Some Run Problems in Markov Chains

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Abstract

Some run models based on Markov chains and semi-Markov processes are discussed, with particular emphasis on their numerical analysis.

These models have applications in learning theory and in the study of certain competitive games.

Information on Tables

More detailed tables of the mean absorption times and the probabilities of eventual absorption in the success-run or the failure run are available upon request by writing to:

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1. Introduction

There is no need for further discussion of the many practical uses and the easy implementation of statistical and probabilistic procedures based on runs. For an introductory treatment of these we refer the reader to Feller, Vol I. [3].

Properties of runs are customarily discussed in connection with sequences of Bernoulli, or sometimes, multinomial trials. With very little extra effort however, the analogous properties for runs in Markov chains can be obtained. This adds some extra generality and realism to the models, and this is desirable in a variety of applications such as some educational or psychological testing procedures or certain types of competitive games.

The theory of runs also provides an excellent example of a common phenomenon in probability models. The explicit analytic formulas that are obtainable, are quite often ill-suited for numerical computations. For instance some of the quantities which we shall discuss below are of definite practical importance, but cannot be expressed in a usable closed form. At this stage we shall use the interplay between the analytic results and the numerical analysis to arrive at accurate tables of the probabilities and related quantities of interest.

In this paper we specifically consider problems associated with longest runs. In order to keep the notation simple and to concentrate only on the main ideas involved, we give ample details only for two-state Markov chains. In a subsequent section we then discuss the generalizations to m-state Markov chains, or to m-state semi-Markov processes.

2. The two-state case.

Let the sequence of random variables X_o, X_1,... form a Markov chain with two states 1 and 2 and with the transition probability matrix P given by:

$$P = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{pmatrix}$$

where $0 < \alpha < 1$ and $0 < \beta < 1$. The initial probabilities are $(p_o, 1-p_o)$, where $p_o = P\{X_o = 1\}$.

The Markov chain spends alternating random lengths of time in states 1 and 2. These time-lengths are referred to as the sojourn times. It is easy to see that the sojourn times are independent geometric random variables. Given that the chain is in the state 1 at any given time, the conditional probability that it moves to state 2 for the first time at the k-th transition thereafter is given by:

$$(1-\alpha)\alpha^{k-1}, \qquad \text{for } k \geq 1$$

and similarly, by:

$$(1-\beta)\beta^{k-1}, \qquad \text{for } k \geq 1$$

if the role of the states is reversed.

A realization of the Markov chain is therefore simply a sequence of alternating runs in the states 1 and 2 respectively.

We now specify two positive integers k_1 and k_2, and define the random variable T as the waiting time until either a run of length k_1 in the state 1 is completed or a run of length k_2 in the state 2, whichever comes first.

Remark

We note that if $\alpha + \beta = 1$ and $p_o = \alpha$, the Markov chain reduces to a sequence of Bernoulli trials with probability α of success (state 1) and probability $1-\alpha$ of failure. (state 2)

Before we discuss the random variable T, we mention some situations where this problem arises.

Some Applications

(a) Cup Tournaments

Some competitive games, such as football are played each year say, between two teams A and B. A trophy is awarded to the team that wins the match a certain number of consecutive times (often three).

If we assume that the probabilities of winning for each team remain constant over successive years, but depend on whether that team won or lost the previous year then we obtain a two-state Markov chain model describing the outcomes of the successive matches.

In some generalizations the probabilities of winning depend on the number of successive wins already scored. These are instances of the semi-Markov model discussed below.

(b) Learning Models

The two-state Markov chain model for learning behavior has received considerable attention in mathematical psychology. If the outcomes at each test are either success and failure, and we identify these with the states 1 and 2 respectively, then α is the probability of a success, given that the preceding trial was also a success. A similar interpretation in terms of failure is given to β.

The subject now performs a sequence of trials with the objective of classifying him either as fully trained or as untrainable. He is pronounced trained as soon as he completes a run of k_1 successes and alternatively as untrainable if he has a run of k_2 failures. The test continues until one or the other decision is reached.

A matter of practical importance in assessing the merits of this classification procedure in relation to others is the probability distribution of the number of trials until a decision is reached.

Equally important are the probabilities that a subject with given α and β is eventually put in one or the other category. Both the number of trials required and the probabilities of eventually reaching one or the other decision vary considerably with the values of k_1 and k_2.

The associated Markov Chain

Most problems in the theory of runs can be equivalently expressed as problems in Markov chains. A number of illustrative examples are given in Feller [3].

In order to study T conveniently, we define the random variable J_n as follows:

$J_n = j$, where $1 \leq j \leq k_1-1$, if and only if the two-state chain is in state 1 for the j-th consecutive time at time n

$J_n = k_1$, if and only if at time n, the two-state chain has been in state 1 for at least k_1 consecutive times

$J_n = -j$, where $1 \leq j \leq k_2-1$, if and only if the two-state chain is in state 2 for the j-th <u>consecutive</u> time at time n.

$J_n = -k_2$, if and only if at time n, the two-state chain has been in state 2 for at least k_2 <u>consecutive</u> times

Briefly, J_n indicates how much of a run of either type is currently built up at time n. The sign indicates whether we have a run in state 1 or in state 2 current at time n.

It is easy to see that $\{J_n\}$ is itself a Markov chain with the state space.

$$\{-k_2,\ -k_2+1, \ldots, -1, +1, \ldots, k_1-1, k_1\}$$

and transition probability matrix R given by:

$$
\begin{array}{c|cccccccccc|}
-k_2 & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
-k_2+1 & \beta & 0 & 0 & \cdots & 0 & 1-\beta & \cdots & 0 & 0 & 0 \\
-k_2+2 & 0 & \beta & 0 & \cdots & 0 & 1-\beta & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & & & & & & \\
-1 & 0 & 0 & 0 & \cdots\ \beta & 0 & 1-\beta & \cdots & 0 & 0 & 0 \\
+1 & 0 & 0 & 0 & \cdots & 1-\alpha & 0 & \alpha\ \cdots & 0 & 0 & 0 \\
+2 & 0 & 0 & 0 & \cdots & 1-\alpha & 0 & 0\ \cdots & 0 & 0 & 0 \\
 & \vdots & \vdots & \vdots & & & & & \ddots & & \\
k_1-2 & 0 & 0 & 0 & \cdots & 1-\alpha & 0 & \cdots & 0 & \alpha & 0 \\
k_1-1 & 0 & 0 & 0 & \cdots & 1-\alpha & 0 & \cdots & 0 & 0 & \alpha \\
k_1 & 0 & 0 & 0 & \cdots & 1-\alpha & 0 & \cdots & 0 & 0 & 1
\end{array}
$$

In order to distinguish this chain from the previous one we shall call

The initial state in the chain R is +1 with probability p_o and -1 with probability $1-p_o$.

In the Markov chain R, the states k_1 and $-k_2$ are **absorbing. The random** variable T is also the time till absorption in the chain R, starting from state +1 or state -1 as the case may be.

The absorption probabilities

The absorption probabilities in the chain R are studied by classical methods.

We define the following quantities:

$H_i^{(n)}$ is the probability of absorption into the state k_1, starting in the state i, with $1 \leq i \leq k_1$, no later than time n.

$R_i^{(n)}$ is the probability of absorption into the state $-k_2$, starting in the state i, with $1 \leq i \leq k_1$, no later than time n.

$G_i^{(n)}$ is the probability of absorption into the state k_1, starting in the state -i, with $1 \leq i \leq k_2$, no later than time n.

$S_i^{(n)}$ is the probability of absorption into the state $-k_2$, starting in the state -i, with $1 \leq i \leq k_2$, no later than time n.

By considering all possibilities for the first transition, we see that the sequences $\{H_i^{(n)}\}$, $\{R_i^{(n)}\}$, $\{G_i^{(n)}\}$ and $\{S_i^{(n)}\}$ satisfy the recurrence relations for $n \geq o$.

(1) (a) $H_i^{(n+1)} = \alpha H_{i+1}^{(n)} + (1-\alpha)G_1^{(n)}$, for $1 \leq i \leq k_1-1$

(b) $R_i^{(n+1)} = \alpha R_{i+1}^{(n)} + (1-\alpha)S_1^{(n)}$, for $1 \leq i \leq k_1-1$

(c) $G_i^{(n+1)} = \beta G_{i+1}^{(n)} + (1-\beta)H_1^{(n)}$, for $1 \leq i \leq k_2-1$

(d) $S_i^{(n+1)} = \beta S_{i+1}^{(n)} + (1-\beta)R_1^{(n)}$, for $1 \leq i \leq k_2-1$

with in addition:

(2) $$H_{k_1}^{(n)} = S_{k_2}^{(n)} = 1,$$

$$R_{k_1}^{(n)} = G_{k_2}^{(n)} = 0.$$

The sequences $\{H_i^{(n)}\}$, $\{R_i^{(n)}\}$, $\{G_i^{(n)}\}$ and $\{S_i^{(n)}\}$ are clearly non decreasing in n.

Their limits as $n \to \infty$ are the conditional probabilities of eventual absorption in either the state k_1 or alternatively the state $-k_2$. We shall denote the limit for each sequence by deleting the superscript (n). Explicitly, H_i is the probability that absorption occurs in the state k_1, rather than in the state $-k_2$, given that the initial state is i $(1 \leq i \leq k_1)$. Similar interpretations hold for R_i, G_i and S_i.

By passing to the limit in (1) and (2) we see immediately that the probabilities of eventual absorption satisfy the system of equations:

(3) (a) $H_i = \alpha H_{i+1} + (1-\alpha)G_1,$ for $1 \leq i \leq k_1-1$

(b) $R_i = \alpha R_{i+1} + (1-\alpha)S_1,$ for $1 \leq i \leq k_1-1$

(c) $G_i = \beta G_{i+1} + (1-\beta)H_1,$ for $1 \leq i \leq k_2-1$

(d) $S_i = \beta S_{i+1} + (1-\beta)R_1,$ for $1 \leq i \leq k_2-1$

with

$$H_{k_1} = S_{k_2} = 1 \text{ and } R_{k_1} = G_{k_2} = 0.$$

By successive substitutions, the equations (3) may be written as:

(4) (a) $H_j = \alpha^{k_1-j} + (1-\alpha^{k_1-j})G_1,$ for $1 \leq j \leq k_1$

(b) $R_j = (1-\alpha^{k_1-j})S_1,$ for $1 \leq j \leq k_1$

(c) $G_j = (1-\beta^{k_2-j})H_1,$ for $1 \leq j \leq k_2$

(d) $S_j = \beta^{k_2-j} + (1-\beta^{k_2-j})R_1,$ for $1 \leq j \leq k_2$.

and therefore:

(5) (a) $H_1 = \alpha^{k_1-1}\left[1 - (1-\alpha^{k_1-1})(1-\beta^{k_2-1})\right]^{-1}$

(b) $R_1 = (1-\alpha^{k_1-1})\,\beta^{k_2-1}\left[1 - (1-\alpha^{k_1-1})(1-\beta^{k_2-1})\right]^{-1}$

(c) $G_1 = \alpha^{k_1-1}(1-\beta^{k_2-1})\left[1 -(1-\alpha^{k_1-1})(1-\beta^{k_2-1})\right]^{-1}$

(d) $S_1 = \beta^{k_2-1}\left[1 - (1-\alpha^{k_1-1})(1-\beta^{k_2-1})\right]^{-1}$

As **is to be expected, we see that:**

$$H_1 + R_1 = G_1 + S_1 = 1,$$

The absorption probabilities for indices j other than one, are immediately found by substitution in the formulas (4). In view of the original application the formulas (5) have the most interest, since from them a number of interesting features of a testing procedure based on runs may be deduced.

