A Class of Non-Eliminating Sequential Multiple Decision Procedures *

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1. Introduction and Summary

This paper is concerned with the multiple decision (selection and ranking) problem for k independent normal populations having unknown means $\theta_1, \ldots, \theta_k$ and a common known variance θ^2. The formulation of the selection and ranking problems has been based on the following two approaches: (i) the"indifference zone" approach and (ii) the "subset-selection" approach. (A brief discussion of the two approachs is given in [7].) Most of the work on the subset-selection problems deals with the fixed-sample size procedures. In this paper a class of sequential and multi-stage procedures using the "subset-selection" approach is defined and investigated. This class consists of rules of a non-eleminating type; a rule belonging to this class selects and rejects populations at various stages but continues taking samples from all populations until the procedure terminates. The sequential subset selection rule investigated in this paper assumes that the successive differences between the ordered θ_i's are known.

Section 2 of this paper deals with the definition of a general class of selection procedures, while Section 3 investigates monotonicity properties of the class. The remaining sections of this paper investigate a particular linear sub-class of the class defined in Section 2. Sections 4 and 5 use a random walk approach to find exact and approximate expressions for the probability of selecting a population and the expected number of stages to reach a decision. An approximate minimax rule for choosing a specific procedure that minimizes the maximum number of samples needed to make a decision on each population, is discussed in Section 6. Finally, the last section offers some comparisons with a fixed-sample size procedure for slippage and equally spaced means configurations.

2. Definition of the General Class of Procedures

In this section the general nature of a non-elimating sequential multiple decision procedure will be outlined. Let $\pi_1, \pi_2, \ldots, \pi_k$ denote k given normal populations with means $\theta_1, \theta_2, \ldots, \theta_k$ respectively and common known variance σ^2. Let $\theta_{[1]} \leq \theta_{[2]} \leq \cdots \leq \theta_{[k]}$ be the ranked means, and $\pi_{(j)}$ (unknown) be the population with mean $\theta_{[j]}$. The object is to select a small subset of $\pi_1, \ldots, \pi_k$ so as to guarantee, with a prescribed probability P*, that the population with the largest (or equivalently the smallest) mean is included in the selected subset. We denote this event by CS (correct selection). If there are more than one population with mean $\theta_{[k]}$ ($\theta_{[1]}$) then one of them will be assumed to have been tagged as the best population. The sequential procedure will be a modification of the following (see [5], [6], and [7]) fixed sample-size procedure R(n).

R(n): Take a sample of size n from each of the k populations π_i, $i = 1,2,\ldots,k$ and select π_i if and only if $\bar{x}_i \geq \bar{x}_{max} - \sigma d/\sqrt{n}$ where d is chosen such that $\inf_{\Omega} P\{CS|R(n)\} = P^*$ and $\Omega = \{\underline{\theta}:\underline{\theta} = (\theta_1, \ldots, \theta_k), -\infty < \theta_i < \infty, i = 1,2,\ldots,k\}$. $\bar{x}_i$, $i = 1,2,\ldots,k$ denotes the sample mean from π_i and $\bar{x}_{max} = \max_{1 \leq i \leq k} \bar{x}_i$

It has been shown in [7] that under this formulation,

$$p_i(n) = P(\text{selecting } \pi_{(i)}|R(n)) = P(\bar{x}_{(i)} \geq \bar{x}_{max} - \sigma d\, n^{-\frac{1}{2}}) \tag{2.1}$$

$$= \int_{-\infty}^{\infty} \Big[\prod_{\substack{j=1 \\ j \neq i}}^{k} \Phi(x + d + (\theta_{[i]} - \theta_{[j]})n^{\frac{1}{2}}/\sigma)\Big] \varphi(x)dx$$

where $\Phi(\cdot)$ and $\varphi(\cdot)$ refer to the cdf and the density of the standard normal random variable. Then $P\{cs|R(n)\} = \int_{-\infty}^{\infty}\Big[\prod_{j=1}^{k-1} \Phi(x+d+(\theta_{[k]}-\theta_{[j]})n^{\frac{1}{2}}/\sigma\Big]\varphi(x)dx$

and thus the infimum of the probability of a correct selection occurs when $\theta_1 = \cdots = \theta_k = \theta$ and is independent of the common value θ. Hence d is

$\eta \equiv \eta_{b,c} = (\{b_m\}, \{c_m\})$ such that for all $m \geq 1$,

(2.2) (i) $b_m \leq b_{m+1}$, $c_m \leq c_{m+1}$, (ii) $b_m < c_m$, (iii) $\lim_{m\to\infty} b_m = \infty$,

(iv) $P\{\bigcap_{m=1}^{\infty} [b_m < S_{im} < c_m]\} = 0, \forall\, i = 1,2,\ldots,k$.

The existance of such sequences is well known and one such pair will be discussed in Section 3.

The sequential procedure $\mathscr{S}$ can now be defined. Since the procedure is sequential, at the m^{th} stage $(m \geq 1)$ there are three possible choices for the experimenter:

(1) accept π_i, that is, choose π_i as one of the members of the selected subset,

(2) reject π_i, do not include it in the selected subset, or

(3) make no decision about concerning π_i and continue onto the $(m+1)^{st}$ stage.

It should be pointed out that the procedure is non-eliminating in that samples are taken from all populations until all have been accepted or rejected. This is done to keep the values $p_1,\ldots,p_k$ constant throughout the procedure. A population will be called tagged whenever it falls into the acceptance or rejectance region. The procedure $\mathscr{S}$ is as follows.

$\mathscr{S}$: Tag population π_i, $i = 1,2,\ldots,k$ at the first stage $m \geq 1$ such that $S_{im} \notin (a_m,b_m)$ and mark it "rejected if $S_{im} \leq a_m$ and "accepted" if $S_{im} \geq b_m$. Continue sampling from all k populations until each has been tagged, then accept those marked "accepted" and reject those marked "rejected."

It should be pointed out that corresponding to any given fixed sample size procedure R for any k populations with densities $f(x;\theta_i)$ $i = 1,2,\ldots,k$ belonging to any general family we can define the class $\mathcal{S}$ of sequential procedures provided the probabilities $p_1,p_2,\ldots p_k$ are known and form a monotone sequence.

The following notation will be used throughout the sequel. Let $a_i(m) \equiv a_i(m,\eta_{b,c}) = P\{\text{accepting } \pi_{(i)} \text{ at stage } m \mid \mathcal{S}(\eta_{b,c})\}$, $r_i(m) \equiv r_i(m,\eta_{b,c}) = P\{\text{rejecting } \pi_{(i)} \text{ at stage } m \mid \mathcal{S}(\eta_{b,c})\}$, $a_i \equiv a_i(\eta_{b,c}) = \sum_{m=1}^{\infty} a_i(m) = P\{\text{accepting } \pi_{(i)} \mid \mathcal{S}(\eta_{b,c})\}$, $r_i \equiv r_i(\eta_{b,c}) = \sum_{m=1}^{\infty} r_i(m) = P\{\text{rejecting } \pi_{(i)} \mid \mathcal{S}(\eta_{b,c})\}$, where $\mathcal{S}(\eta_{b,c})$ is the procedure using the pair of sequences $\eta_{b,c}$. Where there is no ambiguity we shall use $\mathcal{S}(\eta)$ for $\mathcal{S}(\eta_{b,c})$.

In addition let $m_i = 1^{st}$ $m \geq 1$ such that $\pi_{(i)}$ is accepted or rejected, and $M_i = E\{m_i \mid \mathcal{S}(\eta_{b,c})\}$. Condition (iv) of (2.2) guarantees that $P\{m_i < \infty, i = 1,2,\ldots,k\} = 1$ and thus for all $i = 1,\ldots,k$

(2.3) $$a_i + r_i = 1$$

It is also noted that $P\{cs \mid \mathcal{S}\} = a_k$

3. Some Monotonic Properties of the Procedure $\mathcal{S}$.

In the previous section it was shown that each population π_i gave rise to a sequence of zeros and ones which were summed to provide the test statistics. Consider a fixed population π_i and its associated sequences of random variables $\{Y_{im}, m \geq 1\}$ and $\{S_{im} = \sum_{j=1}^{m} Y_{ij}, m \geq 1\}$.

Let $\eta = (\{b_m\}, \{c_m\})$ and $\eta' = (\{b_m'\}, \{c_m'\})$ be two pairs of sequences satisfying (2.2) Two sequences $\{b_m\}$ and $\{b_m'\}$ are said to be pairwise ordered if and only if $b_m \leq b_m'$, $\forall\, m \geq 1$. We denote this by $\{b_m\} \prec \{b_m'\}$.

We also denote the ordering $\eta < \eta'$ to mean $\{b_m\} < \{b'_m\}$ and $\{c_m\} < \{c'_m\}$. A class $\mathcal{C}$ of pairs of sequences satisfying (2.2) is said to be ordered if for all $\eta, \eta' \in \mathcal{C}$ either $\eta < \eta'$ or $\eta' < \eta$.

Theorem 3.1. If $\eta' < \eta$ then $a_i(\eta') \geq a_i(\eta)$ and $r_i(\eta') \leq r_i(\eta)$, $i=1,2,\ldots,k$. In particular $P\{CS|\delta(\eta)\}$.

