The Handicap in a Competitive Game

A Numerical Investigation

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Abstract.

The problem of finding integers n_1 and n_2 such that a billiards game between two players of unequal skills is approximately even, is treated.

The integers n_1 and n_2 are the numbers of points to be scored by the respective players.

Introduction.

In a number of competitive games — billiards is our best known example— the winner is that player who scores a certain number of points before his competitors do. We shall limit our attention to a game with two players only. The extension to more players is routine.

We consider a game with players I and II. We assume that the successive attempts to score points form a sequence of Bernoulli trials and that player I has probability p_1, $0 < p_1 < 1$ of scoring a point at one of his trials. p_2 is the corresponding probability for player II. $0 < p_2 < 1$.

In order to win the game, player I must accumulate n_1 points before player II accumulates n_2. In the alternate case player II wins.

It clearly matters which player goes first. We shall therefore denote by $P_1(n_1,n_2)$ the probability that player I wins, given that he gets the first turn. Similarly $P_2(n_1,n_2)$ will denote the probability that player II wins, given that he gets the first turn.

In many cases a toss-up is performed to decide who goes first. In particular, in the sequel we shall denote by $P(n_1,n_2)$ the probability that player I wins, if he starts the game with probability 1/2 and his opponent starts with probability 1/2.

Clearly we have that:

$$P(n_1,n_2) = \frac{1}{2}[P_1(n_1,n_2) + 1 - P_2(n_1,n_2)] \tag{1}$$

for all n_1 and n_2.

The problem of setting the handicap is the following. For given values of p_1 and p_2, we wish to determine the values of n_1 and n_2 for which $P_1(n_1,n_2)$ or $P_2(n_1,n_2)$ or $P(n_1,n_2)$ is approximately 1/2.

Without loss of generality we may assume that $p_1 \geq p_2$, i.e. that the better player is designated as player I. In reality there may be further restrictions. Often the number n_1 is fixed by the rules of the game and the question is then to determine how many points the better player should "give" to the weaker one to make the game about even.

Although we shall not do so here, it is possible to express $P_1(n_1,n_2)$ and $P_2(n_1,n_2)$ explicitly in terms of p_1,p_2,n_1 and n_2. The resulting expressions are complicated **series** which are of no help at all in determining what the handicap should be.

The most detailed discussion of probability aspects of the game of billiards was given in two papers in Dutch by O. Bottema and S. C. Van Veen (Kansberekeningen bÿ het biljartspel I, II, Nieuw Archief voor Wiskunde, 22, 1943, 16-33 and 123-158).

They obtained involved exact and also approximate expressions for the probabilities $P_1(n_1,n_2)$ and $P_2(n_1,n_2)$ in terms of certain hypergeometric series and their limiting formulas.

They also discuss the interesting variant in which both players always get the same number of turns. If the player who goes first reaches n_1 or n_2, as the case may be, first, then his opponent gets one more turn. If he also completes his alotted number of points the game is a draw. The numerical investigation of the handicap problem is considerably more involved for this case and we will not consider it here.

A number of interesting random variables, such as the duration of play, the number of turns per match and the longest run of successful shots for each player, were also investigated by these authors.

Before we discuss the problem and its numerical analysis formally, we note the following geometric representation of the problem.

If X_k and Y_k denote the numbers of points accumulated by I and II respectively in the first k attempts, then the point (X_k,Y_k) performs a simple type of random walk on the lattice points of the non-negative orthant in the plane. (Fig. 1). The walk starts at $(0,0)$ — we set $X_o = Y_o = 0$ — and proceeds horizontally or vertically by unit steps whenever a point is scored and depending on whether player I or II is playing. Whenever a failure occurs the walk remains in its previous position.

Player I wins if the set of points (n_1,m_2), $0 \leq m_2 < n_2$ is reached eventually; player II wins if the set of points (m_1,n_2), $0 \leq m_1 < n_1$ is reached eventually.

