Optimal Search

Strategies for Wiener Processes

by

E. M. Klimko and James Yackel

Department of Statistics

Division of Mathematical Sciences

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Abstract

Let there be n Wiener processes with common variance $\sigma^2 t$ all of which have zero drift except for one which has drift μt. Further let there be a prior probability distribution over the n processes for the one with drift μt. The authors show that search time is minimized over a certain restricted class of search strategies if one searches that process with the maximum posterior probability of being the process with drift.

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Introduction. The problem we wish to consider here is the same search problem considered by Posner and Rumsey [4], [5]. In [3] we have pointed out some serious errors in their optimality arguments. We also note that this same error was made by Zigangarov [6].

Our purpose here is to partially salvage their results by showing that maximum likelihood strategies are optimal over certain restricted classes of strategies of the type considered by Posner and Rumsey.

First consider a description of the search problem. Let $y_1(t),\ldots,y_n(t)$ be n Wiener processes each with known variance $\sigma^2 t$, $n-1$ of them have zero drift while the remaining one has drift μt where μ is known. Our problem is to determine, with probability at least $1-\varepsilon$ of correct selection, the process with drift μt. Henceforth, this process will be known as the correct process. In addition, we are given a prior distribution $p_1,p_2,\ldots,p_n$ where p_i is the probability that the ith process is the correct one.

In [4], Posner and Rumsey tried to use weak limits of the class of lattice time strategies, for which they claimed optimality, and weak limits of another class of strategies called δ perturbed strategies, for which computations were more tractable, to determine "expected search time". A similar approach was also used in [6]. In [3] it was shown that neither the class of lattice time nor the class of δ perturbed strategies admit weak limits.

The δ perturbed strategies are defined as follows. Observe the process corresponding to the largest prior probability until the first time the posterior probability has decreased by δ/n, then observe the process with maximum posterior probability at that time. We now describe the class of strategies with which we shall deal. As in [3], we call the process which is being searched the target and the act of changing targets a switch. $\mathcal{D}_{\delta\gamma}$ will be the class of strategies which require that the posterior probability of the target being searched must change by at least $+\delta$ or $-\gamma$ before a switch is permitted. Furthermore, the strategies in $\mathcal{D}_{\delta\gamma}$ are determined solely by the values of the posterior probabilities. This class will be more rigorously defined in section 2. Clearly, Posner and Rumsey's δ perturbed strategies belong to $\mathcal{D}_{\delta\gamma}$.

Our main result may now be described. For any δ, $\gamma > 0$, the expected search time is minimized if at any time a switch is permitted, one always searches the target with the highest posterior probability of being the correct one.

2. Preliminaries. We begin with some definitions and structure of the problem. Basic is a probability space (Ω, A, P) on which are defined the n Wiener processes $y_1(t),\ldots, y_n(t)$ for $t \geq 0$, discussed in section 1. We note that from the initial probabilities $p_1(0),\ldots, p_n(0)$, the knowledge that i is the target being searched and the value of $p_i(t)$, we may determine $p_1(t),\ldots, p_n(t)$. Therefore, the strategy I being used and $p_{I(t)}(t)$ are sufficient statistics for the posterior processes. For any α, $\beta > 0$, let $T(p_1,\ldots,p_n,\alpha,\beta,i)$ be the first time that the posterior probability $p_i(t)$ crosses one of the boundaries $p_i(0) + \alpha$, $p_i(0) - \beta$ given that the ith target is searched during the entire time $0 \leq t \leq T$.

For $\delta, \gamma > 0$, we define $\mathcal{D}_{\delta\gamma}$ to be the collection of all strategies $I_{\delta\gamma}$ determined by the conditions:

$$\begin{aligned}
&\text{(i)} \quad \alpha_m \geq \delta,\ \beta_m \geq \gamma \qquad m = 1,2,\ldots \\
&\text{(ii)} \quad T_{m+1} = T(p_1(T_m),\ldots,\, p_n(T_m),\, \alpha_m,\, \beta_m,\, i_m) \\
&\text{(iii)} \quad I_{\delta\gamma} = i_m \quad \text{for} \quad T_m \leq t < T_{m+1} \\
&\text{(iv)} \quad p_i(T_m) + \alpha \leq 1-\varepsilon;\quad p_i(T_m) - \beta > 0,
\end{aligned} \tag{2.1}$$

where α_m, β_m, i_m are arbitrary for $m = 0,1,\ldots$, $T_0 = 0$, and ε is given.

The posterior process $p_i(t|I_{\delta\gamma})$ is the posterior probability that at time t, the target i is the correct one (the one with drift rate μ) when the search strategy $I_{\delta\gamma}$ is used. We will need the following formulae for $p_i(t) = p_i(t|I_{\delta\gamma})$ when $I_{\delta\gamma}(\tau) = j$ for $0 \leq \tau \leq t$ (see [4]):

$$\begin{aligned}
p_j(t) &= \frac{p_j(0)}{p_j(0)+(1-p_j(0))\exp(\frac{\mu}{2\sigma^2}(\mu t-2y_j(t)))} \\
p_i(t) &= \frac{p_i(0)}{(1-p_j(0))+ p_j(0)\exp(\frac{-\mu}{2\sigma^2}(\mu t-2y_j(t)))} = \\
&\qquad p_j(t) \cdot \frac{p_i(0)}{p_j(0)} \exp(\frac{-\mu}{2\sigma^2}(\mu t-2y_j(t))), \quad i \neq j \;.
\end{aligned} \tag{2.2}$$

Throughout the exposition we will abbreviate $p_i(t|I_{\delta\gamma})$ to $p_i(t)$ when there is no possibility of confusion.