For example, for any initial probabilities $(p_o, 1-p_o)$ in the two-state chain, the quantity

$$p_o\, H_1 + (1-p_o)\, G_1$$

is the probability that a run of length k_1 in state 1 occurs earlier than a run of length k_2 in state 2. In terms of the learning model, this is the probability, as a function of α and β, that the subject will eventually be pronounced "trained".

In particular, if we assume that the two-state chain is stationary, we have the initial probabilities $(p^*, 1-p^*)$ where and:

(9) $$H_{k_1}(z) = S_{k_2}(z) = 1,$$

$$R_{k_1}(z) = G_{k_2}(z) = 0.$$

By successive substitutions we obtain:

(10) (a) $$H_i(z) = \alpha^{k_1-i} z^{k_1-i} + \frac{1-\alpha^{k_1-i} z^{k_1-i}}{1-\alpha z} (1-\alpha)z\, G_1(z),$$

for $1 \leq i \leq k_1$

(b) $$R_i(z) = \frac{1-\alpha^{k_1-i} z^{k_1-i}}{1-\alpha z} (1-\alpha)z\, S_1(z),$$

for $1 \leq i \leq k_1$

(c) $$G_i(z) = \frac{1-\beta^{k_2-i} z^{k_2-i}}{1-\beta z} (1-\beta)z\, H_1(z),$$

for $1 \leq i \leq k_2$

(d) $$S_i(z) = \beta^{k_2-i} z^{k_2-i} + \frac{1-\beta^{k_2-i} z^{k_2-i}}{1-\beta z} (1-\beta)z\, R_1(z),$$

for $1 \leq i \leq k_2$.

The four equations corresponding to i = 1 in (10a, b, c, d) are a simple system of four linear equations in four unknowns. From it we readily obtain that:

(11) (a) $$H_1(z) = (\alpha z)^{k_1-1} \left[1 - \frac{1-\alpha^{k_1-1} z^{k_1-1}}{1-\alpha z} \cdot \frac{1-\beta^{k_2-1} z^{k_2-1}}{1-\beta z} (1-\alpha)(1-\beta)z^2\right]^{-1}$$

(b) $R_1(z) = \dfrac{1-\alpha^{k_1-1} z^{k_1-1}}{1-\alpha z} \; (1-\alpha)z \;\; S_1(z)$

(c) $G_1(z) = \dfrac{1-\beta^{k_2-1} z^{k_2-1}}{1-\beta z} \; (1-\beta)z \;\; H_1(z)$

(d) $S_1(z) = (\beta z)^{k_2-1}\left[1 - \dfrac{1-\alpha^{k_1-1} z^{k_1-1}}{1-\alpha z} \cdot \dfrac{1-\beta^{k_2-1} z^{k_2-1}}{1-\beta z} \; (1-\alpha)(1-\beta)z^2\right]^{-1}$

For other values of i the corresponding generating functions are obtained by direct substitution in the equations (10). All these generating functions are rational functions in z. The probabilities $H_i^{(n)}$, $R_i^{(n)}$, $G_i^{(n)}$, and $S_i^{(n)}$ may be obtained from them, in principle at least, by the technique of partial fraction expansions. This technique is discussed in many texts on probability, in particular Feller, Vol I, pp. 275 ff.

For all cases of practical interest they may be obtained much more easily by direct numerical computation from the recurrence relations (1). We return to this below.

For simplicity of notation we set

$$\theta(z) = \left[1 - \frac{1-\alpha^{k_1-1} z^{k_1-1}}{1-\alpha z} \cdot \frac{1-\beta^{k_2-1} z^{k_2-1}}{1-\beta z} \; (1-\alpha)(1-\beta)z^2\right]^{-1}$$

and obtain in this manner:

(12) (a) $H_1(z) = (\alpha z)^{k_1-1} \; \theta(z)$

(b) $R_1(z) = \dfrac{1-(\alpha z)^{k_1-1}}{1-\alpha z} \; (1-\alpha)z \; (\beta z)^{k_2-1} \; \theta(z)$

(c) $G_1(z) = \dfrac{1-(\beta z)^{k_2-1}}{1-\beta z} \; (1-\beta)z \; (\alpha z)^{k_1-1} \; \theta(z)$

(d) $S_1(z) = (\beta z)^{k_2-1} \; \theta(z)$

The distribution of the waiting time till absorption starting from the state 1 has the generating function:

(13) $\Psi_1(z) = H_1(z) + R_1(z) =$

$$\theta(z)\left\{(\alpha z)^{k_1-1} + \left[1-(\alpha z)^{k_1-1}\right] \frac{(1-\alpha)z}{1-\alpha z} \quad (\beta z)^{k_2-1}\right\}$$

Similarly, starting from the state -1 (or equivalently from the state 2 in the two-state chain) the waiting time till absorption has the generating function $\Psi_2(z)$ obtained from $\Psi_1(z)$ by interchanging the roles of k_1 and k_2, α and β.

<u>The conditional mean time till absorption</u>, starting from the state 1, is found by differentiation:

(14) $\mu_1 = \Psi_1'(1) =$

$$\left[1 - (1-\alpha^{k_1-1})(1-\beta^{k_2-1})\right]^{-1} (1-\alpha^{k_1-1}) \left[\frac{1-\beta^{k_2-1}}{1-\beta} + \frac{1}{1-\alpha}\right]$$

Similary the conditional mean time μ_2 till absorption, starting from the state -1, is found to be:

(15) $\mu_2 = \Psi_2'(1) =$

$$\left[1 - (1-\alpha^{k_1-1})(1-\beta^{k_2-1})\right] (1-\beta^{k_2-1}) \left[\frac{1-\alpha^{k_1-1}}{1-\alpha} + \frac{1}{1-\beta}\right]$$

It is also of interest to consider the mean time till absorption if the initial state in the two-state chain is chosen with the stationary probability vector $(p_o^*, 1-p_o^*)$. We denote this mean time till absorption by μ^*. It is given by:

$$(16) \qquad \mu^* = p_o^* \mu_1 + (1-p_o^*) \mu_2$$

Numerical values of μ_1, μ_2 and μ^* are also given in table 1.

Remark

The practical worker should note the following. In many concrete situations, such as the learning model or the competitive game, each of the successive random variables $X_o, X_1, \ldots, X_m, \ldots$ expresses the result of a trial. We note that X_n is the result of the (n+1)-st trial. Some care should therefore be given when the language of trials is used instead of that of Markov chains.

For example, for $k_1 = 2$, $k_2 = 4$, $\alpha = .4$ and $\beta = .1$, the mean $\mu^* = 4.60$. In terms of trials this means that the mean number of trials to reach absorption is $\mu^* + 1 = 5.60$.

Similarly, suppose that we need the probability that absorption in the success-run occurs at the 10-th trial. This is given by:

$$p_o^* \left[H_1^{(9)} - H_1^{(8)}\right] + (1-p_o^*) \left[G_1^{(9)} - G_1^{(8)}\right] =$$

$$(0.6)(.01913) + (0.4)(.06301) = 0.036682.$$

The appropriate probabilities correspond to L = 9 in table 2.

Similarly X_o, the initial state of the Markov chain is the result of the first experiment. In the absence of other information, the most appropriate choice for $P\{X_o = 1\}$ is p_c^*. This is equivalent to assuming that the chance mechanism that generates the successes or failures is stationary. The unconditional probability of a success at time n = o or at any other time is then p_o^*.

As noted before, in the special case where $\alpha + \beta = 1$, the two-state Markov chain reduces to a sequence of Bernoulli trials in which the probability of a success is $p_o^* = \alpha$.

3. Numerical Analysis of the Model

This model offers an example of the use of the interplay between the theoretically tractable results and others of pr ctical importance, which are not analytically simple, but are computed numerically.

The author has prepared an extensively **documented FORTRAN program,** which for any values of α and β and any values of $k_1 \leq 8$, $k_2 \leq 8$(*) computes a number of quantities of interest.

The computational work is organized along the following lines:

(a) First the equations (3) are used to compute the absorption probabilities H_i, R_i for $1 \leq i \leq k_1$ and G_i, S_i for $1 \leq i \leq k_2$.

(b) Next the probabilities $H_i^{(n)}$, $R_i^{(n)}$, $G_i^{(n)}$, $S_i^{(n)}$ are computed recursively. The important probabilities $H_1^{(n)}$, $R_1^{(n)}$, $G_1^{(n)}$, $S_1^{(n)}$, are compared each time to the values of H_1, R_1, G_1 and S_1 computed in (a).

An error bound ERR(**) is to be specified by the user, as well as a maximum value NN for n. Successive four-tuples $H_i^{(n)}$, $R_i^{(n)}$, $G_i^{(n)}$, $S_i^{(n)}$, are computed until either

$$\max \{H_1 - H_1^{(n)}, R_1 - R_1^{(n)}, G_1 - G_1^{(n)}, S_1 - S_1^{(n)}\} < \text{ERR} \quad \text{or until n equals NN.}$$

(*) It is easy to modify the program if larger values of k_1 and k_2 are needed. For the applications the author had in mind, it seems that $k_1 \leq 8$, $k_2 \leq 8$ are sufficient.

(**) In the numerical example given at the end, ERR was set equal to 10^{-6} so as to insure that all probabilities found are correct to within 0.000005. This is more than adequate for applied purposes.

(c) The densities of the waiting time distributions **are obtained by** computing at each step the differences

$$H_i^{(n)} - H_i^{(n-1)},\ R_i^{(n)} - R_i^{(n-1)},\ G_i^{(n)} - G_i^{(n-1)},\ S_i^{(n)} - S_i^{(n-1)}.$$

(d) Finally the conditional mean absorption times and the probability p_o^* are computed using the analytic expressions.

Examples of the output are given in Table 2. A listing of the FORTRAN program is also given at the end of this paper.

4. Generalizations

a. The m-state Markov chain

There is no theoretical difficulty in extending the discussion of the two-state Markov chain to the m-state case.

We denote the matrix of the m-state chain by P and assume that m integers $k_1, \ldots, k_m$ are given. The random variable T is the waitingtime until a sojourn time (or run) of length at least k_j in the state j occurs for one of $j = 1, \ldots, m$.

If the Markov chain P is reducible or irreducible but periodic, the random variable T may be infinite with positive probability. In this case it is indicated to analyze the class structure of the Markov chain first and possibly to reformulate the waiting time problem so as to take advantage of the class structure.

