SOME ASPECTS OF SEARCH STRATEGIES FOR WIENER PROCESSES

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1. *Introduction.* The problem we wish to consider here is the same search problem considered by Posner and Rumsey, [2]. Our purpose here is to point out some serious errors in their optimality arguments and to discuss some aspects of the search problem which they did not consider.

A brief description of the search problem follows. Let $y_1(t),\ldots, y_n(t)$ be n Wiener processes each with variance $\sigma^2 t$; $n-1$ of them have zero drift and the remaining process has drift μt where μ is known. Our problem is to locate the process with drift μt with probability $1-\varepsilon$ of correct selection. In addition, we are given a prior distribution $p_1, p_2, \ldots, p_n$ where p_i is the probability that the ith process is the correct one.

In Section 2 we discuss specifically the difficulty with Posner and Rumsey's argument for optimality. They used weak limits of the class of lattice time strategies for which they claimed optimality and weak limits of another class of strategies called δ perturbed strategies, for which computations were more tractable, to determine the "optimal expected

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search time". We show that neither the class of lattice time nor the class of δ perturbed strategies are tight and hence weak limits do not exist.

The δ perturbed strategies are defined as follows. Observe the process corresponding to the largest prior probability until for the first time the posterior probability has decreased by δ/n, then observe the process with the maximum posterior probability at that time. We call the process which is being searched the *target* and the act of changing targets a *switch*.

Section 3 discusses the expected search time and the expected number of switches of another class of strategies called τ strategies. These strategies are shown to have the same limiting expected search time as the δ-perturbed strategies.

Section 4 discusses the merits of the two types of strategies and points out the simplicity of the τ strategy.

2. *Lack of tightness.* In this section we consider the two classes of strategies which were considered by Posner and Rumsey. We show that it is not possible to consider the weak limits of strategies in these classes which Posner and Rumsey studied since these limits do not exist.

It is necessary to begin with some definitions and structure for the problem. Basic to the situation is a probability space (Ω, A, P) on which are defined the n Wiener processes $y_1(t), \ldots, y_n(t)$ for $t \geq 0$, discussed in Section 1. The strategies which we discuss are functionals of $y_1(t), \ldots, y_n(t)$ whose value at any time t denotes the subscript of that Wiener process which is observed at time t. For example, the discrete time strategy $i_d(\cdot)$ is a functional which is constant over intervals $[kd, (k+1)d)$, $k = 0,1,2,\ldots$.

To study these strategies we choose to use the space $D[0,1]$ of all right continuous real valued functions on $[0,1]$ which have only discontinuities of the first kind. That will be our space of sample functions, the probability measures which we consider on $D[0,1]$ will be those induced by (Ω, A, P) through the functionals i.e. $[i_d(t_0) = k]$ is an event in $D[0,1]$ and also determines an event in Ω whose P probability we assign to that event. Consideration of the strategies as determined for $t \in [0,1]$ is sufficient for our purposes. Since the space $D[0,1]$ contains functions whose discontinuities are only of the first kind we will refer to a discontinuity as a *jump*.

We will show that the sequences of strategies considered by Posner and Rumsey do not converge in the Skorohod D topology, (see [1], p. 109 ff.). Since this topology gives a complete separable metric on the space $D[0,1]$, tightness of the measures is a necessary and sufficient condition for weak convergence (Prohorov's theorem).

Let $\tau_n = (0=t_{0n} < t_{1n} < \ldots < 1)$ be a sequence of partitions of the real line with the time increment going to zero (for simplicity, we assume that each partition is a refinement of its predecessor). We choose the sequence $t_n = t_{1n} \to 0$ (the number 0 plays no special role here). The following lemma reduces our problem to calculating the probability of the set of paths whose first discontinuity is at time t_n.

