# Asymptotic Expansions for the Distributions of Characteristic Roots When the Parameter Matrix has Several Multiple Roots*

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Department of Statistics
Division of Mathematical Sciences
Mimeograph Series #251
March 1971

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* This research was supported by National Science Foundation Grant No. GP-11473

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0. Introduction. Asymptotic expansions for the distributions of characteristic roots of matrices arising in different situations in multivariate analysis were obtained by the authors [3], [4] when the parameter matrix has a single multiple root, extreme or intermediate. These asymptotic expansions generalized the results of earlier authors [1], [2], [5], [6], [7]. However, in extending the work further to the case of several multiple population roots, the method used in [5] was not found to be suitable in view of the fact that the invariance of a function with respect to the choice of a sub-matrix in the orthogonal (unitary) matrix used there does not extend to the simultaneous invariance with respect to the choices of several submatrices as is needed to extend that method. In order to overcome this difficulty we proceed in a different manner without recourse to the invariance property and restate lemma 3.1 of [3] in a more detailed fashion before demonstrating the new approach. Here we use the same notation and symbols as used in [3] as far as possible.

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* This research was supported by National Science Foundation Grant No. GP-11473

1. <u>The maximization procedures</u>. Let us define $\underset{\sim}{R} = \text{diag}\,(r_1, \ldots, r_p)$; $\infty > r_1 > r_2 > \ldots > r_p > 0$.

$$(1) \quad \underset{\sim}{A} = \text{diag}\,(\overset{k_1}{\ell_1, \ldots, \ell_1}, \overset{k_2}{\ell_2, \ldots,} \ell_{k_1+k_2+1}, \ldots, \ell_p),$$

$$\infty > \ell_1 > \ell_2 > \ell_{k_1+k_2+1} > \ldots > \ell_p \geq 0,$$

and let $\underset{\sim}{H} \epsilon O(p)$, where $O(p)$ is the group of orthogonal matrices of order p. Then we state the following lemma.

<u>Lemma 1</u>. If (1) holds, then for all variations of $\underset{\sim}{R} > 0$

$$\max_{\underset{\sim}{H} \epsilon O(p)} C_\kappa(\underset{\sim}{H}'\,\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}) = \max_{\underset{\sim}{H} \epsilon O(p)} C_\kappa(\underset{\sim}{H}\,\underset{\sim}{R}\,\underset{\sim}{H}'\,\underset{\sim}{A}) = C_\kappa(\underset{\sim}{A}\,\underset{\sim}{R})$$

and if the ordering of the elements of $\underset{\sim}{A}$ is reversed then

$$\min_{\underset{\sim}{H} \epsilon O(p)} C_\kappa(\underset{\sim}{H}'\,\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}) = \min_{\underset{\sim}{H} \epsilon O(p)} C_\kappa(\underset{\sim}{H}\,\underset{\sim}{R}\,\underset{\sim}{H}'\,\underset{\sim}{A}) = C_\kappa(\underset{\sim}{A}\,\underset{\sim}{R}) \cdot$$

The optimum values are attained iff $\underset{\sim}{H}$ has the form

$$\underset{\sim}{H} = \begin{array}{c c} \begin{array}{ccc} k_1 & k_2 & q \end{array} & \\ \begin{pmatrix} \underset{\sim}{H_1} & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{H_2} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{I_\circ} \end{pmatrix} & \begin{array}{c} k_1 \\ k_2 \\ q \end{array} \end{array} ,$$

where $q = p - k_1 - k_2$ and $I_\circ(q) = \text{diag}\,(\pm 1, \ldots, \pm 1)$.

<u>Proof</u>. By lemma 3.2 [3], we get $\underset{\sim}{H}$ must have the form

$$\underset{\sim}{H} = \begin{pmatrix} \underset{\sim}{H_1} & \underset{\sim}{H_1} & \underset{\sim}{H_{11}} \\ \underset{\sim}{0} & \underset{\sim}{H_2} & \underset{\sim}{H_{22}} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{I_\circ} \end{pmatrix}$$

But, because of the orthogonality of $\utilde{H}$, we get, $\utilde{H}_{12} = \utilde{H}_{22} = \utilde{0}$ and which in turn gives $\utilde{H}_2 \utilde{H}_2' = \utilde{H}_2' \utilde{H}_2 = I(k_2)$. Thus, $\utilde{H}_{11} = \utilde{0}$, and hence the proof.

The proof just outlined is general and also goes through in the complex analogue of this problem when $\utilde{H}$ is replaced by $\utilde{U}$, where $\utilde{U} \epsilon U(p)$, and $U(p)$ is the group of unitary matrices.

Thus let us consider the following formalization.

$$\utilde{R} = \text{diag}\,(r_1, \ldots, r_p), \quad \infty > r_1 > r_2 > \ldots > r_p > 0$$

$$(2) \quad \utilde{A} = \text{diag}\,(\overset{k_1}{\ell_1, \ldots, \ell_1}, \ldots, \overset{k_m}{\ell_m, \ldots, \ell_m}, \ell_{k_1+\cdots+k_m+1}, \ldots, \ell_p),$$

$$\infty > \ell_1 > \ell_2 > \ldots > \ell_m > \ell_{k_1+\cdots+k_m+1} > \ldots > \ell_p \geq 0,$$

and let $\utilde{H} \epsilon O(p)$. Then

<u>Lemma 1.1</u>. If (2) holds, then for all variations of $\utilde{R} > 0$

$$\max_{\utilde{H} \epsilon O(p)} C_\kappa(\utilde{H}' \utilde{A} \utilde{H} \utilde{R}) = \max_{\utilde{H} \epsilon O(p)} C_\kappa(\utilde{H} \utilde{R} \utilde{H}' \utilde{A}) = C_\kappa(\utilde{A} \utilde{R})$$

and if the ordering of the elements of $\utilde{A}$ is reversed then

$$\min_{\utilde{H} \epsilon O(p)} C_\kappa(\utilde{H}' \utilde{A} \utilde{H} \utilde{R}) = \min_{\utilde{H} \epsilon O(p)} C_\kappa(\utilde{H} \utilde{R} \utilde{H}' \utilde{A}) = C_\kappa(\utilde{A} \utilde{R})$$

and the optimum values are attained iff $\utilde{H}$ has the form

$$\utilde{H} = \text{diag}\,(\utilde{H}_1, \ldots, \utilde{H}_m, \utilde{I}_\circ(p-k_1-\ldots-k_m)),$$

where $\utilde{H}_j$ $(k_j \times k_j)$ is an orthogonal matrix of order $k_j$, $j=1, \ldots, m$, and $\utilde{I}_\circ(p-k_1-\ldots-k_m) = \text{diag}\,(\pm 1, \ldots, \pm 1)$.