Our results depend on the boundary crossing probabilities and exit times for Wiener processes ([2], p. 632) which we list here for convenience. Let $W(t)$ be a Wiener process with drift λt and variance $\tau^2 t$. Then if the process is started at x $(W(0) = x)$ and observed until it crosses one of the boundaries $a > x > b$, exit occurs with probability 1 and the probability $P(x,a,b)$ of exit at a is given by

$$P(x,a,b) = \frac{\exp(\frac{-2\lambda b}{\tau^2}) - \exp(\frac{-2\lambda x}{\tau^2})}{\exp(\frac{-2\lambda b}{\tau^2}) - \exp(\frac{-2\lambda a}{\tau^2})} \tag{2.3}$$

and the expected exit time $M(x,a,b)$ is given by

$$M(x,a,b) = \frac{1}{\lambda}[(a-x)P(x,a,b)-(x-b)Q(x,a,b)] \tag{2.4}$$

where $Q(x,a,b) = 1 - P(x,a,b)$.

From (2.2) we see that the posterior process $p_i(t|I_{\delta\gamma})$ is continuous in t and is a Markov process because of the independent increments of the Wiener process. The following lemmas show that we may study the posterior process $p_i(t|I_{\delta\gamma})$ by means of an appropriate Wiener process.

<u>Lemma</u> 2.1. Let $I_{\delta\gamma} = j$. Then $p_i(t)$ crosses the boundary $p_i(0) + \alpha$ if and only if $p_j(t)$ crosses the boundary $p_j(0) - \alpha(1-p_j(0))/p_i(0)$.

<u>Proof</u>. For $i \neq j$, it follows from (2.2) that

$$p_i(t) = p_i(0)\ p_j(t)\ K(t) \tag{2.4}$$

where $K(t)$ is some process. When $p_i(t) = p_i(0) + \alpha$, we have

$$p_i(0) + \alpha = p_i(0)\ p_j(t)\ K(t) \quad . \tag{2.5}$$

Summing (2.4) over the range $i, i \neq j$, we find that $p_j(t)K(t) = (1-p_j(t))/(1-p_j(0))$. Substituting this value into (2.5), we see that

$$p_j(t) = p_j(0) - \alpha(1-p_j(0))/p_i(0) \quad .$$

Central to our discussion will be the Wiener processes $W(t)$ and $\overline{W}(t)$ where both processes have variance $\mu^2 t/\sigma^2$ and $\overline{W}(t)$ has drift rate $\mu^2/2\sigma^2$ while $W(t)$ has drift rate $-\mu^2/2\sigma^2$. In general, we will use a bar over a quantity to denote conditioning with respect to the correct target while the absence of a bar denotes conditioning with respect to an incorrect one. As the next lemma shows, searching the correct posterior process is equivalent to observing $\overline{W}(t)$ while searching the incorrect process is equivalent to observing $W(t)$.

Lemma 2.2. Let $I_{\delta\gamma}(t) = j$ (= const.) and let α be such that $0 < p_j(0) + \alpha < 1$. Then $p_j(t)$ crosses the boundary $p_j(0) + \alpha$ if and only if the Wiener process $W^*(t)$ crosses the boundary $\log((p_j(0)+\alpha)/(1-p_j(0)-\alpha))$ where $W^*(0) = \log(p_j(0)/(1-p_j(0)))$ and $W^*(t)$ is one of the two processes $\overline{W}(t)$ or $W(t)$ depending on whether j is the correct process or not.

Proof. By (2.2), $p_j(t) = p_j(0) + \alpha$ if and only if

$$p_j(0) = p_j(0)(p_j(0)+\alpha) + (1-p_j(0))(p_j(0)+\alpha)\exp(\frac{\mu}{2\sigma^2}(\mu t-2y_j(t))) \quad .$$

Taking logs, we obtain

$$\log \frac{p_j(0)+\alpha}{1-p_j(0)-\alpha} = \log \frac{p_j(0)}{1-p_j(0)} + \frac{\mu}{2\sigma^2}(2y_j(t) - \mu t) \quad .$$

The process $\frac{\mu}{2\sigma^2}(2y_j(t)-\mu t)$ is a Wiener process with drift rate $\mu^2/2\sigma^2$ if j is the correct target and $-\mu^2/2\sigma^2$ if j is an incorrect target. The variance is easily seen to be $\mu^2 t/\sigma^2$ in either case. This proves the lemma.

We now turn our attention to the unconditioned process. The following somewhat surprising theorem is basic to our results.