Lemma 2.1. *Let* $t_n \to 0$ *and let* A_n *be the collection of sample paths which are* 0 *for* $t < t_n$ *and which have jumps* *on* $[t_n, t_{n-1})$. *Let* P_n *be a sequence of probability measures on* $D[0,1]$ *for which* $P_n(A_n) \geq c > 0$ *for* $n \geq N$. *Then the sequence* P_n *is not tight.*

Before proving the lemma, we introduce some concepts and notations of the D topology. These particulars may be found e.g. in [1] p. 109 ff. Let $i(t)$ denote the sample functions of $D[0,1]$. We define a modulus similar to the modulus of continuity. For $0 < \delta < 1$,

$$w_i'(\delta) = \inf_{\{t_k\}} \max_{0<k<r} \sup_{s,t} \{|i(s)-i(t)| \; s,t \in [t_{i-1}, t_k)\}$$

where the infimum extends over all finite sets $\{t_k\}$ of points satisfying the condition

$$0 = t_0 < t_1 < \ldots < t_r = 1$$
$$t_k - t_{k-1} > \delta \qquad k = 1,2,\ldots,r \quad .$$

The following are necessary and sufficient conditions for tightness (cf. [1] p. 125):

(i) for each $\eta > 0$, there is an a such that

$$P_n\{i: \sup_t |i(t)| > a\} \leq \eta \qquad n \geq 1$$

(ii) for each positive ε and η, there exists δ, $0 < \delta < 1$ and an integer n_0 such that

$$P_n\{i: w_i'(\delta) \geq \varepsilon\} \leq \eta \qquad n \geq n_0.$$

Proof of the lemma. Condition (i) is always satisfied since there are only a finite number of processes being searched. We show that (ii) fails. Indeed, for each sample path i belonging to A_n, an easy computation shows that $w_i'(\delta) \geq 1$ for $\delta > t_n$. Therefore, for any $\delta > 0$ and $n \geq \max(N, 1/\delta)$ we have $P_n\{w_i'(\delta) \geq 1\} \geq c$ which contradicts (ii).

To show that the sequence i_{τ_n} is not tight, it remains to show that $P_n(A_n) \geq c$. This is the content of the next lemma.

Lemma 2.2. *Under the hypotheses of* Lemma 2.1, $\lim_n P_n(A_n) = 1/2$.

Proof. The probability of a jump at t_n may be computed from the posterior distribution of j being correct given j is being searched.

$$p_j(t_n) = \frac{p_j(0)}{p_j(0) + (1-p_j(0)) \exp(\frac{\mu}{2\sigma^2}(\mu t_n - 2y(t)))} .$$

The rule of searching the most likely process at t_n translates (see Posner [2]) into the rule that a switch occurs at time t_n if and only if the likelihood ratio

$$Z(t_n) = \log \frac{p_j(t_n)}{1 - p_j(t_n)} \geq Z(0) \quad ,$$

which is equivalent to

$$-\frac{\mu}{2\sigma^2}(\mu t_n - 2y(t_n)) \geq 0$$

and focuses our attention to the boundary where

$$y(t_n) = 2\mu t_n \quad .$$

The above process at t_n is a normal random variable with mean $(\mu^2/2\sigma^2)t_n$ if j is correct and mean $-(\mu^2/2\sigma^2)t_n$ if j is not correct. In either case, the variance is $\mu^2 t_n/\sigma^2$. The probability of switching at time t_n is

$$p(\text{switch}|j \text{ correct})\ p(j \text{ correct}) + p(\text{switch}|j \text{ not correct})\ p(j \text{ not correct})$$

which equals

$$\Phi\left(-\frac{(\mu^2/2\sigma^2)t_n}{\mu\sqrt{t_n}/\sigma}\right) p_j(0) + \Phi\left(\frac{(\mu^2/2\sigma^2)t_n}{\mu\sqrt{t_n}/\sigma}\right) (1-p_j(0))$$

$$= p_j(0) + \Phi\left(\frac{\mu\sqrt{t_n}}{2\sigma}\right) - 2p_j(0) \quad \Phi\left(\frac{\mu\sqrt{t_n}}{2\sigma}\right)$$

where $\Phi(x)$ is the cdf of a normal distribution with mean zero and variance one. Therefore,

$$\lim_{t_n \to 0} p(\text{switch at } t_n) = p_j(0) + \frac{1}{2} - p_j(0) = \frac{1}{2} \quad .$$

We note that this is independent of whether j is the correct process or not.

We now turn our attention to the δ perturbed strategies, as defined in [2].

Lemma 2.3. *The δ perturbed strategies are not tight.*

Proof. Let P_n denote the measure on D determined by δ_n. We will produce a sequence δ_n and a corresponding sequence of times t_n such that $P_n(A_n) \geq c$ where A_n is the set of sample paths which are 0 for $t < t_n$ and different from 0 for $t_n \leq t < t_{n-1}$. Lack of tightness will then follow from Lemma 2.1. For simplicity, we omit the subscript 0 from $p_0(t)$.