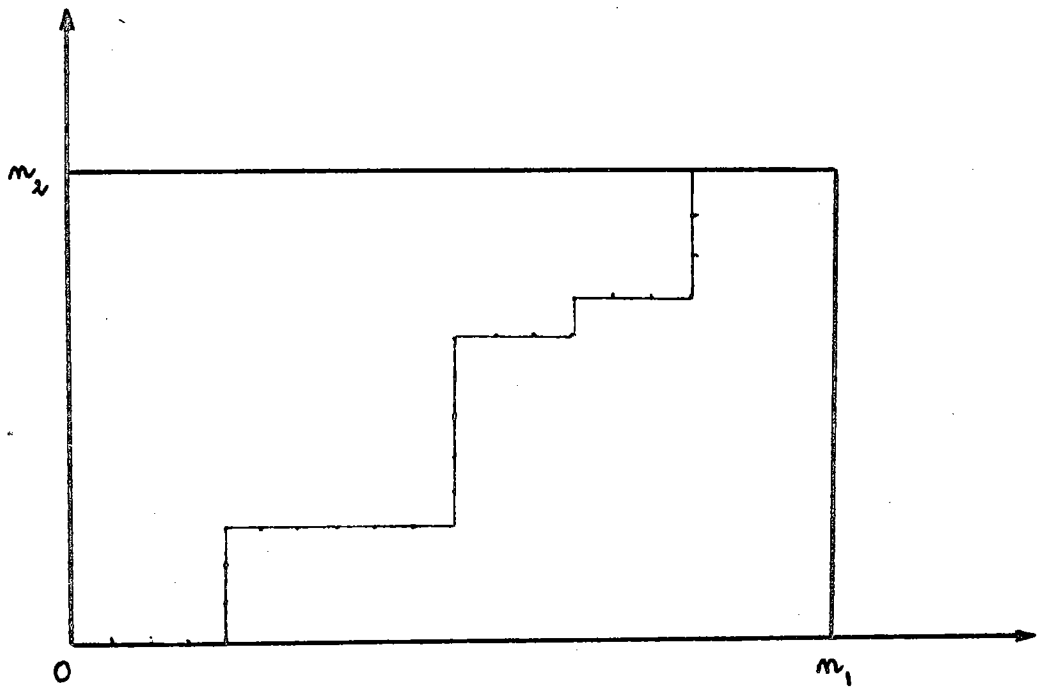

Fig. 1: $n_1 = 20$, $n_2 = 13$. A possible path of (X_k, Y_k).

The Recurrence Relations.

If player I starts, then we find by considering all possibilities after the first trial, that:

$$(2) \qquad P_1(n_1,n_2) = p_1 \; P_1(n_1 - 1,n_2) + q_1[1 - P_2(n_1,n_2)]$$

for $n_1 \geq 1$, $n_2 \geq 1$ and $q_1 = 1 - p_1$.

Similarly, if player II starts, then we find by the same argument that:

$$(3) \qquad P_2(n_1,n_2) = p_2 \; P_2(n_1,n_2 - 1) + q_2[1 - P_1(n_1,n_2)]$$

for $n_1 \geq 1$, $n_2 \geq 1$ and $q_2 = 1 - p_2$.

It is furthermore obvious that we may set:

(4) $\quad P_1(0,n_2) = P_2(n_1,0) = 1, \qquad \text{for} \quad n_1 \geq 1,\ n_2 \geq 1.$

$$P_1(n_1,0) = P_2(0,n_2) = 0,$$

We solve the equations (2) and (3) for $P_1(n_1,n_2)$ and $P_2(n_1,n_2)$ and obtain:

(5)
$$P_1(n_1,n_2) = p_1(1-q_1q_2)^{-1}\,P_1(n_1-1,n_2) + q_1p_2(1-q_1q_2)^{-1}[1-P_2(n_1,n_2-1)] \quad ,$$

$$P_2(n_1,n_2) = p_2(1-q_1q_2)^{-1}\,P_2(n_1,n_2-1) + q_2p_1(1-q_1q_2)^{-1}[1-P_1(n_1-1,n_2)] \quad .$$

The recurrence relations (5) are ideally suited for numerical computation. We see that the quantities $P_1(n_1,n_2)$ and $P_2(n_1,n_2)$ for which $n_1 + n_2 = k+1$ are very simply related to the corresponding string of quantities for which $n_1 + n_2 = k$. We so obtain an iterative scheme similar to Pascal's triangle for the binomial coefficients.