In order to facilitate the subsequent generalization to the complex case we state an analogue of lemma 1.1, the proof being self-evident from previous discussion.

<u>Lemma 1.2</u>. If (2) holds, then for all variations of $\underset{\sim}{R} > 0$

$$\max_{\underset{\sim}{U} \epsilon U(p)} C_\kappa(\underset{\sim}{U}^* \underset{\sim}{A}\, \underset{\sim}{U}\, \underset{\sim}{R}) = \max_{\underset{\sim}{U} \epsilon U(p)} C_\kappa(\underset{\sim}{U}\, \underset{\sim}{R}\, \underset{\sim}{U}^* \underset{\sim}{A}) = C_\kappa(\underset{\sim}{A}\, \underset{\sim}{R})$$

and if the ordering of the elements of $\underset{\sim}{A}$ is reversed then

$$\min_{\underset{\sim}{U} \epsilon U(p)} C_\kappa(\underset{\sim}{U}^* \underset{\sim}{A}\, \underset{\sim}{U}\, \underset{\sim}{R}) = \min_{\underset{\sim}{U} \epsilon U(p)} C_\kappa(\underset{\sim}{U}\, \underset{\sim}{R}\, \underset{\sim}{U}^* \underset{\sim}{A}) = C_\kappa(\underset{\sim}{A}\, \underset{\sim}{R})$$

and the optimum values are attained iff $\underset{\sim}{U}$ has the form

$\underset{\sim}{U} = \text{diag}\ (\underset{\sim}{U}_1, \dots, \underset{\sim}{U}_m, \underset{\sim}{U}_{m+1})$, where $\underset{\sim}{U}_j(k_j x k_j)$ is an unitary matrix of order $k_j$, $j=1, \dots, m$ and $U_{m+1} = \text{diag}\ (e^{\sqrt{-1}\,\theta_1}, \dots, e^{\sqrt{-1}\,\theta_q})$ , $0 \leq \theta_j < 2\pi$, $j=1, \dots, q$, and $q = p-k_1-\dots\dots -k_m$.

Using the above results we get corresponding results for theorem 1.2 and its complex analogue theorem 1.3 of [3].

## 2. <u>Asymptotic expansion for the distribution of the latent roots of the estimated covariance matrix -- several multiple population roots</u>.

Let $\underset{\sim}{R} = \text{diag}\ (r_1, \dots, r_p)$, $\infty > r_1 > \dots > r_p > 0$, where $r_i$'s are the latent roots in discending order, of a sample covariance matrix $\underset{\sim}{C}$ with n d.f. calculated from a sample from a normal population with covariance matrix $\underset{\sim}{\Sigma}$. Let the diagonal matrix of the latent roots of $\underset{\sim}{\Sigma}^{-1}$ be $\underset{\sim}{A}$ and $\underset{\sim}{A}$ has the form

$$\underset{\sim}{A} = \text{diag}\ (\ell_1, \dots, \ell_q, \overbrace{\ell_{q+1}, \dots, \ell_{q+1}}^{k_1}, \dots, \overbrace{\ell_{q+m}, \dots, \ell_{q+m}}^{k_m}),$$

$$\underset{\sim}{R} = \text{diag}\ (r_1, \dots, r_p),\ \infty > r_1 > r_2 > , \dots, > r_p > 0, \tag{3}$$

(4) $$\infty > \ell_{q+m} >, \ldots, > \ell_{q+1} > \ell_q >, \ldots, > \ell_1 \geq 0$$

where $p = k_1 + \ldots + k_m + q$. Then the joint distribution of $r_1, \ldots, r_p$, is

(5) $$C_1 \int_{O(p)} \exp\left(-\frac{n}{2} \operatorname{tr} \underset{\sim}{H}' \underset{\sim}{A} \underset{\sim}{H} \underset{\sim}{R}\right) \quad d(\underset{\sim}{H})$$

where

$$C_1 = \{n^{\frac{np}{2}} \pi^{\frac{p^2}{2}} / [2^{\frac{np}{2}} \Gamma_p(\tfrac{n}{2}) \Gamma_p(\tfrac{p}{2}) ]\} \prod_{i=1}^{q} \ell_i^{\frac{n}{2}}$$

$$\prod_{j=1}^{m} \ell_{q+j}^{\frac{nk_j}{2}} \prod_{i=1}^{p} r_i^{\frac{(n-p-1)}{2}} \prod_{i<j} (r_i - r_j) \prod_{i=1}^{p} dr_i .$$

Now by lemma 1.1 and as shown in [3], the integrand in (5) is maximized for all variations of $\underset{\sim}{R} > 0$, when $\underset{\sim}{H}$ has the following form

(6) $\underset{\sim}{H} = \operatorname{diag}(\underset{\sim}{I}_{\circ}\ (q), \underset{\sim}{H}_1, \ldots, \underset{\sim}{H}_m)$,

where
$\underset{\sim}{H}_i$ $(k_i x k_i)$, $i = 1, .., m$, are orthoganal matrices.

