An Asymptotic Distribution Theory and Applications in Multivariate Analysis

by

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ABSTRACT

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This thesis is essentially in three parts. 1) A maximization problem 2) Applications to multivariate distributions and asymptotic expansions 3) Some asymptotic distributions using perturbation theory. In 1), a theory is developed which generalizes the work of earlier authors: Anderson, Chang, James, Li and Pillai. The first chapter presents this theory based on the maximization of an integral over the group of orthogonal matrices, the integrand being a hypergeometric function of symmetric matrix variates. The theory is further extended to the complex case where the group is over unitary matrices and the matrix variates are Hermitian.

In 2), asymptotic distributions are obtained for the first time in the light of the above maximization theory, in a) MANOVA and b) Canonical correlation. The results of the earlier authors were in the case of covariance matrices for c) one-sample or d) two-sample cases. However, while their results were only for distinct population roots or for one extreme multiple population root, chapters II and III present the asymptotic distributions for several multiple population roots for a) to d) in the real and complex cases.

In 3), asymptotic formulae for the cdf and percentiles are derived for the max-U ratio criterion suggested by Pillai for the test of equality of several covariance matrices but studied here for the two-sample case. Chapter IV presents this asymptotic study using perturbation techniques and since Pillai's criterion involves ratios of independent Hotelling's T_0^2's, the work of this chapter generalizes the work of Ito, Siotani and others. Finally chapter V gives a summary.

CHAPTER I

ON THE MAXIMIZATION OF AN INTEGRAL OF A MATRIX FUNCTION OVER THE GROUP OF ORTHOGONAL MATRICES

1.1. Introduction

In multivariate analysis, the distribution of characteristic roots arising in testing the equality of two covariance matrices, in MANOVA, or in the canonical correlation problem, involves the integration of a hypergeometric function of the following form

$$(1.1)\quad I = \int_{O(p)} {}_sF_t(a_1,\ldots, a_s;b_1,\ldots, b_t, \underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}\,\underset{\sim}{H}')\, d(\underset{\sim}{H}),$$

where $O(p)$ is the group of orthogonal matrices $\underset{\sim}{H}(p \times p)$, $\underset{\sim}{A} = \text{diag}(\ell_1,\ldots, \ell_p)$, $\underset{\sim}{R} = \text{diag}(r_1,\ldots, r_p)$, $d(\underset{\sim}{H})$ is the invariant or Haar measure over the group $O(p)$ normalized so that the measure of the whole group $O(p)$ is unity ${}_sF_t$ is a hypergeometric function of matrix variates (James [9]) and $a_1,\ldots, a_s, b_1,\ldots, b_t,$ are functions of d.f. and are positive real numbers. In the one sample (covariance matrix) case, Anderson [1] has shown that the maximum of the integrand $(s=t=0)$ for all possible variation of $\underset{\sim}{R}$, the sample characteristic root matrix, is attained when $\underset{\sim}{H}$ takes a special form. Chang [3], Li and Pillai [13], [14], found the same form for $\underset{\sim}{H}$ when maximizing the integrand in the two sample (two covariance matrices) problem $(s=1, t=0)$. In the complex analogue of both one sample and two sample cases, Li and Pillai, [13], [14], obtained a similar form of the unitary matrix $\underset{\sim}{U}$. The purpose of this paper is to generalize their results both in the complex and real situations with $a_1, \ldots, a_s$, $b_1, \ldots, b_t$, satisfying some suitable conditions.

We develop the idea in a series of lemmas and theorems and show that the results of Anderson [1], Chang [3], James [10], Li and Pillai, [13], [14], are special cases of our results. The generalization has not only been in regard to ${}_sF_t$ - hypergeometric functions but also when ℓ_i's are equal within each of several sets. We have further proved that the integral under different forms of the matrix $\underset{\sim}{A}$ is invariant under choice of different submatrices of $\underset{\sim}{H}$ and our general results cover some earlier ones of the above authors.

1.2. Maximization of Some Special Functions

First we prove the following lemma.

Lemma 1.1. Let $f(\underset{\sim}{T})$ be a real valued function of the elements of the matrix $\underset{\sim}{T}(p \times p) = (t_{ij})$. Then

$$d\, f(\underset{\sim}{T}) = \mathrm{tr}(\underset{\sim}{Q}' d\, \underset{\sim}{T})$$

where

$$\underset{\sim}{Q} = \begin{bmatrix} \frac{\partial f}{\partial t_{11}}, & \cdots & \frac{\partial f}{\partial t_{1p}} \\ & & \\ \frac{\partial f}{\partial t_{p1}}, & \cdots & \frac{\partial f}{\partial t_{pp}} \end{bmatrix} \quad \text{and} \quad d\, \underset{\sim}{T} = \begin{bmatrix} dt_{11} & \cdots & dt_{1p} \\ & & \\ dt_{p1} & \cdots & dt_{pp} \end{bmatrix}.$$

Proof follows directly from the definition. We give below some special cases.

Case 1. If $\underset{\sim}{B}$ be a nonsingular square matrix, then

$$d|\underset{\sim}{B}| = |\underset{\sim}{B}| \operatorname{tr}[\underset{\sim}{B}^{-1}(d \underset{\sim}{B})] \ . \tag{1.2}$$

This is Hsu's result as reported by Deemer and Olkin [5], but we proved in a different way.

Case 2. If in (1.2) we put $\underset{\sim}{B} = \underset{\sim}{I} + \underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}\,\underset{\sim}{H}'$ where

(1.3) $\underset{\sim}{A} = \operatorname{diag}(\ell_1, \ldots, \ell_p)$ and $\underset{\sim}{R} = \operatorname{diag}(r_1, \ldots, r_p)$ and

$\infty > \ell_1 > \ldots > \ell_p > 0,\ \infty > r_p > r_{p-1} > \ldots > r_1 > 0$ and

$\underset{\sim}{H} \in O(p)$

then we get lemma 1 of [3].

Case 3. Let us now take $f(\underset{\sim}{T})$ in lemma 1.1 to be of the following form

$$f(\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}\,\underset{\sim}{H}') = \exp[-\operatorname{tr}(\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}\,\underset{\sim}{H}')] \tag{1.4}$$

where $\underset{\sim}{A}$, $\underset{\sim}{R}$ and $\underset{\sim}{H}$ satisfy (1.3), and $\underset{\sim}{H}$ is the only variable matrix.

By lemma 1.1

$$d\, f(\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}\,\underset{\sim}{H}') = \operatorname{tr}[\underset{\sim}{Q}' d(\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}\,\underset{\sim}{H}')] \ .$$

But $\underset{\sim}{Q}$ in this case is a non-zero scalar matrix. Hence,

$$\begin{aligned} & d\, f(\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}\,\underset{\sim}{H}') = 0 \\ \Rightarrow\ & \operatorname{tr}[d\,(\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}\,\underset{\sim}{H}')] = 0 \\ \Rightarrow\ & \operatorname{tr}[\underset{\sim}{A}(d\underset{\sim}{H})\underset{\sim}{R}\,\underset{\sim}{H}' + \underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}(d\underset{\sim}{H}')] = 0 \\ \Rightarrow\ & 2\operatorname{tr}[\underset{\sim}{R}\,\underset{\sim}{H}'\underset{\sim}{A}\,\underset{\sim}{H}(\underset{\sim}{H}'d\underset{\sim}{H})] = o \end{aligned} \tag{1.5}$$

But $(\underset{\sim}{H}'d\underset{\sim}{H})$ is a skew symmetric matrix. Hence for all $\underset{\sim}{R} > 0$

$$(1.5) \qquad \Rightarrow \underset{\sim}{R}\,\underset{\sim}{H}'\underset{\sim}{A}\,\underset{\sim}{H} \text{ is symmetric}$$

$$\Rightarrow \underset{\sim}{R}\,\underset{\sim}{H}'\underset{\sim}{A}\,\underset{\sim}{H} = \underset{\sim}{H}'\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}$$

$$\Rightarrow \underset{\sim}{H}'\underset{\sim}{A}\,\underset{\sim}{H} = \text{diag}(\mu_1, \ldots, \mu_p)$$

as $\underset{\sim}{R}$ in (1.3) is diagonal with distinct roots

$$\Rightarrow \underset{\sim}{H} \text{ has the form}$$

(i) $\underset{\sim}{H}$ has ± 1 in each row and column once and once only and zero elsewhere.

Now taking $\underset{\sim}{H}$ of the form (i) after some algebra gives Anderson's result [1].

In the above two corollaries, the functions we have considered though not exactly special forms of the integrand in (1.1) but are equivalent forms. Hence the parallel results in both the cases suggest a similar approach for this general integral (1.1) but unfortunately attempt in this direction proved futile. Hence we give an alternative approach to handle this general problem and give special results as occasions arise.

1.3. Maximization of I when ℓ_i's are all distinct

At first we prove a lemma which will be used in the sequel. Let $\underset{\sim}{S}(p \times p)$ be a symmetric matrix, $Ch_i(\underset{\sim}{S})$ denote the ith characteristic root of $\underset{\sim}{S}$ and $C_\kappa(\underset{\sim}{S})$ stand for the zonal polynomial of the matrix $\underset{\sim}{S}$ corresponding to the partition κ as defined by James [9]. Then we state the following lemma.

Lemma 1.2. Let $ch_i(\underset{\sim}{S}) \geq 0$, $i = 1,\ldots, p$. Then $C_\kappa(\underset{\sim}{S})$ is nonnegative and increasing in each characteristic root of $\underset{\sim}{S}$.

This may be shown by using the differential equation given by James [11], since a) the coefficients of all terms of a zonal polynomial when expressed in terms of monomial symmetric functions of the characteristic roots of the matrix are positive and b) zonal polynomials are themselves symmetric functions of the characteristic roots.

Now let us consider the integrand in (1.1), i.e. let

$$(1.6) \qquad f(\underset{\sim}{H}) = {}_sF_t\,(a_1,\ldots, a_s;\, b_1,\ldots, b_t,\, \underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}\,\underset{\sim}{H}') \ .$$

Also let

$$(1.7) \qquad a_i \geq (1/2)(p-1),\ b_j \geq (1/2)(p-1),\ i = 1,\ldots, s,\ j = 1,\ldots, t \ .$$

Now, by James [9]

$$f(\underset{\sim}{H}) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_\kappa \cdots (a_s)_\kappa}{(b_1)_\kappa \cdots (b_t)_\kappa}\ \frac{C_\kappa(\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}\,\underset{\sim}{H}')}{k!} ,$$

where $\kappa = (k_1,\ldots, k_p)$ is a partition of k and the multivariate hypergeometric coefficient $(a)_\kappa$ is given by

$$(a)_\kappa = \prod_{i=1}^{p} (a - (1/2)(i-1))_{k_i}$$

and

$$(a)_k = a(a+1) \ldots (a+k-1) \ .$$

Under (1.7)

$$(1.8) \quad \max_{\underset{\sim}{H}\varepsilon O(p)} f(\underset{\sim}{H}) = \max_{\underset{\sim}{H}\varepsilon O(p)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_\kappa \cdots (a_s)_\kappa}{(b_1)_\kappa \cdots (b_t)_\kappa} \frac{C_\kappa(\underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{R}\ \underset{\sim}{H}')}{k!}$$

$$\leq \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_\kappa \cdots (a_s)_\kappa}{(b_1)_\kappa \cdots (b_t)_\kappa} \max_{\underset{\sim}{H}\varepsilon O(p)} \frac{C_\kappa(\underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{R}\ \underset{\sim}{H}')}{k!} .$$

Now to proceed any further we have to consider the maximization problem involved in (1.8). To this end we proceed as follows.

For $\underset{\sim}{A}$ and $\underset{\sim}{R}$ in (1.1) let us take (unlike in (1.3))

$$(1.9) \quad \infty > \ell_1 > \ell_2 > \cdots > \ell_p \geq 0 \text{ and } \infty > r_1 > r_2 > \cdots > r_p > 0 .$$

(The ordering and labeling of ℓ_i's and r_i's will be done in different ways as may be necessary. $\underset{\sim}{A}$ and $\underset{\sim}{R}$ are more or less used in a generic sense in order to avoid the use of too many symbols.)

Let us consider $C_\kappa(\underset{\sim}{H}'\underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{R}) = C_\kappa(\underset{\sim}{H}\ \underset{\sim}{R}\ \underset{\sim}{H}'\underset{\sim}{A})$, where $C_\kappa(\underset{\sim}{Z})$ is the zonal polynomial corresponding to the partition κ as defined by James [9] and $\underset{\sim}{H}\varepsilon O(p)$. Let $Ch_i(\underset{\sim}{Z})$ denote the ith characteristic root of $\underset{\sim}{Z}$.

Then

$$\begin{aligned} Ch_1(\underset{\sim}{H}\ \underset{\sim}{R}\ \underset{\sim}{H}'\underset{\sim}{A}) &\leq Ch_1(\underset{\sim}{H}\ \underset{\sim}{R}\ \underset{\sim}{H}')\ Ch_1(\underset{\sim}{A}) \\ &= Ch_1(\underset{\sim}{R})\ Ch_1(\underset{\sim}{A}) \\ &= Ch_1(\underset{\sim}{R}\ \underset{\sim}{A}) \\ &= r_1\ell_1 . \end{aligned}$$

Now if we take

$$(1.10) \qquad H = \begin{bmatrix} \pm 1 & 0 \\ & \\ 0 & H_2 \end{bmatrix},$$

where $H_2(p-1)$ is an orthogonal matrix of order $p-1$, then

$$Ch_1(H\ R\ H'A) = Ch_1 \begin{pmatrix} r_1\ell_1 & 0 \\ 0 & B \end{pmatrix} = r_1\ell_1$$

where

$$B = H_2\ R_2\ H_2'\ A_2$$

and

$$(1.11) \qquad R_2 = \text{diag}(r_2, \ldots, r_p),\ A_2 = \text{diag}(\ell_2, \ldots, \ell_p)\ .$$

Again let

$$(1.12) \qquad S^2 = R \quad \text{and} \quad S = \text{diag}(s_1, \ldots, s_p)\ .$$

Hence we consider the matrix

$$S\ H'\ A\ H\ S = C = (c_{ij}) \quad \text{(say)}\ .$$

Let

$$(1.13) \qquad H = (h_{ij})\ .$$

Then

$$\underset{\sim}{C} = (c_{ij}) = (s_i s_j \sum_{k=1}^{p} h_{ki}\, h_{kj}\, \ell_k) .$$

Let us now take $\underset{\sim}{e}_i' = (0,\ldots, 0,1,0,\ldots,0)$, $i = 1,\ldots, p$ (in the standard notation, i.e. 1 in the ith place and zero elsewhere).

Then

$$\underset{\sim}{e}_1' \underset{\sim}{C} \underset{\sim}{e}_1 = r_1 \sum_{k=1}^{p} h_{k1}^2 \ \ell_k < r_1 \ell_1 \quad ,$$

(1.14)

$$\text{iff} \qquad (h_{21},\ldots, h_{p1}) \neq 0 .$$

Also $\quad \underset{\sim}{e}_i' \underset{\sim}{C} \underset{\sim}{e}_i = r_i \sum_{k=1}^{p} h_{ki}^2\, \ell_k \leq r_i \ell_1,\ i = 2,\ldots, p .$

Let

$$\text{(1.15)} \qquad \underset{\sim}{X}' = (x_1,\ldots, x_p) \ni \underset{\sim}{X}' \underset{\sim}{X} = 1 .$$

Then

$$\text{(1.16)} \quad |\underset{\sim}{X}' \underset{\sim}{C} \underset{\sim}{X}| \leq \sum_{i,j} |x_i|\ |x_j|\ |c_{ij}|$$

$$= \sum_i |x_i|^2 |c_{ii}| + \sum_{i \neq j} |x_i|\ |x_j|\ |c_{ij}|$$

$$\leq \sum_i |x_i|^2 |c_{ii}| + \sum_{i \neq j} |x_i|\ |x_j| \sqrt{|c_{ii}|\ |c_{jj}|}$$

$$= \left(\sum_{i=1}^{p} |x_i| \sqrt{|c_{ii}|} \right)^2$$

$$\leq \ell_1 \left(\sum_{i=1}^{p} |x_i| \sqrt{r_i} \right)^2$$

$$= r_1 \ell_1 \left(|x_1| + |x_2| \sqrt{\frac{r_2}{r_1}} + \ldots + |x_p| \sqrt{\frac{r_2}{r_1}} \right)^2 .$$

Now for further reference we quote a theorem.

<u>Theorem</u> (Courant-Fischer). Let $\underset{\sim}{D}(p \times p)$ be a symmetric matrix with characteristic roots $\lambda_1 \geq \ldots \geq \lambda_p$. Then they may be defined as

$$\lambda_1 = \max_{\underset{\sim}{X}} (\underset{\sim}{X}' \underset{\sim}{D} \underset{\sim}{X}) / (\underset{\sim}{X}' \underset{\sim}{X}) ,$$

$$\lambda_j = \min_{\substack{(\underset{\sim}{Y}_i' \underset{\sim}{Y}_i) = 1 \\ (i=1,\ldots, j-1)}} \ \max_{(\underset{\sim}{X}' \underset{\sim}{Y}_i) = 0} (\underset{\sim}{X}' \underset{\sim}{D} \underset{\sim}{X}) / (\underset{\sim}{X}' \underset{\sim}{X})$$

$$j = 2,\ldots, p$$

or equivalently

$$\lambda_p = \min_{\underset{\sim}{X}} (\underset{\sim}{X}' \underset{\sim}{D} \underset{\sim}{X}) / (\underset{\sim}{X}' \underset{\sim}{X}) ,$$

$$\lambda_j = \max_{\substack{(\underset{\sim}{Y}_i' \underset{\sim}{Y}_i) = 1 \\ i=1,\ldots, j-1}} \ \min_{(\underset{\sim}{X}' \underset{\sim}{Y}_i) = 0} (\underset{\sim}{X}' \underset{\sim}{D} \underset{\sim}{X}) / (\underset{\sim}{X}' \underset{\sim}{X})$$

$$j = 1,\ldots, p-1 ,$$

where $\underset{\sim}{X}, \underset{\sim}{Y}_i$ are column vectors in E_p, the Euclidean space of p dimensions.

Now if $\underset{\sim}{R} > 0$ is such that r_i/r_1 is negligibly small $(i = 2,\ldots, p)$ then applying the above theorem and from (1.16), we get for all $\underset{\sim}{R} > 0$

$$\max_{\underset{\sim}{H}\varepsilon O(p)} Ch_1(\underset{\sim}{H}\,\underset{\sim}{R}\,\underset{\sim}{A}\,\underset{\sim}{H}') = \ell_1 r_1$$

iff equality holds in (1.14) i.e. iff

(1.17) $$(h_{21},\ldots, h_{p1}) = 0 \ .$$

Now since $\underset{\sim}{H}$ is orthogonal

(1.17) $$\Rightarrow h_{11} = \pm 1 \ .$$

Thus for all $\underset{\sim}{R} > 0$

$$\max_{\underset{\sim}{H}\varepsilon O(p)} Ch_1(\underset{\sim}{H}\,\underset{\sim}{R}\,\underset{\sim}{H}'\,\underset{\sim}{A}) = \ell_1 r_1$$

iff

$\underset{\sim}{H}$ has the form (1.10) .

If $\underset{\sim}{H}$ has the form (1.10)

(1.18) $$\underset{\sim}{S}\,\underset{\sim}{H}\,\underset{\sim}{A}\,\underset{\sim}{H}'\,\underset{\sim}{S} = \begin{bmatrix} r_1\ell_1 & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{B}_1 \end{bmatrix}$$

where

$$\underset{\sim}{B}_1 = \underset{\sim}{S}_2\underset{\sim}{H}_2'\underset{\sim}{A}_2\underset{\sim}{H}_2\underset{\sim}{S}_2 \quad \text{and} \quad \underset{\sim}{S}_2 = \text{diag}\,(s_2,\ldots, s_p) \ .$$

Thus characteristic vector corresponding to the root $r_1\ell_1$ is proportional to $\underset{\sim}{e}_1$ and hence any vector belonging to E_p, Euclidean space of p dimensions, and orthogonal to $\underset{\sim}{e}_1$ is generated by $\underset{\sim}{e}_2,\ldots,\underset{\sim}{e}_p$. Thus when $\underset{\sim}{H}$ has the form (1.10), the problem of finding the second maximum characteristic root of (1.18) simply reduces to finding the maximum characteristic root of $\underset{\sim}{B}_1$ for $\underset{\sim}{H}_2 \varepsilon O(p-1)$. Thus we proceed step by step as before, only that we are now dealing with matrices of one less dimension. We also note the following

$$
\begin{aligned}
(1.19)\qquad Ch_i(\underset{\sim}{H}\,\underset{\sim}{R}\,\underset{\sim}{H}'\,\underset{\sim}{A}) &\leq Ch_1(\underset{\sim}{A})\; Ch_i(\underset{\sim}{H}\,\underset{\sim}{R}\,\underset{\sim}{H}') \\
&= Ch_1(\underset{\sim}{A})\; Ch_i(\underset{\sim}{R}) \\
&= \ell_1 r_i \qquad i = 1,\ldots,p\ .
\end{aligned}
$$

Thus from the above discussion and from (1.19), using the fact that zonal polynomials are symmetric functions of the characteristic roots of the matrix and the monotonicity property of zonal polynomial as proved in lemma 1.2, we get the following lemma.

Lemma 1.3. When (1.9) holds, for all $\underset{\sim}{R} > 0$

$$
(1.20)\qquad \max_{\underset{\sim}{H}\varepsilon O(p)} C_\kappa(\underset{\sim}{H}'\,\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}) = \max_{\underset{\sim}{H}\varepsilon O(p)} C_\kappa(\underset{\sim}{H}\,\underset{\sim}{R}\,\underset{\sim}{H}'\,\underset{\sim}{A}) = C_\kappa(\underset{\sim}{A}\,\underset{\sim}{R})\ .
$$

If $\underset{\sim}{A}$ is as in (1.1) and

$$
(1.21)\qquad \infty > \ell_p > \ell_{p-1} > \ldots > \ell_1 > 0, \text{ then}
$$

$$
(1.22)\qquad \min_{\underset{\sim}{H}\varepsilon O(p)} C_\kappa(\underset{\sim}{H}'\,\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}) = \min_{\underset{\sim}{H}\varepsilon O(p)} C_\kappa(\underset{\sim}{H}\,\underset{\sim}{R}\,\underset{\sim}{H}'\,\underset{\sim}{A}) = C_\kappa(\underset{\sim}{A}\,\underset{\sim}{R})\ ,
$$

and

$$(1.23) \quad \max_{\underset{\sim}{H} \varepsilon O(p)} C_\kappa(\underset{\sim}{H}' \underset{\sim}{A} \underset{\sim}{H} \underset{\sim}{R}) = \max_{\underset{\sim}{H} \varepsilon O(p)} C_\kappa(\underset{\sim}{H} \underset{\sim}{R} \underset{\sim}{H}' \underset{\sim}{A}) = C_\kappa(\underset{\sim}{A} \underset{\sim}{E}) ,$$

where

$$\underset{\sim}{E} = \text{diag } (r_p, \ldots, r_1) .$$

The maximum in (1.20) and minimum in (1.22) are attained when and only when $\underset{\sim}{H}$ has the form

$$\text{(ii)} \quad \underset{\sim}{H}(p \times p) = \text{diag } (\pm 1, \ldots, \pm 1) .$$

In (1.23) $\underset{\sim}{H}$ has the form: $\underset{\sim}{H} = \underset{\sim}{H}_1 \underset{\sim}{D}$ where $\underset{\sim}{H}_1$ has the form (ii) and $\underset{\sim}{D} = (\underset{\sim}{e}_p, \ldots, \underset{\sim}{e}_1)$.

In proving (1.22) regarding the minimum value of the zonal polynomial we used the following relations:

$$\begin{aligned}
(1.24) \quad Ch_i(\underset{\sim}{H}' \underset{\sim}{A} \underset{\sim}{H} \underset{\sim}{R}) &= Ch_i(\underset{\sim}{H} \underset{\sim}{R} \underset{\sim}{H}' \underset{\sim}{A}) \\
&\geq Ch_i(\underset{\sim}{H} \underset{\sim}{R} \underset{\sim}{H}') \, Ch_p(\underset{\sim}{A}) \\
&= Ch_i(\underset{\sim}{R}) \, Ch_p(\underset{\sim}{A}) \\
&= r_i \ell_p, \quad i = 1, \ldots, p ,
\end{aligned}$$

and we proceed exactly as in the maximization problem but in this case using the maxi-mini characterization of Courant-Fischer theorem. In case of (1.23) we note that as $\underset{\sim}{H} \varepsilon O(p)$, $\underset{\sim}{H}\underset{\sim}{D} \varepsilon O(p)$ where $\underset{\sim}{D} = (\underset{\sim}{e}_p, \ldots, \underset{\sim}{e}_1)$ and $\underset{\sim}{e}_i$'s are defined earlier.

Also the mapping $\underset{\sim}{H} \to \underset{\sim}{H}\,\underset{\sim}{D}$ is one-to-one and onto and hence in (1.23) instead of considering $C_\kappa(\underset{\sim}{H}\,\underset{\sim}{R}\,\underset{\sim}{H}'\underset{\sim}{A})$ as $\underset{\sim}{H}\varepsilon O(p)$ we consider $C_\kappa((\underset{\sim}{H}\,\underset{\sim}{D})\,\underset{\sim}{R}\,(\underset{\sim}{H}\,\underset{\sim}{D})'\underset{\sim}{A})$ as $\underset{\sim}{H}\varepsilon O(p)$ or equivalently we consider

$$C_\kappa(\underset{\sim}{H}\,\underset{\sim}{E}\,\underset{\sim}{H}'\,\underset{\sim}{A}) \quad \text{as} \quad \underset{\sim}{H}\ \varepsilon\ O(p) \quad \text{where}$$

$$\underset{\sim}{E} = \underset{\sim}{D}\,\underset{\sim}{R}\,\underset{\sim}{D}'\ .$$

In the above discussions we note that if $\underset{\sim}{A}$ and $\underset{\sim}{R}$ has the same ordering of the element then $\underset{\sim}{A}\,\underset{\sim}{R}$ corresponds to the maximization problem and when their ordering is reversed it gives the minimum formulation.

Thus we use (1.20) in (1.8) and we get the following theorem.

Theorem 1.1. If $\underset{\sim}{A}$ and $\underset{\sim}{R}$ are as given in (1.9), the class of orthogonal matrices for which $f(\underset{\sim}{H})$ in (1.6) subject to (1.7) and for all $\underset{\sim}{R} > 0$ is a maximum, is given by (ii) and

$$(1.25) \qquad \max_{\underset{\sim}{H}\varepsilon O(p)} f(\underset{\sim}{H}) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_\kappa \cdots (a_s)_\kappa}{(b_1)_\kappa \cdots (b_t)_\kappa} \frac{C_\kappa(\underset{\sim}{A}\,\underset{\sim}{R})}{k!}\ .$$

If the ordering of ℓ_i's in (1.9) is replaced by (1.21)

$$(1.26) \qquad \min_{\underset{\sim}{H}\varepsilon O(p)} f(\underset{\sim}{H}) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_\kappa \cdots (a_s)_\kappa}{(b_1)_\kappa \cdots (b_t)_\kappa} \frac{C_\kappa(\underset{\sim}{A}\,\underset{\sim}{R})}{k!}$$

with $\underset{\sim}{H}$ again taking the same form (ii).

This is one of the basic results in the chapter and we will subsequently generalize it to more complex cases. But first we give some special cases.

Corollary 1.1.1 If $s=t=0$ in (1.6) then

$$f(\underset{\sim}{H}) = {}_0F_0(\underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{R}\ \underset{\sim}{H}') = \exp[\operatorname{tr} \underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{R}\ \underset{\sim}{H}'] ,$$

and under (1.21) and $\underset{\sim}{R}$ as in (1.9), we get for all $\underset{\sim}{R} > 0$

$$\max_{\underset{\sim}{H}\in O(p)} f(\underset{\sim}{H}) = \exp[\operatorname{tr} \underset{\sim}{E}\ \underset{\sim}{A}] ,$$

where

$$\underset{\sim}{E} = \operatorname{diag}(r_p, \ldots, r_1) .$$

This is Anderson's result [1] mentioned earlier as case 3.

As a second application of our theorem 1.1, let us consider the following. Let

$$g(\underset{\sim}{H}) = |\underset{\sim}{I} + \underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{R}\ \underset{\sim}{H}'|^{-n} = {}_1F_0(n, -\underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{R}\ \underset{\sim}{H}') .$$

where $\underset{\sim}{A}$ and $\underset{\sim}{R}$ are as defined in (1.21) and (1.9) respectively and $n > (1/2)(p-1)$.

As it stands, theorem 1.1 is not directly applicable to this function. So we write, following Khatri [12],

$$|\underset{\sim}{I} + \underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{R}\ \underset{\sim}{H}'| = |\underset{\sim}{I} + \underset{\sim}{R}|\ |\underset{\sim}{I} - (\underset{\sim}{I} - \underset{\sim}{A})\ \underset{\sim}{H}\ \underset{\sim}{R}(\underset{\sim}{I} + \underset{\sim}{R})^{-1}\ \underset{\sim}{H}'| .$$

We now assume $Ch_i(\underset{\sim}{A}) < 1$, $i = 1, \ldots, p$. This is no loss of generality, since for $k > 0$, $|\underset{\sim}{I} + k\ \underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{R}\ \underset{\sim}{H}'| = \prod_{i=1}^{p} (1 + k\alpha_i)$ where $\alpha_i = Ch_i(\underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{R}\ \underset{\sim}{H}') > 0$, $i = 1, \ldots, p$. Thus the problem of finding the maximum or minimum of $|\underset{\sim}{I} + \underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{R}\ \underset{\sim}{H}'|$ with respect to $\underset{\sim}{H}\in O(p)$ is the same as that of $|\underset{\sim}{I} + k\ \underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{R}\ \underset{\sim}{H}'|$. Thus

$$|\underset{\sim}{I} + \underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}\,\underset{\sim}{H}'|^{-n} = |\underset{\sim}{I} + \underset{\sim}{R}|^{-n}|\underset{\sim}{I} - (\underset{\sim}{I} - \underset{\sim}{A})\,\underset{\sim}{H}\,\underset{\sim}{R}\,(\underset{\sim}{I} + \underset{\sim}{R})^{-1}\underset{\sim}{H}'|^{-n}$$

$$= |\underset{\sim}{I} + \underset{\sim}{R}|^{-n}\ {}_1F_0(n,\ \underset{\sim}{B}\,\underset{\sim}{H}\,\underset{\sim}{C}\,\underset{\sim}{H}')\ ,$$

where

$$\underset{\sim}{B} = (\underset{\sim}{I} - \underset{\sim}{A}) = \mathrm{diag}(b_1, \ldots, b_p)\ ,$$

and

$$\underset{\sim}{C} = \underset{\sim}{R}(\underset{\sim}{I} + \underset{\sim}{R})^{-1} = \mathrm{diag}(c_1, \ldots, c_p)\ .$$

Hence from (1.21) and (1.9) we get

$$1 > b_1 > \ldots > b_p > 0 \quad \text{and} \quad 1 > c_1 > c_2 > \ldots > c_p > 0\ .$$

Thus $g(\underset{\sim}{H}) = |\underset{\sim}{I} + \underset{\sim}{R}|^{-n}\ {}_1F_0(n,\ \underset{\sim}{B}\,\underset{\sim}{H}\,\underset{\sim}{C}\,\underset{\sim}{H}')$ and now we can apply the theorem 1.1 and get the following corollary.

