln(b/a)$$

$$h_{21}(z) = 25/16\,H + 17/24\,H^3 + H^5/16 - z\,H - 2z\ln(b/a)$$

$$h_{02}(z) = (H+H^3)/4$$

$$h_{50}(z) = (165/32 - 10z)H + (195/32 - 10z)H^3 + (35H^5 + 5H^7)/32 + (1-20z + 30z^2)\ln(b/a)$$

$$h_{31}(z) = (125/64 - 4z)H + (95/64 - z)H^3 + (19H^5 + H^7)/64 + (-3z + 3z^2)\ln(b/a)$$

$$h_{12}(z) = (5/16 - z)H + 11/24\,H^3 + H^5/16$$

$$h_{60}(z) = (4081/512 - 483/16z + 15/2\,z^2)H + (16615/1536 - 501/16z + 15/2z^2)H^3 + (873/256 - 21/16z)H^5 + (153/256 - 3/16z)H^7 + (55H^9 + 3H^{11})/1536 + (1-30z + 90z^2 - 20z^3)\ln(b/a)$$

$$h_{41}(z) = (625/256 - 163/16z + 2z^2)H + (175/64 - 37/8z)H^3 + (583/640 - 3/16z)H^5 + (28H^7 + H^9)/256 + (-4z + 12z^2)\ln(b/a)$$

$$h_{22}(z) = (25/64 - 5/2\ z)H + (149/192 - z/2)H^3 + (15H^5 + H^7)/64 + z^2 \ln (b/a)$$

$$h_{03}(z) = H/16 + 5/24\ H^3 + H^5/16$$

VITA

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