As stated earlier we do not resort to the invariance technique as used by earlier authors. Now following Anderson [1] we use the transformation

(7) $$\underset{\sim}{H} = \exp [\underset{\sim}{S}] ,$$

where $\underset{\sim}{S}$ is a p x p skew symmetric matrix. Now under (4), the transformation (7) reduces the integrand in (5) to a form which does not yield to direct evaluation. Hence to avoid this difficulty we note that if (4) holds, then for all $\underset{\sim}{R} > 0$ the integrand in (5) is maximized when $\underset{\sim}{H}$ has the form (6). Also when $\tilde{n}$ is large the whole integral is concentrated around its unique maximum value. Thus, instead of (7) we use the transformation

(8) $$\underset{\sim}{H} = \exp [\underset{\sim}{S}_1] ,$$

where $\underset{\sim}{S}_1$ is a p x p skew symmetric matrix but has the following form

$$\underset{\sim}{S}_1 = \begin{pmatrix} \underset{\sim}{S}_0 \\ \underset{\sim}{S}_1 \\ \underset{\sim}{S}_m \end{pmatrix}$$

$\underset{\sim}{S}_0$ $(q \times p)$; $\underset{\sim}{S}_i = (\underset{\sim}{S}_{i1}, \underset{\sim}{S}_{i2}, \underset{\sim}{S}_{i3})$;

$\underset{\sim}{S}_{i1}$ $(k_i \times (q + k_1 + \ldots + k_{i-1}))$, $\underset{\sim}{S}_{i2}$ $(k_i \times k_i) = \underset{\sim}{0}$, $\underset{\sim}{S}_{i3}$ $(k_i \times (k_{i+1} + \ldots + k_m))$,

$i=1,\ldots, m-1$ and $\underset{\sim}{S}_m = (\underset{\sim}{S}_{m1}, \underset{\sim}{S}_{m2})$; $\underset{\sim}{S}_{m1}$ $(k_m \times (p-k_m))$, $\underset{\sim}{S}_{m2}$ $(k_m \times k_m) = \underset{\sim}{0}$.

This is no loss of generality provided the constant factor is adjusted, as for large n the integrand is concentrated around its unique maximum and at least one maximizing set is covered by this substitution. Let $t = p-k_m$.
Then

$$\begin{aligned} \mathrm{tr}\,(\underset{\sim}{H}'\,\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}) &= \mathrm{tr}\,(\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}\,\underset{\sim}{H}') \\ &= \sum_{ij} h_{ij}^2\, \ell_i\, r_j \\ &= \sum_{i=1}^{t} \sum_{j=1}^{p} \ell_i\, r_j\, h_{ij}^2 + \ell_{q+m} \sum_{j=1}^{p} r_j \sum_{i=t+1}^{p} h_{ij}^2 \\ &= \ell_{q+m} \sum_{j=1}^{p} r_j + \sum_{i=1}^{t} \sum_{j=1}^{p} (\ell_i - \ell_{q+m})\, r_j\, h_{ij}^2 , \end{aligned}$$

since

$$\sum_{i=t+1}^{p} h_{ij}^2 = 1 - \sum_{i=1}^{t} h_{ij}^2 \qquad \text{for } j = 1,\ldots, p.$$

For large n and $\ell_i$'s and $r_j$'s well spaced, most of the integrand in (5) will be given by small values of $\underset{\sim}{S}_1$.

Now under (8)

$$h_{ii} = 1 - \tfrac{1}{2} \sum_{j=1}^{p} s_{ij}^{2} + \text{higher order terms in } s_j\text{'s}$$

$$h_{ij} = s_{ij} + \text{higher order terms in } s_{ij}\text{'s.}$$

Thus we get

$$(9) \quad \operatorname{tr} (\underset{\sim}{H}' \underset{\sim}{A} \underset{\sim}{H} \underset{\sim}{R}) = \ell_{q+m} \sum_{j=1}^{p} r_j + \sum_{i=1}^{t} (\ell_i - \ell_{q+m}) r_i (1 - \sum_{j=1}^{p} s_{ij}^{2})$$

$$+ \sum_{i=1}^{t} \sum_{j=1}^{p} (\ell_i - \ell_{q+m}) r_j s_{ij}^{2} + \text{higher order terms in } s_{ij}\text{'s.}$$

$$= \ell_{q+m} \sum_{j=1}^{p} r_j + \sum_{i=1}^{t} (\ell_i - \ell_{q+m}) r_i - \sum_{i=t}^{t} (\ell_i - \ell_{q+m}) r_i \sum_{j=1}^{p} s_{ij}^{2}$$

$$+ \sum_{i=1}^{t} \sum_{j=1}^{p} (\ell_i - \ell_{q+m}) r_j s_{ij}^{2} + \text{higher order terms involving } s_{ij}\text{'s.}$$

$$= \sum_{i=1}^{p} \ell_i r_i - \sum_{i=1}^{q} (\ell_i - \ell_{q+m}) r_i \sum_{j=1}^{p} s_{ij}^{2} - \sum_{i=q+1}^{t} (\ell_i - \ell_{q+m}) r_i \sum_{j=1}^{p} s_{ij}^{2}$$

$$+ \sum_{i=1}^{q} \sum_{j=1}^{p} (\ell_i - \ell_{q+m}) r_j s_{ij}^{2} + \sum_{i=q+1}^{t} \sum_{j=1}^{p} (\ell_i - \ell_{q+m}) r_j s_{ij}^{2} + \text{higher order}$$

terms involving $s_{ij}$'s.

$$= \sum_{i=1}^{p} \ell_i r_i + \sum_{\substack{i=1 \\ i < j}}^{q} \sum_{j=1}^{p} (\ell_i - \ell_j)(r_j - r_i) s_{ij}^{2} + \sum_{u=1}^{m} \sum_{i=q_u}^{q_{u+1}-1} \sum_{j=q_{u+1}}^{p} (\ell_i - \ell_j)(r_j - r_i) s_{ij}^{2}$$

$$+ \text{higher order terms involving } s_{ij}^{2},$$

$$(10) \qquad \text{where } q_1 = q + 1, \; q_i = q + \sum_{j=1}^{i-1} k_j + 1, \quad i = 2, \ldots, m + 1,$$

Substituting (9) in (5) we note that the integrand tends to zero as each $s_{ij} \to \infty$. Also for large n and for $\ell_i$'s and $r_j$'s well spaced we can approximate the integral over N $(\underset{\sim}{S}_1 = \underset{\sim}{0})$ by varying each $s_{ij}$ over the whole real line i.e. $-\infty < s_{ij} < \infty$ for each pair (i,j) which involves in our representation (8).

