Asymptotically Robust Tests in Unbalanced Variance Component Models

by

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Abstract

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Spjøtvoll (Ann. Math. Statist. 38 422-428) has obtained a test associated with an unbalanced one-way layout for Model II ANOVA. Under the assumption of normality, his test possesses several optimum properties. Without the normality assumption, the significance level is (in general) highly non-robust. An attempt to remedy this situation using a test based on the jack-knife technique, appears in Arvesen (Ann. Math. Statist. 40 2076-2100). The present paper proposes as an alternative a jackknifed version of Spjøtvoll's test. The new test is not sensitive to departures from normality, and Monte Carlo sampling and asymptotic efficiency results suggest that it is more powerful than Arvesen's test. The paper also includes some general results for use of the jackknife technique with non-identically distributed random variables.

1. Summary. We obtain in this paper an asymptotically robust test for the hypothesis $\Delta \leq \Delta_0$ against $\Delta > \Delta_0$, where Δ is the variance ratio in an unbalanced one-way layout for Model II ANOVA. The test is based on an extension of Arvesen [1969] using the jackknife. Theoretical and Monte Carlo results show the robustness of the proposed test for non-normal data, and that it performs similarly to Spjøtvoll's [1967] test if the data are normal. That is, the Pitman ARE of both tests is 1 under normality, and theoretical and Monte Carlo results show the robustness of the proposed test for non-normal data.

It is also possible to obtain a confidence interval for Δ using the proposed test. Section 2 discusses the basic model, while section 3 digresses to discuss some general results concerning use of the jackknife with non-identically distributed random variables. Section 4 applies the results of the previous section to the variance component problem, while sections 5 and 6 discuss asymptotic efficiency results and Monte Carlo results respectively.

2. The model. The basic model assumed in an unbalanced one-way layout for Model II ANOVA is

(2.1) $$Y_{ij} = \mu + a_i + e_{ij}, \; i = 1,\ldots,n, \; j = 1,\ldots,J_i ,$$

where μ is an unknown constant, $\{a_i\}$ and $\{e_{ij}\}$ are all mutually independent normal random variables with zero means and variances σ_A^2 and σ_e^2 respectively. If we let $\Delta = \sigma_A^2/\sigma_e^2$, one hypothesis of interest is

(2.2) $$H_0: \; \Delta \leq \Delta_0 \text{ vs. } H_A: \; \Delta > \Delta_0 .$$

For a specified alternative $\Delta = \Delta_1$, Spjøtvoll [1967] has obtained the most powerful similar α-level test of H_0. The value Δ_1 enters into the test statistic. He also proposes an alternate test letting $\Delta_1 \to \infty$, tantamount to achieving high power for distant alternatives. It is this test we now consider. Letting

$$(2.3) \qquad T = \sum_{i=1}^{n} J_i(\Delta_0 J_i+1)^{-1} (\overline{Y}_{i.}-\overline{Y}_{..})^2 \left(\sum_{i=1}^{n} \sum_{j=1}^{J_i} (Y_{ij}-\overline{Y}_{i.})^2\right)^{-1}$$

where $\overline{Y}_{i.} = J_i^{-1} \sum_{j=1}^{J_i} Y_{ij}$, and

$$\overline{Y}_{..} = \left(\sum_{i=1}^{n} J_i(\Delta_0 J_i+1)^{-1}\right)^{-1} \sum_{i=1}^{n} J_i(\Delta_0 J_i+1)^{-1} \overline{Y}_{i.} ,$$

$$N^* = \sum_{i=1}^{n} J_i ,$$

one rejects H_0 at the α-level if

$$(2.4) \qquad (N^*-n)\,(n-1)^{-1}T > F_{\alpha;\, n-1,\, N^*-n} ,$$

where $F_{\alpha;\, \nu_1,\, \nu_2}$ denotes the upper α point of an F distribution with ν_1, ν_2 degrees of freedom. Spjøtvoll also obtains a confidence interval for Δ, although it is subject to the same criticism concerning its non-robust character.

Note that when $J_i = J$, or a balanced model results, the test as given by (2.4) is the same as the standard F-test as given in Scheffé [1959]. It is well-known (see Scheffé [1959]) that this standard F-test is not robust if the observations are non-normal, especially the random effect terms. The significance levels are invalid except in the case $\Delta_0 = 0$.

In the balanced case, a competitor based on the jackknife has been proposed in Arvesen [1969], and its moderate sample size properties were examined by a Monte Carlo computer simulation in Arvesen and Schmitz [1970].

Also, in the unbalanced case, a test based on the usual jackknife was proposed in Arvesen [1969]. This test will be further discussed in sections 5 and 6, where evidence is presented suggesting that in terms of power it is inferior to the test proposed in section 4.

3. The jackknife for non-identically distributed random variables.

(a) Background. First let us describe the jackknife procedure. For a more detailed discussion the reader is referred to Miller [1964]. Let $X_1,\ldots,X_N$ be independent identically distributed observations from the cdf F_θ. Partition these N observations into n groups with k observations in each group $(N = nk)$. Then if $\hat{\theta}_n^0$ is some estimate based on all n groups of observations (all N observations), let $\hat{\theta}_{n-1}^i$, $i = 1,\ldots,n$ denote the estimate obtained after deletion of the ith group of observations.

The jackknife estimate of θ is

(3.1)
$$\hat{\theta} = n^{-1} \sum_{i=1}^{n} \hat{\theta}_i ,$$

where

(3.2)
$$\hat{\theta}_i = n\, \hat{\theta}_n^0 - (n-1)\, \hat{\theta}_{n-1}^i,\ i = 1,\ldots, n .$$

If

(3.3)
$$s_{\hat{\theta}}^2 = (n-1)^{-1} \sum_{i=1}^{n} (\hat{\theta}_i - \hat{\theta})^2 ,$$

it is interesting to conjecture that if n is held fixed,

$$t = \sqrt{n}\,(\hat{\theta}-\theta)/s_{\hat{\theta}} \xrightarrow[N\to\infty]{\mathcal{L}} t_{n-1} \qquad (3.4)$$

as in Tukey [1958]. Miller [1964], [1968], and Arvesen [1969] give a large class of situations where this conjecture proves valid. In what follows, we will assume $k = 1$, and hence $N = n$, and the convergence in (3.4) is to a standard normal distribution.