Corollory 1.1.2. Under the conditions stated immediately above

$$\max_{\underset{\sim}{H}\varepsilon O(p)} g(\underset{\sim}{H}) = |\underset{\sim}{I} + \underset{\sim}{R}|^{-n} \max_{\underset{\sim}{H}\varepsilon O(p)}\ {}_1F_0(n,\ \underset{\sim}{B}\,\underset{\sim}{H}\,\underset{\sim}{C}\,\underset{\sim}{H}')$$

$$= |\underset{\sim}{I} + \underset{\sim}{R}|^{-n}\ {}_1F_0(n,\ \underset{\sim}{B}\,\underset{\sim}{C})$$

$$= |\underset{\sim}{I} + \underset{\sim}{A}\,\underset{\sim}{R}|^{-n}\ .$$

This corresponds to Chang's result [3]. We now restate the above two results in a different form.

Corollary 1.1.3. Let (1.3) hold. Then $\sum_{i=1}^{p} \ell_i r_{i_j}$ and $\prod_{i=1}^{p} (1+\ell_i r_{i_j})$ are both minimized when $r_{i_j} = r_i$, $i = 1, \ldots, p$. They are both maximized when $r_{i_j} = r_{p-i+1}$, $i = 1, \ldots, p$.

The latter two results are implicitly assumed in Anderson [1] and Chang [3].

In fact, we can go a step further and get the following. Let f be a non-negative, non-decreasing function defined on $[0,\infty]$. Then

$$f(\sum_{i=1}^{p} \ell_i r_{i_j}) \leq f(\sum_{i=1}^{p} \ell_i r_i) \text{ and } f(\prod_{i=1}^{p} (1+\ell_i r_{i_j})) \leq f(\prod_{i=1}^{p} (1+\ell_i r_i))$$

under (1.9). These results follow directly from the above discussion but are mentioned separately since they cover a broader ground in the sense that with modification, the results apply to positive convex combinations of two symmetric matrix functions.

1.4. Maximization of I when the ℓ_i's are equal in set

To this end let us consider the following form for $\underset{\sim}{A}$.

$$(1.27) \quad \underset{\sim}{A} = \mathrm{diag}(\overbrace{\ell_1,\ldots,\ell_1}^{k_1}, \overbrace{\ell_2,\ldots,\ell_2}^{k_2}, \overbrace{\ell_i,\ldots,\ell_i}^{k_i}, \ell_{k_1+\ldots+k_i+1},\ldots,\ell_p)$$

and $\infty > \ell_1 > \ell_2 > \ldots > \ell_i > \ell_{k_1+\ldots+k_i+1} > \ell_p \geq 0$ and $\underset{\sim}{R}$ is as given in (1.9).

For the sake of simplicity in presentation we consider the case when $i = 1$, and $k_1 = 2$, i.e. let

$$(1.28) \quad \underset{\sim}{A} = \mathrm{diag}(\ell_1,\ell_1,\ell_3,\ldots,\ell_p), \text{ and } \infty > \ell_1 > \ell_3 > \ldots > \ell_p \geq 0 .$$

This is no loss of generality, as will be seen from our discussion - the more general case corresponding to (1.27) is a straight forward generalization of the same technique.

Proceeding exactly as in the earlier case of all unequal roots we get,

$$(1.29) \qquad \underset{\sim}{e}_1' \, \underset{\sim}{C} \, \underset{\sim}{e}_1 = r_1 \sum_{k=1}^{p} h_{k1}^2 \, \ell_k < r_1 \ell_1$$

iff

$$(h_{31}, \ldots, h_{p1}) \neq 0 ,$$

where

$$\underset{\sim}{C} = \underset{\sim}{S} \, \underset{\sim}{H}' \, \underset{\sim}{A} \, \underset{\sim}{H} \, \underset{\sim}{S} \quad \text{and} \quad \underset{\sim}{S}, \underset{\sim}{H} \quad \text{as}$$

defined earlier and $\underset{\sim}{A}$ as in (1.28)

$$\text{i.e.} \qquad \underset{\sim}{S}^2 = \underset{\sim}{R} = \operatorname{diag}(r_1, \ldots, r_p) .$$

Hence proceeding exactly as in the earlier case we get for all $\underset{\sim}{R} > 0$

$$Ch_1(\underset{\sim}{S} \, \underset{\sim}{H}' \, \underset{\sim}{A} \, \underset{\sim}{H} \, \underset{\sim}{S}) < r_1 \ell_1 .$$

Now equality holds in (1.29)

$$\text{i.e.} \qquad Ch_1(\underset{\sim}{S} \, \underset{\sim}{H}' \, \underset{\sim}{A} \, \underset{\sim}{H} \, \underset{\sim}{S}) = r_1 \ell_1$$

$$(1.30) \qquad \text{iff} \quad (h_{31}, \ldots, h_{p1}) = 0 .$$

Now when (1.30) is satisfied we get by actual matrix multiplication

$$(1.31) \qquad \underset{\sim}{S}\,\underset{\sim}{H}'\,\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{S} = \begin{bmatrix} r_1\ell_1 & \underset{\sim}{0} \\ & \\ \underset{\sim}{0} & \underset{\sim}{G}_2 \end{bmatrix},$$

where $\underset{\sim}{G}_2 = \underset{\sim}{S}_2\,\underset{\sim}{H}_2'\,\underset{\sim}{A}_2\,\underset{\sim}{H}_2\,\underset{\sim}{S}_2$, $\underset{\sim}{S}_2$, $\underset{\sim}{H}_2$, $\underset{\sim}{A}_2$ are defined earlier and $\underset{\sim}{H}_2 \varepsilon O(p-1)$

As is clear from (1.31), the characteristic vector corresponding to the root $r_1\ell_1$ is proportional to $\underset{\sim}{e}_1$ and hence any vector belonging to E_p - the Euclidean p-space and orthogonal to $\underset{\sim}{e}_1$ is generated by $\underset{\sim}{e}_2, \ldots, \underset{\sim}{e}_p$.

Now the problem of finding the $Ch_2(\underset{\sim}{S}\,\underset{\sim}{H}'\,\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{S})$ when (1.30) holds is reduced to finding $Ch_1(\underset{\sim}{G}_2)$.

Again

$$(1.32) \qquad \begin{aligned} Ch_1(\underset{\sim}{G}_2) &= Ch_1(\underset{\sim}{H}_2'\,\underset{\sim}{A}_2\,\underset{\sim}{H}_2\,\underset{\sim}{R}_2) \\ &= Ch_1(\underset{\sim}{H}_2\,\underset{\sim}{R}_2\,\underset{\sim}{H}_2'\,\underset{\sim}{A}_2) \\ &\leq Ch_1(\underset{\sim}{A}_2)\;Ch_1(\underset{\sim}{H}_2\underset{\sim}{R}_2\underset{\sim}{H}_2') \\ &= Ch_1(\underset{\sim}{A}_2)\;Ch_1(\underset{\sim}{R}_2) \\ &= \ell_1 r_2\,. \end{aligned}$$

Proceeding exactly as earlier we get that equality in (1.32) is achieved for all variations of $\underset{\sim}{R}_2 > 0$

$$(1.33) \qquad \text{iff} \quad (h_{32}, \ldots, h_{p2}) = 0.$$

Again when (1.30) and (1.33) are satisfied we get by actual matrix multiplication

$$(1.34) \qquad \underset{\sim}{S}\,\underset{\sim}{H}'\,\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{S} = \begin{bmatrix} r_1\ell_1 & 0 & 0 \\ 0 & r_2\ell_1 & 0 \\ 0 & 0 & \underset{\sim}{G}_3 \end{bmatrix}$$

where

$$\underset{\sim}{G}_3 = \underset{\sim}{S}_3\,\underset{\sim}{H}_3'\,\underset{\sim}{A}_3\,\underset{\sim}{H}_3\,\underset{\sim}{S}_3 , \qquad \underset{\sim}{H}_3 \varepsilon O(p-2)$$

and $\qquad \underset{\sim}{A}_3 = \mathrm{diag}(\ell_3,\ldots,\ell_p), \quad \underset{\sim}{S}_3^2 = \underset{\sim}{R}_3 = \mathrm{diag}(r_3,\ldots,r_p)$.

Now in order to find the form of $\underset{\sim}{H}_3$ so that $Ch_3(\underset{\sim}{S}\,\underset{\sim}{H}\,\underset{\sim}{A}\,\underset{\sim}{H}'\,\underset{\sim}{S})$ is maximized subject to (1.30) and (1.33), we find from (1.39) that we are back to the problem of all distinct roots in $\underset{\sim}{A}_3$ with the dimension of $\underset{\sim}{H}$ reduced by 2. Hence following our earlier technique step by step we get

Lemma 1.3.1. When (1.28) holds and $\underset{\sim}{R}$ is as in (1.9), then for all variations of $\underset{\sim}{R} > 0$, we get

$$\max_{\underset{\sim}{H}\varepsilon O(p)} C_\kappa(\underset{\sim}{H}'\,\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}) = \max_{\underset{\sim}{H}\varepsilon O(p)} C_\kappa(\underset{\sim}{H}\,\underset{\sim}{R}\,\underset{\sim}{H}'\,\underset{\sim}{A}) = C_\kappa(\underset{\sim}{A}\,\underset{\sim}{R})$$

If $\underset{\sim}{R}$ is as in (1.3), then

$$(1.35) \qquad \min_{\underset{\sim}{H}\varepsilon O(p)} C_\kappa(\underset{\sim}{H}'\,\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}) = \min_{\underset{\sim}{H}\varepsilon O(p)} C_\kappa(\underset{\sim}{H}\,\underset{\sim}{R}\,\underset{\sim}{H}'\,\underset{\sim}{A}) = C_\kappa(\underset{\sim}{A}\,\underset{\sim}{R})$$

and this maximum or minimum value of the function in respective cases is achieved when $\underset{\sim}{H}$ has the form

$$(1.36) \qquad \underset{\sim}{H} = \begin{bmatrix} \underset{\sim}{H}_1 \\ \\ \underset{\sim}{H}_2 \end{bmatrix},$$

where $\underset{\sim}{H}_1(2 \times p)$ is arbitrary but otherwise satisfying the orthogonality relation of $\underset{\sim}{H}$ and

$$\underset{\sim}{H}_2 = (\underset{\sim}{0}, \underset{\sim}{H}_{22}),$$

where $\underset{\sim}{0} = \underset{\sim}{0}((p-2) \times 2)$ and $\underset{\sim}{H}_{22}((p-2) \times (p-2))$ satisfies the condition (ii) of lemma 1.3.

In practice it is more frequently useful that $\underset{\sim}{A}$ instead of taking form (1.28) often satisfies the following

$$(1.37) \qquad \underset{\sim}{A} = \mathrm{diag}(\ell_1, \ell_1, \ell_3, \ldots, \ell_p) \text{ and } \infty > \ell_p > \ldots > \ell_3 > \ell_1 \geq 0$$

and

$$\underset{\sim}{R} = \mathrm{diag}(r_1, \ldots, r_p), \ \infty > r_1 > r_2 > \ldots > 0 .$$

The problem in this case more or less remains the same with the following changes.

Now instead of considering $C_\kappa(\underset{\sim}{H}\,\underset{\sim}{R}\,\underset{\sim}{H}'\,\underset{\sim}{A})$ we consider $C_\kappa((\underset{\sim}{H}\,\underset{\sim}{D})\,\underset{\sim}{R}\,(\underset{\sim}{H}\,\underset{\sim}{D})'\,\underset{\sim}{A})$ as $\underset{\sim}{H}\varepsilon O(p)$, where $\underset{\sim}{D} = (\underset{\sim}{e}_p, \ldots, \underset{\sim}{e}_1)$ i.e. we consider

$$C_\kappa(\underset{\sim}{H}\,\underset{\sim}{E}\,\underset{\sim}{H}'\,\underset{\sim}{A}) \text{ as } \underset{\sim}{H}\varepsilon O(p),$$

where $\underset{\sim}{E} = \mathrm{diag}(r_p, \ldots, r_1)$. Also as $\underset{\sim}{H}\varepsilon O(p)$, $\underset{\sim}{H}\,\underset{\sim}{D}\varepsilon O(p)$, by earlier argument we get our results. Thus considering $C_\kappa(\underset{\sim}{H}\,\underset{\sim}{R}\,\underset{\sim}{H}'\,\underset{\sim}{A})$ as $\underset{\sim}{H}\varepsilon O(p)$ we note that the form of $\underset{\sim}{H}$ in this case is $\underset{\sim}{H}\,\underset{\sim}{D}$ where $\underset{\sim}{H}$

satisfies the form (1.36), or more explicitly

$$(1.38) \qquad \max_{\underset{\sim}{H}\varepsilon O(p)} C_\kappa(\underset{\sim}{H}\,\underset{\sim}{R}\,\underset{\sim}{H}'\,\underset{\sim}{A}) = \max_{\underset{\sim}{H}\varepsilon O(p)} C_\kappa((\underset{\sim}{H}\,\underset{\sim}{D})\,\underset{\sim}{R}\,(\underset{\sim}{H}\,\underset{\sim}{D})'\,\underset{\sim}{A})$$

$$= \max_{\underset{\sim}{H}\varepsilon O(p)} C_\kappa(\underset{\sim}{H}\,\underset{\sim}{E}\,\underset{\sim}{H}'\,\underset{\sim}{A})$$

$$= C_\kappa(\underset{\sim}{E}\,\underset{\sim}{A})$$

and it is attained when $\underset{\sim}{H}$ has the form

$$(1.39) \qquad \underset{\sim}{H} = \begin{bmatrix} \underset{\sim}{H}_1 \\ \cdot \\ \underset{\sim}{H}_2 \end{bmatrix} \underset{\sim}{D} ,$$

where the left hand matrix is defined in (1.36). Here of course we note that zonal polynomials are symmetric functions of the characteristic roots of the defining matrices and so long as the characteristic roots of a matrix are unchanged, zonal polynomials defined on them are also unchanged.

Now as a further remark we note that the above proof though stated for only one set of two equal roots is quite general, for at each step we just consider one root at a time and as can be noted that had there been three equal roots in a set, then after (1.34), we should have gotten a corresponding reduction and that it will work generally. Thus our earlier technique shows us that each set of equal roots divides the orthogonal matrix in groups of rows and thus we get more generally

Lemma 1.3.2. When (1.27) holds and $\underset{\sim}{R}$ is as in (1.9), then for all variations of $k > 0$

$$\max_{\underset{\sim}{H}\varepsilon O(p)} C_\kappa(\underset{\sim}{H}'\underset{\sim}{A}\underset{\sim}{H}\underset{\sim}{R}) = \max_{\underset{\sim}{H}\varepsilon O(p)} C_\kappa(\underset{\sim}{H}\underset{\sim}{R}\underset{\sim}{H}'\underset{\sim}{A}) = C_\kappa(\underset{\sim}{A}\underset{\sim}{R})$$

and if $\underset{\sim}{R}$ satisfies (1.3)

$$\min_{\underset{\sim}{H}\varepsilon O(p)} C_\kappa(\underset{\sim}{H}'\underset{\sim}{A}\underset{\sim}{H}\underset{\sim}{R}) = \min_{\underset{\sim}{H}\varepsilon O(p)} C_\kappa(\underset{\sim}{H}\underset{\sim}{R}\underset{\sim}{H}'\underset{\sim}{A}) = C_\kappa(\underset{\sim}{A}\underset{\sim}{R}) .$$

The optimum values are attained iff $\underset{\sim}{H}$ has the following form

$$(1.40) \qquad \underset{\sim}{H} = \begin{bmatrix} \underset{\sim}{H}_1 \\ \vdots \\ \underset{\sim}{H}_i \\ \underset{\sim}{H}_{i+1} \end{bmatrix} ,$$

where

$$\underset{\sim}{H}_1(k_1 \times p), \qquad \underset{\sim}{H}_j = (\underset{\sim}{O}_j, \underset{\sim}{H}_{j1}) ,$$

$$\underset{\sim}{O}_j = \underset{\sim}{O}(k_j \times (k_1 + \dots + k_{j-1})), \ \underset{\sim}{H}_{j1} = \underset{\sim}{H}_{j1}(k_j \times (p-k_1 \dots - k_{j-1})),$$

$$j = 2,\dots, i ,$$

and

$$\underset{\sim}{H}_{i+1} = (\underset{\sim}{O}_{i+1}, \underset{\sim}{H}_{i+1,1}) ,$$

$$\underset{\sim}{O}_{i+1} = \underset{\sim}{O}((p-(k_1 + \dots + k_i)) \times (k_1 + \dots + k_i)),$$

$$\underset{\sim}{H}_{i+1,1} = \underset{\sim}{H}_{i+1,1}((p-(k_1 + \dots + k_i)) \times (p-(k_1 + \dots + k_i)))$$

and $\underset{\sim}{H}_{i+1,1}$ satisfies (ii) of lemma 1.3.

Hence from the above form of $\underset{\sim}{H}$ we note that the optimal values of the zonal polynomials are invariant under changes of $\underset{\sim}{H}_j$, $j = 1,\ldots, i$, up to the extent to which they are already determined.

Now with these discussions we get

Theorem 1.1.2. If $\underset{\sim}{A}$ is given by (1.27) and $\underset{\sim}{R}$ by (1.9), the class of orthogonal matrices for which $f(\underset{\sim}{H})$ in (1.6), subject to (1.7) and for all $\underset{\sim}{R} > 0$, is a maximum, is given by (1.40) and

$$\max_{\underset{\sim}{H}\varepsilon O(p)} f(\underset{\sim}{H}) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_\kappa \cdots (a_s)_\kappa}{(b_1)_\kappa \cdots (b_t)_\kappa} \frac{C_\kappa(\underset{\sim}{A}\ \underset{\sim}{R})}{k!} .$$

If (1.9) is replaced by (1.3) but $\underset{\sim}{A}$ is as in (1.27) then

$$\min_{\underset{\sim}{H}\varepsilon O(p)} f(\underset{\sim}{H}) = \sum_{k=0} \sum_{\kappa} \frac{(a_1)_\kappa \cdots (a_s)_\kappa}{(b_1)_\kappa \cdots (b_t)_\kappa} \frac{C_\kappa(\underset{\sim}{A}\ \underset{\sim}{R})}{k!} .$$

The maximum above is attained when $\underset{\sim}{H}$ is of the form (1.40) and the minimum when $\underset{\sim}{H}$ is of the form $\underset{\sim}{H}\ \underset{\sim}{D}$ where $\underset{\sim}{H}$ is given by (1.40) and $\underset{\sim}{D} = (\underset{\sim}{e}_p,\ldots, \underset{\sim}{e}_1)$.

Remark 1. In case of equality of smaller ℓ_i's in $\underset{\sim}{A}$ in several sets, we get a similar result as in (1.39) and $\underset{\sim}{H}$ takes the form $\underset{\sim}{H}\ \underset{\sim}{D}$ where $\underset{\sim}{H}$ is of the form (1.40) and $\underset{\sim}{D} = (\underset{\sim}{e}_p,\ldots, \underset{\sim}{e}_1)$. Of course, as earlier, we must note that zonal polynomials are symmetric functions of the characteristic roots of the defining matrix.

Remark 2. In the case of one set of equal elements in $\underset{\sim}{A}$, while considering the optimal values of the zonal polynomial $C_\kappa(\underset{\sim}{H}'\ \underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{R})$, we can get a direct proof of invariance of $\underset{\sim}{H}_1$ in (1.39) but subject to the condition that $\underset{\sim}{H}$ is orthogonal. We give this proof as a direct varification of certain results of James [10], Li and Pillai [13], [14].

Let us consider $f(\underset{\sim}{H}) = {}_sF_t(a_1,\ldots, a_s;b_1,\ldots, b_t, \underset{\sim}{H}' \underset{\sim}{A} \underset{\sim}{H} \underset{\sim}{R})$ where a_i's, b_j's, $\underset{\sim}{H}$, $\underset{\sim}{R}$ are as defined earlier and let

(1.41) $\underset{\sim}{A} = \mathrm{diag}(\ell,\ldots, \ell, \ell_{k+1},\ldots, \ell_p)$ and

$$\infty > \ell > \ell_{k+1} > \ldots > \ell_p > 0.$$

Partition $\underset{\sim}{H}$ as

$$\underset{\sim}{H} = \begin{bmatrix} \underset{\sim}{H}_1 \\ \\ \underset{\sim}{H}_2 \end{bmatrix}, \tag{1.42}$$

where $\underset{\sim}{H}_1(k \times (p-k))$ and $\underset{\sim}{H}_2$ is the completion of $\underset{\sim}{H}_1$. Then

$$\underset{\sim}{H}' \underset{\sim}{A} \underset{\sim}{H} \underset{\sim}{R} = \ell \underset{\sim}{R} + \underset{\sim}{H}' \underset{\sim}{B} \underset{\sim}{H} \underset{\sim}{R},$$

where

$$\underset{\sim}{B} = \mathrm{diag}\ (0,\ldots, 0, \ell_{k+1} - \ell,\ldots, \ell_p-\ell) = \begin{bmatrix} \underset{\sim}{0} & \underset{\sim}{0} \\ \\ \underset{\sim}{0} & \underset{\sim}{B}_4 \end{bmatrix},$$

and

$$\underset{\sim}{B}_4 = \mathrm{diag}\ (\ell_{k+1}-\ell,\ldots, \ell_p-\ell).$$

Also we partition $\underset{\sim}{H}$ in (1.42) and $\underset{\sim}{R}$ as follows:

$$(1.43) \qquad H = \begin{bmatrix} H_1 \\ \\ H_2 \end{bmatrix} = \begin{bmatrix} P_1 & P_2 \\ \\ P_3 & P_4 \end{bmatrix}, \quad R = \begin{bmatrix} R_1 & 0 \\ \\ 0 & R_4 \end{bmatrix},$$

where the partitions are appropriately done so that the following matrix products are defined.

$$H' A H R = \ell R + \begin{bmatrix} P_1' & P_3' \\ \\ P_2' & P_4' \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \\ 0 & B_4 \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ \\ P_3 & P_4 \end{bmatrix} \begin{bmatrix} R_1 & 0 \\ \\ 0 & R_4 \end{bmatrix}$$

$$= \ell R + \begin{bmatrix} P_3' B_4 P_3 R_1 & P_3' B_4 P_4 R_4 \\ \\ P_4' B_4 P_3 R_1 & P_4' B_4 P_4 R_4 \end{bmatrix}$$

$$= \ell R + \begin{bmatrix} X \\ \\ H_2 \end{bmatrix}' B \begin{bmatrix} X \\ \\ H_2 \end{bmatrix} R$$

where $X(k \times p)$ is arbitrary but otherwise is a completion of H_2.

Thus under (1.41), $f(H)$ is invariant of the choice of H_1 in H. This result with suitable modifications gives the results of James [10], Li and Pillai [13], [14].

1.5. Complex analogue of previous results

The complex analogue of the previous problems arises from the following consideration. Here instead of the problem of evaluating I in (1.1) in an asymptotic sense we have the parallel problem of finding an asymptotic expansion of

$$(1.44) \qquad I_1 = \int_{U(p)} {}_s\tilde{F}_t(a_1,\ldots, a_s; b_1,\ldots, b_t, \underset{\sim}{U}^* \underset{\sim}{A}\, \underset{\sim}{U}\, \underset{\sim}{R})\, d(\underset{\sim}{U}) ,$$

where U(p) is the group of p x p unitary matrices and $d(\underset{\sim}{U})$ is the invariant measure on U(p), normalized to make the total measure of the whole group unity, $\underset{\sim}{A}$, $\underset{\sim}{R}$ as defined earlier, a_i's and b_j's are still functions of d.f. and hence are positive real numbers. But here considering the definition of hypergeometric functions as given by James [9] we will put the following restrictions on a_i's and b_j's.

$$(1.45) \quad a_i > (p-1), \quad b_j > (p-1) \quad i = 1,\ldots, s,\ j = 1,\ldots, t ,$$

$\underset{\sim}{U} \varepsilon U(p)$, i.e. an element of the group of p x p unitary matrix such that $\underset{\sim}{U}^* \underset{\sim}{U} = \underset{\sim}{U}\, \underset{\sim}{U}^* = \underset{\sim}{I}(p)$.

In this context we have the following lemma.

<u>Lemma 1.3.3</u>. Let $\underset{\sim}{A}$ and $\underset{\sim}{R}$ be as defined in (1.9). Then $\tilde{C}_\kappa(\underset{\sim}{U}^*\underset{\sim}{A}\, \underset{\sim}{U}\, \underset{\sim}{R})$ is real and for all variations of $\underset{\sim}{R} > 0$, we have

$$\max_{\underset{\sim}{U}\varepsilon U(p)} \tilde{C}_\kappa(\underset{\sim}{U}^*\underset{\sim}{A}\, \underset{\sim}{U}\, \underset{\sim}{R}) = \max_{\underset{\sim}{U}\varepsilon U(p)} \tilde{C}_\kappa(\underset{\sim}{U}\, \underset{\sim}{R}\, \underset{\sim}{U}^*\underset{\sim}{A}) = \tilde{C}_\kappa(\underset{\sim}{A}\, \underset{\sim}{R}) .$$

If $\underset{\sim}{A}$ is defined by (1.21) instead of (1.9) ,

$$\min_{\underset{\sim}{U}\varepsilon U(p)} \tilde{C}_\kappa(\underset{\sim}{U}^*\underset{\sim}{A}\, \underset{\sim}{U}\, \underset{\sim}{R}) = \min_{\underset{\sim}{U}\varepsilon U(p)} \tilde{C}_\kappa(\underset{\sim}{U}\, \underset{\sim}{R}\, \underset{\sim}{U}^*\underset{\sim}{A}) = \tilde{C}_\kappa(\underset{\sim}{A}\, \underset{\sim}{R}) .$$

<u>Proof</u>. To prove that $\tilde{C}_\kappa(\underset{\sim}{U}^*\underset{\sim}{A}\, \underset{\sim}{U}\, \underset{\sim}{R})$ is real, we note

$$Ch_p(U^*A\,U)\ Ch_p(R) \le Ch_i(U^*A\,U\,R) \le Ch_1(U^*A\,U)\ Ch_1(R)$$

i.e. $Ch_p(A)\ Ch_p(R) \le Ch_i(U^*A\,U\,R) \le Ch_1(A)\ Ch_1(R)$, $i = 1,\ldots, p$.

For the rest we put as earlier $S^2 = R$ where

$$S = \mathrm{diag}(s_1,\ldots, s_p),\ R = \mathrm{diag}(r_1,\ldots, r_p) \text{ i.e. } s_i^2 = r_i,$$

$$i = 1,\ldots, p .$$

Then let $C = S\,U\,A\,U\,S = (c_{ij})$ and $U = (u_{ij})$. Hence

$$c_{ij} = s_i\ s_j \sum_{k=1}^{p} \bar{u}_{ki} u_{kj} \ell_k\ , \qquad (i,j = 1,\ldots, p) .$$

Now we proceed exactly as in the real case replacing H by U and get

Theorem 1.1.3. Let R be as given in (1.9) and A as in (1.27), then the class of unitary matrices for which $f(U) =$ ${}_sF_t\ (a_1,\ldots, a_s;\ b_1,\ldots, b_t,\ U^*A\,U\,R)$ subject to (1.45) and for all $R > 0$ is a maximum, is given by

(1.46) $$U = \begin{bmatrix} U_1 \\ \vdots \\ U_i \\ U_{i+1} \end{bmatrix},$$

where

$$\underset{\sim}{U}_1(k_1 \times p), \quad \underset{\sim}{U}_j = (\underset{\sim}{O}_j, \underset{\sim}{U}_{j1}) ,$$

$$\underset{\sim}{O}_j = O(k_j \times (k_1 + \dots + k_{j-1})), \ \underset{\sim}{U}_{j1} =$$

$$\underset{\sim}{U}_{j1}(k_j \times (p - k_1 - \dots - k_{j-1})), \quad j = 2, \dots, i ,$$

and

$$\underset{\sim}{U}_{i+1} = (\underset{\sim}{O}_{i+1}, \underset{\sim}{U}_{i+1,1}) ,$$

where

$$\underset{\sim}{O}_{i+1,1} = \underset{\sim}{O}((p-(k_1 + \dots + k_i)) \times (k_1 + \dots + k_i)) ,$$

$$\underset{\sim}{U}_{i+1,1} = \underset{\sim}{U}_{i+1,1}((p-(k_1 + \dots + k_i)) \times (p-(k_1 + \dots + k_i))),$$

$\underset{\sim}{O}$ stands for null matrix and $\underset{\sim}{U}_{i+1,1}$ is a diagonal matrix of the order indicated in the parentheses and with diagonal elements $e^{\sqrt{-1}\theta_j}$, $0 \leq \theta_j < 2\pi$, $\underset{\sim}{U}_{k1}$, $k = 2, \dots, i$, and $\underset{\sim}{U}_1$ are arbitrary matrices subject to the condition that $\underset{\sim}{U}$ in (1.46) is unitary.

Also the maximum in this case is given by

$$(1.47) \quad \max_{\underset{\sim}{U} \in U(p)} f(\underset{\sim}{U}) = \max_{\underset{\sim}{U} \in U(p)} {}_s\tilde{F}_t(a_1, \dots, a_s; b_1, \dots, b_t, \underset{\sim}{U}^* \underset{\sim}{A} \underset{\sim}{U} \underset{\sim}{R})$$
$$= {}_s\tilde{F}_t(a_1, \dots, a_s; b_1, \dots, b_t, \underset{\sim}{A} \underset{\sim}{R}) ,$$

where ${}_s\tilde{F}_t(a_1, \dots, a_s; b_1, \dots, b_t, \underset{\sim}{Z})$ is defined in James [9],

$${}_s\tilde{F}_t(a_1, \dots, a_s; b_1, \dots, b_t, \underset{\sim}{Z}) = \sum_{k=0} \sum_{\kappa} \frac{[a_1]_\kappa \cdots [a_s]_\kappa}{[b_1]_\kappa \cdots [b_t]_\kappa} \frac{\tilde{C}_\kappa(\underset{\sim}{Z})}{k!}$$

and the complex multivariate hypergeometric coefficients are as follows

$$[a]_\kappa = \prod_{i=1}^{p} (a - i+1)_{\kappa_i} ,$$

where $\kappa = (k_1, \ldots, k_p)$ is a partition of the integer κ. When the ordering of the elements of $\underset{\sim}{A}$ is reversed a result parallel to that of (1.26) holds. This last one is the most general result we have had so far. As special cases it gives some already known results. For example putting $s = t = 0$ and $k_1 = \ldots = k_i = 1$ we get, for all $\underset{\sim}{R} > 0$

$$\max_{\underset{\sim}{U} \varepsilon U(p)} {}_0\tilde{F}_0(\underset{\sim}{U}^*\underset{\sim}{A}\ \underset{\sim}{U}\ \underset{\sim}{R}) = {}_0\tilde{F}_0(\underset{\sim}{A}\ \underset{\sim}{R}) ,$$

which is a variant of the result of Li and Pillai [13], [14]. Similarly putting $s = 1$, $t = 0$ and after some manipulation as shown in the corresponding real part we get for all variation of $\underset{\sim}{R} > 0$

$$\max_{\underset{\sim}{U} \varepsilon U(p)} {}_1\tilde{F}_0\ (n, - \underset{\sim}{U}^*\underset{\sim}{A}\ \underset{\sim}{U}\ \underset{\sim}{R}) = {}_1\tilde{F}_0\ (n, - \underset{\sim}{A}\ \underset{\sim}{R}) .$$

Here, of course, n satisfies (1.45) and $\underset{\sim}{A}$ and $\underset{\sim}{R}$ are given by (1.9). This is also a variant of the result of Li and Pillai [13], [14].