Thus for large n, noting that the maximum of the integrand in (5) is attained when $\underset{\sim}{H}$ has the form (6), we get, following Anderson [1]

$$\int_{O(p)} \exp\,[-\tfrac{n}{2}\,\mathrm{tr}\,\underset{\sim}{H}'\,\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}]\,d(\underset{\sim}{H}) = 2^q \prod_{i=1}^{m} \omega_i\,\omega_{m+1}^{-1}$$

$$\exp\,[-\tfrac{n}{2}\,\mathrm{tr}\,\underset{\sim}{A}\,\underset{\sim}{R}] \prod_{\substack{i=1\\ i<j}}^{q} \prod_{j=1}^{p} \left(\frac{2\pi}{nc_{ij}}\right)^{\frac{1}{2}} \prod_{u=1}^{m} \prod_{i=q_u}^{{q_{u+1}}^{-1}} \prod_{j=q_{u+1}}^{p} \left(\frac{2\pi}{nc^\circ_{ij}}\right)^{\frac{1}{2}}$$

$$[1 + \frac{1}{2n}\left(\sum_{\substack{i=1\\ i<j}}^{q} \sum_{j=1}^{p} c_{ij}^{-1} + \sum_{u=1}^{m} \sum_{i=q_u}^{{q_{u+1}}^{-1}} \sum_{j=q_{u+1}}^{p} c^{\circ\,-1}_{ij}\right) + ,\ldots,],$$

where

(11) $c_{ij} = (\ell_i - \ell_j)(r_j - r_i)$, i=1,..., q, j=1,..., p

and

$c^\circ_{ij} = (\ell_i - \ell_j)(r_j - r_i)$, i and j varying over the indicated set where it is non-zero.

$$\omega_i = \pi^{\frac{k_i^2}{2}} \{\Gamma_{k_i}(\tfrac{k_i}{2})\}^{-1},\ i=1,\ldots,m,\ \omega_{m+1} = \pi^{\frac{p^2}{2}} \{\Gamma_p(\tfrac{p}{2})\}^{-1}.$$

The factor involving $\omega_i$ accounts for the fact that integrand in (5) is maximized when $\underset{\sim}{H}$ has the form (6).

Thus we get the following theorem

Theorem 1. An asymptotic expansion of the distribution of the roots $r_1, \ldots, r_p$, of the sample covariance matrix $\underset{\sim}{C}$ for large degrees of freedom n, when the population roots satisfy (4) is given by

$$\omega_{m+1}^{-1} \prod_{i=1}^{m} \omega_i \; 2^q \; C_1 \prod_{\substack{i=1 \\ i<j}}^{q} \prod_{j=1}^{p} \left(\frac{2\pi}{nc_{ij}}\right)^{\frac{1}{2}} \prod_{u=1}^{m} \prod_{i=q_u}^{q_{u+1}-1} \prod_{j=q_{u+1}}^{p} \left(\frac{2\pi}{c^\circ_{ij}}\right)^{\frac{1}{2}}$$

$$\left(1+\frac{1}{2n}\left(\sum_{\substack{i=1 \\ i<j}}^{q} \sum_{j=1}^{p} c_{ij}^{-1} + \sum_{u=1}^{m} \sum_{i=q_u}^{q_{u+1}-1} \sum_{j=q_{u+1}}^{p} c^{\circ\,-1}_{ij}\right) + \ldots\right) \exp\left[-\frac{n}{2} \operatorname{tr} \underset{\sim}{A}\,\underset{\sim}{R}\right]$$

where $q = p - k_1 - \ldots - k_m$, and $\underset{\sim}{A}$, $\underset{\sim}{R}$, $q_i$, $(i=1,\ldots, m+1)$, $c_{ij}$'s, $c^\circ_{ij}$'s are defined by (4), (3), (10) and (11) respectively.

## 3. Asymptotic expansion for the distribution of the latent roots of $\underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$ — several multiple population roots.

The problem of finding the asymptotic expansion of the roots of $\underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$ in case of one extreme multiple population root has been studied by Li and Pillai [6], [7], we here extend their results to the case when there are several multiple population roots.

Let $\underset{\sim}{S}_i$ be independently distributed as Wishart $(n_i, p, \underset{\sim}{\Sigma}_i)$, i=1, 2, and let

$r_i = ch_i\,(\underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1})$, $\quad \ell_i > ch_i\,(\underset{\sim}{\Sigma}_1 \underset{\sim}{\Sigma}_2^{-1})$, i=1,..., p, and let

$$\underset{\sim}{R} = \text{diag}\,(r_1, \ldots, r_p); \quad \infty > r_1 > \ldots > r_p > 0,$$

$$(12) \quad \underset{\sim}{A} = \text{diag}\,(\ell_1, \ldots, \ell_q, \overbrace{\ell_{q+1}, \ldots, \ell_{q+1}}^{k_1}, \ldots, \overbrace{\ell_{q+m}, \ldots, \ell_{q+m}}^{k_m}),$$

and $\infty > \ell_{q+m} > \ell_{q+m-1} > \ldots > \ell_{q+1} > \ell_q > \ldots > \ell_1 \geq 0$,

where $p = k_1 +, \ldots + k_m + q$. Then the joint distribution of the roots $r_1, \ldots, r_p$, is given by

$$(13)\quad c_2 \int_{O(p)} |\underset{\sim}{I} + \underset{\sim}{H}' \underset{\sim}{A}\, \underset{\sim}{H}\, \underset{\sim}{R}|^{-\frac{n}{2}}\, d(\underset{\sim}{H}) \,,$$

where
$$c_2 = \pi^{\frac{p^2}{2}}\, \Gamma_p\left(\frac{n_1+n_2}{2}\right) \{\Gamma_p\left(\frac{n_1}{2}\right) \Gamma_p\left(\frac{n_2}{2}\right) \Gamma_p\left(\frac{p}{2}\right)\}^{-1} |\underset{\sim}{A}|^{\frac{n_1}{2}}$$

$$|\underset{\sim}{R}|^{\frac{(n_1-p-1)}{2}} \prod_{i<j} (\ell_i - \ell_j)$$

and $n = n_1 + n_2$, $\Gamma_p(x) = \pi^{p\frac{(p-1)}{4}} \prod_{j=1}^{p} ((x-\frac{1}{2}(j-1))$,