Theorem 2.1. Let T_0 be the first time the posterior process crosses one of the boundaries $p_j(0) + \alpha$ or $p_j(0) - \beta$, where $p_j(0) + \alpha \leq 1$, $\alpha > 0$, $p_j(0) - \beta > 0$, $\beta > 0$. Furthermore, let $I_{\delta\gamma}(t) = k$ (= const.) for $0 \leq t \leq T_0$. Then whatever be k,

$$\begin{aligned} \Pr[p_j(T_0|I_{\delta\gamma}) &= p_j(0) + \alpha] = \beta/(\alpha+\beta) \\ \Pr[p_j(T_0|I_{\delta\gamma}) &= p_j(0) - \beta] = \alpha/(\alpha+\beta) \quad , \end{aligned} \tag{2.6}$$

provided the posterior process $p_j(t|I_{\delta\gamma})$ can attain the boundaries $p_j(0) + \alpha$ or $p_j(0) - \beta$ at all.

Proof. We first consider the case $k = j$. For ease of notation, we let $p_j(0) = p$. By Lemma 2.2, the Wiener process $\overline{W}(t)$ or $W(t)$ (depending on whether j is the correct or incorrect target) must begin at x and cross one of the two boundaries a,b where

$$\begin{aligned} a &= \log \frac{p+\alpha}{1-p-\alpha} \\ b &= \log \frac{p-\beta}{1-p+\beta} \\ x &= \log \frac{p}{1-p} \quad . \end{aligned} \tag{2.7}$$

Let $\overline{P}(x,a,b)$ be probability that $\overline{W}$ crosses a and let $P(x,a,b)$ the probability that W crosses the boundary a at time T_0. Then the probability that $p_j(t|I_{\delta\gamma})$ crosses the boundary a is $p\overline{P}(x,a,b) + (1-p)P(x,a,b)$. We use (2.4) with $\lambda = \mu^2/2\sigma^2$ for $\overline{W}$, $\lambda = -\mu^2/2\sigma^2$ for W and $\tau^2 = \mu^2/\sigma^2$. Therefore

$$\overline{P}(x,a,b) = \frac{e^{-b} - e^{-x}}{e^{-b} - e^{-a}} , \qquad P(x,a,b) = \frac{e^{b} - e^{x}}{e^{b} - e^{a}} \tag{2.8}$$

From (2.7) we obtain

$$\overline{P}(x,a,b) = \frac{(p+\alpha)\beta}{p(\alpha+\beta)}\,, \qquad P(x,a,b) = \frac{(1-p-\beta)}{(1-p)(\alpha+\beta)} \quad . \tag{2.9}$$

Therefore

$$\begin{aligned} &\Pr[p_j(T_0|I_{\delta\gamma}) = p + \alpha] = \\ &\qquad \frac{p(p+\alpha)\beta}{p(\alpha+\beta)} + \frac{(1-p)(1-p-\alpha)\beta}{(1-p)(\alpha+\beta)} = \frac{\beta}{\alpha+\beta} \quad . \end{aligned} \tag{2.10}$$

Since the posterior process must cross one of the boundaries $p + \alpha$ or $p - \beta$, (2.6) follows for the case $k = j$. If $k \neq j$, then the process $p_k(t|I_{\delta\gamma} = k)$ must cross one of the boundaries $p_k(0) + \alpha'$ or $p_k(0) - \beta'$ where α' and β' are chosen in such a way that the posterior process $p_j(t|I_{\delta\gamma} = k)$ crosses $p_j(0) + \alpha$ or $p_j(0) - \beta$. By Lemma 2.1, $\alpha = \beta'/\psi$ and $\beta = \alpha'/\psi$ where $\psi = p_j/(1-p_k)$. We have

$$\begin{aligned} \Pr[p_j(T_0|I_{\delta\gamma} = k) = p_j(0) + \alpha] &= \Pr[p_k(T_0|I_{\delta\gamma} = k) = p_k(0) - \beta] \\ &= \frac{\alpha'}{\alpha'+\beta'} = \frac{\beta\psi}{\alpha\psi+\beta\psi} = \frac{\beta}{\alpha+\beta} \end{aligned} \tag{2.11}$$

Since one of the two boundaries must be crossed (2.7) follows and the theorem is proved.

<u>Remark</u>. The condition that $p_j(t|I_{\delta\gamma} = k)$ attains the boundaries $p_j(0) + \alpha$ or $p_j(0) - \beta$ is equivalent to the conditions

$$\begin{aligned} &\text{(i)} \quad p_k + \beta/\psi < 1 \\ &\text{(ii)} \quad p_k - \alpha/\psi > 0 \quad . \end{aligned} \tag{2.12}$$

Condition (2.12) (i) is satisfied for any p_k while (2.12) (ii) is satisfied for

$$p_k > \frac{\alpha}{p_j + \alpha} \quad . \tag{2.13}$$

In terms of our search problem, this condition means that it will be difficult for the maximum posterior probability to exceed $1 - \varepsilon$ when searching a target with small prior probability.