The strategy i_δ switches if and only if the posterior probability has decreased by an amount $\delta/n = \delta'$ (assume $\delta < 1$). We will compute the probability of the event

$$\{P(t) > P(0) - \delta',\ 0 \leq t \leq (\delta')^4;\ P(s) < P(0) - \delta',$$

$$\text{for some } s \text{ such that } (\delta')^4 \leq s \leq (\delta')^2\}$$

i.e. the probability of a switch between times $(\delta')^4$ and

$(\delta')^2$. For $P(t) < P(0) - \delta'$ we must have

$$\frac{P(0)}{P(0) + (1-P(0))\exp\left[\frac{\mu}{2\sigma^2}(\mu t - 2y(t))\right]} < P(0) - \delta'$$

or equivalently, the target $y(t)$ must satisfy

$$(1) \qquad y(t) > -\frac{\mu t}{2} + \frac{2\sigma^2}{\mu} \log \left(1 + \frac{\delta'}{[1-P(0)][P(0)-\delta']}\right) .$$

Thus the desired probability is that of the first crossing of the boundary in (1) occurring between times $(\delta')^4$ and $(\delta')^2$.

By the result of Shepp [3, p.348] this probability for a target with zero drift is

$$(2) \qquad \begin{aligned} &\Phi\left(-\frac{\mu}{2\sigma}(\delta')^2 + \frac{2\sigma}{\mu(\delta')^2}c(\delta')\right) - e^{c(\delta')}\Phi\left(-\frac{\mu}{2\sigma}(\delta')^2 - \frac{2\sigma}{\mu}\frac{c(\delta')}{(\delta')^2}\right) \\ &-\Phi\left(-\frac{\mu}{2\sigma}\delta' + \frac{2\sigma}{\mu\delta'}c(\delta')\right) + e^{c(\delta')}\Phi\left(-\frac{\mu}{2\sigma}\delta' - \frac{2\sigma}{\mu}\frac{c(\delta')}{\delta'}\right) \end{aligned}$$

where $c(\delta') = \log\left[1 + \frac{\delta'}{(1-P(0))(P(0)-\delta')}\right]$.

Since $c(\delta') = 0(\delta')$ this probability is bounded away from zero as $\delta \to 0$.

Now if δ is any positive number less than 1 and we define the sequence

$$(3) \qquad \delta'_1 = \delta'; \quad \delta'_n = (\delta'_{n-1})^2 \quad \text{for } n \geq 2$$

then the events

$$A_n = \{P(t) > P(0)-\delta_n' \quad \text{for} \quad 0 \leq t \leq (\delta_n')^4 \quad \text{and}$$

(4) $$P(s) < P(0)-\delta' \quad \text{for some} \quad s \quad \text{such that}$$

$$(\delta_n')^4 \leq s \leq (\delta_n')^2\}$$

are pairwise disjoint and have P_n measure bounded away from zero. Similarly if the target has drift μt we can generate the same sequence of sets.

Next we see that the Prokoroff distance cannot go to zero as the necessary and sufficient conditions of Billingsley are violated as shown in Lemma 2.1 by this sequence of times in (3) and subsets A_n found in (4)

3. τ *Strategies*. We deal with a search strategy i_τ which chooses the target with the highest posterior probability at each switching time and which does not allow for switching (selecting a different target) before a specified elapsed time τ. We will define the strategy only for the case of a uniform prior distribution. The results obtained in that case clearly indicate what happens with a non-uniform prior.

This strategy should not be confused with the discrete time strategies considered by Posner and Rumsey. Note that after time τ has elapsed this strategy has random switching times determined by boundary crossing times.

Specifically this strategy is described as follows. Let S' be a switching time or time 0. If no posterior probability is $\geq 1 - \varepsilon$ then we select at random a target, set all posterior probabilities equal to $1/n$. The next possible switching time S'' is determined by the rule:

$$S'' = \begin{cases} S' + \tau & \text{if the target has posterior probability } \le 1/n. \\ s & \text{where } s = \inf\{t : t > S' + \tau \text{ and the target has posterior } \le 1/n \text{ or } \ge 1 - \varepsilon\}. \end{cases}$$

Thus the strategy i_τ requires at least time τ to switch.