Again considering only one set of equal elements in $\underset{\sim}{A}$ we partition $\underset{\sim}{U}$ as follows:

(1.48) $$\underset{\sim}{U} = \begin{bmatrix} \underset{\sim}{U}_1 \\ \\ \underset{\sim}{U}_2 \end{bmatrix} ,$$

where $\underset{\sim}{U}_1(k \times p)$ and $\underset{\sim}{U}_2((p-k) \times p)$, $\underset{\sim}{\Lambda} = \text{diag}(\ell, \ldots, \ell, \ell_{k+1}, \ldots, \ell_p)$ and $\infty > \ell > \ell_{k+1} > \ldots > \ell_p \geq 0$, and as earlier we get for all variations of $k > 0$

$$\max_{\underset{\sim}{U} \varepsilon U(p)} {}_s\tilde{F}_t\,(a_1, \ldots, a_s; b_1, \ldots, b_t, \underset{\sim}{U}^* \underset{\sim}{A}\, \underset{\sim}{U}\, \underset{\sim}{R})$$

$$= {}_s\tilde{F}_t\,(a_1, \ldots, a_s; b_1, \ldots, b_t, \underset{\sim}{A}\, \underset{\sim}{R})$$

iff $\underset{\sim}{U}_2$ in (1.48) takes the form

$\underset{\sim}{U}_2 = (\underset{\sim}{O}, \underset{\sim}{D})$, $\underset{\sim}{O}((p-k) \times k)$ and $\underset{\sim}{D} = \text{diag}(e^{\sqrt{-1}\theta_1}, \ldots, e^{\sqrt{-1}\theta_q})$ where $q = p-k$ and $0 \leq \theta_j < 2\pi$, $j = 1, \ldots, q$, a_i's and b_j's are assumed to satisfy (1.45). This form of $\underset{\sim}{U}$ in (1.48) immediately asserts invariance results of Li and Pillai [13], [14]. In fact obtaining the form of $\underset{\sim}{U}$ as in (1.46) is in short the most general result obtained in this dissertation.

CHAPTER II

ASYMPTOTIC EXPANSIONS OF THE DISTRIBUTIONS OF CHARACTERISTIC ROOTS IN MANOVA AND CANONICAL CORRELATION

2.1. Introduction

An asymptotic expansion of the distribution of a sample covariance matrix (one-sample case) was studied by Anderson [1] and James [10], and extending their work, an asymptotic representation was obtained by Chang [3] in the two-sample case when the population roots are all distinct. Li, Pillai and Chang [14] generalized Chang's results [3] to cover the case of a single extreme multiple population root. Li and Pillai [13], [14], have further obtained the second term of the expansion in the two-sample case and also extended the results to the complex case. In this part, asymptotic expansions are derived in the MANOVA and canonical correlation situations both in the real and complex cases.

2.2. Asymptotic Expansion for Canonical Correlation -- Population Roots all Distinct

Let $x_1, \ldots, x_p, x_{p+1}, \ldots, x_{p+q}$, $p \leq q$ be distributed $N(\underset{\sim}{0}, \underset{\sim}{\Sigma})$, where

$$(2.1)\qquad \Sigma = \begin{array}{c|cc|} & p & q \\ \hline p & \Sigma_{11} & \Sigma_{12} \\ q & \Sigma'_{12} & \Sigma_{22} \\ \hline \end{array} \; .$$

Let $P^2 = \text{diag}(\rho_1^2, \ldots, \rho_p^2)$, where ρ_i^2, $i = 1, \ldots, p$, be the roots of

$$(2.2)\qquad |\Sigma_{12}\, \Sigma_{22}^{-1}\, \Sigma'_{12} - \rho^2\, \Sigma_{11}| = 0$$

and let $\hat{P}^2 = \text{diag}(\hat{\rho}_1^2, \ldots, \hat{\rho}_p^2)$, where $\hat{\rho}_i^2$, $i = 1, \ldots, p$, be the maximum likelihood estimates of ρ_i^2, $i = 1, \ldots, p$, from a sample of size $n \geq p+q$ from the above population. Then the joint density of

$$(2.3)\qquad \hat{P}^2 = R = \text{diag}(r_1, \ldots, r_p)$$

is given by [4]

$$(2.4)\qquad D_1 \int_{O(p)} {}_2F_1\left(\tfrac{1}{2}n, \tfrac{1}{2}n; \tfrac{1}{2}q, H' A H R\right) d(H) ,$$

where

$$(2.5)\qquad \hat{P}^2 = A = \text{diag}(\ell_1, \ldots, \ell_p), \quad 1 > \ell_1 > \ldots > \ell_p \geq 0 ,$$

$d(H)$ is the invariant or Haar measure defined on the group $O(p)$ of $p \times p$ orthogonal matrices

$$(2.6)\qquad D_1 = \{\Pi^{\frac{p^2}{2}}\; \Gamma_p(\tfrac{1}{2}n)/\Gamma_p(\tfrac{1}{2}q)\; \Gamma_p(\tfrac{1}{2}(n-q))\; \Gamma_p(\tfrac{1}{2}p)\}\; |I-A|^{\frac{n}{2}}$$
$$|R|^{\frac{1}{2}(q-p-1)}\; |I - R|^{\frac{1}{2}(n-q-p-1)} \prod_{i<j} (r_i - r_j) ,$$

where

$$\Gamma_p(t) = \pi^{\frac{p(p-1)}{4}} \prod_{j=1}^{p} \Gamma\left(t - \frac{1}{2}(j-1)\right) ,$$

and the hypergeometric function of the symmetric matrix $\underset{\sim}{Q}$ is given by [4]

$$ {}_\mu F_\eta(a_1, \ldots, a_\mu; b_1, \ldots, b_\eta, \underset{\sim}{Q}) = \sum_{k=0} \sum_{\kappa} \frac{(a_1)_\kappa \ldots (a_\mu)_\kappa}{(b_1)_\kappa \ldots (b_\eta)_\kappa} \frac{C_\kappa(\underset{\sim}{Q})}{k!} ,$$

where $a_1, \ldots, a_\mu, b_1, \ldots, b_\eta$ are real or complex constants and the multivariate hypergeometric coefficient $(a)_\kappa$ is given by

$$(a)_\kappa = \prod_{i=1}^{p} \left(a - \frac{1}{2}(i-1)\right)_{k_i} ,$$

where

$$(a)_k = a(a+1) \ldots (a+k-1) .$$

The group $O(p)$ has volume

$$v(p) = \int_{O(p)} d(\underset{\sim}{H}) = 2^p \pi^{\frac{p^2}{2}} \{\Gamma_p(\tfrac{1}{2}p)\}^{-1} .$$

Let us order the r_i's in $\underset{\sim}{R}$ as

(2.7) $$1 > r_1 > \ldots > r_p > 0 .$$

The density (2.4) involves an integral and following Anderson [1], Chang [3], Li and Pillai, [13], [14], our main objective is to maximize this integral. Let us now denote the integral by

$$(2.8) \qquad E = \int_{O(p)} {}_2F_1\,(s,s;t,H'A\,H\,R)\;d(H) \quad ,$$

where, for notational simplicity we put $s = \frac{1}{2}n$ and $t = \frac{1}{2}q$.

Now with mild restrictions on s by theorem 1.1 we find that for variation of $H \epsilon O(p)$, ${}_2F_1(s,s;t,H'A\,H\,R)$ is maximized when H is given by (ii) in lemma 1.3 namely, ${}_2F_1(s,s;t,A\,R)$. But here we proceed to obtain an alternate form which is more useful. First we use Kummer's formula and get

$$(2.9) \qquad {}_2F_1\,(s,s;t,H'A\,H\,R) = |I - H'A\,H\,R|^{-(2s-t)}\;{}_2F_1((t-s),(t-s);t,H'A\,H\,R) \;.$$

Now following results of the previous chapter, varying H over $N(I)$, the neighborhood of $I(p \times p)$, i.e. varying $H'A\,H\,R$ around $A\,R$, we get

$$(2.10) \qquad {}_2F_1\,(t-s,t-s;t,H'A\,H\,R) = {}_2F_1(t-s,t-s;t,A\,R) + O(\varepsilon) \;.$$

We prove below a more general result.

<u>Lemma 2.1</u>. If $H \epsilon N(I)$, $a_i \geq \frac{1}{2}(p-1)$, $b_j \geq \frac{1}{2}(p-1)$, $i = 1,\ldots,\mu$ $j = 1,\ldots,\eta$, then

$$\begin{aligned} &{}_\mu F_\eta(a_1,\ldots,a_\mu;\,b_1,\ldots,b_\eta,\,H'A\,H\,R) = \\ &{}_\mu F_\eta(a_1,\ldots,a_\mu;\,b_1,\ldots,b_\eta,A\,R) + O(\varepsilon)\;, \end{aligned}$$

provided

$$t_j - \varepsilon \leq ch_j(H'A\;H\;R) \leq t_j+\varepsilon, \quad \text{where}$$

$$t_i = ch_i(A\;R),\; i = 1,\ldots,p\;.$$

Proof. Let $f(H) = {}_{\mu}F_{\eta}(a_1,\ldots,a_{\mu};\,b_1,\ldots,b_{\eta},\,H'A\;H\;R)$. Then by lemma 1.2 $f(H)$ is an increasing function in each of its characteristic roots. Thus varying $H \varepsilon N(I)$, we note that first partial derivatives of $f(H)$ with respect to each characteristic root exist, except possibly over a set of zero measure. Again as $f(H)|_{H=I}$ exists, the mean value theorem applies and hence the lemma.

Now application of lemma 2.1 in conjunction with Kummer's formula (2.9) gives (2.10). Following Anderson [1], Chang [3], Li and Pillai [13], [14], and using (2.10) we get for large values of (2s-t)

$$E \simeq 2^p \int_{N(I)} |\,I-H'A\;H\;R|^{-(2s-t)} d(H)\,{}_2F_1(t-s,t-s;t,A\;R) + O(\varepsilon)\;.$$

Further we consider

$$(2.11)\qquad F = 2^p \int_{N(I)} |I - H'A\;H\;R|^{-(2s-t)} d(H)\;.$$

The integrand in (2.11) is quite similar to that of Chang [3] and hence what follows is essentially his technique as modified by Li and Pillai, [13], [14]. For the sake of continuity we write down the essential steps as applied in our case omitting the details to the above references with suitable modification.

Let us use the transformation

$$(2.12)\qquad H = \exp[S]\;,$$

where $S(p \times p)$ is a skew symmetric matrix. Then by Anderson [1]

$$(2.13) \quad J(\underset{\sim}{S};\underset{\sim}{H}) = 1 + [(p-2)/4!]\, \mathrm{tr}\underset{\sim}{S}^2 + [(8-p)/(4\cdot 6!)](\mathrm{tr}\underset{\sim}{S}^4) + [(5p^2 - 20p + 14)/(8\cdot 6!)](\mathrm{tr}\underset{\sim}{S}^2)^2 + \ldots .$$

Under this transformation $N(\underset{\sim}{I}) \to \underset{\sim}{N}(\underset{\sim}{S} = \underset{\sim}{0})$. However as shown by Anderson [1] and Chang [3], for large $(2s-t)$ we can approximate F in (2.11) by integrating not exactly on $N(\underset{\sim}{S} = \underset{\sim}{0})$ but simply over intervals $-\infty < s_{ij} < \infty$ for each s_{ij}. Under the transformation (2.12) we have

$$|\underset{\sim}{I} - \underset{\sim}{H}'\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}| = |\underset{\sim}{I} - \underset{\sim}{A}\,\underset{\sim}{R}|\,|\underset{\sim}{I} + \{\underset{\sim}{S}\} + \{\underset{\sim}{S}^2\} + \{\underset{\sim}{S}^3\} + \ldots|,$$

i.e.

$$(2.14) \quad |\underset{\sim}{I} - \underset{\sim}{H}'\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}|^{-(2s-t)} = |\underset{\sim}{I} - \underset{\sim}{A}\,\underset{\sim}{R}|^{-(2s-t)}\,|\underset{\sim}{I} + \underset{\sim}{G}|^{-(2s-t)},$$

where

$$\underset{\sim}{G} = \{\underset{\sim}{S}\} + \{\underset{\sim}{S}^2\} + \{\underset{\sim}{S}^3\} + \ldots .$$

Henceforth for notational ease we will write $2s-t = v$ i.e. (2.14) is rewritten as

$$|\underset{\sim}{I} - \underset{\sim}{H}'\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}|^{-v} = |\underset{\sim}{I} - \underset{\sim}{A}\,\underset{\sim}{R}|^{-v}\,|\underset{\sim}{I} + \underset{\sim}{G}|^{-v} .$$

Let $\underset{\sim}{T} = (\underset{\sim}{I} - \underset{\sim}{A}\,\underset{\sim}{R})^{-1}$. Since $\underset{\sim}{A}$ in our case is a fixed diagonal matrix and $\underset{\sim}{R}$ has random entries corresponding to sample canonical correlations, we neglect the set in which $\underset{\sim}{T}$ is undefined as at most it will contribute a set of measure zero. Thus without loss of generality we can write

$$(2.15) \qquad \underset{\sim}{T} = (\underset{\sim}{I} - \underset{\sim}{A}\,\underset{\sim}{R})^{-1}\,\underset{\sim}{A} = \begin{bmatrix} t_1 & 0 & \dots & 0 \\ 0 & t_2 & \dots & 0 \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ 0 & \dots & \dots & t_p \end{bmatrix}$$

where

$$t_j = \ell_j(1-\ell_j r_j)^{-1}, \quad j = 1,\dots, p\ .$$

Then

$$\{\underset{\sim}{S}\} = \underset{\sim}{T}\ (\underset{\sim}{R}\,\underset{\sim}{S} - \underset{\sim}{S}\,\underset{\sim}{R})\ ,$$

$$\{\underset{\sim}{S}^2\} = \tfrac{1}{2}\underset{\sim}{T}\ (2\underset{\sim}{S}\,\underset{\sim}{R}\,\underset{\sim}{S} - \underset{\sim}{S}^2\underset{\sim}{R} - \underset{\sim}{R}\,\underset{\sim}{S}^2)\ ,$$

and $\{\underset{\sim}{S}^3\}$ and other terms are obtainable with modification from Li and Pillai [13], [14]. Further we quote a lemma.

<u>Lemma 2.2</u>. Let b_j be the jth characteristic root of $\underset{\sim}{B}(p \times p)$ such that

$$\max_{1 \le j \le p} |b_j| < 1\ ,$$

then

$$|\underset{\sim}{I} + \underset{\sim}{B}|^v = \exp[v\ \mathrm{tr}(\underset{\sim}{B} - \tfrac{1}{2}\underset{\sim}{B}^2 + \tfrac{1}{3}\underset{\sim}{B}^3\ \dots)]\ .$$

For proof see Chang [3].

Under transformation (2.12), $N(I) \to N(S = 0)$ and taking S sufficiently close to 0 we can take the maximum characteristic root of G to be less than unity Hence applying lemma 2.2 we get under (2.12)

$$|I - H'A\,H\,R|^{-v} = |I - A\,R|^{-v}\,|I + G|^{-v}$$

$$= |I - A\,R|^{-v}\exp[-v\,\mathrm{tr}([S] + [S^2] + [S^3] + \dots)]\ ,$$

where

$$[S] = \{S\}\ ,$$

$$[S^2] = \{S^2\} - \tfrac{1}{2}\{S\}^2\ ,$$

and $[S^3]$ and other terms are available from Li and Pillai, [13], [14], with obvious modification. Now putting $S = (s_{ij})$, and $S' = -S$ we have

$$\mathrm{tr}[S] = 0,$$

$$\mathrm{tr}[S^2] = \sum_{i<j} c_{ij}\ s_{ij}^2\ ,$$

where

$$c_{ij} = (t_{ji} - t_i t_j r_{ij}) r_{ij} = c_{ji}\ , \tag{2.16}$$

$$t_{ij} = t_i - t_j,\ r_{ij} = r_i - r_j\ .$$

Thus we note that the above and other expressions follow from those of Li and Pillai [13], [14] changing R to $-R$ and with accompanying change of notation. Hence following Li and Pillai [13], [14] we get after some lengthy algebra

$$F = 2^p |I - A R|^{-\nu} \prod_{i<j} \left(\frac{\pi}{\nu c_{ij}}\right)^{\frac{1}{2}} \{1 + \frac{1}{4\nu}[\sum_{i<j} c_{ij}^{-1} + \alpha(p)] + \ldots\} ,$$

where

(2.17) $$\alpha(p) = p(p-1)(2p+5)/12 .$$

Thus substituting back this value in E we get the theorem:

Theorem 2.1. For large n, an asymptotic expansion of the distribution of $r_1, \ldots, r_p$ (the squares of the canonical correlation coefficients) where $1 > r_1 > \ldots > r_p > 0$ and the population parameters from (2.2) are such that $1 > \ell_1 > \ldots > \ell_p > 0$, is given by

(2.18) $$D_1 \; B \prod_{i<j} \left(\frac{2\pi}{(2n-q)c_{ij}}\right)^{\frac{1}{2}} |I - A R|^{-\frac{1}{2}(2n-q)} \{1 + \frac{1}{2(2n-q)} [\sum_{i<j} c_{ij}^{-1} + \alpha(p)] + \ldots\} \; {}_2F_1(\tfrac{1}{2}(q-n), \tfrac{1}{2}(q-n); \tfrac{1}{2}q, A R) + O(\varepsilon) ,$$

where R, D_1, c_{ij} and $\alpha(p)$ are given by (2.3), (2.6), (2.16) and (2.17) respectively and $B = 2^p c_1^{-1}$.

2.3. The Asymptotic Expansion for Canonical Correlation-One Extreme Population Multiple Root

James [10] has studied the distribution of smaller roots given the larger roots of a sample covariance matrix and has found a gamma type approximation with linkage factors between sample roots corresponding to smaller and larger population roots. In their study of the two-sample case, Chang [3], Li and Pillai [13], [14], have found a beta type approximation in the same context. We obtain below a similar beta type approximation in the canonical correlation case. Let us assume

(2.19) $\underset{\sim}{A} = \mathrm{diag}(\ell_1, \ldots, \ell_p),\ 1 > \ell_1 > \ell_2 > \ldots > \ell_k > \ell_{k+1} = \ldots = \ell_p = \ell \geq 0$

and $\underset{\sim}{R} = \mathrm{diag}(r_1, \ldots, r_p),\ 1 > r_1 > \ldots > r_p > 0$.

The joint distribution of $r_1, \ldots, r_p$ in this case is given by (2.4) with appropriate changes in definition of $\underset{\sim}{A}$ and as earlier we consider (2.8). Here we partition $\underset{\sim}{H}$ as follows

(2.20) $$\underset{\sim}{H} = \begin{array}{c} p \\ \begin{bmatrix} \underset{\sim}{H}_1 \\ \underset{\sim}{H}_2 \end{bmatrix} \end{array} \begin{array}{l} \\ k \\ p-k \end{array} ,$$

i.e. $\underset{\sim}{H}_1(k \times p)$ and $\underset{\sim}{H}_2((p-k) \times p)$. Under (2.19) we note that our integrand in (2.8) is invariant under choice of $\underset{\sim}{H}_2$ up to the restriction that the matrix $\underset{\sim}{H}$ is orthogonal. Because of the above we can integrate out $\underset{\sim}{H}_2$ in (2.20) using the formula

(2.21) $$c_1 \int_{\underset{\sim}{H}_2} d(\underset{\sim}{H}) = c_2\, d(\underset{\sim}{H}_1) ,$$

where

(2.22) $$c_1 = \pi^{p^2/2} \{\Gamma_p(\tfrac{p}{2})\}^{-1} \text{ and } c_2 = \pi^{kp/2} \{\Gamma_k(\tfrac{p}{2})\}^{-1} ,$$

where $d(\underset{\sim}{H}_1)$ denotes the invariant volume element of the Stiefel-manifold of orthonormal k-frames in p-space normalized to make its integral unity. Now following Chang [3], Li and Pillai [13], [14], the integrand in (2.8) can be closely approximated for large $\underset{\sim}{S}$ when $\underset{\sim}{H}$ has the following form

$$(2.23) \qquad \underset{\sim}{H} = \begin{bmatrix} \underset{\sim}{I}_o(k) & \underset{\sim}{0} \\ & \underset{\sim}{H}_2 \end{bmatrix} ,$$

where $\underset{\sim}{I}_o(k) = \mathrm{diag}(\pm 1, \ldots, \pm 1)$ and is of dimension k. Now restricting ourselves to orthogonal matrices we apply the following transformations

$$(2.24) \qquad \underset{\sim}{H} = \exp[\underset{\sim}{S}] ,$$

where

$$(2.25) \qquad \underset{\sim}{S} = \begin{bmatrix} \underset{\sim}{S}_{11} & \underset{\sim}{S}_{12} \\ \underset{\sim}{S}'_{12} & \underset{\sim}{0} \end{bmatrix} ,$$

and $\underset{\sim}{S}_{11}(k \times k)$ is a skew symmetric matrix and $\underset{\sim}{S}_{12}(k \times (p-k))$ is a rectangular matrix. The jacobian of transformation (2.24) is given by (2.13). Also we have by analogy with Anderson [1], (James [10]),

$$(2.26) \qquad c_2 d(\underset{\sim}{H}_1) = c_3 d(\underset{\sim}{S}_{11})\, d(\underset{\sim}{S}_{12})(1+0(\text{squares of } s_{ij}\text{'s})) ,$$

where

$$(2.27) \qquad c_3 = \Pi^{\frac{q^2}{2}} \{\Gamma_q(\tfrac{1}{2}q)\}^{-1}, \qquad q = p-k ,$$

$d(\underset{\sim}{S}_{11})$ and $d(\underset{\sim}{S}_{12})$ stand for $\prod_{i<j=1}^{k} ds_{ij}$ and $\prod_{i=1}^{k} \prod_{j=k+1}^{p} ds_{ij}$ respectively. From equations (2.24)' and (2.25) we get

$$h_{ii} = 1 - \frac{1}{2} \sum_{j=1}^{p} s_{ij}^2 . \qquad i \leq k .$$

and

$$h_{ij} = s_{ij} + \text{higher order terms}, \ (i \neq j), \ s_{ij} = -s_{ji} .$$

Now using the transformation (2.24) and following the technique used earlier and remembering that $q = p-k$ of the roots of $\underset{\sim}{A}$ are equal, we get

$$tr[\underset{\sim}{S}^2] = \sum_{i<j} c_{ij} s_{ij}^2 + \sum_{i=1}^{k} \sum_{j=k+1}^{p} c_{ij}^o \ s_{ij}^2 ,$$

where

$$(2.28) \quad c_{ij} = (t_{ji} - t_i t_j r_{ij}) r_{ij} = c_{ji}, \ i,j=1,\ldots, k, \ i < j ,$$

$$(2.29) \quad c_{ij}^o = (t_{ji} - t_i t_j r_{ij}) r_{ij}, \qquad i = 1,\ldots, k, \ j = k+1,\ldots, p ,$$

and

$$t_i = \begin{cases} \ell_i/(1-r_i\ell_i), & i = 1,\ldots, k \\ \ell/(1-r_i\ell), & i = k+1,\ldots, p , \end{cases}$$

$$t_{ij} = t_i - t_j \quad \text{and} \quad r_{ij} = r_i - r_j .$$

Thus following Li and Pillai [13], [14], we get

$$|\underset{\sim}{I} - \underset{\sim}{H}'\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}|^{-(2s-t)} = |\underset{\sim}{I} - \underset{\sim}{A}\,\underset{\sim}{R}|^{-(2s-t)} \prod_{i<j=1}^{k} \exp[-(2s-t)c_{ij}s_{ij}^2]$$

$$\prod_{i=1}^{k} \prod_{j=k+1}^{p} \exp[-(2s-t)c_{ij}^{o}s_{ij}^2]\ \{1 + O(s_{ij}^2)\} \ .$$

Now for large $(2s-t)$, and remembering that in the present context the integrand (2.11) is invariant of the choice of $\underset{\sim}{H}_2$ in (2.20) and using (2.21) and (2.26) we get

$$(2.30) \quad F \approx 2^k c_3 c_1^{-1} |\underset{\sim}{I} - \underset{\sim}{A}\,\underset{\sim}{R}|^{-(2s-t)} \int_{S_{11}} \int_{S_{12}} \prod_{i<j=1}^{k} \exp[-(2s-t)c_{ij}s_{ij}^2]\, ds_{ij}$$

$$\prod_{i=1}^{k} \prod_{j=k+1}^{p} \exp[-(2s-t)c_{ij}^{o}s_{ij}^2]\, ds_{ij}\ \{1 + O(\frac{1}{(2s-t)})\} \ .$$

Again when $(2s-t)$ is large and ℓ_i's and r_i's are well spaced $(i = 1,\dots, p)$, most of the integral in (2.30) will be obtained from small values of the elements of $\underset{\sim}{S}_{11}$, $\underset{\sim}{S}_{12}$. Hence to obtain an asymptotic expansion, we can replace the range of elements of s_{ij} for all real values of them. With this stipulation, following Li and Pillai, [13], [14], we get after some lengthy algebra

$$(2.31)\qquad F \simeq 2^k c_3 c_1^{-1} |\underset{\sim}{I} - \underset{\sim}{A}\,\underset{\sim}{R}|^{-(2s-t)} \prod_{i<j=1}^{k} \left(\frac{\Pi}{(2s-t)c_{ij}}\right)^{\frac{1}{2}}$$

$$\prod_{i=1}^{k} \prod_{j=k+1}^{p} \left(\frac{\Pi}{(2s-t)c_{ij}^{o}}\right)^{\frac{1}{2}}$$

$$\{1 + \frac{1}{4(2s-t)} [\sum_{i<j=1}^{k} c_{ij}^{-1} + \sum_{i=1}^{k} \sum_{j=k+1}^{p} c_{ij}^{o\,-1} + \alpha(p,k)] + \ldots \},$$

where $\alpha(p,k)$ is defined below. Now using this value of F as in (2.31) and proceeding exactly as in the case of distinct roots in the matrix $\underset{\sim}{A}$ we get the following theorem.

Theorem 2.1.1. For large n, an asymptotic expansion of the distribution or $r_1,\ldots, r_p$, where $1 > r_1 > \ldots > r_p > 0$ and the parameters from (2.2) are such that $1 > \ell_1 > \ldots > \ell_k > \ell_{k+1} = \ldots = \ell_p \geq 0$, is given by

$$(2.32)\qquad D_1 c_3 c_1^{-1} 2^k |\underset{\sim}{I} - \underset{\sim}{A}\,\underset{\sim}{R}|^{-\frac{1}{2}(2n-q)} \prod_{i<j=1}^{k} \left(\frac{2\Pi}{(2n-q)c_{ij}}\right)^{\frac{1}{2}}$$

$$\prod_{i=1}^{k} \prod_{j=k+1}^{p} \left(\frac{2\Pi}{(2n-q)c_{ij}^{o}}\right)^{\frac{1}{2}}$$

$$\{1 + \frac{1}{2(2n-q)} [\sum_{i<j=1}^{k} c_{ij}^{-1} + \sum_{i=1}^{k} \sum_{j=k+1}^{p} c_{ij}^{o\,-1} + \alpha(p,k)] + \ldots\}$$

$${}_2F_1\left(\tfrac{1}{2}(q-n),\ \tfrac{1}{2}(q-n);\ \tfrac{1}{2}q,\ \underset{\sim}{A}\,\underset{\sim}{R}\right) + O(\varepsilon),$$

where D_1, c_1, c_3, c_{ij} and c_{ij}^{o} are defined in (2.6), (2.22), (2.27), (2.28), (2.29) respectively and

$$\alpha(p,k) = \frac{k}{12}\{(k-1)(4k+1) + 6(p^2-k^2)\}.$$

2.4. Asymptotic Expansion for MANOVA - Population Roots All Distinct.

Let $\underset{\sim}{B}$ be the Between S.P. matrix and $\underset{\sim}{W}$ the Within S.P. matrix. Then $\underset{\sim}{B}(p \times p)$ has a non-central Wishart distribution with s d.f. and matrix of non-centrality parameter $\underset{\sim}{A}$, and $\underset{\sim}{W}$ has a central Wishart distribution on t d.f., the covariance matrix in each case being $\underset{\sim}{\Sigma}$, and

$$(2.33) \qquad \underset{\sim}{A} = \frac{1}{2} \underset{\sim}{\mu} \underset{\sim}{\mu}' \underset{\sim}{\Sigma}^{-1} ,$$

where $\underset{\sim}{\mu}(p \times s)$ is the matrix of the mean vectors. Then the probability distribution function of the roots of the matrix

$$(2.34) \qquad \underset{\sim}{R} = \underset{\sim}{B}(\underset{\sim}{B} + \underset{\sim}{W})^{-1} ,$$

is given by [4]

$$(2.35) \qquad T_1 \int_{O(p)} {}_1F_1(\tfrac{1}{2}(s+t); \tfrac{1}{2}s, \underset{\sim}{H}'\underset{\sim}{A}\, \underset{\sim}{H}\, \underset{\sim}{R})\, d(\underset{\sim}{H}) ;$$

where

$$T_1 = \Pi^{\frac{p^2}{2}} \Gamma_p(\tfrac{1}{2}(s+t)) \{\Gamma_p(\tfrac{1}{2}t)\, \Gamma_p(\tfrac{1}{2}s)\, \Gamma_p(\tfrac{1}{2}p)\}^{-1} \exp[-\mathrm{tr}\, \underset{\sim}{A}]$$

$$(\prod_{i=1}^{p} r_i)^{\frac{1}{2}(s-p-1)} \prod_{i=1}^{p} (1-r_i)^{\frac{1}{2}(t-p-1)} \prod_{i<j} (r_i - r_j) .$$

Let

$$(2.36) \qquad \underset{\sim}{R} = \mathrm{diag}(r_1, \ldots, r_p),\ 1 > r_1 > r_2 > \ldots > r_p > 0 ,$$

$$\underset{\sim}{A} = \mathrm{diag}(\ell_1, \ldots, \ell_p),\ \infty > \ell_1 > \ell_2 > \ldots > \ell_p \geq 0 ,$$

where $\underset{\sim}{R}$ and $\underset{\sim}{A}$ are otherwise specified in (2.34) and (2.33), $O(p)$ and $d(\underset{\sim}{H})$ are as specified in the earlier problem.

