On the Use of Analytic Matric Functions in Queueing Theory

by

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INTRODUCTION

Kendall [7] originally suggested the use of a branching process argument for the analysis of the M|G|1 queue, however he did not develop the idea in detail. Independently of Kendall, Neuts also considered and developed this technique, [16] and showed how it can be used in the analysis of several queueing models. The idea of the branching process argument is to group the customers into "generations" in a certain way. To be specific, suppose we consider a single server queueing system with $k \geqslant 1$ customers present at time $t = 0$, and suppose one customer is just starting service. These k customers form the first generation in the branching process and their total service time is termed the lifetime of the first generation. The customers who arrive during this lifetime form the second generation and so on. This technique was applied in [16] to the M|G|1 queue. Further results using the same idea were reported by Neuts [19] and by Neuts and Yadin [20].

When we analyze more complex queueing systems we see how useful the technique is. Neuts [17] studied the system having two servers in series by use of a branching process argument and, using a class of matric functions defined for the first time in [17], found a matric function analogue of Takács' equation [24] for the busy period. In [18] time dependent results for the M|G|1 queue subject to an extraneous phase process were obtained using the same methods.

In both [17] and [18] an equation of the form,

$$(1) \qquad Z = \sum_{n=0}^{\infty} A_n Z^n \quad \|Z\| \leqslant 1$$

where Z, A_n are mxm matrices, was obtained. The existence of a solution to equation (1) was shown in [17, 18] but the uniqueness of the solution was shown only under slightly restrictive conditions by using techniques of complex analysis and matric theory.

Chapter 1 contains the necessary mathematical definitions and theorems for the remainder of this thesis. The main idea is to treat equation (1) as a non-linear operator equation in a certain Banach space. Conditions are then given so that the non-linear operator equation has a unique fixed point.

In Chapter 2 we give some results on semi-Markov matrices - in particular results concerning the Perron-Frobenius eigenvalue [12]. These results were used in [18] and also in Chapter 6 of this thesis.

Chapters 3 and 4 use the results of Chapter 1 to show that equation (1) has a unique solution. In Chapters 5 and 6 we show how the branching process method can be used in two more queueing models. In Chapter 5 we consider a single server queue with Poisson input and semi-Markov service times. This model has been discussed by Neuts [15] and Çinlar [2] but the approach via the branching process is presented here for the first time. We obtain a matrix analogue of Takaçs' Equation and prove that this equation has a unique solution. The time dependent behavior of a Birth and Death process defined on a continuous time Markov chain is discussed in Chapter 6. Naor and Yechiali [14] and Yechiali [25] initiated work on this model and obtained its steady state solution.

CHAPTER I

ANALYTIC MATRIC FUNCTIONS

Neuts [17] introduced a class of matric functions which we will call here "analytic matric functions." These functions arise quite naturally in some queueing models. In this chapter we examine some of the properties of these functions.

Definition 1.1. Let A(z) be an mxm matrix all of whose entries are analytic functions over the unit disc. By A_n we denote the matrix of coefficients of z^n in the Maclaurin expansion of the functions $A_{ij}(z)$. Let B be an mxm matrix such that $\|B\| = \max_i \sum_{j=1}^{m} |B_{ij}| \leq 1$. Then the analytic matric function A[B] is defined by

$$(1) \qquad A[B] = \sum_{n=0}^{\infty} A_n B^n.$$

This definition is consistent since the series in (1) is absolutely convergent.

Such functions do not appear to have been studied in the literature. In fact, Reinhart [23] in his survey of existing definitions of a matric function gives no reference to functions of this type.

Properties of Analytic Matric Functions.

Since our main interest in analytic matric functions is their use in queueing theory, we restrict our discussion primarily to those properties needed therein. However the following theorem is of interest in its own right.

Definition 1.2. Suppose a matrix A has eigenvalues $\lambda_1, \lambda_2 \ldots \lambda_r$. We define the spectral radius of A, $\mu(A)$, by

(2) $$\mu(A) = \max_{1 \leqslant i \leqslant r} |\lambda_i|.$$

We use the symbol I for the unit matrix.

Lemma 1.1.

Let A be an mxm matrix with $\mu(A) < 1$. Then $I - A$ is non-singular and

(3) $$(I-A)^{-1} = \sum_{n=0}^{\infty} A^n .$$

Proof: The proof of this lemma is given in any standard textbook of matrix theory.

Theorem 1.1.

Let A(z) be a matrix whose entries are analytic functions over the unit disc. Let B be an mxm matrix with $\|B\| \leqslant 1$ and let γ be a closed contour lying inside the region $|z| > \mu(B)$ then,

(4) $$A[B] = \frac{1}{2\pi i} \int_\gamma A(z)\ [z\,I-B]^{-1}\, dz.$$

Proof: We note that

(5) $$[z\,I - B]^{-1} = \frac{1}{z}\,[I - \frac{1}{z}B]^{-1}$$

Then, since $\mu(\frac{1}{z}B) = \frac{1}{|z|}\ \mu(B) < 1$, for $z \epsilon \gamma$ we have by lemma 1.1

(6) $$[z\,I-B]^{-1} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{1}{z^n} B^n.$$

Consequently we have,

(7)
$$\frac{1}{2\pi i}\int_\gamma A(z)\,[z\,I-B]^{-1}\,dz$$

$$= \frac{1}{2\pi i}\int_\gamma A(z)\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\,B^n\,dz$$

$$= \sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_\gamma \frac{A(z)dz}{z^{n+1}}\,B^n$$

$$= \sum_{n=0}^{\infty} A_n\,B^n.$$

This completes the proof.

We could now investigate how many of the properties of analytic functions carry over to these matric analytic functions. From a queueing theory viewpoint we would be most interested in extending Rouché's theorem and, hopefully, Lagrange's expansion. We choose not to pursue this approach here. Instead we look upon $A[\cdot]$ as a non-linear operator on a Banach space–the space of mxm complex matrices.

The non-linear operator $A[\cdot]$.

Let,

(8)
$$M = \{Z:\ Z \text{ is an mxm complex matrix}\}.$$

M is a linear space and we can define a norm on M by

(9)
$$\|Z\| = \max_i \sum_{j=1}^{m} |Z_{ij}|.$$

Under this norm, M is a Banach space. Let,

(10)
$$D = \{Z: \|Z\| \leqslant 1,\ Z \epsilon M\}$$

(11)
$$\delta D = \{Z: \|Z\| = 1, Z \epsilon M\}\ .$$

The $D \subset M$ and δD is the boundary of D.

Let A be the non-linear operator defined by,

$$A[Z] = \sum_{n=0}^{\infty} A_n Z^n, \quad Z, A_n \in M. \tag{12}$$

As usual we will define

$$\|A\| = \sup_{\|Z\|=1} \|A[Z]\|. \tag{13}$$

We will assume in the rest of this discussion that $\|A\| \leqslant 1$. This being so we have,

$$A: D \to D. \tag{14}$$

Lemma 1.2.

D is a convex subset of M.

Proof:

Let $X, Y \in D$ and let α and β be real numbers with $\alpha \geqslant 0$, $\beta \geqslant 0$, $\alpha + \beta = 1$. Then,

$$\|\alpha X + \beta Y\| \leqslant \alpha \|X\| + \beta \|Y\| \leqslant \tag{15}$$
$$\leqslant \alpha + \beta = 1.$$

So, $\alpha X + \beta Y \in D$.

Let D^0 denote the interior of D.

Definition 1.3. The Gateaux derivative of A. Suppose there exists a linear mapping $LA(X_0)$, $X_0 \in D^0$ such that

$$\lim_{t \to 0} \frac{1}{t} \{A[X_0 + tX] - A[X_0]\} = LA(X_0) X, \quad X \in M. \tag{16}$$

Then $LA(X_0)$ is called the Gateaux derivative of A at X_0.

Definition 1.4. The Fréchet Derivative of A. If at some point $X_0 \in D^0$,

(17) $$A[X_0 + H] - A[X_0] = dA(X_0,H) + \omega(X_0H)$$

where $dA(X_0,H)$ is a linear operator in H and

(18) $$\lim_{\|H\| \to 0} \frac{\|\omega(X_0,H)\|}{\|H\|} = 0$$

then $dA(X_0,H)$ is called the Fréchet differential of A at X_0. The linear operator $A'[X_0]$ defined by

(19) $$A'[X_0]\,H = dA(X_0,H)$$

is called the Fréchet derivative of A at X_0.

Lemma 1.3.

If the limit in (16) holds uniformly in X, for all X such that $\|X\| = 1$ then

(20) $$LA(X_0) = A'(X_0).$$

Proof: See Vainberg [26] p. 42.

Definition 1.5.

A is <u>continuously Fréchet differentiable</u> if the mapping $X_0 \rightarrow A'(X_0)$, from D to the space of continuous linear operators on M, is continuous.

Theorem 1.2.

The non-linear operator A is continuously Fréchet differentiable at each point $X_0 \in D^0$. The Fréchet derivative is given by

(21) $$A'(X_0)X = \sum_{n=1}^{\infty} A_n \, P_{X_0}^{(n)}(X)$$

where

(22) $$P_{X_0}^{(n)}(X) = \sum_{k=1}^{n} X_0^{n-k} \, X \, X_0^{k-1}.$$

Proof: Consider the limit,

(23) $$\lim_{t\to 0} \frac{1}{t} \{A[X_0 + tX] - A[X_0]\}$$
$$= \lim_{t\to 0} \frac{1}{t} \sum_{n=0}^{\infty} A_n[(X_0 + tX)^n - X_0^n].$$

For t sufficiently small, $X_0 + tX \in D^0$. This implies that the series on the right hand side of (23) is absolutely convergent. The n-th term in this series can be written as,

(24) $$t\, P_{X_0}^{(n)}(X) + \text{terms involving higher powers of } t.$$

We can interchange the order of summation and taking limits. Doing so we get,

(25) $$\lim_{t\to 0} \frac{1}{t} \{A[X_0 + tX] - A[X_0]\}$$
$$= \sum_{n=1}^{\infty} A_n \, P_{X_0}^{(n)}(X).$$

We now define the operator $A'(X_0)$ by

(26) $$A'(X_0)X = \sum_{n=1}^{\infty} A_n \, P_{X_0}^{(n)}(X).$$

$A'(X_0)$ is then a linear operator on M and so is the Gateaux derivative of A at X_0. It remains to be shown that $A'(X_0)$ is the Fréchet derivative. Suppose $\|X\| = 1$. Then we have

(27) $$\|X_0 + tX\| \leqslant \|X_0\| + |t|.$$

So we have that for $t < 1 - \|X_0\|$, $X_0 + tX \in D^0$ for all X such that $\|X\| = 1$. Hence the limit in (23) is uniform in X, $\|X\| = 1$ and so by lemma 1.3., $A'(X_0)$ is the Fréchet derivative of A at X_0. We must show that the mapping $X_0 \to A'(X_0)$ is continuous. Let

(28) $$L(X_0) = A'(X_0) \quad X_0 \in D^0.$$

We prove the following Lemma

Lemma 1.4.

Let $\{X_m, m \geqslant 1\}$ be a sequence of points in D^0 which converge to a point $X_0 \in D^0$. Then

(29) $$\lim_{m\to\infty} L(X_m) = L(X_0).$$

Proof: Let Y be such that $\|Y\| = 1$. Then

(30) $$\lim_{m\to\infty} L(X_m)Y = \lim_{m\to\infty} \sum_{n=1}^{\infty} A_n \sum_{k=1}^{n} X_m^{n-k}\, Y\, X_m^{k-1}.$$

We have the following relationship

(31) $$\left\| \sum_{n=1}^{\infty} A_n \sum_{k=1}^{n} X_m^{n-k}\, Y\, X_m^{k-1} \right\|$$

$$\leqslant \sum_{n=1}^{\infty} m\|A_n\| \quad \|X_m\|^{n-1}.$$

Now for any $\epsilon > 0$ and for sufficiently large m, $\|X_m\| \leqslant \|X_0\| + \epsilon$. By taking now ϵ sufficiently small we have $\|X_0\| + \epsilon < 1$. Therefore the series in (31) is uniformly and absolutely convergent. We can then interchange the limit and summation operation in (30). We obtain

$$\lim_{m\to\infty} L(X_m)Y = \sum_{n=1}^{\infty} A_n \sum_{k=1}^{n} X_0^{n-k}\, Y\, X_0^{k-1} \tag{32}$$

$$= L(X_0)\,Y.$$

Since (32) is true for all Y, $\|Y\| = 1$ it follows that equation (29) holds. Thus we have shown that A is continually differentiable on D^0.

Generalized Contraction mappings and a fixed point theorem.

Following Kirk [9] we define a generalized contraction mapping by,

Definition 1.6. Let B and M be Banach spaces and let A be a non-linear mapping of B into M. Then A is a generalized contraction mapping if for each $X \in B$ there exists a real number $\alpha(X)$, $\alpha(X) < 1$ such that for each $Y \in B$,

$$\|A(X) - A(Y)\| \leq \alpha(X)\, \|X - Y\|. \tag{32}$$

The following two theorems are due to Kirk [9]. Suppose E is a bounded open subset of B and let A map E into B.

Theorem 1.3.

Suppose A is continuously Fréchet differentiable on E. Then A is a generalized contraction mapping if and only if, for each $X_0 \in E$,

$$\|A'(X_0)\| < 1. \tag{33}$$

The next theorem is the key result needed in our subsequent examination of queueing systems.

Let U be an open ball centered at the origin and having positive radius. Let δU denote the boundary of U.

Theorem 1.4.

Suppose A is a generalized contraction mapping on U which maps δU into U. Then A has a unique fixed point in U.

In subsequent chapters we will show how, in a number of queueing models, these theorems may be applied to yield the uniqueness of the solution of interest for the equation for the busy period distribution.

CHAPTER II

SEMI–MARKOV MATRICES

1. Introduction

A natural combination of the theories of stochastic matrices and of distribution functions, which arises in a large number of problems of analytic Probability theory, is the theory of semi-Markov matrices.

Here we wish to consider properties of semi-Markov matrices involving multivariate distributions.

Definition 2.1. k-variate semi-Markov matrix.

Let $Q(\underline{x})$ be an mxm matrix, whose entries are real valued functions defined on R^k such that every entry $Q_{ij}(\underline{x})$ may be written as:

$$Q_{ij}(\underline{x}) = p_{ij}\, F_{ij}(x_1,\ldots,x_k)\,, \tag{1}$$

where $F_{ij}(x_1,\ldots,x_k)$ is a k-variate probability distribution and where $p_{ij} \geqslant 0$, $\sum_{j=1}^{m} p_{ij} = 1$, $i=1,\ldots,m$, then $Q(\underline{x})$ is a k-variate semi-Markov matrix.

We note that if $p_{ij} = 0$, the probability distribution $F_{ij}(\cdot)$ may be arbitrarily chosen.

Definition 2.2. Irreducible semi-Markov matrix.