If the Markov chain P is irreducible and aperiodic, then the only states i in which runs of length greater than one can occur are those for which $P_{ii} > 0$. This limits the sets of states i in which a run of length k_i can occur, at least if $k_i > 1$.

Limiting our attention to the irreducible, aperiodic case, we see that every realization of the Markov chain may be visualized as follows.

Given that the chain enters the state i, it sojourns there for a length of time which has a geometric distribution with parameter P_{ii}. At the end of its sojourn in state i it moves to state j ($i \neq j$) with (conditional) probability $P_{ij}\ (1-P_{ii})^{-1}$.

If we denote the successive distinct states visited by the Markov chain by $J_o, J_1, \ldots, J_n, \ldots$ and the corresponding number of visits (the sojourn times) to each of them by $Y_1, Y_2, \ldots, Y_n, \ldots$ respectively, then the vector sequence

$$\{(J_n, Y_n),\ n \geq 0\}$$

in which we set $Y_o = 0$ for convenience, is an example of a semi-Markov sequence. For expository discussions of semi-Markov sequences and related processes, we refer to Pyke [4,5] or Cinlar [2]. An early reference to them in mathematical psychology is V. Cane [1].

The semi-Markov sequence obtained here has a particularly simple transition probability matrix $Q_{ij}(k)$ given by:

$$(17) \qquad Q_{ij}(k) = P\,\{J_{n+1} = j,\ y_{n+1} \leq k \,|\, J_m = i\}$$

$$= 0, \qquad \text{for } i = j$$

$$= \frac{1-P_{ii}^k}{1-P_{ii}}\ P_{ij}, \qquad \text{for } i \neq j,\ k \geq 1,$$

and $Q_{ij}\ (o) = 0$ for all i and j.

It is easy but tedious to write down the recurrence relations corresponding to the formulae (1) for this case. These are again well suited for numerical computation. In programming them for the digital computer one should make more economical use of computer memory storage than we did for the two-state case. This leads to a less elegant display of the results, but this is of no consequence to the solution of the problem.

b. The m-state semi-Markov process

We now assume that the probability of staying an additional unit of time in a state depends on the length of the run already completed. We are then no longer dealing with a Markov chain but with a semi-Markov process.

Formally we consider an m-state process $\{J_n\}$ with sojourn times $\{Y_n\}$. For convenience we set $Y_o = o$.

The semi-Markov assumption is equivalent to :

$$P\{J_{n+1} = j,\ Y_{n+1} \leq x \mid J_n = i,\ Y_n, \ldots, J_o\} =$$

$$P\{J_{n+1} = j,\ Y_{n+1} \leq x \mid J_n = i\} = Q_{ij}(x)$$

for all $n \geq o$.

The matrix of distribution functions $Q(x)$ is the transition probability matrix of the semi-Markov process. In the run models of interest here, the sojourn times are integer-valued. It is hence more convenient to consider the density matrix with entries:

$$q_{ij}(k) = Q_{ij}(k) - Q_{ij}(k-1), \qquad k \geq 1.$$

More explicitely $q_{ij}(k)$ may also be written as:

$$q_{ij}(k) = \frac{1}{Q_{ij}(\infty)} \quad r_{ij}(k),$$

Note that when $Q_{ij}(\infty) = 0$, the precise definition of $q_{ij}(k)$ is immaterial. We further note that the matrix $Q(\infty) = \{Q_{ij}(\infty)\}$ is an m-th order stochastic matrix, which we henceforth assume to be irreducible.

In the study of runs we may further assume that $Q_{ii}(\infty) = 0$ for $i = 1,\ldots, m$ since we do not consider transitions from a state to itself.

The quantity $r_{ij}(k)$ may be interpreted as the (conditional) probability that the semi-Markov process <u>jumps from state i to state j after exactly k units of time in the state i</u>.

The run problem, discussed for the two-state Markov chain, is a special case of the following more general problem.

<u>General Run Problem</u>

We consider the m-state (irreducible) semi-Markov process with density matrix $\{q_{ij}(\nu),\ i = 1,\ldots, m,\ j = 1,\ldots, m,\ \nu \geq 1\}$.

We specify m positive integers $k_1,\ldots, k_m$. We say that absorption occurs if and only if for some j, $j = 1,\ldots, m$, there is a sojourn time in the state j which is greater than or equal to k_1.

<u>Absorption occurs at time n</u> if and only if at time n the semi-Markov process has been in some state j for exactly k_j units of time and if absorption has not occurred at an earlier time.

We define the waiting time T as the time at which absorption occurs and I as the state of the semi-Markov process at the time of absorption. We shall assume that

$$\sum_{j=1}^{m} Q_{ij}(k_i - 1) < 1$$

for at least one i, which will insure that T is finite with probability one.

T as a first-passage time

There is a simple, but elegant way of identifying T as the first-passage time from one set of states to another in a semi-Markov process.

To do so we "split" every state j of the original semi-Markov process into two states j' and j". We define a new semi-Markov process which has 2 m states 1', 1", 2', 2",..., m', m". We shall say that the new semi-Markov process is in the state j' whenever the old one is currently in the state j and has been there for at most k_j-1 units of time. The new semi-Markov process is in the state j" if and only if absorption into the state j has occurred in the original semi-Markov process. T is therefore the time until absorption into one of the states 1", 2",..., m" and I = j if and only if eventual absorption is into the state j".

The absorption probabilities

We define $t_{ij}(n)$ as the probability:

(18) $P\{T = n, I = j \mid J_o = i\}$, for i = 1,... m; j = 1,.., m and $n \geq 0$

and the matrix $t(n) = \{t_{ij}(n)$. The quantities $t_{ij}(n)$ satisfy the recurrence relations:

(19) $$t_{ij}(n) = \sum_{r=1}^{m} \sum_{\nu=0}^{k_i-1} q_{ir}(\nu)\, t_{rj}(n-\nu), \qquad \text{for } n \leq k_i-1$$

$$t_{ij}(k_i) = \delta_{ij}\,[1 - \sum_{j=1}^{m} Q_{ij}(k_i-1)] + \sum_{r=1}^{m} \sum_{\nu=0}^{k_i-1} q_{ir}(\nu)\, t_{rj}(k_i-\nu)$$

$$t_{ij}(n) = \sum_{r=1}^{m} \sum_{\nu=0}^{k_i-1} q_{ir}(\nu)\, t_{rj}(n-\nu), \qquad \text{for } n \geq k_i + 1$$

for $i = 1, \ldots, m$; $j = 1, \ldots, m$ and $t_{ij}(o) = o$.

These formulae are obtained by considering all possible alternatives up to the time of the first transition out of state i. The expressions (19) are well-suited for numerical computation. In the absence of data referring to concrete situations the author has not performed any such computations.

It is also easy to obtain a system of linear equations for the probabilities $t_j^* = P(I = j)$. These are obtained by completely routine first passage arguments for a finite semi-Markov process and we shall not elaborate further on them here.

In the m-state semi-Markov model there are very few tractable analytic results. The need for detailed numerical investigations in any concrete situation is therefore all the greater.

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Table 1

This table lists the following quantities from left to right.

1. $k_1 = 1\ (1)\ 10$

2. $k_2 = 1\ (1)\ 10$

3. $\alpha = .2\ (.1)\ .9$

4. $\beta = .1\ (.1)\ \alpha$

5. p_o^*, the stationary probability of being in state 1

6. $P_{k_1, k_2}(\alpha,\beta)$, the probability that in the stationary chain absorption into the succes-run (k_1) occurs.

7. μ_1, the mean time till absorption, given that one success has occurred.

8. μ_2, the mean time till absorption, given that one failure has occurred.

9. μ^*, the mean time till absorption for the stationary two-state chain.

Remark

Only the table corresponding to $k_1 = 2$, $k_2 = 4$ is reproduced here. The tables for all k_1 and k_2 up to 10 may be obtained from the author upon request.

k_1	k_2	α	β	P_0^*	$P_{k_1k_2}(\alpha,\beta)$	μ_1	μ_2	μ^*
2	4	.200	.100	.529412	.995547	9.402	10.503	9.920
2	4	.200	.200	.500000	.965116	9.651	10.814	10.233
2	4	.300	.100	.562500	.997236	5.910	7.014	6.393
2	4	.300	.200	.533333	.978010	6.113	7.304	6.668
2	4	.300	.300	.500000	.928034	6.187	7.410	6.798
2	4	.400	.100	.600000	.998103	4.159	5.265	4.601
2	4	.400	.200	.571429	.984754	4.308	5.514	4.825
2	4	.400	.300	.538462	.949100	4.407	5.678	4.993
2	4	.400	.400	.500000	.883212	4.416	5.693	5.055
2	4	.500	.100	.642857	.998644	3.107	4.214	3.502
2	4	.500	.200	.615385	.989011	3.214	4.429	3.681
2	4	.500	.300	.583333	.962756	3.301	4.602	3.843
2	4	.500	.400	.545455	.912509	3.346	4.692	3.958
2	4	.500	.500	.500000	.833333	3.333	4.667	4.000
2	4	.600	.100	.692308	.999026	2.405	3.513	2.746
2	4	.600	.200	.666667	.992042	2.480	3.700	2.887
2	4	.600	.300	.636364	.972674	2.547	3.869	3.028
2	4	.600	.400	.600000	.934527	2.596	3.990	3.153
2	4	.600	.500	.555556	.871795	2.615	4.038	3.248
2	4	.600	.600	.500000	.779720	2.599	3.998	3.298
2	4	.700	.100	.750000	.999322	1.903	3.012	2.180
2	4	.700	.200	.727273	.994409	1.953	3.178	2.287
2	4	.700	.300	.700000	.980554	2.001	3.337	2.402
2	4	.700	.400	.666667	.952540	2.041	3.471	2.518
2	4	.700	.500	.625000	.904661	2.068	3.559	2.627
2	4	.700	.600	.571429	.830544	2.076	3.588	2.724
2	4	.700	.700	.500000	.722319	2.064	3.546	2.805
2	4	.800	.100	.818182	.999568	1.527	2.636	1.729
2	4	.800	.200	.800000	.996407	1.557	2.784	1.802
2	4	.800	.300	.777778	.987335	1.587	2.934	1.886
2	4	.800	.400	.750000	.968504	1.614	3.071	1.978
2	4	.800	.500	.714286	.935065	1.636	3.182	2.078
2	4	.800	.600	.666667	.880455	1.651	3.254	2.185
2	4	.800	.700	.600000	.794658	1.656	3.278	2.304
2	4	.800	.800	.500000	.659574	1.649	3.245	2.447
2	4	.900	.100	.900000	.999789	1.234	2.343	1.345
2	4	.900	.200	.888889	.998224	1.248	2.478	1.384
2	4	.900	.300	.875000	.993644	1.262	2.618	1.431
2	4	.900	.400	.857143	.983861	1.275	2.754	1.487
2	4	.900	.500	.833333	.965753	1.288	2.877	1.553
2	4	.900	.600	.800000	.934375	1.298	2.977	1.634
2	4	.900	.700	.750000	.880686	1.305	3.047	1.740
2	4	.900	.800	.666667	.784693	1.308	3.078	1.898
2	4	.900	.900	.500000	.587882	1.306	3.064	2.185

Computer Program

This is a listing of the FORTRAN program which computes the waiting time distribution and other quantities of interest for given values of k_1, k_2, α and β.