Unfortunately, if one starts with $X_1,\ldots,X_n$ as only independent, and not necessarily identically distributed, the situation becomes more complicated. In fact, the results given in Arvesen [1969] required $\hat{\theta}_n^0$ to be of a very special restrictive form. In the notation of that paper, let $f^*(X_{\alpha_1},\ldots,X_{\alpha_m})$ be a symmetric kernel with <u>the same expectation</u> $E[f^*(X_{\alpha_1},\ldots,X_{\alpha_m})] = \eta$ for all $\alpha_1,\ldots,\alpha_m$. Then the U-statistic (see Hoeffding [1948]) for estimating η is

$$U_n = \binom{n}{m}^{-1} \sum_{C_n} f^*(X_{\alpha_1},\ldots,X_{\alpha_m}) \qquad (3.5)$$

where C_n indicates the summation is over all combinations $\alpha_1,\ldots,\alpha_m$ of m integers chosen from $1,\ldots,n$. Then theorems 10 and 11 of Arvesen [1969] show that under mild regularity conditions, the conjecture of (3.4) is valid. However for many purposes, including those to be discussed in section 4 below, the restriction to kernels with the same expectation is too restrictive.

(b) A modified jackknife estimate. To modify this restriction, let $X_1,\ldots,X_n$ be independent (not necessarily identically distributed) random variables, and assume

$$E[f^*(X_{\alpha_1},\ldots,X_{\alpha_m})] = \eta_{\alpha_1\ldots\alpha_m}\,\mu \qquad (3.6)$$

where $\eta_{\alpha_1\ldots\alpha_m}$ is a <u>known</u> constant, and μ is an <u>unknown</u> parameter. Also, let $\eta_n^0 = \binom{n}{m}^{-1} \sum_{C_n} \eta_{\alpha_1 \cdots \alpha_m}$, $\bar{\eta} = \lim_{n\to\infty} \eta_n^0$ (which we assume exists, is finite and and non-zero), and $\eta_{n-1}^i = \binom{n-1}{m}^{-1} \sum_{C_{n-1}^i} \eta_{\beta_1^i \cdots \beta_m^i}$ where $\sum_{C_{n-1}^i}$ indicates the sum is over all combinations of m integers $(\beta_1^i, \ldots, \beta_m^i)$ chosen from $(1, \ldots, i-1, i+1, \ldots, n)$. Let

$$U_n^0 = \binom{n}{m}^{-1} \sum_{C_n} f^*(X_{\alpha_1}, \ldots, X_{\alpha_m}) ,$$

$$U_{n-1}^i = \binom{n-1}{m}^{-1} \sum_{C_{n-1}^i} f^*(X_{\beta_1^i}, \ldots, X_{\beta_m^i}) ,$$

$$f^*_{c;\beta_1,\ldots,\beta_{m-c}}(X_1, \ldots, X_c) = E\{f^*(X_1, \ldots, X_c, X_{\beta_1}, \ldots, X_{\beta_{m-c}}) | X_1 = x_1, \ldots, X_c = x_c\} ,$$

$$\zeta_{c;(\alpha_1 \ldots \alpha_c)\beta_1,\ldots,\beta_{m-c};\gamma_1,\ldots,\gamma_{m-c}} =$$

$$\mathrm{Cov}\{f^*_{c;\beta_1,\ldots,\beta_{m-c}}(X_{\alpha_1}, \ldots, X_{\alpha_c}), f^*_{c;\gamma_1,\ldots,\gamma_{m-c}}(X_{\alpha_1}, \ldots, X_{\alpha_c})\},$$

and

$$\zeta_{c(\alpha_1,\ldots,\alpha_c)} = \left[\binom{n-c}{m-c}\binom{n-m}{m-c} \right]^{-1} \sum \zeta_{c;(\alpha_1,\ldots,\alpha_c)\beta_1,\ldots,\beta_{m-c};\gamma_1,\ldots,\gamma_{m-c}}$$

where the sum is extended over all disjoint sets $(\beta_1, \ldots, \beta_{m-c})$, $(\gamma_1, \ldots, \gamma_{m-c})$ chosen from $(1, \ldots, n)$ excluding $(\alpha_1, \ldots, \alpha_c)$. Let

$\zeta_{c,n} = \binom{n}{c}^{-1} \sum \zeta_{c(\alpha_1,\ldots,\alpha_c)}$ where this sum is over all combinations $(\alpha_1, \ldots, \alpha_c)$ of c integers chosen from $(1, \ldots, n)$. Then one can show that var $(U_n^0) = \binom{n}{m}^{-1} \sum_{c=1}^{m} \binom{m}{c} \binom{n-m}{m-c} \zeta_{c,n}$. Let

(3.7) $$h_{1(\nu)}(X) = \binom{n-1}{m-1}^{-1} \sum_{\neq\nu} (f^*_{1;\beta_1,\ldots,\beta_{m-1}}(X) - \eta_{\nu\beta_1\ldots\beta_{m-1}}\mu)$$

where the sum is over all sets $(\beta_1,\ldots,\beta_{m-1})$ chosen from the first n integers excluding the integer ν.

Now let

$$\hat{\theta}_n^0 = g(U_n^0/\eta_n^0),\ \hat{\theta}_{n-1}^i = g(U_{n-1}^i/\eta_{n-1}^i),\ i = 1,\ldots,n\ ,$$

(3.8)

$$\theta = g(\mu)\ .$$

Then following (3.1), (3.2) and (3.3), we will be interested in the statistic $\sqrt{n}\,(\hat{\theta} - \theta)/s_{\hat{\theta}}$, and conditions under which it is asymptotically standard normal.

First, the following two lemmas will be helpful.

Lemma 1. Let $X_1,\ldots,X_n$ be independent random variables, and assume for any $\delta > 0$, and $0 < A < \infty$, $E[f^*(X_{\alpha_1},\ldots,X_{\alpha_m})]^{2+\delta} < A$. Then if $\zeta_{1,n} \to \zeta_1$ as $n \to \infty$, $0 < \zeta_1 < +\infty$, $Z_n = n^{-1}\sum_{i=1}^{n} (h_{1(i)}(X_i))^2 \xrightarrow{a.s.} \zeta_1$.

PROOF. First note that

$$E(Z_n) = n^{-1}\binom{n-1}{m-1}^{-2} \sum_{i=1}^{n} \sum_{(i)} \zeta_{1(i)\beta_1,\ldots,\beta_{m-1};\gamma_1,\ldots,\gamma_{m-1}}$$

where $\sum_{(i)}$ denotes the sum is over all combinations $(\beta_1,\ldots,\beta_{m-1})$ of $m-1$ integers chosen from $(1,\ldots,i-1,\ i+1,\ldots,n)$ and all combinations $(\gamma_1,\ldots,\gamma_{m-1})$

of m-1 integers chosen from $(1,\ldots,i-1, i+1,\ldots,n)$. However, since $E[f^*(X_{\alpha_1},\ldots,X_{\alpha_m})]^{2+\delta} < A$, one obtains

$$E(Z_n) = \binom{n-m}{m-1}\binom{n-1}{m-1}^{-1} n^{-1} \sum_{i=1}^{n} \zeta_{1(i)} + O(1)$$

$$\rightarrow \binom{n-m}{m-1}\binom{n-1}{m-1}^{-1} \zeta_{1,n} \rightarrow \zeta_1 .$$

From theorem A, p. 241 of Loève, the result now follows.