The semi-Markov matrix $Q(\underline{x})$ is called irreducible if and only if the stochastic matrix $P = \{p_{ij}\}$ is irreducible.

Definition 2.3. Nondegenerate k-variate semi-Markov matrix.

The semi-Markov matrix $Q(\underline{x})$ is nondegenerate k-variate if and only if for every $\nu = 1,\ldots,k$ there exists a pair of indices (i,j) such that $p_{ij} > 0$ and the corresponding distribution $F_{ij}(x_1,\dots,x_k)$ has a marginal distribution $F_{ij}(+\infty,\dots,x_\nu,\dots,+\infty)$ which is not degenerate at zero.

The nondegeneracy condition eliminates the case where one or more of the k-variables $x_1,\dots,x_k$ are actually redundant.

Henceforth we assume that $Q(\underline{x})$ is an irreducible and nondegenerate k-variate semi-Markov matrix.

We now consider the k-dimensional Lebesgue-Stieltjes integrals:

$$\text{(2)} \qquad q_{ij}(\xi_1,\dots,\xi_k) = q_{ij}(\underline{\xi}) = \int_{R^k} \exp[-\sum_{\nu=1}^{k} \xi_\nu x_\nu]\, d_{x_1,\dots,x_k} Q_{ij}(x_1,\dots,x_k),$$

which we refer to as the Laplace-Stieltjes transforms of the entries $Q_{ij}(x_1,\dots,x_k)$ of $Q(\underline{x})$.

The functions $q_{ij}(\xi_1,\dots,\xi_k)$ are obviously defined for Re $\xi_1 = 0,\dots,$ Re $\xi_k = 0$, but they may not be defined anywhere else. We are mainly interested in the cases where the domain of definition of the $q_{ij}(\underline{\xi})$ is larger, as is the case in most applications.

We distinguish the unilateral and the bilateral cases.

In the unilateral case, we assume that all $F_{ij}(x_1,\dots,x_k)$ corresponding to indices i, j such that $p_{ij} > 0$, concentrate all their mass on the positive orthant $x_1 \geqslant 0,\dots,x_k \geqslant 0$. In this case all integrals in (2) exist for all $\underline{\xi}$ with Re $\xi_1 \geqslant 0,\dots,$ Re $\xi_k \geqslant 0$. Moreover all the functions $q_{ij}(\xi_1,\dots,\xi_k)$ are jointly analytic in Re $\xi_1 > 0,\dots,$Re $\xi_k > 0$ and any function obtained by setting some but not all of its variables equal to zero is analytic inside the corresponding part of the boundary of the set Re $\xi_1 > 0,\dots,$ Re $\xi_k > 0$. The latter statement is obvious if we realize that setting one or more, but not all of the ξ–variables equal to zero, corresponds to taking the Laplace-Stieltjes transforms of suitable "marginal" distributions of $Q_{ij}(x_1,\dots,x_k)$.

The bilateral case encompasses all distributions not in the unilateral case.

In our discussion of the bilateral case we shall assume that there exist 2k real numbers ξ_i' and ξ_i'', $i = 1,\dots,k$ such that:

$$\text{(3)} \qquad -\infty \leqslant \xi_i'' < 0 < \xi_i' \leqslant +\infty, \quad i = 1,\dots,k$$

and such that in the "box":

(4) $$\xi_i'' \leqslant \text{Re } \xi_i \leqslant \xi_i', \quad i = 1,\ldots,k,$$

all functions $q_{ij}(\xi_1,\ldots,\xi_k)$ are analytic in $\xi_1,\ldots,\xi_k$.

In order to discuss both cases simultaneously, we shall refer to the domain D in the unilateral case as the open positive orthant $\xi_1 > 0,\ldots,\xi_k > 0$ and in the bilateral case as the box $\xi_1'' \leqslant \xi_1 \leqslant \xi_1',\ldots,\xi_k'' \leqslant \xi_k \leqslant \xi_k'$.

2. The Perron-Frobenius Eigenvalue of $q(\underline{\xi})$.

The matrix $q(\underline{\xi})$ with entries $q_{ij}(\xi_1,\ldots,\xi_k)$ is an irreducible, nonnegative matrix for every real point $\underline{\xi}$ in the domain D or on its boundary. It follows from the classical theory of nonnegative matrices, [6, 12], that $q(\underline{\xi})$ has an eigenvalue of maximum modulus, which is real, positive and of geometric and algebraic multiplicity one. Denoting this, the Perron-Frobenius eigenvalue, by $\rho(\underline{\xi}) = \rho(\xi_1,\ldots,\xi_k)$, we set out to discuss the properties of $\rho(\underline{\xi})$ as a function of $\underline{\xi}$ over the domain D. In the simpler case where k = 1, this was done by H. D. Miller [13].

Lemma 2.1.

All functions $q_{ij}(\underline{\xi})$, $i,j = 1,\ldots,m$ are convex functions over the domain D and its boundary, i.e. for $\underline{\xi}$ and $\underline{\eta}$ in the closure $\bar{D}$, we have:

(5) $$q_{ij}[\alpha\underline{\xi} + (1-\alpha)\,\underline{\eta}] \leqslant \alpha q_{ij}(\underline{\xi}) + (1-\alpha)\, q_{ij}(\underline{\eta})$$

for all $0 \leqslant \alpha \leqslant 1$, and all $i,j = 1,\ldots,m$.

Moreover if $\underline{\xi} \neq \underline{\eta}$ and $0 < \alpha < 1$, strict inequality must hold in (5) for at least one pair (i,j).

Proof:

Since for all real k-tuples $(x_1,\ldots,x_k)$, the function $\exp[-\sum_{\nu=1}^{k} \xi_\nu x_\nu]$ is strictly convex over the domain $\bar{D}$, the inequality (5) follows immediately from the definition of $q_{ij}(\underline{\xi})$.

To prove the next statement we must clearly consider only those pairs (i,j) for which $p_{ij} > 0$. The corresponding Laplace-Stieltjes transform $q_{ij}(\xi_1,\ldots,\xi_k)$ is strictly convex with respect to all the variables which explicitly occur in it. The variables ξ_r which do not explicitly occur in $q_{ij}(\xi_1,\ldots,\xi_k)$ correspond to variables x_r in $F_{ij}(x_1,\ldots,x_k)$ with respect to which the marginal distributions are degenerate at zero.

The nondegeneracy assumption may be restated as saying that every variable ξ_ν, $\nu = 1,\ldots,k$ must occur explicitly in at least one of the functions $q_{ij}(\xi_1,\ldots,\xi_k)$.

Let now $\underline{\xi} \neq \underline{\eta}$. In particular $\xi_\nu \neq \eta_\nu$. Let (i,j) be a pair such that $q_{ij}(\xi_1,\ldots,\xi_k)$ contains ξ_ν explicitly, then for $0 < \alpha < 1$

$$q_{ij}[(1-\alpha)\,\underline{\eta} + \alpha\underline{\xi}] < \alpha q_{ij}(\underline{\xi}) + (1-\alpha)\, q_{ij}(\underline{\eta}),$$

since $q_{ij}(\cdot)$ is jointly strictly convex in all variables upon which it explicitly depends.

Superconvex Matrices.

Let f be a positive function defined on the convex set $\Gamma \epsilon K$. Then f is <u>superconvex</u> if log f is a convex function on Γ. Clearly, f is superconvex if and only if for each $\underline{\xi},\ \underline{\eta} \epsilon \Gamma$,

$$f(\alpha\underline{\xi} + \beta\underline{\eta}) \leqslant [f(\underline{\xi})]^\alpha [f(\underline{\eta})]^\beta \;;\quad \alpha + \beta = 1,\ \alpha \geqslant 0,\ \beta \geqslant 0.$$

Definition 2.4.

A matrix $A(\underline{\xi}) = [A_{ij}(\underline{\xi})]$ is <u>superconvex</u> if for each (i,j), $A_{ij}(\underline{\xi})$ is superconvex on Γ.

The proofs of the following lemmas can be found in reference (8) <u>or</u> (10).

Lemma 2.2.

If f is superconvex on Γ, then it is convex there.

Lemma 2.3.

Let $\gamma(\underline{\xi})$ be any <u>non constant</u> positive linear function on Γ. Then $\gamma(\underline{\xi})$ is not superconvex.

Following Kingman (8) we let C denote the class of all superconvex functions along with the constant functions on Γ.

Lemma 2.4.

$\mathcal{C}$ is closed under addition, multiplication and raising to any positive power. If for each n, $f_n \in \mathcal{C}$, so does $\limsup_{n\to\infty} f_n$

Lemma 2.5.

Let $A(\underline{\xi})$ be a superconvex matrix on Γ and let $\rho(\underline{\xi})$ denote its largest eigenvalue. Then $\rho(\underline{\xi}) \in \mathcal{C}$.

Lemma 2.6.

Let $A(\underline{\xi})$ be a superconvex matrix on Γ and suppose $\rho(\underline{\xi})$ is not a constant function. Then $\rho(\underline{\xi})$ is strictly convex on Γ.

Proof:

By lemma 5, $\rho(\underline{\xi})$ is superconvex on Γ and so it follows from lemma 2 that $\rho(\underline{\xi})$ is also convex on Γ. Suppose now that $\rho(\underline{\xi})$ is in fact linear. Then by lemma 3, since $\rho(\underline{\xi})$ is not constant, $\rho(\underline{\xi})$ is not superconvex. This contradiction implies that $\rho(\underline{\xi})$ is strictly convex on Γ.

Theorem 2.1.

Let $\underline{\xi} = \underline{\sigma} + i\,\underline{\tau}$ where $\underline{\xi} \in D$.

(a) The Perron-Frobenius eigenvalue, $\rho(\underline{\xi})$ is analytic at $\underline{\xi} = \underline{\sigma}$ in the domain D.

(b) $\rho(\underline{\sigma})$ is a strictly convex function of $\underline{\sigma}$ in $\bar{D}$, suitably continuous on the boundary.

Proof:

(a) As in the univariate case, Miller [13], for each real $\underline{\sigma}$, $\rho(\underline{\sigma})$ is a simple root of the determinantal equation $|zI - g(\underline{\sigma})| = 0$. Since $|zI - g(\underline{\sigma})|$ is an analytic function of the k+1 complex variables, z, $\sigma_1, \ldots, \sigma_k$, the result follows from the implicit function theorem for analytic functions.

(b) We need only show that $q_{ij}(\underline{\sigma})$ is a superconvex function for each (i, j). This follows at once since

$$\int_D \exp[-(\alpha\underline{\sigma} + \beta\underline{\sigma}')\cdot\underline{X}]\ dQ(\underline{X}) \leqslant [\int_D \exp[-\underline{\sigma}\cdot\underline{X}]\ dQ(\underline{X})]^{\alpha}\ [\int_D \exp[-\underline{\sigma}'\cdot\underline{X}]\ dQ(\underline{X})]^{\beta}$$

for $\underline{\xi} = \underline{\sigma} + i\,\underline{\tau}$, $\underline{\xi}' = \underline{\sigma}' + i\,\underline{\tau}'$, $\underline{\xi}, \underline{\xi}' \in D$, and $\underline{\sigma}\cdot\underline{X} = \sigma_1 X_1 + \cdots + \sigma_k X_k$. This is just Hölder's inequality for a Banach space with a finite measure. Consequently $q(\underline{\sigma})$ is a superconvex matrix and so $\rho(\underline{\sigma})$ is convex. By lemma 1, $\rho(\underline{\sigma})$ is not constant and so by lemma 6, $\rho(\underline{\sigma})$ is strictly convex on D.

By suitably convex an the boundary $\bar{D}$ we mean that if $\underline{\xi}^* = \underline{\sigma}^* + i\,\underline{\tau}^* \in \bar{D}$ and if $\underline{\xi}_n \rightarrow \underline{\xi}^*$ where $\underline{\xi}_n \in D$ then $\rho(\underline{\sigma}_n) \rightarrow \rho(\underline{\sigma}^*)$. Hence we have $\rho(\underline{\sigma})$ is strictly convex on $\bar{D}$.

The entries of $q(\underline{\xi})$ are all suitably continuous on the boundary and hence $\rho(\underline{\xi})$ is suitably continuous on the boundary, since convergence of a sequence of positive matrices entails convergence of their Perron-Frobenius eigenvalues to that of the limit matrix.

The theorem 2.1 implies in particular that $\rho(\underline{\xi})$ is a continuously differentiable function of $\underline{\xi}$ in D. In the unilateral case one may easily verify that $\rho(\underline{\xi})$ is also suitably differentiable at all boundary points of the positive orthant D, with the possible exception of the origin.

In many applications, see Neuts [15], the quantities

$$M_j = [\frac{\partial}{\partial \xi_j}\rho(\xi_1,\ldots,\xi_k)]_{\underline{\xi}=\underline{0}} \tag{11}$$

play a fundamental role. In the unilateral case, the derivatives at $\underline{0}$ are to be understood in the same "suitable" sense as in theorem 1.

We denote by $\alpha_i^{(\nu)}$, the mean with respect to the variable x_ν of the probability distribution $H_i(x_1,\ldots,x_k)$ defined by:

$$H_i(x_1,\ldots,x_k) = \sum_{j=1}^{m} p_{ij}\, F_{ij}(x_1,\ldots,x_k),\quad i = 1,\ldots,m \tag{12}$$

i.e. $\alpha_i^{(\nu)}$ is given by:

$$\alpha_i^{(\nu)} = \int_{R^k} x_\nu \, d_{x_1,\ldots,x_k} H_i(x_1,\ldots,x_k) \,, \tag{13}$$

provided the integral (13) converges absolutely. In this case $\alpha_i^{(\nu)}$ is also given by:

$$\alpha_i^{(\nu)} = -\left[\frac{\partial}{\partial \xi_\nu} \sum_{j=1}^{m} q_{ij}(\xi_1,\ldots,\xi_k)\right]_{\underline{\xi}=\underline{0}} \tag{14}$$

where the derivative is in the suitable sense in the unilateral case.

Furthermore, let $\pi_1,\ldots,\pi_m$ be the stationary probabilities associated with the matrix P, i.e. the row-vector $\underline{\pi} = (\pi_1,\ldots,\pi_m)$ is the unique solution to the equations:

$$\underline{\pi} = \underline{\pi}P, \quad \underline{\pi}\cdot\underline{e} = 1 \,, \tag{15}$$

where $\underline{e}$ is the columnvector with all its components equal to one.

Theorem 2.2.

The quantities M_j are given by:

$$M_j = -\sum_{i=1}^{m} \pi_i \alpha_i^{(j)} \,. \tag{16}$$

In the unilateral case, this is provided the means $\alpha_i^{(j)}$, $i=1,\ldots,m$ exist. In the bilateral case, our earlier assumptions encompass the existence of these means.