PROOF: Let $A_{im}(\eta) = [\bigcap_{\nu=1}^{m-1} [b_\nu < S_{i\nu} < c_\nu] \cap [S_{im} \geq c_m]]$. Since $\eta' < \eta$, $[S_{im} \geq c_m] \subset [S_{im} \geq c'_m]$ and also $[\bigcap_{\nu=1}^{m-1} [b_\nu < S_{i\nu} < c_\nu]]$ $\subset [\bigcap_{\nu=1}^{m-1} [b'_\nu < S_{i\nu} < c_\nu]]$. But this implies that either $[\bigcap_{\nu=1}^{m-1} [b'_\nu < S_{i\nu} < c'_\nu]]$ or there exists an $n \leq m-1$ such that $[\bigcap_{\nu=1}^{n-1} [b'_\nu < S_{i\nu} < c'_\nu] \cap [S_{in} \geq c'_n]]$ holds. Thus it follows that $A_{im}(\eta) \subset A_{in}(\eta')$ for some $n = 1,2,\ldots,m$ so that $A_{im}(\eta) \subset \bigcup_{n=1}^{m} A_{in}(\eta')$ and so $\bigcup_{m=1}^{\infty} A_{im}(\eta) \subset \bigcup_{m=1}^{\infty} A_{im}(\eta')$, $P(\bigcup_{m=1}^{\infty} A_{im}(\eta)) \leq P(\bigcup_{m=1}^{\infty} A_{im}(\eta'))$. Now it is clear that $A_{im}(\eta) \cap A_{in}(\eta) = \emptyset$ for $m \neq n$ and for all η satisfying (2.2) since $A_{im}(\eta)$ is the first time the sequence $\{S_{im}, m \geq 1\}$ leaves the bounds (b_m, c_m) and crosses the upper one. Thus from the previous implication $a_i(\eta) = \sum_{m=1}^{\infty} P(A_{im}(\eta)) = P(\bigcup_{m=1}^{\infty} A_{im}(\eta))$ $\leq P(\bigcup_{m=1}^{\infty} A_{im}(\eta')) = a_i(\eta')$. From (2.3), $r_i(\eta) = 1 - a_i(\eta)$. Since $a_i(\eta) \leq a_i(\eta')$ it follows $r_i(\eta) \geq r_i(\eta')$. Applying the first result to $\pi_{(k)}$ we get $P\{CS|\delta(\eta')\} \geq P\{CS|\delta(\eta)\}$ and complete the proof.

Consider two populations $\pi_{(i)}$ and $\pi_{(j)}$ with $1 \leq i < j \leq k$. This implies $p_i < p_j$.

Lemma 3.1. There exists a sequence of independent identically distributed random variables $\{U_m; m \geq 1\}$ such that for all $m \geq 1$

(i) $P(U_m \leq u) = P(Y_{im} \leq u)$ for all real u, and

(ii) $P(U_m \leq Y_{im}) = 1$.

PROOF: Define a sequence of independent random variables $Z_m = 0$ or 1 such that Z_m is independent of $Y_{i\ell}$, $\ell \neq m$ and for $\ell = m$, $P(Z_m = 1 | Y_{jm} = 1) = p_i/p_j$, $P(Z_m = 0|Y_{jm} = 0) = 1$ so that $\{Y_{jm}Z_m;\ m \geq 1\}$ is a sequence of independent and identically distributed random variables. Then let $U_m = Z_m Y_{jm}$, $m \geq 1$. $P(U_m = 1) = P(Y_{jm}, Z_m = 1) = P(Z_m = 1| Y_{jm} = 1)P(Y_{jm} = 1) = p_i$ and $P(U_m = 0) = 1 - p_i$, $m \geq 1$. Clearly then the sequence of U_m's and the sequence of Y_{im}'s have the same distribution and $P(U_m \leq Y_{jm}) = 1$, which completes the proof.

Theorem 3.2. The procedure $\mathcal{S}(\eta)$ is monotone and unbiased, i.e., $a_k \geq a_{k-1} \geq \cdots \geq a_1$ and $r_k \leq r_i$, $i = 1,2,\ldots,k-1$.

PROOF: It will suffice to show $a_1 \leq a_2$. As in Theorem 3.1

$$a_1 = \sum_{m=1}^{\infty} P(A_{(1)m}(\eta)) = \sum_{m=1}^{\infty} P\{\bigcap_{\nu=1}^{m-1}[b_\nu < S_{(1)\nu} < c_\nu] \cap [S_{(1)m} \geq c_m]\}.$$ Let

$S_{um} = \sum_{\nu=1}^{m} U_\nu$ where $\{U_m\}$ is the sequence defined in Lemma 3.1, and

$a_u = \sum_{m=1}^{\infty} P\{\bigcap_{\nu=1}^{m-1}[a_\nu < S_{u\nu} < c_\nu] \cap [S_{um} \geq c_m]\}$. By Lemma 3.1 $P\{S_{um} \leq S_{(2)m}\} = 1$, $m \geq 1$. Then $[S_{um} \geq c_m] \subset [S_{(2)m} \geq c_m]$ and also $[\bigcap_{\nu=1}^{m-1}[b_\nu < S_{um} < c_\nu]]$

implies either $[\bigcap_{\nu=1}^{m-1}[b_\nu < S_{(2)}r < c_\nu]]$ or there exists an $n \leq m-1$ such that $[\bigcap_{\nu=1}^{n-1}[b_\nu < S_{(2)\nu} < c_\nu] \cap [S_{(2)n} \geq c_n]]$. It then follows that $A_{um}(\eta) \subset \bigcup_{n=1}^{m} A_{(2)n}(\eta)$ and thus $\bigcup_{m=1}^{\infty} A_{um}(\eta) \subset \bigcup_{m=1}^{\infty} A_{(2)m}(\eta)$. This implies $a_u \leq a_2$. Since by Lemma 3.1 $P\{Y_{(1)m} \leq u\} = P\{U_m \leq u\}$ for all real u, it follows that $a_1 = a_u$ and thus $a_1 \leq a_2$ which completes the proof of monotonicity. Unbiasedness follows from (2.1).

4. Exact Results

In this and the remaining sections of this paper, we will investigate the procedure $\mathcal{S}(\eta)$ using the following class $\mathcal{C}_1$ of pairs of sequences. Let $b_m = \delta m - \gamma_1$, $c_m = \delta m + \gamma_2$ where δ is a rational number in (0,1) and γ_1, γ_2 are positive integers. It is clear that for γ_1 and γ_2 fixed the class $\mathcal{C}_1$ is ordered in δ, and the results of Section 3 hold. That $\eta \in \mathcal{C}_1$, satisfy (iv) of (2.2) will be shown in this section.

Consider a fixed population π_i and its associated sequence of test statistics $\{S_{im}, m \geq 1\}$ where $S_{im} = \sum_{j=1}^{m} Y_{ij}$; with Y_{ij}, $j \geq 1$ a sequence of independent identically distributed random variables with $P\{Y_{ij}=1\} = 1 - P\{Y_{ij} = 0\} = p_i$. In this section exact expressions are found for $a_i(n)$, $r_i(n)$, a_i, r_i, and M_i. Whenever no ambiquity arises we shall drop the population subscript i.

For δ rational in (0,1) set $Z_j = Y_j - \delta$ for $j \geq 1$. Then $R_m = \sum_{j=1}^{m} Z_j = S_m - \delta m$. Thus for any $\eta \in \mathcal{C}_1$ the events $[\delta m - \gamma_1 < S_m < \delta m + \gamma_2]$, $[S_m \geq \delta m + \gamma_2]$, and $[S_m \leq \delta m - \gamma_1]$ are equivalent to $[-\gamma_1 < R_m < \gamma_2]$, $[R_m \geq \gamma_2]$, and $[R_m \leq -\gamma_1]$ respectively. So the various probabilities and expectations can be evaluated as solutions to a one-dimensional random walk on a finite interval. Further, if we take $\delta = t/s$ where t and s are relatively prime integers with $t < s$ then the state space of the walk is all points of the form $(Ns - Mt)/s$ for all integers $M > N > 0$. It is a well-known theorem of number theory that $xs - yt = J$ has non-negative integer solutions $x = N$, $y = M$ with $M > N$, for any integer J provided t and s are relatively prime. In general, then, the state space is of the form J/s, J an integer. Thus the correspondence $J/s \to J$ enables one to consider the walk $\{R_m\}$ on the space of integers with transition function $P(x,y)$ defined by,

(4.1) $P(x, x-t) = 1-p$, $P(x, x+s-t) = p$, $P(x,y) = 0$ elsewhere.

For positive integers γ_1, γ_2 we define the following:

(4.2) $B_1 = [-s\gamma_1 - t+1, \ldots, -s\gamma_1]$, $B_2 = [s\gamma_2, s\gamma_2+1, \ldots, s\gamma_2+s-t-1]$,
$B = [-\infty, s\gamma_1] \cup [s\gamma_2, \infty]$,

For any set B, let $\overline{B}$ be the complement with respect to the integers. Let $R_o = x$, for $x \in \overline{B}$ and,

(4.3) $m_B = \min\{m \geq 1, R_m \in B\}$

(4.4) $Q_n(x,y) = P\{[R_m = y] \cap [m_B > n]\}$ for $y \in \overline{B}$, $n \geq 0$

(4.5) $H_B^{(n)}(x,y) = P\{[R_m = y] \cap [m_B = n]\}$, for $y \in B_1 \cup B_2$, $n \geq 1$.

At any step the random walk R_m can only move s-t steps to the right or t steps to the left, so $B_1 \cup B_2$ are the only absorption points of the walk. It is clear that m_B is the stopping time of the walk, $Q_n(x,y)$ is the probability of going from x to y in n steps without leaving $\overline{B}$ and $H_B^{(n)}(x,y)$ is the probability of starting at $x \in \overline{B}$ and leaving $\overline{B}$ at the nth step entering $B_1 \cup B_2$ at y. Analytically, (4.4) and (4.5) can be described as follows (See [8], pg 107).