Lemma 2. Let $X_1,\ldots,X_n$ be independent random variables, and assume for some $\delta > 0$ and some $0 < A < \infty$,

(3.9) $$0 < |\eta_{\alpha_1\ldots\alpha_m}| < A,\ 0 < E\,|f^*(X_{\alpha_1},\ldots,X_{\alpha_m})|^{2+\delta} < A$$

for all $(\alpha_1,\ldots,\alpha_m)$,

(3.10) $$E\,|h_{1(\nu)}(X_\nu)|^3 < \infty \text{ for } \nu = 1,\ldots,n, \text{ and}$$

(3.11) $$\lim_{n\to\infty} \sum_{\nu=1}^{n} E\{|h_{1(\nu)}(X_\nu)|^3\}/[\sum_{\nu=1}^{n} E\{h_{1(\nu)}(X_\nu)\}^2]^{3/2} = 0 .$$

If $\zeta_{1,n} \to \zeta_1$ as $n \to \infty$, $0 < \zeta_1 < \infty$, then $(n-1)\sum_{i=1}^{n}(U_{n-1}^{i} - (\eta_{n-1}^{i}/\eta_n^0)U_n^0)^2 \xrightarrow{P} m^2\zeta_1$.

PROOF. Let

(3.12) $$T_n = (n-1)\sum_{i=1}^{n} (U_{n-1}^{i} - (\eta_{n-1}^{i}/\eta_n^0)\, U_n^0)^2, \text{ and since}$$

$E(U_{n-1}^{i}) = \eta_{n-1}^{i}\ \mu$, we may assume $\mu = 0$.

Let $a_i = (n\,\eta_n^0 - (n-1)\,\eta_{n-1}^{i})/\eta_n^0$, then

$$T_n = (n-1) \sum_{i=1}^{n} (U_{n-1}^{i} - (n-a_i) U_n^0/(n-1))^2$$

(3.13)
$$= (n-1) \sum_{i=1}^{n} (U_{n-1}^{i} - U_n^0 + (a_i-1) U_n^0/(n-1))^2$$

$$= (n-1) \sum_{i=1}^{n} (U_{n-1}^{i} - U_n^0)^2 + (U_n^0)^2 \sum_{i=1}^{n} (a_i-1)^2/(n-1)$$

$$+ X - \text{product term.}$$

It will be shown that the first term of (3.13) converges in probability to $m^2 \zeta_1$, the second term converges in probability to zero, and hence by the Cauchy-Schwarz inequality, the cross-product term converges to zero in probability.

Note that if

(3.14)
$$V_n = n^{-1} \sum_{i=1}^{n} (mh_{1(i)}(X_i) - (n(n-i))^{1/2}(U_n^0 - U_{n-1}^{i}))^2 ,$$

and $E(V_n) \to 0$, this fact and lemma 1 suffice to show that the first term of (3.13) converges in probability to $m^2 \zeta_1$.

Let

(3.15)
$$S_n = (n-1) \sum_{i=1}^{n} (U_{n-1}^{i} - U_n^0)^2$$

$$= (n-1) \{\sum_{i=1}^{n} (U_{n-1}^{i})^2 - n(U_n^0)^2\}$$

$$= (n-1)\, n^{-1} \binom{n-1}{m}^{-2} \sum_{c=0}^{m} (cn-m^2) \sum_c f^*(X_{\alpha_1},\ldots,X_{\alpha_m})\, f^*(X_{\beta_1},\ldots,X_{\beta_m})$$

as in (21) of Arvesen [1969] where $\sum_c$ indicates that the sum is over all combinations $(\alpha_1,\ldots,\alpha_m)$ of m integers from $(1,\ldots,n)$ and all combinations $(\beta_1,\ldots,\beta_m)$ of m integers from $(1,\ldots,n)$ having exactly c common members. But now, as in the expression immediately before (47) of Arvesen [1969] , (3.16) $$E(S_n) = m^2\,\zeta_{1,n} + O(1) = m^2\,\zeta_1 + O(1)\ .$$

We are done with the first term of (3.13) if it can be shown that

(3.17) $$E[\textstyle\sum_{i=1}^{n} h_{1(i)}(X_i)(U_n^0 - U_{n-1}^{\,i})] = m\,\zeta_1 + O(1)\ .$$

To this end, note that

$$\begin{aligned} &U_n^0 - U_{n-1}^{\,i} \\ &= \binom{n}{m}^{-1} \textstyle\sum_{D_{n-1}^{\,i}} f^*(X_i, X_{\alpha_1^i},\ldots,X_{\alpha_{m-1}^{\,i}}) \\ &\qquad -\left[\binom{n}{m}^{-1} - \binom{n-1}{m}^{-1}\right] \textstyle\sum_{C_{n-1}^{\,i}} f^*(X_{\beta_1^i},\ldots,X_{\beta_m^i}) \end{aligned}$$

where $\sum_{D_{n-1}^{\,i}}$ indicates that the sum is over all combinations of $m-1$ integers $(\alpha_1^i,\ldots,\alpha_{m-1}^{\,i})$ chosen from $(1,\ldots,i-1,\ i+1,\ldots,n)$. Hence

$$E\,[h_{1(i)}(X_i)(U_n^0 - U_{n-1}^{\,i})]$$

(3.18) $$\begin{aligned} &= \binom{n-1}{m-1}^{-1} \binom{n}{m}^{-1} E[(\textstyle\sum\neq_i f^*_{1;\beta_1,\ldots,\beta_{m-1}}(X_i))(\sum_{D_{n-1}^{\,i}} f^*(X_i,X_{\alpha_1^i},\ldots,X_{\alpha_m^i}))] \\ &= \binom{n-1}{m-1}^{-1}\binom{n}{m}^{-1} \textstyle\sum^{*}\neq_i\ \zeta_{1(i);\beta_1,\ldots,\beta_{m-1};\gamma_1,\ldots,\gamma_{m-1}} \end{aligned}$$

where $\sum^{*}\neq_i$ denotes the sum is over all combinations of $m-1$ integers $(\beta_1,\ldots,\beta_{m-1})$ chosen from $(1,\ldots,i-1,\ i+1,\ldots,n)$, and all combinations of $m-1$ integers $(\gamma_1,\ldots,\gamma_{m-1})$ chosen from $(1,\ldots,i-1,\ i+1,\ldots,n)$. Summing (3.18) over i, and using (3.9) one obtains (3.17), and hence $E(U_n) \to 0$.