Proof:

Let $\underline{x}(\underline{\xi})$ and $\underline{y}(\underline{\xi})$ be right and left eigenvectors of $q(\underline{\xi})$ corresponding to $\rho(\underline{\xi})$, normalized such that $\underline{y}(\underline{\xi})\cdot\underline{x}(\underline{\xi}) = 1$, and $\underline{y}(\underline{\xi})\cdot\underline{e} = 1$. It is known that such a normalization is possible and uniquely determines $\underline{x}$ and $\underline{y}$ for every $\underline{\xi}$. Moreover as $\underline{\xi}$ tends (suitably) to $\underline{0}$, we have that $\underline{y}(\underline{\xi}) \to \underline{\pi}$ and $\underline{x}(\underline{\xi}) \to \underline{e}$, componentwise. The components of $\underline{x}(\underline{\xi})$ and $\underline{y}(\underline{\xi})$ are (suitably) continuously differentiable functions of $\underline{\xi}$ in $\overline{D}$.

We have that:

$$(17)\qquad \sum_{j=1}^{m} q_{\nu j}(\xi_1,\ldots,\xi_k)\, x_j(\xi_1,\ldots,\xi_k) = \rho(\xi_1,\ldots,\xi_k)\, x_\nu(\xi_1,\ldots,\xi_k),$$

for $\nu = 1,\ldots,m$ and all $\underline{\xi}$ in $\overline{D}$.

Differentiation with respect to ξ_i yields:

$$(18)\qquad \rho(\xi_1,\ldots,\xi_k)\frac{\partial}{\partial \xi_i} x_\nu(\xi_1,\ldots,\xi_k) + x_\nu(\xi_1,\ldots,\xi_k)\frac{\partial}{\partial \xi_i}\rho(\xi_1,\ldots,\xi_k)$$

$$= \sum_{j=1}^{m} x_j(\xi_1,\ldots,\xi_k)\frac{\partial}{\partial \xi_i} q_{\nu j}(\xi_1,\ldots,\xi_k) + \sum_{j=1}^{m} q_{\nu j}(\xi_1,\ldots,\xi_k)\frac{\partial}{\partial \xi_i} x_j(\xi_1,\ldots,\xi_k).$$

Upon letting $\underline{\xi} \to \underline{0}$ (suitably) and noting that $\rho(\underline{0}) = 1$, we obtain:

$$(19)\qquad \left[\frac{\partial}{\partial \xi_i} x_\nu(\underline{\xi})\right]_{\underline{\xi}=\underline{0}} + M_i = -\alpha_\nu^{(i)} + \sum_{j=1}^{m} P_{\nu j}\left[\frac{\partial}{\partial \xi_i} x_j(\underline{\xi})\right]_{\underline{\xi}=\underline{0}}$$

for $\nu = 1,\ldots,m$.

Multiplying by π_ν in (19), summing on ν and applying (15), it follows that:

$$(20)\qquad M_i = -\sum_{\nu=1}^{m} \pi_\nu \alpha_\nu^{(i)}.$$

Remarks:

(1) Formally, the quantities M_i appear in the same manner as the first moment does from the Laplace-Stieltjes transform of a probability distribution. A natural question to ask is whether $\rho(\xi_1,\ldots,\xi_k)$ is itself the transform of a probability distribution. The answer is negative in general. Consider the following example of a 2x2 univariate semi-Markov matrix

$$p_{11} = p_{22} = 0,\ p_{12} = p_{21} = 1.$$

It is easy to see that:

$$\rho(\xi) = [f_1(\xi) \cdot f_2(\xi)]^{1/2},$$

where $f_1(\xi)$ and $f_2(\xi)$ are the Laplace-Stieltjes transforms of the probability distributions $F_{12}(\cdot)$ and $F_{21}(\cdot)$. It is well-known that $f_1(\xi)$ and $f_2(\xi)$ can be chosen so that their product is not the square of a Laplace-Stieltjes transform of a probability distribution, e.g.:

$$f_1(\xi) = e^{-\xi}, \quad f_2(\xi) = \frac{1}{2} + \frac{1}{2} e^{-\xi} .$$

(2) The Results of this chapter represent work done jointly by M. F. Neuts and the author.

CHAPTER III

THE M|G|1 QUEUE SUBJECT TO EXTRANEOUS PHASE CHANGES

1. Introduction

Neuts [18] discussed the M|G|1 queue subject to an extraneous phase change process and, using the branching process argument of [16], showed that the Laplace transform of the busy period matrix must satisfy an equation of the form

$$(1) \qquad Z = A[Z] \qquad \|Z\| \leqslant 1$$

where $A[Z]$ is an analytic matric function. Equation (1) is the transformed version of the system of non-linear integral equations,

$$(2) \qquad \Psi(\theta) = \sum_{n=0}^{\infty} \int_0^{\theta} \Psi^{(n)} (\theta - u)\, d\chi_n(u)$$

where $\Psi(\theta)$ is the busy period matrix and for each n, $\chi_n(u)$ is a sub-stochastic S-M matrix Our interest is in finding a solution which is a (possibly substochastic) semi-Markov matrix. Consequently we require that a solution to equation (1) have all its entries analytic in the right half plane. By applying the results of Chapter 1 we show here that equation (1) has a unique solution in this class of matrices.

2. The Queueing Model

The model under discussion here is an M|G|1 queue governed by an m-state extraneous phase process. The phase process is assumed to be an irreducible, m-state Markov chain in continuous time with only stable states. This Markov chain is fully characterized by its state at t = 0, by the transition probability matrix P of its embedded discrete parameter chain, which is irreducible and stochastic and by the parameters $\sigma_1,\ldots,\sigma_m$ of the negative exponential sojourn times in the states $1,\ldots,m$ respectively - Pyke [22].

The arrival process to the queue is assumed to be an homogeneous Poisson process of rate $\lambda_i \geqslant 0$, during any interval of time that the phase process is in state i, $i = 1,\ldots,m$. The successive service times are conditionally independent given the phase process. A customer who begins service during a phase of type i has a service time distribution $H_i(\cdot)$ of finite mean α_i. A key assumption is this, if a customer begins service during a phase of type i, his service is governed by $H_i(\cdot)$ for its total duration regardless of phase changes occurring during the service time.

The Busy Period.

Suppose that at $t = 0$ a single customer is just entering service and that there are no other customers waiting in line. The length of time until the queue becomes empty again is called the busy period. We now define,

(a) $\psi_{ij}(r,x)$ as the probability that a busy period initiated by a single customer while the phase process is in state i ends before time x, that the phase process is then in state j and that exactly r services are dispensed.

(b) $$\gamma_{ij}(s,w) = \sum_{r=1}^{\infty} \int_0^{\infty} e^{-sx}\, d\, \psi_{ij}(r,x) w^r$$

(c) $P_{ij}(n,x)$ as the probability that we have n arrivals in $(0,x)$ and that after the n-th arrival the phase process is in state j given that the phase process is in state i at $t = 0$.

We let $\psi(r,x)$ denote the matrix $[\psi_{ij}(r,x)]$ and $\gamma(s,w)$ the matrix $[\gamma_{ij}(s,w)]$. We introduce the generating function,

$$\psi(w,x) = \sum_{r=1}^{\infty} \psi(r,x) w^r \tag{3}$$

and the probability,

(4) $$[\chi_n(u)]_{ij} = \int_0^u P_{ij}(n,\theta)\, d\, H_i(\theta).$$

Theorem 3.1.

The generating function $\psi(w,x)$ of the probabilities $\psi_{ij}(r,x)$ must satisfy the matrix non-linear integral equation

(5) $$\psi(w,x) = w \sum_{n=0}^{\infty} \int_0^x \psi(w,x-u)\, d\, \chi_n(u)$$

where $\psi^{(k)}(\cdot,\cdot)$ denotes the k-th convolution of $\psi(\cdot,\cdot)$ with itself.

Proof:

The proof of this is given in Neuts [18].

If we denote $\psi(1,x)$ by $\psi(x)$ then for $w = 1$, equation (5) is a system of non-linear matrix integral equations of Volterra type,

(6) $$\psi(x) = \sum_{n=0}^{\infty} \int_0^x \psi^{(n)}(x-u)\, d\, \chi_n(u).$$

If we now let

(7) $$A_n(s) = \int_0^{\infty} e^{-sx}\, d\chi_n(x)$$

we can write (6) in transformed form as

(8) $$\gamma(s) = \sum_{n=0}^{\infty} A_n(s)\, \gamma^{(n)}(s).$$

We wish to show that equation (6) has a unique solution in the class of (possibly sub-stochastic) semi-Markov matrices. As pointed out in the introduction to this chapter, it suffices to prove the following theorem.

Theorem 3.2.

The equation,

(9) $$Z(s) = A[Z,s] \quad \|Z\| \leqslant 1$$

has a unique solution $Z = \gamma(s)$ with entries analytic in s for Re $s \geqslant 0$.

Proof:

We recall, Chapter 1, that the Fréchet derivative of A at $X_0 \epsilon D^0$ is given by

(10) $$A'(X_0)X = \sum_{n=1}^{\infty} A_n P_{X_0}^{(n)}(X).$$

To show that A is a generalized contraction mapping it sufficies to show that $\|A'(X_0)\| < 1$ for all $X_0 \epsilon D^0$.

Now $$\|A'(X_0)\| = \sup_{\|X\|=1} \|A'(X_0)X\|.$$

However,

(11) $$\|A'(X_0)X\| = \|\sum_{n=1}^{\infty} A_n P_{X_0}^{(n)}(X)\|$$

$$< \sum_{n=1}^{\infty} \|A_n\| \quad \|P_{X_0}^{(n)}(X)\|.$$

But,

(12) $$\|P_{X_0}^{(n)}(X)\| \leqslant n\|X_0\|^{n-1}\|X\|.$$

Consequently we get

(13) $$\|A'(X_0)\| \leqslant \sum_{n=1}^{\infty} n\|A_n\| \quad \|X_0\|^{n-1}.$$

However, since for all $X_0 \epsilon D^0$ we have $\|X_0\| < 1$ it follows that

(14) $$\|A'(X_0)\| \leqslant \sum_{n=1}^{\infty} n\|A_n\|.$$

By equation (4) we have

(15) $$\|A_n(s)\| = \max_i \sum_{j=1}^{m} \int_0^{\infty} e^{-\sigma x} P_{ij}(n,x)\, dH_i(x)$$

where $\sigma = \text{Re } s$.

In order that $\|A'(X_0)\|$ be less than 1 for all $X_0 \epsilon D^0$ it then suffices that

(16) $$\sum_{n=1}^{\infty} n \sum_{i=1}^{m} \sum_{j=1}^{m} \int_0^{\infty} e^{-\sigma x} P_{ij}(n,x)\, dH_i(x) < 1.$$

From which in turn it suffices that

(17) $$\sum_{n=1}^{\infty} n \int_0^{\infty} \sum_{j=1}^{m} e^{-\sigma x} P_{ij}(n,x)\, dH_i(x) < \frac{1}{m} \quad \forall \ i = 1,\ldots,m.$$

Let now

(18) $$K_i(t) = \sum_{j=1}^{m} K_{ij}(t)$$

where:

(19) $$K_{ij}(t) = \sum_{n=1}^{\infty} n\, P_{ij}(n,t).$$

Clearly then $\widetilde{K}_i(t)$ is the expected number of arrivals in a time interval of length t which begins when the phase process is in state i. $\widetilde{K}_i(t)$ is then majorized by t $\max(\lambda_1,\ldots,\lambda_m)$. But this is just the expected number of arrivals during an interval of length t in a Poisson process of rate $\max(\lambda_1,\ldots,\lambda_m)$. By assumption, $H_i(\cdot)$ has a finite mean. So we have

(20) $$\int_0^\infty \tilde{K}_i(t)\, dH_i(t) < \infty \qquad i = 1,\ldots,m.$$

Now let

(21) $$\rho_i(\sigma) = \int_0^\infty e^{-\sigma t}\, \tilde{K}_i(t)\; dH_i(t).$$

Then we have

(a) $$\rho_i(\sigma_1) > \rho_i(\sigma_2), \quad \text{for} \quad \sigma_1 < \sigma_2$$

(b) $$\rho_i(\sigma) \to 0 \qquad \text{as} \quad \sigma \to \infty.$$

So, by choosing σ sufficiently large, we can make $\rho_i(\sigma) < 1/m \quad i = 1,\ldots,m.$

However we note that

(22) $$\rho_i(\sigma) = \sum_{n=1}^{\infty} n \int_0^\infty \sum_{j=1}^{m} e^{-\sigma x}\, P_{ij}(n,x)\, dH_i(x).$$

Consequently there exists σ_1 such that whenever Re $s > \sigma_1$,

(23) $$\|A'(X_0)\| < 1 \qquad \|X_0\| < 1.$$

This means, by Theorem 3, Chapter 1, that $A[Z,s]$ is a generalized contraction mapping for all s such that Re $s > \sigma_1$.

To show that $A[Z,s]$ has a unique fixed point we need only show that $\|A[X,s]\| < 1$ for $\|X\| = 1$. However, for all s with Re $s > 0$

(24) $$A: \; \delta D \to D^0.$$

By Theorem 4, Chapter 1 it then follows that for all s such that Re $s > \sigma_1$ the equation $Z = A[Z,s]$ has a unique solution $Z = \hat{\gamma}(s)$.

We have seen already that $\gamma(s)$, Re $s \geqslant 0$, is a solution whose entries are analytic in s. Since $\gamma(s) = \hat{\gamma}(s)$ for all s such that Re $s > \sigma$ it follows by analytic continuation that $\gamma(s) = \hat{\gamma}(s)$ for all s with Re $s > 0$. For Re $s = 0$, the solution is defined by continuity. So, $\gamma(s)$ is the unique analytic solution for Re $s \geqslant 0$ of the equation

$$Z = A[Z,s] \qquad \|Z\| \leqslant 1 .$$

Outside of its mathematical interest, this result should be most useful in the numerical solution of equations of this type.

CHAPTER IV

TWO SERVERS IN SERIES WITH A FINITE INTERMEDIATE WAITING ROOM

1. Introduction

Neuts [17] used a branching process argument to analyze a queueing system having two servers in series with a finite intermediate waiting room. Using the matric functions in Chapter 1 he found that the distribution of the busy period must satisfy a matrix form of Takaçs' equation. We show here that the equation found by Neuts has a unique solution.

2. The Queueing Model

We summarize here the relevant details of the model in question. For a fuller description we refer to [17].

The system is made up of two servers in series with a finite waiting room in between. Customers arrive at unit I under a homogeneous Poisson process of parametric λ and are served in order of arrival. The successive service times are independent identically distributed random variables with distribution $H(\cdot)$ having finite mean α. The output of the first service unit is fed into the second service unit via the waiting room. A customer who has not yet finished service in unit I is called a I–customer; a customer who has finished service in I but not in unit II is called a II–customer. The finiteness of the waiting room is expressed by the condition – there can never be more than k+1 II–customers in the system at any time. Whenever the number of II–customers reaches k+1 the system becomes blocked.

The service mechanism in unit II depends upon whether or not the system is blocked.

(a) Service in II when the system is unblocked.