$d(\underset{\sim}{H})$ is the invariant measure on the group $O(p)$. Again, as earlier, by lemma 1.1 and as is shown in [3], the integrand in (13) is maximized for all variation of $\underset{\sim}{R} > 0$ when $\underset{\sim}{H}$ has the form (6).

Again we make a substitution of the form (8) and after legnthy algebra similar in line to that of Li and Pillai [6], [7], we get for large n, and $\ell_i$'s and $r_j$'s well spaced $(i,j = 1,\ldots, p)$

$$\int_{O(p)} |\underset{\sim}{I} + \underset{\sim}{H}'\, \underset{\sim}{A}\, \underset{\sim}{H}\, \underset{\sim}{R}|^{-\frac{n}{2}}\, d(\underset{\sim}{H}) = 2^q \prod_{i=1}^{m} \omega_i\, \omega_{m+1}^{-1}\, |\underset{\sim}{I} + \underset{\sim}{A}\, \underset{\sim}{R}|^{-\frac{n}{2}}$$

$$\prod_{i=1}^{q} \prod_{j=1}^{p} \left(\frac{2\pi}{nc_{ij}}\right)^{\frac{1}{2}} \prod_{u=1}^{m} \prod_{i=q_u}^{q_{u+1}-1} \prod_{j=q_{u+1}}^{p} \left(\frac{2\pi}{nc^{\circ}_{ij}}\right)^{\frac{1}{2}}$$

$$[1 + \frac{1}{2n} \Big( \sum_{i=1}^{q} \sum_{j=1}^{p} c_{ij}^{-1} + \sum_{u=1}^{m} \sum_{i=q_u}^{q_{u+1}-1} \sum_{j=q_{u+1}}^{p} c^{\circ\,-1}_{ij} + \alpha_1(p,q)$$

$$+ \alpha_2\,(p,\, q,\, k_1,\ldots,\, k_m)\Big) + \ldots],$$

where $c_{ij} = (t_{ji} - t_i\, t_j\, r_{ij})\, r_{ij}$,

(14) $$t_{ij} = t_i - t_j, \quad t_i = \ell_i(1+\ell_i r_i)^{-1},$$

$$r_{ij} = r_i - r_j, \quad i=1,\ldots, q, \quad j=1,\ldots, p, \quad i < j,$$

and $c^{\circ}_{ij}$ is similarly defined as $c_{ij}$ **but subscripts varying over the indicated** set where it is non-zero,

$$\alpha_1 (p, q) = \frac{q}{12} \quad \{(q-1)(4q+1) + 6(p^2-q^2)\}$$

and

$$\alpha_2 (p, q, k_1,\ldots, k_m) = \frac{1}{2} \sum_{i=1}^{m} k_i(p-q-k_1-\ldots\ldots-k_1)(p-q-k_1-\ldots\ldots-k_i-1)$$

$$+ \sum_{i<j<\ell=3}^{m} k_i k_j k_\ell + \frac{3}{2} \sum_{i<j=2}^{m} k_i k_j \,.$$

Thus we have the following theorem.

Theorem 2. For large degrees of freedom $n = n_1 + n_2$, an asymptotic expansion for the distribution of the roots $\infty > r_1 >, \dots, > r_p > 0$ when the population roots satisfy (12) is given by

$$2^q \prod_{i=1}^{m} \omega_i \, \omega_{m+1}^{-1} \, c_2 \, |\underset{\sim}{I} + \underset{\sim}{A}\,\underset{\sim}{R}|^{-\frac{n}{2}} \prod_{i=1}^{q} \prod_{\substack{j=1 \\ i<j}}^{p} \left(\frac{2\pi}{nc_{ij}}\right)^{\frac{1}{2}}$$

$$\prod_{u=1}^{m} \prod_{i=q_u}^{q_{u+1}-1} \prod_{j=q_{u+1}}^{p} \left(\frac{2\pi}{nc^{\circ}_{ij}}\right)^{\frac{1}{2}} [1 + \frac{1}{2n} ( \sum_{i=1}^{q} \sum_{\substack{j=1 \\ i<j}}^{p} c_{ij}^{-1} +$$

$$\sum_{u=1}^{m} \sum_{i=q_u}^{q_{u+1}-1} \sum_{j=q_{u+1}}^{p} c^{\circ -1}_{ij} + \alpha(p,q) + \alpha_2(p,q,k_1,\dots, k_m)) + \dots],$$

where the constants are defined by (14).

In the following we give the asymptotic expansions for the roots of relevant matrices for MANOVA and Canonical correlation cases and for complex analogue of all these problems. Detailed ground work being already done in [3], [4], [6], [7] and above we just state the problems and the corresponding solutions omitting the details.

4. Asymptotic expansion for MANOVA - several multiple population roots. Let $\underset{\sim}{B}$ be the between S. P. matrix and $\underset{\sim}{W}$ the within S. P. matrix. Then $\underset{\sim}{B}$ (pxp) has a non-central Wishart distribution with s d.f. and matrix of non-centrality parameter $\underset{\sim}{A}$, and $\underset{\sim}{W}$ has the central Wishart distribution on t d.f., the co-variance matrix in each case being $\underset{\sim}{\Sigma}$.