It is also easy to obtain, from formula (5), the expressions for the generating functions:

(6)
$$P_1(z_1,z_2) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} P_1(n_1,n_2)\, z_1^{n_1} z_2^{n_2}$$

$$P_2(z_1,z_2) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} P_2(n_1,n_2)\, z_1^{n_1} z_2^{n_2} \quad .$$

We shall omit these as they are not very useful in obtaining numerical results for the handicap problem which we are interested in.

The Numerical Solution to the Handicap Problem.

Depending on whether n_1 and n_2 are set before or after the player to go first is selected, we shall wish to find those pairs n_1 and n_2 for which $P_1(n_1,n_2)$, $P_2(n_1,n_2)$ or $P(n_1,n_2)$ are close to 0.5. Except for very special values of p_1 and p_2, we may not be able to find pairs (n_1,n_2) within the range of interest, for which these probabilities are exactly 1/2.

We therefore specify an interval (α,β), say (.495, .505), about 1/2 and write the computer program to print out the values of $n_1,n_2,P_1(n_1,n_2)$, $P_2(n_1,n_2)$, $P(n_1,n_2)$ for which at least one of the latter three quantities belongs to (α,β).

In order to make efficient use of computer memory storage, we compute all the quantities $P_1(n_1,n_2)$ and $P_2(n_1,n_2)$ for which $n_1+n_2=k+1$ recursively from those for which $n_1+n_2=k$, by use of the recurrence relations (5), starting with $k=1$, $P_1(1,0) = P_2(0,1) = 1 - P_1(0,1) = 1 - P_2(1,0) = 0$.

Only those $n_1,n_2,P_1(n_1,n_2)$, $P_2(n_1,n_2)$ and $P(n_1,n_2)$ for which one of the latter three quantities lies in (α,β) are stored in arrays for later printout.

In the examples computed by the author the bounds $\alpha = .495$ and $\beta = .505$ were chosen and k ran from 1 to 600. The latter bound is larger than required for most practical purposes. Even so, the central processing time on the CDC 6500 at Purdue was only 32 seconds (approx.) per set of values p_1 and p_2. With k up to 200, the central processing time was 7.5 seconds (approx.).

As far as the computer program is concerned, there is of course nothing particular about the values α and β chosen. If one wished to concentrate attention on any other sub interval of $[0,1]$, it would suffice to choose the values of α and β accordingly. In particular for $\alpha = 0$ and $\beta = 1$ one would obtain a complete table of the three probabilities of interest.

It is possible to obtain a theoretical approximate solution to the handicap problem by using the bivariate central limit theorem for the random walk (X_k, Y_k). We shall not pursue this matter here as the exact numerical solution can be carried out with such great ease, especially in the range of values for n_1 and n_2 which are encountered in applications.

<u>Generalizations</u>.

A. The claim is commonly made by billiard players that if a good player misses a shot he is likely to leave a favorable configuration for his opponent at the next trial, whereas when a mediocre player misses a shot, the configuration that results is usually not particularly favorable to his opponent, or may even be markedly unfavorable.

To add an extra touch of realism to our model, we now assume that the probability of success for a player is p_i, $i = 1,2$ at each trial, except for those trials that follow a missed shot for which the probability of success will be denoted by p_i', $i = 1,2$. We assume that the first shot of the game is an "ordinary" shot for either player, i.e. the probability of success is p_i, $i = 1,2$.