Definition 2.1. For $0 < x < 1$,

$$R(x) = (2x-1) \log \frac{x}{1-x} \quad . \tag{2.14}$$

This function R permits us to simplify analytic expressions for expected search time.

Let $M(p,\alpha,\beta)$ be the expected time for the posterior process of the target being searched to start at p and cross one of the boundaries $p + \alpha$, $p - \beta$. We note that this time is independent of the target.

Theorem 2.2. Let $\alpha,\beta > 0$ be such that $p + \alpha < 1$, $p - \beta > 0$. Then

$$M(p,\alpha,\beta) = \frac{\mu^2}{\sigma^2} \left[\frac{\beta}{\alpha+\beta} R(p+\alpha) + \frac{\alpha}{\alpha+\beta} R(p-\beta) - R(p)\right] \tag{2.15}$$

Proof. Let $\overline{M}$ be the exit time given that the target being searched is correct and M the exit time given that the target is not the correct one. Then

$$M(p,\alpha,\beta) = p\overline{M} + (1-p)M \quad . \tag{2.16}$$

Since $\overline{M}$ is the expected time for the process $\overline{W}$ to begin at x and cross one of the boundaries a,b where a,b,x are as in (2.7), we obtain from (2.4)

$$\overline{M} = \frac{\sigma^2}{\mu^2}\,[(a-x)\overline{P}(x,a,b) + (b-x)(1-\overline{P}(x,a,b))]$$
$$M = \frac{-\sigma^2}{\mu^2}[(a-x)P(x,a,b) + (b-x)(1-P(x,a,b))] \quad . \tag{2.17}$$

From (2.9) we have

$$p\,\overline{P}(x,a,b) - (1-p)P(x,a,b) = \frac{(2(p+\alpha)-1)\beta}{\alpha+\beta}$$
$$p(1-\overline{P}(x,a,b)) - (1-p)(1-P(x,a,b)) = \frac{2(p+\beta)-1)\alpha}{\alpha+\beta} \tag{2.18}$$

Putting together (2.16), (2.17) and (2.18), we obtain

$$M(p,\alpha,\beta) = \frac{\sigma^2}{\mu^2}\,[\frac{(2(p+\alpha)-1)\beta a}{\alpha+\beta} + \frac{(2(p-\beta)-1)\alpha b}{\alpha+\beta} - (2p-1)x] \tag{2.19}$$

which yields (2.15) if we utilize the definition of R.

We conclude this section with a tabulation of the properties of the function R.

<u>Lemma</u> 2.3. For $0 \leq x \leq 1$, the function R has the properties:

$$\begin{array}{lll} \text{(i)} & R^{(n)}(x) = R^{(n)}(1-x) & n \text{ even} \\ \text{(ii)} & R^{n}(x) = -R^{n}(1-x) & \\ \text{(iii)} & R^{n}(1/2) = 0 & \left.\right\} n \text{ odd} \\ \text{(iv)} & R^{n}(x) \text{ is increasing,} & \end{array} \tag{2.20}$$

where $R^{(n)}(x)$ is the nth derivative of R.

<u>Proof</u>. For $n = 0$, (i) follows directly from (2.14). For $n > 0$, (i) and (ii) follow by induction and a direct differentiation; (iii) follows from (ii). To show (iv), we observe that

$$R^{(1)}(x) = \frac{1}{1-x} - \frac{1}{x} + 2 \log \frac{x}{1-x}$$

$$R^{(2)}(x) = \frac{1}{(x(1-x))^2} \quad . \tag{2.21}$$

From this, (iv) follows for $n = 1$. Using the change of variables $x = (u+1)/2$, we have $R^{(2)}(x) = S(u)$ where $S(u) = (4/(1-u^2))^2$. From the power series expansion for $S(u)$, we see that $S^{(n)}(u)$ is increasing for $0 < u < 1$. From this (iv) follows.

3. Main Results. In this section, we present our main result which may be stated loosely by saying that it is optimal to search the target with largest posterior probability. We first prove the following.

Lemma 3.1. Let T be the time at which the posterior process $p_i(t|I_{\delta\gamma})$ first crosses one of the boundaries $p_i(0) + \alpha$, $p_i(0) - \beta$ where $\alpha,\beta > 0$, $p_i(0) + \alpha < 1$ and $p_i(0) - \beta > 0$. Further assume that the strategy $I_{\delta\gamma} = j$ (= const.) for $0 \leq t \leq T$, $p_j(0) \leq p_i(0)$ and $p_j(0) \leq (1+\alpha-\beta)/2$. Then the expected search time is smallest if $j = i$.

Remark. Implicit in the above assumptions is that (2.13): $p_j(0)\alpha/(p_i(0)+\alpha)$ is satisfied; otherwise exit from the boundary $p_i(0)+\alpha$ would not be possible.