(4.6) $Q_o(x,y) = \delta(x,y)$, $Q(x,y) = P(x,y)$,

$$Q_{n+1}(x,y) = \sum_{z \in \overline{B}} Q_n(x,z)Q_1(z,y), \quad x,y \in \overline{B},\ n \geq 1.$$

(4.7) $$H_B^{(n)}(x,y) = \sum_{z \in \overline{B}} Q_{n-1}(x,z)P(z,y), \quad x \in \overline{B},\ y \in B_1 \cup B_2,\ n \geq 1.$$

Since $\overline{B} = [-s\gamma_1 + 1, \ldots, s\gamma_2 - 1]$ then (4.6) says we can express $Q(x,y)$ as the nth power of an NxN matrix $Q = (q_{ij})$ where $N = s(\gamma_1 + \gamma_2) - 1$ and $q_{ij} = P(i - s\gamma_1, j - s\gamma_1)$ for $i - s\gamma_1$, $j - s\gamma_1 \in \overline{B}$. Equation (4.6) expresses the fact that $Q_n(i,j) = (q^{(n)}_{i+s\gamma_1, j+s\gamma_2})$ where $q^{(n)}_{ij}$ is the (ij)th entry in Q^n. By the nature of the walk absorption can take place at $y \in B_2$ at stage n if and only if at stage n-1, $y - (s-t) \in \bar{B}$. Similarly, absorption at $y \in B_1$ at stage n can take place if and only if at stage n-1, $y + t \in \bar{B}$. From (4.7) then

$$H_B^{(n)}(x,y) = \sum_{z=-s\gamma_1+1}^{s\gamma_2-1} Q_{n-1}(x,z)P(z,y) \text{ for } y \in B_1 \cup B_2.$$

However, as stated above if $y \in B_2$, the $P(z,y) > 0$ if and only if $z = y -(s-t)$, and if $y \in B_1$ then $P(z,y) > 0$ if and only if $z = y + t$. Therefore,

$$(4.8)\quad H_B^{(n)}(x,y) = \begin{cases} p\, q^{(n-1)}_{x+s\gamma_1,\; y-(s-t)+s\gamma_1} & \text{for } y \in B_2 \\ q\, q^{(n-1)}_{x+s\gamma_1,\; y+t+s\gamma_1} & \text{for } y \in B_1. \end{cases}$$

<u>Theorem 4.1.</u> $a(n) = p \sum_{j=N-(s-t)+1}^{N} q^{(n-1)}_{s\gamma_1,j}$, and $r(n) = q \sum_{j=1}^{t} q^{(n-1)}_{s\gamma_1,j}$.

PROOF: From (4.5) $a(n) = \sum_{y \in B_2} H_B^{(n)}(0,y)$, since we select π only at points of B_2. Substituting from (4.8) then gives the first result. Similarly for the second result $r(n) = \sum_{y \in B_1} H_B^{(n)}(0,y)$ and again substituting from (4.8) produces the second equality and completes the proof.

We then define $H_B(x,y) = \sum_{m=1}^{\infty} H_B^{(m)}(x,y)$ for $x \in \bar{B}$, the probability of starting at x and being absorped at y.

From (4.8) for $x \in \bar{B}$

$$(4.9)\quad H_B(x,y) = \begin{cases} p \sum_{m=1}^{\infty} q^{(m-1)}_{x+s\gamma_1, y-(s-t)+s\gamma_1} & \text{for } y \in B_2 \\ q \sum_{m=1}^{\infty} q^{(m-1)}_{x+s\gamma_1, y+t+s\gamma_1} & \text{for } y \in B_1. \end{cases}$$

The matrix $Q = (q_{ij})$ is the transition matrix of the random walk restricted to states in $\bar{B}$. It has the following elements $q_{i,i+s-t} = p$, for $i = 1,2,\ldots,N-s+t$, $q_{i,i+s-t} = q$, for $i = t+1,\ldots,N$ and $q_{ij} = 0$ elsewhere. Thus Q is a sub-stochastic matrix. It can be shown (see Gantmacher [4]) that a sub-stochastic matrix has all its characteristic roots inside the unit circle in the complex plane, and so the series expansion $(I - Q)^{-1} = \sum_{n=0}^{\infty} Q^n$, where I is the NxN unit matrix, is valid. Then we get the following theorem.

Theorem 4.2. $a = p \sum_{j=N-s+t+1}^{N} q^{s\gamma_1, j}$, and $r = q \sum_{j=1}^{t} q^{s\gamma_1, j}$ where $q^{i,j}$ is the (i,j)th entry in $[I-Q]^{-1}$.

PROOF: Forming the sum in (4.9) we get

$$H_B(x,y) = \begin{cases} p\, q^{x+s\gamma_1,\ y+s-t+s\gamma_1} & \text{for } y \in B_2 \\ q\, q^{x+s\gamma_1,\ y+t+s\gamma_1} & \text{for } y \in B_1. \end{cases}$$

However,

$$a = \sum_{y \in B_2} H_B(0,y) = p \sum_{j=N-s+t+1}^{N} q^{s\gamma_1, j} \text{ and similarly } r = q \sum_{j=1}^{t} q^{s\gamma_1, j}$$

which completes the proof.

$R_m = \sum_{j=1}^{m} Z_j$ is the summ of independent identically distributed random variables with $E\, Z_i = (s-t)p - tq$ and $E\, Z_1^2 = (s-t)^2 p + t^2(1-p)$. Thus if $E\, Z_1 \neq 0$, then $ER_{m_B} = E\, Z_1 \cdot Em_B$, and if $E\, Z_1 = 0$, $E\, R_{m_B}^2 = E\, Z_1^2\, Em_B$.

Thus we prove the following theorem.

Theorem 4.3. $M_B = E\{m_B \mid \mathcal{S}(\eta)\} = \frac{1}{(s-t)p-tq} [p \sum_{j=N-s+t+1}^{N} (j+s-t-s\gamma_1) q^{s\gamma_1, j}$

$$+ q \sum_{j=1}^{t} (j-t-s\gamma_1) q^{s\gamma_1, j}] \text{ if } p \neq t/s$$

(4.10) $$= E\{m_B \mid \mathcal{S}(\eta)\} = \frac{1}{(s-t)t} [p \sum_{j=N-s+t+1}^{N} (j+s-t-s\gamma_1)^2 q^{s\gamma_1, j}$$

$$+ q \sum_{j=1}^{t} (j-t-s\gamma_1)^2 q^{s\gamma_1, j}] \text{ if } p = t/s$$

PROOF: If $p \neq t/s$ $M_B = (EZ_1)^{-1} E\, R_{m_B}$. However, $P(R_{m_B} = y) = H_B(0,y)$

for $y \in B_1 \cup B_2$, thus $ER_{m_B} = \sum_{y \in B_1 \cup B_2} y\, H_B(0,y)$ and from (4.9)

$$E\, R_{m_B} = p \sum_{y \in B_2} y\, q^{s\gamma_1, y-s+t+s\gamma_1} + q \sum_{y \in B_1} y\, q^{s\gamma_1, y+t+s\gamma_1}$$

$$= p \sum_{j=N-s+t+1}^{N} (j + s - t - s\gamma_1) q^{s\gamma_1, j} + q \sum_{j=1}^{t} (j - t - s\gamma_1) q^{s\gamma_1, j}$$

and the first equality of (4.10) holds. For $p = t/s$, $E\, Z_1 = 0$ and $E\, Z_1^2 = (s - t)t$ and again using (4.9) one can write $E\, R^2_{m_B}$ from which the second equality of (4.10) holds.

It should be noted here that the condition (iv) of (2.2) holds for all $\Pi \in \mathcal{C}_1$ since this condition for a random walk on a finite interval is well-known, (See p.297, [3]).

5. Bounds and Approximations.

This section will deal with bounds and approximations on the various probabilities and expectations derived in Section 4. These bounds are often easier to compute than the corresponding exact expressions and in addition give a better insight into the nature of the procedure. Feller (p. 303, [3]) discusses bounds for the probability of a more general random walk leaving $\overline{B}$ at one end, and the expected number of steps to do so.

Following Feller let $U_{t/s}(x)$ be the probability of the walk starting at $x \in \overline{B}$ and reaching or crossing $s\gamma_2$ before $-s\gamma_1$. Then $a = U_{t/s}(0)$. Conditioning on the first step, $U(x) = U_{t/s}(x)$ satisfies the following homogeneous difference equation and boundary conditions:

$$U(x) = pU(x + s - t) + q\, U(x - t), \quad -s\gamma_1 < x < s\gamma_2$$

$$(5.1) \qquad U(x) = 0, \quad x = -s\gamma_1 - t + 1, \ldots, -s\gamma_1$$

$$U(x) = 1, \quad x = s\gamma_2, \ldots, s\gamma_2 + s - t - 1$$

The characteristic equation of the generating random variables of this walk is $f(x) = (1-p)x^{-t} + px^{s-t}$, setting $f(x) = 1$, we get

$$(5.2) \qquad px^s - x^t + 1 - p = 0.$$

Suppose $p \neq t/s$. Equation (5.2) has unity as a single root, and exactly one more positive root y. For, consider $g(x) = px^s - x^t + 1 - p$, then $g'(x) = x^{t-1}(psx^{s-t}-t)$ so that for $x > 0$, $g(x)$ is decreasing on $(0,[t/ps]^{1/(s-t)})$ and an increasing function on $([\frac{t}{ps}]^{1/s-t}, \infty)$. Further $g(0) = 1-p > 0$ and $g(1) = 0$ hence if $p > t/s$ $g(x)$ crosses the X-axis at $0 < y < 1$, and if $p < t/s$ $g(x)$ crosses at $y > 1$. If $p = t/s$ there is a double root at $y = 1$.