As before we set $1 - p_i = q_i$ and $1 - p_i' = q_i'$ and assume that $0 < p_i < 1$, $0 < p_i' < 1$ for $i = 1,2$. $P_1(n_1,n_2)$, $P_2(n_1,n_2)$ and $P(n_1,n_2)$ have the same interpretations as before.

The recurrence relations for $P_1(n_1,n_2)$ and $P_2(n_1,n_2)$ are now obtained as follows. If player I goes first, he either makes the first point or else II and I will try alternatingly until one of them makes a point.

The conditional probability that I makes the first successful shot, given that he starts is given by:

$$(6)\qquad p_1 + q_1q_2'p_1' + q_1q_2'q_1'q_2'p_1' + q_1q_2'(q_1'q_2')^2 p_1' + \dots + q_1q_2'(q_1'q_2')^{\nu} p_1' + \dots$$

$$= p_1 + q_1q_2'p_1'(1 - q_1'q_2')^{-1} = \theta_1 \quad .$$

The conditional probability that player II makes the first successful shot is given by:

$$(7)\qquad 1 - p_1 - q_1q_2'p_1'(1 - q_1'q_2')^{-1} = q_1[1 - q_2'p_1'(1-q_1'q_2')^{-1}]$$

$$= q_1p_2'(1-q_1'q_2')^{-1} = 1 - \theta_1 \quad .$$

If player I makes the first successful shot the remaining game is similar to the original one except that now (n_1-1, n_2) points remain to be played. Similarly if player II makes the first successful shot.

The recurrence relations now become:

$$(8)\qquad P_1(n_1,n_2) = \theta_1 P_1(n_1-1,n_2) + (1-\theta_1)[1 - P_2(n_1,n_2-1)]$$

$$P_2(n_1,n_2) = \theta_2 P_2(n_1,n_2-1) + (1-\theta_2)[1 - P_1(n_1-1,n_2)]$$

where:

$$(9)\qquad \theta_2 = p_2 + q_2q_1'p_2'(1 - q_1'q_2')^{-1} \quad .$$

The recurrence relations (8) are similar to those in formula (5). With a minor modification in the coefficients, the same computer program may be used to find the handicap.

We see that for $p_1' = p_1$ and $p_2' = p_2$ the formulae (8) reduce to those given in (5).

B. In an alternate generalization of the original model, we may assume that the probability of a success depends only on the number of consecutive successes scored during that turn. Specifically let $p_1(\nu)$ be the probability that player I has a success if he has already made exactly $\nu-1$ successes during his current turn. Similarly we define $p_2(\nu)$ for the second player. We assume that $0 < p_1(\nu) < 1$ and $0 < p_2(\nu) < 1$ for all $\nu \geq 1$.

The probabilities $P_1(n_1,n_2)$, $P_2(n_1,n_2)$ and $P(n_1,n_2)$ are defined as above. They no longer satisfy simple first order recurrence relations. Instead we obtain:

$$(10) \quad P_1(n_1,n_2) = q_1(1)[1-P_2(n_1,n_2)] + \sum_{\nu=1}^{n_1} \prod_{r=1}^{\nu} p_1(r)\; q_1(\nu+1)[1-P_2(n_1-\nu,n_2)]$$

$$P_2(n_1,n_2) = q_2(1)[1-P_1(n_1,n_2)] + \sum_{\nu=1}^{n_2} \prod_{r=1}^{\nu} p_2(r)\; q_2(\nu+1)[1-P_1(n_1,n_2-\nu)]$$

by conditioning on the number of points scored by the player who goes first, during his first turn. By solving for $P_1(n_1,n_2)$ and $P_2(n_1,n_2)$ in (10), we see that the quantities $P_1(n_1,n_2)$ and $P_2(n_1,n_2)$ for which $n_1 + n_2 = k+1$ may be computed recursively in terms of those for which $n_1 + n_2 \leq k$. Although the computer program to solve the handicap problem for this case is easy to write, substantially more memory storage and computation time will be required to obtain the numerical answers for this model. In the absence of concrete data, the author has not performed any such computations.