It remains to show that the second term of (3.13) converges to zero in probability. First note that by Hoeffding's [1948] U-statistic Central Limit Theorem for the non-identically distributed case, one obtains

(3.19) $$(U_n^0)^2 \xrightarrow{P} 0 .$$

Also note that since $|\eta_{\alpha_1 \ldots \alpha_m}| < A$ for all $(\alpha_1, \ldots, \alpha_m)$

$$|a_i - 1| = (n-1) \mid \binom{n}{m}^{-1} \sum_{D^i_{n-1}} \eta_{i\alpha_1^i \cdots \alpha_{m-1}^i} + (\binom{n}{m}^{-1} - \binom{n-1}{m}^{-1}) \sum_{C^i_{n-1}} \eta_{\beta_1^i \ldots \beta_m^i} | / |\eta_n^0|$$

$$< (n-1)[\binom{n}{m}^{-1}\binom{n-1}{m-1} A + (\binom{n-1}{m}^{-1} - \binom{n}{m}^{-1})\binom{n-1}{m} A] \ / \ |\eta_n^0|$$

$$< 2Am/|\eta_n^0| .$$

Since η_n^0 converges to non-zero $\bar{\eta}$,

(3.20) $$\lim_{n\to\infty} (n-1)^{-1} \sum_{i=1}^n (a_i - 1)^2 \leq A^2(2_m)^2 / \bar{\eta}^2 .$$

Combining (3.19) and (3.20), the second term of (3.13) converges to zero in probability, and the lemma follows.

Theorem 1. Let $X_1, \ldots, X_n$ be independent random variables, and assume that (3.9), (3.10), and (3.11) hold. Let g be a function defined on the real line, which in a neighborhood of μ has a bounded second derivative. Let $\hat{\theta}$, the jackknife estimate of $\theta = g(\mu)$ be as defined in (3.10) with $\hat{\theta}_n^0 = g(U_n^0/\eta_n^0)$. Then if

(3.21) $$\zeta_{1,n} \to \zeta_1 \text{ as } n \to \infty, \ 0 < \zeta_1 < \infty ,$$

the distribution of $(\hat{\theta} - \theta)/(g'(\mu)(\mathrm{Var}(U_n^0/\eta_n^0))^{1/2})$ is asymptotically normal with mean zero and variance one.

PROOF. Without loss of generality, let $\mu = 0$, and also let $Y_i = \binom{n-1}{m-1}^{-1} \sum_{D_{n-1}^i} f^*(X_i, X_{\alpha_1^i}, \ldots, X_{\alpha_m^i})$. Noting that

$E(Y_i^2) = \mathrm{Var}(Y_i) + (E(Y_i))^2 \leq \zeta_{1(i)} + A + 1$ for n sufficiently large, the proof follows from that of theorem 10 of Arvesen [1969] until we expand terms in a power series to obtain

$$(\hat{\theta} - \theta) = (ng(U_n^0/\eta_n^0) - (n-1)n^{-1} \textstyle\sum_{i=1}^n g(U_{n-1}^i/\eta_{n-1}^i) - g(0))$$

(3.22)
$$= [g(U_n^0/\eta_n^0) - g(0)] - (n-1)n^{-1}[g'(\tfrac{U_n^0}{\eta_n^0}) \textstyle\sum_{i=1}^n (U_{n-1}^i/\eta_{n-1}^i - U_n^0/\eta_n^0)$$

$$+ \textstyle\sum_{i=1}^n (U_{n-1}^i/\eta_{n-1}^i - U_n^0/\eta_n^0)^2 g''(\xi_i)/2]$$

where ξ_i lies between U_{n-1}^i/η_{n-1}^i and U_n^0/η_n^0. First note that

(3.23)
$$(n-1)\{\textstyle\sum_{i=1}^n (\eta_n^0)^2 (U_{n-1}^i/\eta_{n-1}^i - U_n^0/\eta_n^0)^2 - \sum_{i=1}^n (U_{n-1}^i - (\eta_{n-1}^i/\eta_n^0)U_n^0)^2\} \xrightarrow{P} 0$$

since $((\eta_n^0)^2 - (\eta_{n-1}^i)^2)/(\eta_{n-1}^i)^2$

$$= (\eta_n^0 - \eta_{n-1}^i)(\eta_n^0 + \eta_{n-1}^i)/(\eta_{n-1}^i)^2$$

$$< 2mA(n-1)^{-1} (\eta_n^0 + \eta_{n-1}^i)/(\eta_{n-1}^i)^2 \text{ by (3.20)}$$

and (3.9) assures there is a $0 < M < \infty$ such that

$$|\eta_n^0 + \eta_{n-1}^i|/(\eta_{n-1}^i)^2 < M,$$ and using lemma 2, (3.23) follows.

Since $\lim_{n\to\infty} \eta_n^0 = \bar{\eta} \neq 0$, the second and third terms of (3.22) converge to zero using the proof of Arvesen [1969] (note that the Cauchy-Schwarz inequality handles the second term).

Theorem 2. Let $X_1,\ldots,X_n$ be independent random variables, and assume the hypotheses of theorem 1. Then

$$s_{\hat{\theta}}^2 \xrightarrow{P} m^2 \zeta_1 (g'(\mu))^2/(\bar{\eta})^2 . \tag{3.24}$$

where $s_{\hat{\theta}}^2$ is given by (3.3).

PROOF. The proof follows theorem 11 of Arvesen [1969]. However note that

$$s_{\hat{\theta}}^2 = (n-1)^{-1} \sum_{i=1}^n (\hat{\theta}_i - \hat{\theta})^2$$

$$= (n-1) \sum_{i=1}^n (g(U_{n-1}^i/\eta_{n-1}^i) - n^{-1} \sum_{j=1}^n g(U_{n-1}^j/\eta_{n-1}^j))^2$$

$$= (n-1) \sum_{i=1}^n [(U_{n-1}^i/\eta_{n-1}^i - U_n^0/\eta_n^0) g'(\tau_i) - n^{-1} \sum_{j=1}^n (U_{n-1}^j/\eta_{n-1}^j - U_n^0/\eta_n^0) g'(\tau_j)]^2$$

(3.25)

$$= (n-1)\sum_{i=1}^n [(U_{n-1}^i/\eta_{n-1}^i - U_n^0/\eta_n^0) g'(0) + (U_{n-1}^i/\eta_{n-1}^i - U_n^0/\eta_n^0)(g'(\tau_i) - g'(0)) - n^{-1} \sum_{j=1}^n (U_{n-1}^j/\eta_{n-1}^j - U_n^0/\eta_n^0) g'(\tau_j)]^2$$

$$= (n-1) \sum_{i=1}^{n} (U_{n-1}^{i}/\eta_{n-1}^{i} - U_n^0/\eta_n^0)^2 (g'(0))^2$$

$$+ (n-1)[\sum_{i=1}^{n} (U_{n-1}^{i}/\eta_{n-1}^{i} - U_n^0/\eta_n^0)(g'(\tau_i) - g'(0)) - n^{-1}\sum_{j=1}^{n} (U_{n-1}^{j}/\eta_{n-1}^{j}$$

$$- U_n^0/ {}_n^0)\, g'(\tau_j)]^2 + \text{X-product term}$$

where τ_i lies between $U_{n-1}^{i}/\eta_{n-1}^{i}$ and U_n^0/η_n^0. Now from (3.23), and lemma 2, the first term of (3.25) converges to $m^2\, \zeta_1 (g'(0))^2/(\bar{\eta})^2$. The second term may also be readily shown to converge to zero in probability. Hence the result follows.

