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On the Asymptotic Distribution of the Maximum of Sums of a Random Number of I.I.D. Random Variables. II

by

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1. INTRODUCTION. Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of independent and identically distributed (i.i.d.) random variables taking on real values. Let for $n = 0,1,2,\ldots$, $S_n = \sum_{i=0}^{n} X_i$, where $S_0 = X_0 = 0$. We shall be concerned here with the random variables

$$\eta_n = \max(0, S_1, S_2, \ldots, S_n),\ n = 0,1,2\ldots \quad . \tag{1}$$

We assume that $E|X_n| < \infty$ and write $a = EX_n$. Let

$$\eta = \lim_{n\to\infty} \eta_n = \sup_{0\leq n<\infty} S_n \quad . \tag{2}$$

The random variable η is nonnegative, but possibly improper. We shall call the process $\{\eta_n\}$ subcritical, critical and supercritical according as $a < 0$, equal to zero and $a > 0$, respectively. We shall assume that $P(X_n = 0) < 1$, for in the trivial case where $P(X_n = 0) = 1$, we have $P(\eta_n = 0) = 1$, for all n. We summarize in the following few known asymptotic results converning η_n. The exact distribution of η_n is of course covered by the celebrated Spitzer's identity [12].

(i) In the subcritical case, $P(\eta < \infty) = 1$, whereas in the remaining cases

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(7) $$\xi_n = \max(0, S_n - S_{n-1}, \ldots, S_n) = S_n + \max(0, S_1, -S_2, \ldots, -S_n),$$

so that

(8) $$\frac{\xi_n - na}{\sqrt{n}} = \frac{S_n - na}{\sqrt{n}} + \frac{\max(0, -S_1, -S_2, \ldots, -S_n)}{\sqrt{n}} .$$

Now using the fact that $\max(0,-S_1,-S_2,\ldots,-S_n)$ corresponds to a subcritical process, (taking $-X_n$'s instead of X_n's) it follows from (i) that $\max(0,-S_1,-S_2,\ldots,-S_n)$ tends in law to a proper random variables, so that the last term of (8) tends to zero in probability as $n \to \infty$. Hence the theorem follows from (8) by using the central limit theorem.

The aim of the present paper is to establish the above asymptotic results for $\eta_{\nu(n)}$ as $n \to \infty$, where $\nu(n)$ is a positive integer-valued random variable for $n \geq 1$, such that $\nu(n)/f(n)$ converges in probability to a positive random variable ν as $n \to \infty$, for some positive sequence $\{f(n)\}$ with $f(n) \to \infty$, as $n \to \infty$. For the case, where it is assumed that for any $n \geq 1$, $\nu(n)$ is independent of the random variables η_n $(n = 1,2,\ldots)$ the above results are easy to establish. However, in the present work we make no such assumption.

A central limit theorem type result in this direction was originally established by Anscombe [1] under a condition of uniform continuity in probability of the random variables involved. However there it was assumed that $\nu(n)/n$ tends in probability to a positive constant. Rényi [9] gave a simpler proof of Anscombe's theorem for the special case of simple sum of i.i.d. random variables and established a central limit theorem. Later in [11], Rényi generalized his result to the case where $\nu(n)/n$ tends in probability to a positive discrete random variable. He, however, conjectured that his result holds more generally even when $\nu(n)/n$ tends in probability to an arbitrary positive random variable. The validity of the conjecture was almost simultaneously established by Blum, Hanson and Rosenblatt [3] and by Mogyoródi [6]. Reader may also like to refer in this connection to the papers of Barndorff-Nielson [2] and Mogyoródi [7].

The author is grateful to Professor LeCam for some helpful discussions and in particular for drawing his attention to Rényi's work. He is equally grateful to the referee for providing other references, in particular [3] and [6], which led the author to strengthen his earlier results, where it was assumed that $\nu(n)/f(n)$ tends in probability to a positive constant (See Puri [8]).

2. SOME PRELIMINARY RESULTS. During the course of developments of later sections, we shall need the notion of strongly mixing sequences of events (see Rényi [10]). Let $(\Omega,\mathcal{Q},P)$ be a probability space. Then a sequence A_n $(n = 0,1,2,...)$ of events is called strongly mixing with density $\alpha (0 < \alpha < 1)$ if for any event $B \varepsilon \mathcal{Q}$, we have

$$\lim_{n\to\infty} P(A_n B) = \alpha P(B) . \tag{9}$$

However it is evidently sufficient to restrict only to those events B with $P(B) > 0$, in which case the sequence A_n shall be strongly mixing with density α if

$$\lim_{n\to\infty} P(A_n|B) = \alpha, \tag{10}$$

for every B with $P(B) > 0$. Following theorem, due to Rényi [10], gives a necessary and sufficient condition for a sequence of events to be strongly mixing.