A listing of the FORTRAN IV program for the handicap problem and tables of the numerical values corresponding to the parameter values:

$p_1 = 0.1\ (0.2)\ 0.9$

$p_2 = 0.1\ (0.2)\ p_1$

$\alpha = 0.495$

$\beta = 0.505$

$n_1 + n_2 \leq 600$

may be obtained from the author upon request, by writing to:

Department of Statistics
Purdue University
West Lafayette, IN
47907
USA

Table 1

This table illustrates the numerical results for the handicap problem corresponding to:

$$p_1 = .85 \qquad p_2 = .75 \quad .$$

The numerical values of $n_1, n_2, P_1(n_1,n_2)$, $P_2(n_1,n_2)$ and $P(n_1,n_2)$ are given for all cases where at least one of the latter three probabilities belongs to the interval (α,β), where:

$$\alpha = .498 \qquad \beta = .502$$

and where $n_1 + n_2 \leq 600$.

n_1	n_2	P_1	P_2	P	n_1	n_2	P_1	P_2	P	n_1	n_2	P_1	P_2	P
6	5	.740	.501	.619	8	6	.708	.501	.604	13	7	.589	.589	.500
16	7	.500	.663	.418	27	16	.616	.498	.559	29	17	.610	.501	.555
30	16	.556	.556	.500	31	15	.500	.612	.444	44	25	.592	.498	.547
45	24	.547	.544	.502	46	23	.501	.589	.456	46	26	.588	.500	.544
47	25	.544	.545	.500	48	24	.500	.589	.455	48	27	.585	.502	.541
49	26	.542	.545	.498	61	34	.578	.498	.540	62	33	.540	.537	.501
63	32	.501	.576	.463	63	35	.575	.500	.538	64	34	.538	.538	.500
65	33	.500	.576	.462	65	36	.573	.501	.536	66	35	.536	.539	.498
67	34	.498	.576	.461	78	43	.569	.498	.535	79	42	.535	.533	.501
80	41	.501	.567	.467	80	44	.567	.500	.534	81	43	.533	.534	.500
82	42	.499	.568	.466	82	45	.565	.501	.532	83	44	.532	.535	.499
84	43	.498	.568	.465	95	49	.502	.560	.471	95	52	.563	.499	.532
96	51	.532	.530	.501	97	50	.501	.561	.470	97	53	.561	.500	.531
98	52	.530	.531	.500	99	51	.499	.561	.469	99	54	.559	.501	.529
100	53	.529	.532	.499	101	52	.498	.562	.468	112	58	.502	.556	.473
112	61	.558	.499	.530	113	60	.529	.527	.501	114	59	.501	.556	.472
114	62	.556	.500	.528	115	61	.528	.528	.500	116	60	.499	.557	.471
116	63	.555	.501	.527	117	62	.527	.529	.499	118	61	.498	.557	.471
118	64	.553	.502	.526	128	68	.529	.525	.502	129	67	.502	.552	.475
129	70	.554	.499	.528	130	69	.527	.526	.501	131	68	.501	.552	.474
131	71	.553	.500	.526	132	70	.526	.526	.500	133	69	.499	.553	.473
133	72	.551	.501	.525	134	71	.525	.527	.499	135	70	.498	.554	.472
135	73	.550	.502	.524	145	77	.527	.523	.502	146	76	.501	.549	.476
146	79	.551	.499	.526	147	78	.526	.524	.501	148	77	.500	.549	.476