Combining theorems 1 and 2, one obtains the result that $\sqrt{n}(\hat{\theta}-\theta)/s_{\hat{\theta}}$ is asymptotically standard normal. In the original grouping $N = nk$, if n remains finite as $N \to \infty$, theorem 7 of Arvesen [1969] can be readily extended to obtain convergence to a t distribution with $n-1$ degrees of freedom. Again, in what follows, we will assume $k = 1$.

(c) Functions of several U-statistics. The generalization of theorems 1 and 2 to functions of several U-statistics is straightforward, proceeding along the lines of theorems 12 and 13 of Arvesen [1969]. First however, let us define some notation. Let $X_1,\ldots,X_n$ be independent random vectors of p components, and $U^{(1)},\ldots,U^{(q)}$ be such that

$$U^{(j)} = \binom{n}{m_j}^{-1} \sum_{C_n} f^{*(j)} (X_{\alpha_1},\ldots,X_{\alpha_{m_j}}),\ j = 1,\ldots,q$$

where as in (3.6),

$$E[f^{*(j)}(X_{\alpha_1},\ldots,X_{\alpha_{m_j}})] = \eta_{\alpha_1\ldots\alpha_{m_j}}\ \mu^{(j)} ,$$

$$\binom{n}{m_j}^{-1} \sum_{C_n} \eta_{\alpha_1\ldots\alpha_{m_j}} = \eta^{(j)},\ \binom{n-1}{m_j}^{-1} \sum_{C^i_{n-1}} \eta_{\beta^i_1\ldots\beta^i_{m_j}} = \eta_{\sim i}^{(j)} ,$$

and $\eta_{\alpha_1\ldots\alpha_{m_j}}$ is known, $\mu^{(j)}$ is unknown. Let g be a real-valued function of q arguments and

$$\theta = g(\mu^{(1)},\ldots,\mu^{(q)}) ,$$

(3.26) $$\hat{\theta}^0_n = g(U^{(1)}/\eta^{(1)},\ldots,U^{(q)}/\eta^{(q)}) ,$$

$$\hat{\theta}^i_{n-1} = g(U^{(1)}_{\sim i}/\eta_{\sim i}^{(1)},\ldots,U_{\sim i}^{(q)}/\eta_{\sim i}^{(q)}) ,$$

where $U_{\sim i}(j) = \binom{n-1}{m_j}^{-1} \sum_{C^i_{n-1}} f^{*(j)}(X_{\beta^i_1},\ldots,X_{\beta^i_{m_j}})$. Also, let the jackknife based on $\hat{\theta}^0_n$ be as defined in (3.1), (3.2) and (3.3). Let $\zeta_{1,n}^{(i,j)}$ be as defined in Arvesen [1969] immediately before theorem 12.

Theorem 3. Let $X_1,\ldots,X_n$ be independent random vectors of p components. Assume that (3.9), (3.10), (3.11) and (3.25) hold for each $f^{*(j)}(X_1,\ldots,X_{m_j})$, $j = 1,\ldots,q$, and that

$$\zeta_{1,n}^{(i,1)} \longrightarrow \zeta_1^{(i,1)} > 0 \quad \text{for } i = 1,\ldots,q, \text{ and}$$

$$\zeta_{1,n}^{(i,j)} \longrightarrow \zeta_1^{(i,j)} \quad \text{for } i,\ j = 1,\ldots,q .$$

Let q be a real-valued function defined on R^q, which in a neighborhood of $(\mu^{(1)},\ldots,\mu^{(q)})$ has bounded second partial derivatives. Then

$$\sqrt{n}\,(\hat{\theta} - \theta)/s_{\hat{\theta}} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1) \text{ as } n \longrightarrow \infty .$$

PROOF. The proof is identical to theorems 12 and 13 of Arvesen [1969] with the modifications of theorems 1 and 2 of the present paper.

4. An asymptotically robust test. The results of the previous section will now be used to obtain an asymptotically robust test of (2.2). Consider the model specified in (2.1) without the normality assumption, but assuming moments of order at least four plus δ for any $\delta > 0$.

Temporarily let us assume that we are on the boundary of H_0 as given in (2.2), that is $\Delta_0 = \sigma_A^2/\sigma_e^2$. Let

$$X_i = \begin{pmatrix} \overline{Y}_i \\ \\ \sum_{j=1}^{J_i} (Y_{ij} - \overline{Y}_{i.})^2 \end{pmatrix}, \; \overline{Y}_{i.} = J_i^{-1} \sum_{j=1}^{J_i} Y_{ij}, \; w_i = J_i(\Delta_0 J_i + 1)^{-1} ,$$

$$i=1,\ldots,n, \; W = \sum_{i=1}^{n} w_i ,$$

$$f^{*(1)}(X_{\alpha_1}, X_{\alpha_2}) = w_{\alpha_1} w_{\alpha_2} (\overline{Y}_{\alpha_1.} - \overline{Y}_{\alpha_2.})^2/2 ,$$

$$f^{*(2)}(X_{\alpha_1}) = \sum_{j=1}^{J_{\alpha_1}} (Y_{\alpha_1 j} - \overline{Y}_{\alpha_1.})^2 . \text{ Note}$$

$$E\,[f^{*(1)}(X_{\alpha_1}, X_{\alpha_2})] = w_{\alpha_1} w_{\alpha_2} (\sigma_A^2 + \sigma_e^2 (J_{\alpha_1}^{-1} + J_{\alpha_2}^{-1})/2)$$

$$= \sigma_e^2\, w_{\alpha_1} w_{\alpha_2} (\Delta_0 + (J_{\alpha_1}^{-1} + J_{\alpha_2}^{-1})/2) ,$$

$$E\,[f^{*(2)}(X_{\alpha_1})] = (J_{\alpha_1} - 1)\,\sigma_e^2 . \text{ Thus if}$$

$$U^{(1)} = \binom{n}{2}^{-1} \sum_{\alpha_1<\alpha_2} f^{*(1)}(X_{\alpha_1}, X_{\alpha_2})$$

$$= \binom{n}{2}^{-1} W(\sum_{i=1}^{n} w_i(\overline{Y}_{i}.-\overline{Y}..)^2/2) ,$$

$$\overline{Y}.. = W^{-1} \sum_{i=1}^{n} w_i \overline{Y}_i. ,$$

$$U^{(2)} = \binom{n}{1}^{-1} \sum_{\alpha_1} f^{*}(X_{\alpha_1})$$

$$= n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{J_i} (Y_{ij}-\overline{Y}_i.)^2 ,$$

$$E\ (U^{(1)}) = \binom{n}{2}^{-1} \sigma_e^2 \sum_{\alpha_1<\alpha_2} w_{\alpha_1} w_{\alpha_2} (\Delta_0 + (J_{\alpha_1}^{-1} + J_{\alpha_2}^{-1})/2)$$

$$= \binom{n}{2}^{-1} \sigma_e^2 [\Delta_0(W^2 - \sum_{i=1}^{n} w_i^2) + \sum_{i=1}^{n}(W-w_i)w_i J_i^{-1}]/2$$