THEOREM 2: The sequence A_n of events, such that $A_0 = \Omega$ and $P(A_n) > 0$ $(n = 1,2,\ldots)$, is strongly mixing with density α if and only if

$$\lim_{n\to\infty} P(A_n|A_k) = \alpha, \tag{11}$$

for $k = 0,1,2,\ldots$, where $0 < \alpha < 1$ and α does not depend on k.

As also observed by Rényi (see footnote, page 217 [10]), the assumption $P(A_n) > 0$, $n = 1,2,\ldots$, made in the above theorem, is not an essential restriction. In fact, according to the definition given above, in a strongly mixing sequence of events there can occur only a finite number of events having the probability zero, which can be omitted without any loss of the strongly mixing character of the sequence.

We shall call a sequence $\xi_n (n = 1,2,\ldots)$ of random variables a mixing sequence with the limiting distribution function $F(\cdot)$ if for every $B \in \mathcal{G}$ with $P(B) > 0$ and for every real x, which is a point of continuity of $F(\cdot)$, we have

$$\lim_{n\to\infty} P(\xi_n \le x|B) = F(x) . \tag{12}$$

We now have the following theorem to be needed later.

THEOREM 3: Let $a = 0$ and $EX_n^2 = 1$. Then the sequence $\{\eta_n/n^{1/2}\}$ is a mixing sequence with the limiting distribution given by (3).

PROOF. In view of theorem 2, it is sufficient to prove that for any $x > 0$, and for any $k \ge 1$, with $P(\eta_k \le x\, k^{1/2}) > 0$,

$$\lim_{n\to\infty} P(\eta_n \le x\, n^{1/2}|\eta_k \le x\, k^{1/2}) = 2\,\Phi(x) - 1 . \tag{13}$$

Note that because of (3), $P(\eta_k \leq x\, k^{1/2}) > 0$ holds for all $k \geq 1$, except possibly for a finite number of them, which can be ignored in view of the remark following theorem 2. For this we first note that as $n \to \infty$

$$(14) \qquad \{\eta_n - \max(0, S_{k+1}-S_k, \ldots, S_n-S_k)\}/n^{1/2} \overset{P}{\to} 0.$$

This follows from the fact that

$$(15) \qquad |\eta_n - \max(0, S_{k+1}-S_k, \ldots, S_n-S_k)| \leq |\eta_n - \max(S_k, \ldots, S_n)| + |S_k| \leq \eta_{k-1} + |S_k|$$

and that $(\eta_{k-1}+|S_k|)/n^{1/2} \overset{P}{\to} 0$, as $n \to \infty$.

Using (3) and (14) and the fact that η_k and $\max(0, S_{k+1}-S_k, \ldots, S_n-S_k)$ are mutually independent, it follows that for $x > 0$,

$$(16) \qquad \lim_{n\to\infty} P(\max(0, S_{k+1}-S_k, \ldots, S_n-S_k) \leq x\, n^{1/2} | \eta_k \leq x\, k^{1/2})$$

$$= \lim_{n\to\infty} P(\max(0, S_{k+1}-S_k, \ldots, S_n-S_k) \leq x\, n^{1/2})$$

$$= \lim_{n\to\infty} P(\eta_n \leq x\, n^{1/2}) = 2\Phi(x) - 1.$$

Consider now the probability space $(\Omega, \mathcal{G}, P')$, where

$$(17) \qquad P'(A) = P(A | \eta_k \leq x \cdot k^{1/2}), \; A \in \mathcal{G},$$

and $P(\eta_k \leq x\, k^{1/2}) > 0$, by assumption. For any sequence A_n of events, with indicator function I_{A_n}, since $I_{A_n} \overset{P}{\to} 0$ implies $I_{A_n} \overset{P'}{\to} 0$, (13) follows in view of (14) and (16). This completes the proof of theorem 3.

The following theorem gives an analogous result for the supercritical case and is given here without proof.

THEOREM 4: Let $a > 0$ and $EX_n^2 = 1$. Then the sequence $\{(\eta_n - na)/n^{1/2}\}$ is a mixing sequence with the limiting distribution given by (5).