As stated earlier, as in relation to the canonical correlation problem we consider the following:

$$(2.37)\quad E_1 = \int_{O(p)} {}_1F_1\left(\tfrac{1}{2}(s+t);\ \tfrac{1}{2}s,\ \underset{\sim}{H}'\underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{R}\right)\ d(\underset{\sim}{H}) \quad .$$

The integrand as it stands is not easy to work with, hence we apply the confluence relation (James [9]).

$$(2.38)\qquad \underset{c \to \infty}{\mathrm{Lt}}\ {}_2F_1(a,c;b,c^{-1}\underset{\sim}{S}) = {}_1F_1(a;b,\underset{\sim}{S}) \quad .$$

Applying the dominated convergence theorem, since the functions involved are well defined, we get using (2.38)

$$\underset{a \to \infty}{\mathrm{Lt}} \int_{O(p)} {}_2F_1\left(\tfrac{1}{2}(s+t),\ a;\ \tfrac{1}{2}s,\ a^{-1}\underset{\sim}{H}'\underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{R}\right)\ d(\underset{\sim}{H})$$

$$= \int_{O(p)} \underset{a \to \infty}{\mathrm{Lt}}\ {}_2F_1\left(\tfrac{1}{2}(s+t),\ a;\ \tfrac{1}{2}s,\ a^{-1}\ \underset{\sim}{H}'\underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{R}\right)\ d(\underset{\sim}{H})$$

$$= \int_{O(p)} {}_1F_1\left(\tfrac{1}{2}(s+t);\ \tfrac{1}{2}s,\ \underset{\sim}{H}'\underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{R}\right)\ d(\underset{\sim}{H}) = E_1 \quad .$$

Thus, for evaluating E_1, we consider, for large a,

$$(2.39)\qquad E_2 = \int_{O(p)} {}_2F_1\left(\tfrac{1}{2}(s+t),a;\tfrac{1}{2}s,a^{-1}\underset{\sim}{H}'\underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{R}\right)\ d(\underset{\sim}{H}) \quad .$$

Thus we note that we can apply the earlier technique but with slight modification as would be noted in the process. For notational simplicity we use

$$(2.40) \qquad m = \frac{1}{2}(s+t) \quad \text{and} \quad n = \frac{1}{2}s \quad .$$

Now, using Kummer's relation given by James [9] we get

$$(2.41) \quad {}_2F_1(m,a;n,a^{-1}\underset{\sim}{H}'\underset{\sim}{A}\;\underset{\sim}{H}\;\underset{\sim}{R}) =$$

$$|\underset{\sim}{I}-a^{-1}\underset{\sim}{H}'\underset{\sim}{A}\;\underset{\sim}{H}\;\underset{\sim}{R}|^{n-m-a}\;{}_2F_1(n-m,n-a;n,a^{-1}\underset{\sim}{H}'\underset{\sim}{A}\;\underset{\sim}{H}\;\underset{\sim}{R}) \;.$$

Again

$$|\underset{\sim}{I}-a^{-1}\underset{\sim}{H}'\underset{\sim}{A}\;\underset{\sim}{H}\;\underset{\sim}{R}| =$$

$$|\underset{\sim}{I} + \underset{\sim}{H}'\underset{\sim}{A}\;\underset{\sim}{H}\;\underset{\sim}{D}|\;|\underset{\sim}{I}-(\underset{\sim}{I}+\underset{\sim}{H}'\underset{\sim}{A}\;\underset{\sim}{D}\;\underset{\sim}{H})^{-1}(\underset{\sim}{H}'\underset{\sim}{A}\;\underset{\sim}{H}\;\underset{\sim}{D} + a^{-1}\underset{\sim}{H}'\underset{\sim}{A}\;\underset{\sim}{H}\;\underset{\sim}{R})|$$

and

$$(2.42) \qquad \underset{\sim}{D} = \underset{\sim}{R}^{-1} \quad .$$

Thus we get from (2.41)

$$(2.43) \quad {}_2F_1(m,a;n,a^{-1}\underset{\sim}{H}'\underset{\sim}{A}\;\underset{\sim}{H}\;\underset{\sim}{R}) =$$

$$|\underset{\sim}{I} + \underset{\sim}{H}'\underset{\sim}{A}\;\underset{\sim}{H}\;\underset{\sim}{D}|^{n-m}|\underset{\sim}{I}-a^{-1}\underset{\sim}{H}'\underset{\sim}{A}\;\underset{\sim}{H}\;\underset{\sim}{R}|^{-a}|\underset{\sim}{I}-(\underset{\sim}{I}+\underset{\sim}{H}'\underset{\sim}{A}\;\underset{\sim}{H}\;\underset{\sim}{D})^{-1}$$

$$(\underset{\sim}{H}'\underset{\sim}{A}\;\underset{\sim}{H}\;\underset{\sim}{D} + a^{-1}\underset{\sim}{H}'\underset{\sim}{A}\;\underset{\sim}{H}\;\underset{\sim}{R})|^{n-m}\;{}_2F_1(n-m,n-a;n,a^{-1}\underset{\sim}{H}'\underset{\sim}{A}\;\underset{\sim}{H}\;\underset{\sim}{R}) \quad .$$

Also the integrand in (2.37) is maximized under the present set up when $\underset{\sim}{H}$ has the form (ii) in lemma 1.3. Now if we expand the last three factors in (2.43) around $\underset{\sim}{H}\varepsilon N(\underset{\sim}{I})$, applying lemma 2.1, we get

$$(2.44) \quad {}_2F_1(m,a;n,a^{-1}\underset{\sim}{H}'\underset{\sim}{A}\;\underset{\sim}{H}\;\underset{\sim}{R}) =$$

$$|\underset{\sim}{I} + \underset{\sim}{H}'\underset{\sim}{A}\;\underset{\sim}{H}\;\underset{\sim}{D}|^{-(m-n)}\;\Phi(m,n,a,\underset{\sim}{A},\underset{\sim}{D},\underset{\sim}{R}) + O(\varepsilon) \quad ,$$

where

$$\Phi(m,n,a,\underset{\sim}{A},\underset{\sim}{D},\underset{\sim}{R}) = |\underset{\sim}{I}-a^{-1}\underset{\sim}{A}\ \underset{\sim}{R}|^{-a}|\underset{\sim}{I}-(\underset{\sim}{I}+\underset{\sim}{A}\ \underset{\sim}{D})^{-1}$$

$$(\underset{\sim}{A}\ \underset{\sim}{D} + a^{-1}\underset{\sim}{A}\ \underset{\sim}{R})|^{-(m-n)}\ {}_2F_1(n-m,n-a;n,a^{-1}\underset{\sim}{A}\ \underset{\sim}{R}) .$$

Thus using (2.44) in (2.39) we get for large m

$$E_2 \simeq 2^p \int_{N(\underset{\sim}{I})} |\underset{\sim}{I} + \underset{\sim}{H}'\underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{D}|^{-(m-n)}\Phi(m,n,a,\underset{\sim}{A},\underset{\sim}{D},\underset{\sim}{R})d(\underset{\sim}{H}) + O(\underset{\sim}{\varepsilon}) .$$

Further we consider the following

$$E_3 = 2^p \int_{N(\underset{\sim}{I})} |I + \underset{\sim}{H}'\underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{D}|^{-(m-n)}d(\underset{\sim}{H}) ,$$

where

$$\underset{\sim}{D} = \mathrm{diag}(d_1,\ldots, d_p),\ \infty > d_p > \ldots > d_1 > 1 ,$$

and $\underset{\sim}{A}$ is as defined in (2.36). The integrand as it stands corresponds to that in Li and Pillai [13], [14], and hence following them as $m > n$ and for large m we get ,

$$E_3 \simeq 2^p|\underset{\sim}{I} + \underset{\sim}{A}\ \underset{\sim}{D}|^{-(m-n)} \prod_{i<j=1}^{p} \left(\frac{\Pi}{(m-n)c_{ij}}\right)^{\frac{1}{2}}$$

$$\{1 + \frac{1}{4(m-n)} [\sum_{i<j} c_{ij}^{-1} + \alpha(p)] + \ldots\} ,$$

where

$$(2.45) \qquad c_{ij} = \frac{(d_i-d_j)(\ell_j-\ell_i)}{(1+\ell_i d_i)(1+\ell_j d_j)} , \qquad i < j ,$$

and $\alpha(p)$ given in (2.17).

Thus putting all these results together we get the theorem:

Theorem 2.2. For large t (and hence for large sample size), an asymptotic expansion for the distribution of the characteristic roots of $\underset{\sim}{R}$ in (2.34) with parameter matrix $\underset{\sim}{A}$ as in (2.33), where $\underset{\sim}{R}$ and $\underset{\sim}{A}$ satisfy (2.36) is given by

$$T_1\, 2^p\, c_1^{-1} \prod_{i<j=1}^{p} \left(\frac{2\Pi}{t\, c_{ij}}\right)^{\frac{1}{2}} \{1 + \frac{1}{2t}[\sum_{i<j} c_{ij}^{-1} + \alpha(p)] + \dots\}$$

$$\exp[\operatorname{tr} \underset{\sim}{A}\,\underset{\sim}{R}]\ {}_1F_1(-\tfrac{1}{2}t;\ \tfrac{1}{2}s,\ -\underset{\sim}{A}\,\underset{\sim}{R}) + O(\varepsilon)\ ,$$

where T_1 is given by (2.35) and c_{ij} by (2.45).

2.5. Asymptotic Expansion for MANOVA - One Extreme Population Multiple Root

The problem involved here is quite similar to the previous problem with the difference that the matrix $\underset{\sim}{A} = \operatorname{diag}(\ell_1, \dots, \ell_p)$ defined in (2.33) now satisfies (2.46) instead of (2.36)

$$(2.46) \qquad \infty > \ell_1 > \ell_2 > \dots > \ell_k > \ell_{k+1} = \dots = \ell_p = \ell \geq 0\ .$$

Thus every step of the previous problem in canonical correlation, Section 2.3, follows smoothly and we come to the consideration of (2.39). But now as in Section 2.3 we get by lemma 1.3.2, the integrand in (2.33) is invariant of the choice of $\underset{\sim}{H}_2$ in (2.20). Thus, again, following the arguments and algebra as in Section 2.3, we get the following theorem (details of algebra are available from Li and Pillai [13], [14], with slight changes.).

Theorem 2.2.1. For large t (and hence for large sample size) an asymptotic expansion for the distribution of the characteristic roots of $\underset{\sim}{R}$ in (2.34) with parameter matrix satisfying (2.46), is given by

$$T_1 c_3 c_1^{-\frac{1}{2}k} \prod_{i<j=1}^{k} \left(\frac{2\pi}{t\, c_{ij}}\right)^{\frac{1}{2}} \prod_{i=1}^{k} \prod_{j=k+1}^{p} \left(\frac{2\pi}{t c_{ij}^{o}}\right)^{\frac{1}{2}} \exp[\mathrm{tr}\, \underset{\sim}{A}\, \underset{\sim}{R}]$$

$$\cdot\ {}_1F_1(-\tfrac{1}{2}t; \tfrac{1}{2}s, -\underset{\sim}{A}\,\underset{\sim}{R}) \{1 + \frac{1}{2t} [\sum_{i<j=1}^{k} c_{ij}^{-1} + \sum_{i=1}^{t} \sum_{j=k+1}^{p} c_{ij}^{o\,-1}$$

$$+ \alpha(p,k)] + \dots \} + O(\varepsilon) ,$$

where c_1, c_3 and T_1 are given by (2.22), (2.27) and (2.35) respectively, and c_{ij} and c_{ij}^{o} are defined as follows:

(2.47) $$c_{ij} = (t_{ji} - t_i t_j r_{ij}) r_{ij},\ i,j=1,\dots, k,\ i < j,\ (c_{ij} = c_{ji}) ,$$

$$c_{ij}^{o} = (t_{ji} - t_i t_j r_{ij}) r_{ij},\ i=1,\dots, k,$$

$$j=k+1,\dots, p,\ (c_{ij}^{o} = c_{ji}^{o}) ,$$

where

$$t_{ij} = t_i - t_j, \qquad r_{ij} = r_i - r_j ,$$

$$t_i = \begin{cases} \ell_i/(1+\ell_i d_i), & i = 1,\dots, k \\ \\ \ell/(1+\ell\, d_i), & i=k+1,\dots, p . \end{cases}$$

$$\alpha(p,\kappa) = \frac{k}{12} \{(k-1)(4k+1) + 6(p^2 - k^2)\}$$

and

$$\underset{\sim}{D} = \mathrm{diag}(d_1, \ldots, d_p) = \underset{\sim}{R}^{-1} .$$

2.6. Asymptotic Expansion for Canonical Correlation in the Complex Case - Population Roots all Distinct

Let $x_1, \ldots, x_p, x_{p+1}, \ldots, x_{p+q}$, $p \leq q$ be distributed complex normal $N_c(\underset{\sim}{0}, \underset{\sim}{\Sigma}_c)$, where

$$\underset{\sim}{\Sigma}_c = \begin{array}{cc} & \begin{array}{cc} p & q \end{array} \\ \begin{array}{c} p \\ q \end{array} & \begin{bmatrix} \underset{\sim}{\Sigma}_{c11} & \underset{\sim}{\Sigma}_{c12} \\ \underset{\sim}{\overline{\Sigma}}'_{c12} & \underset{\sim}{\Sigma}_{c22} \end{bmatrix} \end{array} .$$

Let $\underset{\sim}{P}_c^2 = \mathrm{diag}(\rho_{c_1}^2, \ldots, \rho_{c_p}^2)$, where $\rho_{c_i}^2$, $i = 1, \ldots, p$ are the roots of

$$(2.48) \qquad |\underset{\sim}{\Sigma}_{c12}\, \underset{\sim}{\Sigma}_{c22}^{-1}\, \underset{\sim}{\overline{\Sigma}}'_{c12} - \rho_c^2\, \underset{\sim}{\Sigma}_{c11}| = 0 ,$$

and let $\hat{\underset{\sim}{\rho}}_c^2 = \mathrm{diag}(\hat{\rho}_{c_1}^2, \ldots, \hat{\rho}_{c_p}^2)$, where $\hat{\rho}_{c_i}^2$, $i = 1, \ldots, p$, are the maximum likelihood estimators of $\rho_{c_i}^2$, $i = 1, \ldots, p$, from a sample of size $n \geq p+q$ from the above population. Then the joint density of

$$(2.49) \qquad \hat{\underset{\sim}{\rho}}_c^2 = \underset{\sim}{R} = \mathrm{diag}(r_1, \ldots, r_p) ,$$

is given by James [9], as

$$D_2 \int_{U(p)} {}_2\tilde{F}_1(n, n, q, \underset{\sim}{U}^* \underset{\sim}{A}\, \underset{\sim}{U}\, \underset{\sim}{R})\, d(\underset{\sim}{U}) ,$$

where

$$(2.50) \qquad \underset{\sim}{P}_c^2 = \underset{\sim}{A} = \mathrm{diag}(\ell_1,\ldots,\ell_p),\ 1 > \ell_1 > \ell_2 > \ldots > \ell_p \geq 0\ ,$$

$d(\underset{\sim}{U})$ is the invariant measure or the Haar measure defined on the group $U(p)$ of unitary matrices of order p,

$$(2.51) \qquad D_2 = [\pi^{p(p-1)}\tilde{\Gamma}_p(n)/\tilde{\Gamma}_p(n-q)\tilde{\Gamma}_p(q)\tilde{\Gamma}_p(p)]\,|\underset{\sim}{I}-\underset{\sim}{A}|^n|\underset{\sim}{R}|^{q-p}$$

$$|\underset{\sim}{I} - \underset{\sim}{R}|^{m-q-p} \prod_{i<j} (r_i - r_j)^2\ ,$$

$$\tilde{\Gamma}_p(t) = \pi^{\frac{1}{2}p(p-1)} \prod_{j=1}^{p} \Gamma(t-j+1)\ ,$$

where the hypergeometric function of the Hermitian matrix $\underset{\sim}{Z}$ is defined in [9] as

$$(2.52) \qquad {}_\mu\tilde{F}_\eta(a_1,\ldots,a_\mu;\ b_1,\ldots,b_\eta,\ \underset{\sim}{Z}) = \sum_{k=0}^{\infty}\sum_{\kappa} \frac{[a_1]_\kappa \cdots [a_\mu]_\kappa}{[b_1]_\kappa \cdots [b_\eta]_\kappa}\frac{\tilde{C}_\kappa(\underset{\sim}{Z})}{k!}$$

where

$$[a]_\kappa = \prod_{i=1}^{p} (a - i+1)_{k_i}\ ,$$

and $\kappa = (k_1,\ldots,k_p)$ is a partition of the integer k.

Let us now consider the elements of $\underset{\sim}{R}$ and $\underset{\sim}{A}$ as in (2.7) and (2.50) respectively. Then as in the real case we consider the following integral:

$$\int_{U(p)} {}_2\tilde{F}_1(n,n;a,\underset{\sim}{U}^*\underset{\sim}{A}\ \underset{\sim}{U}\ \underset{\sim}{R})d(\underset{\sim}{U})\ .$$

To study the above integral let us consider a lemma analogous to lemma 2.1 in the real case.

Lemma 2.1.1. If $\underset{\sim}{U} \epsilon U(p)$, and a_i's and b_j's are real ($i = 1,\ldots, \mu$, $j = 1,\ldots, \eta$), then

(i) $\quad {}_{\mu}\tilde{F}_{\eta}(a_1,\ldots, a_{\mu}; b_1,\ldots, b_{\eta}, \underset{\sim}{U}^*\underset{\sim}{A}\,\underset{\sim}{U}\,\underset{\sim}{R})$ is real and if

(2.53) $\quad \underset{\sim}{U} \epsilon N(\underset{\sim}{I}),\ a_i \geq (p-1),\ b_j \geq (p-1),\ (i = 1,\ldots, \mu,\ j=1,\ldots, \eta)$

then

(ii) $$ {}_{\mu}\tilde{F}_{\eta}(a_1,\ldots, a_{\mu}; b_1,\ldots, b_{\eta}, \underset{\sim}{U}^*\underset{\sim}{A}\,\underset{\sim}{U}\,\underset{\sim}{R}) = $$

$$ {}_{\mu}\tilde{F}_{\eta}(a_1,\ldots, a_{\mu}; b_1,\ldots, b_{\eta}, \underset{\sim}{A}\,\underset{\sim}{R}) + O(\varepsilon)\ , $$

provided

$$ t_i - \varepsilon \leq ch_i(\underset{\sim}{U}^*\underset{\sim}{A}\,\underset{\sim}{U}\,\underset{\sim}{R}) \leq t_i + \varepsilon\ , $$

where

$$ t_i = ch_i(\underset{\sim}{A}\,\underset{\sim}{R}), \qquad i = 1,\ldots, p\ . $$

For proving (i), in view of (2.52), and a_i's and b_j's being real, it will suffice if we can show that $\tilde{C}_{\kappa}(\underset{\sim}{U}^*\underset{\sim}{A}\,\underset{\sim}{U}\,\underset{\sim}{R})$ is real. This has been shown in lemma 1.3.3.

Now as $ch_i(\underset{\sim}{U}^*\underset{\sim}{A}\,\underset{\sim}{U}\,\underset{\sim}{R})$ is real ($i = 1,\ldots, p$), and nonnegative in this case, under (2.53), we get (2.52) is an increasing function in each characteristic root. Result (ii) now follows by arguments similar to those in lemma 2.1.

Now as is done in the real case using Kummer's formula we get

$$ {}_2\tilde{F}_1(n,n;q,\underset{\sim}{U}^*\underset{\sim}{A}\ \underset{\sim}{U}\ \underset{\sim}{R}) = |\underset{\sim}{I} - \underset{\sim}{U}^*\underset{\sim}{A}\ \underset{\sim}{U}\ \underset{\sim}{R}|^{-(2n-q)}\ {}_2\tilde{F}_1((q-n),(q-n);q,\underset{\sim}{U}^*\underset{\sim}{A}\ \underset{\sim}{U}\ \underset{\sim}{R}) $$

and using lemma 2.1.1 and following Li and Pillai, [13], [14], we get the following theorem.

<u>Theorem 2.1.2</u>. For large n, the asymptotic expansion for the distribution of $r_1,\ldots, r_p$, in (2.49) where $1 > r_1 > \ldots > r_p > 0$ and the parameters from (2.48) satisfy (2.50), is given by

$$ D_4^{-1} D_2 \prod_{i<j=1}^{p} \left(\frac{\pi}{(2n-q)c_{ij}}\right) \{1 + \frac{1}{3(2n-q)} [\sum_{i<j} c_{ij}^{-1} + \beta(p)] + \ldots \} $$

$$ |\underset{\sim}{I} - \underset{\sim}{A}\ \underset{\sim}{R}|^{-(2n-q)}\ {}_2\tilde{F}_1((q-n),(q-n);\ q,\ \underset{\sim}{A}\ \underset{\sim}{R}) + O(\varepsilon) , $$

where $\beta(p) = p(p-1)(2p-1)/12$, $\underset{\sim}{R}$, $\underset{\sim}{A}$ and D_2 are given by (2.49), (2.50) and (2.51) respectively and c_{ij} are defined as in (2.16) with r_i and ℓ_j being replaced by corresponding elements of (2.49) and (2.50), $D_4 = \pi^{p(p-1)}\{\tilde{\Gamma}_p(p)\}^{-1}$.

<u>2.7. Asymptotic Expansion for Canonical Correlation in the Complex Case - One Extreme Multiple Population Root</u>

In this case we have the same model as in the distinct root case with the change that $\underset{\sim}{A}$ defined in (2.50) has the form

$$ (2.54)\quad \underset{\sim}{P}_c^2 = \underset{\sim}{A} = \operatorname{diag}(\ell_1,\ldots, \ell_p),\ 1 > \ell_1 > \ell_2 > \ldots > \ell_k > \ell_{k+1} = \ldots = \ell_p = \ell \geq 0 , $$

where $\underset{\sim}{P}_c^2$ is as defined in (2.48). Now, with necessary modifications in our procedure in the real case, and using lemma 2.1.1, and following Li and Pillai [15] we get the theorem:

Theorem 2.1.3. For large n, the asymptotic expansion for the distribution of $r_1, \ldots, r_p$, where $1 > r_1 > \ldots > r_p > 0$ and the parameters from (2.48) satisfy (2.54), is given by

$$D_4^{-1} D_2 D_3 \prod_{i<j=1}^{k} \left(\frac{\pi}{(2n-q)c_{ij}}\right) \prod_{i=1}^{k} \prod_{j=k+1}^{p} \left(\frac{\pi}{(2n-q)c_{ij}^{o}}\right) \{1 + \frac{1}{3(2n-q)} [\sum_{i<j=1}^{k} c_{ij}^{-1}$$

$$+ \sum_{i=1}^{k} \sum_{j=k+1}^{p} c_{ij}^{o^{-1}} + \beta(p,k)] + \ldots \} |\underset{\sim}{I} - \underset{\sim}{A}\, \underset{\sim}{R}|^{-(2n-q)}$$

$$_2\tilde{F}_1 \;((q-n),\, (q-n);q,\underset{\sim}{A}\, \underset{\sim}{R}) + O(\varepsilon) \;,$$

where

$$D_3 = \pi^{q(q-1)} \{\tilde{\Gamma}_q(q)\}^{-1}$$

and $q=p-k$, D_2 is as in (2.51), c_{ij} and c_{ij}^{o} are as in the real case with r_i's and ℓ_j's substituted from (2.49) and (2.54) and $\beta(p,k) = \frac{k}{2}\{(k-1)(2k-1) + 3(p-k)(p+k-1)\}$.

2.8. Asymptotic Expansion for MANOVA in the Complex Case - Population Roots all Distinct

Let $\underset{\sim}{B}_1$ be the Between S.P. matrix and $\underset{\sim}{W}_1$ the Within S.P. matrix in a complex multivariate normal case. Then $\underset{\sim}{B}_1$ $(p \times p)$ has a complex non-central Wishart distribution with s d.f. and matrix of noncentrality parameter $\underset{\sim}{A}$ and $\underset{\sim}{W}_1$ has a complex central Wishart distribution on t d.f., the covariance matrix in each case being $\underset{\sim}{\Sigma}_1$,

and

(2.55) $$\underset{\sim}{A} = \underset{\sim}{\mu}_1 \underset{\sim}{\mu}_1' \underset{\sim}{\Sigma}_1^{-1}$$

where $\underset{\sim}{\mu}_1 (p \times s)$ is the matrix of the mean vectors. Then the density of the roots of the matrix

(2.56) $$\underset{\sim}{R} = \underset{\sim}{B}_1 (\underset{\sim}{W}_1 + \underset{\sim}{B}_1)^{-1}$$

is given by [9]

$$T_2 \int_{U(p)} {}_1\tilde{F}_1((s+t);s, \underset{\sim}{U}^* \underset{\sim}{A}\, \underset{\sim}{U}\, \underset{\sim}{R}) d(\underset{\sim}{U}) ,$$

where

(2.57) $$T_2 = [\pi^{p(p-1)} \tilde{\Gamma}_p(s+t)/\tilde{\Gamma}_p(p)\tilde{\Gamma}_p(s)\tilde{\Gamma}_p(t)]\,|\underset{\sim}{I}-\underset{\sim}{R}|^{t-p}$$
$$|\underset{\sim}{R}|^{(s-p)} \prod_{i<j} (r_i - r_j)^2 \exp[-\mathrm{tr}\, \underset{\sim}{A}] ,$$

where

(2.58) $$\underset{\sim}{R} = \mathrm{diag}(r_1, \ldots, r_p),\ 1 > r_1 > \ldots > r_p > 0$$
$$\underset{\sim}{A} = \mathrm{diag}\ (\ell_1, \ldots, \ell_p),\ \infty > \ell_1 > \ldots > \ell_p > 0$$

and $d(\underset{\sim}{U})$ is as defined earlier. Now as in real case we consider

$$\int_{U(p)} {}_1\tilde{F}_1((s+t);\ s,\ \underset{\sim}{U}^* \underset{\sim}{A}\, \underset{\sim}{U}\, \underset{\sim}{R}) d(\underset{\sim}{U}) .$$

But as stated in the real case, the integrand as it stands is difficult to work with and as such we consider, instead, for large a

$$\int_{U(p)} {}_2\tilde{F}_1\,((s+t),\ a;\ s,\ a^{-1}\underset{\sim}{U}^*\underset{\sim}{A}\,\underset{\sim}{U}\,\underset{\sim}{R})d(\underset{\sim}{U})\ .$$

Thus proceeding as before and by Li and Pillai [13], [14], we get the theorem:

Theorem 2.2.2. For large t (and hence for large sample size), the asymptotic expansion for the distribution of the characteristic roots of $\underset{\sim}{R}$ in (2.56) where $\underset{\sim}{R}$ and the parameter matrix $\underset{\sim}{A}$ in (2.55) satisfy (2.58), is given by

$$D_4^{-1}\ T_2 \prod_{i<j} \left(\frac{\pi}{t\ c_{ij}}\right) \{1 + \frac{1}{3t}\,[\sum_{i<j} c_{ij}^{-1} + \beta(p)] + \ldots\}\ \exp\,[\mathrm{tr}\ \underset{\sim}{A}\ \underset{\sim}{R}]$$

$${}_1\tilde{F}_1\ (-t;\ s,\ -\underset{\sim}{A}\ \underset{\sim}{R}) + O(\varepsilon)\ ,$$

where c_{ij} is given by (2.45) using $\underset{\sim}{A}$ and $\underset{\sim}{R}$ from (2.55) and (2.56) respectively, T_2 is given by (2.57) and $\beta(p) = p(p-1)(2p-1)/12$.

2.9. Asymptotic Expansion for MANOVA in the Complex Case - One Extreme Multiple Population Root

As in the canonical correlation case with one extreme multiple population root, here again the model is the same as in the distinct root case with the change that $\underset{\sim}{A}$ defined in (2.55) has the following form

$$(2.59)\quad \underset{\sim}{\mu}_1\underset{\sim}{\mu}_1'\underset{\sim}{\Sigma}_1^{-1} = \underset{\sim}{A} = \mathrm{diag}(\ell_1,\ldots,\ \ell_p);\ \infty > \ell_1 > \ldots > \ell_k > \ell_{k+1} = \ldots = \ell_p \geq 0\ ,$$

where $\underset{\sim}{\mu}_1$, $\underset{\sim}{\Sigma}_1$ are defined as in the distinct root case. Now proceeding as in the earlier case with necessary changes and following Li and Pillai [15] we get the theorem:

Theorem 2.2.5. For large t (and hence for large sample size), the asymptotic expansion for the distribution of the characteristic roots of $\underset{\sim}{R}$ in (2.56) where $1 > r_1 > \ldots > r_p > 0$ and parameter matrix $\underset{\sim}{A}$ in (2.55) satisfy (2.59), is given by

$$D_4^{-1} T_2 D_3 \prod_{i<j=1}^{k} \left(\frac{\Pi}{t\, c_{ij}}\right) \prod_{i=1}^{k} \prod_{j=k+1}^{p} \left(\frac{\Pi}{t\, c_{ij}^{o}}\right) \{ 1 + \frac{1}{3t} \left[\sum_{i<j=1}^{k} c_{ij}^{-1} \right.$$

$$\left. + \sum_{i=1}^{k} \sum_{j=k+1}^{p} c_{ij}^{o\,-1} + \beta(p,k) \right] + \ldots \} \exp[\operatorname{tr} \underset{\sim}{A}\, \underset{\sim}{R}]$$

$${}_1\tilde{F}_1\,(-t;\, s,\, - \underset{\sim}{A}\, \underset{\sim}{R}) + O(\varepsilon)\ ,$$

where

$$D_3 = \Pi^{q(q-1)} \{ \tilde{\Gamma}_q(q) \}^{-1},\ \beta(p,k) = \frac{k}{2}\{ (k-1)(2k-1) + 3(p-k)(p+k-1) \}$$

and $q = p-k$, c_{ij} and c_{ij}^{o} are as defined in (2.47) but the r_i and ℓ_j's being taken from (2.58) and (2.59) respectively.

2.10. Remarks

As will be noted from the following there are some general remarks which apply to all cases discussed above and some others which pertain only to special cases.

1. The method as outlined above is a generalization of Anderson's result [1] and all his comments are applicable here also. Note especially the following one.

2. No proof has been given to show that we have an asymptotic expansion of the integrals involved in each case, but application of an extension of Laplace's method as given by Hsu [6] can be utilized to show that in each case the first term gives an asymptotic representation and has been explicitly shown by Chang [3] and hence we just refer to his result.

3. In approximating ${}_{\mu}F_{\eta}$ or ${}_{\mu}\tilde{F}_{\eta}$ by Kummer's formula we note that if we take $N(\underset{\sim}{I})$ involved in each case to be sufficiently close to $\underset{\sim}{I}$, which is possible for large enough sample size, we can neglect $O(\varepsilon)$ in each case for good enough approximation.

4. The direction of ordering of roots in each problem is immaterial and as such the only restriction is that the roots of the sample and population matrices should be ordered in the same direction.

5. From remark 4 it may be seen that the expansion for one extreme multiple population root covers the largest root although the results given in the paper are for the smallest.

6. Each formula, as given, gives a considerable simplification in the ${}_{\mu}F_{\eta}$ function since each population root goes along with its sample counterpart.

7. In the real case when a in $(a)_k$ is a negative integer the hypergeometric function involved reduces to a polynomial. In the complex situation a constant being negative ${}_{\mu}\tilde{F}_{\eta}$ always reduces to a polynomial expansion.

8. When all the population roots are equal we see that $O(\varepsilon)$ term in our expansion is identically zero. Here we have to take any empty product to be unity.

9. Though in Chapter 1 the limits of the elements of the matrices $\underset{\sim}{A}$ and $\underset{\sim}{R}$ are taken to be the whole real line, it does not matter even if we take it to be any interval [a,b], where a and b are two distinct real numbers.