(4.1)

$$= W\sigma_e^2/n ,$$

$$E\ (U^{(2)}) = (N^*-n)\sigma_e^2/n .$$

Also, note that

$$\eta^{(1)} = \binom{n}{2}^{-1} \sum_{\alpha_1<\alpha_2} w_{\alpha_1} w_{\alpha_2} (\Delta_0 + (J_{\alpha_1}^{-1} + J_{\alpha_2}^{-1})/2)$$

(4.2) $$= W/n ,$$

$$\eta^{(2)} = (N^*-n)/n .$$

Hence letting

$$g(U^{(1)}, U^{(2)}) = \frac{U^{(1)}/n^{(1)}}{U^{(2)}/n^{(2)}}, \tag{4.3}$$

the hypotheses of theorem 3 are satisfied as long as σ_e^2 is non-zero and max $(J_1,\ldots,J_n)$ remains bounded as $n \to \infty$. Note that (2.4) and (4.3) are identical.

Finally, note that under the assumption of normality in (2.1), the numerator and denominator of (4.3) are independent random variables, each distributed as a constant times a chi-square random variable. This was shown by Spjøtvoll. Thus, use of the jackknife in conjunction with the log transformation is suggested, and hence let

$$\hat{\theta}_n^0 = \log \{ \frac{U^{(1)}/n^{(1)}}{U^{(2)}/n^{(2)}} \} . \tag{4.4}$$

Note that the variances of $U^{(1)}$, $U^{(2)}$ go to zero as $n \to \infty$ as shown by Tukey [1957]. Moreover, for σ_A^2, σ_e^2 arbitrary, following Tukey

$$\begin{aligned} E[U^{(1)}] &= [(W^2 - \textstyle\sum_{i=1}^n w_i^2)\, \sigma_A^2 + \sum_{i=1}^n w_i(W-w_i)\, J_i^{-1}\, \sigma_e^2]/(n(n-1)) \\ &= W\, \sigma_e^2/n + (\Delta-\Delta_0)(W^2 - \textstyle\sum_{i=1}^n w_i^2)\, \sigma_e^2/(n(n-1)) , \end{aligned} \tag{4.5}$$

and hence under H_A, $U^{(1)}$ converges in probability to a quantity greater than $W\sigma_e^2/n$, on the boundary of H_0 and H_A, $U^{(1)}$ converges in probability to $W\sigma_e^2/n$, and in the interior of H_A, $U^{(1)}$ converges in probability to a quantity less than $W\sigma_e^2/n$. Thus, applying (3.26), one obtains an asymptotically robust and unbiased test of (2.2) by rejecting H_0 at the α-level if

$$\sqrt{n}\ \hat{\theta}_n^0 / s_{\hat{\theta}} > Z_\alpha \ , \tag{4.6}$$

where Z_α is the upper α point of a standard normal distribution.

Finally, in practice (and to be conservative), one might replace the cutoff point in (4.6) by the upper α point of a t distribution with $(n-1)$ degrees of freedom. This point will be discussed again in section 6.

Unfortunately, it is not possible to readily use (4.6) to obtain a lower confidence bound for Δ since the cutoff value Δ_0 of (2.2) appears in the original statistic $\hat{\theta}_n^0(1)$. However, note that in testing

$$H_0: \ \Delta \leq \Delta^*$$

$$H_A: \ \Delta > \Delta^*$$

at the α-level using the proposed jackknife technique, if

$\Delta_{Acc} = \{\Delta^*: \ H_0 \text{ is accepted}\}$, and $\Delta_L = \inf \Delta_{Acc}$, then $\Delta_L < \Delta$ forms an asymptotic lower $(1-\alpha) \times 100\%$ confidence bound for Δ. Spjøtvoll's proposed confidence interval also has this unpleasant property. Note this technique can be readily used to obtain a two-sided confidence interval, or an upper confidence bound for Δ.

5. Asymptotic efficiency results. Section 4 of this paper suggests that H_0: $\Delta \leq \Delta_0$ be tested by applying the jackknife procedure to the logarithm of Spjøtvoll's F statistic ($F = \frac{U^{(1)}/n^{(1)}}{U^{(2)}/n^{(2)}}$ in the notation of section 4). Section 3(c) of Arvesen [1969] proposed the use of the jackknife with the statistic $H = MSA/MSE$, where

$$MSA = (n-1)^{-1} \sum_{i=1}^{n} (Y_{i.} - n^{-1} \sum_{i'=1}^{n} Y_{i'.})^2 ,$$

$$MSE = n^{-1} \sum_{i=1}^{n} (J_i - 1)^{-1} \sum_{j=1}^{J_i} (Y_{ij} - Y_{i.})^2 ,$$

and $Y_{i.} = J_i^{-1} \sum_{j=1}^{J_i} Y_{ij}$. In this section we find the Pitman ARE of F versus H when all effects are assumed to be normally distributed. The log transformation and jackknifing do not affect the ARE, so the result holds for the comparison of tests based on the jackknifed versions of log F and log H respectively (these tests were used in the Monte Carlo study discussed in the next section).

Recalling that F is the ratio of independent random variables, each distributed as a constant times a χ^2 variable (in particular, for $\Delta = \Delta_0$, $F \sim F_{n-1,\, N^*-n}$), and using (4.5), we readily find that the Pitman efficacy of F is

$$n \; \frac{\lim (n-1)^{-2} (W - \sum_{i=1}^{n} w_i^2/w)^2}{\lim 2\bar{J}/(\bar{J} - 1)} \tag{5.1}$$

where $w_i = J_i/(\Delta_0 J_i + 1)$, $W = \sum_{i=1}^{n} w_i$ and $\bar{J} = n^{-1} \sum_{i=1}^{n} J_i$.