The approach adopted in Sections 4 and 5 to prove our results is that of Mogyoródi [6]. In this connection, the following theorem, due to Mogyoródi [6] and an extension of the celebrated Kolmogorov inequality, will be found useful.

THEOREM 5: Let $\tau_1, \tau_2, \ldots, \tau_n, \ldots,$ be a sequence of independent random variables with $E(\tau_i) = \mu_i$ and $\mathrm{Var}(\tau_i) = \sigma_i^2$. Let further

$$T_n = \frac{\tau_1 + \ldots + \tau_n - (\mu_1 + \ldots + \mu_n)}{V_n}$$

where

$$V_n = [\sum_{i=1}^{n} \sigma_i^2]^{1/2} .$$

Let us suppose that the distribution function $G_n(\cdot)$ of random variable T_n converges to the nondegenerate distribution function $G(\cdot)$ with variance 1. Let further C be an arbitrary random event having positive probability. Then there exists an integer $n_0 = n_0(C)$ such that for $n \geq n_0$,

$$P(\max_{1 \leq k \leq n} |\sum_{i=1}^{k} (\tau_i - \mu_i)| \geq \lambda V_n, C) \leq 3[P(C)]^{1/2}/\lambda^2 , \tag{18}$$

where λ is an arbitrary positive number.

3. SUBCRITICAL CASE.

Here we assume $a < 0$, and prove the following theorem.

THEOREM 6: Let $a < 0$. Let $\nu(n)$ denote a positive integer valued random variable for every $n = 1,2,\ldots,$ such that as $n \to \infty$, $\nu(n)/f(n)$ converges in probability to a positive random variable ν, for some arbitrary sequence of positive numbers $f(n)$ with $f(n) \to \infty$, as $n \to \infty$. Then $\eta_{\nu(n)} \xrightarrow{P} \eta$, as $n \to \infty$.

PROOF. Since $\nu(n)/f(n) \xrightarrow{P} \nu$, there exists a nondecreasing sequence $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ as $n \to \infty$, such that

$$(19) \qquad P\left(\left|\frac{\nu(n)}{f(n)} - \nu\right| > \varepsilon_n\right) \leq \varepsilon_n, \; n = 1,2,\ldots .$$

Let I_A denote the indicator function of a set A and $\bar{A}$ stand for its complement. Also, let B(n) denote the set

$$B(n) = \{-\varepsilon_n \leq \frac{\nu(n)}{f(n)} - \nu \leq \varepsilon_n\} .$$

For an arbitrarily small $\delta > 0$, choose $\alpha > 0$ such that $P(\nu \leq \alpha) \leq \delta$. Let $N(n) = [(\alpha-\varepsilon_n)f(n)]$, where [...] denotes the integral part of the number in the square bracket. Since $\eta_{N(n)} \xrightarrow{a.s.} \eta$, it suffices to show that $|\eta_{\nu(n)}-\eta_{N(n)}| \xrightarrow{P} 0$, as $n \to \infty$. However,

$$(20) \qquad |\eta_{\nu(n)}-\eta_{N(n)}| = |\eta_{\nu(n)}-\eta_{N(n)}| I_{[\nu\leq\alpha]} + |\eta_{\nu(n)}-\eta_{N(n)}| I_{[\nu>\alpha]\cap\bar{B}(n)} + |\eta_{\nu(n)}-\eta_{N(n)}| I_{[\nu>\alpha]\cap B(n)}.$$

Since $I_{\bar{B}(n)} \xrightarrow{P} 0$, as $n \to \infty$, the second term on the right side tends to zero in probability. Also, since the probability of the first term on the right side being positive, can be made arbitrarily small by choosing δ and α accordingly, it is sufficient to show that the last term of (20) tends to zero in probability. On the other hand,

$$(21) \qquad |\eta_{\nu(n)}-\eta_{N(n)}| I_{[\nu>\alpha]\cap B(n)} \leq (\eta_{\nu(n)}-\eta_{N(n)}) I_{[(\alpha-\varepsilon_n)f(n) < \nu(n)]}$$
$$\leq \sup_{k>N(n)} (\eta_k-\eta_{N(n)})$$
$$\leq (\eta-\eta_{N(n)}),$$

and since the last quantity tends to zero in probability, the theorem follows.