We can easily compute the expected time $E(S'' - S')$, that is, the expected time the strategy i_τ searches the same target before switching or terminating the search. Let $a = \ell n(\frac{1-\varepsilon}{\varepsilon})(n-1)$, and $\lambda = \mu^2/2\sigma^2$, for the process with drift, we find

$$E_c(S''-S') = \tau + \int_0^a M(x)\, dP$$

where $M(x)$ is the expected time for the Wiener process to either reach the switching boundary or the terminating boundary from the point x. More explicitly this expression is

$$E_c(S''-S') = \tau - \frac{1}{\lambda(1-e^{-a})} \{ a\psi(-) - (1-e^{-a})(\frac{\lambda\tau}{2}\psi(-) + \sqrt{\frac{\lambda\tau}{2\Pi}}\,[e^{-\frac{\lambda\tau}{8}} - e^{-\frac{(a-\frac{\lambda\tau}{2})^2}{2\lambda\tau}}] - a\psi(+)\}$$

wherein $\psi(-) = \Phi(\frac{a-\frac{\lambda\tau}{2}}{\sqrt{\lambda\tau}}) - \Phi(-\frac{\sqrt{\lambda\tau}}{2})$,

$$\psi(+) = \Phi(\frac{a+\frac{\lambda\tau}{2}}{\sqrt{\lambda\tau}}) - \Phi(\frac{\sqrt{\lambda\tau}}{2}) ,$$

Φ is the standard normal c.d.f. and $\lambda = \frac{\mu^2}{\sigma^2}$.

When the target process has zero drift this is

$$E_{inc}(S''-S') = \tau - \frac{1}{\lambda(1-e^{a})} \{a(\psi(+)-\psi(-)) - (1-e^{a})(- \frac{\lambda\tau}{2}\psi(+) + \sqrt{\frac{\lambda\tau}{2\Pi}} [e^{-\frac{\lambda\tau}{8}} - e^{-\frac{(a+\frac{\lambda\tau}{2})^2}{2\lambda\tau}}])\}.$$

Similarly the probability of switching before stopping is

$$q_c = \int_{-\infty}^{0} \frac{\phi(\frac{x-\frac{\lambda\tau}{2}}{\sqrt{\lambda\tau}})}{\sqrt{\lambda\tau}} dx + \int_0^a \frac{(e^x - e^{-a})}{(1 - e^{-a})} \frac{\phi(\frac{x-\frac{\lambda\tau}{2}}{\sqrt{\lambda\tau}})}{\sqrt{\lambda\tau}} dx$$

for the process with drift and for any process wwth zero drift

$$q_{inc} = \int_{-\infty}^{0} \frac{\phi(\frac{x+\frac{\lambda\tau}{2}}{\sqrt{\lambda\tau}})}{\sqrt{\lambda\tau}} dx + \int_0^a \frac{(e^x - e^{a})}{(1-e^{-a})} \frac{\phi(\frac{x-\frac{\lambda\tau}{2}}{\sqrt{\lambda\tau}})}{\sqrt{\lambda\tau}} dx .$$

From these expressions, the expected search time M and the expected number of switches S must satisfy the relations respectively

$$M = \frac{1}{n} E_c(S'' - S') + \frac{n-1}{n} E_{inc}(S'' - S') + qM$$

and

$$S = 1 + qS = 1 + \frac{1}{n} q_c + \frac{n-1}{n} q_{inc}$$

so that both M and S are found in closed form.

Proposition: *As* $\tau \to 0$ *the expected search time of the* τ *strategy is* $\frac{1}{\lambda} \{(1-2\varepsilon)\ln(\frac{1-\varepsilon}{\varepsilon})(n-1) + (n-2)(\frac{n-1-\varepsilon n}{n-1})\}$ *and the expected number of switches* $S \sim \frac{c}{\sqrt{\tau}}$ *as* $\tau \to 0$.

Proof: Note that all three expressions $E_c(S''-S')$, $E_{inc}(S''-S')$ and $1 - q$ approach zero at the rate $\sqrt{\tau}$ when $\tau \to 0$.

L'Hospital's rule applied to the expression for M gives its limiting value as $\tau \to 0$ and the equation $S = \frac{1}{1-q}$ immediately shows the limit behavior of S.