CHAPTER III

ASYMPTOTIC EXPANSIONS FOR THE DISTRIBUTIONS OF CHARACTERISTIC ROOTS WHEN THE PARAMETER MATRIX HAS SEVERAL MULTIPLE ROOTS

3.1. Introduction

In the preceding chapter asymptotic expansions for the distributions of characteristic roots of matrices arising in MANOVA and canonical correlation case are obtained when the parameter matrix has a single multiple root, extreme or intermediate. However, in extending the work further to the case of several multiple population roots, the method used in [10] was not found to be suitable in view of the fact that the invariance of a function with respect to the choice of a sub-matrix in the orthogonal (unitary) matrix used there does not extend to the simultaneous invariance with respect to the choices of several submatrices as is needed to extend that method. In order to overcome this difficulty we proceed in a different manner without recourse to the invariance property and restate lemma 1.3.1 in a more detailed fashion before demonstrating the new approach.

3.2. The Maximization Procedures

Let us define $\underset{\sim}{R} = \mathrm{diag}(r_1, \ldots, r_p)$; $\infty > r_1 > r_2 > \ldots > r_p > 0$.

(3.1) $\quad \underset{\sim}{A} = \mathrm{diag}\,(\overbrace{\ell_1, \ldots, \ell_1}^{k_1}, \overbrace{\ell_2, \ldots,}^{k_2} \ell_{k_1+k_2+1}, \ldots, \ell_p)$,

$$\infty > \ell_1 > \ell_2 > \ell_{k_1+k_2+1} > \cdots > \ell_p \geq 0 \, ,$$

and let $\underset{\sim}{H}\varepsilon O(p)$, where $O(p)$ is the group of orthogonal matrices of order p. Then we state the following lemma.

<u>Lemma 3.1</u>. If (3.1) holds, then for all variations of $\underset{\sim}{R} > 0$

$$\max_{\underset{\sim}{H}\varepsilon O(p)} C_\kappa(\underset{\sim}{H}'\underset{\sim}{A}\underset{\sim}{H}\underset{\sim}{R}) = \max_{\underset{\sim}{H}\varepsilon O(p)} C_\kappa(\underset{\sim}{H}\underset{\sim}{R}\underset{\sim}{H}'\underset{\sim}{A}) = C_\kappa(\underset{\sim}{A}\underset{\sim}{R})$$

and if the ordering of the elements of $\underset{\sim}{A}$ is reversed then

$$\min_{\underset{\sim}{H}\varepsilon O(p)} C_\kappa(\underset{\sim}{H}'\underset{\sim}{A}\underset{\sim}{H}\underset{\sim}{R}) = \min_{\underset{\sim}{H}\varepsilon O(p)} C_\kappa(\underset{\sim}{H}\underset{\sim}{R}\underset{\sim}{H}'\underset{\sim}{A}) = C_\kappa(\underset{\sim}{A}\underset{\sim}{R}) \, .$$

The optimum values are attained iff $\underset{\sim}{H}$ has the form

$$\underset{\sim}{H} = \begin{array}{cc} \begin{array}{ccc} k_1 & k_2 & q \end{array} & \\ \begin{bmatrix} \underset{\sim}{H}_1 & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{H}_2 & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{I}_o \end{bmatrix} & \begin{array}{c} k_1 \\ k_2 \\ q \end{array} \end{array} ,$$

where $q = p - k_1 - k_2$ and $\underset{\sim}{I}_o(q) = \text{diag}\,(\pm 1, \ldots, \pm 1)$.

<u>Proof</u>. By lemma 1.3.2, we get $\underset{\sim}{H}$ must have the form

$$\underset{\sim}{H} = \begin{bmatrix} \underset{\sim}{H}_1 & \underset{\sim}{H}_{11} & \underset{\sim}{H}_{12} \\ \underset{\sim}{0} & \underset{\sim}{H}_2 & \underset{\sim}{H}_{22} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{I}_o \end{bmatrix}$$

But, because of the orthogonality of $\underset{\sim}{H}$, we get, $\underset{\sim}{H}_{12} = \underset{\sim}{H}_{22} = \underset{\sim}{0}$ and which in turn gives $\underset{\sim}{H}_2 \underset{\sim}{H}_2' = \underset{\sim}{H}_2' \underset{\sim}{H}_2 = \underset{\sim}{I}(k_2)$. Thus, $\underset{\sim}{H}_{11} = \underset{\sim}{0}$, and hence the proof.

The proof just outlined is general and also goes through in the complex analogue of this problem when $\underset{\sim}{H}$ is replaced by $\underset{\sim}{U}$, where $\underset{\sim}{U}\varepsilon U(p)$, and $U(p)$ is the group of unitary matrices. Thus let us consider the following formalization.

$$\underset{\sim}{R} = \mathrm{diag}(r_1, \ldots, r_p),\ \infty > r_1 > r_2 > \ldots > r_p > 0$$

$$(3.2)\quad \underset{\sim}{A} = \mathrm{diag}(\overbrace{\ell_1, \ldots, \ell_1}^{k_1}, \ldots, \overbrace{\ell_m, \ldots, \ell_m}^{k_m}, \ell_{k_1+\ldots+k_m+1}, \ldots, \ell_p)\ ,$$

$$\infty > \ell_1 > \ell_2 > \ldots > \ell_m > \ell_{k_1+\ldots+k_m+1} > \ldots > \ell_p \geq 0\ ,$$

and let $\underset{\sim}{H}\varepsilon O(p)$. Then

<u>Lemma 3.1.1</u>. If (3.2) holds, then for all variations of $\underset{\sim}{R} > 0$

$$\max_{\underset{\sim}{H}\varepsilon O(p)} C_\kappa(\underset{\sim}{H}' \underset{\sim}{A} \underset{\sim}{H} \underset{\sim}{R}) = \max_{\underset{\sim}{H}\varepsilon O(p)} C_\kappa(\underset{\sim}{H} \underset{\sim}{R} \underset{\sim}{H}' \underset{\sim}{A}) = C_\kappa(\underset{\sim}{A} \underset{\sim}{R})$$

and if the ordering of the elements of $\underset{\sim}{A}$ is reversed then

$$\min_{\underset{\sim}{H}\varepsilon O(p)} C_\kappa(\underset{\sim}{H}' \underset{\sim}{A} \underset{\sim}{H} \underset{\sim}{R}) = \min_{\underset{\sim}{H}\varepsilon O(p)} C_\kappa(\underset{\sim}{H} \underset{\sim}{R} \underset{\sim}{H}' \underset{\sim}{A}) = C_\kappa(\underset{\sim}{A} \underset{\sim}{R})$$

and the optimum values are attained iff $\underset{\sim}{H}$ has the form

$$\underset{\sim}{H} = \mathrm{diag}(\underset{\sim}{H}_1, \ldots, \underset{\sim}{H}_m, \underset{\sim}{I}_o(p-k_1 - \ldots - k_m))\ ,$$

where $\underset{\sim}{H}_j$ $(k_j \times k_j)$ is an orthogonal matrix of order k_j, $j=1,\ldots, m$, and

$$\tilde{I}_o(p-k_1 - \dots - k_m) = \text{diag}\ (\pm 1, \dots, \pm 1)\ .$$

In order to facilitate the subsequent generalization to the complex case we state an analogue of lemma 3.1.1, the proof being self-evident from previous discussion.

Lemma 3.2.2. If (3.2) holds, then for all variations of $\tilde{R} > 0$

$$\max_{\tilde{U}\epsilon U(p)} \tilde{C}_\kappa(\tilde{U}^* \tilde{A} \tilde{U} \tilde{R}) = \max_{\tilde{U}\epsilon U(p)} \tilde{C}_\kappa(\tilde{U} \tilde{R} \tilde{U}^* \tilde{A}) = \tilde{C}_\kappa(\tilde{A} \tilde{R})$$

and if the ordering of the elements of $\tilde{A}$ is reversed then

$$\min_{\tilde{U}\epsilon U(p)} \tilde{C}_\kappa(\tilde{U}^* \tilde{A} \tilde{U} \tilde{R}) = \min_{\tilde{U}\epsilon U(p)} \tilde{C}_\kappa(\tilde{U} \tilde{R} \tilde{U}^* \tilde{A}) = \tilde{C}_\kappa(\tilde{A} \tilde{R})$$

and the optimum values are attained iff $\tilde{U}$ has the form $\tilde{U} = \text{diag}\ (\tilde{U}_1, \dots, \tilde{U}_m, \tilde{U}_{m+1})$, where $\tilde{U}_j(k_j \times k_j)$ is an unitary matrix of order k_j, $j=1,\dots,m$ and $\tilde{U}_{m+1} = \text{diag}\ (e^{\sqrt{-1}\ \theta_1}, \dots, e^{\sqrt{-1}\ \theta_q})$, $0 \leq \theta_j < 2\pi$, $j=1,\dots,q$, and $q = p-k_1 - \dots - k_m$.

Using the above results we get corresponding results for theorem 1.1.2 and its complex analogue theorem 1.1.3.

3.3. Asymptotic Expansion for the Distribution of the Latent Roots of the Estimated Covariance Matrix -- Several Multiple Population Roots

Let $\tilde{R} = \text{diag}\ (r_1, \dots, r_p)$, $\infty > r_1 > \dots > r_p > 0$, where r_i's are the latent roots in discending order, of a sample covariance matrix $\tilde{C}$ with n d.f. calculated from a sample from a normal population with covariance matrix $\tilde{\Sigma}$. Let the diagonal matrix of the latent roots of $\tilde{\Sigma}^{-1}$ be $\tilde{A}$ and $\tilde{A}$ has the form

(3.3) $$\underset{\sim}{A} = \text{diag}\,(\ell_1, \ldots, \ell_q, \overbrace{\ell_{q+1}, \ldots, \ell_{q+1}}^{k_1}, \ldots, \overbrace{\ell_{q+m}, \ldots, \ell_{q+m}}^{k_m}),$$
$$\underset{\sim}{R} = \text{diag}\,(r_1, \ldots, r_p), \ \infty > r_1 > r_2 > \ldots\ldots > r_p > 0,$$

(3.4) $$\infty > \ell_{q+m} > \ldots\ldots > \ell_{q+1} > \ell_q > \ldots\ldots > \ell_1 \geq 0$$

where $p = k_1 + \ldots + k_m + q$. Then the joint distribution of $r_1, \ldots, r_p$, is

(3.5) $$C_1 \int_{O(p)} \exp\left(-\frac{n}{2}\,\text{tr}\,\underset{\sim}{H}'\,\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}\right)\ d(\underset{\sim}{H})$$

where

$$C_1 = \{n^{\frac{np}{2}}\ \pi^{\frac{p^2}{2}}\ /\ [2^{\frac{np}{2}}\ \Gamma_p(\tfrac{n}{2})\ \Gamma_p(\tfrac{p}{2})\ ]\}\ \prod_{i=1}^{q} \ell_i^{\frac{n}{2}}$$

$$\prod_{j=1}^{m} \ell_{q+j}^{\frac{nk_j}{2}} \prod_{i=1}^{p} r_i^{\frac{(n-p-1)}{2}} \prod_{i<j} (r_i - r_j) \prod_{i=1}^{p} dr_i\ .$$

Now by lemma 3.1.1. and as shown in chapter one, the integrand in (3.5) is maximized for all variations of $\underset{\sim}{R} > 0$, when $\underset{\sim}{H}$ has the following form

(3.6) $$\underset{\sim}{H} = \text{diag}\,(\underset{\sim}{I}_o\ (q), \underset{\sim}{H}_1, \ldots, \underset{\sim}{H}_m),$$

where

$\underset{\sim}{H}_i\ (k_i \times k_i)$, $i = 1, \ldots, m$, are orthoganal matrices.

As stated earlier we do not resort to the invariance technique as used by earlier authors. Now following Anderson [1] we use the transformation

$$H = \exp [S] , \tag{3.7}$$

where S is a $p \times p$ skew symmetric matrix. Now under (3.4), the transformation (3.7) reduces the integrand in (3.5) to a form which does not yield to direct evaluation. Hence to avoid this difficulty we note that if (3.4) holds, then for all $R > 0$ the integrand in (3.5) is maximized when H has the form (3.6). Also when n is large the whole integral is concentrated around its unique maximum value. Thus, instead of (3.7) we use the transformation

$$H = \exp [S_1] , \tag{3.8}$$

where S_1 is a $p \times p$ skew symmetric matrix but has the following form

$$S_1 = \begin{bmatrix} S_0 \\ S_1 \\ S_m \end{bmatrix} /$$

S_0 $(q \times p)$; $S_i = (S_{i1}, S_{i2}, S_{i3})$;

S_{i1} $(k_i \times (q + k_1 + \dots + k_{i-1}))$, S_{i2} $(k_i \times k_i) =$

0, S_{i3} $(k_i \times (k_{i+1} + \dots + k_m))$, $i = 1,\dots, m-1$ and $S_m =$

(S_{m1}, S_{m2}); S_{m1} $(k_m \times (p-k_m))$, S_{m2} $(k_m \times k_m) = 0$.

This is no loss of generality provided the constant factor is adjusted, as for large n the integrand is concentrated around its unique maximum and at least one maximizing set is covered by this substitution. Let $g = p-k_m$. Then

$$\begin{aligned}
\operatorname{tr}(\underset{\sim}{H}' \underset{\sim}{A} \underset{\sim}{H} \underset{\sim}{R}) &= \operatorname{tr}(\underset{\sim}{A} \underset{\sim}{H} \underset{\sim}{R} \underset{\sim}{H}') \\
&= \sum_{ij} h_{ij}^2 \, \ell_i \, r_j \\
&= \sum_{i=1}^{g} \sum_{j=1}^{p} \ell_i \, r_j \, h_{ij}^2 + \ell_{q+m} \sum_{j=1}^{p} r_j \sum_{i=g+1}^{p} h_{ij}^2 \\
&= \ell_{q+m} \sum_{j=1}^{p} r_j + \sum_{i=1}^{g} \sum_{j=1}^{p} (\ell_i - \ell_{q+m}) \, r_j \, h_{ij}^2 \, ,
\end{aligned}$$

since

$$\sum_{i=g+1}^{p} h_{ij}^2 = 1 - \sum_{i=1}^{g} h_{ij}^2 \qquad \text{for} \qquad j = 1, \ldots, p \, .$$

For large n and ℓ_i's and r_j's well spaced, most of the integrand in (3.5) will be given by small values of $\underset{\sim}{S}_1$.

Now under (3.8)

$$h_{ii} = 1 - \frac{1}{2} \sum_{j=1}^{p} s_{ij}^2 + \text{higher order terms in } s_{ij}\text{'s}$$

$$h_{ij} = S_{ij} + \text{higher order terms in } s_{ij}\text{'s.}$$

Thus we get

$$\begin{aligned}
(3.9) \quad \operatorname{tr}(\underset{\sim}{H}' \underset{\sim}{A} \underset{\sim}{H} \underset{\sim}{R}) &= \ell_{q+m} \sum_{j=1}^{p} r_j + \sum_{i=1}^{g} (\ell_i - \ell_{q+m}) \, r_i \, (1 - \sum_{j=1}^{p} s_{ij}^2) \\
&\quad + \sum_{i=1}^{g} \sum_{j=1}^{p} (\ell_i - \ell_{q+m}) \, r_j \, s_{ij}^2 + \text{higher order terms in } s_{ij}\text{'s.}
\end{aligned}$$

$$= \ell_{q+m} \sum_{j=1}^{p} r_j + \sum_{i=1}^{g} (\ell_i - \ell_{q+m}) r_i - \sum_{i=1}^{g} (\ell_i - \ell_{q+m}) r_i \sum_{j=1}^{p} s_{ij}^2$$

$$+ \sum_{i=1}^{g} \sum_{j=1}^{p} (\ell_i - \ell_{q+m}) r_j s_{ij}^2 + \text{higher order terms involving } s_{ij}\text{'s} .$$

$$= \sum_{i=1}^{p} \ell_i r_i - \sum_{i=1}^{q} (\ell_i - \ell_{q+m}) r_i \sum_{j=1}^{p} s_{ij}^2 - \sum_{i=q+1}^{g} (\ell_i - \ell_{q+m}) r_i \sum_{j=1}^{p} s_{ij}^2$$

$$+ \sum_{i=1}^{q} \sum_{j=1}^{p} (\ell_i - \ell_{q+m}) r_j s_{ij}^2 + \sum_{i=q+1}^{g} \sum_{j=1}^{p} (\ell_i - \ell_{q+m}) r_j s_{ij}^2 +$$

higher order terms involving s_{ij}'s .

$$= \sum_{i=1}^{p} \ell_i r_i + \sum_{\substack{i=1 \\ i<j}}^{q} \sum_{j=1}^{p} (\ell_i - \ell_j)(r_j - r_i) s_{ij}^2 +$$

$$\sum_{u=1}^{m} \sum_{i=q_u}^{q_{u+1}-1} \sum_{j=q_{u+1}}^{p} (\ell_i - \ell_j)(r_j - r_i) s_{ij}^2$$

$$+ \text{higher order terms involving } s_{ij}^2 ,$$

(3.10) where $q_1 = q + 1$, $q_i = q + \sum_{j=1}^{i-1} k_j + 1$, $i = 2, \ldots, m + 1$.

Substituting (3.9) in (3.5) we note that the integrand tends to zero as each $s_{ij} \to \infty$. Also for large n and for ℓ_i's and r_j's well spaced we can approximate the integral over $\underset{\sim}{N}$ ($\underset{\sim}{S}_1 = 0$) by varying each s_{ij} over the whole real line i.e. $-\infty < s_{ij} < \infty$ for each pair (i,j) which involves in our representation (3.8). Thus for large n, noting that the maximum of the integrand in (3.5) is attained when $\underset{\sim}{H}$ has the form (3.6), we get, following Anderson [1]

$$\int_{O(p)} \exp\left[-\frac{n}{2} \operatorname{tr} \underset{\sim}{H}' \underset{\sim}{A} \underset{\sim}{H} \underset{\sim}{R}\right] d(\underset{\sim}{H}) = 2^q \prod_{i=1}^{m} \omega_i\, \omega_{m+1}^{-1}$$

$$\exp\left[-\frac{n}{2} \operatorname{tr} \underset{\sim}{A} \underset{\sim}{R}\right] \prod_{\substack{i=1\\ i<j}}^{q} \prod_{j=1}^{p} \left(\frac{2\pi}{nc_{ij}}\right)^{\frac{1}{2}} \prod_{u=1}^{m} \prod_{i=q_u}^{q_{u+1}-1} \prod_{j=q_{u+1}}^{p} \left(\frac{2\pi}{nc^o_{ij}}\right)^{\frac{1}{2}}$$

$$\left[1 + \frac{1}{2n}\left(\sum_{\substack{i=1\\ i<j}}^{q} \sum_{j=1}^{p} c_{ij}^{-1} + \sum_{u=1}^{m} \sum_{i=q_u}^{q_{u+1}-1} \sum_{j=q_{u+1}}^{p} c^{o\,-1}_{ij}\right) + \ldots\ldots\right],$$

where

$$(3.11) \quad c_{ij} = (\ell_i - \ell_j)(r_j - r_i),\ i = 1,\ldots, q,\ j = 1,\ldots, p$$

and

$c^o_{ij} = (\ell_i - \ell_j)(r_j - r_i)$, i and j varying over the indicated set where it is non-zero.

$$\omega_i = \Pi^{\frac{k_i^2}{2}} \{\Gamma_{k_i}(\tfrac{k_i}{2})\}^{-1},\ i=1,\ldots, m,\ \omega_{m+1} = \Pi^{\frac{p^2}{2}} \{\Gamma_p(\tfrac{p}{2})\}^{-1}.$$

The factor involving ω_i accounts for the fact that integrand in (3.5) is maximized when $\underset{\sim}{H}$ has the form (3.6). Thus we get the following theorem:

Theorem 3.1. An asymptotic expansion of the distribution of the roots $r_1,\ldots, r_p$, of the sample covariance matrix $\underset{\sim}{C}$ for large degrees of freedom n, when the population roots satisfy (3.4) is given by

$$\omega_{m+1}^{-1} \prod_{i=1}^{m} c_i \, 2^q C_1 \prod_{\substack{i=1 \\ i<j}}^{q} \prod_{j=1}^{p} \left(\frac{2\pi}{nc_{ij}}\right)^{\frac{1}{2}} \prod_{u=1}^{m} \prod_{i=q_u}^{q_{u+1}-1} \prod_{j=q_{u+1}}^{p} \left(\frac{2\pi}{nc^o_{ij}}\right)^{\frac{1}{2}}$$

$$\left(1+\frac{1}{2n}\left(\sum_{\substack{i=1 \\ i<j}}^{q} \sum_{j=1}^{p} c_{ij}^{-1} + \sum_{u=1}^{m} \sum_{i=q_u}^{q_{u+1}-1} \sum_{j=q_{u+1}}^{p} c^{o\,-1}_{ij}\right) + \ldots\right) \exp[-\frac{n}{2} \operatorname{tr} \underset{\sim}{A}\, \underset{\sim}{R}]$$

where $q = p - k_1 - \ldots - k_m$, and $\underset{\sim}{A}$, $\underset{\sim}{R}$, q_i, $(i=1,\ldots, m+1)$, c^o_{ij}'s, c_{ij}'s are defined by (3.4), (3.3), (3.10) and (3.11) respectively.

3.4. Asymptotic Expansion for the Distribution of the Latent Roots of $\underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$ - Several Multiple Population Roots

The problem of finding the asymptotic expansion of the roots of $\underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$ in case of one extreme multiple population root has been studied by Li and Pillai [14], [15], we here extend their results to the case when there are several multiple population roots.

Let $\underset{\sim}{S}_i$ be independently distributed as Wishart $(n_i, p, \underset{\sim}{\Sigma}_i)$, $i = 1, 2$, and let $r_i = ch_i\ (\underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1})$, $\ell_i = ch_i\ (\underset{\sim}{\Sigma}_1 \underset{\sim}{\Sigma}_2^{-1})$, $i = 1,\ldots, p$, and let $\underset{\sim}{R} = \operatorname{diag}(r_1,\ldots, r_p)$; $\infty > r_1 > \ldots > r_p > 0$,

$$(3.12)\quad \underset{\sim}{A} = \operatorname{diag}(\ell_1,\ldots, \ell_q, \overbrace{\ell_{q+1},\ldots, \ell_{q+1}}^{k_1},\ldots, \overbrace{\ell_{q+m},\ldots, \ell_{q+m}}^{k_m}),$$

and $\infty > \ell_{q+m} > \ell_{q+m-1} > \ldots > \ell_{q+1} > \ell_q > \ldots > \ell_1 \geq 0$, where $p = k_1 + \dots + k_m + q$. Then the joint distribution of the roots $r_1, \dots, r_p$, is given by

(3.13) $$c_2 \int_{O(p)} |\underset{\sim}{I} + \underset{\sim}{H}' \underset{\sim}{A} \underset{\sim}{H} \underset{\sim}{R}|^{-\frac{n}{2}} d(\underset{\sim}{H}) ,$$

where

$$c_2 = \pi^{\frac{p^2}{2}} \Gamma_p\left(\frac{n_1+n_2}{2}\right) \{\Gamma_p\left(\frac{n_1}{2}\right) \Gamma_p\left(\frac{n_2}{2}\right) \Gamma_p\left(\frac{p}{2}\right)\}^{-1} |\underset{\sim}{A}|^{\frac{n_1}{2}}$$

$$|\underset{\sim}{R}|^{\left(\frac{n_1-p-1}{2}\right)} \prod_{i<j} (\ell_i - \ell_j)$$

and $$n = n_1 + n_2, \; \Gamma_p(x) = \pi^{\frac{p(p-1)}{4}} \prod_{j=1}^{p} \Gamma((x - \tfrac{1}{2}(j-1))) .$$

$d(\underset{\sim}{H})$ is the invariant measure on the group $O(p)$. Again, as earlier, by lemma 3.1.1 and as is shown in Chapter One, the integrand in (3.13) is maximized for all variation of $\underset{\sim}{R} > 0$ when $\underset{\sim}{H}$ has the form (3.6).

Again we make a substitution of the form (3.8) and after lengthy algebra similar in line to that of Li and Pillai [14], [15], we get for large n, and ℓ_i's and r_j's well spaced $(i, j = 1, \dots, p)$

$$\int_{O(p)} |\underset{\sim}{I} + \underset{\sim}{H}' \underset{\sim}{A} \underset{\sim}{H} \underset{\sim}{R}|^{-\frac{n}{2}} d(\underset{\sim}{H}) = 2^q \prod_{i=1}^{m} \omega_i \, \omega_{m+1}^{-1} |\underset{\sim}{I} + \underset{\sim}{A} \underset{\sim}{R}|^{-\frac{n}{2}}$$

$$\prod_{\substack{i=1 \\ i<j}}^{q} \prod_{j=1}^{p} \left(\frac{2\pi}{nc_{ij}}\right)^{\frac{1}{2}} \prod_{u=1}^{m} \prod_{i=q_u}^{q_{u+1}-1} \prod_{j=q_{u+1}}^{p} \left(\frac{2\pi}{nc^o_{ij}}\right)^{\frac{1}{2}}$$

$$[1 + \frac{1}{2n} \Big(\sum_{\substack{i=1 \\ i<j}}^{q} \sum_{j=1}^{p} c_{ij}^{-1} + \sum_{u=1}^{m} \sum_{i=q_u}^{q_{u+1}-1} \sum_{j=q_{u+1}}^{p} c^{o\,-1}_{ij} + \alpha_1(p,q)$$

$$+ \alpha_2 (p, q, k_1, \dots, k_m)\Big) + \dots] ,$$

where $c_{ij} = (t_{ji} - t_i\, t_j\, r_{ij})\, r_{ij}$,

(3.14) $t_{ij} = t_i - t_j$, $t_i = \ell_i (1+\ell_i r_i)^{-1}$,

$r_{ij} = r_i - r_j$, $i = 1,\ldots, q$, $j = 1,\ldots, p$, $i < j$,

and c^o_{ij} is similarly defined as c_{ij} but subscripts varying over the indicated set where it is non-zero,

$$\alpha_1\ (p,\ q) = \frac{q}{12}\{(q-1)(4q+1) + 6(p^2-q^2)\}$$

and

$$\alpha_2\ (p,\ q,\ k_1,\ldots,\ k_m) = \frac{1}{2}\sum_{i=1}^{m}$$

$$k_i(p-q-k_1 - \cdots\cdots - k_i)(p-q-k_1 - \cdots\cdots - k_i-1)$$

$$+ \sum_{i<j<\ell=3}^{m} k_i k_j k_\ell + \frac{3}{2}\sum_{i<j=2}^{m} k_i k_j \ .$$

Thus we have the following theorem:

Theorem 3.2. For large degrees of freedom $n = n_1 + n_2$, an asymptotic expansion for the distribution of the roots $\infty > r_1 > \cdots\cdots > r_p > 0$ when the population roots satisfy (3.12) is given by

$$2^q \prod_{i=1}^{m} \omega_i\, \omega_{m+1}^{-1}\, c_2\, |\underset{\sim}{I} + \underset{\sim}{A}\, \underset{\sim}{R}|^{-\frac{n}{2}} \prod_{\substack{i=1 \\ i<j}}^{q} \prod_{j=1}^{p} \left(\frac{2\pi}{nc_{ij}}\right)^{\frac{1}{2}}$$

$$\prod_{u=1}^{m} \prod_{i=q_u}^{q_{u+1}-1} \prod_{j=q_{u+1}}^{p} \left(\frac{2\pi}{nc^o_{ij}}\right)^{\frac{1}{2}} [1 + \frac{1}{2n} (\sum_{\substack{i=1 \\ i<j}}^{q} \sum_{j=1}^{p} c_{ij}^{-1} +$$

$$\sum_{u=1}^{m} \sum_{i=q_u}^{q_{u+1}-1} \sum_{j=q_{u+1}}^{p} c_{ij}^{o^{-1}} + \alpha_1(p,q) + \alpha_2(p,q,k_1,\ldots,k_m)) + \ldots] ,$$

where the constants are defined by (3.14).

In the following we give the asymptotic expansions for the roots of relevant matrices for MANOVA and Canonical correlation cases and for complex analogue of all these problems. Detailed ground work being already done in earlier chapters and above we just state the problems and the corresponding solutions omitting the details.

3.5. Asymptotic Expansion for MANOVA - Several Multiple Population Roots

Let $\underset{\sim}{B}$ be the between S. P. matrix and $\underset{\sim}{W}$ the within S. P. matrix. Then $\underset{\sim}{B}(p \times p)$ has a non-central Wishart distribution with s d.f. and matrix of non-centrality parameter $\underset{\sim}{A}$, and $\underset{\sim}{W}$ has the central Wishart distribution on t d.f., the co-variance matrix in each case being $\underset{\sim}{\Sigma}$. Let $\underset{\sim}{A} = \underset{\sim}{\mu}\,\underset{\sim}{\mu}'\,\underset{\sim}{\Sigma}^{-1}$ and $\underset{\sim}{R} = \underset{\sim}{B}(\underset{\sim}{W} + \underset{\sim}{B})^{-1}$ and in terms of the characteristic roots let

$$(3.15) \quad \underset{\sim}{R} = \mathrm{diag}\,(r_1,\ldots, r_p),\ 1 > r_1 > r_2,\ldots, > r_p > 0 ,$$

$$(3.16) \quad \underset{\sim}{A} = \mathrm{diag}\,(\ell_1,\ldots, \ell_q, \overbrace{\ell_{q+1},\ldots, \ell_{q+1}}^{k_1},\ldots, \overbrace{\ell_{q+m},\ldots, \ell_{q+m}}^{k_m}) ,$$

where

$$\infty > \ell_1 > \ldots\ldots > \ell_q > \ell_{q+1} > \ldots\ldots > \ell_{q+m} \geq 0 .$$

Then we have the following theorem:

Theorem 3.3. For large t (and hence for large sample size) an asymptotic expansion for the distribution of the characteristic roots of $\underset{\sim}{R}$ in (3.15), when the parameter matrix $\underset{\sim}{A}$ satisfies (3.16) is given by

$$c_3\, 2^q \prod_{i=1}^{m} \omega_i\, \omega_{m+1}^{-1} \prod_{\substack{i=1\\ i<j}}^{q} \prod_{j=1}^{p} \left(\frac{2\pi}{tc_{ij}}\right)^{\frac{1}{2}} \prod_{u=1}^{m} \prod_{i=q_u}^{q_{u+1}-1} \prod_{j=q_{u+1}}^{p} \left(\frac{2\pi}{tc^o_{ij}}\right)^{\frac{1}{2}}$$

$$\{1 + \frac{1}{2t} \Big(\sum_{\substack{i=1\\ i<j}}^{q} \sum_{j=1}^{p} c_{ij}^{-1} + \sum_{u=1}^{m} \sum_{i=q_u}^{q_{u+1}-1} \sum_{j=q_{u+1}}^{p} c^{o\,-1}_{ij} + \alpha_1(p,q)$$

$$+ \alpha_2(p,q,k_1,\ldots,k_m)\Big) + \ldots\ldots\} \exp[\mathrm{tr}\, \underset{\sim}{A}\, \underset{\sim}{R}]\, {}_1F_1(-\tfrac{t}{2}; \tfrac{s}{2}, -\underset{\sim}{A}\, \underset{\sim}{R}) + O(\varepsilon) ,$$

where

$$c_3 = \pi^{\frac{p^2}{2}}\, \Gamma_p(\tfrac{1}{2}(s+t))\, \{\Gamma_p(\tfrac{t}{2})\, \Gamma_p(\tfrac{s}{2})\, \Gamma_p(\tfrac{p}{2})\}^{-1} \exp[\mathrm{tr}\, \underset{\sim}{A}]\, |\underset{\sim}{R}|^{\frac{1}{2}(s-p-1)}$$

$$|\underset{\sim}{I} - \underset{\sim}{R}|^{\frac{1}{2}(t-p-1)} \prod_{i>j} (r_i - r_j)$$

and

$$c_{ij} = \frac{(d_i - d_j)(\ell_j - \ell_i)}{(1+\ell_i d_i)(1+\ell_j d_j)} , \quad i = 1,\ldots, q,\ j=1,\ldots, p,\ i < j .$$

c^o_{ij}'s are similarly defined as c_{ij} but the subscripts varying over the indicated set such that c^o_{ij} is non-zero and $\underset{\sim}{D} = \underset{\sim}{R}^{-1}$, i.e., $d_i = r_i^{-1}$, $i = 1,\ldots, p$.