Suppose T_1 and T_2 are epochs of arrival in the waiting room and suppose the system is not blocked at T_1. Clearly then no blocking occurs in $(T_1\ T_2)$. In the interval $(T_1\ T_2)$ we assume that the departure process from unit II is a Markovian death process with, possibly, state dependent death rates. Specifically, let $T_1 < t < t+dt < T_2$ and let there be j II–customers present at time t. Then the probability that a II–customer will leave the system in (t, t+dt) is $\sigma_j\, dt$ with $\sigma_0 = 0$ and $\sigma_j > 0 \quad j = 1,\ldots,k$.

(b) Service in unit II when the system is blocked.

Suppose that T is an epoch at which the number of II–customers reaches k+1. The system then becomes blocked. Clearly T is the time of a service completion in unit I. The system remains blocked until some time $T' > T$, where $T' - T$ is stochastically independent of (a) the arrival process, (b) the service process in unit I and (c) conditionally independent of the service process in II prior to T.

We let $H_j(x)$ be the probability that $T' - T$ is at most x and that the number of II–customers at $T' + 0$ is j. We also let

$$(1) \qquad \widetilde{H}(x) = \sum_{j=1}^{k} H_j(x)$$

$\widetilde{H}(x)$ is an honest probability distribution with finite mean $\widetilde{\alpha}$. At $T' + 0$ we assume that the service mechanism in unit II becomes again as outline in (a).

<u>A (k+1) state Markov renewal process, related to the service process in unit II.</u>

Suppose at some instant we have $i \geqslant 1$ customers in unit I, one of which is just beginning service. Let $\tau_0\ \tau_1,\ldots,\tau_{i-1}$ be the epochs at which these customers begin service and let τ_i be the epoch corresponding to the end of the i–th service. Let ζ_n, $n = 0,1,\ldots,i$ be the number of II–customers at $\tau_n + 0$. Then the random variables

$$(2) \qquad (\zeta_0,\ \tau_0 = 0),\ \ (\zeta_1,\ \tau_1 - \tau_0) \cdots (\zeta_i,\ \tau_i - \tau_{i-1})$$

may be regarded as the first i+1 states and sojourn times in a Markov renewal process with (k+1) states. For details of Markov renewal processes the reader is referred to

Pyke [21] and [22] and Çinlar [4]. For a simple Markovian death process, let $P_{ij}(t)$ be the conditional probability that there are j customers present at time t given that there were i present at time 0.

We can now write down the transition matrix of the (k+1) state Markov renewal process. Let

(3) $$R_{rv}(x) = P[\zeta_{n+1} = v,\ \tau_{n+1} - \tau_n \leq x \mid \zeta_n = r].$$

Explicitly then we have

(4) $$R_{rv}(x) = \int_0^x P_{r,v-1}(u)\, dH(u), \quad 0 \leq v-1 \leq r \leq k.$$

(5) $$R_{k+1,v}(x) = \sum_{\nu=v-1}^{k} \int_0^x dH_\nu(x-u) \int_0^u P_{\nu,v-1}(w)\, dH(w).$$

We note that

(6) $$\sum_{v=1}^{k+1} R_{rv}(x) = H(x),\ r = 1,\ldots,k.$$

(7) $$\sum_{v=1}^{k+1} R_{k+1,v}(x) = H * \widetilde{H}(x).$$

We will need these results later on.

The Busy Period for Unit I.

We describe briefly the branching process argument used to analyze the behavior of the queue during a busy period for unit I. At time t = 0 suppose there are i I–customers and r, $1 \leq r \leq k+1$, II–customers in the system and let t = 0 correspond to a service completion in I. The i I–customers are called the 1st generation of customers and the time until they have all become II–customers is the lifetime of the 1st generation. The customers, if any, who join the queue in front of unit I during this lifetime form the 2nd generation. We proceed thus until a generation is reached during whose lifetime no customers arrive. At the end of this generation's lifetime the initial busy period for unit I has come to an end.

Let $\tilde{\xi}_0 = i$ and $\tilde{\zeta}_0 = r$. We define $T_0 = 0$ and T_1 as the total time spent in unit I by the initial i customers. $\tilde{\xi}_1$ will denote the number of I customers present at $T_1 + 0$. If $\tilde{\xi}_1 = 0$ then T_2 is undefined, if $\tilde{\xi}_1 \geq 1$ T_2 is defined as for T_1. ζ_1 is the number of II–customers present at $T_1 + 0$. Proceeding this way we generate the sequence of triples $(\tilde{\xi}_n, \tilde{\zeta}_n, T_n)$ which is a Markov renewal process defined on the state space $(0,1,\ldots) \times (1,2,\ldots,k+1)$. The states $(0,1) \ldots (0,k+1)$ are absorbing states.

The transition matrix is given by

$$(8) \quad Q_{rv}(i,j,x) = P[\tilde{\xi}_{n+1}=j, \tilde{\zeta}_{n+1}=r, T_{n+1} \leq x \mid \tilde{\xi}_n=i, \tilde{\zeta}_n=r] = \int_0^x e^{-\lambda y} \frac{(\lambda y)^j}{j!} d\, R_{rv}^{(i)}(y)$$

for $i \geq 1$, $1 \leq r \leq k+1$, $j \geq 0$, $1 \leq v \leq k+1$. Where

$$R_{rv}^{(i)}(x) = \sum_{\nu=1}^{k+1} \int_0^x R_{r\nu}^{(i-1)}(x-u)\, d\, R_{\nu v}(u).$$

Denoting the corresponding Laplace-Stieljes transforms by $q_{rv}(i,j,s)$ and introducing generating functions we get

$$(9) \qquad q_{rv}^{(i)}(z,s) = \sum_{j=0}^{\infty} q_{rv}(i,j,s)\, z^j = \tilde{R}_{rv}^{(i)}(s+\lambda-\lambda z)$$

where $\tilde{R}_{rv}^{(i)}(s)$ is the Laplace-Stieljes of $R_{rv}^{(i)}(\cdot)$.

We now let $g_{rv}(x)$ be the probability that a busy period initiated in unit I by a single I customer with r II customers present ends before time x with v II customers present. Let $\gamma_{rv}(s)$ be the Laplace transform of $g_{rv}(x)$.

Neuts [17] has shown that $\gamma(s)$, the matrix whose (r,v)-th entry is $\gamma_{rv}(s)$ must satisfy the equation

$$(10) \qquad \gamma(s) = \tilde{R}[(s+\lambda)I - \lambda\gamma(s)] \quad \text{Re } s \geq 0.$$

Our aim is to show that the solution to (10) is unique.

Theorem 1.

The equation

$$(11) \qquad Z = \tilde{R}[(s+\lambda)I-\lambda Z)] \quad \|Z\| \leqslant 1$$

has a unique solution $Z = \gamma(s)$ with entries analytic in s for Re $s \geqslant 0$.

Proof:

Let $\tilde{R}^{(n)}(s)$ be the matrix of coefficients of z^n in the series expansions of the functions $\tilde{R}_{rv}(s+\lambda-\lambda z)$, then equation (11) can be written as

$$(12) \qquad Z = \sum_{n=0}^{\infty} \tilde{R}^{(n)}(s)\, Z^n = \tilde{R}[Z,s].$$

But we know that,

$$(13) \qquad \tilde{R}^{(n)}_{rv}(s) = \int_0^\infty e^{-(s+\lambda)y} \frac{(\lambda y)^n}{n!} d\, R_{rv}(y).$$

We now apply the fixed point theorem of Chapter 1. We begin by showing that $\|\tilde{R}'\| < 1$. As in Chapter 3 it suffices to show that

$$(14) \qquad \sum_{n=1}^{\infty} n \int_0^\infty \sum_{v=1}^{k+1} e^{-(\sigma+\lambda)y} \frac{(\lambda y)^n}{n!} d\, R_{rv}(y) < \frac{1}{m} \quad r = 1,\ldots,k+1$$

where σ = Re s.

Now for $r = 1,\ldots,k$ we have that the left hand side of (14) is

$$(15) \qquad \sum_{n=1}^{\infty} n \int_0^\infty e^{-\sigma y} e^{-\lambda y} \frac{(\lambda y)^n}{n!} dH(y).$$

And for $r = k+1$ we have

(16)
$$\sum_{n=1}^{\infty} n \int_0^{\infty} e^{-\sigma y} e^{-\lambda y} \frac{(\lambda y)^n}{n!} dH * \tilde{H}(y).$$

However, since both $H(\cdot)$ and $H * H(\cdot)$ have finite means, it follows as in Chapter 3 that we can find σ_1 such that for $\sigma > \sigma_1$, $\|A'\| < 1$, i.e. $R[Z,s]$ is a generalized contraction mapping. Again $R: \delta D \to D^0$ for all σ such that $\text{Re}\,\sigma > 0$.

So by Theorem 4, Chapter 1, equation (11) has a unique solution for $\text{Re}\,s > \sigma_1$.

However we already know that $\gamma(s)$ is a solution to (11) with entries analytic in s for $\text{Re}\,s \geqslant 0$. So, by analytic continuation, $\gamma(s)$ is the unique analytic solution to equation (11).

CHAPTER V

THE SINGLE SERVER QUEUE WITH POISSON INPUT AND SEMI–MARKOV SERVICE TIMES

1. Introduction

In this chapter we discuss a variation of a model studied by Neuts [13] and Çinlar [2]. We approach the problem via the branching process argument of [18] and obtain a matrix form of Takaçs' equation. We discuss also the equilibrium conditions for the model.

2. The Queueing Model

We consider a single server system in which there are m customer types. We assume that during a service of type i the arrivals are Poisson with rate λ_i and that a customer arrives at an empty queue after a negative exponential time with parameter λ_0. The customers are served in order of arrival and the successive customer types form an m-state Markov chain which we assume to be irreducible. The successive service times are conditionally independent given the Markov chain and depend only on the transitions occurring within the chain. We let J_n denote the type of the (n+1)st customer and X_n denote the service time of the n-th customer. The sequence

$$\{(J_n, X_n);\ n \geq 0\} \tag{1}$$

forms a Markov renewal sequence with transition probability matrix given by

$$R_{ij}(x) = P[J_n = j,\ X_n \leq x \mid J_{n-1} = i]. \tag{2}$$

For further reference we introduce the matrix A(z,s) with entries defined by

(3) $$A_{ij}(z,s) = \sum_{n=0}^{\infty} \int_0^{\infty} e^{-(s+\lambda_i)u} \frac{(\lambda_i u)^n}{n!} dR_{ij}(u) z^n.$$

We note that $A_{ij}(z,s)$ is just the transform of the probability that, starting at time 0 with a customer of type i entering service, the service ends before time x that we have n arrivals during this service time and that the next customer to enter service is type j. The matrix $A(z,s)$ can be written as $\Gamma(sI + \Lambda - \Lambda z)$ where,

(a) $\Lambda = \text{diag.} (\lambda_1 \ \ldots \lambda_m)$

(b) $\Gamma(sI + \Lambda - \Lambda z) = \int_0^{\infty} e^{-(sI+\Lambda-\Lambda z)u} dR(u).$

A first embedded Markov renewal process.

Let $t_0 = 0, t_1, t_2 \ldots$ be successive epochs at which departures from the queue take place; $J_0, J_1 \ldots$ be the type of the next customer to enter service after $t_0, t_1 \ldots$ and $\xi_0, \xi_1 \ldots$ be the queue length at $t_0 + 0$, $t_1 + 0 \cdots$. As usual we have

(4) $$\xi_{n+1} = (\xi_n - 1 + \gamma_n)^+$$

where γ_n is the number of customers joining the queue during the n-th service time. The sequence

(5) $$\{(J_n, \xi_n, t_n - t_{n-1});\ n \geqslant 0\}$$

is a Markov renewal sequence defined on the state space $(1, \ldots, m) \times (0, 1, 2, \ldots)$. The transition probability matrix for the sequence is given by

(6) $$Q_{ij}(k,k',x) = P[J_{n+1} = j,\ \xi_{n+1} = k',\ t_{n+1} - t_n \leqslant x \mid J_n = i,\ \xi_n = k].$$

These probabilities can be given explicit forms as follows,

(a) For $k' \geqslant k-1 \geqslant 0$

(7) $$Q_{ij}(k,k',x) = \int_0^x e^{-\lambda_i u} \frac{(\lambda_i u)^{k'-k+1}}{(k'-k+1)!} \, d\, R_{ij}(u).$$

To see this we just note that at some time $u \leqslant x$ the first service comes to an end, during $(0,u)$ we have $k'-k+1$ arrivals and the second customer to enter service is of type j.

(b) For $k' < k-1$

(8) $$Q_{ij}(k, k', x) = 0$$

(c) For $k = 0$

(9) $$Q_{ij}(0,k',x) = \int_0^x \lambda_0 e^{-\lambda_0 u} \, Q_{ij}(1,k',x-u)du.$$

Suppose now that at time $t = 0$ we have k customers present and that a customer of type i is just entering service. Let ${}_0Q_{ij}^{(n)}(k,k',x)$ be the probability that the initial busy period involves at least n services, that the n-th service is completed no later than time x, that at the end of the n-th service there are k′ customers in the system and that the (n+1)st customer to enter service is a type j customer, all this conditional on the first customer being of type i and k customers being present initially. We set

(10) $${}_0Q_{ij}^{(0)}(k,k',x) = \delta_{ij} \, \delta_{k,k'} \, U(x)$$

where $U(\cdot)$ is the distribution which is degenerate at zero.

The following relationships are immediate

(11) $${}_0Q_{ij}^{(n+1)}(k,k',x) = \sum_{h=1}^{m} \sum_{\nu=1}^{k+1} \int_0^x {}_0Q_{ih}^{(n)}(k,\nu,x-u) \, d\, Q_{hj}(\nu,k',u).$$

If we let ${}_0q_{ij}^{(n)}(k,k',s)$ denote the Laplace transform of ${}_0Q_{ij}^{(n)}(k,k',s)$ we can write

$$(12)\qquad {}_0q_{ij}^{(n+1)}(k,k',s) = \sum_{h=1}^{m}\sum_{\nu=1}^{k'+1} {}_0q_{ih}^{(n)}(k,\nu,s)\int_0^\infty e^{-(s+\lambda_h)u}\frac{(\lambda_h u)^{\nu-k+1}}{(\nu-k+1)!}\,dR_{hj}(u).$$

We shall need equation (12) in the next section.

The Markov renewal branching process.

In this section we show how the analytic matric functions of Chapter 1 again occur in a queueing model.