Applying Feller's method to R_m it can be shown that if $p \neq t/s$,

$$(5.3) \qquad \frac{y^x - y^{s\gamma_1}}{1 - y^{s(\gamma_1 + \gamma_2)+s-t-1}} \leq U(x) \leq \frac{1 - y^{s\gamma_1 + t - 1}}{1 - y^{s(\gamma_1 + \gamma_2) + t - 1}}$$

where $y \neq 1$ is a positive root of (5.2). Thus,

$$(5.4) \qquad \frac{1 - y^{s\gamma_1}}{1 - y^{s(\gamma_1 + \gamma_2) + s - t - 1}} \leq a \leq \frac{1 - y^{s\gamma_1 + t - 1}}{1 - y^{s(\gamma_1 + \gamma_2) + t - 1}}$$

If $p = t/s$, in a similar manner we get

$$(5.5) \qquad \frac{s\gamma_1}{s(\gamma_1 + \gamma_2) + s - t - 1} \leq a \leq \frac{s\gamma_1 + t - 1}{s(\gamma_1 + \gamma_2) + t - 1}$$

If we make the assumption that $s\gamma_1 \gg t$, $s(\gamma_1 + \gamma_2) \gg s-t$ we can write the following approximations using (5.4) and (5.5)

$$(5.6) \qquad a \cong \frac{1 - y^{s\gamma_1}}{1 - y^{s(\gamma_1 + \gamma_2)}} = \tilde{a} \quad \text{if } p \neq t/s$$

$$(5.7) \qquad a \cong \frac{\gamma_1}{\gamma_1 + \gamma_2} \qquad \text{if } p = t/s.$$

The symbol $\cong$ in (5.6) and (5.7) and in the sequel will be taken to mean that the ratio of the left hand side to the right side tends to 1 as γ_1 and γ_2 tend to ∞. In (5.6) it can be shown that $\lim_{\gamma_1,\gamma\to\infty} a/\tilde{a} = 1$ if $p > t/s$ aor $p < t/t+1$. If $p < t/s$ for $s \geq t+2$, $\tilde{a}$ is actually an asymptotic upper bound for a, that is, $c \leq \lim_{\gamma_1,\gamma_2\to\infty} a/\tilde{a} \leq 1$ where $c = y^{-s+t+1}$. However, as $p \to 0$ or $p \to t/s$, then $c \to 1$, and for values of p other than the extremes, numerical evidence (see [1]) shows $\tilde{a}$ a good approximation of a for even small values of γ_1, γ_2. Thus (5.6) will be taken as the approximation to a for all $p \neq t/s$.

Using (5.6) and (5.7) we can get approximations for M. For $p \neq t/s$ $M = 1/EZ_1\ ER_{m_B}$, and if we assume we leave $\bar{B}$ at the boundary points $-s\gamma_1$ or $s\gamma_2$, we get

$$ER_{m_B} = s\gamma_2 a - s\gamma_1 r = s(\gamma_1 + \gamma_2)a - s\gamma_1$$

$$\cong \frac{s(\gamma_1 + \gamma_2)(1 - y^{s\gamma_1}) - s\gamma_1(1 - y^{s(\gamma_1 + \gamma_2)})}{1 - y^{s(\gamma_1 + \gamma_2)}}$$

Thus if $p \neq t/s$,

$$(5.8)\quad M_B \cong \frac{s(\gamma_1 + \gamma_2)(1 - y^{s\gamma_1}) - s\gamma_1(1 - y^{s(\gamma_1 + \gamma_2)})}{(ps-t)(1 - y^{s(\gamma_1 + \gamma_2)})}$$

For $p = t/s$ we use $M_B = (EZ_1^2)^{-1}ER_{m_B}^2$ which with (5.7) gives

$$(5.9)\quad M_B \cong \frac{s^2\gamma_1\gamma_2}{(s-t)t}$$

In the symetric boundaries case, $\gamma_1 = \gamma_2 = \gamma$ formulae (5.6), (5.7), (5.8), and (5.9) simplify to produce a more complete theory,

(5.10)
$$a \cong \begin{cases} \dfrac{1}{1 + y^{s}\gamma} & \text{If } p \neq t/s \\ 1/2 & \text{If } p = t/s \end{cases}$$

(5.11)
$$M_B \cong \begin{cases} \dfrac{s_\gamma}{ps - t} \cdot \dfrac{1 - y^{s\gamma}}{1 + y^{s\gamma}} & \text{If } p \neq t/s \\ \dfrac{s^2\gamma^2}{t(s-t)} & \text{If } p = t/s \end{cases}$$

Theorem 5.1. Let p be the acceptance probability of any population π when the rule R(1) is used. Then for the sequential procedure $\mathcal{S}(\eta)$ where $\eta = (\{\delta m - \gamma\}, \{\delta m + \gamma\})$ and $\delta = t/s, > 0$

$$\lim_{\gamma \to \infty} a(\delta, \gamma) = \begin{cases} 0 & \text{if } p < t/s \\ 1/2 & \text{if } p = t/s \\ 1 & \text{if } p > t/s \end{cases}$$

PROOF: Suppose $p < t/s$, then from (5.4) with $\gamma_1 = \gamma_2 = \gamma$, $a = a(\delta,\gamma)$

$\leq (1 - y^{s\gamma + t-1})/(1 - y^{2s\gamma + t-1})$, where $y > 1$ is a root of (5.2).

Then clearly as $\gamma \to \infty$ $(1 - y^{s\gamma + t-1})/(1 - y^{2s\gamma + t-1}) \to 0$ and thus $a(\delta, \gamma) \to 0$. Similarly if $p > t/s$ (5.4) gives $a(\delta, \gamma) > (1 - y^{s\gamma})/(1 - y^{2s\gamma + s-t-1})$ where $0 < y < 1$ is a root of (5.2). As $\gamma \to \infty$, $y^{s\gamma} \to 0$ and, therefore, $a(\delta, \gamma) \to 1$. Finally, if $p = t/s$ (5.5) shows that,

$$\frac{s\gamma}{2s\gamma + s - t - 1} \leq a(\delta,\gamma) \leq \frac{s\gamma + t - 1}{2s\gamma + t - 1},$$

hence as $\gamma \to \infty$, $a(\delta, \gamma) \to 1/2$ which completes the proof.

Theorem 5.2. For any population π under the conditions of the previous theorem, for large γ and $p \neq t/s$ we have,

$$M_B \cong \frac{s\gamma}{|ps - t|} = \frac{\gamma}{|p - t/s|}.$$

PROOF: It is clear that $\lim_{\gamma \to \infty} \frac{1-y^{s\gamma}}{1+y^{s\gamma}} = 1$ and that the sequence approaches one through positive numbers if $0 < y < 1$, that is, if $ps - t > 0$, and through negative numbers if $y > 1$, that is, if $ps - t < 0$. Hence for large γ the result follows from (5.11).

Numerically the approximations given by (5.10) and (5.11) to Theorem 4.2 and Theorem 4.3, respectively, are very good even for samll values of γ. Tables comparing those values for several values of γ, δ, and p have been tabulated in [1]. An example of which is given in Tables 1 or 2, for $\delta = .75$ and various values of γ and p. In Table 1 the upper value gives the exact probability as defined in Theorem 4.2 and the lower value gives the approximate probability as defined in (5.10). It can be seen that the approximation is good for all values of γ chosen, and that it improves as γ increases. The conclusions of Theorem 5.1 are also apparent; for if $p < .75 = \delta$ then $a \to 0$ as γ increases, and if $p > .75 = \delta$ then $a \to 1$ as γ increases.

Table 1

Comparisons of Exact and Approximate Values of the Probability of Selecting a Population Using $\mathcal{S}(\pi)$ for $\pi \in C_1$

$\delta = .75$

p \ γ	3	4	5	6	7	8	9	10
.40	.00003 .00003	.00000 .00000	.00000 .00000	.00000 .00000	.00000 .00000	.00000 .00000	.00000 .00000	.00000 .00000
.60	.01183 .01180	.00271 .00272	.00062 .00062	.00014 .00014	.00003 .00003	.00001 .00001	.00000 .00000	.00000 .00000
.80	.85823 .84378	.91345 .90455	.94835 .94327	.96902 .96686	.98254 .98084	.98997 .98899	.99426 .99369	.99671 .99640
.90	.99797 .99773	.99982 .99970	.99997 .99996	.99999 .99999	1.00000 1.00000	1.00000 1.00000	1.00000 1.00000	1.00000 1.00000

In Table 2 the upper value gives the exact expectations and the lower value gives the approximate expectations as defined in Theorem (4.3) and (5.11) respectively. Again the approximations appear quite good for all values of γ chosen.

Table 2

Comparisons of Exact and Approximate Values of the Expected Number of Stages to Tag a Population Using $\mathcal{S}(\eta)$ for $\eta \in \mathcal{C}_1$.