4. CRITICAL CASE. We prove here the following theorem, the analogue of (ii), section 1.

THEOREM 7: *Let* $a = 0$ *and* EX_n^2 *exist. Also without loss of generality let* $EX_n^2 = 1$. *Let* $\nu(n)$ *be as defined in theorem* 6. *Then*

$$\lim_{n\to\infty} P(\eta_{\nu(n)} \leq x(\nu(n))^{1/2}) = \Psi(x) \; , \tag{22}$$

where $\Psi(\cdot)$ *is as defined in* (4).

PROOF. For any arbitrary $\varepsilon > 0$, choose $\alpha > 0$ small enough and $\beta > 0$ large enough, both continuity points of the distribution function of ν, such that $P(\alpha < \nu \leq \beta) > 1-\varepsilon$. Now since $\nu(n)/f(n) \xrightarrow{P} \nu$, there exists an n_0 such that for $n \geq n_0$, $P(A_n) > 1-2\varepsilon$, where the event $A_n = \{\alpha < \nu(n)/f(n) \leq \beta\}$. Let $\alpha = a_0 < a_1 < \ldots . < a_{k-1} < a_k = \beta$, be continuity points of the distribution function of ν. Also, let $A_n^{(i)} = \{a_{i-1} < \nu(n)/f(n) \leq a_i\}$, $i = 1,2,\ldots,k$, which are mutually exclusive events, so that $A_n = \sum_{i=1}^{k} A_n^{(i)}$. Similarly let $A^{(i)} = \{a_{i-1} < \nu \leq a_i\}$, $i = 1,2,\ldots,k$. Then for $x > 0$,

$$P(\eta_{\nu(n)} \leq x(\nu(n))^{1/2}) = \sum_{i=1}^{k} P(\eta_{\nu(n)} \leq x(\nu(n))^{1/2}, A_n^{(i)}) + P(\eta_{\nu(n)} \leq x(\nu(n))^{1/2}, \bar{A}_n). \tag{23}$$

The last term is smaller than 2ε for $n \geq n_0$, so that we only need to consider the first term on the right side of (23). Again, we may write

$$\eta_{\nu(n)}(\nu(n))^{-1/2} = \eta_{N_{i-1}}(N_{i-1})^{-1/2} + R(n,k,i) - T(n,k,i), \tag{24}$$

where for $i = 1,\ldots,k$, $[f(n)a_{i-1}] = N_{i-1}$, the integral part of $f(n)a_{i-1}$, and

$$R(n,k,i) = \left[\frac{N_{i-1}}{\nu(n)}\right]^{1/2}\left(\frac{\eta_{\nu(n)} - \eta_{N_{i-1}}}{\sqrt{N_{i-1}}}\right),$$

$$T(n,k,i)=\frac{\eta_{N_{i-1}}}{\sqrt{N_{i-1}}}\left(1-\left(\frac{N_{i-1}}{\nu(n)}\right)^{1/2}\right).$$

Now choose $\alpha(\alpha > \delta > 0)$ such that

(25) $$|\Psi(x)-\Psi(x\pm\delta)| < \varepsilon ,$$

and let $C(n,k,i,\delta)$ denote the event

(26) $$C(n,k,i,\delta)=\{|R(n,k,i)-T(n,k,i)| < \delta\},\ i = 1,2,\ldots,k,$$

with $\bar{C}(\ )$ denoting its complement. Then

(27) $$\sum_{i=1}^{k} P(\eta_{\nu(n)}\leq x(\nu(n))^{1/2},A_n^{(i)})=\sum_{i=1}^{k} P(\eta_{\nu(n)}\leq x(\nu(n))^{1/2},A_n^{(i)},C(n,k,i,\delta))$$
$$+\sum_{i=1}^{k} P(\eta_{\nu(n)}\leq x(\nu(n))^{1/2},A_n^{(i)},\bar{C}(n,k,i,\delta)).$$

On the other hand it is easy to establish using (24) that

(28) $$\sum_{i=1}^{k} P(\eta_{\nu(n)}\leq x(\nu(n))^{1/2},A_n^{(i)},C(n,k,i,\delta))\leq \sum_{i=1}^{k} P(\eta_{N_{i-1}}\leq (x+\delta)(N_{i-1})^{1/2},A_n^{(i)}),$$

and also

(29) $$\sum_{i=1}^{k} P(\eta_{\nu(n)}\leq x(\nu(n))^{1/2},A_n^{(i)},C(n,k,i,\delta))$$
$$\geq \sum_{i=1}^{k} P(\eta_{N_{i-1}}\leq (x-\delta)(N_{i-1})^{1/2},A_n^{(i)},C(n,k,i,\delta))$$
$$\geq \sum_{i=1}^{k} P(\eta_{N_{i-1}}\leq (x-\delta)\ (N_{i-1})^{1/2},A_n^{(i)})-\sum_{i=1}^{k} P(A_n^{(i)},\bar{C}(n,k,i,\delta)).$$

Here the last inequality follows from the fact that for any three events A, B, and C, $P(ABC) \geq P(AB)-P(A\bar{C})$.