Suppose that at time $t = 0$ we have $k \geqslant 1$ customers present and that a type i customer is just entering service. Let T_1 be the epoch when all customers present at time 0 complete service. Let ξ_1 be the number of arrivals during this time period and let I_1 be the type of the next customer to begin service after T_1. If $\xi_1 = 0$ then T_1 is the end of the initial busy period, otherwise we define T_2, ξ_2, I_2 as for T_1, ξ_1, I_1. Continuing in this way we generate the sequence,

$$(13)\qquad \{(I_n, \xi_n, T_n - T_{n-1}); n \geqslant 0\},\ T_0 = 0,\ \xi_0 = k,\ I_0 = i.$$

This is a Markov renewal sequence on the state space $(1,2,\ldots,m) \times (0,1,2,\ldots)$ with states $(j,0)$, $j = 1,\ldots,m$ as absorbing states. The transition probabilities for this sequence are

$$(14)\qquad \tilde{Q}_{ij}(k,k',x) = P[I_{n+1}=j,\ \xi_{n+1}=k',\ T_{n+1}-T_n \leqslant x \mid I_n=i,\ \xi_n=k].$$

We see immediately that

$$(15)\qquad \tilde{Q}_{ij}(k,k',x) = {}_0Q_{ij}^{(k)}(k,k',x).$$

Lemma 5.1.

For $1 \leqslant i, j \leqslant m$, $k \geqslant 1$ and $\text{Re}\, s > 0$, $|z| \leqslant 1$ or $\text{Re}\, s \geqslant 0$, $|z| < 1$ we have,

(16) $$\sum_{k'=0}^{\infty} \tilde{q}_{ij}(k,k's)\,z^{k'} = [A^k(z,s)]_{ij}\,.$$

Proof:

By (15) we have

(17) $$\sum_{k'=0}^{\infty} \tilde{q}_{ij}(k,k',s)\,z^{k'} = \sum_{k'=0}^{\infty} {}_0q_{ij}^{(k)}(k,k',s)\,z^{k'}.$$

Using equation (12) we can write

(18) $$\sum_{k'=0}^{\infty} {}_0q_{ij}^{(n+1)}(k,k',s)\,z^{k'}$$

$$= \sum_{k'=0}^{\infty} \sum_{h=1}^{m} \sum_{\nu=1}^{k'+1} {}_0q_{ih}^{(n)}(k,\nu,s) \int_0^{\infty} e^{-(s+\lambda_i)u} \frac{(\lambda_i u)^{k'-\nu+1}}{(k'-\nu+1)!} dR_{hj}(u)\,z^{k'}$$

$$= \sum_{h=1}^{m} \frac{1}{z} [\sum_{\nu=0}^{\infty} {}_0q_{ih}^{(n)}(k,\nu,s) - {}_0q_{ih}^{(n)}(k,0,s)]\; A_{hj}(z,s).$$

We recall now that

(19) $${}_0q_{ij}^{(1)}(k,k',s) = \int_0^{\infty} e^{-sx}\, d_0Q_{ij}^{(1)}(k,k',x) = \int_0^{\infty} e^{-(s+\lambda_i)x} \frac{(\lambda_i x)}{(k'-k+1)!} dR_{ij}(u).$$

From equation (18) we deduce

(20) $$\sum_{k'=0}^{\infty} {}_0q_{ij}^{(1)}(k,k',s)\,z^{k'} = \sum_{k'=0}^{\infty} \int_0^{\infty} e^{-(s+\lambda_i)x} \frac{(\lambda_i x)}{(k'-k+1)!} dR_{ij}(u)\,z^{k'}$$

$$= z^{k-1} \sum_{k'=0}^{\infty} \int_0^{\infty} e^{-(s+\lambda_i)x} \frac{(\lambda_i x)^{k'-k+1}}{(k'-k+1)!} z^{k'-k+1}\, dR_{ij}(u)$$

$$= z^{k-1} A_{ij}(z,s),$$

Equation (18) along with equation (20) then gives

$$\sum_{k'=0}^{\infty} {}_0q_{ij}^{(n)}(k,k',s)z^{k'} = z^{k-n}[A^n(z,s)]_{ij}\,. \tag{21}$$

The lemma now follows by taking k = n.

Following Neuts [18], we introduce the random variable θ_n defined to be the total number of customers served up to and including the n-th generation. We now define a new probability mass function by

$$ {}_0\tilde{Q}_{ij}^{(n)}(k,k',r,x) = P[I_n=j,\ \xi_n=k',\ \theta_n=r,\ T_n \leqslant x,\ \xi_\nu \neq 0,\ \nu=0,\ldots,n-1 \mid I_0=i,\ \xi_0=k] \tag{22}$$

and denote its Laplace–Stieljes transform by ${}_0\tilde{q}_{ij}^{(n)}(k,k',r,s)$. Let us introduce the generating function

$$ {}_0\phi_{ij}^{(n)}(z,w,s) = \sum_{r=1}^{\infty}\sum_{k'=0}^{\infty} {}_0\tilde{q}_{ij}^{(n)}(k,k',r,s)Z^{k'}W^r\,. \tag{23}$$

Finally we introduce the matrix-functional iterates $A_{(n)}(z,s,w)$ defined by

$$A_{(0)}(z,s,w) = zI \tag{24}$$

$$A_{(n)}(z,s,w) = w\,A[A_{(n-1)}(z,s,w),s]\,.$$

Theorem 5.1.

The matrices ${}_0\phi^{(n)}(z,s,w)$ and $A_{(n)}(z,s,w)$ are related by the equation

$${}_0\phi^{(0)}(z,s,w) = z^k\,I \tag{25}$$

$${}_0\phi^{(n)}(z,s,w) = A^k_{(u)}(z,s,w) - A^k_{(n-1)}(0,s,w)\,.$$

Proof:

The Laplace-Stieljes transforms of the probabilities defined by equation (22) satisfy the relationship

(26) $$ {}_0\tilde{q}^{(n+1)}(k,k',s) = \sum_{h=1}^{m} \sum_{\nu=1}^{r} {}_0\tilde{q}_{ih}^{(n)}(k,\nu,r-\nu,s)\ \tilde{q}_{hj}(\nu,k',s) $$

providing we set ${}_0\tilde{q}_{ij}^{(0)}(k,k',r,s) = \delta_{ij}\ \delta_{kk'}\ \delta_{or}$. Using Lemma 5.1 we may write this more compactly in terms of the generating functions ${}_0\phi_{ij}^{(n)}(z,s,w)$ as

(27) $$ {}_0\phi_{ij}^{(n+1)}(z,s,w) = \sum_{h=1}^{m} \sum_{\nu=1}^{\infty} \sum_{r=1}^{\infty} {}_0\tilde{q}_{ih}(k,\nu,r,s) w^r [w\ A\ (z,s)]_{hj}\ . $$

Equation (27) can be written in more compact form as

(28) $$ {}_0\phi^{(n+1)}(z,s,w) = {}_0\phi^{(n)}[w\ A(z,s),s,w] - {}_0\phi^{(n)}(0,s,w). $$

By induction from equation (28) we finally have

(29) $$ {}_0\phi^{(0)}(z,s,w) = z^k\ I $$

$$ {}_0\phi^{(n)}(z,s,w) = A_{(n)}^{k}(z,s,w) - A_{(n-1)}^{k}(0,s,w). $$

Since the $A_{(n)}$ are, in principle, known functions we have exhibited ${}_0\phi^{(n)}(z,s,w)$ in terms of known quantities.

Corollaries to Theorem 5.1.

Let $\psi_{ij}^{(N)}(k,r,x)$ be the probability that the initial busy period beginning with a customer of type i and having k customers initially present lasts for N generations, involves exactly r services and ends before time x and that the next customer to enter service is of type j. As N tends to infinity, $\psi_{ij}^{(N)}(k,r,x)$ converges to the probability $\psi_{ij}(k,r,x)$ that the busy period with the initial conditions $\xi_0=k$, $I_0=i$ ends before time X, involves a total of r services and that the customer who initiates the next busy period is a type j customer.

Let now

(30) $$ \gamma_{ij}^{(k)}(s,w) = \sum_{r=1}^{\infty} w^r \int_0^{\infty} e^{-sx}\ d\ \psi_{ij}(k,r,x). $$

Corollary 1

The probabilities $\psi(k,r,x)$ have a generating function given by

$$(31) \qquad \gamma_{ij}^{(k)}(s,w) = \lim_{N\to\infty} [A_{(N)}^{k}(0,s,w)]_{ij}.$$

Proof:

Letting z=0 in equation (25) leads to

$$(32) \qquad \sum_{n=1}^{N} {}_{o}\phi^{(n)}(0,s,w) = A_{(N)}^{k}(0,s,w).$$

From which we get

$$(33) \qquad \sum_{r=1}^{\infty} w^{r} \int_{0}^{\infty} e^{-sx}\, d\,\psi_{ij}^{(N)}(k,r,x) = \sum_{n=1}^{N} {}_{o}\phi_{ij}^{(n)}(0,s,w) = [A_{(N)}^{k}(0,s,w)]_{ij}\,.$$

Since

$$(34) \qquad \psi_{ij}(k,r,x) = \lim_{N\to\infty} \psi_{ij}^{(N)}(k,r,x)$$

the result follows from equation (33). We note from Corollary 1 that

$$(35) \qquad \gamma^{(k)}(s,w) = [\gamma^{(1)}(s,w)]^{k}.$$

We will henceforth write $\gamma^{(1)}(s,w)$ simply as $\gamma(s,w)$. We note further that $\gamma(s,1)$ which we will denote by $\gamma(s)$ is the transform of the distribution matrix of the busy period.

Corollary 2

The transform $\gamma(s)$ must satisfy the equation

$$(36) \qquad \gamma(s) = \sum_{n=0}^{\infty} \int_{0}^{\infty} e^{-(sI+\Lambda)u}\, \frac{(\Lambda u)^{n}}{n!}\, dR(u)\, \gamma_{(s)}^{n}, \quad \text{Re } s \geqslant 0.$$

Proof:

Since $A[Z,s]$ is continuous in Z for $\|Z\| \leqslant 1$ this result follows immediately from equation (28).

Theorem 5.2.

The equation

$$(37) \qquad Z = A[Z,s] \qquad \|Z\| \leqslant 1$$

has a unique solution $\gamma(s)$ with entries analytic in s for Re $s \geqslant 0$. This solution is the Laplace–Stieljes transform of the busy period distribution.

Proof:

We recall that

$$(38) \qquad A_{ij}(z,s) = \sum_{n=0}^{\infty} \int_0^{\infty} e^{-(s+\lambda_i)x} \frac{(\lambda_i x)^n}{n!} dR_{ij}(x) z^n.$$

Setting

$$(39) \qquad [A_n(s)]_{ij} = \int_0^{\infty} e^{-(s+\lambda_i)x} \frac{(\lambda_i x)^n}{n!} dR_{ij}(x)$$

we can write equation (36) as

$$(40) \qquad Z = \sum_{n=0}^{\infty} A_n(s) Z^n = A[Z,s].$$

Once again we wish to use Kirk's fixed point theorem of Chapter 1. As before it suffices, for A to be a generalized contraction mapping to show that $\|A'\| < 1$. For which it is sufficient that

$$(41) \qquad \sum_{n=1}^{\infty} n \sum_{j=1}^{m} \int_0^{\infty} e^{-\sigma x} e^{-\lambda_i x} \frac{(\lambda_i x)^n}{n!} dR_{ij}(x) < \frac{1}{m} \text{ for } i = 1,\ldots,m.$$

However we observe that

(42) $$\sum_{n=1}^{\infty} n \int_0^{\infty} e^{-\lambda_i x} \frac{(\lambda_i x)^n}{n!} \sum_{j=1}^{m} dR_{ij}(x) < \infty.$$

(42) holds since

(a) $\sum_{n=1}^{\infty} n\, e^{-\lambda_i x} \frac{(\lambda_i x)^n}{n!}$ is just the expected number of arrivals in $(0,x)$ under a Poisson process of rate λ_i.

(b) $\sum_{j=1}^{m} R_{ij}(x)$ is the probability that the service time of the i-th kind of customer ends before time x. We assume as usual that all the service time distributions have finite means.

The rest of the proof is exactly the same as in Chapter 4 Theorem 1.

Comments

Here again we have shown how the branching type argument can be used to solve some queueing problems. We also note that the queue with semi-Markovian arrivals, see Cinlar [3], can be treated in a similar fashion, since the crucial step in its discussion is the solution of a system of integral equations of similar type.

The Equilibrium Condition.

Let $\pi = (\pi_1, \ldots, \pi_m)$ be the stationary probability distribution for the Markov chain with transition matrix $R(+\infty)$ and let α_i be defined as

(43) $$\alpha_i = \sum_{j=1}^{m} \int_0^{\infty} x \; dR_{ij}(x), \quad i = 1, \ldots, m.$$

α_i is then the mean service time of the i-th type customer. We assume that α_i is finite for $i = 1, \ldots, m$. We now introduce the quantity ρ^* defined by

(44) $$\rho^* = \sum_{i=1}^{m} \pi_i \int_0^{\infty} \lambda_i t \; d\Big(\sum_{j=1}^{m} R_{ij}(t)\Big).$$

As we shall see, ρ^* is the traffic intensity for this model. We note that ρ^* is clearly finite.

Both $A(Z,s)$ and $\gamma(s,w)$ are irreducible, non-negative, (sub)stochastic matrices for $\text{Re } s \geq 0$, $0 \leq z \leq 1$, $0 \leq w \leq 1$. We denote the Perron-Frobenius eigenvalue of $A(Z,s)$ by $\eta(z,s)$, $0 \leq z \leq 1$, $s \geq 0$ and the Perron-Frobenius eigenvalue of $\gamma(s,w)$ by $x(s,w)$, $s \geq 0$, $0 \leq w \leq 1$. The proof of the following lemma can be found in Neuts [18].

Lemma 5.2.

(i) For every $0 \leq z \leq 1$, $s \geq 0$, $\eta(Z,s)$ is uniquely determined and is analytic in (Z,s) for $0 \leq z \leq 1$, $s > 0$ or $0 \leq z < 1$, $s \geq 0$.

(ii) $\eta(Z,s)$ is a convex function of both z and s. For every z, $0 \leq z \leq 1$ it is strictly decreasing in $s \geq 0$ and for every $s \geq 0$ it is strictly increasing in z, $0 \leq z \leq 1$.

(iii) $$\lim_{\substack{z\to 1^- \\ s\to 0^+}} \frac{\partial \eta(z,s)}{\partial z} = \rho^*$$

$$\lim_{\substack{z\to 1^- \\ s\to 0^+}} \frac{\partial \eta(z,s)}{\partial s} = -\sum_{j=1}^{m} \pi_j \alpha_j .$$

Theorem 5.3.

The queue is in equilibrium or equivalently $\gamma(0^+,\Gamma)$ is stochastic if and only if

(45) $$\rho^* = \sum_{i=1}^{m} \pi_i \lambda_i \alpha_i \leq 1.$$

Proof:

The proof is the same as that given in Neuts [18].