4. *Comparison of δ perturbed and τ strategies.* An important advantage of the strategy i_τ is its inherent simplicity in implementation. This simplicity results from the fact that one need never compute posterior probabilities. Each switching time is determined by a pair of linear stopping boundaries for the target process and these boundaries remain unchanged throughout the search. For a given ε, n they are simply:

$$\ell(t) = \frac{\mu t}{2} \quad \text{and} \quad u(t) = \frac{\mu t}{2} + \frac{a\sigma^2}{\mu}$$

for the lower and upper boundaries respectively.

As noted by the proposition of Section 3 the strategy i_τ has the same limiting expected search time when $\tau \to 0$ as the strategy i_δ as $\delta \to 0$, c.f. [2]. This is not completely obvious since the strategy i_τ switches targets "infinitely often as $\tau \to 0$" and hence disregards the actual posteriors "infinitely often as $\tau \to 0$".

The question of switching is an important one which has not been previously considered. We now compare the switching behavior of these two strategies.

Proposition: *Let S_τ and S_δ be the expected number of switches for the two strategies i_τ and i_δ respectively. Then for $\delta = \sqrt{\tau}$*

$$\lim_{n \to \infty} \lim_{\delta,\tau \to 0} \frac{S_\delta}{S_\tau} = \frac{1}{\varepsilon}\sqrt{\frac{\lambda}{2\Pi}} \quad .$$

Proof: An expression for S_δ can be obtained by considering the possible events until a first return to the uniform distribution i.e.

$$S_\delta = 1 + q_1(1 + q_2(\ldots + q_n S_\delta)$$

where $q_1, q_2, \ldots, q_n$ are the probabilities of switching the 1st time, 2nd time, etc. Thus

$$S_\delta = 1 + q_1 + q_1 q_2 + \ldots + q_1 q_2 \cdots q_n S_\delta$$

but each of these terms $q_i = 1 - \frac{n\delta\varepsilon}{(n-1)(n-1-n\varepsilon)} + o(\delta)$, [2], so that we can say

$$S_\delta = \frac{1 - q^n}{(1-q)(1-q^n)} + o(\delta)$$

$$= \frac{1}{\frac{n\delta\varepsilon}{(n-1)(n-1-n\varepsilon)}} + o(\delta) \quad .$$

We have previously seen that $S_\tau = \sqrt{\frac{2\Pi}{\lambda}} \; \frac{(n-1)(1-\varepsilon)}{\sqrt{\tau}} + o(\sqrt{\tau})$ and the result follows.

For the sake of comparing some explicit expected search times and expected number of switches, Table I below gives some representative values. Note that appropriate comparison values are for $\delta = \sqrt{\tau}$.

Table 1

Comparisons of δ and τ Strategies

Epsilon = .100 Delta = .0010 N = 10 $TN(1-\varepsilon)$ = 10.6267

Delta	Expected Time	Expected Switches	Tau	Expected Time	Expected Switches
.0010	10.63	35985.95	.000016	10.65	4981.22
.0020	10.63	17985.95	.00002	10.66	3521.96
.0030	10.63	11985.95	.00003	10.67	2875.49
.0040	10.63	8985.95	.00004	10.68	2490.12
.0050	10.63	7185.95	.00005	10.68	2227.12
.0060	10.63	5985.95	.00006	10.69	2032.99
.0070	10.63	5128.81	.00007	10.69	1882.11
.0080	10.63	4485.95	.00008	10.70	1760.49
.0090	10.63	3985.95	.00009	10.70	1659.75
.0100	10.63	3585.95	.0001	10.71	1574.53
.0200	10.63	1785.96	.0002	10.74	1113.08
.0300	10.64	1185.98	.0003	10.76	908.65
.0400	10.64	885.99	.0004	10.79	786.78
.0500	10.65	706.00	.0005	10.80	703.62
.0600	10.66	586.01	.0006	10.82	642.23
.0700	10.66	500.30	.0007	10.84	594.52
.0800	10.67	436.03	.0008	10.85	556.07
.0900	10.68	386.04	.0009	10.87	524.21
.1000	10.69	346.05	.0010	10.88	497.26
.2000	10.82	166.15	.0020	10.98	351.35
.3000	11.03	106.25	.0030	11.07	286.72
.4000	11.34	76.35	.0040	11.13	248.19
.5000	11.80	58.45	.0050	11.20	221.90
			.0060	11.25	202.50
			.0070	11.30	187.42
			.0080	11.35	175.27
			.0100	11.44	156.68
			.0400	12.30	78.04
			.0900	13.22	51.91
			.1600	14.20	38.91
			.2500	15.25	31.17

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