Proof. For brevity, we let $p_j = p_j(0)$, $p_i = p_i(0)$. We let $j \neq i$. If the jth target is searched, the posterior process $p_j(t|I_{\delta\gamma} = j)$ must cross the boundaries $p_j + \alpha'$ or $p_j - \beta'$ where $\alpha' = \beta\psi$, $\beta' = \alpha\psi$ and $\psi = (1-p_j)/p_i$. We note that $\psi \geq 1$. The lemma will be proved if we show that

$$M(p_j, \alpha', \beta') - M(p_i, \alpha, \beta) \geq 0. \tag{3.1}$$

If $p_j = 1-p_i$, $\psi = 1$, $M(p_j, \alpha', \beta') = M(1-p_i, \beta, \alpha) = M(p_i, \alpha, \beta)$ and the difference in (3.1) is zero. Since $p_j \leq 1-p_i$, we need only show that the difference in (3.1) is decreasing in p_j for $0 \le p_j \le 1-p_i$. Indeed, the derivative of the left side of (3.1) with respect to p_j is

$$\frac{\alpha}{\alpha+\beta} R'(p_j+\beta\psi) + \frac{\beta}{\alpha+\beta} R'(p_j-\alpha\psi) - R'(p_j). \tag{3.2}$$

From Lemma 2.3, it follows that R' is concave on $(0,\frac{1}{2}]$ and convex on $[\frac{1}{2},1)$. Therefore, if $p_j + \beta\psi \le 1/2$, we have by concavity of $R'(x)$ on $(0\ \frac{1}{2}]$,

$$\frac{\alpha}{\alpha+\beta} R'(p_j + \beta\psi) + \frac{\beta}{\alpha+\beta} R'(p_j-\alpha\psi) \le R'(p_j) . \tag{3.3}$$

If $p_j + \beta\psi > 1/2$, we proceed as follows. Let $\mathcal{L}_1(x)$ and $\mathcal{L}_2(x)$ be linear functions for which

$$\begin{aligned}
\mathcal{L}_1(p_j-\alpha\psi) &= R'(p_j-\alpha\psi) \\
\mathcal{L}_1(1-p_j+\alpha\psi) &= R'(1-p_j+\alpha\psi) \\
\mathcal{L}_2(p_j-\alpha\psi) &= R'(p_j-\alpha\psi) \\
\mathcal{L}_2(p_j+\beta\psi) &= R'(p_j+\beta\psi) \quad .
\end{aligned} \tag{3.4}$$

Then

$$\frac{\alpha}{\alpha+\beta} R'(p_j+\beta\psi) + \frac{\beta}{\alpha+\beta} R'(p_j-\alpha\psi) = \mathcal{L}_2(p_j) \quad . \tag{3.5}$$

By the antisymmetry (2.20) (ii) of R', $\mathcal{L}_1(\frac{1}{2}) = 0$. Therefore, from the concavity of R' on $(0,\frac{1}{2}]$, we have $R'(x) \ge \mathcal{L}_1(x)$ for $p_j - \alpha\psi \le x \le \frac{1}{2}$. It follows that (3.3) will remain valid if we show that $\mathcal{L}_1(x) \ge \mathcal{L}_2(x)$ for $p_j - \alpha\psi \le x \le 1$. Let

$$t^* = -\frac{2p_j + 2\beta\psi - 1}{2p_j - 2\alpha\psi - 1} \quad .$$

As we will show below, $p_j \le (1+\psi(\alpha-\beta))/2$ which implies that $0 \le t^* \le 1$ which together with the convexity of R' on $[\frac{1}{2},1)$ yields

$$\mathcal{L}_1(p_j+\beta\psi) = (1-t^*)R'(\tfrac{1}{2}) + t^*R'(1-p_j+\alpha\psi) \geq R'(p_j+\beta\psi) = \mathcal{L}_2(p_j+\beta\psi) \quad , \tag{3.6}$$

since $(1-t^*)/2 + t^*(1-p_j+\alpha\psi) = p_j+\beta\psi$. The conditions $\mathcal{L}_1(p_j-\alpha\psi) = \mathcal{L}_2(p_j-\alpha\psi)$ and $\mathcal{L}_1(p_j+\beta\psi) \geq \mathcal{L}_2(p_j+\beta\psi)$ imply that $\mathcal{L}_1(x) \geq \mathcal{L}_2(x)$ for $x \geq p_j-\alpha\psi$, which proves (3.3). Therefore, the difference (3.1) is decreasing as long as $p_j \leq (1+\psi(\alpha-\beta))/2$ (and $p_j \leq 1/2$ which is always true since $p_j \leq p_i$ and $p_j \leq 1-p_i$). Using the fact that $\psi = (1-p_j)/p_i$, we may write this condition as

$$p_j(1 + \frac{\alpha-\beta}{2p_i}) \leq \frac{1}{2} + \frac{\alpha-\beta}{2p_i} \tag{3.7}$$

Since $p_i - \beta \geq 0$ by assumption, we have $1 + (\alpha-\beta)/2p_i \geq 0$ which implies that the left hand side of (3.7) is increasing in p_j. For $p_j = 1-p_i$, $\psi = 1$ and the condition becomes $p_j \leq (1+\alpha-\beta)/2$, which is true by assumption. Therefore, the condition holds for $0 \leq p_j \leq 1-p_i$, which completes the proof of the lemma.