Two examples of the output generated are given in Table 2.

```
      PROGRAM SCCS (INPUT,OUTPUT,TAPE5=INPUT)
C
C
C THIS PROGRAM COMPUTES A NUMBER OF PROBABILISTIC FEATURES
C OF THE FOLLOWING MODEL.
C A TWO-STATE MARKOV CHAIN IS CONSIDERED,WHOSE STATES ARE
C SUCCESS AND FAILURE. THE CONDITIONAL PROBABILITY THAT A
C SUCCESS IS FOLLOWED BY A SUCCESS IS ALPHA AND THE PROBA-
C BILITY THAT A FAILURE IS FOLLOWED BY A FAILURE IS BETA.
C WE SAY THAT ABSORPTION IN THE SUCCESS-RUN OCCURS WHEN
C AN UNINTERRUPTED STRING OF K1 SUCCESSES OCCURS BEFORE
C AN UNINTERRUPTED STRING OF K2 FAILURES.
C WE SAY THAT ABSORPTION IN THE FAILURE-RUN OCCURS IF
C AN UNINTERRUPTED STRING OF K2 FAILURES OCCURS
C EARLIER THAN AN UNINTERRUPTED STRING OF K1 SUCCESSES.
C
C THE FOLLOWING QUANTITIES ARE COMPUTED AND PRINTED OUT.
C ** THE ABSORPTION PROBABILITIES.
C ** THE ABSORPTION TIME DISTRIBUTIONS.
C ** THE DENSITIES OF THE ABSORPTION TIME DISTRIBUTIONS.
C ** THE MEAN TIMES TILL ABSORPTION.
C
C
      DIMENSION H(8),R(8),G(8),S(8),GG(8),SS(8),JJ(10),JH(10)
      DIMENSION HX(8,150),RX(8,150),GX(8,150),SX(8,150)
      DIMENSION HY(8,150),RY(8,150),GY(8,150),SY(8,150)
      DIMENSION G1(8,150),G2(8,150),S1(8,150),S2(8,150)
C
C THE FOLLOWING DATA NEED TO BE SUPPLIED:
C *** K1 IS THE NUMBER OF CONSECUTIVE SUCCESSES REQUIRED FOR
C ABSORPTION IN THE SUCCESS-RUN.
C *** K2 IS THE NUMBER OF CONSECUTIVE FAILURES REQUIRED FOR
C ABSORPTION IN THE FAILURE-RUN.
C *** AL IS ALPHA, THE CONDITIONAL PROBABILITY THAT A
C SUCCESS FOLLOWS A SUCCESS.
C *** BET IS BETA, THE CONDITIONAL PROBABILITY THAT A
C FAILURE FOLLOWS A FAILURE.
C *** ERR IS THE MAXIMUM TAILPROBABILITY WHICH IS
C NEGLECTED IN THE COMPUTATION OF THE ABSORPTION TIME
C DISTRIBUTIONS.
   10 READ (5,1) K1,K2,NNN
      IF(EOF,5) 11,12
   12 READ(5,2) AL,BET
      READ (5,5) ERR
C
C THIS PORTION OF THE PROGRAM COMPUTES THE ABSORPTION
C PROBABILITIES AND THE STATIONARY PROBABILITY PP OF
C A SUCCESS.
C
C H(I) IS THE PROBABILITY OF ABSORPTION IN THE SUCCESS-
C RUN, STARTING FROM A SUCCESS-RUN OF LENGTH I.
C P(I) IS THE PROBABILITY OF ABSORPTION IN THE FAILURE-
C RUN, STARTING FROM A SUCCESS-RUN OF LENGTH I.
C S(I) IS THE PROBABILITY OF ABSORPTION IN THE FAILURE-
C RUN, STARTING FROM A FAILURE RUN OF LENGTH I.
C G(I) IS THE PROBABILITY OF ABSORPTION IN THE SUCCESS-
C RUN, STARTING FROM A FAILURE-RUN OF LENGTH I.
```

```
C
C
      Y1=AL**(K1-1)
      Y2=BET**(K2-1)
      Y=1./(Y1+Y2-Y1*Y2)
      H(1)=Y1*Y
      G(1)=(1.-Y2)*H(1)
      S(1)=Y2*Y
      R(1)=(1.-Y1)*S(1)
      TH=(1.-AL)/AL
      TJ=(1.-BET)/BET
      H1=H(1)*TJ
      G1=G(1)*TH
      S1=S(1)*TH
      R1=R(1)*TJ
      PP=(1.-BET)/(2.-AL-BET)
      K11=K1-1
      K21=K2-1
      DO 3 I=1,K11
      H(I+1)=H(I)/AL-G1
      R(I+1)=R(I)/AL-S1
    3 CONTINUE
      DO 4 I=1,K21
      G(I+1)=G(I)/BET-H1
      S(I+1)=S(I)/BET-R1
    4 CONTINUE
C
C
C THIS PORTION OF THE PROGRAM COMPUTES THE ABSORPTION
C TIME DENSITY AND DISTRIBUTIONS RECURSIVELY.
C
C
      M=1
      HX(K1,1)=1.
      RX(K1,1)=0.
      GX(K2,1)=0.
      SX(K2,1)=1.
      K11=K1-1
      K21=K2-1
      DO 20 I=1,K11
      HX(I,1)=0.
      RX(I,1)=0.
      HY(I,1)=0.
      RY(I,1)=0.
   20 CONTINUE
      DO 21 I=1,K21
      GX(I,1)=0.
      SX(I,1)=0.
      GY(I,1)=0.
      SY(I,1)=0.
   21 CONTINUE
      V1=H(1)
      V2=G(1)
      V3=R(1)
      V4=S(1)
    6 M=M+1
      IF(M.GT.NNN) GO TO 17
      HX(K1,M)=1.
```

```
      RX(K1,M)=0.
      GX(K2,M)=0.
      SX(K2,M)=1.
      DO 22 J=1,K11
      HX(J,M)=HX(J+1,M-1)*AL+(1.-AL)*GX(1,M-1)
      RX(J,M)=RX(J+1,M-1)*AL+(1.-AL)*SX(1,M-1)
      HY(J,M)=HX(J,M)-HX(J,M-1)
      RY(J,M)=RX(J,M)-RX(J,M-1)
      DO 22 I=1,K21
      GX(I,M)=GX(I+1,M-1)*BET+(1.-BET)*HX(1,M-1)
      SX(I,M)=SX(I+1,M-1)*BET+(1.-BET)*RX(1,M-1)
      GY(I,M)=GX(I,M)-GX(I,M-1)
      SY(I,M)=SX(I,M)-SX(I,M-1)
   22 CONTINUE
C
C
C THIS PORTION OF THE PROGRAM COMPARES THE TOTAL PROBA-
C BILITY FOR THE ABSORPTION TIME DISTRIBUTIONS ALREADY
C COMPUTED TO THE ABSORPTION PROBABILITIES FOUND IN THE
C FIRST PART OF THE PROGRAM AND LOOPS BACK TO COMPUTE
C ADDITIONAL TERMS IF NECESSARY. THE FIRST MOMENTS OF THE
C ABSORPTION TIME DISTRIBUTIONS ARE ALSO COMPUTED.
C
C
      U1=H(1)-HX(1,M)
      U2=G(1)-GX(1,M)
      U3=R(1)-RX(1,M)
      U4=S(1)-SX(1,M)
      U=AMAX1(U1,U2,U3,U4)
      IF(U.GE.ERR) GO TO 6
      GO TO 18
   17 MM=M-1
      GO TO 19
   18 MM=M
   19 CONTINUE
      XK1=K11
      XK2=K21
      Q=((1.-Y1)*(1.-Y2))/((1.-AL)*PP)
      Q=Q-XK1*Y1*(1.-Y2)-XK2*Y2*(1.-Y1)
      Q=Q*Y
      W1=XK1+Q
      W4=XK2+Q
      W2=W1+1./(1.-BET)-XK2*Y2/(1.-Y2)
      W3=W4+1./(1.-AL)-XK1*Y1/(1.-Y1)
      PRINT 31
      PRINT 23,K1,K2
      PRINT 24,AL,BET
      PRINT 32,PP
      PRINT 25,NNN
      PRINT 26,W1,W2,W3,W4
      DO 7 J=1,K2
      JJ(J)=-K2-1+J
      JH(J)=-JJ(J)
      GG(J)=G(K2+1-J)
      SS(J)=S(K2+1-J)
      DO 7 L=1,MM
      G1(J,L)=GX(K2+1-J,L)
      G2(J,L)=GY(K2+1-J,L)
```

```
      S1(J,L)=SX(K2+1-J,L)
      S2(J,L)=SY(K2+1-J,L)
    7 CONTINUE
C
C
C THIS LATTER PORTION OF THE PROGRAM CONTAINS ALL PRINT
C AND FORMAT INSTRUCTIONS.
C
C
      PRINT 31
      PRINT 34
      PRINT 27,(JJ(J),GG(J),SS(J),J=1,K2)
      PRINT 27,(I,H(I),R(I),I=1,K1)
      PRINT 31
      DO 8 L=1,MM
      PRINT 28,L
      PRINT 30,(JJ(J),G1(J,L),G2(J,L),S1(J,L),S2(J,L),J=1,K2)
      PRINT 30,(I,HX(I,L),HY(I,L),RX(I,L),RY(I,L),I=1,K1)
    8 CONTINUE
      PRINT 31
      PRINT 33
      PRINT 29,(L,HX(1,L),GX(1,L),RX(1,L),SX(1,L),L=1,MM)
      GO TO 10
   11 CONTINUE
    1 FORMAT(2I1,I3)
    2 FORMAT(2F8.6)
    5 FORMAT(F8.6)
   23 FORMAT(X,≠THE NUMBER K1 OF SUCCESSES REQUIRED FOR ≠
     1≠ABSORPTION IS ≠,I1/,X,≠THE NUMBER K2 OF FAILURES ≠
     2≠REQUIRED FOR ABSORPTION IS ≠,I1/)
   24 FORMAT(X,≠THE PROBABILITY THAT≠,/,3X,≠A SUCCESS IS ≠
     1≠FOLLOWED BY A SUCCESS IS ALPHA = ≠,F8.6/,3X,
     2≠A FAILURE IS FOLLOWED BY A FAILURE IS BETA = ≠,
     3F8.6/)
   25 FORMAT(X,≠AT MOST ≠,I3, ≠ POINTS OF THE ABSORPTION ≠
     1≠TIME DISTRIBUTION ARE COMPUTED.≠/)
   26 FORMAT(X,≠THE CONDITIONAL MEAN ABSORPTION TIME INTO ≠
     1≠THE SUCCESS RUN AFTER ONE SUCCESS IS≠/,
     25X,F12.4/,X,≠AND AFTER ONE FAILURE IS≠/,5X,F12.4/,
     3≠ THE CONDITIONAL MEAN ABSORPTION TIME INTO THE ≠
     4≠FAILURE RUN AFTER ONE SUCCESS IS≠/,5X,F12.4/,
     5≠ AND AFTER ONE FAILURE IS≠/,5X,F12.4/)
   27 FORMAT(2X,4(I3,2X,F7.5,2X,F7.5,2X))
   34 FORMAT(X,≠THE FOLLOWING IS A TABLE OF THE ABSORPT≠
     1≠ION PROBABILITIES IN THE SUCCESS-RUN OR IN≠
     2≠ THE≠/,X,≠FAILURE RUN, GIVEN THAT ALREADY THE ≠
     3≠INDICATED NUMBER OF SUCCESSIVE SUCCESSES OR FAILURES≠/
     4,X,≠HAVE OCCURRED.≠/)
   28 FORMAT(/X,≠L = ≠,I4/)
   30 FORMAT(X,2(I4,4F8.5,2X))
   29 FORMAT(/(X,2(I4,4F8.5)))
   31 FORMAT(≠1≠)
   32 FORMAT(X,≠THE STATIONARY PROBABILITY OF A SUCCESS≠,
     1≠ PP IS EQUAL TO:≠/,10X,F7.5)
   33 FORMAT(X,≠ TABLE OF THE ABSORPTION TIME DISTRIBUTION≠
     1≠ FOLLOWING A SUCCESS, RESP. A FAILURE.≠/,3X,≠ THE ≠
     2≠FIRST TWO QUANTITIES ARE THE PROBABILITIES OF ≠
     3≠ABSORPTION IN THE SUCCESS-RUN≠/,4X,≠GIVEN THAT ≠
```

```
4≠A SUCCESS, RESP. A FAILURE HAVE OCCURRED.≠/,3X,
5≠ THE NEXT TWO QUANTITIES ARE THE PROBABILITIES OF ≠
6≠ABSORPTION IN THE FAILURE-RUN≠/,4X,≠GIVEN THAT A ≠
7≠SUCCESS, RESP. A FAILURE HAVE OCCURRED.≠)
 END
```