$\delta = .75$

$p \backslash \gamma$	3	4	5	6	7	8	9	10
.40	9.17 8.57	12.05 11.43	14.89 14.29	17.75 17.14	20.61 20.00	23.47 22.86	26.32 25.71	29.19 28.57
.60	20.78 19.53	27.79 26.52	34.56 32.29	41.25 39.99	47.93 46.66	54.60 53.33	61.27 60.60	67.93 66.67
.80	42.53 41.25	65.83 64.73	89.51 88.65	112.66 112.05	135.06 135.63	156.76 156.48	177.91 177.73	198.66 198.68
.90	19.93 19.91	26.66 26.65	33.33 33.33	40.00 40.00	46.67 46.67	53.33 53.33	60.00 60.00	66.67 66.67

6. A Minimax Approach

A class of procedures $\mathcal{S}(\eta)$, $\eta \in \mathcal{C}_1$, has been proposed and certain probabilities and expectations concerning the procedure have been obtained. The experimenter now faces the problem of choosing two specific constants δ and γ. Theorem 5.1 guarantees that for any choice of $\delta \in (p_{k-1}, p_k)$ there exists a $\gamma = \gamma(\delta, \epsilon)$ such that for any $\epsilon > 0$.

$$\text{(6.1)} \qquad \begin{aligned} &\text{(i)} \quad a_k(\delta, \gamma) \geq 1 - \epsilon \\ &\text{(ii)} \quad a_{k-1}(\delta, \gamma) \leq \epsilon, \end{aligned}$$

regardless of the configuration of $p_1 \leq p_2 \leq \cdots \leq p_k$ and hence the configuration of $\theta_{[1]} \leq \theta_{[2]} \leq \cdots \leq \theta_{[k]}$. So that for a small enough ϵ the P* condition can always be satisfied by choosing an appropriate $\eta \in \mathcal{C}_1$. If we define S to be the size of the selected subset when the procedure terminates, then from Theorem 3.2

$$ES = \sum_{i=1}^{k} a_i \leq 1 + (k-1)a_{k-1}$$

Thus we can replace (6.1) by

$$(6.2) \qquad \begin{array}{ll} \text{(i)} & a_k(\delta, \gamma) \geq 1 - \epsilon \\ \text{(ii)} & 1 - \epsilon < ES \leq 1 + (k-1)\epsilon \end{array}$$

regardless of the configuration of the means $\theta_1, \theta_2, \ldots, \theta_k$.

Obviously if for a fixed $\delta \in (p_{k-1}, p_k)$, γ is choosen such that (6.2) holds, any choice of $\gamma' > \gamma$ will also satisfy (6.2). So that the experimenter has for any $\delta \in (p_{k-1}, p_k)$ a countably infinite number of procedures η which guarantee (6.2). It is also clear that (6.2) are desirable properties in that the larger the bound on $P\{CS|\mathcal{S}(\eta)\}$ the smaller the expected number of populations selected. Given two procedures $\eta, \eta' \in \mathcal{C}_1$ which satisfy (6.2), the procedure which has the smallest expected number of stages is in some sense preferable. Therefore, the experimenter will want to use η if it exists which minimizes

$$(6.3) \qquad M = \max_{1 \leq i \leq k} M_i = \max_{1 \leq i \leq k} E\{m_i | \mathcal{S}(\eta)\}$$

over the subclass $\mathcal{C}_2 \subset \mathcal{C}_1$ of procedures satisfying (6.2) and where m_1 is the number of stages until population $\pi_{(i)}$(unknown) is tagged. In this section we will show two bounds $\bar{\delta}$ and δ^* between which the δ minimizing the approximation to (6.3) given in Theorem 5.2 is found.

Theorem 5.2 shows that M_i is asymptotically proportional to γ, so that for a given $\delta \in (p_{k-1}, p_k)$ in order to minimize (6.3) over all γ such that (6.2) is satisfied the experimenter would choose the smallest γ. Thus the problem is reduced to finding which $\delta \in (p_{k-1}, p_k)$ produces a $\eta \in \mathcal{C}_2$ that minimizes (6.3).

Definition 6.1. For any rational $\delta \in (p_{k-1}, p_k)$ let $\gamma_1(\delta)$ be the first positive integer such that $a_k \geq 1 - \epsilon$, and $\gamma_2(\delta)$ be the first positive integer such that $a_{k-1} \leq \epsilon$. Finally let $\gamma(\delta) = \max(\gamma_1(\delta), \gamma_2(\delta))$.

The existance of $\gamma_1(\delta)$ and $\gamma_2(\delta)$ is guaranteed by Theorem 5.1. Then we have the following lemma.

Lemma 6.1. $\gamma_1(\delta)$ is a non-decreasing function, and $\gamma_2(\delta)$ is a non-increasing function of δ.

PROOF. For any fixed γ and $\delta' < \delta$, Theorem (3.1) implies that $a_k(\delta', \gamma) \geq a_k(\delta, \gamma)$. Now for $\gamma = \gamma_1(\delta)$, $a_k(\delta, \gamma_1(\delta)) \geq 1 - \epsilon$ thus $a_k(\delta', \gamma(\delta)) \geq 1 - \epsilon$. However, $\gamma_1(\delta')$ is the smallest positive integer such that $a_k(\delta', \gamma) \geq 1 - \epsilon$, thus $\gamma_1(\delta') \leq \gamma_1(\delta)$. Similarly for fixed $\gamma = \gamma_2(\delta')$, $a_{k-1}(\delta', \gamma_2(\delta')) \leq \epsilon$ so by Theorem 3.1 $a_{k-1}(\delta, \gamma) \leq \epsilon$. But $\gamma_2(\delta)$ is the smallest integer such that $a_{k-1}(\delta, \gamma) \leq \epsilon$ thus $\gamma_2(\delta\ ) \leq \gamma(\delta')$ which completes the proof.

Approximate values for $\gamma_1(\delta)$ and $\gamma_2(\delta)$ can be obtained from (5.10) by setting $1/(1 + y_k^{s\gamma_1}) = 1 - \epsilon$ and $1/(1 + y_{k-1}^{s\gamma_2}) = \epsilon$. Thus,

$$\gamma_1(\delta) = \frac{\ln\frac{\epsilon}{1-\epsilon}}{s \ln y_k(\delta)} \quad \text{for } \delta \in (p_{k-1}, p_k) \tag{6.4}$$

and

$$\gamma_2(\delta) = \frac{-\ln y_k(\delta)}{\ln y_{k-1}(\delta)}\ \gamma_1(\delta) = \frac{\ln \frac{1-\epsilon}{\epsilon}}{s \ln y_{k-1}(\delta)} \quad \text{for } \delta \in (p_{k-1}, p_k) \tag{6.5}$$

The approximate unique value δ^* such that $\gamma_1(\delta) = \gamma_2(\delta)$ is given in the following lemma.

Lemma 6.2.

$$\delta^* = \frac{\ln \frac{1-p_{k-1}}{1-p_k}}{\ln \frac{p_k(1-p_{k-1})}{p_{k-1}(1-p_k)}} \qquad \text{if } p_{k-1} + p_k \neq 1 \tag{6.6}$$

(6.7) $\delta^* = 1/2$ if $p_{k-1} + p_k = 1$

PROOF. At δ^*, $\gamma(\delta^*) = \gamma_1(\delta^*) = \gamma_2(\delta^*)$. From (6.5) then $y_k y_{k-1} = 1$. From (5.2) $p_k y_k^s - y_k^t + 1 - p_k = 0$ for $\delta^* = t/s$. Thus,

$$p_k = \frac{1 - y_k^t}{1 - y_k^s}.$$

Again from (5.2) noting that $y_{k-1} = y_k^{-1}$, we get $p_{k-1} y_k^{-s} + y_k^{-t} + 1 - p_{k-1} = 0$

or

$$p_{k-1} = y_k^{s-t} \cdot \frac{1 - y_k^t}{1 - y_k^s} = y_k^{s-t} p_k.$$

therefore, $y_k = [p_{k-1}/p_k]^{1/s-t}$. Substituting back in (5.2) for y_k,

$$p_k \left(\frac{p_{k-1}}{p_k}\right)^{s/s-t} - \left(\frac{p_{k-1}}{p_k}\right)^{t/s-t} + 1 - p_k = 0, \text{ or}$$

(6.8)
$$\left(\frac{p_{k-1}}{p_k}\right)^t = \left(\frac{1 - p_k}{1 - p_{k-1}}\right)^{s-t}.$$

If $p_k + p_{k-1} = 1$ then (6.8) is satisfied only for $t = s-t$ or $\delta^* = 1/2$. If $p_k + p_{k-1} \neq 1$ then taking logarithms of both sides of (6.8) and solving for t/s completes the proof.

Lemma 6.3. For δ^* as given in (6.6) and (6.7)

$$\gamma(\delta) = \begin{cases} \gamma_1(\delta), & \text{when } \delta \geq \delta^* \\ \gamma_2(\delta), & \text{when } \delta \leq \delta^*. \end{cases}$$

PROOF. Suppose that for some δ, $\gamma(\delta) = \gamma_1(\delta)$. If $\delta' \geq \delta$ then by Lemma (6.1), $\gamma_1(\delta') \geq \gamma_1(\delta)$ and $\gamma_2(\delta') \leq \gamma_2(\delta)$. By assumption $\gamma_1(\delta) \geq \gamma_2(\delta)$ so that $\gamma_1(\delta') \geq \gamma_2(\delta')$ and so $\gamma(\delta') = \gamma_1(\delta')$.