148	80	.549	.500	.525	149	79	.525	.525	.500	150	78	.500	.550	.475
150	81	.548	.501	.524	151	80	.523	.526	.499	152	79	.499	.550	.474
152	82	.547	.502	.523	162	86	.525	.522	.502	163	85	.501	.546	.478
163	88	.548	.499	.525	164	87	.524	.523	.501	165	86	.500	.547	.477
165	89	.547	.500	.524	166	88	.523	.523	.500	167	87	.500	.547	.476
167	90	.546	.501	.523	168	89	.522	.524	.499	169	88	.499	.548	.475
169	91	.545	.502	.521	170	90	.521	.525	.498	178	96	.547	.498	.524
179	95	.524	.521	.502	180	94	.501	.544	.479	180	97	.546	.499	.523
181	96	.523	.522	.501	182	95	.500	.544	.478	182	98	.545	.500	.522
183	97	.522	.522	.500	184	96	.500	.545	.477	184	99	.544	.501	.521
185	98	.521	.523	.499	186	97	.499	.546	.477	186	100	.542	.501	.520
187	99	.520	.524	.498	195	105	.545	.498	.523	196	104	.523	.520	.502
197	103	.501	.542	.480	197	106	.544	.499	.522	198	105	.522	.521	.501
199	104	.500	.542	.479	199	107	.543	.500	.521	200	106	.521	.521	.500
201	105	.500	.543	.478	201	108	.542	.501	.521	202	107	.520	.522	.499
203	106	.499	.544	.478	203	109	.541	.501	.520	204	108	.519	.523	.498
212	111	.502	.539	.481	212	114	.543	.498	.522	213	113	.522	.519	.502
214	112	.501	.540	.481	214	115	.542	.499	.521	215	114	.521	.520	.501
216	113	.500	.541	.480	216	116	.541	.500	.521	217	115	.520	.521	.500
218	114	.500	.541	.479	218	117	.540	.501	.520	219	116	.519	.521	.499
220	115	.499	.542	.478	220	118	.539	.501	.519	221	117	.519	.522	.498
229	120	.502	.538	.482	229	123	.541	.498	.522	230	122	.521	.518	.501
231	121	.501	.539	.481	231	124	.540	.499	.521	232	123	.520	.519	.501
233	122	.500	.539	.481	233	125	.539	.500	.520	234	124	.520	.520	.500
235	123	.500	.540	.480	235	126	.539	.501	.519	236	125	.519	.520	.499
237	124	.499	.540	.479	237	127	.538	.501	.518	238	126	.518	.521	.498
239	125	.498	.541	.479	246	129	.502	.537	.483	246	132	.540	.498	.521
247	131	.521	.518	.501	248	130	.501	.537	.482	248	133	.539	.499	.520
249	132	.520	.518	.501	250	131	.500	.538	.481	250	134	.538	.500	.519
251	133	.519	.519	.500	252	132	.500	.538	.481	252	135	.537	.501	.518
253	134	.518	.520	.499	254	133	.499	.539	.480	254	136	.536	.501	.518
255	135	.517	.520	.498	256	134	.498	.540	.479	256	137	.535	.502	.517
263	138	.502	.535	.483	263	141	.539	.498	.520	264	140	.520	.517	.501
265	139	.501	.536	.483	265	142	.538	.499	.519	266	141	.519	.518	.501