The next theorem is a key result in our optimality arguments. This rather surprising result is a generalization of Theorem 2.1.

<u>Theorem</u> 3.1. Let $I_{\delta\gamma}$ be any search strategy in $\mathcal{D}_{\delta\gamma}$. Let $I_{\delta\gamma}(0) = i$, $\alpha > 0$, $\beta > 0$, $p_i(0) + \alpha \leq 1$, $p_i(0) - \beta \geq 0$ and let T be the time at which the process $p_{I_{\delta\gamma}}(t|I_{\delta\gamma})$ crosses one of the boundaries $p_i(0) + \alpha$, $p_i(0) - \beta$. Then

$$\begin{aligned} \Pr(p_{I_{\delta\gamma}}(T|I_{\delta\gamma}) = p_i + \alpha) &= \frac{\beta}{\alpha+\beta} \\ \Pr(p_{I_{\delta\gamma}}(T|I_{\delta\gamma}) = p_i - \beta) &= \frac{\alpha}{\alpha+\beta} \end{aligned} \quad . \tag{3.8}$$

<u>Proof</u>. We first consider the case when $I_{\delta\gamma}$ has one switch; i.e., $I_{\delta\gamma}$ remains constant until $p_{I_{\delta\gamma}}(t|I_{\delta\gamma})$ crosses $p_i+\alpha_1$ or $p_i-\beta_1$ and then remains

constant until $p_{I_{\delta\gamma}}(t|I_{\delta\gamma})$ crosses $p_i + \alpha$ or $p_i - \beta$. By Theorem 2.1 applied twice, the probabilities are

$$
\begin{aligned}
&\Pr(p_{I_{\delta\gamma}}(T|I_{\delta\gamma}) = p_i(0) + \alpha) = \\
&\frac{\beta_1}{\alpha_1+\beta_1} \Pr(p_{I_{\delta\gamma}}(T|I_{\delta\gamma}) = p_i(0) + \alpha - \alpha_1) \\
&+ \frac{\alpha_1}{\alpha_1+\beta_1} \Pr(p_{I_{\delta\gamma}}(T|I_{\delta\gamma}) = p_{I_{\delta\gamma}} + \alpha + \beta) \\
&= \frac{\beta_1}{\alpha_1+\beta_1} \cdot \frac{\beta+\alpha_1}{\alpha+\beta} + \frac{\alpha_1}{\alpha_1+\beta_1} \frac{\beta-\beta_1}{\alpha+\beta} = \frac{\beta}{\alpha+\beta} \quad .
\end{aligned}
\tag{3.9}
$$

Similarly for the other boundary. We proceed by finite induction on the number of switches. Let T_n be the time of the nth switch. Then $I_{\delta\gamma}(t)$ is constant for $T_{n-1} \leq t < T_n$. Let $p_{I_{\delta\gamma}}(T_{n-1}|I_{\delta\gamma})$ take on the values $p^*_{k,n-1}$ with probability $\Pi_{k,n-1}$, $k = 1,\ldots, 2^{n-1}$ (there are at most 2^{n-1} values $p_{I_{\delta\gamma}}(T_{n-1}|I_{\delta\gamma})$ may have). Then

$$
\begin{aligned}
&\Pr(p_{I_{\delta\gamma}}(T_n|I_{\delta\gamma}) = p_i(0) + \alpha) = \\
&\sum_{k=1}^{2^{n-1}} \Pi^*_{k,n-1} \frac{\beta+(p^*_{k,n-1} - p_i(0))}{\beta+\alpha} = \\
&\frac{\beta}{\alpha+\beta} + \sum_{k=1}^{2^{n-1}} \frac{(p^*_{k,n-1} - p_i(0))}{\alpha+\beta} \Pi_{k,n-1} \quad .
\end{aligned}
\tag{3.10}
$$

Here we have used the Markov property of the posterior process $p_i(t|I_{\delta\gamma})$. The proof will be complete if we show that

$$
\sum_{k=1}^{2^{n-1}} (p^*_{k,n-1} - p_i(0)) \Pi_{k,n-1} = 0 \tag{3.11}
$$

For $n = 1$, $p^*_{k,0} = p_i(0)$ and (3.11) is trivial. There is a sequence of constants α_k, β_k, such that $p^*_{k,n}$ may be written in the form $p^*_{j,n-1}+\alpha_j$ and $p^*_{j,n-1}-\beta_j$. Thus (3.11) becomes

$$\sum_{k=1}^{2^{n-1}} \left(\frac{p^*_{k,n-1}+\alpha_k-p_i(0)}{\alpha+\beta}\, \Pi_{k,n-1}\, \frac{\beta_k}{\alpha_k+\beta_k} + \frac{p^*_{k,n-1}-\beta_k-p_i(0)}{\alpha+\beta}\, r_{k,n-1}\, \frac{\alpha_k}{\alpha_k+\beta_k}\right)$$

$$= \sum_{k=1}^{2^{n-1}} \frac{p^*_{k,n-1}-p_i(0)}{\alpha+\beta}\, \Pi_{k,n-1} + \sum_{k=1}^{2^{n-1}} \frac{\alpha_k\beta_k}{\alpha_k+\beta_k}\, \Pi_{k,n-1} - \frac{\alpha_k\beta_k}{\alpha_k+\beta_k}\, \Pi_{k,n-1} \tag{3.12}$$

$$= 0 \ .$$

We have used the induction hypothesis on the first term of the right hand side of (3.12). This completes the proof of the theorem.