CHAPTER VI

A BIRTH AND DEATH PROCESS DEFINED ON A MARKOV CHAIN

1. Introduction

We discuss here an $M|M|1$ queue subject to an extraneous phase process. The phase process is assumed to be an m-state irreducible Markov chain in continuous time. This chain is fully characterized by its initial conditions, by an irreducible stochastic matrix P and by the mean sojourn times $\sigma_1 \ldots \sigma_m$ in each state. During any interval when the phase process is in state i the arrival process is Poisson with parameter λ_i and the service time distribution is exponential with parameter μ_i. Both the service and arrival rates change when the phase process changes state. This feature distinguishes the present model from that discussed in Chapter 3.

Naor and Yechiali [14] initiated work on this model for the case m=2, $p_{11} = p_{22} = 0$ and obtained its steady solution. Yechiali [25] further obtained the steady state solution for general m under the assumption $p_{ii} = 0$, i=1,. . .m. Here we discuss the time dependent behavior of the model without these restrictions. The work is divided into two parts. In part I we give, very briefly, an approach using matrix differential-difference equations while in part 2 we use the more elegant branching process method.

Part 1

Let X(t) be the state of the phase process and Y(t) the number of customers in the system at time t. We define the quantity $P_{ij}(r,t)$ by

$$(1) \qquad P_{ij}(r,t) = P[X(t)=j,\ Y(t)=r \mid X(0)=i].$$

When the phase process is in state i the queueing model forms a birth and death process which has birth rate λ_i and death rate μ_i. For the queueing model a "birth" corresponds to the arrival of a customer while a "death" is the departure of a customer from the system. Because of the properties of birth and death processes, for sufficiently small h the only transitions possible in (0,h) are

(a) a single birth

(b) a single death

(c) neither a birth nor a death.

We let $\widetilde{P}_{ij}(n,h)$ be the probability that in (0,h) we have n arrivals and that at epoch h the phase process is in state j given that X(0)=i. We allow n to be negative in this definition - a negative arrival meaning a departure from the system. The Chapman-Kolmogorov equations may be written

$$(2)\quad \widetilde{P}_{ij}(r,t+h) = \sum_{\nu=1}^{m} [P_{i\nu}(r,t)\widetilde{P}_{\nu j}(0,h) + P_{i\nu}(r+1,t)\widetilde{P}_{\nu j}(-1,h) + P_{i\nu}(r-1,t)\widetilde{P}_{\nu j}(1,h)] + o(h).$$

The following lemmas follow by standard birth and death process arguments and their proofs are therefore omitted.

Lemma 6.1.

For $1 \leqslant i, j \leqslant m$ we have,

$$(3)\qquad \lim_{h\to 0} \frac{1}{h} \widetilde{P}_{ij}(1,h) = \lambda_i \,\delta_{ij}$$

$$(4)\qquad \lim_{h\to 0} \frac{1}{h} \widetilde{P}_{ij}(-1,h) = \mu_i \,\delta_{ij}.$$

Lemma 6.2.

For $1 \leqslant i, j \leqslant m$ we have,

$$(5)\quad \widetilde{P}_{ij}(0,h) = \delta_{ij}[1-(\lambda_i+\mu_i+\sigma_i)h] + \sum_{\rho=1}^{m} \sigma_i P_{i\rho} \widetilde{P}_{\rho j}(h-h\theta,0)h + o(h)$$

where $0 < \theta < 1$.

We introduce now the following mxm matrices,

$$P(r,t) = [P_{ij}(r,t)]$$

$$\Lambda = [\delta_{ij}\ \lambda_j]$$

$$M = [\delta_{ij}\ \mu_j]$$

$$\Sigma = [\delta_{ij}\ \sigma_j]\ .$$

Theorem 6.1.

The differential-difference equations for this system are,

(6) $$\frac{\partial P(0,t)}{\partial t} = P(1,t)M - P(0,t)[\Sigma(I-P)+\Lambda]$$

(7) $$\frac{\partial P(r,t)}{\partial t} = P(r+1,t)M + P(r-1,t)\Lambda - P(r,t)[\Sigma(I-P)+\Lambda+M], \quad r \geqslant 1\ .$$

Proof:

By equations (2), Lemma 6.1 and Lemma 6.2 it follows, for $r \geqslant 1$, that

(8) $$\lim_{h\to 0} \frac{1}{h} [P_{ij}(r,t+h) - P_{ij}(r,t)] = \lambda_j P_{ij}(r-1,t)+\mu_j P_{ij}(r+1,t)-(\lambda_j+\mu_j+\sigma_j)P_{ij}(r,t) + \sum_{\nu=1}^{m} \sigma_\nu P_{\nu j} P_{i\nu}(r,t).$$

Equation (7) is just equation (8) written in matrix form.

We now note that when r=0 the death rate is zero. Hence,

(9) $$P_{ij}(0,t+h) = \sum_{\nu=1}^{m} [P_{i\nu}(0,t)\widetilde{P}_{\nu j}(0,h) + P_{i\nu}(1,t)\widetilde{P}_{\nu j}(-1,h)] + o(h).$$

Equation (6) now follows by the same argument used for equation (7).

The steady-state conditions.

As pointed out in [14], $\lim_{t\to\infty} P(r,t)$ exists for all r. We let now,

(10) $$P(r) = \lim_{t\to\infty} P(r,t).$$

The steady state equations for the system are

(11) $$P(0)[\Sigma(P-I)-\Lambda] + P(1)M = 0$$

(12) $$P(r)[\Sigma(P-I)-\Lambda-M] + P(r-1)\Lambda+P(r+1)M=0, \quad r \geqslant 1.$$

Equations (11) and (12) are the matrix forms of the steady-state equations of Yechiali [25].

Let $\pi_1^*,\ldots,\pi_m^*$ be the steady state probabilities for the matrix P and let

$$P_j(r) = \lim_{t\to\infty} P[X(t)=j, \ Y(t)=r].$$

Then we have (Yechiali, [25])

(13) $$\sum_{r=0}^{\infty} P_j(r) = {}^{\pi_j^*/\sigma_j}/ \sum_{k=1}^{m} (\pi_k^*/\sigma_k).$$

Finally, let $\hat{\mu} = \sum_{j=1}^{m} \mu_j \, (\pi_j^*/\sigma_j)$

$$\hat{\lambda} = \sum_{j=1}^{m} \lambda_j \, (\pi_j^*/\sigma_j).$$

Then we have

Theorem 6.2.

The steady-state condition for this system under the assumption $p_{ii} = 0$, $i=1,\ldots,m$ is

$$\hat{\mu} - \hat{\lambda} > 0. \tag{14}$$

Proof:

This is just a re-statement of Yechiali's condition.

Solution of the differential-difference equations (6) and (7).

Let $\quad \pi(z,t) = \sum_{r=0}^{\infty} P(r,t)\, z^r, \qquad |z| \leqslant 1.$

$$A(z) = z\Lambda + \frac{1}{z} M - [\Sigma(I-P)+\Lambda+M] .$$

We recall here the definition of $e^{A(z)t}$

$$e^{A(z)t} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n(z). \tag{15}$$

Theorem 6.3.

The system of equations defined by equations (6) and (7) has a solution whose generating function is

$$\pi(z,t) = z^k e^{A(z)t} - \left(\frac{1}{z} - 1\right) \int_0^t e^{A(z)(t-u)} P(0,u)\, du \cdot M . \tag{16}$$

The solution is given in terms of P(0,u) which consequently we must find in some other way.

Proof:

From the definition of $\pi(z,t)$ it follows that

$$\frac{\partial \pi(z,t)}{\partial t} = \sum_{r=0}^{\infty} \frac{\partial P(r,t)}{\partial t} z^r . \tag{17}$$

So, by using equations (6) and (7) we obtain

$$(18)\quad \frac{\partial\pi(z,t)}{\partial t} = \frac{1}{z}[\pi(z,t) - P(0,t)]M + Z\pi(z,t)M - \pi(z,t)[\Sigma(I-P)+\Lambda+M] + P(0,t)M .$$

Which we can rewrite as

$$(19)\quad \frac{\partial\pi(z,t)}{\partial t} = \pi(z,t)A(z) - (\frac{1}{z} - 1)P(0,t)M.$$

Following Loynes [11], the solution to equation (19) is

$$(20)\quad \pi(z,t) = \pi(z,0)e^{A(z)t} - (\frac{1}{z} - 1)\int_0^t e^{A(z)(t-u)}P(0,u)du\, M.$$

Now $\pi(z,0) = \sum_{r=0}^{\infty} P(r,0)z^r$ and if we suppose that k customers are initially present it follows that

$$(21)\quad \pi(z,0) = z^k I.$$

Consequently,

$$(22)\quad \pi(z,t) = z^k e^{A(z)t} - (\frac{1}{z} - 1)\int_0^t e^{A(z)(t-u)}P(0,u)du\, M.$$

In principle we could proceed to find P(0,t) by conditioning on the epochs of phase changes in (0,t). However, we will not pursue this here since the methods of part 2 enable us to discuss this model more elegantly.

Part II

1. The Arrival and Service Process.

Let $S_{ij}(n,t)$ be the probability that a service which begins at time 0 ends before time t, that n arrivals take place during this service time and that the phase process is in state j at the completion of the service, given that X(0)=i. These probabilities satisfy the equation,

$$(23)\qquad S_{ij}(n,t) = \delta_{ij} \int_0^t \mu_i e^{-(\lambda_i+\sigma_i+\mu_i)u} \frac{(\lambda_i u)^n}{n!} du$$

$$+ \sum_{\nu=1}^{m} \sum_{h=0}^{n} \sigma_i P_{i\nu} \int_0^t e^{-(\sigma_i+\lambda_i+\mu_i)u} \frac{(\lambda_i u)^{n-h}}{(n-h)!} S_{\nu j}(h,t-u)du.$$

Our first problem is to show that this system of equations has a unique bounded solution.

Lemma 6.3.

The system of equations defined by equation (23) has a unique bounded solution.

Proof:

We define the generating functions $S_{ij}(z,t)$ by

$$(24)\qquad S_{ij}(z,t) = \sum_{n=0}^{\infty} S_{ij}(n,t)z^n \quad |z| \leq 1$$

and the Laplace transform of $S_{ij}(z,t)$ by

$$(25)\qquad S'_{ij}(z,s) = \int_0^{\infty} e^{-st}\, dS_{ij}(z,t).$$

In terms of these transforms we may write equation (23) in transformed version as

$$(26)\qquad S'_{ij}(z,s) = \delta_{ij}\ \mu_i(\lambda_i+\mu_i+\sigma_i-\lambda_i z+s)^{-1}$$

$$+ \sum_{\nu=1}^{m} \sigma_i P_{i\nu}(\lambda_i+\mu_i+\sigma_i-\lambda_i z+s)^{-1} S'_{\nu j}(z,s) .$$

We now introduce the following mxm matrices

$$(27)\qquad S'(z,s) = [S'_{ij}(z,s)]$$

(28) $$\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_m)$$

(29) $$\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_m)$$

(30) $$M = \text{diag}(\mu_1, \ldots, \mu_m).$$

In terms of these matrices we can write equation (26) as

(31) $$S'(z,s) = M[\Lambda+\Sigma+M-z\Lambda + zI]^{-1} + [\Lambda+\Sigma+M-z\Lambda + sI]^{-1}\Sigma\ P\ S'(z,s).$$

Rearranging terms we obtain

(32) $$\{I-[\Lambda+\Sigma+M-z\Lambda + sI]^{-1}\Sigma\ P\}\ S'(z,s) = M[\Lambda+\Sigma+M-z\Lambda+ sI]^{-1}.$$

In order to solve equation (32) for $S'(z,s)$ we need to show that $I-[\Lambda+\Sigma+M-z\Lambda+sI]^{-1}\Sigma\ P$ is non-singular. We note to begin that

$$\left|[(sI+\Sigma+M+\Lambda-z\Lambda)^{-1}\Sigma\ P]_{ij}\right| = \frac{\sigma_i\ P_{ij}}{|s+\sigma_i+\lambda_i+\mu_i-\lambda_i z|} \leqslant P_{ij}$$

with strict inequality holding for some (i, j). Consequently the matrix of interest has a spectral radius which is less than the spectras radius of P. Since $\mu(P) = 1$ we then have

(33) $$\mu[(sI+\Sigma+M+\Lambda-z\Lambda)^{-1}\Sigma\ P] < 1.$$

So, by lemma 1.1, $I-[sI+\Lambda+\Sigma+M-z\Lambda]^{-1}\Sigma\ P$ is non-singular. It then follows that

(34) $$S'(z,s) = [I-(sI+\Lambda+\Sigma+M-z\Lambda)^{-1}\Sigma\ P]^{-1}\cdot(sI+\Lambda+\Sigma+M-z\Lambda)^{-1}M.$$

Rearranging equation (34) leads to

$$(35) \qquad S'(z,s) = (sI+\Sigma+M+(1-z)\Lambda-\Sigma P)^{-1}M.$$

Equation (35) implies that the entries $S'_{ij}(z,s)$ are analytic functions of z and s in the region Re $s \geq 0$, $|z| < 1$ or Re $s > 0$, $|z| \leq 1$. This completes the proof of lemma 6.3.

2. A first embedded Markov renewal process.

Let $t_0=0, t_1, t_2, \ldots$ be the successive epochs in which departures from the queue occur. Let $\xi_0, \xi_1, \ldots, J_0, J_1, \ldots$ be respectively the queue lengths and the states of the phase process at $t_0+, t_1+\cdots$. If we let ν_n be the number of customers joining the queue during the service time of the n-th customer it follows that

$$(36) \qquad \xi_{n+1} = (\xi_n - 1 + \nu_n)^+ .$$

The random variables ν_n depend upon the past only through the phase state at t_n+. It follows that the sequence

$$(37) \qquad \{(J_n, \xi_n, t_n - t_{n-1}), n \geq 0\} \quad t_{-1} = 0.$$

is a Markov renewal sequence on the state space $(1,2,\ldots,m) \times (0,1,\ldots)$. The transition matrix of this sequence is defined by

$$(38) \qquad Q_{ij}(k,k',x) = P[J_{n+1}=j, \xi_{n+1}=k', t_{n+1}-t_n \leq x \mid J_n=i, \xi_n=k].$$

For $k' \geq k-1 \geq 0$ we have

$$(39) \qquad Q_{ij}(k,k',x) = \int_0^x d\,S_{ij}(k'-k+1,u).$$

While for $k' < k$ we obtain

$$(40) \qquad Q_{ij}(k,k',x) = 0.$$

3. Transitions within a busy period.

Suppose we have $k \geqslant 1$ customers present at $t=0$. Let ${}_0Q_{ij}^{(n)}(k,k',x)$ be the probability that the initial busy period involves at least n services that the n-th service is completed not later than time x, that at the time of the n-th departure k′ customers are in the system and the phase process is in state j; all this conditional on $J_0=i$, $\xi_0=k$. For convenience we set ${}_0Q_{ij}^{(0)}(k,k',x) = \delta_{ij}\,\delta_{kk'}\,U(x)$. We immediately have the following relationship

$$(41) \qquad {}_0Q_{ij}^{(n+1)}(k,k',x) = \sum_{h=1}^{m} \sum_{\nu=1}^{k'+1} \int_0^x {}_0Q_{ih}^{(n)}(k,\nu,x-u)\,d\,Q_{hj}(\nu,k',u).$$

If we denote the Laplace–Stieljes transform of ${}_0Q_{ij}^{(n)}(k,k',x)$ by ${}_0q_{ij}^{(n)}(k,k',s)$ it follows that

$$(42) \qquad {}_0q_{ij}^{(n+1)}(k,k',s) = \sum_{h=1}^{m} \sum_{\nu=1}^{k'+1} {}_0q_{ih}^{(n)}(k,\nu,s) \int_0^\infty e^{-su}\,d\,S_{hj}(k'-\nu+1,u)\,.$$

This relationship is required in the next section.