Table 2

The following numerical information is supplied.

1. The values of k_1, k_2, α and β.

2. The upper limit to the number of points of the absorption time distribution that are computed.

3. The stationary probability p_o^* of a success. (state 1)

4. The conditional mean absorption times.

5. The conditional absorption probabilities into the states k_1 and $-k_2$ in the chain R from any given initial state.

6. The conditional absorption time distributions and densities into the states k_1 and $-k_2$ in the chain R from the initial states +1 and -1.

7. A table of the conditional absorption time distributions and densities into the states k_1 and $-k_2$ in the chain R from any given initial state.

```
THE NUMBER K1 OF SUCCESSES REQUIRED FOR ABSORPTION IS 2
THE NUMBER K2 OF FAILURES REQUIRED FOR ABSORPTION IS 4

THE PROBABILITY THAT
  A SUCCESS IS FOLLOWED BY A SUCCESS IS ALPHA =  .400000
  A FAILURE IS FOLLOWED BY A FAILURE IS BETA =  .100000

THE STATIONARY PROBABILITY OF A SUCCESS PP IS EQUAL TO:
         .60000
AT MOST 150 POINTS OF THE ABSORPTION TIME DISTRIBUTION ARE COMPUTED.

THE CONDITIONAL MEAN ABSORPTION TIME INTO THE SUCCESS RUN AFTER ONE SUCCESS IS
         4.1543
AND AFTER ONE FAILURE IS
         5.2624
THE CONDITIONAL MEAN ABSORPTION TIME INTO THE FAILURE RUN AFTER ONE SUCCESS IS
         7.1543
AND AFTER ONE FAILURE IS
         6.1543
```

THE FOLLOWING IS A TABLE OF THE ABSORPTION PROBABILITIES IN THE SUCCESS-RUN OR IN THE FAILURE RUN, GIVEN THAT ALREADY THE INDICATED NUMBER OF SUCCESSIVE SUCCESSES OR FAILURES HAVE OCCURRED.

-4	-.00000	1.00000	-3	.89865	.10135	-2	.98852	.01148	-1	.99750	.00250
1	.99850	.00150	2	1.00000	.00000						

TABLE OF THE ABSORPTION TIME DISTRIBUTION FOLLOWING A SUCCESS, RESP. A FAILURE.
THE FIRST TWO QUANTITIES ARE THE PROBABILITIES OF ABSORPTION IN THE SUCCESS-RUN GIVEN THAT A SUCCESS, RESP. A FAILURE HAVE OCCURRED.
THE NEXT TWO QUANTITIES ARE THE PROBABILITIES OF ABSORPTION IN THE FAILURE-RUN GIVEN THAT A SUCCESS, RESP. A FAILURE HAVE OCCURRED.

1	0.00000	0.00000	0.00000	0.00000	2	.40000	0.00000	0.00000	0.00000
3	.40000	.36000	0.00000	0.00000	4	.61600	.39600	0.00000	.00100
5	.63760	.59400	.00060	.00100	6	.75640	.63288	.00060	.00154
7	.77973	.74369	.00092	.00159	8	.84621	.77557	.00096	.00189
9	.86534	.83857	.00113	.00195	10	.90314	.86198	.00117	.00212
11	.91719	.89833	.00127	.00216	12	.93900	.91454	.00130	.00226
13	.94873	.93577	.00135	.00229	14	.96146	.94662	.00138	.00235
15	.96797	.95915	.00141	.00237	16	.97549	.96624	.00142	.00240
17	.97975	.97371	.00144	.00242	18	.98423	.97828	.00145	.00244
19	.98697	.98276	.00146	.00245	20	.98966	.98567	.00147	.00246
21	.99140	.98838	.00148	.00247	22	.99303	.99021	.00148	.00247
23	.99413	.99186	.00148	.00248	24	.99511	.99301	.00149	.00248
25	.99581	.99401	.00149	.00249	26	.99641	.99473	.00149	.00249
27	.99684	.99534	.00149	.00249	28	.99721	.99579	.00149	.00249
29	.99748	.99617	.00149	.00249	30	.99770	.99645	.00150	.00249
31	.99787	.99668	.00150	.00249	32	.99801	.99685	.00150	.00249
33	.99811	.99699	.00150	.00249	34	.99820	.99710	.00150	.00250
35	.99826	.99719	.00150	.00250	36	.99831	.99726	.00150	.00250
37	.99835	.99731	.00150	.00250	38	.99839	.99735	.00150	.00250
39	.99841	.99738	.00150	.00250	40	.99843	.99741	.00150	.00250
41	.99845	.99743	.00150	.00250	42	.99846	.99745	.00150	.00250
43	.99847	.99746	.00150	.00250	44	.99847	.99747	.00150	.00250
45	.99848	.99748	.00150	.00250	46	.99849	.99748	.00150	.00250
47	.99849	.99749	.00150	.00250	48	.99849	.99749	.00150	.00250
49	.99849	.99749	.00150	.00250	50	.99850	.99750	.00150	.00250
51	.99850	.99750	.00150	.00250	52	.99850	.99750	.00150	.00250
53	.99850	.99750	.00150	.00250	54	.99850	.99750	.00150	.00250
55	.99850	.99750	.00150	.00250	56	.99850	.99750	.00150	.00250
57	.99850	.99750	.00150	.00250	58	.99850	.99750	.00150	.00250
59	.99850	.99750	.00150	.00250					

Absorption time probabilities

The following table lists the conditional absorption probabilities as follows.

For each value $L = n$ and each given initial state i in the chain R, $-k_2 \leq i \leq k_1$, four numbers are given.

(a) The left-most number is the conditional probability that starting in state i, absorption in the state k_1 (the success-run) occurs no later than time n.

(b) The second number is the conditional probability that, starting in state i, absorption in the state k_1 (the success-run) occurs at time n.

(c) The third number is the conditional probability that, starting in state i, absorption in the state $-k_2$ (the failure-run) occurs no later than time n.

(d) The fourth number is the conditional probability that, starting in state i, absorption in the state $-k_2$ (the failure-run) occurs at time n.

L = 1

-4	0.00000	0.00000	1.00000	0.00000	-3	0.00000	0.00000	0.00000	0.00000
-2	0.00000	0.00000	0.00000	0.00000	-1	0.00000	0.00000	0.00000	0.00000
1	0.00000	0.00000	0.00000	0.00000	2	1.00000	0.00000	0.00000	0.00000

L = 2

-4	0.00000	0.00000	1.00000	0.00000	-3	0.00000	0.00000	.10000	.10000
-2	0.00000	0.00000	0.00000	0.00000	-1	0.00000	0.00000	0.00000	0.00000
1	.40000	.40000	0.00000	0.00000	2	1.00000	0.00000	0.00000	0.00000

L = 3

-4	0.00000	0.00000	1.00000	0.00000	-3	.36000	.36000	.10000	0.00000
-2	.36000	.36000	.01000	.01000	-1	.36000	.36000	0.00000	0.00000
1	.40000	0.00000	0.00000	0.00000	2	1.00000	0.00000	0.00000	0.00000

L = 4

-4	0.00000	0.00000	1.00000	0.00000	-3	.36000	0.00000	.10000	0.00000
-2	.39600	.03600	.01000	0.00000	-1	.39600	.03600	.00100	.00100
1	.61600	.21600	0.00000	0.00000	2	1.00000	0.00000	0.00000	0.00000

L = 5

-4	0.00000	0.00000	1.00000	0.00000	-3	.55440	.19440	.10000	0.00000
-2	.59040	.19440	.01000	0.00000	-1	.59400	.19800	.00100	0.00000
1	.63760	.02160	.00060	.00060	2	1.00000	0.00000	0.00000	0.00000

L = 6

-4	0.00000	0.00000	1.00000	0.00000	-3	.57384	.01944	.10054	.00054
-2	.62928	.03888	.01054	.00054	-1	.63288	.03888	.00154	.00054
1	.75640	.11880	.00060	0.00000	2	1.00000	0.00000	0.00000	0.00000

L = 7

-4	0.00000	0.00000	1.00000	0.00000	-3	.68076	.10692	.10054	0.00000
-2	.73814	.10886	.01059	.00005	-1	.74369	.11081	.00159	.00005
1	.77973	.02333	.00092	.00032	2	1.00000	0.00000	0.00000	0.00000

L = 8

-4	0.00000	0.00000	1.00000	0.00000	-3	.70176	.02100	.10083	.00029
-2	.76983	.03169	.01089	.00029	-1	.77557	.03188	.00189	.00030
1	.84621	.06648	.00096	.00003	2	1.00000	0.00000	0.00000	0.00000

L = 9

-4	0.00000	0.00000	1.00000	0.00000	-3	.76159	.05984	.10086	.00003
-2	.83177	.06194	.01094	.00006	-1	.83857	.06301	.00195	.00006
1	.86534	.01913	.00113	.00018	2	1.00000	0.00000	0.00000	0.00000