We shall have occasion to use the quantity q_i which is the value of the posterior probability for which the i largest posterior probabilities are equal. The q_i are given by (see [5])

$$q_i = \frac{p_{(i)}}{1+(i-1)p_{(i)} - \sum_{j=1}^{i-1} p_{(j)}} \qquad 1 \le i \le n,$$

where $p_{(i)}$ denotes the ordered values of $p_i(0)$. In particular $q_1 = p_{(1)}$ and $q_n = 1/n$. We shall also use the notation $p_{(i)}(t|I_{\delta\gamma})$ to denote the ordered values of the posterior processes.

<u>Lemma</u> 3.3. Let $p_{(1)} > q_2$ and let T be the time at which $p_1(t|I_{\delta\gamma})$ crosses either of the boundaries $1 - \varepsilon$ or q_2. Then $E(T)$ is minimized if $I_{\delta\gamma}$ is constant on $0 \le t \le T$ and equals i^*, the index of the target with largest posterior probability.

Proof. We let $\alpha = 1 - \varepsilon - p_{(1)}$, $\beta = p_{(1)} - q_2$. Then we have $p_{(2)} \leq 1 - p_{(1)}$ and $p_{(2)} \leq q_2$. Therefore, for any $j = i^*$ we have

$$p_{(j)} \leq p_{(2)} \leq \frac{1-p_{(1)}-p_{(2)}}{2} \leq \frac{1-p_{(1)}+q_2}{2} + \frac{1-\varepsilon-p_{(1)}}{2} = \frac{1+\alpha-\beta}{2} \quad . \qquad (3.13)$$

We consider searching only those targets for which $p_j(0) - \psi\alpha \geq 0$ where $\psi = (1-p_j(0))/p_{(1)}$; otherwise exit at the top boundary is not possible and the expected exit time is infinite. Since $I_{\delta\gamma} \in \mathcal{D}_{\delta\gamma}$, it is constant, say k, over a random interval $[0,T_1]$ where $p_k(T_1|I_{\delta\gamma})$ is either $p_k(0) + \delta$ or $p_k(0) - \gamma$. There exist $\delta' > \delta$, $\gamma' > \gamma$ such that $p_{(1)}(t|I_{\delta\gamma})$ crosses $p_{(1)} + \delta'$, $p_{(1)} - \gamma'$ when $p_k(t|I_{\delta\gamma})$ crosses $p_k(0) + \delta$, $p_k(0) - \gamma$. From (3.13) we see that the conditions of Lemma 3.1 are satisfied. Therefore $E(T_1)$ is minimized if $k = i^*$. Since the posterior processes are Markovian,

$$E(T) = E(T_1) + \frac{\gamma'}{\delta'+\gamma'} E(T_\delta) + \frac{\delta'}{\delta'+\gamma'} E(T_\gamma) \qquad (3.14)$$

where T_δ is the exit time starting from $p_{(1)} + \delta'$ and T_γ is the exit time starting from $p_{(1)} - \gamma'$. Next let T_1 be the first time $I_{\delta\gamma} \neq i^*$ and let T_2 be the time after T_1 at which $I_{\delta\gamma}$ switches again. Then the rule $I'_{\delta\gamma}$ where

$$I'_{\delta\gamma}(t) = \begin{cases} i^* & 0 \leq t < T_2 \\ I_{\delta\gamma}(t) & T_2 \leq t \leq T \end{cases} \qquad (3.15)$$

will have a shorter expected exit time than $I_{\delta\gamma}$. This follows from arguments similar to the preceeding ones. Therefore $I_{\delta\gamma}(t) = i^*$ for $0 \leq t < T$ which proves the lemma.

We now state and prove our main result.

Theorem 3.2. The optimal strategy $I_{\delta\gamma} \in \mathcal{D}_{\delta\gamma}$ searches the target i for which $p_i(t) = \max_j p_j(t)$, whenever a switch is permitted.

Proof. We consider three cases:

$$p_{(1)}(0)-\gamma \geq p_2(0), \; p_{(1)}(0) = p_{(2)}(0), \quad \text{and} \quad p_{(2)}(0) > p_{(1)}(0)-\gamma \quad .$$

In the case $p_{(1)}(0)-\gamma \geq p_2(0)$, it follows by Theorem 3.1 that the posterior process $p_{(1)}(t|I_{\delta\gamma})$ will arrive at one of the two boundaries $1 - \varepsilon$ and q_2 with the same probability whatever rule in $\mathcal{D}_{\delta\gamma}$ is used. Furthermore, the posterior distributions at exit time are also independent of the rule used. The Markovian property of the posterior process ensures that the future decisions depend only on the value of the posteriors at exit time. Therefore, we may choose the rule which minimizes the expected exit time. This choice is given by Lemma 3.3 as search the target with the maximum posterior probability.