(6.7) $\delta^* = 1/2$ if $p_{k-1} + p_k = 1$

PROOF. At δ^*, $\gamma(\delta^*) = \gamma_1(\delta^*) = \gamma_2(\delta^*)$. From (6.5) then $y_k y_{k-1} = 1$. From (5.2) $p_k y_k^s - y_k^t + 1 - p_k = 0$ for $\delta^* = t/s$. Thus,

$$p_k = \frac{1 - y_k^t}{1 - y_k^s}.$$

Again from (5.2) noting that $y_{k-1} = y_k^{-1}$, we get $p_{k-1} y_k^{-s} + y_k^{-t} + 1 - p_{k-1} = 0$

or

$$p_{k-1} = y_k^{s-t} \cdot \frac{1 - y_k^t}{1 - y_k^s} = y_k^{s-t} p_k.$$

therefore, $y_k = [p_{k-1}/p_k]^{1/s-t}$. Substituting back in (5.2) for y_k,

$$p_k \left(\frac{p_{k-1}}{p_k}\right)^{s/s-t} - \left(\frac{p_{k-1}}{p_k}\right)^{t/s-t} + 1 - p_k = 0, \text{ or}$$

(6.8) $$\left(\frac{p_{k-1}}{p_k}\right)^t = \left(\frac{1 - p_k}{1 - p_{k-1}}\right)^{s-t}.$$

If $p_k + p_{k-1} = 1$ then (6.8) is satisfied only for $t = s-t$ or $\delta^* = 1/2$. If $p_k + p_{k-1} \neq 1$ then taking logarithms of both sides of (6.8) and solving for t/s completes the proof.

Lemma 6.3. For δ^* as given in (6.6) and (6.7)

$$\gamma(\delta) = \begin{cases} \gamma_1(\delta), & \text{when } \delta \geq \delta^* \\ \gamma_2(\delta), & \text{when } \delta \leq \delta^*. \end{cases}$$

PROOF. Suppose that for some δ, $\gamma(\delta) = \gamma_1(\delta)$. If $\delta' \geq \delta$ then by Lemma (6.1), $\gamma_1(\delta') \geq \gamma_1(\delta)$ and $\gamma_2(\delta') \leq \gamma_2(\delta)$. By assumption $\gamma_1(\delta) \geq \gamma_2(\delta)$ so that $\gamma_1(\delta') \geq \gamma_2(\delta')$ and so $\gamma(\delta') = \gamma_1(\delta')$.

(6.7) $\quad \delta^* = 1/2 \qquad\qquad$ if $p_{k-1} + p_k = 1$

PROOF. At δ^*, $\gamma(\delta^*) = \gamma_1(\delta^*) = \gamma_2(\delta^*)$. From (6.5) then $y_k y_{k-1} = 1$. From (5.2) $p_k y_k^s - y_k^t + 1 - p_k = 0$ for $\delta^* = t/s$. Thus,

$$p_k = \frac{1 - y_k^t}{1 - y_k^s}.$$

Again from (5.2) noting that $y_{k-1} = y_k^{-1}$, we get $p_{k-1} y_k^{-s} + y_k^{-t} + 1 - p_{k-1} = 0$

or

$$p_{k-1} = y_k^{s-t} \cdot \frac{1 - y_k^t}{1 - y_k^s} = y_k^{s-t} p_k.$$

therefore, $y_k = [p_{k-1}/p_k]^{1/s-t}$. Substituting back in (5.2) for y_k,

$$p_k \left(\frac{p_{k-1}}{p_k}\right)^{s/s-t} - \left(\frac{p_{k-1}}{p_k}\right)^{t/s-t} + 1 - p_k = 0, \text{ or}$$

(6.8)
$$\left(\frac{p_{k-1}}{p_k}\right)^t = \left(\frac{1 - p_k}{1 - p_{k-1}}\right)^{s-t}.$$

If $p_k + p_{k-1} = 1$ then (6.8) is satisfied only for $t = s-t$ or $\delta^* = 1/2$. If $p_k + p_{k-1} \neq 1$ then taking logarithms of both sides of (6.8) and solving for t/s completes the proof.

Lemma 6.3. For δ^* as given in (6.6) and (6.7)

$$\gamma(\delta) = \begin{cases} \gamma_1(\delta), & \text{when } \delta \geq \delta^* \\ \gamma_2(\delta), & \text{when } \delta \leq \delta^*. \end{cases}$$

PROOF. Suppose that for some δ, $\gamma(\delta) = \gamma_1(\delta)$. If $\delta' \geq \delta$ then by Lemma (6.1), $\gamma_1(\delta') \geq \gamma_1(\delta)$ and $\gamma_2(\delta') \leq \gamma_2(\delta)$. By assumption $\gamma_1(\delta) \geq \gamma_2(\delta)$ so that $\gamma_1(\delta') \geq \gamma_2(\delta')$ and so $\gamma(\delta') = \gamma_1(\delta')$.

Suppose now $\gamma(\delta) = \gamma_2(\delta)$. If $\delta'' < \delta$ then again by Lemma 6.1 $\gamma_1(\delta) \geq \gamma_1(\delta'')$ and $\gamma_2(\delta) \leq \gamma_2(\delta'')$. Since $\gamma(\delta) = \gamma_2(\delta) \geq \gamma_1(\delta)$ it follows that $\gamma_2(\delta'') \geq \gamma_1(\delta'')$ and so $\gamma(\delta'') = \gamma_2(\delta'')$. Thus it has been shown that if $\gamma(\delta) = \gamma_1(\delta)$, for some $\delta \in (p_{k-1}, p_k)$ then $\gamma(\delta') = \gamma_1(\delta'')$ for all $\gamma' \in (\delta, p_k)$ and if $\gamma(\delta) = \gamma_2(\delta)$ for some $\delta \in (p_{k-1}, p_k)$ then $\gamma(\delta') = \gamma_2(\delta')$ for all $\delta' = (p_{k-1}, \delta)$.

Since $\gamma_1(\delta^*) = \gamma_2(\delta^*)$ the lemma follows.

Corollary 6.1. $\gamma(\delta^*) = \min \gamma(\delta)$ for $\delta \in (p_{k-1}, p_k)$.

PROOF. At δ^*, $\gamma_1(\delta^*) = \gamma_2(\delta^*)$. Suppose $\delta > \delta^*$ then from lemma (6.3) $\gamma(\delta) = \gamma_1(\delta)$. But by lemma (6.1) $\gamma_1(\delta) \geq \gamma_1(\delta^*)$. Similarly for $\delta < \delta^*$, $\gamma(\delta) = \gamma_2(\delta) \geq \gamma_2(\delta^*)$ but $\gamma(\delta^*) = \gamma_1(\delta^*) = \gamma_2(\delta^*)$ so the corollary follows.

Lemma 6.4. For $\delta \in (p_{k-1}, p_k)$

$$M \cong \begin{cases} \dfrac{\gamma(\delta)}{\delta - p_{k-1}}, & \text{for } \delta \leq \bar{\delta} \\[2ex] \dfrac{\gamma(\delta)}{p_k - \delta}, & \text{for } \delta \geq \bar{\delta} \end{cases}$$

where $\bar{\delta} = \dfrac{p_k + p_{k-1}}{2}$

PROOF. From Theorem 5.2 $M_i \cong \dfrac{s\gamma(\delta)}{|p_i s - r|} = \dfrac{\gamma(\delta)}{|p_i - \delta|}$ for $\delta = r/s$.

It is clear that since $p_i \leq p_{k-1} < \delta$ for $i = 1,2,\ldots, k$ then $|p_i - \delta| = \delta - p_i \geq \delta - p_{k-1} = |p_{k-1} - \delta|$. Thus $M = \max(M_{k-1}, M_k)$. Now $|p_k - \delta| = p_k - \delta \geq \delta - p_{k-1}$ if and only if $\delta \leq \bar{\delta}$. Hence, for $\delta \leq \bar{\delta}$

$$M = \frac{\gamma(\delta)}{(\delta - p_{k-1})}$$ and for $\delta \geq \bar{\delta}$ $M = \gamma(\delta)/(p_k - \delta)$ which completes the proof.

Lemma 6.5. (1) $\frac{\gamma_1(\delta)}{p_k-\delta}$ is an increasing function, and (2) $\frac{\gamma_2(\delta)}{\delta - p_{k-1}}$ is a decreasing function of $\delta \in (p_{k-1}, p_k)$.

PROOF. Lemma 6.1 shows that $\gamma_1(\delta)$ is a non-decreasing function and $\gamma_2(\delta)$ is a non-increasing function of $\delta \in (p_{k-1}, p_k)$. Now $p_k - \delta$ decreases monotonically to 0 as $\delta \to p_k$ and $\delta - p_{k-1}$ increases monotonically as $\delta \to p_k$. Thus in (1) the numerator increases and the denominator decreases hence the fraction increases as δ increases; and in (2) the numerator decreases and the denominator increases as δ increases hence the fraction decreases as δ increases. This completes the proof.

Theorem 6.1 For $\delta \in (p_{k-1}, p_k)$,

$$\min_{\delta} M = \begin{cases} \min\limits_{\delta^* \leq \delta \leq \bar{\delta}} \dfrac{\gamma_1(\delta)}{\delta - p_{k-1}}, & \text{for } \delta^* \leq \bar{\delta} \\ \min\limits_{\bar{\delta} \leq \delta \leq \delta^*} \dfrac{\gamma_2(\delta)}{p_k - \delta}, & \text{for } \bar{\delta} \leq \delta^*. \end{cases}$$

PROOF. Suppose $\delta^* \leq \bar{\delta}$, then for $\delta \geq \bar{\delta}$ Lemma 6.3 and Lemma 6.4 imply

$M \cong \frac{\gamma_1(\delta)}{(p_k - \delta)}$. However, by Lemma 6.5 this is an increasing function of δ, and thus the minimum for $\delta \geq \bar{\delta}$ occurs at $\delta = \bar{\delta}$. For $\delta \leq \delta^*$ Lemma 6.3 and Lemma 6.4 show $M = \frac{\gamma_2(\delta)}{(\delta - p_{k-1})}$ which by Lemma 6.5 decreases as δ increases so that the minimum for $\delta \leq \delta^*$ occurs at $\delta = \delta^*$. Thus it follows that the min M for $\delta \in (p_{k-1}, p_k)$ occurs for some $\delta \in [\delta^*, \bar{\delta}]$. Since $\delta \leq \bar{\delta}$ by Lemma 6.4,

$$\min_{\delta^*\leq \delta\leq \overline{\delta}} M = \min_{\delta^*\leq\delta\leq\overline{\delta}} \frac{\gamma(\delta)}{(\delta-p_{k-1})}, \text{ since } \delta \geq \delta^* \quad \gamma(\delta) = \gamma_1(\delta)$$

and the first approximation of the theorem follows. A similar argument for $\delta^* \geq \overline{\delta}$ will provide the second approximation and the theorem.