Ignoring terms of order n^{-1} and smaller, $E(H) = \Delta + \bar{\bar{J}}$, where $\bar{\bar{J}} = n^{-1} \sum_{i=1}^{n} J_i^{-1}$. Also

$$\text{var}(MSA) = 2\sigma_e^4[(n-1)^{-1}(\Delta + \bar{J})^2 + (n-2)n^{-1}(n-1)^{-2} \textstyle\sum_{i=1}^n (J_i^{-1} - \bar{J})^2] ,$$

$$\text{var}(MSE) = 2\sigma_e^4\, n^{-2} \textstyle\sum_{i=1}^n (J_i - 1)^{-1}, \text{ and}$$

cov(MSA, MSE) = 0 (Tukey [1957]). Hence, ignoring terms of order n^{-1} and smaller,

$$n\,\text{var}(H) = 2[(\Delta + \bar{J})^2 (1 + n^{-1} \textstyle\sum_{i=1}^n (J_i - 1)^{-1}) + n^{-1} \textstyle\sum_{i=1}^n (J_i^{-1} - \bar{J})^2] .$$

The Pitman efficacy of H is therefore

$$(5.2) \qquad n/\lim 2[(\Delta_0 + \bar{J})^2(1 + n^{-1} \textstyle\sum_{i=1}^n (J_i - 1)^{-1} + n^{-1} \textstyle\sum_{i=1}^n (J_i^{-1} - \bar{J})^2] .$$

If all $J_i = J$, both (5.1) and (5.2) reduce to $nJ(J-1)/2(\Delta_0 J+1)^2$, and the Pitman ARE of F versus H is 1. If the J_i are not all equal (and $\lim n^{-1} \sum_{i=1}^n (J_i^{-1} - \bar{J})^2 > 0$), then the ARE is > 1. This can be seen by noting that

$$\text{ARE} > \lim \frac{(n-1)^{-2}(W - \sum_{i=1}^n w_i^2/w)^2(\Delta_0 + \bar{J})^2(1 + n^{-1} \sum_{i=1}^n (J_i - 1)^{-1})}{\bar{J}/(\bar{J}-1)}$$

$$= \lim \left\{\frac{n^{-1} \sum_{i=1}^n (\Delta_0 + J_i^{-1})^{-1}}{(\Delta_0 + n^{-1} \sum_{i=1}^n J_i^{-1})^{-1}}\right\}^2 \left\{\frac{n - n^{-1} \sum_{i=1}^n (\Delta_0 + J_i^{-1})^{-2} \, (n^{-1} \sum_{i=1}^n (\Delta_0 + J_i^{-1})^{-1})^{-2}}{n-1}\right\}^2$$

$$\cdot \frac{1 + n^{-1} \sum_{i=1}^n (J_i - 1)^{-1}}{1 + (n^{-1} \sum_{i=1}^n J_i - 1)^{-1}} .$$

The first and third factors of this expression are ≥ 1 by Jensen's inequality, and the limit of the second factor is 1, since the J_i are assumed to be bounded. Suppose that the J_i have values 2, 3, and 4 in equal proportions, and that $\Delta_0 = 1$ (which is the case in the Monte Carlo simulation of section 6). Then the ARE of F vs. H is 1.1.

6. Monte Carlo simulation. To obtain some information about the small sample behavior of the tests discussed in the preceding sections, a Monte Carlo simulation study was made. The program was run on the CDC6500 at Purdue University using procedures described by Rubin [1971]. The model selected to test the jackknife procedure was:

$$Y_{ij} = a_i + e_{ij},\ i = 1,\ldots,15,\ j = 1,\ldots,J_i\ , \tag{6.1}$$

$$J_1 = \ldots = J_5 = 2,\ J_6 = \ldots = J_{10} = 3,\ J_{11} = \ldots = J_{15} = 4,$$

and the $\{a_i\}$, $\{e_{ij}\}$ are mutually independent random variables with mean zero, variance σ_A^2, σ_e^2 respectively. As in Arvesen and Schmitz [1970], three distributions were considered for the $\{a_i\}$, $\{e_{ij}\}$: both sets normal random variables, both sets double exponential random variables (kurtosis of 3), and both sets uniform random variables (kurtosis of -1.2).

The Monte Carlo study compares the empirical power functions of the Spjøtvoll F test to the jackknife procedure in testing

$$H_0:\ \Delta = \sigma_A^2/\sigma_e^2 \le 1 \text{ vs. } \Delta > 1\ . \tag{6.2}$$

The jackknife was used with

$$\hat{\theta}_{15}^{0}(1) = \log\left(\frac{U^{(1)}/\eta^{(1)}}{U^{(2)}/\eta^{(2)}}\right) \text{ where} \tag{6.3}$$

$U^{(i)}$, $\eta^{(i)}$, $i = 1,2$ are as given in section 4, and with

(6.4) $\hat{\theta}_{15}^{0}(2) = \log(\mathrm{MSA}/\mathrm{MSE})$ where

$$\mathrm{MSA} = (n-1)^{-1} \sum_{i=1}^{n} (Y_{i\cdot} - n^{-1} \sum_{i'=1}^{n} Y_{i'\cdot})^2 ,$$

$$\mathrm{MSE} = n^{-1} \sum_{i=1}^{n} (J_i - 1)^{-1} \sum_{j=1}^{J_i} (Y_{ij} - Y_{i\cdot})^2 ,$$

and $Y_{i\cdot} = J_i^{-1} \sum_{j=1}^{J_i} Y_{ij}$.

Of course $n = 15$, as stated above. In terms of the decomposition $N = nk$, $k = 1$ for this study.

There were 1000 sets of $\{a_i\}$, $\{e_{ij}\}$ generated according to the three distributions. They were first generated with $\Delta = 1$, and then scaled so that $\Delta = .5, 1.5, 2.5, 4, 6, 9$. Hence 180,000 pseudo-random numbers were generated in all. As mentioned in section 4, there is some confusion as to whether the t_{n-1} distribution (t_{14} in this case) or the standard normal should be used in practice with moderate samples. The latter seems to be preferable for reasonable significance levels as the results in Table 1 demonstrate. Results are given separately for $\alpha = .10$, $\alpha = .05$, $\alpha = .01$. Finally, $J(\hat{\theta}_{15}^{0}(1))$ $J(\hat{\theta}_{15}^{0}(2))$ denotes the jackknife procedure using (6.3) and (6.4) respectively, and they are used either with the t_{14} distribution (w/t) or with the standard normal distribution (w/z) to obtain critical values.