Let $\underset{\sim}{A} = \underset{\sim}{\mu}\,\underset{\sim}{\mu}'\,\underset{\sim}{\Sigma}^{-1}$ and $\underset{\sim}{R} = \underset{\sim}{B}(\underset{\sim}{W} + \underset{\sim}{B})^{-1}$ and in terms of the characteristic roots let

(15) $\underset{\sim}{R} = \mathrm{diag}\,(r_1, \ldots, r_p),\ 1 > r_1 > r_2, \ldots, > r_p > 0,$

(16) $\underset{\sim}{A} = \mathrm{diag}\,(\ell_1, \ldots, \ell_q, \overbrace{\ell_{q+1}, \ldots, \ell_{q+1}}^{k_1}, \ldots, \overbrace{\ell_{q+m}, \ldots, \ell_{q+m}}^{k_m}),$

where

$$\infty > \ell_1 >, \ldots, > \ell_q > \ell_{q+1} > , \ldots, > \ell_{q+m} \geq 0.$$

Then we have the following theorem.

Theorem 3. For large t (and hence for large sample size) an asymptotic expansion for the distribution of the characteristic roots of $\underset{\sim}{R}$ in (15), when the parameter matrix $\underset{\sim}{A}$ satisfies (16) is given by

$$c_3\ 2^q \prod_{i=1}^{m} \omega_i\ \omega_{m+1}^{-1} \prod_{\substack{i=1\\ i<j}}^{q} \prod_{j=1}^{p} \left(\frac{2\pi}{tc_{ij}}\right)^{\frac{1}{2}} \prod_{u=1}^{m} \prod_{i=q_u}^{q_{u+1}-1} \prod_{j=q_{u+1}}^{p} \left(\frac{2\pi}{tc^\circ_{ij}}\right)^{\frac{1}{2}}$$

$$\{1 + \frac{1}{2t} \Big( \sum_{\substack{i=1\\ i<j}}^{q} \sum_{j=1}^{p} c_{ij}^{-1} + \sum_{u=1}^{m} \sum_{i=q_u}^{q_{u+1}-1} \sum_{j=q_{u+1}}^{p} c^{\circ\,-1}_{ij} + \alpha_1\,(p,q)$$

$$+ \alpha_2\,(p,q,k_1, \ldots, k_m)\Big) +, \ldots, \}\quad \exp\,[\mathrm{tr}\ \underset{\sim}{A}\ \underset{\sim}{R}]\ {}_1F_1(-\tfrac{t}{2}, \tfrac{s}{2}, -\underset{\sim}{A}\ \underset{\sim}{R}) + O(\epsilon),$$

where

$$c_3 = \pi^{\frac{p^2}{2}}\ \Gamma_p(\tfrac{1}{2}\,(s+t))\ \{\Gamma_p(\tfrac{t}{2})\ \Gamma_p(\tfrac{s}{2})\ \Gamma_p(\tfrac{p}{2})\}^{-1} \quad \exp\,[\mathrm{tr}\ \underset{\sim}{A}]\ \ |\underset{\sim}{R}|^{\frac{1}{2}(s-p-1)}$$

$$|\underset{\sim}{I} - \underset{\sim}{R}|^{\frac{1}{2}(t-p-1)} \prod_{i>j} (r_i - r_j)$$

and

$$c_{ij} = \frac{(d_i - d_j)(\ell_j - \ell_i)}{(1+\ell_i d_i)(1+\ell_j d_j)}, \qquad i=1, \ldots, q,\ j=1, \ldots, p,\ i < j.$$

$c^\circ_{ij}$'s are similarly defined as $c_{ij}$ but the subscripts varying over the indicated set such that $c^\circ_{ij}$ is non-zero and $\underset{\sim}{D} = \underset{\sim}{R}^{-1}$, i.e., $d_i = r_i^{-1}$, $i=1,\dots, p$.

## 5. Asymptotic expansion for canonical correlation-several multiple population roots.

Let $x_1,\dots, x_p, x_{p+1},\dots, x_{p+t}$, $p \leq t$ be distributed $\underset{\sim}{N}(\underset{\sim}{0}, \underset{\sim}{\Sigma})$, where

$$\underset{\sim}{\Sigma} = \begin{array}{cc} p & t & \\ \left(\begin{array}{cc} \underset{\sim}{\Sigma}_{11} & \underset{\sim}{\Sigma}_{12} \\ \underset{\sim}{\Sigma}'_{12} & \underset{\sim}{\Sigma}_{22} \end{array}\right) & \begin{array}{c} p \\ t \end{array} \end{array}$$

Let $\underset{\sim}{P}^2 = \text{diag}\,(\rho_1^2,\dots, \rho_p^2)$, where $\rho_i^2$, $i=1,\dots, p$, be the roots of

$$|\underset{\sim}{\Sigma}_{12} \ \underset{\sim}{\Sigma}_{22}^{-1} \ \underset{\sim}{\Sigma}'_{12} - \rho^2 \underset{\sim}{\Sigma}_{11}| = 0$$

and let $\hat{\underset{\sim}{P}}^2 = \text{diag}\,(\hat{\rho}_1^{\,2},\dots, \hat{\rho}_p^{\,2})$, where $\hat{\rho}_i^{\,2}$, $i=1,\dots, p$, be the maximum likelihood estimates.

Also let

$$\hat{\underset{\sim}{P}}^2 = \underset{\sim}{R} = \text{diag}\,(r_1,\dots, r_p) \tag{17}$$

$$\underset{\sim}{P}^2 = \underset{\sim}{A} = \text{diag}\,(\ell_1,\dots, \ell_q, \overbrace{\ell_{q+1},\dots, \ell_{q+1}}^{k_1},\dots, \overbrace{\ell_{q+m},\dots, \ell_{q+m}}^{k_m})$$

where

$$1 > r_1 > ,\dots, > r_p > 0,\ 1 > \ell_1 > \ell_2 > ,\dots, > \ell_q > \ell_{q+1} > ,\dots,$$
$$> \ell_{q+m} \geq 0.$$

Then we have the following theorem.