L = 10

-4	0.00000	0.00000	1.00000	0.00000	-3	.77881	.01722	.10102	.00016
-2	.85497	.02320	.01111	.00016	-1	.86198	.02341	.00212	.00017
1	.90314	.03780	.00117	.00003	2	1.00000	0.00000	0.00000	0.00000

L = 11

-4	0.00000	0.00000	1.00000	0.00000	-3	.81283	.03402	.10105	.00003
-2	.89071	.03574	.01115	.00005	-1	.89833	.03634	.00216	.00005
1	.91719	.01405	.00127	.00010	2	1.00000	0.00000	0.00000	0.00000

L = 12

-4	0.00000	0.00000	1.00000	0.00000	-3	.82547	.01264	.10114	.00009
-2	.90675	.01604	.01125	.00009	-1	.91454	.01622	.00226	.00009
1	.93900	.02181	.00130	.00003	2	1.00000	0.00000	0.00000	0.00000

L = 13

-4	0.00000	0.00000	1.00000	0.00000	-3	.84510	.01963	.10117	.00003
-2	.92764	.02089	.01128	.00003	-1	.93577	.02123	.00229	.00004
1	.94873	.00973	.00135	.00006	2	1.00000	0.00000	0.00000	0.00000

L = 14

-4	0.00000	0.00000	1.00000	0.00000	-3	.85385	.00876	.10122	.00005
-2	.93836	.01072	.01134	.00005	-1	.94662	.01085	.00235	.00005
1	.96146	.01274	.00138	.00002	2	1.00000	0.00000	0.00000	0.00000

L = 15

-4	0.00000	0.00000	1.00000	0.00000	-3	.86532	.01146	.10124	.00002
-2	.95070	.01234	.01136	.00002	-1	.95915	.01254	.00237	.00002
1	.96797	.00651	.00141	.00003	2	1.00000	0.00000	0.00000	0.00000

L = 16

-4	0.00000	0.00000	1.00000	0.00000	-3	.87117	.00586	.10127	.00003
-2	.95771	.00700	.01139	.00003	-1	.96624	.00709	.00240	.00003
1	.97549	.00752	.00142	.00001	2	1.00000	0.00000	0.00000	0.00000

L = 17

-4	0.00000	0.00000	1.00000	0.00000	-3	.87794	.00677	.10128	.00001
-2	.96506	.00735	.01141	.00002	-1	.97371	.00747	.00242	.00002
1	.97975	.00425	.00144	.00002	2	1.00000	0.00000	0.00000	0.00000

L = 18

-4	0.00000	0.00000	1.00000	0.00000	-3	.88177	.00383	.10130	.00002
-2	.96957	.00451	.01143	.00002	-1	.97828	.00456	.00244	.00002
1	.98423	.00448	.00145	.00001	2	1.00000	0.00000	0.00000	0.00000

L = 19

-4	0.00000	0.00000	1.00000	0.00000	-3	.88581	.00403	.10131	.00001
-2	.97398	.00442	.01144	.00001	-1	.98276	.00448	.00245	.00001
1	.98697	.00274	.00146	.00001	2	1.00000	0.00000	0.00000	0.00000

L = 20

-4	0.00000	0.00000	1.00000	0.00000	-3	.88827	.00246	.10132	.00001
-2	.97685	.00287	.01145	.00001	-1	.98567	.00291	.00246	.00001
1	.98966	.00269	.00147	.00001	2	1.00000	0.00000	0.00000	0.00000

L = 21

-4	0.00000	0.00000	1.00000	0.00000	-3	.89069	.00242	.10132	.00001
-2	.97952	.00267	.01145	.00001	-1	.98838	.00271	.00247	.00001
1	.99140	.00174	.00148	.00001	2	1.00000	0.00000	0.00000	0.00000

L = 22

-4	0.00000	0.00000	1.00000	0.00000	-3	.89226	.00157	.10133	.00001
-2	.98133	.00181	.01146	.00001	-1	.99021	.00184	.00247	.00001
1	.99303	.00162	.00148	.00000	2	1.00000	0.00000	0.00000	0.00000

L = 23

-4	0.00000	0.00000	1.00000	0.00000	-3	.89372	.00146	.10133	.00000
-2	.98295	.00162	.01147	.00000	-1	.99186	.00164	.00248	.00000
1	.99413	.00110	.00148	.00000	2	1.00000	0.00000	0.00000	0.00000

L = 24

-4	0.00000	0.00000	1.00000	0.00000	-3	.89471	.00099	.10134	.00000
-2	.98409	.00114	.01147	.00000	-1	.99301	.00115	.00248	.00000
1	.99511	.00099	.00149	.00000	2	1.00000	0.00000	0.00000	0.00000

L = 25

-4	0.00000	0.00000	1.00000	0.00000	-3	.89560	.00089	.10134	.00000
-2	.98507	.00099	.01147	.00000	-1	.99401	.00100	.00249	.00000
1	.99581	.00069	.00149	.00000	2	1.00000	0.00000	0.00000	0.00000

L = 26

-4	0.00000	0.00000	1.00000	0.00000	-3	.89623	.00062	.10134	.00000
-2	.98579	.00071	.01147	.00000	-1	.99473	.00072	.00249	.00000
1	.99641	.00060	.00149	.00000	2	1.00000	0.00000	0.00000	0.00000

L = 27

-4	0.00000	0.00000	1.00000	0.00000	-3	.89677	.00054	.10134	.00000
-2	.98639	.00060	.01148	.00000	-1	.99534	.00061	.00249	.00000
1	.99684	.00043	.00149	.00000	2	1.00000	0.00000	0.00000	0.00000

L = 28

-4	0.00000	0.00000	1.00000	0.00000	-3	.89716	.00039	.10134	.00000
-2	.98683	.00044	.01148	.00000	-1	.99579	.00045	.00249	.00000
1	.99721	.00037	.00149	.00000	2	1.00000	0.00000	0.00000	0.00000

L = 29

-4	0.00000	0.00000	1.00000	0.00000	-3	.89749	.00033	.10134	.00000
-2	.98720	.00037	.01148	.00000	-1	.99617	.00037	.00249	.00000
1	.99748	.00027	.00149	.00000	2	1.00000	0.00000	0.00000	0.00000

L = 30

-4	0.00000	0.00000	1.00000	0.00000	-3	.89773	.00024	.10135	.00000
-2	.98748	.00028	.01148	.00000	-1	.99645	.00028	.00249	.00000
1	.99770	.00022	.00150	.00000	2	1.00000	0.00000	0.00000	0.00000

L = 31

-4	0.00000	0.00000	1.00000	0.00000	-3	.89793	.00020	.10135	.00000
-2	.98770	.00023	.01148	.00000	-1	.99668	.00023	.00249	.00000
1	.99787	.00017	.00150	.00000	2	1.00000	0.00000	0.00000	0.00000

L = 32

-4	0.00000	0.00000	1.00000	0.00000	-3	.89808	.00015	.10135	.00000
-2	.98788	.00017	.01148	.00000	-1	.99685	.00017	.00249	.00000
1	.99801	.00014	.00150	.00000	2	1.00000	0.00000	0.00000	0.00000

L = 33

-4	0.00000	0.00000	1.00000	0.00000	-3	.89821	.00012	.10135	.00000
-2	.98802	.00014	.01148	.00000	-1	.99699	.00014	.00249	.00000
1	.99811	.00010	.00150	.00000	2	1.00000	0.00000	0.00000	0.00000

L = 34

-4	0.00000	0.00000	1.00000	0.00000	-3	.89830	.00009	.10135	.00000
-2	.98812	.00011	.01148	.00000	-1	.99710	.00011	.00250	.00000
1	.99820	.00008	.00150	.00000	2	1.00000	0.00000	0.00000	0.00000

L = 35

-4	0.00000	0.00000	1.00000	0.00000	-3	.89838	.00008	.10135	.00000
-2	.98821	.00009	.01148	.00000	-1	.99719	.00009	.00250	.00000
1	.99826	.00006	.00150	.00000	2	1.00000	0.00000	0.00000	0.00000

L = 36

-4	0.00000	0.00000	1.00000	0.00000	-3	.89844	.00006	.10135	.00000
-2	.98827	.00007	.01148	.00000	-1	.99726	.00007	.00250	.00000
1	.99831	.00005	.00150	.00000	2	1.00000	0.00000	0.00000	0.00000

L = 37

-4	0.00000	0.00000	1.00000	0.00000	-3	.89848	.00005	.10135	.00000
-2	.98833	.00005	.01148	.00000	-1	.99731	.00005	.00250	.00000
1	.99835	.00004	.00150	.00000	2	1.00000	0.00000	0.00000	0.00000

L = 38

-4	0.00000	0.00000	1.00000	0.00000	-3	.89852	.00004	.10135	.00000
-2	.98837	.00004	.01148	.00000	-1	.99735	.00004	.00250	.00000
1	.99839	.00003	.00150	.00000	2	1.00000	0.00000	0.00000	0.00000

L = 39

-4	0.00000	0.00000	1.00000	0.00000	-3	.89855	.00003	.10135	.00000
-2	.98840	.00003	.01148	.00000	-1	.99738	.00003	.00250	.00000
1	.99841	.00002	.00150	.00000	2	1.00000	0.00000	0.00000	0.00000

L = 40

-4	0.00000	0.00000	1.00000	0.00000	-3	.89857	.00002	.10135	.00000
-2	.98842	.00003	.01148	.00000	-1	.99741	.00003	.00250	.00000
1	.99843	.00002	.00150	.00000	2	1.00000	0.00000	0.00000	0.00000

L = 41

-4	0.00000	0.00000	1.00000	0.00000	-3	.89859	.00002	.10135	.00000
-2	.98844	.00002	.01148	.00000	-1	.99743	.00002	.00250	.00000
1	.99845	.00002	.00150	.00000	2	1.00000	0.00000	0.00000	0.00000

L = 42

-4	0.00000	0.00000	1.00000	0.00000	-3	.89860	.00001	.10135	.00000
-2	.98846	.00002	.01148	.00000	-1	.99745	.00002	.00250	.00000
1	.99846	.00001	.00150	.00000	2	1.00000	0.00000	0.00000	0.00000

L = 43

-4	0.00000	0.00000	1.00000	0.00000	-3	.89861	.00001	.10135	.00000
-2	.98847	.00001	.01148	.00000	-1	.99746	.00001	.00250	.00000
1	.99847	.00001	.00150	.00000	2	1.00000	0.00000	0.00000	0.00000

L = 44

-4	0.00000	0.00000	1.00000	0.00000	-3	.89862	.00001	.10135	.00000
-2	.98848	.00001	.01148	.00000	-1	.99747	.00001	.00250	.00000
1	.99847	.00001	.00150	.00000	2	1.00000	0.00000	0.00000	0.00000

L = 45

-4	0.00000	0.00000	1.00000	0.00000	-3	.89863	.00001	.10135	.00000
-2	.98849	.00001	.01148	.00000	-1	.99748	.00001	.00250	.00000
1	.99848	.00001	.00150	.00000	2	1.00000	0.00000	0.00000	0.00000