4. The Markov renewal branching process.

The queueing model under consideration can be analyzed by the argument detailed in Neuts [18]. We describe an imbedded discrete parameter process as follows. Suppose there are initially k customers present one of whom is just beginning service. Let T_1 be the time when all of these customers have completed service. These k customers form the first generation in a branching process and T_1 is their total lifetime. We denote the number of arrivals in $(0,T_1)$ by ξ'_1 and the state of the phase process at T_1+ by J'_1. We also set $\xi'_0=k$, $T_0=0$ and $J'_0=i$. If $\xi'_1=0$ then T_1 is the end of the initial busy period. If $\xi'_1 > 0$ we let T_2 be the time when all customers present at T_1+ have departed and J'_2, ξ'_2 are respectively the state of the phase process and the queue length at that time. Continuing in this way we define the sequence

$$(43) \qquad \{J'_n, \xi'_n, T_n - T_{n-1});\ n \geqslant 0,\ T_{-1}=0\}$$

which is a Markov renewal sequence on the state space $(1,\ldots,m) \times (0,1,\ldots,)$. The state $(j,0)$, $j = 1,\ldots,m$ are absorbing.

The transition probabilities for this sequence are

$$(44) \qquad \tilde{Q}_{ij}(k,k',x) = P[J'_{n+1}=j, \xi'_{n+1}=k', T_{n+1}-T_n \leq x \mid J'_n=i, \xi_n=k].$$

These probabilities may also be written

$$(45) \qquad \tilde{Q}_{ij}(k,k',x) = {}_0Q_{ij}^{(k)}(k,k',x).$$

The following lemma expresses the generating function of the Laplace–Stieljes transforms $\tilde{q}_{ij}(k,k',s)$ in terms of the matrix $S'(z,s)$.

Lemma 6.4.

For $1 \leq i, j \leq m$, $k \geq 1$ and $\operatorname{Re} s > 0$, $|z| \leq 1$ or $\operatorname{Re} s \geq 0$ $|z| < 1$

$$(46) \qquad \sum_{k'=0}^{\infty} \tilde{q}_{ij}(k,k',s)z^{k'} = \{[S'(z,s)]^k\}_{ij}.$$

Proof:

By (45) we obtain

$$(47) \qquad \sum_{k'=0}^{\infty} \tilde{q}_{ij}(k,k',s)z^{k'} = \sum_{k'=0}^{\infty} {}_0q_{ij}^{(k)}(k,k',s)z^{k'}.$$

From equation (42) it follows that

$$(48) \qquad \sum_{k'=0}^{\infty} {}_0q_{ij}^{(n+1)}(k,k',s)z^{k'}$$

$$= \sum_{h=1}^{m} \sum_{\nu=1}^{\infty} \sum_{k'=\nu-1}^{\infty} {}_0q_{ih}^{(n)}(k,\nu,s) \int_0^{\infty} e^{-su}\, d\,S_{hj}(k'-\nu+1,u)z^{k'}$$

$$= \sum_{h=1}^{m} \sum_{\nu=1}^{\infty} {}_0q_{ih}^{(n)}(k,\nu,s) \sum_{k'=\nu-1}^{\infty} \int_0^{\infty} e^{-su}\, d\, S_{hj}(k'-\nu+1,u) z^{k'-\nu+1} z^{\nu-1}$$

$$= \sum_{h=1}^{m} \delta_{hj}(z,s) \frac{1}{z} \{ \sum_{\nu=0}^{\infty} {}_0q_{ih}^{(n)}(k,\nu,s) z^{\nu} - {}_0q_{ih}^{(n)}(k,0,s) \}.$$

We will now show that

(49) $$\sum_{k'=0}^{\infty} {}_0q_{ij}^{(1)}(k,k',s) z^{k'} = z^{k-1} S'_{ij}(z,s).$$

Proof:

Since

(50) $${}_0q_{ij}^{(1)}(k,k',s) = \int_0^{\infty} e^{-sx}\, d\, S_{ij}(k'-k+1,x)$$

it follows that

(51) $$\sum_{k'=0}^{\infty} {}_0q_{ij}^{(1)}(k,k',s) z^{k'} = \sum_{k'=0}^{\infty} \int_0^{\infty} e^{-sx}\, d\, S_{ij}(k'-k+1,x) z^{k'} = z^{k-1} S'_{ij}(z,s).$$

It then follows by equation (48) that

(52) $$\sum_{k'=0}^{\infty} {}_0q_{ij}^{(n)}(k,k',s) z^{k'} = z^{k-n} \{ [S'(z,s)]^n \}_{ij}.$$

The lemma then follows by setting k=n in (52).

Let now θ_n denote the total number of customers served up to time T_n, provided T_n is defined. We introduce the probability,

(53) $${}_0\widetilde{Q}_{ij}^{(n)}(k,k',r,x) = P[J'_n=j,\ \xi'_n=k',\ \theta_n=r,\ T_n \leq x,\ \xi'_\nu \neq 0,\ \nu \leq n-1 \mid J'_0=i,\ \xi'_0=k]$$

and denote its Laplace–Stieljes transform by ${}_0\tilde{q}_{ij}^{(n)}(k,k',r,s)$. Let

(54) $${}_0\phi_{ij}^{(n)}(z,s,w) = \sum_{r=1}^{\infty} \sum_{k'=0}^{\infty} {}_0\tilde{q}_{ij}^{(n)}(k,k',r,s) z^{k'} w^r$$

and let ${}_0\phi^{(n)}(z,s,w)$ be the matrix whose entries are defined by equation (54). Finally we define the matrix–functional iterates $S_{(n)}(z,s,w)$ by

(55) $$S_{(0)}(z,s,w) = z\,I$$

$$S_{(n+1)}(z,s,w) = w\,S'[S_{(n)}(z,s,w),s] \ .$$

We recall here the definition on $S[Z,s]$

(56) $$S'[Z,s] = \sum_{n=0}^{\infty} S_n(s) Z^n$$

where

(57) $$S'(z,s) = \sum_{n=0}^{\infty} S_n(s) z^n \ .$$

Theorem 6.4.

For $n \geqslant 1$, $k \geqslant 1$,

(58) $${}_0\phi^{(0)}(z,s,w) = z^k\,I$$

$${}_0\phi^{(n)}(z,s,w) = S_{(n)}^{k}(z,s,w) - S_{(n-1)}^{k}(0,s,w).$$

Proof:

We begin by noting that

(59) $${}_0\tilde{q}_{ij}^{(n+1)}(k,k',r,s) = \sum_{h=1}^{m} \sum_{\nu=1}^{r} {}_0\tilde{q}_{ih}^{(n)}(k,\nu,r-\nu,s)\, \tilde{q}_{hj}(\nu,k',s)$$

if we let ${}_0\tilde{q}_{ij}^{(0)}(k,k',r,s) = \delta_{ij}\,\delta_{0r}\,\delta_{kk'}$.

In terms of the generating functions ${}_0\phi_{ij}^{(n)}(z,s,w)$ we can write

(60) $${}_0\phi_{ij}^{(n+1)}(z,s,w) = \sum_{h=1}^{m} \sum_{\nu=1}^{\infty} \sum_{r=1}^{\infty} {}_0\tilde{q}_{ih}^{(n)}(k,\nu,r,s) w^r \{w^\nu [S'(z,s)]^\nu\}_{hj} \, ;$$

In matrix form equation (60) becomes

(61) $${}_0\phi^{(n+1)}(z,s,w) = {}_0\phi^{(n)}[w S'(z,s),s,w] - {}_0\phi^{(n)}[0,s,w]$$

and also

(62) $${}_0\phi^{(0)}(z,s,w) = z^k \, I \, .$$

The theorem now follows by induction.

This is the key result needed for the remainder of this ayalysis.

5. The joint distribution of the busy period and the number of customers served during it.

Let $\psi_{ij}(k,r,x)$ be the probability that the busy period with initial conditions $\xi_0'=k$, $J_0'=i$, ends before time x during a phase of type j and involves a total of r services. The transform of this probability we will define by

(63) $$\gamma_{ij}^{(k)}(s,w) = \sum_{r=1}^{\infty} w^r \int_0^{\infty} e^{-sx} d\, \psi_{ij}(k,r,x).$$

Theorem 6.5.

The matrix $\gamma^{(k)}(s,w)$ with entries $\gamma_{ij}^{(k)}(s,w)$ Re $s \geqslant 0$, $|w| < 1$ <u>or</u> Re $s > 0$ $|w| \leqslant 1$, is given by

(64) $$\gamma^{(k)}(s,w) = \lim_{N\to\infty} S_{(N)}^{k}(0,s,w).$$

Proof:

On setting z=0 in equation (58) it follows that

$$(65) \qquad \sum_{n=1}^{N} {}_0\phi^{(n)}(0,s,w) = S^k_{(N)}(0,s,w).$$

We denote by $\psi_{ij}^{(N)}(k,r,x)$ the probability that the initial busy period with k customers and in state i initially lasts for N generations and ends before time x during a phase of type j and involves exactly r services. Then,

$$(66) \qquad \sum_{r=k}^{\infty} w^r \int_0^{\infty} e^{-sx} d\, \psi_{ij}^{(N)}(k,r,x) = \sum_{n=1}^{N} {}_0\phi_{ij}^{(n)}(0,s,w) = [S^k_{(N)}(0,s,w)]_{ij}.$$

Now we also have

$$(67) \qquad \lim_{N \to \infty} \psi_{ij}^{(N)}(k,r,x) = \psi_{ij}(k,r,x).$$

On taking limits in equation (66) the results follows.

We note that equation (64) implies that

$$(68) \qquad \gamma^{(k)}(s,w) = [\gamma^{(1)}(s,w)]^k.$$

We will denote $\gamma^{(1)}(s,w)$ simply by $\gamma(s,w)$ henceforth.

Corollary 1.

The matrix $\gamma(s,w)$ must satisfy the equation

$$(69) \qquad \gamma(s,w) = w\, S[\gamma(s,w),s]$$

for $\text{Re}\, s > 0$, $|w| \leq 1$ <u>or</u> $\text{Re}\, s \geq 0$, $|w| < 1$.

Proof:

The proof follows immediately by equations (58) and (64).

If we let $\gamma(s)$ be the matrix whose entries are $\gamma_{ij}(s)$ where $\gamma_{ij}(s)$ is $\gamma_{ij}(s,1-)$ we have,

Corollary 2.

The matrix $\gamma(s)$ must satisfy the equations

(70) $$\gamma(s) = S[\gamma(s),s].$$

Proof:

This follows immediately by Corollary 1.

We note here that these corollaries are the matrix analogues of the corresponding results for the M|M|1 queue. Indeed we obtain the M|M|1 queue as a special case on taking m=1. For m=1 we have, by equation (35)

(71) $$S[\gamma(s,w),s] = \{s+\mu+[1-\gamma(s,w)]\lambda\}^{-1}\mu .$$

(72) $$S[\gamma(s),s] = \{s+\mu+[1-\gamma(s,w)]\lambda\}^{-1}\mu .$$

Then we have

Corollary 1A.

For the M|M|1 queue the transform $\gamma(s,w)$ must satisfy the equation

(73) $$\gamma(s,w) = w\ \mu\ \{s+\mu+[1-\gamma(s,w)]\lambda\}^{-1} .$$

Corollary 2A.

For the M|M|1 queue the transform $\gamma(s)$, of the busy period must satisfy the equation

(74) $$\gamma(s) = \mu\{s+\mu+[1-\gamma(s)]\lambda\}^{-1} .$$

We now turn our attention to the matrix $S(1-,0+)$. We show that this is a stochastic matrix.

Lemma 6.5.

The matrix S(1−,0+) is a stochastic matrix.

Proof:

By equation (35),

(75) $$S(1-,0+) = [\Sigma+M-\Sigma P]^{-1}M.$$

As is easily seen by writting out the entries,

(76) $$[\Sigma+M-\Sigma P]\ \underline{e} = M\ \underline{e}$$

so, consequently we have

(77) $$[\Sigma+M-\Sigma P]^{-1}M\ \underline{e} = \underline{e}\,.$$

Hence,

(78) $$S(1-,0+)\ \underline{e} = \underline{e}$$

and since all of its entries are non-negative S(1−,0+) is therefore a stochastic matrix.

6. The Equilibrium Condition.

Let $\Pi = (\pi_1, \ldots, \pi_m)$ be the vector of stationary probabilities corresponding to the irreducible stochastic matrix S(1−,0+). We define the quantity ρ^* by,

(79) $$\rho^* = \sum_{n=1}^{\infty} n \int_0^{\infty} \sum_{i=1}^{m} \pi_i\, d\, S_{i.}\,(n,x).$$

On examination we see that ρ^* may be interpreted as the mean number of arrivals during a suitably defined "average" service. Theorem 6.6 gives us an explicit expression for ρ^* in terms of the original parameters of the queue.