**Theorem 4.** For large n, an asymptotic expansion of the distribution of $r_1, \ldots, r_p$, (squares of the canonical correlation coeffeients) when population parameters satisfy (17) is given by

$$c_4 \, 2^q \prod_{i=1}^{m} \omega_i \; \omega_{m+1}^{-1} \; |\underset{\sim}{I} - \underset{\sim}{A} \, \underset{\sim}{R}|^{-\frac{1}{2}(2n-t)} \{1 + \frac{1}{2(2n-t)} \; ( \sum_{i=1}^{q} \sum_{j=1}^{p} c_{ij}^{-1} \; +$$

$$\sum_{u=1}^{m} \sum_{i=q_u}^{q_{u+1}-1} \sum_{j=q_{u+1}}^{p} c\circ_{ij}^{-1} + \alpha_1(p,q) + \alpha_2(p,q,k_1, \ldots, k_m))+, \ldots, \} \; {}_2F_1(\tfrac{1}{2}(t-n), \tfrac{1}{2}(t-m$$

$$\tfrac{1}{2}t, \underset{\sim}{A} \, \underset{\sim}{R}) + O(\epsilon),$$

where

$$c_{ij} = (t_{ji} - t_i t_j \, r_{ij}) \, r_{ij} = c_{ji},$$

$$t_{ij} = t_i - t_j, \; r_{ij} = r_i - r_j,$$

$$t_i = \ell_i \, (1 - r_i \ell_i)^{-1}, \qquad i=1, \ldots, q, \; j=1, \ldots, p, \;\; i < j,$$

and $c^\circ_{ij}$ is similarly defined as $c_{ij}$ but the subscripts vary over the indicated set such that $c^\circ_{ij}$ is non-zero.

And

$$c_4 = \{\pi^{\frac{p^2}{2}} \; \Gamma_p \, (\tfrac{n}{2}) / \Gamma_p \, (\tfrac{t}{2}) \; \Gamma_p \, (\tfrac{n-t}{2}) \; \Gamma_p \, (\tfrac{p}{2})\} |\underset{\sim}{I} - \underset{\sim}{A}|^{\frac{n}{2}} \; |\underset{\sim}{R}|^{\frac{1}{2}(t-p-1)} \prod_{i>j} (r_i - r_j).$$

6. **Complex analogues of previous results.** In the following generalization of the above results to the complex case we refer to lemma 1.2 and the corresponding results of theorem 1.3 of [3] and proceed as above, the details of algebra obtainable from Li and Pillai [6], [7] with suitable changes. Complex analogues of theorem 1-4 are as follows.

**Theorem 1.1.** For large degrees of freedom n, an asymptotic expansion of the distribution of the roots of the covariance matrix $\underset{\sim}{S}$ when the parameter matrix

$\underset{\sim}{\Sigma}^{-1}$ has roots $\ell_i$'s and satisfy (4), is given by

$$D_1 \prod_{i=1}^{m} \theta_i \; \theta_{m+1}^{-1} \prod_{\substack{i=1\\ i<j}}^{q} \prod_{j=1}^{p} \left(\frac{2\pi}{nc_{ij}}\right) \prod_{u=1}^{m} \prod_{i=q_u}^{q_{u+1}-1} \prod_{j=q_{u+1}}^{p} \left(\frac{2\pi}{nc^{\circ}_{ij}}\right)$$

$$\{1 + \frac{1}{3n} \Big( \sum_{\substack{i=1\\ i<j}}^{q} \sum_{j=1}^{p} c_{ij}^{-1} + \sum_{u=1}^{m} \sum_{i=q_u}^{q_{u+1}-1} \sum_{j=q_{u+1}}^{p} c^{\circ\,-1}_{ij} \Big) +,\ldots, \} \exp\left[-\frac{n}{2} \operatorname{tr} \underset{\sim}{A}\,\underset{\sim}{R}\right],$$

where

$$D_1 = \pi^{p(p-1)} \{\tilde{\Gamma}_p(p)\, \tilde{\Gamma}_p(n)\, |\underset{\sim}{\Sigma}|^{n}\}^{-1}\, |\underset{\sim}{R}|^{n-p} \prod_{i<j=2}^{p} (\ell_i-\ell_j)^2$$

and

$$\theta_i = \pi^{k_i(k_i-1)} \{\tilde{\Gamma}k_i\ (k_i)\}^{-1},\ i=1,\ldots,m,$$

$$\theta_{m+1} = \pi^{p(p-1)} \{\tilde{\Gamma}_p(p)\}^{-1},\ \tilde{\Gamma}_p(n) = \pi^{p\frac{(p-1)}{2}} \prod_{i=1}^{p} (n-i+1),$$

$c_{ij}$ and $c^{\circ}_{ij}$ are similarly defined as in the corresponding real case.

Theorem 2.1. For large degrees of freedom $n = n_1+n_2$, an asymptotic expansion of the distribution of the roots of $\underset{\sim}{S}_1\,\underset{\sim}{S}_2^{-1}$ in the complex case when the population roots satisfy the form (4) is given by

$$D_2 \prod_{i=1}^{m} \theta_i \; \theta_{m+1}^{-1} \prod_{\substack{i=1\\ i<j}}^{q} \prod_{j=1}^{p} \left(\frac{2\pi}{nc_{ij}}\right) \prod_{n=1}^{m} \prod_{i=q_u}^{q_{u+1}-1} \prod_{j=q_{u+1}}^{p} \left(\frac{2\pi}{nc^{\circ}_{ij}}\right)$$

$$|\underset{\sim}{I} + \underset{\sim}{A}\,\underset{\sim}{R}|^{-n} \; \{1 + \frac{1}{3n} \Big( \sum_{\substack{i=1 \\ i < j}}^{q} \; \sum_{j=1}^{p} \; c_{ij}^{-1} \; + \; \sum_{u=1}^{m} \; \sum_{i=q_n}^{q_{u+1}-1} \; \sum_{j=q_{n+1}}^{p} \; c_{ij}^{\bullet\,-1}$$

$$+ \beta_1 \, (p,q) + \beta_2 \, (p,q,k_1,\ldots, k_m) \Big) \, \Phi, \ldots, \},$$