L = 46

-4	0.00000	0.00000	1.00000	0.00000	-3	.89863	.00001	.10135	.00000
-2	.98850	.00001	.01148	.00000	-1	.99748	.00001	.00250	.00000
1	.99849	.00000	.00150	.00000	2	1.00000	0.00000	0.00000	0.00000

L = 47

-4	0.00000	0.00000	1.00000	0.00000	-3	.89864	.00000	.10135	.00000
-2	.98850	.00000	.01148	.00000	-1	.99749	.00000	.00250	.00000
1	.99849	.00000	.00150	.00000	2	1.00000	0.00000	0.00000	0.00000

L = 48

-4	0.00000	0.00000	1.00000	0.00000	-3	.89864	.00000	.10135	.00000
-2	.98850	.00000	.01148	.00000	-1	.99749	.00000	.00250	.00000
1	.99849	.00000	.00150	.00000	2	1.00000	0.00000	0.00000	0.00000

L = 49

-4	0.00000	0.00000	1.00000	0.00000	-3	.89864	.00000	.10135	.00000
-2	.98851	.00000	.01148	.00000	-1	.99749	.00000	.00250	.00000
1	.99849	.00000	.00150	.00000	2	1.00000	0.00000	0.00000	0.00000

L = 50

-4	0.00000	0.00000	1.00000	0.00000	-3	.89864	.00000	.10135	.00000
-2	.98851	.00000	.01148	.00000	-1	.99750	.00000	.00250	.00000
1	.99850	.00000	.00150	.00000	2	1.00000	0.00000	0.00000	0.00000

L = 51

-4	0.00000	0.00000	1.00000	0.00000	-3	.89865	.00000	.10135	.00000
-2	.98851	.00000	.01148	.00000	-1	.99750	.00000	.00250	.00000
1	.99850	.00000	.00150	.00000	2	1.00000	0.00000	0.00000	0.00000

L = 52

-4	0.00000	0.00000	1.00000	0.00000	-3	.89865	.00000	.10135	.00000
-2	.98851	.00000	.01148	.00000	-1	.99750	.00000	.00250	.00000
1	.99850	.00000	.00150	.00000	2	1.00000	0.00000	0.00000	0.00000

L = 53

-4	0.00000	0.00000	1.00000	0.00000	-3	.89865	.00000	.10135	.00000
-2	.98851	.00000	.01148	.00000	-1	.99750	.00000	.00250	.00000
1	.99850	.00000	.00150	.00000	2	1.00000	0.00000	0.00000	0.00000

L = 54

-4	0.00000	0.00000	1.00000	0.00000	-3	.89865	.00000	.10135	.00000
-2	.98851	.00000	.01148	.00000	-1	.99750	.00000	.00250	.00000
1	.99850	.00000	.00150	.00000	2	1.00000	0.00000	0.00000	0.00000

L = 55

-4	0.00000	0.00000	1.00000	0.00000	-3	.89865	.00000	.10135	.00000
-2	.98851	.00000	.01148	.00000	-1	.99750	.00000	.00250	.00000
1	.99850	.00000	.00150	.00000	2	1.00000	0.00000	0.00000	0.00000

L = 56

-4	0.00000	0.00000	1.00000	0.00000	-3	.89865	.00000	.10135	.00000
-2	.98852	.00000	.01148	.00000	-1	.99750	.00000	.00250	.00000
1	.99850	.00000	.00150	.00000	2	1.00000	0.00000	0.00000	0.00000

L = 57

-4	0.00000	0.00000	1.00000	0.00000	-3	.89865	.00000	.10135	.00000
-2	.98852	.00000	.01148	.00000	-1	.99750	.00000	.00250	.00000
1	.99850	.00000	.00150	.00000	2	1.00000	0.00000	0.00000	0.00000

L = 58

-4	0.00000	0.00000	1.00000	0.00000	-3	.89865	.00000	.10135	.00000
-2	.98852	.00000	.01148	.00000	-1	.99750	.00000	.00250	.00000
1	.99850	.00000	.00150	.00000	2	1.00000	0.00000	0.00000	0.00000

L = 59

-4	0.00000	0.00000	1.00000	0.00000	-3	.89865	.00000	.10135	.00000
-2	.98852	.00000	.01148	.00000	-1	.99750	.00000	.00250	.00000
1	.99850	.00000	.00150	.00000	2	1.00000	0.00000	0.00000	0.00000

```
THE NUMBER K1 OF SUCCESSES REQUIRED FOR ABSORPTION IS 3
THE NUMBER K2 OF FAILURES REQUIRED FOR ABSORPTION IS 3

THE PROBABILITY THAT
  A SUCCESS IS FOLLOWED BY A SUCCESS IS ALPHA =  .750000
  A FAILURE IS FOLLOWED BY A FAILURE IS BETA =  .250000

THE STATIONARY PROBABILITY OF A SUCCESS PP IS EQUAL TO:
          .75000
AT MOST 150 POINTS OF THE ABSORPTION TIME DISTRIBUTION ARE COMPUTED.

THE CONDITIONAL MEAN ABSORPTION TIME INTO THE SUCCESS RUN AFTER ONE SUCCESS IS
          3.8278
AND AFTER ONE FAILURE IS
          5.0278
THE CONDITIONAL MEAN ABSORPTION TIME INTO THE FAILURE RUN AFTER ONE SUCCESS IS
          5.2564
AND AFTER ONE FAILURE IS
          3.8278
```

THE FOLLOWING IS A TABLE OF THE ABSORPTION PROBABILITIES IN THE SUCCESS-RUN OR IN THE FAILURE RUN, GIVEN THAT ALREADY THE INDICATED NUMBER OF SUCCESSIVE SUCCESSES OR FAILURES HAVE OCCURRED.

-3	0.00000	1.00000	-2	.71523	.28477	-1	.89404	.10596
1	.95364	.04636	2	.97351	.02649	3	1.00000	0.00000

TABLE OF THE ABSORPTION TIME DISTRIBUTION FOLLOWING A SUCCESS, RESP. A FAILURE.
THE FIRST TWO QUANTITIES ARE THE PROBABILITIES OF ABSORPTION IN THE SUCCESS-RUN GIVEN THAT A SUCCESS, RESP. A FAILURE HAVE OCCURRED.
THE NEXT TWO QUANTITIES ARE THE PROBABILITIES OF ABSORPTION IN THE FAILURE-RUN GIVEN THAT A SUCCESS, RESP. A FAILURE HAVE OCCURRED.

1	0.00000	0.00000	0.00000	0.00000	2	0.00000	0.00000	0.00000	0.00000
3	.56250	0.00000	0.00000	.06250	4	.56250	.42188	.01563	.06250
5	.66797	.52734	.02734	.07422	6	.77344	.60645	.03027	.08594
7	.81299	.70532	.03540	.09033	8	.85254	.75476	.03870	.09473
9	.88344	.79184	.04062	.09816	10	.90198	.82243	.04230	.10022
11	.91658	.84213	.04346	.10184	12	.92724	.85655	.04425	.10303
13	.93454	.86729	.04485	.10384	14	.93993	.87476	.04528	.10444
15	.94381	.88017	.04558	.10487	16	.94656	.88409	.04580	.10517
17	.94855	.88688	.04596	.10539	18	.94999	.88890	.04607	.10555
19	.95101	.89034	.04615	.10567	20	.95175	.89138	.04621	.10575
21	.95229	.89213	.04625	.10581	22	.95267	.89267	.04628	.10585
23	.95294	.89305	.04630	.10588	24	.95314	.89333	.04632	.10590
25	.95328	.89353	.04633	.10592	26	.95338	.89367	.04634	.10593
27	.95346	.89378	.04634	.10594	28	.95351	.89385	.04635	.10595
29	.95355	.89390	.04635	.10595	30	.95357	.89394	.04635	.10595
31	.95359	.89397	.04635	.10595	32	.95361	.89399	.04635	.10596
33	.95362	.89400	.04636	.10596	34	.95362	.89401	.04636	.10596
35	.95363	.89402	.04636	.10596	36	.95363	.89403	.04636	.10596
37	.95364	.89403	.04636	.10596	38	.95364	.89403	.04636	.10596
39	.95364	.89403	.04636	.10596	40	.95364	.89404	.04636	.10596
41	.95364	.89404	.04636	.10596	42	.95364	.89404	.04636	.10596
43	.95364	.89404	.04636	.10596	44	.95364	.89404	.04636	.10596

L = 1

-3	0.00000	0.00000	1.00000	0.00000
-1	0.00000	0.00000	0.00000	0.00000
1	0.00000	0.00000	0.00000	0.00000
3	1.00000	0.00000	0.00000	0.00000
-2	0.00000	0.00000	0.00000	0.00000
2	0.00000	0.00000	0.00000	0.00000

L = 2

-3	0.00000	0.00000	1.00000	0.00000
-1	0.00000	0.00000	0.00000	0.00000
1	0.00000	0.00000	0.00000	0.00000
3	1.00000	0.00000	0.00000	0.00000
-2	0.00000	0.00000	.25000	.25000
2	.75000	.75000	0.00000	0.00000

L = 3

-3	0.00000	0.00000	1.00000	0.00000
-1	0.00000	0.00000	.06250	.06250
1	.56250	.56250	0.00000	0.00000
3	1.00000	0.00000	0.00000	0.00000
-2	0.00000	0.00000	.25000	0.00000
2	.75000	0.00000	0.00000	0.00000

L = 4

-3	0.00000	0.00000	1.00000	0.00000
-1	.42188	.42188	.06250	0.00000
1	.56250	0.00000	.01563	.01563
3	1.00000	0.00000	0.00000	0.00000
-2	.42188	.42188	.25000	0.00000
2	.75000	0.00000	.01563	.01563

L = 5

-3	0.00000	0.00000	1.00000	0.00000
-1	.52734	.10547	.07422	.01172
1	.66797	.10547	.02734	.01172
3	1.00000	0.00000	0.00000	0.00000
-2	.42188	0.00000	.26172	.01172
2	.85547	.10547	.01563	0.00000

L = 6

-3	0.00000	0.00000	1.00000	0.00000
-1	.60645	.07910	.08594	.01172
1	.77344	.10547	.03027	.00293
3	1.00000	0.00000	0.00000	0.00000
-2	.50098	.07910	.27051	.00879
2	.88184	.02637	.01855	.00293

L = 7

-3	0.00000	0.00000	1.00000	0.00000
-1	.70532	.09888	.09033	.00439
1	.81299	.03955	.03540	.00513
3	1.00000	0.00000	0.00000	0.00000
-2	.58008	.07910	.27271	.00220
2	.90161	.01978	.02148	.00293

L = 8

-3	0.00000	0.00000	1.00000	0.00000
-1	.75476	.04944	.09473	.00439
1	.85254	.03955	.03870	.00330
3	1.00000	0.00000	0.00000	0.00000
-2	.60974	.02966	.27655	.00385
2	.92633	.02472	.02258	.00110