We consider the case when $p_{(1)}(0) = p_{(2)}(0)$. Henceforth, we abbreviate $p_j(0)$ to p_j. By arguments similar to those of Lemma 3.1, $M(p,\delta,\gamma)$ is decreasing for $p \leq (1+\delta-\gamma)/2$. For any $p_j < p_{(1)} = p_{(2)}$, we have $p_j \leq p_{(1)}$ and $p_j \leq 1 - 2p_{(1)}$. We show that $p_j \leq (1+\delta-\gamma)/2$ by considering the cases $p_{(1)} \leq 1/3$ and $1/3 < p_{(1)} \leq 1/2$. In the first case we have

$$p_j \leq p_{(1)} \leq \frac{1-p_{(1)}}{2} \leq \frac{1+\delta-\gamma}{2}$$

while in the second case, we have

$$p_j \leq 1-2p_{(1)} \leq \frac{1-p_{(1)}}{2} \leq \frac{1+\delta-\gamma}{2} \quad .$$

Therefore, the expected time to the next permissible switch is smallest if we search the target with index i^* whose probability is $p_{(1)}$ (we randomize since there is a tie). Moreover let $I_{\delta\gamma} = j \neq i^*$ and $I^*_{\delta\gamma} = i^*$ until the permissible switching times T_1 and T^*_1 respectively. Then

$$\Pr(p_{(1)}(T_1^*|I_{\delta\gamma}^*) = p_{(1)} + \delta) = \frac{\gamma}{\delta+\gamma}$$

$$\Pr(p_1(T_1^*|I_{\delta\gamma}^*) = p_{(1)} + \gamma\frac{p_{(1)}}{1-p_{(1)}}) = \frac{\delta}{\delta+\gamma}$$

$$\Pr(p_1(T_1|I_{\delta\gamma}) = p_{(1)} + \gamma\frac{p_{(1)}}{1-p_k}) = \frac{\delta}{\delta+\gamma} \tag{3.16}$$

$$\Pr(p_{(1)}(T_1|I_{\delta\gamma}) = p_{(1)} - \delta\frac{p_{(1)}}{1-p_j}) = \frac{\gamma}{\delta+\gamma} \quad .$$

Let T_{11} be the first passage time required for the process $p_{(1)}$ to go from $p_{(1)} + \gamma p_{(1)}/(1-p_{(j)})$ to $p_{(1)} + \gamma\, p_{(1)}/(1-p_{(1)})$ and T_{12} the first passage time required to go from $p_{(1)}(0) - \delta p_{(1)}/(1-p_j)$ to $p_{(1)} + \delta$, T_{21} the first passage time from $p_{(1)} + p_{(1)}/(1-p_{(1)})$ to $1 - \varepsilon$ and T_{22} the first passage time from $p_{(1)} + \delta$ to $1 - \varepsilon$. Now

$$E(T|I_{\delta\gamma}^*) = E(T_1^*) + \frac{\gamma}{\delta+\gamma}\; E(T_{22}) + \frac{\delta}{\delta+\gamma}\, E(T_{21})$$

$$\begin{aligned} E(T|I_{\delta\gamma}) = E(T_1) &+ \frac{\gamma}{\delta+\gamma}\,(E(T_{11}) + E(T_{21})) \\ &+ \frac{\delta}{\delta+\gamma}\,(E(T_{12}) + E(T_{22})) \quad , \end{aligned} \tag{3.17}$$

where T is the total search time. From (3.17) we see that $E(T|I_{\delta\gamma}) \geq E(T|I_{\delta\gamma}^*)$. Considerations similar to the preceeding show that in case $p_{(2)}(0) > p_{(1)}(0) - \gamma$, it is optimal to search the i*th target. The main difference between this case and the case $p_{(1)}(0) - \gamma > p_{(2)}$ is that a switch is not permitted when $p_{(1)}(t) = p_{(2)}(t)$ since $p_{(1)}(t|I_{\delta\gamma})$ will not have decreased by γ. To complete the proof, we note that the three cases cover all possible starting positions and each case terminates with posterior probabilities which are covered by the three cases. We use the

Markovian nature of the posterior process to independently determine the rules at successive switching times. If $u_1, u_2, \ldots$ are the successive switching times, then the expected search time is

$$\sum_{i=1}^{\infty} E(u_i)\, \Pi_i \tag{3.18}$$

where Π_i is the probability of continuing the search after the ith switch. This probability is independent of the rule, by Theorem 3.1. Therefore the search time is minimized by termwise minimizing (3.18). This completes the proof of the theorem.

4. Remarks and Extensions. Our results cover only $\delta\gamma$ strategies. This leaves open the question of optimality for discrete time strategies. It is hoped that this optimality may be obtained as a convergence of the results for $\delta\gamma$ strategies via some sort of convergence.

It also remains to investigate the applications to sequential design of experiments [1]. These results will very likely solve some optimality problems in ranking and selection problems. This relationship however remains to be investigated.

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