3.6. Asymptotic Expansion for Canonical Correlation - Several Multiple Population Roots

Let $x_1, \ldots, x_p, x_{p+1}, \ldots, x_{p+f}$, $p \leq f$ be distributed $N(\underset{\sim}{0}, \underset{\sim}{\Sigma})$, where

$$\underset{\sim}{\Sigma} = \begin{array}{c} \begin{array}{cc} p & f \end{array} \\ \begin{bmatrix} \underset{\sim}{\Sigma}_{11} & \underset{\sim}{\Sigma}_{12} \\ \\ \underset{\sim}{\Sigma}'_{12} & \underset{\sim}{\Sigma}_{22} \end{bmatrix} \end{array} \begin{array}{c} \\ p \\ \\ f \end{array}$$

Let $\underset{\sim}{P}^2 = \operatorname{diag}(\rho_1^2, \ldots, \rho_p^2)$, where ρ_i^2, $i=1, \ldots, p$, be the roots of

$$|\underset{\sim}{\Sigma}_{12}\, \underset{\sim}{\Sigma}_{22}^{-1}\, \underset{\sim}{\Sigma}'_{12} - \rho^2\, \underset{\sim}{\Sigma}_{11}| = 0$$

and let $\hat{\underset{\sim}{P}}{}^2 = \operatorname{diag}(\hat{\rho}_1{}^2, \ldots, \hat{\rho}_p{}^2)$, where $\hat{\rho}_i{}^2$, $i=1, \ldots, p$, be the maximum likelihood estimates. Also let

$$\hat{\underset{\sim}{P}}{}^2 = \underset{\sim}{R} = \operatorname{diag}(r_1, \ldots, r_p) \tag{3.17}$$

$$\underset{\sim}{P}^2 = \underset{\sim}{A} = \operatorname{diag}(\ell_1, \ldots, \ell_q, \overbrace{\ell_{q+1}, \ldots, \ell_{q+1}}^{k_1}, \ldots, \overbrace{\ell_{q+m}, \ldots, \ell_{q+m}}^{k_m})$$

where

$$1 > r_1 > \ldots\ldots > r_p > 0,\ 1 > \ell_1 > \ell_2 > \ldots\ldots > \ell_q > \ell_{q+1} > \ldots\ldots > \ell_{q+m} \geq 0 .$$

Then we have the following theorem:

Theorem 3.4. For large n, an asymptotic expansion of the distribution of $r_1, \ldots, r_p$, (squares of the canonical correlation coefficients) when population parameters satisfy (3.17) is given by

$$c_4\, 2^q \prod_{i=1}^{m} \omega_i\, \omega_{m+1}^{-1}\, |\underset{\sim}{I} - \underset{\sim}{A}\,\underset{\sim}{R}|^{-\frac{1}{2}(2n-f)} \{1 + \frac{1}{2(2n-f)} (\sum_{\substack{i=1 \\ i<j}}^{q} \sum_{j=1}^{p} c_{ij}^{-1} + \sum_{u=1}^{m} \sum_{i=q_u}^{q_{u+1}-1} \sum_{j=q_{u+1}}^{p} c_{ij}^{o^{-1}} + \alpha_1(p,q) + \alpha_2(p,q,k_1,\ldots,k_m)) + \ldots\ldots\}\, {}_2F_1(\tfrac{1}{2}(f-n), \tfrac{1}{2}(f-n); \tfrac{1}{2}f, \underset{\sim}{A}\,\underset{\sim}{R}) + O(\varepsilon) ,$$

where

$$c_{ij} = (t_{ji} - t_i t_j\, r_{ij})\, r_{ij} = c_{ji} ,$$

$$t_{ij} = t_i - t_j,\; r_{ij} = r_i - r_j ,$$

$$t_i = \ell_i\, (1 - r_i \ell_i)^{-1},\; i=1,\ldots, q,\; j=1,\ldots, p,\; i < j ,$$

and c_{ij}^o is similarly defined as c_{ij} but the subscripts vary over the indicated set such that c_{ij}^o is non-zero. And

$$c_4 = \{\Pi^{\frac{p^2}{2}}\, \Gamma_p(\tfrac{n}{2})/\Gamma_p(\tfrac{f}{2})\, \Gamma_p(\tfrac{n-f}{2})\, \Gamma_p(\tfrac{p}{2})\}\, |\underset{\sim}{I} - \underset{\sim}{A}|^{\frac{n}{2}}\, |\underset{\sim}{R}|^{\frac{1}{2}(f-p-1)} \prod_{i>j} (r_i - r_j) \prod_{\substack{i=1 \\ i<j}}^{q} \prod_{j=1}^{p} (\frac{2\pi}{(2n-f)c_{ij}})^{\frac{1}{2}} \prod_{u=1}^{m} \prod_{i=q_u}^{q_{u+1}-1} \prod_{j=q_{u+1}}^{p} (\frac{2\pi}{(2n-f)c_{ij}^o})^{\frac{1}{2}}$$

3.7. Complex Analogues of Previous Results

In the following generalization of the above results to the complex case we refer to lemma 3.1.2 and the corresponding results of theorem 1.1.3 and proceed as above, the details of algebra obtainable from Li and Pillai [14], [15] with suitable changes. Complex analogues of theorem 3.1 - 3.4 are as follows:

Theorem 3.1.1. For large degrees of freedom n, an asymptotic expansion of the distribution of the roots of the covariance matrix $\underset{\sim}{S}$ when the parameter matrix $\underset{\sim}{\Sigma}^{-1}$ has roots ℓ_i's and satisfy (3.4), is given by

$$D_1 \prod_{i=1}^{m} \theta_i \, \theta_{m+1}^{-1} \prod_{\substack{i=1 \\ i<j}}^{q} \prod_{j=1}^{p} \left(\frac{\pi}{nc_{ij}}\right) \prod_{u=1}^{m} \prod_{i=q_u}^{q_{u+1}-1} \prod_{j=q_{u+1}}^{p} \left(\frac{\pi}{nc^{o}_{ij}}\right)$$

$$\{1 + \frac{1}{3n} \Big(\sum_{\substack{i=1 \\ i<j}}^{q} \sum_{j=1}^{p} c_{ij}^{-1} + \sum_{u=1}^{m} \sum_{i=q_u}^{q_{u+1}-1} \sum_{j=q_{u+1}}^{p} c^{o\,-1}_{ij} \Big) +$$

$$\ldots\ldots\} \exp [- \frac{n}{2} \operatorname{tr} \underset{\sim}{A} \, \underset{\sim}{R}] ,$$

where

$$D_1 = \pi^{p(p-1)} \{\tilde{\Gamma}_p(p) \, \tilde{\Gamma}_p(n) \, |\underset{\sim}{\Sigma}|^{n}\}^{-1} \, |\underset{\sim}{R}|^{n-p} \prod_{i<j=2}^{p} (\ell_i - \ell_j)^2$$

and

$$\theta_i = \pi^{k_i(k_i-1)} \{\tilde{\Gamma}_{k_i}(k_i)\}^{-1}, \; i=1,\ldots, m,$$

$$\theta_{m+1} = \pi^{p(p-1)} \{\tilde{\Gamma}_p(p)\}^{-1}, \; \tilde{\Gamma}_p(n) = \pi^{p\left(\frac{p-1}{2}\right)} \prod_{i=1}^{p} \Gamma(n-i+1) ,$$

c_{ij} and c^{o}_{ij} are similarly defined as in the corresponding real case.

Theorem 3.2.1. For large degrees of freedom $n = n_1 + n_2$, an asymptotic expansion of the distribution of the roots of $\underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$ in the complex case when the population roots satisfy the form (3.4) is given by

$$D_2 \prod_{i=1}^{m} \theta_i \, \theta_{m+1}^{-1} \prod_{\substack{i=1 \\ i<j}}^{q} \prod_{j=1}^{p} \left(\frac{\pi}{nc_{ij}}\right) \prod_{u=1}^{m} \prod_{i=q_u}^{q_{u+1}-1} \prod_{j=q_{u+1}}^{p} \left(\frac{\pi}{nc^o_{ij}}\right)$$

$$|\underset{\sim}{I} + \underset{\sim}{A}\,\underset{\sim}{R}|^{-n} \{1 + \frac{1}{3n} \Big(\sum_{\substack{i=1 \\ i<j}}^{q} \sum_{j=1}^{p} c_{ij}^{-1} + \sum_{u=1}^{m} \sum_{i=q_u}^{q_{u+1}-1} \sum_{j=q_{u+1}}^{p} {c^o_{ij}}^{-1}$$

$$+ \beta_1 (p,q) + \beta_2 (p,q,k_1,\ldots, k_m)\Big) + \ldots\ldots \} ,$$

where

$$D_2 = \pi^{\frac{p(p-1)}{2}} \tilde{\Gamma}_p(n_1 + n_2)/\{\tilde{\Gamma}_p(p)\, \tilde{\Gamma}_p(n_1)\, \tilde{\Gamma}_p(n_2)\}\, |\underset{\sim}{A}|^{-p}\, |\underset{\sim}{R}|^{n_2-p}$$

$$\prod_{i<j} (r_i-r_j)^2 ,$$

where c_{ij} and c^o_{ij} are defined as in the corresponding real case,

$$\beta_1(p,q) = \frac{q}{2} \{(q-1)(2q-1) + 3\,(p-q)(p+q-1)\} ,$$

and

$$\beta_2(p,q,k_1,\ldots, k_m) = \frac{3}{2} \sum_{i=1}^{m} k_i \,(p-q-k_1 - \ldots\ldots - k_i)$$

$$(p-q - \ldots\ldots - k_i-1)$$

$$+ 3 \sum_{i<j<\ell=3}^{m} k_i k_j k_\ell + 3 \sum_{i<j=2}^{m} k_i k_j .$$

Theorem 3.3.1. For large t (and hence for large sample size) an asymptotic expansion of the distribution of the sample roots in the complex MANOVA case when population roots satisfy the form (3.16) is given by

$$D_3 \prod_{i=1}^{m} \theta_i \, \theta_{m+1}^{-1} \prod_{\substack{i=1 \\ i<j}}^{q} \prod_{j=1}^{p} \left(\frac{\pi}{tc_{ij}}\right) \prod_{u=1}^{m} \prod_{i=q_u}^{q_{u+1}-1} \prod_{j=q_{u+1}}^{p} \left(\frac{\pi}{tc^{o}_{ij}}\right)$$

$$\{1 + \frac{1}{3t} \Big(\sum_{\substack{i=1 \\ i<j}}^{q} \sum_{j=1}^{p} c_{ij}^{-1} + \sum_{u=1}^{m} \sum_{i=q_u}^{q_{u+1}-1} \sum_{j=q_{u+1}}^{p} c^{o\,-1}_{ij} + \beta_1(p,q) +$$

$$\beta_2(p,q,k_1,\ldots, k_m)\Big) + \ldots\ldots\} \exp\,[\mathrm{tr}\, \underset{\sim}{A}\, \underset{\sim}{R}]\ {}_1F_1(-t;s, -\underset{\sim}{A}\, \underset{\sim}{R}) + O(\varepsilon)\ ,$$

where

$$D_3 = [\pi^{p(p-1)}\ \tilde{\Gamma}_p(s+t)/\tilde{\Gamma}_p(p)\ \tilde{\Gamma}_p(s)\ \tilde{\Gamma}_p(t)]\ |\underset{\sim}{I} - \underset{\sim}{R}|^{t-p}$$

$$|\underset{\sim}{R}|^{(s-p)} \prod_{i>j} (r_i - r_j)^2 \exp\,[-\mathrm{tr}\ \underset{\sim}{A}]$$

and c_{ij}, c^{o}_{ij}'s are as defined in corresponding real case.

Theorem 3.4.1. For large n, an asymptotic expansion for the distribution of the canonical correlation coefficients when population coefficients in the complex case satisfy the form (3.17) is given by

$$D_3 \prod_{i=1}^{m} \theta_i \, \theta_{m+1}^{-1} \, |\underset{\sim}{I} - \underset{\sim}{A}\, \underset{\sim}{R}|^{-(2n-f)} \{1 + \frac{1}{3(2n-f)} \Big(\sum_{\substack{i=1 \\ i<j}}^{q} \sum_{j=1}^{p} c_{ij}^{-1} +$$

$$\sum_{u=1}^{m} \sum_{i=q_u}^{q_{u+1}-1} \sum_{j=q_{u+1}}^{p} c^{o\,-1}_{ij} + \beta_1(p,q) + \beta_2(p,q,k_1,\ldots, k_m)\Big) + \ldots\ldots\}$$

$${}_2\tilde{F}_1\,((f-n),\,(f-n);\,f,\,\underset{\sim}{A}\,\underset{\sim}{R}) + O(\varepsilon)\;,\ \text{where}$$

$$D_3 = [\pi^{p(p-1)}\,\tilde{\Gamma}_p(n)/\tilde{\Gamma}_p(n-f)\,\tilde{\Gamma}_p(f)\,\tilde{\Gamma}_p(p)]\;|\underset{\sim}{I} - \underset{\sim}{A}|^{n}\;|\underset{\sim}{R}|^{f-p}$$

$$|I\ [illegible]\ \prod_{i<j}(r_i - r_j)^2$$

$$\prod_{i=1}^{q}\ \prod_{\substack{j=1\\ i<j}}^{p}\ ([illegible])\ \prod_{u=1}^{m}\ \prod_{i=q_u}^{q_{u+1}-1}\ \prod_{j=q_{u+1}}^{p}\ \left(\frac{\pi}{(2n-f)\,c^{o}_{ij}}\right)$$

3.8. Remarks

1. As will be seen from the above formulae, they give the already known results of Anderson [1], Chang [3], James [10], Li and Pillai [14], [15] as special cases.

2. Though we have taken the sets with multiple roots in the population parametric matrix at one extreme, actually it does not matter even if they were otherwise. By pre and post multiplication by suitable permutation matrix, all multiple roots can be brought to one extreme place without affecting our distribution problem but, of course, care should be taken in defining c_{ij} and c^{o}_{ij} coefficients.

3. Since, for all variations of $\underset{\sim}{R} > 0$, the appropriate integral in each case takes the identical maximum when the corresponding orthogonal or unitary matrices take definite special forms, we can take particular transformations like (3.8) or its complex analogue to approximate the integrand around one such optimum and hence adjust for all such optima.

4. As will be evident, our technique being a generalization of techniques of earlier authors, the restrictions made by earlier authors

also apply in our case.

5. As said earlier we tacitly avoided the "invariance" technique used by James and subsequently followed by others. Moreover, our technique gives their result as a special case and hence gives a different interpretation of their results.

CHAPTER IV

ASYMPTOTIC FORMULAE FOR THE DISTRIBUTIONS OF SOME CRITERIA FOR TESTS OF EQUALITY OF COVARIANCE MATRICES

4.1. Introduction

Let $m\, \underset{\sim}{S}_1$ and $n\, \underset{\sim}{S}_2$ be independently distributed $W(m, p, \underset{\sim}{\Sigma}_1)$ and $W(n, p, \underset{\sim}{\Sigma}_2)$ respectively. Then the asymptotic expansions for the distribution and percentiles of $T = m \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$ have been obtained in this paper upto terms of order 1/n. It may be noted that $T = n\, U^{(p)}$, where $U^{(p)}$ is the criterion studied by Pillai [16] for the test of $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$, and the power of this test against alternatives of a one-sided nature was discussed by Pillai and Jayachandran [17]. Further, asymptotic expansions for the distribution and percentile are derived for $F' = (m_1/n_1)(\operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{S}_4^{-1}/\operatorname{tr} \underset{\sim}{S}_3 \underset{\sim}{S}_2^{-1})$, where $m_1 \underset{\sim}{S}_1$, $m_2 \underset{\sim}{S}_2$, $n_1 \underset{\sim}{S}_3$ and $n_2 \underset{\sim}{S}_4$ are independently distributed Wishart matrices with degrees of freedom m_1, m_2, n_1, n_2 respectively and each of the pairs $(\underset{\sim}{S}_1, \underset{\sim}{S}_2)$, $(\underset{\sim}{S}_3, \underset{\sim}{S}_4)$ has a common covariance matrix. Also the asymptotic distribution is obtained for the maximum F' in a special case (see section 4.5). Pillai has suggested F'_{max} [18] for test of equality of the common covariance matrices of the pairs above.

If in F', $\underset{\sim}{S}_1$ and $\underset{\sim}{S}_2$ are computed from independent subsamples of the same sample and so also are $\underset{\sim}{S}_3$ and $\underset{\sim}{S}_4$, F'_{max} is an alternate to the $U^{(p)}$ test above.

4.2. An Asymptotic Expansion for Percentiles of

$$T = m \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$$

In this section we will derive an asymptotic expansion for the percentiles of $T = m \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$.

By definition earlier, let $m \underset{\sim}{S}_1$ and $n \underset{\sim}{S}_2$ be distributed $W(m, p, \underset{\sim}{B}^{-1})$ and $W(n, p, \underset{\sim}{A}^{-1})$ respectively, where $\underset{\sim}{B}^{-1} = \underset{\sim}{\Sigma}_1$ and $\underset{\sim}{A}^{-1} = \underset{\sim}{\Sigma}_2$. Then it is well known [2] that the statistic $y = m \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{A}$ is a linear function of central chi-square variables i.e.

$y = \sum_{j=1}^{p} \lambda_j \chi_j^2(m)$ where $\chi_j^2(m)$'s are independent central chi-square variables with m d.f. and λ_j's, $j = 1, \ldots, p$ are the characteristic roots of $\underset{\sim}{U} = \underset{\sim}{A}\underset{\sim}{B}^{-1}$ i.e. λ_j's satisfy $|\underset{\sim}{A} - \lambda \underset{\sim}{B}| = 0$.

Let $G(\theta) = \Pr\{m \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{A} \leq 2\theta\}$ and replacing $\underset{\sim}{A}$ by $\underset{\sim}{B}$ we get

$$(4.1) \qquad \Pr\{m\operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{B} \leq 2\theta\} = G_\rho(\theta) = [\Gamma(\rho)]^{-1} \int_0^\theta e^{-t} t^{\rho-1}\, dt$$

where $\rho = mp/2$.

Now since for large n we may use $\underset{\sim}{S}_2$ as an approximation of $\underset{\sim}{A}^{-1}$ we can use as a first approximation

$$(4.2) \qquad G(\theta) = \Pr\{m \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1} \leq 2\theta\} .$$

Now we may, as suggested in [7], obtain a function $h(\underset{\sim}{S}_2)$ in the elements of $\underset{\sim}{S}_2$ such that

$$G(\theta) = \Pr\{m \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1} \leq 2\, h(\underset{\sim}{S}_2)\}$$

and then write $h(\underset{\sim}{S}_2)$ as a series with the first term being a linear function of chi-square variables and successive terms of decreasing order of magnitude. We get

$$(4.3) \quad \Pr\{m \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1} \leq 2h(\underset{\sim}{S}_2)\} = \int_R \Pr\{m \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1} \leq 2h(\underset{\sim}{S}_2) \mid \underset{\sim}{S}_2\}$$

$$\Pr\{d\underset{\sim}{S}_2\}$$

where $\Pr\{d\,\underset{\sim}{S}_2\}$ is the probability element of the central Wishart distribution of $\underset{\sim}{S}_2$ and R is the domain of integration of $\underset{\sim}{S}_2$. Now let

$$(4.4) \qquad \underset{\sim}{A}^{-1} = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1p} \\ \vdots & & \vdots \\ \sigma_{p1} & \cdots & \sigma_{pp} \end{bmatrix} .$$

We expand the integrand in (4.3) around $\underset{\sim}{A}^{-1}$ in Taylor's series and get

$$\Pr\{m \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1} \leq h(\underset{\sim}{S}_2) \mid \underset{\sim}{S}_2\}$$

$$= \{\exp [\sum_{i\leq j=1}^{p} (s_{ij} - \sigma_{ij}) \frac{\partial}{\partial\sigma_{ij}}]\} \Pr\{m \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{A} \leq 2h\ (\underset{\sim}{A}^{-1})\}$$

$$= \{\exp [\operatorname{tr} (\underset{\sim}{S}_2 - \underset{\sim}{A}^{-1})\ \underset{\sim}{\partial}]\} \Pr\{m \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{A} \leq 2h\ (\underset{\sim}{A}^{-1})\}$$

where

$$(4.5)\quad \underset{\sim}{\partial}(p \times p) = (1/2\ (1+\delta_{ij})\ \frac{\partial}{\partial\sigma_{ij}}) = \begin{bmatrix} \frac{\partial}{\partial\sigma_{11}} & \frac{1}{2}\frac{\partial}{\partial\sigma_{12}} & \cdots & \frac{1}{2}\frac{\partial}{\partial\sigma_{1p}} \\ \vdots & \vdots & & \vdots \\ \frac{1}{2}\frac{\partial}{\partial\sigma_{p1}} & \frac{1}{2}\frac{\partial}{\partial\sigma_{p2}} & \cdots & \frac{\partial}{\partial\sigma_{pp}} \end{bmatrix}$$

where

$$\delta_{ij} = 1 \text{ if } i = j$$
$$= 0 \text{ if } i \neq j$$

Now integrating term by term for sufficiently large n we get [7]

$$G(\theta) = \int_R \exp\,[\mathrm{tr}\,(\underset{\sim}{S_2} - \underset{\sim}{A}^{-1})\ \underset{\sim}{\partial}]\ \Pr\{m\ \mathrm{tr}\ \underset{\sim}{S_1}\ \underset{\sim}{A} \leq 2\ h\ (\underset{\sim}{A}^{-1})\}$$

$$\Pr\{d\ \underset{\sim}{S_2}\}$$

$$= \Theta\ \Pr\{m\ \mathrm{tr}\ \underset{\sim}{S_1}\ \underset{\sim}{A} \leq 2\ h\ (\underset{\sim}{A}^{-1})\}$$

where

$$(4.6)\qquad \Theta = \exp\,[-\mathrm{tr}\ \underset{\sim}{A}^{-1}\ \underset{\sim}{\partial}]\ (\Gamma_p(n))^{-1}\ |\underset{\sim}{A}|^{n/2} \int_R |\underset{\sim}{S_2}|^{(n-p-1)/2}$$

$$\exp[\mathrm{tr}(\underset{\sim}{S_2}\underset{\sim}{\partial} - (n/2)\underset{\sim}{A}\ \underset{\sim}{S_2})]\ d\underset{\sim}{S_2}$$

$$= \exp\,[-\mathrm{tr}\ \underset{\sim}{A}^{-1}\ \underset{\sim}{\partial}]\ |\underset{\sim}{I} - (2/n)\ \underset{\sim}{A}^{-1}\underset{\sim}{\partial}|^{-(n/2)}$$

$$= \exp\,[-\mathrm{tr}\ \underset{\sim}{A}^{-1}\underset{\sim}{\partial} - (n/2)\ \log\ |\underset{\sim}{I} - (2/n)\ \underset{\sim}{A}^{-1}\ \underset{\sim}{\partial}|\ ]$$

$$= \exp\,[-\mathrm{tr}\ \underset{\sim}{A}^{-1}\underset{\sim}{\partial} + (n/2)\ \{\mathrm{tr}\ (2/n)\ \underset{\sim}{A}^{-1}\underset{\sim}{\partial} +$$

$$1/2\ \mathrm{tr}\ (2/n\ \underset{\sim}{A}^{-1}\ \underset{\sim}{\partial})^2 + O\ (n^{-3})\}]$$

$$= 1 + (1/n)\ \mathrm{tr}\ (\underset{\sim}{A}^{-1}\underset{\sim}{\partial})^2 + O(n^{-2})$$

$$= 1 + (1/n)\ \Sigma\ \sigma_{rs}\ \sigma_{tu}\ \partial_{st}\ \partial_{ur} + O(n^{-2}) \ .$$

Further, we expand $h(\underset{\sim}{S}_2)$ around θ as

$$h(\underset{\sim}{S}_2) = \theta + h_1(\underset{\sim}{S}_2) + h_2(\underset{\sim}{S}_2) + \ldots\ ,$$

where $h_i(\underset{\sim}{S}_2)$ is $O(n^{-i})$.

Hence we get

$$\Pr\{m\ \mathrm{tr}\ \underset{\sim}{S}_1\ \underset{\sim}{A} \le 2h\ (\underset{\sim}{A}^{-1})\} = \exp\ [\{h_1\ (\underset{\sim}{A}^{-1}) +$$

$$h_2(\underset{\sim}{A}^{-1}) + \ldots\}D]\ \Pr\{m\ \mathrm{tr}\ \underset{\sim}{S}_1\ \underset{\sim}{A} \le 2\theta\}$$

$$= [1 + \{h_1(\underset{\sim}{A}^{-1}) + h_2(\underset{\sim}{A}^{-1}) + \ldots\}\ D + \ldots]\ \Pr\{m\ \mathrm{tr}\ \underset{\sim}{S}_1\ \underset{\sim}{A} \le 2\theta\}\ ,$$

where

$$D = \frac{\partial}{\partial\theta}\ . \tag{4.7}$$

Hence we get

$$G(\theta) = [1 + 1/n\ \Sigma\ \sigma_{rs}\ \sigma_{tu}\ \partial_{st}\ \partial_{ur} + O(n^{-2})]$$

$$[1 + h_1(\underset{\sim}{A}^{-1})D + O(n^{-2})]\ \Pr\{m\ \mathrm{tr}\ \underset{\sim}{S}_1\ \underset{\sim}{A} \le 2\theta\}$$

Now equating terms of successive order [7] we have

$$\{h_1\ (\underset{\sim}{A}^{-1})\ D + 1/n\ \Sigma\ \sigma_{rs}\ \sigma_{tu}\ \partial_{st}\ \partial_{ur}\}\ \Pr\{m\ \mathrm{tr}\ \underset{\sim}{S}_1\ \underset{\sim}{A} \le 2\theta\} = 0.$$

Hence to evaluate $h_1(\underset{\sim}{A}^{-1})$ we have to find $\partial_{st}\ \partial_{ur}\ \Pr\{m\ \mathrm{tr}\ \underset{\sim}{S}_1\ \underset{\sim}{A} \le 2\ \theta\}$.

Here we use the perturbation technique [8]. Let

$$J = \Pr\{m \operatorname{tr} \underset{\sim}{S}_1 (\underset{\sim}{A}^{-1} + \underset{\sim}{\varepsilon})^{-1} \leq 2\,\theta\}$$

where $\underset{\sim}{\varepsilon}(p \times p)$ is a symmetric matrix sufficiently close to $\underset{\sim}{O}(p \times p)$. By Taylor's theorem we get

$$(4.8) \quad J = \{1 + \Sigma\, \sigma_{rs}\, \partial_{rs} + 1/2\, \Sigma\, \varepsilon_{rs}\, \varepsilon_{tu}\, \partial_{rs}\, \partial_{tu} + \ldots\} \Pr\{m \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{A} \leq 2\,\theta\} .$$

Also by definition we get

$$J = \frac{|\underset{\sim}{B}|^{(m/2)}}{(2\,\pi)^{(mp)/2}} \int_R \exp\,[-\tfrac{1}{2} \operatorname{tr} \underset{\sim}{B}\, \underset{\sim}{Y}\, \underset{\sim}{Y}']\, d\,\underset{\sim}{Y}$$

where $m\, \underset{\sim}{S}_1 = \underset{\sim}{Y}\, \underset{\sim}{Y}'$, $\underset{\sim}{Y}(p \times m)$ and $R:\{\underset{\sim}{Y}:\ m \operatorname{tr} \underset{\sim}{S}_1(\underset{\sim}{A}^{-1}+\underset{\sim}{\varepsilon})^{-1} \leq 2\,\theta\}$. Now let $\underset{\sim}{\Gamma}(p \times p)$ be a non-singular matrix such that

$$\frac{1}{2}\, \underset{\sim}{\Gamma}'\, \underset{\sim}{B}\, \underset{\sim}{\Gamma} = \underset{\sim}{I}\ (p \times p) - \underset{\sim}{D}_\eta$$

$$\frac{1}{2}\, \underset{\sim}{\Gamma}'\, (\underset{\sim}{A}^{-1} + \underset{\sim}{\varepsilon})^{-1}\, \underset{\sim}{\Gamma} = \underset{\sim}{I}(p \times p)$$

for $\underset{\sim}{\varepsilon}(p \times p)$ sufficiently close to $\underset{\sim}{O}(p \times p)$ and $\underset{\sim}{D}_\eta = \operatorname{diag}(\eta_1, \ldots, \eta_p)$. This is possible as $\underset{\sim}{B}$ and $\underset{\sim}{A}$ are p.d. .