where

$$D_2 = \pi^{\frac{p(p-1)}{2}} \quad \tilde{\Gamma}_p \, (n_1+n_2)/\{\tilde{\Gamma}_p \, (p) \, \tilde{\Gamma}_p(n_1) \;\; \tilde{\Gamma}_p \, (n_2)\} \; |\underset{\sim}{A}|^{-p} \; |\underset{\sim}{R}|^{n_2-p}$$

$$\pi_{i<j} \;\; (r_i-r_j)^2,$$

where $c_{ij}$ and $c^{\bullet}_{ij}$ are defined as in the corresponding real case,

$$\beta_1(p,q) = \frac{q}{2} \, \{(q-1)(2q-1) + 3 \, (p-q)(p+q-1)\},$$

and

$$\beta_2(p,q,k_1,\ldots, k_m) = \frac{3}{2} \;\; \sum_{i=1}^{m} k_i \;\; (p-q-k_1,\ldots\ldots -k_i)(p-q-\ldots\ldots -k_i-1)$$

$$+ \; 3 \sum_{i<j<\ell=3}^{m} \;\; k_i k_j k_i \;\; + \; 3 \sum_{i<j=2}^{m} k_i k_j .$$

Theorem 3.1. For large t (and hence for large sample size) an asymptotic expansion of the distribution of the sample roots in the complex MANOVA case population roots satisfy the form (16) is given by

$$D_3 \prod_{i=1}^{m} \theta_i \; \theta_{m+1}^{-1} \underset{i<j}{\prod_{i=1}^{q} \prod_{j=1}^{p}} \left(\frac{\pi}{tc_{ij}}\right) \prod_{u=1}^{m} \prod_{i=q_u}^{q_{u+1}-1} \prod_{j=q_{u+1}}^{p} \left(\frac{\pi}{tc^{\circ}_{ij}}\right)$$

$$\{1 + \frac{1}{3t} \Big( \underset{i<j}{\sum_{i=1}^{q} \sum_{j=1}^{p}} c_{ij}^{-1} + \sum_{u=1}^{m} \sum_{i=q_u}^{q_{u+1}-1} \sum_{j=q_{u+1}}^{p} c^{\circ -1}_{ij} + \beta_1(p,q) + ,\ldots,+$$

$$\beta_2(p,q,k_1,\ldots,k_m)\Big) + ,\ldots, \} \quad \exp[\mathrm{tr}\, \underset{\sim}{A}\, \underset{\sim}{R}] \; {}_1\tilde{F}_1(-t,s,-\underset{\sim}{A}\, \underset{\sim}{R}) + O(\varepsilon),$$

where

$$D_3 = [\pi^{p(p-1)} \quad \tilde{\Gamma}_p(s+t)/\tilde{\Gamma}_p(p)\, \tilde{\Gamma}_p(s)\, \tilde{\Gamma}_p(t)]\, |\underset{\sim}{I} - \underset{\sim}{R}|^{t-p}$$

$$|\underset{\sim}{R}|^{(s-p)} \quad \prod_{i>j} \; (r_i - r_j)^2 \quad \exp[-\mathrm{tr}\, \underset{\sim}{A}]$$

and $c_{ij}$, $c^{\circ}_{ij}$'s are as defined in corresponding real case.

Theorem 4.1. For large n, an asymptotic expansion for the distribution of the canonical correlation coefficients when population coefficients in the complex case satisfy the form (17) is given by

$$D_q \prod_{i=1}^{m} \theta_i \; \theta_{m+1}^{-1} \; |\underset{\sim}{I} - \underset{\sim}{A}\, \underset{\sim}{R}|^{-(2n-t)} \; \{1 + \frac{1}{3(2n-t)} \Big( \underset{i<j}{\sum_{i=1}^{q} \sum_{j=1}^{p}} c_{ij}^{-1} +$$

$$\sum_{u=1}^{m} \sum_{i=q_u}^{q_{u+1}-1} \sum_{j=q_{u+1}}^{p} c^{\circ -1}_{ij} + \beta_1(p,q) + \beta_2(p,q,k_1,\ldots,k_m)\Big) +,\ldots,\}$$

$${}_2\widetilde{F}_1\ ((t-n),\ (t-n),\ t,\ \underset{\sim}{A}\,\underset{\sim}{R}) + O(\epsilon),$$

where

$$D_3 = [\pi^{p(p-1)}\ \widetilde{\Gamma}_p(n)/\widetilde{\Gamma}_p(n-t)\ \widetilde{\Gamma}_p(t)\ \widetilde{\Gamma}_p(p)]\ |\underset{\sim}{I} - \underset{\sim}{A}|^n\ |\underset{\sim}{R}|^{t-p}\ |\underset{\sim}{I} - \underset{\sim}{R}|^{n-t-p}$$

$$\underset{i>j}{\pi}\ (r_i - r_j)^2$$

and $c_{ij}$, $c^{\circ}_{ij}$'s are as defined in the corresponding real case.

Remarks. 1. As will be seen from the above formulae, they give the already known results of Anderson [1], Chang [2], James [5], Li and Pillai [6], [7] as special cases.

2. Though we have taken the sets with multiple roots in the population parametric matrix at one extreme, actually it does not matter even if they were otherwise. By pre and post multiplication by suitable permutation matrix, all multiple roots can be brought to one extreme place without affecting our distribution problem but, of course, care should be taken in defining $c_{ij}$ and $c^{\circ}_{ij}$ coefficients.

3. Since, for all variations of $\underset{\sim}{R}$, the appropriate integral in each case takes the identical maximum when the corresponding orthogonal or unitary matrices take definite special forms, we can take particualr transformations like (8) or its complex analogue to approximate the integrand around one such optimum and hence adjust for all such optima.

4. As will be evident, our technique being a generalization of techniques of earlier authors, the restrictions made by earlier authors also apply in our case.

5. As said earlier we tacitly avoided the "invariance" technique used by James and subsequently followed by others. Moreover, our technique gives their result as a special case and hence gives a different interpretation of their results.

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