We have shown then that the $\delta \in (p_{k-1}, p_k)$ which asympototically minimizes (6.3) is found between δ^* and $\overline{\delta}$, and that $\gamma(\delta)$ approximated by

$$(6.9) \qquad \gamma(\delta) = \begin{cases} \dfrac{\ln \frac{\epsilon}{1-\epsilon}}{s \ln y_k(\delta)}, & \text{for } \delta \geq \delta^* \\[2ex] \dfrac{\ln \frac{1-\epsilon}{\epsilon}}{s \ln y_{k-1}(\delta)}, & \text{for } \delta \leq \delta^* \end{cases}$$

then provides a $\eta = (\delta, \gamma(\delta)) \in \mathcal{C}_2$, that is, a procedure, satisfying (6.2). This still leaves the experimenter with the problem of choosing a specific δ if $\delta^* \neq \overline{\delta}$ ($p_{k-1} + p \neq 1$). It has been found empirically (see[1]) that often $\delta^* \cong \overline{\delta}$, so that the experimenter will not be "far" from the minimum for any choice of δ between $\overline{\delta}$ and δ^*. Numerical evidence indicates that if $\overline{\delta}$ and δ^* are significantly apart, the minimum takes place near δ^*. Another advantage to using δ^* is that the approximation of $\gamma(\delta^*)$ can be given as a function of p_{k-1}, p_k and ϵ so that the experimenter need not find the roots y_k and y_{k-1} to (5.2). In fact, using δ^* defined in (6.6) and (6.7) then from (6.4),

$$(6.10) \quad \gamma^* = \gamma(\delta^*) = (1 - \delta^*) \ln \frac{\epsilon}{1-\epsilon} \left(\ln \frac{p_{k-1}}{p_k}\right)^{-1} = \ln \frac{1-\epsilon}{\epsilon} \left(\ln \frac{p_k(1-p_{k-1})}{p_{k-1}(1-p_k)}\right)^{-1} .$$

Thus the above discussion suggests that an approximate minimax rule which has certain desirable properties would be $\mathcal{S}(\eta^*)$ where $\eta^* = (\{\delta^* m - \gamma^*\}, \{\delta^* m + \gamma^*\})$. This, of course, is not the only choice of $\eta \in \mathcal{C}_1$ available. It depends on the need of the experimenter who may wish to replace (ii) of (6.1) by some other condition such as $a_i \leq \epsilon$ for some $0 < p < k$ or he may want less stringent requirements on ES than (ii) of (6.2). The use of δ^*, γ^* is only one suggestion toward meeting a practical requirement of a good sequential test.

7. Some Sample Size Comparisons of $\mathcal{S}(\eta^*)$ and R(n).

In this section we offer some numerical comparisons between the procedure $\mathcal{S}(\eta^*)$ and R(n). Comparisons are difficult in general because analytic expressions involving the two procedures are not readily obtainable, and because of the small number of tables available on the performance of R(n). Two special configurations of the means $\theta_1, \theta_2, \ldots, \theta_k$ will be considered. The first is called the "slippage configuration", that is,

$$(7.1) \quad \theta_{[1]} = \theta_{[2]} = \cdots = \theta_{[k-1]} = \theta, \ \theta_{[k]} = \theta + \tau, \quad \tau > 0.$$

Tables of P{selecting π_i |R(n)} have been tabulated in this case for selected values of P*, k,n, and τ, in [2]. The second configuration called the "equally-spaced means" configuration, is

$$(7.2) \quad \theta_{[1]} = \theta, \ \theta_{[2]} = \theta + \tau, \ \theta_{[3]} = \theta + 2\tau, \ \ldots, \ \theta_{[k]} = \theta + (k-1)\tau, \ \tau > 0.$$

Tables of P{selecting π_i |R(n)} have been tabulated in this case for selected values of P*, k,n, and τ in [7].

For any multiple-decision rule R, consider the following inequalities, for $0 < \epsilon < 1$

$$(7.3) \quad \begin{aligned} &\text{(i) } P\{CS \mid R\} \geq 1 - \epsilon \\ &\text{(ii) } 1 - \epsilon < E\{S \mid R\} \leq 1 + (k-1)\epsilon \end{aligned}$$

for $R = \mathcal{S}(\eta)$ we again let $M = \max_{1 \leq i \leq 1} E\{m_i \mid \mathcal{S}(\eta)\}$. As shown in Section 6 by choosing $\eta = \eta^*$ we get the sequential procedure that approximately

TABLE 3

Sample Size Comparisons for the Sequential and
Fixed Sample Size Rules for the Slippage
Configuration for the Normal Population: $P^* = .75$

k \ τ	0.05	0.10	0.20	0.30	0.40	0.50	0.60	1.00	2.00
	5422.7	1315.7	336.6	151.8	87.5	57.3	40.6	16.1	15.1
2	11240.0	2810.0	702.5	312.2	175.6	112.4	78.1	28.1	19.7
	.482	.468	.479	.486	.498	.510	.520	.573	.766
	7553.9	1922.4	491.4	219.1	124.3	82.0	57.7	22.1	6.7
3	13600.0	3460.0	850.0	377.7	212.5	136.0	94.5	34.0	8.5
	.555	.556	.578	.580	.585	.603	.611	.650	.788
	9890.1	2418.0	586.1	259.4	148.0	95.6	67.5	26.1	7.9
4	15120.0	3780.0	945.0	420.0	236.2	151.2	102.9	37.8	9.5
	.654	.640	.620	.618	.627	.632	.656	.690	.832
	10752.1	2485.3	637.2	284.1	161.6	104.9	74.8	29.0	8.7
5	15640.0	3910.0	977.5	434.4	244.4	156.4	108.7	39.1	9.8
	.687	.636	.652	.654	.661	.672	.688	.742	.888
	10752.1	2754.5	679.7	306.9	171.5	111.9	84.5	31.2	9.4
6	15880.0	3970.0	992.5	441.1	248.1	158.8	110.4	39.7	9.9
	.677	.694	.685	.696	.691	.705	.765	.786	.949
	12119.5	2835.3	695.0	317.9	180.1	117.0	84.8	32.8	10.0
7	16400.0	4100.0	1025.0	455.5	256.2	164.0	114.0	41.0	10.2
	.739	.692	.678	.698	.693	.713	.744	.800	.980
	12695.4	2803.7	708.1	325.3	183.9	120.4	86.4	34.2	10.6
8	16920.0	4230.0	1057.5	470.0	264.4	169.2	117.6	42.3	10.6
	.750	.663	.670	.692	.796	.712	.735	.809	1.000
	12695.4	2803.7	749.6	331.1	188.6	126.7	88.9	35.2	10.9
9	17440.0	4360.0	10900.0	484.4	272.5	174.4	121.2	43.6	10.9
	.728	.643	.688	.685	.692	.726	.733	.807	1.000
	13037.6	2960.2	731.4	346.3	192.9	127.2	91.7	36.4	11.4
10	17680.0	4420.0	1105.0	491.1	276.3	176.8	122.9	44.2	11.1
	.737	.670	.662	.705	.698	.719	.746	.824	1.027
	13037.6	3192.4	817.1	369.7	212.7	143.5	103.5	43.7	14.8
25	20440.0	5110.0	1277.5	567.7	319.4	204.4	142.1	51.1	12.8
	.638	.625	.640	.651	.666	.702	.726	.855	1.156
	13695.1	3096.1	786.9	365.1	220.0	148.1	107.2	46.7	17.2
50	21600.0	5400.0	1350.0	599.9	337.5	216.0	150.1	51.0	13.5
	.634	.573	.583	.609	.665	.686	.714	.864	1.27