TABLE 1

Values of the Monte Carlo Power Function for Testing (6.2)

$\Delta = \sigma_A^2/\sigma_e^2$	.5	1.0	1.5	2.5	4	6	9
				Normal distribution			
$\alpha = .10$							
Spjøtvoll F test	.010	.109	.305	.631	.889	.977	.995
$J(\hat{\theta}^0_{15}(1))(w/t)$	.009	.091	.276	.595	.852	.959	.990
$J(\hat{\theta}^0_{15}(2))(w/t)$	.012	.093	.280	.569	.833	.952	.985
$J(\hat{\theta}^0_{15}(1))(w/z)$	.010	.109	.300	.619	.869	.966	.993
$J(\hat{\theta}^0_{15}(2))(w/z)$	.013	.109	.299	.585	.844	.956	.986
$\alpha = .05$							
Spjøtvoll F test	.001	.053	.184	.516	.813	.948	.990
$J(\hat{\theta}^0_{15}(1))(w/t)$	.001	.039	.154	.449	.743	.913	.980
$J(\hat{\theta}^0_{15}(2))(w/t)$	.002	.043	.147	.433	.706	.885	.970
$J(\hat{\theta}^0_{15}(1))(w/z)$	.003	.056	.178	.488	.775	.927	.982
$J(\hat{\theta}^0_{15}(2))(w/z)$	.004	.052	.181	.471	.739	.906	.974
$\alpha = .01$							
Spjøtvoll F Test	.000	.006	.051	.282	.600	.849	.965
$J(\hat{\theta}^0_{15}(1))(w/t)$	.000	.008	.041	.191	.457	.707	.888
$J(\hat{\theta}^0_{15}(2))(w/t)$	.000	.009	.038	.185	.435	.664	.837
$J(\hat{\theta}^0_{15}(1))(w/z)$	.000	.015	.066	.272	.550	.788	.927
$J(\hat{\theta}^0_{15}(2))(w/z)$	.000	.014	.064	.264	.532	.743	.906

TABLE 1 continued

$\Delta = \sigma_A^2/\sigma_e^2$	.5	1.0	1.5	2.5	4	6	9
Double exponential distribution							
$\alpha = .10$							
Spjøtvoll F test	.023	.143	.300	.559	.799	.916	.977
$J(\hat{\theta}^0_{15}(1))(w/t)$	.012	.092	.199	.430	.681	.831	.934
$J(\hat{\theta}^0_{15}(2))(w/t)$	.018	.088	.210	.437	.668	.826	.928
$J(\hat{\theta}^0_{15}(1))(w/z)$	.016	.100	.220	.464	.695	.843	.946
$J(\hat{\theta}^0_{15}(2))(w/z)$	.025	.093	.229	.466	.683	.838	.939
$\alpha = .05$							
Spjøtvoll F test	.012	.084	.219	.461	.725	.861	.958
$J(\hat{\theta}^0_{15}(1))(w/t)$	.004	.045	.108	.289	.536	.718	.868
$J(\hat{\theta}^0_{15}(2))(w/t)$	.006	.046	.107	.297	.527	.713	.860
$J(\hat{\theta}^0_{15}(1))(w/z)$	.007	.055	.128	.330	.574	.749	.886
$J(\hat{\theta}^0_{15}(2))(w/z)$	.011	.057	.135	.331	.566	.748	.877
$\alpha = .01$							
Spjøtvoll F test	.002	.031	.094	.287	.522	.763	.895
$J(\hat{\theta}^0_{15}(1))(w/t)$	.000	.010	.038	.101	.248	.433	.617
$J(\hat{\theta}^0_{15}(2))(w/t)$	.000	.012	.032	.100	.257	.424	.608
$J(\hat{\theta}^0_{15}(1))(w/z)$	.000	.018	.055	.154	.320	.536	.712
$J(\hat{\theta}^0_{15}(2))(w/z)$	.001	.020	.052	.149	.335	.525	.703

TABLE 1 continued

$\Delta = \sigma_A^2/\sigma_e^2$	.5	1.0	1.5	2.5	4	6	9
			Uniform distribution				
$\alpha = .10$							
Spjøtvoll F test	.005	.060	.230	.672	.939	.993	1.000
$J(\hat{\theta}_{15}^0(1))$(w/t)	.005	.074	.277	.728	.952	.994	.998
$J(\hat{\theta}_{15}^0(2))$(w/t)	.007	.084	.279	.693	.930	.989	.998
$J(\hat{\theta}_{15}^0(1))$(w/z)	.006	.088	.299	.749	.956	.995	.999
$J(\hat{\theta}_{15}^0(2))$(w/z)	.008	.097	.299	.719	.934	.992	.999
$\alpha = .05$							
Spjøtvoll F test	.003	.023	.119	.519	.881	.982	.999
$J(\hat{\theta}_{15}^0(1))$(w/t)	.004	.026	.161	.572	.892	.983	.997
$J(\hat{\theta}_{15}^0(2))$(w/t)	.004	.031	.161	.546	.853	.976	.996
$J(\hat{\theta}_{15}^0(1))$(w/z)	.004	.040	.190	.612	.916	.986	.998
$J(\hat{\theta}_{15}^0(2))$(w/z)	.005	.044	.184	.584	.882	.980	.997
$\alpha = .01$							
Spjøtvoll F test	.001	.004	.021	.201	.655	.917	.987
$J(\hat{\theta}_{15}^0(1))$(w/t)	.000	.005	.036	.280	.676	.898	.977
$J(\hat{\theta}_{15}^0(2))$(w/t)	.000	.006	.042	.251	.620	.860	.976
$J(\hat{\theta}_{15}^0(1))$(w/z)	.002	.010	.066	.361	.760	.947	.990
$J(\hat{\theta}_{15}^0(2))$(w/z)	.001	.009	.067	.338	.722	.917	.984

Examination of Table I produces several interesting results.

(i.) The definite non-robustness of the significance level of the Spjøtvoll F test is readily apparent. Actually comparison with an even more leptokurtic distribution than the double exponential could further emphasize this fact.

(ii.) It was felt that the jackknife would work well at $\alpha = .10$, giving poorer results at $\alpha = .01$. Actually, at $\alpha = .10$, the jackknife using $J(\hat{\theta}^0_{15}(1))(w/z)$ is an excellent competitor to the Spjøtvoll F test even if the data are normal, and gives a more appropriate empirical significance level even if the data are double exponential or uniform.

(iii.) At $\alpha = .10$, $J(\hat{\theta}^0_{15}(1))$ appears to be slightly more powerful than $J(\hat{\theta}^0_{15}(w))$ using either t or z critical values. Of course this is also essentially shown by Spjøtvoll and in section 5.

(iv.) At $\alpha = .10$, the use of z critical values appear to be recommended, as t critical values are too conservative. Note that at $\alpha = .01$, t values appear to be recommended, but then the power of the jackknife procedure is too low to recommend it as a competitor to Spjøtvoll's test. Of course, a larger sample size would correct this situation. Thus there appears to be an interesting question as to the connection between sample size and the general asymptotic results.

In conclusion, it appears that if the jackknife works well, it should be used with $J(\hat{\theta}^0_n(1))(w/z)$. A researcher will have to be careful to see that n is large enough to use the normal approximation. In the Monte Carlo study given, for $n = 15$, $\alpha = .10$ results are excellent, $\alpha = .05$ results are good, $\alpha = .01$ results are poor.

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Key Words

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