Let $\underset{\sim}{Y}(p \times m) = \underset{\sim}{\Gamma}(p \times p)\ \underset{\sim}{Z}\ (p \times m)$. Then

$$J = |\underset{\sim}{I} - \underset{\sim}{D}_\eta|^{(m/2)}\ \pi^{-(pm)/2} \int_{[m \operatorname{tr} \underset{\sim}{Z}\, \underset{\sim}{Z}' \leq \theta]} e^{-\operatorname{tr}[\underset{\sim}{Z}\, \underset{\sim}{Z}' - \underset{\sim}{Z}\, \underset{\sim}{D}_\eta\, \underset{\sim}{Z}']}\ d\underset{\sim}{Z}$$

$$= |\underset{\sim}{I} - \underset{\sim}{D}_\eta|^{(m/2)} \pi^{-(pm)/2} \int_{[\sum_{i=1}^{p} \sum_{j=1}^{m} Z_{ij}^2 \le \theta]} \exp - [\sum_{i=1}^{p} (1 - \eta_i) \sum_{j=1}^{m} Z_{ij}^2]$$

$$\prod_{i=1}^{p} \prod_{j=1}^{m} dZ_{ij} .$$

$$= \pi^{-(pm)/2} |\underset{\sim}{I} - \underset{\sim}{D}_\eta|^{(m/2)} \sum_{b_{11},\ldots,b_{pm}=0}^{\infty} (\eta_1^{b_{11}} \ldots \eta_1^{b_{1m}} \ldots \eta_p^{b_{p1}} \ldots \eta_p^{b_{pm}})$$

$$\times \frac{1}{(b_{11}!) \,..(b_{1m}!) \,.. (b_{p1}!) \,.. (b_{pm}!)} \int_{[\sum_{i=1}^{p} \sum_{j=1}^{m} Z_{ij}^2 \le \theta]} \exp - \sum_{i=1}^{p} \sum_{j=1}^{m} Z_{ij}^2$$

$$\times \prod_{i=1}^{p} \prod_{j=1}^{m} Z_{ij}^{2b_{ij}} \prod_{i=1}^{p} \prod_{j=1}^{m} dZ_{ij}$$

$$= \pi^{-(pm)/2} |\underset{\sim}{I} - \underset{\sim}{D}|^{(m/2)} \sum_{b_{11},\ldots,b_{pm}=0}^{\infty} (\eta_1^{b_{11}} \ldots \eta_p^{b_{pm}})$$

$$\frac{\Gamma(b_{11} + 1/2) \ldots \Gamma(b_{pm} + 1/2)}{(b_{11}!) \ldots\ldots\ldots (b_{pm}!)}$$

$$\times \frac{1}{\Gamma(\Sigma_{ij} b_{ij} + \frac{mp}{2})} \int_0^\theta e^{-t} \, t^{\sum_{i=1}^{p} \sum_{j=1}^{m} b_{ij} + (mp)/2-1} dt$$

$$= |\underset{\sim}{I} - \underset{\sim}{D}_\eta|^{(m/2)} \Sigma \prod_{i\,j} [\frac{(1/2) \,..\, (b_{ij} - 1/2)}{(b_{ij}!)} \eta_i^{b_{ij}} E^{b_{ij}} G_\rho(\theta)]$$

$$= |\underset{\sim}{I} - \underset{\sim}{D}_\eta|^{(m/2)} |\underset{\sim}{I} - \underset{\sim}{D}_\eta E|^{-(m/2)} G_\rho(\theta)$$

$$= (\frac{|\underset{\sim}{I} - \underset{\sim}{D}_\eta E|}{|\underset{\sim}{I} - \underset{\sim}{D}_\eta|})^{-(m/2)} G_\rho(\theta) ,$$

where $\rho = mp/2$ and E is an operator such that $EG_\rho(\theta) = G_{\rho+1}(\theta)$. Now let $E = \Delta + 1$. Then

$$|\underset{\sim}{I} - \underset{\sim}{D}_\eta E| / |\underset{\sim}{I} - \underset{\sim}{D}_\eta| = |\underset{\sim}{I} - \underset{\sim}{D}_\eta - \underset{\sim}{D}_\eta \Delta| / |\underset{\sim}{I} - \underset{\sim}{D}_\eta|$$

$$= |\underset{\sim}{I} - [\underset{\sim}{B}^{-1} (\underset{\sim}{A}^{-1} + \underset{\sim}{\varepsilon})^{-1} - \underset{\sim}{I}] \Delta|$$

$$= |\underset{\sim}{I} - \underset{\sim}{X} \Delta| \qquad \text{(say).}$$

Hence

$$J = |\underset{\sim}{I} - \underset{\sim}{X} \Delta|^{-(m/2)} G_\rho(\theta)$$

$$= \text{Exp } [(-m/2) \log |I - \underset{\sim}{X} \Delta|] \; G_\rho(\theta) \; .$$

Now if $\underset{\sim}{B}^{-1} \underset{\sim}{A} = \underset{\sim}{I} + \underset{\sim}{F}$ such that $|Ch_i (\underset{\sim}{F})| < 1$, $i = 1,\ldots, p$, then for $\underset{\sim}{\varepsilon}(p \times p)$ sufficiently close to $\underset{\sim}{0}(p \times p)$ we get $|ch_i (\underset{\sim}{X})| < 1$, $i = 1,\ldots, p$, and

$$J = [1 + (m/2) (\text{tr } \underset{\sim}{X}) \Delta + \{(m^2/8)(\text{tr } \underset{\sim}{X})^2 + (m/4)(\text{tr } \underset{\sim}{X}^2)\} \Delta^2$$

$$+ \{(m/6) \text{ tr } \underset{\sim}{X}^3 + (m^2/8) \text{ tr } \underset{\sim}{X} + (m^3/48)(\text{tr } \underset{\sim}{X})^3\} \Delta^3 + \ldots] \; G_\rho(\theta).$$

Now using Taylor's expansion for $\underset{\sim}{A}^{-1} + \underset{\sim}{\varepsilon}$ we get

$$\underset{\sim}{X} = \underset{\sim}{B}^{-1}(\underset{\sim}{A}^{-1} + \underset{\sim}{\varepsilon})^{-1} - \underset{\sim}{I} = \underset{\sim}{B}^{-1} (\underset{\sim}{A}^{-1} + \Sigma \, \varepsilon_{rs} \underset{\sim}{A}_{rs}^{-1})^{-1} - \underset{\sim}{I}$$

$$= \underset{\sim}{B}^{-1} (\underset{\sim}{I} + \Sigma \, \varepsilon_{rs} \underset{\sim}{A} \underset{\sim}{A}_{rs}^{-1})^{-1} \underset{\sim}{A} - \underset{\sim}{I}$$

$$= \underset{\sim}{B}^{-1} (\underset{\sim}{I} - \Sigma \, \varepsilon_{rs} \underset{\sim}{A} \underset{\sim}{A}_{rs}^{-1} + \Sigma \, \varepsilon_{rs} \, \varepsilon_{tu} (\underset{\sim}{A} \underset{\sim}{A}_{rs}^{-1})(\underset{\sim}{A} \underset{\sim}{A}_{tu}^{-1}) + \ldots)\underset{\sim}{A} - \underset{\sim}{I}$$

$$= (\underset{\sim}{B}^{-1} \underset{\sim}{A} - \underset{\sim}{I}) - \Sigma \, \varepsilon_{rs} (\underset{\sim}{B}^{-1} \underset{\sim}{A}) (\underset{\sim}{A}_{rs}^{-1} \underset{\sim}{A}) +$$

$$\Sigma \, \varepsilon_{rs} \, \varepsilon_{tu} (\underset{\sim}{B}^{-1} \underset{\sim}{A}) (\underset{\sim}{A}_{rs}^{-1} \underset{\sim}{A}) (\underset{\sim}{A}_{tu}^{-1} \underset{\sim}{A}) \ldots .$$

where $\underset{\sim}{A}^{-1}_{rs}$ is the p x p matrix obtained by operating ∂_{rs} on $\underset{\sim}{A}^{-1}$ i.e. it has its $(i, j)^{th}$ element as $\frac{1}{2}(\delta_{ri}\,\delta_{sj} + \delta_{si}\,\delta_{rj})$. Now using the notations

$$
\begin{aligned}
&\mathrm{tr}\ (\underset{\sim}{A}^{-1}_{rs}\ \underset{\sim}{A}) &&= (rs)\\
&\mathrm{tr}\ (\underset{\sim}{A}^{-1}_{rs}\ \underset{\sim}{A})\ (\underset{\sim}{A}^{-1}_{tu}\ \underset{\sim}{A}) &&= (rs|tu)\\
&\mathrm{tr}\ \underset{\sim}{F}\ (\underset{\sim}{A}^{-1}_{rs}\ \underset{\sim}{A})\ (\underset{\sim}{A}^{-1}_{tu}\ \underset{\sim}{A}) &&= (\underset{\sim}{F}|rs|tu)\\
&\mathrm{tr}\ (\underset{\sim}{B}^{-1}\ \underset{\sim}{A})\ (\underset{\sim}{A}^{-1}_{rs}\ \underset{\sim}{A})\ (\underset{\sim}{A}^{-1}_{tu}\ \underset{\sim}{A}) &&= (\underset{\sim}{I} + \underset{\sim}{F}|rs|tu)\\
&\mathrm{tr}\ \underset{\sim}{F} &&= (\underset{\sim}{F})\\
&\mathrm{tr}\ \underset{\sim}{F}^2 &&= (\underset{\sim}{F}|\underset{\sim}{F}) \qquad \text{etc.}
\end{aligned}
$$

we get

$$
\begin{aligned}
(4.9)\quad J = [&1 + (m/2)(\underset{\sim}{F})\ \Delta + \{(m^2/8)(\underset{\sim}{F})^2 + (m/4)(\underset{\sim}{F}/\underset{\sim}{F})\}\ \Delta^2\\
&+ \{(m/6)(\underset{\sim}{F}|\underset{\sim}{F}|\underset{\sim}{F}) + (m^2/8)(\underset{\sim}{F})(\underset{\sim}{F}|\underset{\sim}{F}) + (m^3/48)(\underset{\sim}{F})^3\}\ \Delta^3 + \ldots.\\
&+ \Sigma\ \varepsilon_{rs}\ \{-(m|2)(\underset{\sim}{I} + \underset{\sim}{F}|rs)\ \Delta + (\ (-m/2)(\underset{\sim}{F}|\underset{\sim}{I} + \underset{\sim}{F}|rs)\\
&- (m^2/4)(\underset{\sim}{F})(\underset{\sim}{I} + \underset{\sim}{F}|rs)\ )\ \Delta^2 + (\ -(m/2)(\underset{\sim}{F}|\underset{\sim}{F}|\underset{\sim}{I} + \underset{\sim}{F}|rs)\\
&+ (m^2/8)\ \{-2(\underset{\sim}{F})(\underset{\sim}{F}|\underset{\sim}{I} + \underset{\sim}{F}|rs) - (\underset{\sim}{F}|\underset{\sim}{F})\ (\underset{\sim}{I} + \underset{\sim}{F}|rs)\}\\
&- 3\ m^3/48\ (\underset{\sim}{F})^2\ (\underset{\sim}{I} + \underset{\sim}{F}|rs)\ )\ \Delta^3 + \ldots.\ \ldots.\\
&+ \Sigma\ \varepsilon_{rs}\ \varepsilon_{tu}\ \{(m/2)(\underset{\sim}{I} + \underset{\sim}{F}|rs|tu)\ \Delta + (\ (m/4)\\
&\quad \{(\underset{\sim}{I} + \underset{\sim}{F}|rs|\underset{\sim}{I} + \underset{\sim}{F}|tu)\\
&+ 2\ (\underset{\sim}{F}|\underset{\sim}{I} + \underset{\sim}{F}|rs|tu)\} + (m^2/8)\ \{(\underset{\sim}{I} + \underset{\sim}{F}|rs)(\underset{\sim}{I} + \underset{\sim}{F}|tu)\\
&+ 2\ (\underset{\sim}{F})(\underset{\sim}{I} + \underset{\sim}{F}|rs|tu)\})\ \Delta^2 + ((m/6)\{3(\underset{\sim}{F}|\underset{\sim}{I} + \underset{\sim}{F}|rs|\underset{\sim}{I} + \underset{\sim}{F}|tu)\\
&+ 3\ (\underset{\sim}{F}|\underset{\sim}{F}|\underset{\sim}{I} + \underset{\sim}{F}|rs|tu)\} + (m^2/8)\ \{(\underset{\sim}{F}|\underset{\sim}{F})(\underset{\sim}{I} + \underset{\sim}{F}|rs|tu)
\end{aligned}
$$

$$+ (\underset{\sim}{F})(\underset{\sim}{I} + \underset{\sim}{F}|rs|\underset{\sim}{I} + \underset{\sim}{F}|tu) + 2\ (\underset{\sim}{I} + \underset{\sim}{F}|tu)(\underset{\sim}{F}|\underset{\sim}{I} + \underset{\sim}{F}|rs) + 2\ (\underset{\sim}{F})$$

$$(\underset{\sim}{F}|\underset{\sim}{I} + \underset{\sim}{F}|rs|tu)\} + (m^3/48)\ \{3\ (\underset{\sim}{F})^2\ (\underset{\sim}{I} + \underset{\sim}{F}|rs|tu)$$

$$+ 3\ (\underset{\sim}{F})(\underset{\sim}{I} + \underset{\sim}{F}|rs)(\underset{\sim}{I} + \underset{\sim}{F}|tu)\})\ \Delta^3 + \ldots.\ \ldots.$$

$$\ldots.]\ G_\rho(\theta)\ .$$

Also we note $\Delta G_\rho(\theta) = -E\ g_\rho(\theta)$ where $g_\rho(\theta) = [\Gamma(\rho)]^{-1} e^{-t}\ t^{\rho-1}$. Comparing the two expressions for J in (4.8) and (4.9) and equating coefficients of $\varepsilon_{tu}\ \varepsilon_{rs}$ we get

$$h_1(\underset{\sim}{A}^{-1}) = 1/4n \sum_{r,s,t,u} \sigma_{st}\ \sigma_{ur}\ [4m\ (\underset{\sim}{I} + \underset{\sim}{F}|rs|tu)E$$

$$+ (2\ m\ \{(\underset{\sim}{I} + \underset{\sim}{F}|rs|\underset{\sim}{I} + \underset{\sim}{F}|tu) + 2\ (\underset{\sim}{F}|\underset{\sim}{I} + \underset{\sim}{F}|rs|tu)\}$$

$$+ m^2\ \{(\underset{\sim}{I} + \underset{\sim}{F}|rs)(\underset{\sim}{I} + \underset{\sim}{F}|tu) + 2\ (\underset{\sim}{F})(\underset{\sim}{I} + \underset{\sim}{F}|rs|tu)\})\ E\ (E - 1)$$

$$+ (4m\ \{(\underset{\sim}{F}|\underset{\sim}{I} + \underset{\sim}{F}|rs|\underset{\sim}{I} + \underset{\sim}{F}|tu) + (\underset{\sim}{F}|\underset{\sim}{F}|\underset{\sim}{I} + \underset{\sim}{F}|rs|tu)\}$$

$$+ m^2\ \{(\underset{\sim}{F}|\underset{\sim}{F})(\underset{\sim}{I} + \underset{\sim}{F}|rs|tu) + (\underset{\sim}{F})(\underset{\sim}{I} + \underset{\sim}{F}|rs|\underset{\sim}{I} + \underset{\sim}{F}|tu)$$

$$+ 2\ \ (\underset{\sim}{F}|\underset{\sim}{I} + \underset{\sim}{F}|rs)(\underset{\sim}{I} + \underset{\sim}{F}|tu) + 2\ (\underset{\sim}{F})\ \ (\underset{\sim}{F}|\underset{\sim}{I} + \underset{\sim}{F}|rs|tu)\}$$

$$+ (m^3/2)\ \{(\underset{\sim}{F})^2\ (\underset{\sim}{I} + \underset{\sim}{F}|rs|tu) + (\underset{\sim}{F})(\underset{\sim}{I} + \underset{\sim}{F}|rs)$$

$$(\underset{\sim}{I} + \underset{\sim}{F}|tu)\})\ E(E^2 - 2\ E + 1) + \ldots\ \ldots]\ g_\rho(\theta)\ [G'(\theta)]^{-1}\ .$$

The result as it stands is not convenient for practical use. In order to make some simplification we assume that terms involving $f_{ij}\ f_{kl}$ are negligible, where f_{ij} in the (i,j) element of $\underset{\sim}{F}$.

Also we note

$$\sum_{r,s,t,u} \sigma_{st}\ \sigma_{ur}\ (rs|tu) = \frac{1}{2}\, p\ (p + 1)\ ,$$

$$\sum_{r,s} \sigma_{rs}\ (rs) = p\ ,$$

$$\sum_{r,s,t,u} \sigma_{st}\ \sigma_{ur}\ (rs)(tu) = p\ ,$$

$$\sum_{r,s,t,u} \sigma_{st}\ \sigma_{ur}\ (\underset{\sim}{F}|rs)(tu) = (\underset{\sim}{F})\ ,$$

and $$\sum_{r,s,t,u} \sigma_{st}\ \sigma_{ur}\ (\underset{\sim}{F}|rs|tu) = (\underset{\sim}{F})(p+1)/2\ .$$

Hence under the above assumptions we get

$$h_1\ (\underset{\sim}{A}^{-1}) = 1/4n\ [4m\ (p\ (p+1)/2 + (p+1)\ (\underset{\sim}{F})/2)\ E$$
$$+ (2m\ \{p\ (p+1)/2 + 4\ (\underset{\sim}{F})(p+1)/2\} + m^2\ \{2\ (\underset{\sim}{F}) + (\underset{\sim}{F})\ p\ (p+1)$$
$$+ p\}\ E\ (E-1) + (4m\ (\underset{\sim}{F})\ (p+1)/2 + m^2\ \{(\underset{\sim}{F})\ p\ (p+1)/2$$
$$+ 2\ (\underset{\sim}{F})\} + (m^3/2)\ (\underset{\sim}{F})\ p)\ (E^3 - 2\ E^2 + E)]\ g_\rho(\theta)\ [G'\ (\theta)]^{-1}$$

Recalling that $(mp/2) = \rho$ and putting $2\ \theta = y$ and noting

$$E^i\ g_\rho(\theta) = \frac{\theta^i}{\rho(\rho+1)\ldots(\rho+i-1)}\ g_\rho(\theta)\ ,$$

we get

$$(4.10)\quad T = m\ \mathrm{tr}\ \underset{\sim}{S}_1\ \underset{\sim}{S}_2^{-1} = y + 1/2n\ [2(p+1)(1 + (\underset{\sim}{F})/p)y$$
$$+ (p+1)(1 + 4(\underset{\sim}{F})/p)\ y\ (\frac{y}{mp+2} - 1) + (m/p)$$
$$(2\ (\underset{\sim}{F}) + (\underset{\sim}{F})\ p\ (p+1) + p)\ y\ (\frac{y}{mp+2} - 1) + (2\ (\underset{\sim}{F})\ (p+1)/p$$
$$+ m\ \{(\underset{\sim}{F})(p+1)/2 + 2(\underset{\sim}{F})/p\} + (m^2/2)(\underset{\sim}{F}))\ y\ (\frac{y^2}{(mp+2)(mp+4)}$$
$$- \frac{2y}{(mp+2)} + 1)]\ g_\rho(\theta)\ [G'(\theta)]^{-1} + O\ (n^{-2})\ .$$

Hence we have the following theorem.

__Theorem 4.1.__ Let $m\underset{\sim}{S}_1$ and $n\underset{\sim}{S}_2$ be independently distributed $W(m, p, \underset{\sim}{B}^{-1})$, $W(n, p, \underset{\sim}{A}^{-1})$ respectively and let

(i) $\underset{\sim}{B}^{-1} \underset{\sim}{A} = \underset{\sim}{I} + \underset{\sim}{F}$ and $|ch_i(\underset{\sim}{F})| < 1$, $i = 1, \ldots, p$

(ii) terms involving $f_{ij} f_{kl}$ be negligible, where f_{ij} is the (i,j) element of $\underset{\sim}{F}$.

Then an asymptotic expansion for the percentile of $T = m \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$ is given by (4.10).

Here as defined earlier

$$y = \sum_{j=1}^{p} \lambda_j \chi_j^2(m) \tag{4.11}$$

and $\chi_j^2(m)$'s are independent central chi-square variables with m d.f. and λ_j's are ch. roots of $\underset{\sim}{U} = \underset{\sim}{A}\,\underset{\sim}{B}^{-1}$.

__Special case.__

As a check we put $\underset{\sim}{F}(p \times p) = \underset{\sim}{0}(p \times p)$. Then

$$y = \chi^2(m\,p)$$

is a central chi square variable with $m\,p$ d.f. and $G(\theta) = G_\rho(\theta)$. Hence we get

$$T = \chi^2 + 1/2n\ [2(p+1)\chi^2 + (p+1)\chi^2 (\frac{\chi^2}{mp+2} - 1)$$

$$+ (mp)/p\ \chi^2 (\frac{\chi^2}{mp+2} - 1)]\ g_\rho(\theta)\ [g_\rho(\theta)]^{-1} + O(n^{-2})$$

$$= \chi^2 + 1/2n\ [((p+m+1)/(mp+2))\chi^4 + (p-m+1)\chi^2] + O(n^{-2})$$

where $\chi^2 = \chi^2(mp)$. This agrees with Ito's result [7] up to the indicated order.

4.3. An Asymptotic Expansion for the c.d.f. of $T = m \text{ tr } \underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$

In this section we obtain an asymptotic expansion for the c.d.f. of T following the methods described in the previous section. Again we write

$$\Pr\{m \text{ tr } \underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1} \leq 2\,\theta\} = \int_R \Pr\{m \text{ tr } \underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1} \leq 2\theta | \underset{\sim}{S}_2\} \Pr\{d\, \underset{\sim}{S}_2\}$$

$$= \boldsymbol{\theta} \Pr\{m \text{ tr } \underset{\sim}{S}_1 \underset{\sim}{A} \leq 2\theta\} .$$

where

$$\boldsymbol{\theta} = 1 + 1/2\, \Sigma\, \sigma_{rs}\, \sigma_{tu}\, \partial_{st}\, \partial_{ur} + O(n^{-2})$$

we get

$$\Pr\{m \text{ tr } \underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1} \leq 2\,\theta\} = \Pr\{m \text{ tr } \underset{\sim}{S}_1 \underset{\sim}{A} \leq 2\,\theta\}$$

$$+ 1/2\, \Sigma\, \sigma_{rs}\, \sigma_{tu}\, \partial_{st}\, \partial_{ur} \Pr\{m \text{ tr } \underset{\sim}{S}_1 \underset{\sim}{A} \leq 2\,\theta\} + O(n^{-2}) .$$

Now using the value of $\partial_{st}\, \partial_{ur} \Pr\{m \text{ tr } \underset{\sim}{S}_1 \underset{\sim}{A} \leq 2\,\theta\}$ as before, we get

$$\Pr\{m \text{ tr } \underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1} \leq 2\,\theta\} = G(\theta) - 1/n\, [h_1\, (\underset{\sim}{A}^{-1})]\, G'\,(\theta) + O\,(n^{-2})$$

and under the assumptions (i) and (ii) of theorem 4.1,

$$(4.12) \quad \Pr\{m \text{ tr } \underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1} \leq 2\,\theta\} = G(\theta) - 1/4n\, [4\,(p+1)(p+(\underset{\sim}{F}))\, \frac{\theta}{p}$$

$$+ \frac{2}{p}\,((p+1)(p+4\,(\underset{\sim}{F})) + m\,\{2\,(\underset{\sim}{F}) + (\underset{\sim}{F})\, p\,(p+1) + p\})$$

$$\theta\,(\frac{2\boldsymbol{\theta}}{mp+2} - 1) + \frac{2}{p}\,(2\,(\underset{\sim}{F})(p+1) + m\,(\underset{\sim}{F})\, p\,(p+1)/2$$

$$+ 2m (\underset{\sim}{F}) + m^2 p (\underset{\sim}{F})/2)\ \theta\ (\frac{4\theta^2}{(mp+2)(mp+4)} - \frac{4\theta}{(mp+2)} + 1)]\ g_\rho(\theta)$$

$$+ O(n^{-2})$$

Hence we have the following theorem.

Theorem 4.2. Under the assumptions (i) and (ii) of theorem 1, the asymptotic expansion for the c.d.f. of $T = m \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$ is given by (4.12).

Special case.

Here again as a check we put $\underset{\sim}{F}(p \times p) = \underset{\sim}{O}(p \times p)$ and we get

$$\Pr\{m \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1} \leq 2\ \theta\}$$

$$= G_\rho(\theta) - 1/2n\ [(p-m+1) + 2\ (p+1+m)/(mp+2)\ \theta^2]\ g_\rho(\theta) + O\ (n^{-2})$$

and this agrees with Ito's result [7].

4.4. Asymptotic Expansion for the c.d.f. and Percentiles of $F' = (m_1 \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{S}_4^{-1})/(n_1 \operatorname{tr} \underset{\sim}{S}_3 \underset{\sim}{S}_2^{-1})$

Let $m_1\underset{\sim}{S}_1$, $m_2\underset{\sim}{S}_2$, $n_1\underset{\sim}{S}_3$, $n_2\underset{\sim}{S}_4$ be distributed independently $W(m_1, p, \underset{\sim}{A}^{-1})$, $W(m_2, p, \underset{\sim}{A}^{-1})$, $W(n_1, p, \underset{\sim}{B}^{-1})$, $W(n_2, p, \underset{\sim}{B}^{-1})$ respectively where $\underset{\sim}{S}_1$, $\underset{\sim}{S}_2$, $\underset{\sim}{S}_3$, $\underset{\sim}{S}_4$ are four covariance matrices. Then

$$(4.13) \qquad \Pr\{F' = (m_1 \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{B})/(n_1 \operatorname{tr} \underset{\sim}{S}_3 \underset{\sim}{A}) \leq \theta\} = L\ (\theta)$$

where $L(\theta)$ is the distribution of ratio of two independent quadratic forms. Now when $\underset{\sim}{A} = \underset{\sim}{B}$ we get

$$L(\theta) = G\ (\theta, \rho_1, \rho_2)$$

$(n_1\theta/m_1)$ is the tabled F value with m_1p and n_1p d.f. for a given probability and

$$(4.14) \qquad (m_1\ p/2) = \rho_1,\ (n_1\ p/2) = \rho_2\ .$$

Also

$$G(\theta,\ \rho_1,\ \rho_2) = \int_0^\theta \beta(t,\ \rho_1,\ \rho_2)\ dt$$

where

$$(4.15) \quad \beta(t,\ \rho_1,\ \rho_2) = [B\ (\rho_1,\ \rho_2)]^{-1}\ t^{\rho_1-1}/(1+t)^{\rho_1+\rho_2}\ .$$

As in the earlier part, our problem here is to find a function $h(\underset{\sim}{S}_2,\ \underset{\sim}{S}_4,\ \theta) = h$ (say) such that

$$\Pr\{m_1\ \mathrm{tr}\ \underset{\sim}{S}_1\ \underset{\sim}{S}_4^{-1}/n_1\ \mathrm{tr}\ \underset{\sim}{S}_3\ \underset{\sim}{S}_2^{-1} \leq h\} = L(\theta)$$

As before we now write h as a series with the first term as the ratio of two independent quadratic forms and successive terms of decreasing order of magnitude i.e. we put

$$(4.16) \qquad h = \theta + h_1 + h_2 + \ldots..$$

where

$$(4.17) \qquad h_i = O\ (m^{-2})\ \text{and}\ m = \min\ (n_2,\ m_2)\ .$$

Proceeding as in the first part we get

$$\Pr\{m_1 \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{S}_4^{-1}/n_1 \operatorname{tr} \underset{\sim}{S}_3 \underset{\sim}{S}_2^{-1} \le h\}$$

$$= \int_{R_1} \Pr\left\{\frac{m_1 \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{S}_4^{-1}}{n_1 \operatorname{tr} \underset{\sim}{S}_3 \underset{\sim}{S}_2^{-1}} \le h \,|\, \underset{\sim}{S}_2, \underset{\sim}{S}_4\right\} \Pr\{d\,\underset{\sim}{S}_4\} \Pr\{d\,\underset{\sim}{S}_2\}$$

where $\Pr\{d\,\underset{\sim}{S}_2\}$ and $\Pr\{d\,\underset{\sim}{S}_4\}$ represent the probability element of $\underset{\sim}{S}_2$ and $\underset{\sim}{S}_4$ respectively and R_1 is the appropriate range of integration. Now

$$\Pr\left\{\frac{m_1 \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{S}_4^{-1}}{n_1 \operatorname{tr} \underset{\sim}{S}_3 \underset{\sim}{S}_2^{-1}} \le h \,|\, \underset{\sim}{S}_2, \underset{\sim}{S}_4\right\} =$$

$$\{\exp[\operatorname{tr}(\underset{\sim}{S}_4 - \underset{\sim}{B}^{-1})\, \underset{\sim}{\partial}^{(1)}]\}\, \{\exp[\operatorname{tr}(\underset{\sim}{S}_2 - \underset{\sim}{A}^{-1})\, \underset{\sim}{\partial}^{(2)}]\}$$

$$\Pr\left\{\frac{m_1 \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{B}}{n_1 \operatorname{tr} \underset{\sim}{S}_3 \underset{\sim}{A}} \le h(\underset{\sim}{A}^{-1}, \underset{\sim}{B}^{-1})\right\}$$

where $\underset{\sim}{A}^{-1}$ is given in (4.4) and

$$\underset{\sim}{B}^{-1} = \begin{bmatrix} \lambda_{11} & \cdots & \lambda_{1p} \\ \vdots & & \vdots \\ \lambda_{p1} & \cdots & \lambda_{pp} \end{bmatrix}$$

and

$$\underset{\sim}{\partial}^{(1)} = (1/2\ (1 + \delta_{ij})\ \frac{\partial}{\partial \lambda_{ij}})$$

$$\underset{\sim}{\partial}^{(2)} = (1/2\ (1 + \delta_{ij})\ \frac{\partial}{\partial \sigma_{ij}})$$

Hence we get

$$L(\theta) = \int_{R_1} \{\exp [\operatorname{tr} (\underset{\sim}{S}_4 - \underset{\sim}{B}^{-1}) \underset{\sim}{\partial}^{(1)}]\} \{\exp [\operatorname{tr} (\underset{\sim}{S}_2 - \underset{\sim}{A}^{-1}) \underset{\sim}{\partial}^{(2)}]\}$$

$$\Pr\{\frac{m_1 \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{B}}{n_1 \operatorname{tr} \underset{\sim}{S}_3 \underset{\sim}{A}} \leq h(\underset{\sim}{A}^{-1}, \underset{\sim}{B}^{-1})\} \Pr\{d \underset{\sim}{S}_4\} \Pr\{d \underset{\sim}{S}_2\}$$

$$= \Theta \Pr\{\frac{m_1 \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{B}}{n_1 \operatorname{tr} \underset{\sim}{S}_3 \underset{\sim}{A}} \leq h (\underset{\sim}{A}^{-1}, \underset{\sim}{B}^{-1})\}$$

where

$$\Theta = \{\exp [-\operatorname{tr} \underset{\sim}{B}^{-1} \underset{\sim}{\partial}^{(1)}] \mid \underset{\sim}{I} - (\frac{2}{n_2}) \underset{\sim}{\partial}^{(1)} \mid^{-n_2/2} \}$$

$$\{\exp [-\operatorname{tr} \underset{\sim}{A}^{-1} \underset{\sim}{\partial}^{(2)}] \mid \underset{\sim}{I} - (2/m_2) \underset{\sim}{\partial}^{(2)} \mid^{-m_2/2} \} .$$

Hence we get

$$(4.18) \quad \Theta = 1 + 1/n_2 \operatorname{tr} (\underset{\sim}{B}^{-1} \underset{\sim}{\partial}^{(1)})^2 + 1/m_2 \operatorname{tr} (\underset{\sim}{A}^{-1} \underset{\sim}{\partial}^{(2)})^2 + O(m^{-2})$$

$$= 1 + 1/n_2 \Sigma \lambda_{rs} \lambda_{tu} \partial_{st}^{(1)} \partial_{ur}^{(1)} + 1/m_2 \Sigma \sigma_{rs} \sigma_{tu} \partial_{st}^{(2)} \partial_{ur}^{(2)}$$

$$+ O(m^{-2}) ,$$

where

$$\partial_{rs}^{(1)} = 1/2 (1 + \delta_{rs}) \frac{\partial}{\partial \lambda_{rs}}, \quad \partial_{rs}^{(2)} = 1/2 (1 + \delta_{rs}) \frac{\partial}{\partial \sigma_{rs}} .$$

Hence

$$L(\theta) = \{1 + 1/n_2 \Sigma \lambda_{rs} \lambda_{tu} \partial_{st}^{(1)} \partial_{ur}^{(1)} + 1/m_2 \Sigma \sigma_{rs} \sigma_{tu} \partial_{st}^{(2)} \partial_{ur}^{(2)}$$

$$+ O(m^{-2})\}$$

$$[1 + h_1 D + \{h_2 D + 1/2\, h_1^2 D^2\} + O(m^{-3})] \Pr\{\frac{m_1 \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{B}}{n_1 \operatorname{tr} \underset{\sim}{S}_3 \underset{\sim}{A}} \leq \theta\}$$

where D is defined in (4.7). Now equating terms of successive orders [19], [20], we have

$$(4.19) \quad \{h_1 D + 1/n_2\, \Sigma\, \lambda_{rs} \lambda_{tu}\, \partial_{st}^{(1)} \partial_{ur}^{(1)} + 1/m_2\, \Sigma\, \sigma_{rs} \sigma_{tu}\, \partial_{st}^{(2)} \partial_{ur}^{(2)}\}$$

$$\Pr\{\frac{m_1 \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{B}}{n_1 \operatorname{tr} \underset{\sim}{S}_3 \underset{\sim}{A}} \leq \theta\} = 0$$

etc. Hence to find

$$\{\Sigma\, \lambda_{rs} \lambda_{tu}\, \partial_{st}^{(1)} \partial_{ur}^{(1)}\} \Pr\{\frac{m_1 \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{B}}{n_1 \operatorname{tr} \underset{\sim}{S}_3 \underset{\sim}{A}} \leq \theta\}$$

and $$\{\Sigma\, \sigma_{rs} \sigma_{tu}\, \partial_{st}^{(2)} \partial_{ur}^{(2)}\} \Pr\{\frac{m_1 \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{B}}{n_1 \operatorname{tr} \underset{\sim}{S}_3 \underset{\sim}{A}} \leq \theta\} .$$

We again apply the perturbation technique as in James [8]. Let

$$J_1 = \Pr\{\frac{m_1 \operatorname{tr} \underset{\sim}{S}_1 (\underset{\sim}{B}^{-1} + \underset{\sim}{\varepsilon}^{(1)})^{-1}}{n_1 \operatorname{tr} \underset{\sim}{S}_3 (\underset{\sim}{A}^{-1} + \underset{\sim}{\varepsilon}^{(2)})^{-1}} \leq \theta\}$$

where $\underset{\sim}{\varepsilon}^{(1)}$ (p x p), $\underset{\sim}{\varepsilon}^{(2)}$ (p x p) are symmetric matrices corresponding to small increments $\varepsilon_{ij}^{(1)}$ and $\varepsilon_{ij}^{(2)}$.