Theorem 6.6

The quantity ρ^* is given by,

$$(80) \qquad \rho^* = \sum_{j=1}^{m} \pi_j \left(\frac{\lambda_j}{\mu_j}\right) .$$

Proof:

We begin by showing that,

$$(81) \qquad \rho^* = \Pi\,[\Sigma+M-\Sigma P]^{-1}\,\Lambda\,\,[\Sigma+M-\Sigma P]^{-1}\,M\,\underline{e}\,.$$

From equations (79) and (35) it follows that,

$$(82) \qquad \rho^* = \Pi \left[\left.\frac{\partial S'(z,s)}{\partial z}\right|_{\substack{z=1- \\ s=0+}} \right] \underline{e}\,.$$

In order to evaluate the derivative term in thes expression we write

$$(83) \qquad \left.\frac{\partial S'(z,s)}{\partial z}\right|_{\substack{z=1- \\ s=0+}} = \lim_{z\to 1} \frac{1}{1-z}\,[S'(1-,0+) - S'(z,0+)]$$

$$= \lim_{z\to 1} \frac{1}{1-z}\,\{[\Sigma+M-\Sigma P]^{-1} - [\Sigma+M+(1-z)\Lambda-\Sigma P]^{-1}\}\cdot M$$

We can rearrange this last expression to give,

$$(84) \qquad \lim_{z\to 1} \frac{1}{1-z}\,\{[\Sigma+M-\Sigma P]^{-1}[I-(\Sigma+M-\Sigma P)(\Sigma+M+(1-z)\Lambda-\Sigma P)^{-1}]\}\,.$$

We next note that,

$$(85) \qquad I-(\Sigma+M-\Sigma P)(\Sigma+M+(1-z)\Lambda-\Sigma P]^{-1} = I-\{I+(1-z)\Lambda\,(\Sigma+M-\Sigma P)^{-1}\}^{-1}.$$

To express the second term in equation (85) in terms of a power series we must show that, for z sufficiently close to 1, the spectral radius of $(1-z)\Lambda(\Sigma+M-\Sigma P)^{-1}$ is less than 1. However,

(86) $$\|(1-z)\Lambda(\Sigma+M-\Sigma P)^{-1}\| \leq |1-z|\ \|\Lambda(\Sigma+M-\Sigma P)^{-1}\|.$$

Now, since $\|\Lambda(\Sigma+M-\Sigma P)^{-1}\| < \infty$ by taking z sufficiently close to 1, it follows that,

(87) $$\|(1-z)\Lambda(\Sigma+M-\Sigma P)^{-1}\| < 1.$$

Now for any matrix A, $\mu(A) \leq \|A\|$ and hence the required result follows from equation (87). Expanding as a power series gives,

(88) $$\{I + (1-z)\Lambda(\Sigma+M-\Sigma P)^{-1}\}^{-1} = \sum_{n=0}^{\infty} (1-z)^n \frac{(-1)^n}{n!} \{\Lambda(\Sigma+M-\Sigma P)^{-1}\}^n$$

Returning to equation (83) we now have,

(89) $$\begin{aligned} &\lim_{z\to 1} \frac{1}{1-z} \{I-(\Sigma+M-\Sigma P)[\Sigma+M+(1-z)\Lambda-\Sigma P]^{-1}\} \\ &= \lim_{z\to 1} \frac{1}{1-z} \{I - \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(1-z)^n \{\Lambda(\Sigma+M-\Sigma P)^{-1}\}^n \\ &= \Lambda(\Sigma+M-\Sigma P)^{-1}. \end{aligned}$$

Consequently, by equation (82) we can write,

(90) $$\rho^* = \Pi(\Sigma+M-\Sigma P)^{-1}\Lambda(\Sigma+M-\Sigma P)^{-1}M\,\underline{e}\,.$$

We recall that $S'(1-,0+)$ is given by

(91) $$S'(1-,0+) = (\Sigma+M-\Sigma P)^{-1}M\,.$$

So consequently,

(92) $$(\Sigma+M-\Sigma P)^{-1}M\,\underline{e} = \underline{e}\,.$$

Further, the vector Π is given by

(93) $$\Pi = \Pi[\Sigma + M - \Sigma P]^{-1} M$$

and so,

(94) $$\Pi M^{-1} = \Pi[\Sigma + M - \Sigma P]^{-1}.$$

Thus we can rewrite equation (90) as

(95) $$\rho^* = \Pi M^{-1} \Lambda \underline{e}\ .$$

From equation (95) it follows immediately that,

(96) $$\rho^* = \sum_{j=1}^{m} \pi_j \ \lambda_j/\mu_j\ .$$

The two main results of this chapter are stated in the next two theorems.

Theorem 6.7

The equation,

(97) $$Z = S[Z,s] \qquad \|Z\| \leqslant 1$$

has a unique solution $\gamma(s)$ whose entries are analytic in s for $\mathrm{Re}\ s \geqslant 0$.

Proof:

The proof follows the same lines as the proof of Theorem 3.2 and will therefore be omitted.

Theorem 6.8

The queueing system is in equilibrium or equivalently, the matrix $\gamma(0+,1-)$ is stochastic if and only if,

(98) $$\rho^* = \sum_{j=1}^{m} \pi_j \left(\frac{\lambda_j}{\mu_j}\right) \leqslant 1.$$

Proof:

We denote the Perron–Froberius eigenvalue of

$$S'(z,s) \text{ by } \eta(z,s) \qquad 0 \leqslant z \leqslant 1, \quad s \geqslant 0$$

and the Perron–Froberius eigenvalue of $\gamma(s,w)$ by $\chi(s,w)$. By the results given in chapter 2 we have the following,

(a) For $0 \leqslant z \leqslant 1$, $s \geqslant 0$, $\eta(z,s)$ is uniquely determined and is analytic in (z,s) for $0 \leqslant z < 1$, $s \geqslant 0$ or $0 \leqslant z \leqslant 1$, $s > 0$.

(b) $\eta(z,s)$ is convex jointly in z and s. For every z, $0 \leqslant z \leqslant 1$, it is a strictly decreasing function of $s \geqslant 0$ and for every $s \geqslant 0$ it is a strictly increasing function of z in $0 \leqslant z \leqslant 1$.

(c) $$p^* = \lim_{\substack{z \to 1- \\ s \to 0+}} \frac{\partial \eta(z,s)}{\partial z}.$$

Similar results hold for $\chi(s,w)$. We also have

(d) The limit $\chi(0+,1-)$ exists and is the Perron–Froberius eigenvalue of $\gamma(0+,1-)$.

To complete the proof of the theorem we need only show that $\chi(0+,1-) = 1$ if and only if $\rho^* \leqslant 1$. The proof of this fact follows exactly the same lines as the proof of Theorem 3 given in Neuts [18]. We summarize this proof without repeating all the details.

i) For each s and w with $s > 0$, $0 < w \leqslant 1$ and $s \geqslant 0$, $0 < w < 1$ there is a unique abcissa $z = \chi_o(s,w)$ such that

(99) $$\chi_o(s,w) = \eta[\chi_o(s,w), s] \qquad 0 < \chi_o(s,w) < 1.$$

ii) For $s = 0$, $w = 1$, the value $z = 1$ is always a solution. By continuity $\chi_o(0+,1-)$ is also a solution which may or may not be identical with $z = 1$; this depends upon the value of the derivative of $\eta(z,o)$ at $z = 1-$ i.e. on the value of ρ^*. For $\rho^* > 1$ we have $\chi_o(0+,1-) < 1$ while for $\rho^* \leqslant 1$, $\chi_o(0+,1-) = 1$.

iii) Finally it is established that

$$\chi_0(s,w) = \chi(s,w); \quad s \geqslant 0, \quad 0 \leqslant w \leqslant 1.$$

Thus we have shown that the queue is in equilibrium if and only if $\rho^\circ \leqslant 1$. Theorem 6.8 then follows immediately from Theorem 6.6.

7. Some special cases.

We now show that the equilibrium condition for the $M|M|1$ queue, the Naor and Yechiali model and finally, the Yechiali model all follow from Theorem 6.8.

1. The $M|M|1$ queue.

The $M|M|1$ queue is obtained as a special case of our model for "lxl" matrices since then $\Lambda = \lambda$, $M = \mu$, $\Pi = 1$.

2. The Naor–Yechiali model.

In this case we have only a 2 state Markov chain and our equilibrium condition can then be written as

$$\pi_1\left(\frac{\lambda_1}{\mu_1}\right) + \pi_2\left(\frac{\lambda_2}{\mu_2}\right) \leqslant 1. \tag{100}$$

where (π_1, π_2) is the vector of stationary probabilities for the stochastic matrix $S(1-,0+)$. In this case,

$$S(1-,0+) = \begin{bmatrix} \sigma_1+\mu_1 & -\sigma_1 \\ -\sigma_2 & \sigma_2+\mu_2 \end{bmatrix}^{-1} \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}. \tag{101}$$

Elementary calculations then show that,

$$\pi_1 = \frac{\mu_1\sigma_2}{\mu_1\sigma_2+\mu_2\sigma_1} \tag{102}$$

$$\pi_2 = \frac{\mu_2\sigma_1}{\mu_1\sigma_2+\mu_2\sigma_1}.$$

Equation (100) can now be written as

(103) $$\frac{\lambda_1\sigma_2 + \lambda_2\sigma_1}{\mu_1\sigma_2 + \mu_2\sigma_1} \leqslant 1$$

which is the equilibrium condition given by Naor and Yechiali (14).

3. The Yechiali model.

Yechiali (25) discussed the same model as did Naor and Yechiali (14) in the case of an m–state Markov chain with transition matrix P with $p_{ii} = 0$, $i=1 \ldots m$. If we let Π^* be the vector of stationary probabilities for P, Yechialis' equilibrium condition can be written as,

(104) $$\sum_{i=1}^{m} \pi_i^* \left(\frac{\lambda_i}{\sigma_i}\right) / \sum_{i=1}^{m} \pi_i^* \left(\frac{\mu_i}{\sigma_i}\right) < 1.$$

To show that this inequality reduces to an inequality of the form

(105) $$\sum_{i=1}^{m} \pi_i \left(\frac{\lambda_i}{\mu_i}\right) < 1$$

we will express π^* in terms of π.

Lemma 6.6

The vectors Π^* and Π are connected by

(106) $$\Pi^* = \Pi M^{-1}\Sigma\,(\Pi M^{-1}\Sigma\,\underline{e})^{-1}.$$

Proof:

By definition we have,

(107) $$\Pi = \Pi\,[\Sigma+M-\Sigma P]^{-1}M.$$

• Consequently,

(108) $$\Pi M^{-1}(\Sigma+M-\Sigma P) = \Pi.$$

On expanding the left hand side of equation (108) we get,

(109) $$\Pi M^{-1}\Sigma = \Pi M^{-1}\Sigma P$$

and, since $\Pi M^{-1}\Sigma \underline{e} \neq 0$ we get,

(110) $$(\Pi M^{-1}\Sigma \underline{e})^{-1}\Pi M^{-1}\Sigma = (\Pi M^{-1}\Sigma \underline{e})^{-1}\Pi M^{-1}\Sigma P$$

(111) $$(\Pi M^{-1}\Sigma \underline{e})^{-1} \Pi M^{-1}\Sigma \underline{e} = 1.$$

Therefore,

(112) $$\Pi^* = (\Pi M^{-1}\Sigma \underline{e})^{-1}\Pi M^{-1}\Sigma .$$

This proves the lemma.

We have immediately from Lemma 6.6,

(113) $$\pi_i^* = \pi_i(\frac{\sigma_i}{\mu_i}) [\sum_{i=1}^{m} \pi_i(\frac{\sigma_i}{\mu_i})]^{-1} .$$

On substituting for π_i^*,

(114) $$\sum_{i=1}^{m} \pi_i^* (\frac{\lambda_i}{\mu_i}) / \sum_{i=1}^{m} \pi_i^* (\frac{\lambda_i}{\mu_i}) = \sum_{i=1}^{m} \pi_i (\frac{\lambda_i}{\mu_i}) .$$

We would also comment that the equilibrium condition written in the form of equation (105) is to be preferred to that given in equation (104) since it shows more clearly the relationship of this model to the M|M|1 queue.

BIBLIOGRAPHY

[1] Bailey, N. T. J. A continuous time treatment of a simple queue using generating functions. J. Roy. Stat. Soc. Ser. B **16**, 288–291 (1954).

[2] Çinlar, E. Time dependence of Queues with Semi–Markovian services. J. Appl. Prob. **4**, 356–364 (1967).

[3] Çinlar, E. Queues with Semi–Markovian arrivals. J. Appl. Prob. **4**, 365-379 (1967).

[4] Çinlar, E. Markov Renewal Theory, Adv. in Appl. Prob. **1**, 123–187 (1969).

[5] Conolly, B. W. A difference equation technique applied to the simple queue. J. Roy. Stat. Soc. Ser. B **20**, 165–167 (1958).

[6] Karlin, S. (1966). A first course in Stochastic Processes. Academic Press, New York and London.

[7] Kendall, D. G. Some problems in the theory of queues. J. Roy. Stat. Soc. Ser. B **13**, 151–185 (1951).

[8] Kingman, J. F. C. A convexity property of positive matrices. Quar. J. Math. Ox. (2) **12**, 283–284.

[9] Kirk, W. A. Mappings of generalized contractive type. J. Math. Anal. Appl. **32**, 567–572 (1970).

[10] Klinger, A. and Mangasarian, O. L. Logarithmic convexity and Geometric programming. J. Math. Anal. and Appl. **24**, 388–408.

[11] Loynes, R. M. A continuous time treatment of certain queues and infinite dams. J. Aust. Math. Soc. **2**, 484–498 (1961).

[12] Marcus, M. and Ming, H. (1964). A survey of Matrix theory and Matrix inequalities. Allyn and Bacon, Boston.

[13] Miller, H. D. A convexity property in the theory of random variables defined on a finite Markov chain. Ann. Math. Stat. **32**, 1260–1270.

[14] Naor, P. and Yechiali, U. Queueing problems with heterogeneous arrivals and service. Ops. Res. **19**, November 3, 1971.

[15] Neuts, M. F. The single server queue with Poisson input and semi–Markov service times. J. Appl. Prob. **3**, 202–230 (1966).

[16] Neuts, M. F. The queue with Poisson inputs and general service times, treated as a branching process. Duke M. J. **36**, 215–232 (1969).

[17] Neuts, M. F. Two servers in series, studied in terms of a Markov Renewal branching process. Adv. in Appl. Prob. **2**, 110–149 (1970).

[18] Neuts, M. F. A queue subject to extraneous phase changes. Adv. in Appl. Prob. **3**, 78–119 (1971).

[19] Neuts, M. F. The joint distribution of the virtual waitingtime and the residual busy period for the M|G|1 queue. J. Appl. Prob. **5**, 224–229 (1968).

[20] Neuts, M. F. and Yadin, M. The transient behavior of the queue with alterating priorities, with special reference to the waitingtimes. Bull. Soc. Math. Belg. **20**, 343–376 (1968).

[21] Pyke, R. Markov Renewal Processes: definitions and preliminary properties. Ann. Math. Stat. **32**, 1231–1242.

[22] Pyke, R. Markov Renewal Processes with finitely many states. Ann. Math. Stat. **32**, 1243–1259.

[23] Rinehart, R. F. The equivalence of definitions of a matric function. A.M.M. **62**, 395–413 (1955).

[24] Takács, L. (1962). Introduction to the theory of queues. Oxford Univ. Press.

[25] Yechiali, U. A queueing type birth and death process defined on a continuous time Markov chain. Unpublished manuscript.

[26] Vainberg, M. (1964). Variational methods for the study of non-linear operators. Holden–Day, San Francisco.

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In this thesis we define an analytic matric function $F[\cdot]$ by $F[X] = \sum_{n=0}^{\infty} F_n X^n$ where F_n, X are mxm complex matrices. In Chapter 1 we discuss a fixed point theorem for such functions and then in subsequent chapters we analyze various queueing models where these occur.

The basic problem in each of the queueing models analyzed reduces to the solution of a non-linear matrix integral equation of Volterra type. By use of the fixed point theorem of Chapter 1 we show that the non-linear integral equation has a unique solution. The complete transient behavior of each queue may be expressed in terms of the solution of the integral equation. For each model, the equilibrium condition is also determined.

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