n_1	n_2	P_1	P_2	P	n_1	n_2	P_1	P_2	P	n_1	n_2	P_1	P_2	P
267	140	.500	.537	.482	267	143	.537	.500	.519	268	142	.518	.518	.500
269	141	.500	.537	.481	269	144	.536	.501	.518	270	143	.517	.519	.499
271	142	.499	.538	.481	271	145	.535	.501	.517	272	144	.517	.520	.498
273	143	.498	.538	.480	273	146	.534	.502	.516	280	147	.502	.534	.484
280	150	.537	.498	.520	281	149	.519	.517	.501	282	148	.501	.535	.483
282	151	.537	.499	.519	283	150	.518	.517	.501	284	149	.500	.535	.482
284	152	.536	.500	.518	285	151	.518	.518	.500	286	150	.500	.536	.482
286	153	.535	.500	.517	287	152	.517	.519	.499	288	151	.499	.537	.481
288	154	.534	.501	.516	289	153	.516	.519	.499	290	152	.498	.537	.481
290	155	.533	.502	.516	296	157	.519	.516	.502	297	156	.502	.533	.484
297	159	.536	.498	.519	298	158	.519	.516	.501	299	157	.501	.534	.484
299	160	.536	.499	.518	300	159	.518	.517	.501	301	158	.500	.534	.483
301	161	.535	.500	.517	302	160	.517	.517	.500	303	159	.500	.535	.482
303	162	.534	.500	.517	304	161	.516	.518	.499	305	160	.499	.536	.482
305	163	.533	.501	.516	306	162	.516	.519	.499	307	161	.498	.536	.481
307	164	.532	.502	.515	313	166	.519	.515	.502	314	165	.502	.532	.485
314	168	.535	.498	.518	315	167	.518	.516	.501	316	166	.501	.533	.484
316	169	.535	.499	.518	317	168	.517	.516	.501	318	167	.500	.533	.483
318	170	.534	.500	.517	319	169	.517	.517	.500	320	168	.500	.534	.483
320	171	.533	.500	.516	321	170	.516	.518	.499	322	169	.499	.535	.482
322	172	.532	.501	.516	323	171	.515	.518	.499	324	170	.498	.535	.482
324	173	.532	.502	.515	330	175	.518	.515	.502	331	174	.502	.531	.485
331	177	.534	.499	.518	332	176	.518	.515	.501	333	175	.501	.532	.484
333	178	.534	.499	.517	334	177	.517	.516	.501	335	176	.500	.533	.484
335	179	.533	.500	.517	336	178	.516	.516	.500	337	177	.500	.533	.483
337	180	.532	.500	.516	338	179	.516	.517	.499	339	178	.499	.534	.483
339	181	.531	.501	.515	340	180	.515	.518	.499	341	179	.498	.534	.482
341	182	.531	.502	.515	342	181	.514	.518	.498	347	184	.518	.514	.502
348	183	.502	.531	.485	348	186	.534	.499	.518	349	185	.517	.515	.501
350	184	.501	.531	.485	350	187	.533	.499	.517	351	186	.517	.515	.501
352	185	.500	.532	.484	352	188	.532	.500	.516	353	187	.516	.516	.500
354	186	.500	.532	.484	354	189	.531	.500	.515	355	188	.515	.517	.499
356	187	.499	.533	.483	356	190	.531	.501	.515	357	189	.515	.517	.499
358	188	.498	.533	.482	358	191	.530	.502	.514	359	190	.514	.518	.498
364	193	.518	.514	.502	365	192	.501	.530	.486	365	195	.533	.499	.517
366	194	.517	.515	.501	367	193	.501	.531	.485	367	196	.532	.499	.516
368	195	.516	.515	.501	369	194	.500	.531	.485	369	197	.531	.500	.516
370	196	.516	.516	.500	371	195	.500	.532	.484	371	198	.531	.500	.515
372	197	.515	.516	.499	373	196	.499	.532	.483	373	199	.530	.501	.514
374	198	.514	.517	.499	375	197	.498	.533	.483	375	200	.529	.502	.514
376	199	.514	.517	.498	380	203	.533	.498	.517	381	202	.517	.514	.502
382	201	.501	.529	.486	382	204	.532	.499	.517	383	203	.516	.514	.501
384	202	.501	.530	.486	384	205	.531	.499	.516	385	204	.516	.515	.501
386	203	.500	.530	.485	386	206	.531	.500	.515	387	205	.515	.515	.500
388	204	.500	.531	.484	388	207	.530	.500	.515	389	206	.515	.516	.499
390	205	.499	.531	.484	390	208	.529	.501	.514	391	207	.514	.516	.499
392	206	.498	.532	.483	392	209	.529	.502	.514	393	208	.513	.517	.498