TABLE 4

Sample Size Comparisons for the Sequential and

Fixed Sample Size Rules for the Slippage

Configuration for the Normal Population: P* = .90

k \ τ	0.05	0.10	0.20	0.30	0.40	0.10	0.60	1.00	2.00
	8786.5	2079.3	542.4	257.6	148.0	97.0	69.2	28.3	9.1
2	15360.0	3840.0	960.0	426.6	240.0	153.6	106.5	38.4	9.6
	.572	.541	.565	.604	.608	.632	.650	.737	.948
	12469.0	3069.0	849.1	385.1	217.9	145.6	100.9	46.4	12.2
3	17440.0	4360.0	1090.0	484.4	272.5	174.4	121.2	43.6	10.9
	.716	.704	.779	.795	.800	.835	.833	1.064	1.12
	14282.3	3781.5	1024.6	477.0	260.5	170.2	122.4	49.1	14.5
4	18600.0	4650.0	1162.5	516.6	290.6	186.0	129.3	46.5	11.6
	.786	.813	.881	.923	.896	.915	.947	1.056	1.25
	14282.3	4177.9	1085.0	494.3	290.9	186.7	135.0	51.0	16.2
5	19600.0	4800.0	1225.0	544.3	306.3	196.0	136.2	59.0	12.3
	.729	.853	.886	.908	.950	.953	.991	1.091	1.32
	14282.3	4230.0	1148.4	531.2	300.5	201.2	148.8	59.0	17.8
6	20160.0	5040.0	1260.0	559.9	315.0	201.6	140.1	50.4	12.6
	.708	.839	.911	.949	.954	.998	1.062	1.171	1.413
	18633.7	4463.5	1148.4	550.7	333.2	219.8	152.7	63.3	19.1
7	20720.0	5180.0	1295.0	575.5	323.8	207.2	144.0	51.8	13.0
	.899	.862	.887	.957	1.029	1.061	1.060	1.222	1.469
	18633.7	4653.4	1335.6	566.9	337.9	224.1	161.2	66.8	20.3
8	21040.0	5260.0	1315.0	584.4	328.8	210.4	146.2	52.6	13.2
	.886	.885	1.016	.970	1.028	1.065	1.103	1.270	1.538
	18633.7	4653.4	1335.6	589.0	341.6	232.2	167.5	70.0	21.4
9	21320.0	5330.0	1332.5	592.2	333.1	213.2	148.2	53.3	13.3
	.874	.873	1.002	.995	1.026	1.089	1.130	1.313	1.609
	18633.7	4653.4	1335.6	589.0	341.6	246.6	168.4	72.0	72.4
10	21680.0	5420.0	1355.0	602.2	338.8	216.8	150.7	54.2	13.6
	.859	.859	.986	.978	1.008	1.137	1.117	1.328	1.647
	18633.7	4653.4	1381.8	632.1	383.4	265.6	195.3	86.7	30.4
25	24040.0	6010.0	1520.5	667.7	375.6	240.4	167.1	60.1	15.0
	.775	.774	.909	.947	1.021	1.105	1.169	1.443	2.027
	18801.4	5257.6	1338.0	683.3	403.9	278.1	200.2	104.1	37.8
50	25600.0	6400.0	1600.0	711.0	400.0	256.0	177.9	64.0	16.0
	.734	.821	.836	.961	1.010	1.086	1.125	1.627	2.36

minimizes M over all $\eta \in \mathcal{C}_1$, satisfying (7.3). Similarly let N be the sample size required to satisfy (7.3) when R = R(n). Then clearly $\mathcal{S}(\eta^*)$ will be preferable to R(n) whenever M < N, and R(n) preferable to $\mathcal{S}(\eta^*)$ when N < M.

For the slippage configuration in (7.1) equation (2.1) becomes

$$(7.4) \quad p_i = \int_{-\infty}^{\infty} \Phi^{k-2}(x+d) \; \Phi(x+d + \tau n^{1/2}\sigma) \; \varphi(x)dx, \; i = 1,2,\ldots k-2.$$

$$(7.5) \quad p_k = \int_{-\infty}^{\infty} \Phi^{k-1} (x+ d + \tau n^{1/2}/\sigma) \; \varphi(x)dx.$$

Using tables found in [2], (7.4) and (7.5) were computed assuming $\sigma = 1$. For each p_{k-1} and p_k with $n = 1$, $\eta^* = (\delta^*, \gamma^*)$ were computed using (6.6) and (6.10). For $P^* = .75$, Table 3 compares M with N when $\mathcal{S}(\eta^*)$ and R(n) satisfy (7.3). The upper value is the expected sample size M while the middle value is the fixed-sample size N. The lower value gives the ratio of M to N. The smaller the ratio the more inclined we are to use $\mathcal{S}(\eta^*)$ over R(n). The savings in the number of samples needed to achieve (7.3) with $\epsilon = .001$ using $\mathcal{S}(\eta^*)$ over R(n) vary for different values of k from better than 50% to 25% for $\tau \leq .50$ to less of a saving for $.50 \leq \tau \leq 1$. For larger values of τ, such as $\tau = 2$, the fixed sample procedure R(n) requires less samples than $\mathcal{S}(\eta^*)$ to achieve (7.3) with $\epsilon = .001$ and is a more preferable procedure.

As an example for k = 6 populations and $\tau = 0.4$, the expected number of samples from each population needed to satisfy (7.3) with $\epsilon = .001$ using $\mathcal{S}(\eta^*)$ is 172.5 or a total of 1035 observations, while using R(n) a sample of 248.1 must be taken from each population, a total of 1488.6 observations to satisfy the same conditions. This is better than a 30% savings in using $\mathcal{S}(\eta^*)$.

Table 4 gives the same data as Table 3 but for $P^* = .90$. In this case the savings are generally less when using $\mathcal{S}(\eta^*)$ over R(n), and τ generally must be smaller. In general, in both tables for a fixed k as τ increases the ratio M/N increases. For a fixed τ, M/N is smallest for k = 2, while the maximum increases from k = 7 to k = 50 as τ increases. Thus it appears that $\mathcal{S}(\eta^*)$ is a better procedure when the mean that has

TABLE 5

Sample Size Comparisons for the Sequential and

Fixed Sample Size Rules for the Equally-Spaced Means

Configuration for the Normal Population: P* = .75

k \ τ	0.05	0.10	0.20	0.30	0.40	0.50	0.60
	5422.7	1315.7	336.0	151.8	87.5	57.3	40.6
2	11240.0	2810.0	702.5	312.2	175.6	112.4	78.1
	.482	.468	.479	.486	.498	.510	.520
	7430.9	1841.7	470.9	208.9	118.1	77.2	54.5
3	13440.0	3360.0	840.0	372.3	210.0	134.4	93.4
	.553	.548	.561	.561	.562	.574	.584
	8258.4	2194.8	551.4	280.3	141.2	90.4	64.2
4	14880.0	3720.0	930.0	463.3	232.5	148.8	103.4
	.555	.590	.593	.605	.607	.608	.623
	8824.8	2393.9	619.2	276.9	155.8	101.0	72.4
5	15880.8	3970.0	992.5	441.1	248.1	158.8	110.4
	.556	.603	.624	.628	.628	.636	.656

TABLE 6

Sample Size Comparisons for the Sequential and

Fixed Sample Size Rules for the Equally-Spaced Means

Configuration for the Normal Population: P* = .90

k \ τ	0.05	0.10	0.20	0.30	0.40	0.50	0.60
	8786.5	2079.3	542.4	257.6	146.0	97.0	69.2
2	15360.0	3840.0	960.0	426.6	240.0	153.6	106.5
	.572	.541	.565	.604	.608	.632	.650
	1186.4	3262.7	783.2	372.1	214.1	144.6	100.6
3	17440.0	4360.0	1090.0	484.4	272.5	174.4	121.2
	.680	.748	.719	.768	.786	.829	.830
	14435.0	3792.2	961.9	459.1	259.4	174.5	123.8
4	18640.0	4660.0	1165.0	517.7	291.3	186.4	129.5
	.774	.814	.826	.887	.890	.936	.956
	17436.4	4517.0	1207.6	540.1	310.1	197.0	147.7
5	19520.0	4880.0	1220.0	542.2	305.0	195.2	135.7
	.893	.926	.990	.996	1.02	1.01	1.09

slipped to the right has not slipped far. Close examination of both tables reveals that the ratio M/N does not increase monotonically for very small values of τ, this is due to the fact that p_k and p_{k-1} do not vary greatly as k increases. For example, for P* = .75 and $\tau = 0.05$, to three significant places, $p_k = .763$, $p_{k-1} = .748$ for k = 7, 8 and 9. Thus for small τ rounding errors play a somewhat higher role than for larger τ. Another factor in all tables of this section is that in practice the exact value of δ^* cannot be used to obtain the various results, but a close approximation of δ^* is used instead. This tends to destroy the apparent monotonicity as well.

For the configuration in (7.2) equation (2.1) becomes,

$$(7.6) \quad p_i = \int_{-\infty}^{\infty} [\prod_{\substack{j=1 \\ j \neq i}}^{k} \Phi(x + d - (j-1)\tau n^{1/2}/\sigma)] \,\varphi(x)dx, \quad i = 1,2,\ldots,k.$$

Using tables and extensions of tables in [7], (7.6) was evaluated assuming $\sigma = 1$. A numerical comparison of $\mathcal{S}(\pi^*)$ and R(n) was carried out using the same method as in the slippage configuration. That is, M and N were evaluated so that (7.3) holds for $\mathcal{S}(\pi^*)$ and R(n) with $\epsilon = .001$. For P* = .75, Table 5 gives the values of M, N and the ratio M/N for selected values of k and c. The upper value being M, and middle value being N, and the lower value being the ratio M/N. Table 6 contains the same information for P* = .90.

It can be seen from Table 5 that the behavior of the ratio M/N is similar to that in Table 4 for the slippage configuration. That is, the smaller τ is the smaller the ratio. In fact, for k = 2, there is a 50% or better saving in the expected number of samples using $\mathcal{S}(\pi^*)$ instead of R(n) for $\tau \leq .50$. For a fixed τ as k increases from 2 to 5, M/N increases. Of course, Table 5 only goes to k = 5, and since Table 3 showed erratic behavior as k increased to 50, one cannot make a general statement about this monotonic behavior. Table 6 shows the same basic behavior but the savings using $\mathcal{S}(\pi^*)$ over R(n) are in general, less.

Thus based on the numerical computation for the slippage configuration and the equally spaced means configuration one empirical conclusion is that (π^*) is a more preferable procedure when the means are close and R(n) is a more preferable procedure as any one mean gets significantly larger (or smaller) than the others.

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13. ABSTRACT

This paper is concerned with the multiple decision (selection and ranking) problem for k independent normal populations having unknown means and a known common variance. A class of sequential and multi-stage procedures is defined and investigated. This class consists of rules of a non-eliminating type; a rule belonging to this class selects and rejects populations at various stages but continues taking samples from all populations until the procedure terminates.

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14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, roles, and weights is optional.