L = 9

-3	0.00000	0.00000	1.00000	0.00000	-2	.63940	.02966	.27902	.00247
-1	.79184	.03708	.09816	.00343					
1	.88344	.03090	.04062	.00192	2	.93869	.01236	.02368	.00110
3	1.00000	0.00000	0.00000	0.00000					

L = 10

-3	0.00000	0.00000	1.00000	0.00000	-2	.66258	.02317	.28046	.00144
-1	.82243	.03059	.10022	.00206					
1	.90198	.01854	.04230	.00168	2	.94796	.00927	.02454	.00086
3	1.00000	0.00000	0.00000	0.00000					

L = 11

-3	0.00000	0.00000	1.00000	0.00000	-2	.67648	.01390	.28173	.00126
-1	.84213	.01970	.10184	.00162					
1	.91658	.01460	.04346	.00116	2	.95561	.00765	.02505	.00051
3	1.00000	0.00000	0.00000	0.00000					

L = 12

-3	0.00000	0.00000	1.00000	0.00000	-2	.68743	.01095	.28259	.00087
-1	.85655	.01443	.10303	.00118					
1	.92724	.01066	.04425	.00079	2	.96053	.00492	.02546	.00041
3	1.00000	0.00000	0.00000	0.00000					

L = 13

-3	0.00000	0.00000	1.00000	0.00000	-2	.69543	.00800	.28319	.00059
-1	.86729	.01073	.10384	.00081					
1	.93454	.00730	.04485	.00060	2	.96414	.00361	.02576	.00030
3	1.00000	0.00000	0.00000	0.00000					

L = 14

-3	0.00000	0.00000	1.00000	0.00000	-2	.70090	.00547	.28364	.00045
-1	.87476	.00747	.10444	.00060					
1	.93993	.00539	.04528	.00042	2	.96682	.00268	.02596	.00020
3	1.00000	0.00000	0.00000	0.00000					

L = 15

-3	0.00000	0.00000	1.00000	0.00000	-2	.70494	.00404	.28396	.00032
-1	.88017	.00541	.10487	.00043					
1	.94381	.00388	.04558	.00030	2	.96869	.00187	.02611	.00015
3	1.00000	0.00000	0.00000	0.00000					

L = 16

-3	0.00000	0.00000	1.00000	0.00000	-2	.70785	.00291	.28418	.00023
-1	.88409	.00392	.10517	.00031					
1	.94656	.00275	.04580	.00022	2	.97004	.00135	.02622	.00011
3	1.00000	0.00000	0.00000	0.00000					

L = 17

-3	0.00000	0.00000	1.00000	0.00000	-2	.70992	.00207	.28435	.00017
-1	.88688	.00279	.10539	.00022					
1	.94855	.00199	.04596	.00016	2	.97102	.00098	.02629	.00008
3	1.00000	0.00000	0.00000	0.00000					

L = 18

-3	0.00000	0.00000	1.00000	0.00000	-2	.71142	.00150	.28447	.00012
-1	.88890	.00201	.10555	.00016					
1	.94999	.00143	.04607	.00011	2	.97172	.00070	.02635	.00006
3	1.00000	0.00000	0.00000	0.00000					

L = 19

-3	0.00000	0.00000	1.00000	0.00000	-2	.71249	.00108	.28455	.00008
-1	.89034	.00145	.10567	.00011					
1	.95101	.00103	.04615	.00008	2	.97222	.00050	.02639	.00004
3	1.00000	0.00000	0.00000	0.00000					

L = 20

-3	0.00000	0.00000	1.00000	0.00000	-2	.71326	.00077	.28461	.00006
-1	.89138	.00104	.10575	.00008					
1	.95175	.00074	.04621	.00006	2	.97259	.00036	.02642	.00003
3	1.00000	0.00000	0.00000	0.00000					

L = 21

-3	0.00000	0.00000	1.00000	0.00000	-2	.71382	.00055	.28466	.00004
-1	.89213	.00075	.10581	.00006					
1	.95229	.00053	.04625	.00004	2	.97285	.00026	.02644	.00002
3	1.00000	0.00000	0.00000	0.00000					

L = 22

-3	0.00000	0.00000	1.00000	0.00000	-2	.71421	.00040	.28469	.00003
-1	.89267	.00054	.10585	.00004					
1	.95267	.00038	.04628	.00003	2	.97303	.00019	.02645	.00001
3	1.00000	0.00000	0.00000	0.00000					

L = 23

-3	0.00000	0.00000	1.00000	0.00000	-2	.71450	.00029	.28471	.00002
-1	.89305	.00039	.10588	.00003					
1	.95294	.00027	.04630	.00002	2	.97317	.00013	.02646	.00001
3	1.00000	0.00000	0.00000	0.00000					

L = 24

-3	0.00000	0.00000	1.00000	0.00000	-2	.71471	.00021	.28473	.00002
-1	.89333	.00028	.10590	.00002					
1	.95314	.00020	.04632	.00002	2	.97326	.00010	.02647	.00001
3	1.00000	0.00000	0.00000	0.00000					

L = 25

-3	0.00000	0.00000	1.00000	0.00000	-2	.71485	.00015	.28474	.00001
-1	.89353	.00020	.10592	.00002					
1	.95328	.00014	.04633	.00001	2	.97333	.00007	.02648	.00001
3	1.00000	0.00000	0.00000	0.00000					

L = 26

-3	0.00000	0.00000	1.00000	0.00000	-2	.71496	.00011	.28475	.00001
-1	.89367	.00014	.10593	.00001					
1	.95338	.00010	.04634	.00001	2	.97338	.00005	.02648	.00000
3	1.00000	0.00000	0.00000	0.00000					

L = 27

-3	0.00000	0.00000	1.00000	0.00000	-2	.71504	.00008	.28475	.00001
-1	.89378	.00010	.10594	.00001					
1	.95346	.00007	.04634	.00001	2	.97342	.00004	.02648	.00000
3	1.00000	0.00000	0.00000	0.00000					

L = 28

-3	0.00000	0.00000	1.00000	0.00000	-2	.71509	.00005	.28476	.00000
-1	.89385	.00007	.10595	.00001					
1	.95351	.00005	.04635	.00000	2	.97344	.00003	.02648	.00000
3	1.00000	0.00000	0.00000	0.00000					

L = 29

-3	0.00000	0.00000	1.00000	0.00000	-2	.71513	.00004	.28476	.00000
-1	.89390	.00005	.10595	.00000					
1	.95355	.00004	.04635	.00000	2	.97346	.00002	.02649	.00000
3	1.00000	0.00000	0.00000	0.00000					

L = 30

-3	0.00000	0.00000	1.00000	0.00000	-2	.71516	.00003	.28476	.00000
-1	.89394	.00004	.10595	.00000					
1	.95357	.00003	.04635	.00000	2	.97348	.00001	.02649	.00000
3	1.00000	0.00000	0.00000	0.00000					

L = 31

-3	0.00000	0.00000	1.00000	0.00000	-2	.71518	.00002	.28476	.00000
-1	.89397	.00003	.10595	.00000					
1	.95359	.00002	.04635	.00000	2	.97349	.00001	.02649	.00000
3	1.00000	0.00000	0.00000	0.00000					

L = 32

-3	0.00000	0.00000	1.00000	0.00000	-2	.71519	.00001	.28477	.00000
-1	.89399	.00002	.10596	.00000					
1	.95361	.00001	.04635	.00000	2	.97349	.00001	.02649	.00000
3	1.00000	0.00000	0.00000	0.00000					

L = 33

-3	0.00000	0.00000	1.00000	0.00000	-2	.71520	.00001	.28477	.00000
-1	.89400	.00001	.10596	.00000					
1	.95362	.00001	.04636	.00000	2	.97350	.00000	.02649	.00000
3	1.00000	0.00000	0.00000	0.00000					

L = 34

-3	0.00000	0.00000	1.00000	0.00000	-2	.71521	.00001	.28477	.00000
-1	.89401	.00001	.10596	.00000					
1	.95362	.00001	.04636	.00000	2	.97350	.00000	.02649	.00000
3	1.00000	0.00000	0.00000	0.00000					

L = 35

-3	0.00000	0.00000	1.00000	0.00000	-2	.71522	.00001	.28477	.00000
-1	.89402	.00001	.10596	.00000					
1	.95363	.00001	.04636	.00000	2	.97350	.00000	.02649	.00000
3	1.00000	0.00000	0.00000	0.00000					

L = 36

-3	0.00000	0.00000	1.00000	0.00000	-2	.71522	.00000	.28477	.00000
-1	.89403	.00001	.10596	.00000					
1	.95363	.00000	.04636	.00000	2	.97351	.00000	.02649	.00000
3	1.00000	0.00000	0.00000	0.00000					

L = 37

-3	0.00000	0.00000	1.00000	0.00000	-2	.71522	.00000	.28477	.00000
-1	.89403	.00000	.10596	.00000					
1	.95364	.00000	.04636	.00000	2	.97351	.00000	.02649	.00000
3	1.00000	0.00000	0.00000	0.00000					

L = 38

-3	0.00000	0.00000	1.00000	0.00000	-2	.71523	.00000	.28477	.00000
-1	.89403	.00000	.10596	.00000					
1	.95364	.00000	.04636	.00000	2	.97351	.00000	.02649	.00000
3	1.00000	0.00000	0.00000	0.00000					

L = 39

-3	0.00000	0.00000	1.00000	0.00000	-2	.71523	.00000	.28477	.00000
-1	.89403	.00000	.10596	.00000					
1	.95364	.00000	.04636	.00000	2	.97351	.00000	.02649	.00000
3	1.00000	0.00000	0.00000	0.00000					

L = 40

-3	0.00000	0.00000	1.00000	0.00000	-2	.71523	.00000	.28477	.00000
-1	.89404	.00000	.10596	.00000					
1	.95364	.00000	.04636	.00000	2	.97351	.00000	.02649	.00000
3	1.00000	0.00000	0.00000	0.00000					

L = 41

-3	0.00000	0.00000	1.00000	0.00000	-2	.71523	.00000	.28477	.00000
-1	.89404	.00000	.10596	.00000					
1	.95364	.00000	.04636	.00000	2	.97351	.00000	.02649	.00000
3	1.00000	0.00000	0.00000	0.00000					

L = 42

-3	0.00000	0.00000	1.00000	0.00000	-2	.71523	.00000	.28477	.00000
-1	.89404	.00000	.10596	.00000					
1	.95364	.00000	.04636	.00000	2	.97351	.00000	.02649	.00000
3	1.00000	0.00000	0.00000	0.00000					

L = 43

-3	0.00000	0.00000	1.00000	0.00000	-2	.71523	.00000	.28477	.00000
-1	.89404	.00000	.10596	.00000					
1	.95364	.00000	.04636	.00000	2	.97351	.00000	.02649	.00000
3	1.00000	0.00000	0.00000	0.00000					

L = 44

-3	0.00000	0.00000	1.00000	0.00000	-2	.71523	.00000	.28477	.00000
-1	.89404	.00000	.10596	.00000					
1	.95364	.00000	.04636	.00000	2	.97351	.00000	.02649	.00000
3	1.00000	0.00000	0.00000	0.00000					