As before by Taylor's theorem applied on $\underset{\sim}{B}^{-1} + \underset{\sim}{\varepsilon}^{(1)}$ and $\underset{\sim}{A}^{-1} + \underset{\sim}{\varepsilon}^{(2)}$, we get

$$(4.20)\quad J_1 = [1 + \Sigma\, \varepsilon_{rs}^{(1)}\, \partial_{rs}^{(1)} + \Sigma\, \varepsilon_{rs}^{(2)}\, \partial_{rs}^{(2)}$$

$$+ \Sigma\, \varepsilon_{rs}^{(1)}\, \varepsilon_{tu}^{(2)}\, \partial_{rs}^{(1)}\, \partial_{tu}^{(2)} + 1/2\, \{\Sigma\, \varepsilon_{rs}^{(1)}\, \varepsilon_{tu}^{(1)}\, \partial_{rs}^{(1)}\, \partial_{tu}^{(1)}$$

$$+ \Sigma\, \varepsilon_{rs}^{(2)}\, \varepsilon_{tu}^{(2)}\, \partial_{rs}^{(2)}\, \partial_{tu}^{(2)}\} + \ldots]\ \mathrm{pr}\{\frac{m_1\ \mathrm{tr}\ \underset{\sim}{S}_1\ \underset{\sim}{B}}{n_1\ \mathrm{tr}\ \underset{\sim}{S}_3\ \underset{\sim}{A}} \leq \theta\}\ .$$

We can also find J_1 as follows

$$J_1 = \frac{|\underset{\sim}{A}|^{m_1/2}\ |\underset{\sim}{B}|^{n_1/2}}{(2\pi)^{(m_1+n_1)p/2}} \int_{R_1'} \exp[-1/2\ \mathrm{tr}(\underset{\sim}{A}\ \underset{\sim}{X}_1\ \underset{\sim}{X}_1' + \underset{\sim}{B}\ \underset{\sim}{X}_3\ \underset{\sim}{X}_3')]\ d\underset{\sim}{X}_1\ d\underset{\sim}{X}_3$$

where

$$\underset{\sim}{X}_1(p \times m_1),\ \underset{\sim}{X}_3(p \times n_1),\ \underset{\sim}{S}_1 = \underset{\sim}{X}_1\ \underset{\sim}{X}_1',\ \underset{\sim}{S}_3 = \underset{\sim}{X}_3\ \underset{\sim}{X}_3'$$

and

$$R_1':\ [\underset{\sim}{X}_1,\ \underset{\sim}{X}_3:\ \frac{m_1\ \mathrm{tr}\ \underset{\sim}{S}_1\ (\underset{\sim}{B}^{-1} + \underset{\sim}{\varepsilon}^{(1)})^{-1}}{n_1\ \mathrm{tr}\ \underset{\sim}{S}_3\ (\underset{\sim}{A}^{-1} + \underset{\sim}{\varepsilon}^{(2)})^{-1}} \leq \theta]\ .$$

Now let $\underset{\sim}{G}(p \times p)$ and $\underset{\sim}{H}(p \times p)$ be nonsingular matrices such that

$$1/2\ \underset{\sim}{G}'\ (\underset{\sim}{A}^{-1} + \underset{\sim}{\varepsilon}^{(2)})^{-1}\ \underset{\sim}{G} = \underset{\sim}{I}(p \times p)$$

$$1/2\ \underset{\sim}{G}'\ \underset{\sim}{B}\ \underset{\sim}{G} = \underset{\sim}{I}(p \times p) - \underset{\sim}{\eta}^{(2)}$$

$$1/2\ \underset{\sim}{H}'\ (\underset{\sim}{B}^{-1} + \underset{\sim}{\varepsilon}^{(1)})^{-1}\ \underset{\sim}{H} = \underset{\sim}{I}(p \times p)$$

$$1/2\ \underset{\sim}{H}'\underset{\sim}{A}\ \underset{\sim}{H} = \underset{\sim}{I}(p \times p) - \underset{\sim}{\eta}^{(1)}$$

where $$\eta^{(1)} = \text{diag}\,(\eta_1^{(1)}, \ldots, \eta_p^{(1)})$$

$$\eta^{(2)} = \text{diag}\,(\eta_1^{(2)}, \ldots, \eta_p^{(2)})$$

This is possible as A^{-1}, B^{-1} are p.d. and $\varepsilon^{(1)}$, and $\varepsilon^{(2)}$ are sufficiently close to $O(p \times p)$.

Now let

$$X_1 = H\, Z_1, \quad X_3 = G\, Z_3$$

Then we have

$$J_1 = |I - \eta^{(1)}|^{m_1/2}\, |I - \eta^{(2)}|^{n_1/2}\, \pi^{-(n_1 + m_1)p/2}$$

$$\int_{R_1'} \exp[-\text{tr}\,\{(I - \eta^{(1)})\, Z_1\, Z_1' + (1 - \eta^{(2)})\, Z_3\, Z_3'\}]\, d\, Z_1\, d\, Z_3$$

where

$$R_1' : [Z_1, Z_3 : \frac{m_1\, \text{tr}\, Z_1\, Z_1'}{n_1\, \text{tr}\, Z_3\, Z_3'} \leq \theta] \; .$$

Now following Siotani [19], [20] we get

$$J_1 = \left\{ \frac{|I - \eta^{(1)}\, E_1|}{|I - \eta^{(1)}|} \right\}^{-m_1/2} \left\{ \frac{|I - \eta^{(2)}\, E_2|}{|I - \eta^{(2)}|} \right\}^{-n_1/2} G\,(\theta, \rho_1, \rho_2)$$

where

$$E_1^a\, f(\rho_1, \rho_2) = f(\rho_1 + a, \rho_2) \text{ and } E_2^b\, f(\rho_1, \rho_2) = f(\rho_1, \rho_2 + b)$$

for any positive integer a and b and function $f(\rho_1, \rho_2)$.

Now as in the previous sections we get

$$|\underset{\sim}{I} - \underset{\sim}{\eta}^{(1)} E_1| / |\underset{\sim}{I} - \underset{\sim}{\eta}^{(1)}| = |\underset{\sim}{I} - \underset{\sim}{X} \Delta_1|$$

and $$|\underset{\sim}{I} - \underset{\sim}{\eta}^{(2)} E_2| / |\underset{\sim}{I} - \underset{\sim}{\eta}^{(2)}| = |\underset{\sim}{I} - \underset{\sim}{Y} \Delta_2|$$

where $\underset{\sim}{X} = \underset{\sim}{A}^{-1} (\underset{\sim}{B}^{-1} + \underset{\sim}{\varepsilon}^{(1)})^{-1} - \underset{\sim}{I}$ and $\underset{\sim}{Y} = \underset{\sim}{B}^{-1} (\underset{\sim}{A}^{-1} + \underset{\sim}{\varepsilon}^{(2)})^{-1} - \underset{\sim}{I}$.

Hence

$$J_1 = \{|\underset{\sim}{I} - \underset{\sim}{X} \Delta_1|^{-m_1/2} |\underset{\sim}{I} - \underset{\sim}{Y} \Delta_2|^{-n_1/2}\} G(\theta, \rho_1, \rho_2) .$$

Now if $\underset{\sim}{B}^{-1} \underset{\sim}{A} = \underset{\sim}{I} + \underset{\sim}{F}$ and $\underset{\sim}{A}^{-1} \underset{\sim}{B} = \underset{\sim}{I} + \underset{\sim}{E}$ such that $|ch_i(\underset{\sim}{F})| < 1$, $|ch_i(\underset{\sim}{E})| < 1$, $i = 1, \ldots, p$, then for $\underset{\sim}{\varepsilon}^{(1)}$ and $\underset{\sim}{\varepsilon}^{(2)}$ sufficiently close to $\underset{\sim}{O}(p \times p)$ we get $|ch_i(\underset{\sim}{X})| < 1$, $|ch_i(\underset{\sim}{Y})| < 1$, $i = 1, \ldots, p$. Hence

$$(4.21) \quad J_1 = [1 + m_1/2(\text{tr } \underset{\sim}{X})\Delta_1 + \{m_1/4 \text{ tr}(\underset{\sim}{X}^2) + m_1^2/8(\text{tr } \underset{\sim}{X})^2\} \Delta_1^2 + ..]$$
$$\times [1 + n_1/2(\text{tr } \underset{\sim}{Y})\Delta_2 + \{n_1/4 \text{ tr}(\underset{\sim}{Y}^2) + n_1^2/8(\text{tr } \underset{\sim}{Y})^2\} \Delta_2^2 + ..]$$
$$G(\theta, \rho_1, \rho_2) .$$

Also

$$\underset{\sim}{X} = (\underset{\sim}{A}^{-1} \underset{\sim}{B} - \underset{\sim}{I}) - \Sigma\, \varepsilon_{rs}^{(1)} (\underset{\sim}{A}^{-1} \underset{\sim}{B})(\underset{\sim}{B}_{rs}^{-1} \underset{\sim}{B}) +$$
$$\Sigma\, \varepsilon_{rs}^{(1)} \varepsilon_{tu}^{(1)} (\underset{\sim}{A}^{-1} \underset{\sim}{B})(\underset{\sim}{B}_{rs}^{-1} \underset{\sim}{B})(\underset{\sim}{B}_{tu}^{-1} \underset{\sim}{B}) + \ldots$$

and $$\underset{\sim}{Y} = (\underset{\sim}{B}^{-1} \underset{\sim}{A} - \underset{\sim}{I}) - \Sigma\, \varepsilon_{rs}^{(2)} (\underset{\sim}{B}^{-1} \underset{\sim}{A}) (\underset{\sim}{A}_{rs}^{-1} \underset{\sim}{A}) +$$
$$\Sigma\, \varepsilon_{rs}^{(2)} \varepsilon_{tu}^{(2)} (\underset{\sim}{B}^{-1} \underset{\sim}{A}) (\underset{\sim}{A}_{rs}^{-1} \underset{\sim}{A}) (\underset{\sim}{A}_{tu}^{-1} \underset{\sim}{A}) - \ldots .$$

Now using similar notations as before i.e.

$$\mathrm{tr}\ (\underset{\sim}{A}_{rs}^{-1} \underset{\sim}{A}) = (rs)$$

$$\mathrm{tr}\ (\underset{\sim}{A}_{rs}^{-1} \underset{\sim}{A})(\underset{\sim}{A}_{tu}^{-1} \underset{\sim}{A}) = (rs|tu)$$

$$\mathrm{tr}\ (\underset{\sim}{B}_{rs}^{-1} \underset{\sim}{B}) = [rs]$$

$$\mathrm{tr}\ (\underset{\sim}{B}_{rs}^{-1} \underset{\sim}{B})(\underset{\sim}{B}_{tu}^{-1} \underset{\sim}{B}) = [rs|tu]$$

$$\mathrm{tr}\ (\underset{\sim}{A}^{-1} \underset{\sim}{B}) = (\underset{\sim}{I} + \underset{\sim}{E})$$

$$\mathrm{tr}\ (\underset{\sim}{B}^{-1} \underset{\sim}{A}) = [\underset{\sim}{I} + \underset{\sim}{F}] \qquad \text{etc.}$$

and equating coefficients of $\varepsilon_{rs}^{(1)}\ \varepsilon_{tu}^{(1)}$, $\varepsilon_{rs}^{(2)}\ \varepsilon_{tu}^{(2)}$, $\varepsilon_{rs}^{(1)}\ \varepsilon_{tu}^{(2)}$ in (4.20) and (4.21) with the assumption that terms involving $e_{ij}\ e_{kl}$, $e_{ij}\ f_{kl}$, $f_{ij}\ f_{kl}$ are negligible, where e_{ij} and f_{ij} are (i,j) elements of $\underset{\sim}{E}$ and $\underset{\sim}{F}$ respectively. We get

$$(4.22)\quad \partial_{rs}^{(1)}\ \partial_{tu}^{(1)}\ \Pr\left\{\frac{m_1\ \mathrm{tr}\ \underset{\sim}{S}_1\ \underset{\sim}{B}}{n_1\ \mathrm{tr}\ \underset{\sim}{S}_3\ \underset{\sim}{A}} \le \theta\right\}$$

$$= [1 + (n_1/2)(\underset{\sim}{F})\ \Delta_2][m_1(\underset{\sim}{I} + \underset{\sim}{E}|rs|tu)\ \Delta_1 + ((m_1/2)\ \{(\underset{\sim}{I} + \underset{\sim}{E}|rs$$

$$|\underset{\sim}{I} + \underset{\sim}{E}|tu) + 2\ (\underset{\sim}{E}|\underset{\sim}{I} + \underset{\sim}{E}|rs|tu)\} + (m_1^2/4)\ \{(\underset{\sim}{I} + \underset{\sim}{E}|rs)$$

$$(\underset{\sim}{I} + \underset{\sim}{E}|tu) + 2\ (\underset{\sim}{E})(\underset{\sim}{I} + \underset{\sim}{E}|rs|tu)\})\ \Delta_1^2]\ G(\theta,\ \rho_1,\ \rho_2)\ .$$

$$(4.23)\quad \partial_{rs}^{(2)}\ \partial_{tu}^{(2)}\ \Pr\{\{\frac{m_1\ \mathrm{tr}\ \underset{\sim}{S}_1\ \underset{\sim}{B}}{n_1\ \mathrm{tr}\ \underset{\sim}{S}_3\ \underset{\sim}{A}} \le \theta\}$$

$$= [1 + (m_1/2)\ (\underset{\sim}{E})\ \Delta_1][n_1\ [\underset{\sim}{I} + \underset{\sim}{F}|rs|tu]\ \Delta_2 + ((n_1/2)$$

$$\{[\underset{\sim}{I} + \underset{\sim}{F}|rs|\underset{\sim}{I} + \underset{\sim}{F}|tu] + 2\ [\underset{\sim}{F}|\underset{\sim}{I} + \underset{\sim}{F}|rs|tu]\} + (n_1^2/4)\ \{[\underset{\sim}{I} + \underset{\sim}{F}|rs]$$

$$[\underset{\sim}{I} + \underset{\sim}{F}|tu] + 2\ (\underset{\sim}{F})(\underset{\sim}{I} + \underset{\sim}{F}|rs|tu)\})\ \Delta_2^2]\ G(\theta,\ \rho_1,\ \rho_2)\ .$$

(4.24) $$\partial_{rs}^{(1)}\ \partial_{tu}^{(2)}\ \Pr\ \{\frac{m_1\ \mathrm{tr}\ \underset{\sim}{S}_1\ \underset{\sim}{B}}{n_1\ \mathrm{tr}\ \underset{\sim}{S}_3\ \underset{\sim}{A}} \le \theta\}$$

$$= 1/2\ \{m_1\ n_1\ (\underset{\sim}{I} + \underset{\sim}{E}|rs)\ [\underset{\sim}{I} + \underset{\sim}{F}|tu]\ \Delta_1\ \Delta_2\}\ G(\theta,\ \rho_1,\ \rho_2)\ .$$

Now note that

$$\Delta_1\ G(\theta,\ \rho_1,\ \rho_2) = -\ \frac{\theta}{\rho_1}\ \beta(\theta,\ \rho_1,\ \rho_2)\ ,$$

$$\Delta_2\ G(\theta,\ \rho_1,\ \rho_2) = \frac{\theta}{\rho_2}\ \beta(\theta,\ \rho_1,\ \rho_2)\ ,$$

$$\Delta_1^2\ G(\theta,\ \rho_1,\ \rho_2) = -\ (\frac{(\rho_2 - 1)}{\rho_1(1+\rho_1)}\ \theta - \frac{\rho_1 + \rho_2}{\rho_1(1+\rho_1)}\ \frac{\theta}{1+\theta})\ \beta(\theta,\ \rho_1,\ \rho_2)\ ,$$

$$\Delta_2^2\ G(\theta,\ \rho_1,\ \rho_2) = (\frac{\rho_1 + \rho_2}{\rho_2(1+\rho_2)}\ \frac{\theta}{1+\theta} - \frac{\theta}{\rho_2})\ \beta(\theta,\ \rho_1,\ \rho_2)\ ,$$

$$\Delta_1\ \Delta_2\ G(\theta,\ \rho_1,\ \rho_2) = \Delta_2\ \Delta_1\ G(\theta,\ \rho_1,\ \rho_2) =$$

$$(\frac{\theta}{\rho_1} - \frac{\rho_1+\rho_2}{\rho_1\ \rho_2}\ \frac{\theta}{1+\theta})\ \beta(\theta,\ \rho_1,\ \rho_2)\ ,$$

$$\Delta_1\ \Delta_2^2\ G(\theta,\ \rho_1,\ \rho_2) = \Delta_2^2\ \Delta_1\ G(\theta,\ \rho_1,\ \rho_2) = \Delta_2\ \Delta_1\ \Delta_2\ G(\theta,\ \rho_1,\ \rho_2)$$

$$= (\frac{(\rho_1+\rho_2)(\rho_1+\rho_2+1)}{\rho_2\ \rho_1\ (\rho_2 + 1)}\ \frac{\theta^2}{(1+\theta)^2} - \frac{(\rho_1+\rho_2)}{\rho_1\ \rho_2}\ \frac{\theta^2}{(1+\theta)} - \frac{(\rho_1 + \rho_2)}{\rho_2(1+\rho_2)}\ \frac{\theta}{1+\theta}$$

$$+ \frac{\theta}{\rho_2})\ \beta(\theta,\ \rho_1,\ \rho_2)\ ,$$

and

$$\Delta_2\ \Delta_1^2\ G(\theta,\ \rho_1,\ \rho_2) = \Delta_1^2\ \Delta_2\ G(\theta,\ \rho_1,\ \rho_2) = \Delta_1\Delta_2\Delta_1\ G(\theta,\ \rho_1,\ \rho_2)$$

$$= - \left(\frac{(\rho_1+\rho_2+1)(\rho_1+\rho_2)}{\rho_2\ \rho_1\ (1+\rho_1)} \frac{\theta^2}{(1+\theta)^2} - \frac{(\rho_1+\rho_2)}{\rho_1\ \rho_2} \frac{\theta}{1+\theta} - \frac{(\rho_1+\rho_2)}{\rho_1(1+\rho_1)} \frac{\theta^2}{1+\theta}\right.$$

$$\left. + \frac{\theta}{\rho_1}\right) \beta(\theta,\ \rho_1,\ \rho_2) \ .$$

Hence we have from (4.19)

$$(4.25) \quad - h_1(\underset{\sim}{A}^{-1},\ \underset{\sim}{B}^{-1})\ D[L(\theta)] = - \{[(m_1/2n_2)(\underset{\sim}{I} + \underset{\sim}{E}) \frac{2\theta}{m_1 p}$$

$$- (n_1/2m_2)(\underset{\sim}{I} + \underset{\sim}{F})(p + 1) \frac{2\theta}{n_1 p}] + [(m_1/2n_2)(p(p + 1)/2$$

$$+ 2\ (\underset{\sim}{E})(p + 1)) + m_1^2/4n_2(p + 2\ (\underset{\sim}{E}) + (\underset{\sim}{E})\ p\ (p + 1))]$$

$$[\frac{(n_1 p - 2)}{(2 + m_1 p)} \frac{2\theta}{m_1 p} - \frac{(m_1 + n_1)\ p}{m_1(m_1 p + 2)p} \frac{2\theta}{1+\theta}] - [\ (n_1/2m_2)$$

$$(p(p + 1)/2 + 2\ (\underset{\sim}{F})(p + 1)) + (n_1^2/4m_2)(p + 2\ (\underset{\sim}{F}) + (\underset{\sim}{F})\ p(p+1))]$$

$$[\frac{(m_1+n_1)p}{n_1(n_1 p+2)p} \frac{2\theta}{1+\theta} - \frac{2\theta}{n_1 p}] - [(n_1\ m_1/4n_2\ )\ (\underset{\sim}{F})\ (p + 1)$$

$$+ (m_1\ n_1/4m_2)\ (\underset{\sim}{E})\ (p + 1)]\ [\frac{2\theta}{m_1 p} - \frac{(m_1 + n_1)p}{m_1\ n_1\ p^2} \frac{2\theta}{1+\theta}]$$

$$+ [(n_1/2n_2)\ (\underset{\sim}{F})\ (m_1\ p\ (p + 1)/4 + m_1^2\ p/4)]$$

$$[\frac{(m_1\ p + n_1\ p + 2)(m_1 + n_1)\ p}{m_1\ n_1\ (2 + m_1\ p)\ p^2} \frac{2\theta^2}{(1 + \theta)^2} - \frac{(m_1 + n_1)\ p}{m_1(2 + m_1\ p)} \frac{2\theta^2}{1 + \theta}$$

$$- \frac{(m_1 + n_1)\ p}{m_1\ n_1\ p^2} \frac{2\theta}{1+\theta} + \frac{2\theta}{m_1 p}] - [(m_1\ /2m_2)\ (\underset{\sim}{E})\ (m_1\ p(p + 1)/4$$

$$+ n_1^2\ p/4)][\frac{(m_1\ p + n_1\ p + 2)(m_1 + n_1)\ p}{m_1\ n_1\ (2 + m_1\ p)\ p^2} \frac{2\ \theta^2}{(1 + \theta)^2}$$

$$- \frac{(m_1 + n_1)\, p}{m_1\, n_1\, p^2} \frac{2\, \theta^2}{1 + \theta} + \frac{(m_1 + n_1)\, p}{n_1 (2 + n_1 p)\, p} \frac{2\theta}{1 + \theta} + \frac{2\theta}{n_1 p}]\}$$

$$\beta(\theta;\, \rho_1,\, \rho_2)\ .$$

Using $h_1(\underset{\sim}{A}^{-1}, \underset{\sim}{B}^{-1})$ in (4.25) and the

$$\Pr \{\frac{m_1 \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{S}_4^{-1}}{n_1 \operatorname{tr} \underset{\sim}{S}_3 \underset{\sim}{S}_2^{-1}} \leq \theta\} = \Theta \Pr \{\frac{m_1 \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{B}}{n_1 \operatorname{tr} \underset{\sim}{S}_3 \underset{\sim}{A}} \leq \theta\}$$

where Θ is defined in (4.18) we get

$$(4.26) \quad \Pr \{\frac{m_1 \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{S}_4^{-1}}{n_1 \operatorname{tr} \underset{\sim}{S}_3 \underset{\sim}{S}_2^{-1}} \leq \theta\} = L(\theta) - h_1(\underset{\sim}{A}^{-1}, \underset{\sim}{B}^{-1})\, D[L(\theta)] + O(m^{-2})$$

where $L(\theta)$ is defined in (4.13) and D in (4.7).

Also noting the expansion $h = \theta + h_1 + h_2 + \cdots\cdots$ as in (4.16) we have

$$(4.27) \quad F' = (m_1 \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{S}_4^{-1})/(n_1 \operatorname{tr} \underset{\sim}{S}_3 \underset{\sim}{S}_2^{-1}) = \theta + h_1(\underset{\sim}{A}^{-1}, \underset{\sim}{B}^{-1}) + O(m^{-2}),$$

where θ is the percentile corresponding to the ratio of two quadratic forms. Thus we have the following theorem.

<u>Theorem 4.3</u>. Let $\underset{\sim}{S}_1, \underset{\sim}{S}_2, \underset{\sim}{S}_3, \underset{\sim}{S}_4$ be covariance matrices having $W(m_1, p, \underset{\sim}{A}^{-1})$, $W(m_2, p, \underset{\sim}{A}^{-1})$, $W(n_1, p, \underset{\sim}{B}^{-1})$, $W(n_2, p, \underset{\sim}{B}^{-1})$ respectively. Then the asymptotic expansions for the c.d.f. and the percentile of the statistic

$$F' = (m_1 \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{S}_4^{-1})/(n_1 \operatorname{tr} \underset{\sim}{S}_3 \underset{\sim}{S}_2^{-1})$$

are given by (4.26) and (4.27) respectively.

Special cases.

If $\underset{\sim}{A} \equiv \underset{\sim}{B}$ and $m_2 = n_2 = n$ then noting

$$\partial_{rs}^{(1)} \partial_{tu}^{(1)} \Pr \{\frac{m_1 \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{A}}{n_1 \operatorname{tr} \underset{\sim}{S}_3 \underset{\sim}{A}} \leq \theta\} = \partial_{rs}^{(2)} \partial_{tu}^{(2)} \Pr \{\frac{m_1 \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{A}}{n_1 \operatorname{tr} \underset{\sim}{S}_3 \underset{\sim}{A}} \leq \theta\}$$

$$= \partial_{rs}^{(1)} \partial_{tu}^{(2)} \Pr \{\frac{m_1 \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{A}}{n_1 \operatorname{tr} \underset{\sim}{S}_3 \underset{\sim}{A}} \leq \theta\} \text{ , we get}$$

$$\partial_{rs} \partial_{tu} \Pr \{\frac{m_1 \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{A}}{n_1 \operatorname{tr} \underset{\sim}{S}_3 \underset{\sim}{A}} \leq \theta\} = 1/2 \; [(rs|tu) \; \{m_1(2 \; \Delta_1 + \Delta_1^2)$$

$$+ n_1(2 \; \Delta_2 + \Delta_2^2) + 1/2 \; (rs) \; (tu) \; (m_1 \; \Delta_1 + n_1 \; \Delta_2)^2] \; G(\theta, \rho_1, \rho_2)$$

which agrees with Siotani's result [19], [20].

4.5. Distribution of F' max. (Special Case)

Denoting

$$F'_{max} = \max \left(\frac{m_1 \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{S}_4^{-1}}{n_1 \operatorname{tr} \underset{\sim}{S}_3 \underset{\sim}{S}_2^{-1}} , \frac{n_1 \operatorname{tr} \underset{\sim}{S}_3 \underset{\sim}{S}_2^{-1}}{m_1 \operatorname{tr} \underset{\sim}{S}_1 \underset{\sim}{S}_4^{-1}} \right)$$

we have then the following theorem.

Theorem 4.4. The asymptotic expansion for the distribution of F'_{max} i.e. $\Pr\{F' \max \leq \theta\}$ is given by

$$\begin{aligned} \Pr\{F' \max \leq \theta\} &= 0 \text{ if } \theta < 1 \\ &= H(m_1, n_1, \theta) - H(m_1, n_1, 1) \\ &\quad + H(n_1, m_1, \theta) - H(n_1, m_1, 1) + O(m^{-2}) \end{aligned}$$

if $\theta \geq 1$, where $H(m_1, n_1, \theta) = L(\theta) - h_1(\underset{\sim}{A}^{-1}, \underset{\sim}{B}^{-1}) \; D \; [L(\theta)]$.

Proof. This directly follows from the following two relations.

a) $$1 \leq F' \max \leq \infty ,$$

and

b) $$\Pr \{ F' \max \leq \theta\} = \Pr \{ 1 \leq (m_1 \operatorname{tr} S_1 S_4^{-1}/n_1 \operatorname{tr} S_3 S_2^{-1}) \leq \theta\}$$

$$+ \Pr \{1 \leq (n_1 \operatorname{tr} S_3 S_2^{-1}/m_1 \operatorname{tr} S_1 S_4^{-1}) \leq \theta\} \text{ where } \theta \geq 1.$$

Now we apply the previous results.

CHAPTER V

SUMMARY AND CONCLUSION

In this thesis, a general theory has been developed first for obtaining asymptotic expansions for distributions of sample characteristic roots of matrices from real as well as complex multivariate normal populations. The general theory has been discussed in the light of a maximization problem, namely, that of an integral over the group of orthogonal (unitary) matrices. The maximizing sub-group of the group of orthogonal (unitary) matrices has been obtained for a class of symmetric (Hermitian) matrix-valued functions as integrand. All this work has been presented in the first chapter.

Next, the above theory has been applied in Chapters II and III, to obtain asymptotic expansions of the distributions of the characteristic roots arising in a) MANOVA, b) Canonical correlation and tests concerning covariance matrices either in c) one-sample, or d) two-sample cases. In each of the above situations, our technique has been to find an asymptotic expansion of an integral involved in each problem by using Laplace's method. The integrand in each case consisted of a hypergeometric function with matrix arguments which had to be integrated over the group of orthogonal (unitary) matrices of certain fixed order and with respect to invariant Haar measure defined on it. Using results in the first chapter we have obtained the sub-group of the whole group for which the integrand attains its unique maximum under different assumptions regarding the parameter (population characteristic root) matrix and with mild restriction on the coefficients of the hypergeometric functions involved. The most general results obtained have been under the assumption of several multiple population roots. For large sample size, the integrand is localized around its unique optimum value and by Laplace's method as generalized by L.C. Hsu, the integral has been evaluated around the optimizing set giving terms in decreasing order of monotonic functions of sample sizes.

In Chapter IV we have presented asymptotic formulae for the c.d.f. and percentile of the max-U ratio criterion suggested by Pillai for the test of equality of several covariance matrices but studied here only for the two-sample case. Perturbation technique has been used to obtain these expansions.

In summary, the first three chapters generalize the methods of Anderson, Chang, James, Li and Pillai for the study of one or two covariance matrices to the case of MANOVA and canonical correlation. Though considerable simplification has been obtained by ordering terms, yet the expansions remain complicated enough. This is mainly because of the nature of the hypergeometric functions involved. In some cases, at least, the functions involved reduce to polynomials and simplify the situation further. But the basic point to be made here is that this gives an approach to handle a hypergeometric function of the stated form satisfying some restrictions, and as such can be used in similar cases. The asymptotic expansions derived for MANOVA and Canonical correlation still involve hypergeometric functions and furthcr research is needed to bring them, if possible, in forms similar to those of the covariance matrices.

The 4th chapter is just the first step in solving a more general problem. Much more work has to be done in extending theorem 4.4 to more general case i.e. increasing the number of populations. The mutual dependence of different U-ratios is the main difficulty. Further work on this line could be carried out.

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