Let $A \Delta B$ denote the symmetric difference $(A-B)+(B-A)$, for any two events A and B. Then since $\nu(n)/f(n) \xrightarrow{P} \nu$, as $n \to \infty$, for any $\varepsilon > 0$, there exists a positive integer $n_1 (n_1 \geq n_0)$ such that for $n \geq n_1$,

$$(30) \qquad \sum_{i=1}^{k} P(A_n^{(i)} \Delta A^{(i)}) \leq \varepsilon/2.$$

Thus using this and the fact that for any three events A, B and C

$$|P(AB) - P(AC)| \leq P(B \Delta C) ,$$

we have for $n \geq n_1$,

$$(31) \qquad \Big|\sum_{i=1}^{k} P(n_{N_{i-1}} \leq (x \pm \delta)(N_{i-1})^{1/2}, A_n^{(i)}) - \sum_{i=1}^{k} P(n_{N_{i-1}} \leq (x \pm \delta)(N_{i-1})^{1/2}, A^{(i)})\Big| \leq \varepsilon/2.$$

Now using theorem 3, it is easy to show that

$$(32) \qquad \lim_{n\to\infty} \sum_{i=1}^{k} P(n_{N_{i-1}} \leq (x \pm \delta)(N_{i-1})^{1/2}, A^{(i)}) = \Psi(x \pm \delta) P(\alpha < \nu \leq \beta),$$

so that there exists $n_2 (n_2 \geq n_1)$ such that for $n \geq n_2$,

$$(33) \qquad \Big|\sum_{i=1}^{k} P(n_{N_{i-1}} \leq (x \pm \delta)(N_{i-1})^{1/2}, A^{(i)}) - \Psi(x \pm \delta) P(\alpha < \nu \leq \beta)\Big| \leq \varepsilon/2.$$

Thus using (31)and (33) in (28) and (29) we have for $n \geq n_2$,

$$(34) \qquad \Psi(x-\delta)P(a<\nu\leq b) - \varepsilon - \sum_{i=1}^{k} P(A_n^{(i)}, \bar{C}(n,k,i,\delta))$$
$$\leq \sum_{i=1}^{k} P(n_{\nu(n)} \leq x(\nu(n))^{1/2}, A_n^{(i)}, C(n,k,i,\delta)) \leq \Psi(x+\delta)P(\alpha<\nu\leq\beta) + \varepsilon.$$

On the other hand, since $P(\alpha<\nu\leq\beta) \geq 1-\varepsilon$, on using (25) we have for $n \geq n_2$,

$$(35) \qquad \Psi(x) - 3\varepsilon - \sum_{i=1}^{k} P(A_n^{(i)}, \bar{C}(n,k,i,\delta))$$
$$\leq \sum_{i=1}^{k} P(n_{\nu(n)} \leq x(\nu(n))^{1/2}, A_n^{(i)}, C(n,k,i,\delta)) \leq \Psi(x) + 3\varepsilon.$$

Now using this in (27) we have for $n \geq n_2$,

$$(36)\quad \Psi(x)-3\varepsilon-\sum_{i=1}^{k} P(A_n^{(i)},\bar{C}(n,k,i,\delta)) \leq \sum_{i=1}^{k} P(\eta_{\nu(n)} \leq x(\nu(n))^{1/2}, A_n^{(i)})$$

$$\leq \Psi(x)+3\varepsilon+\sum_{i=1}^{k} P(A_n^{(i)},\bar{C}(n,k,i,\delta)).$$

Thus the theorem follows once we show that for any $\varepsilon > 0$, there exists $n_3 (n_3 \geq n_2)$ such that for $n \geq n_3$,

$$(37)\qquad \sum_{i=1}^{k} P(A_n^{(i)},\bar{C}(n,k,i,\delta)) \leq \varepsilon \ .$$

Again, given $A_n^{(i)}$, since both R and T are nonnegative, it easily follows from (26) that

$$(38)\quad \sum_{i=1}^{k} P(A_n^{(i)},\bar{C}(n,k,i,\delta)) \leq \sum_{i=1}^{k} P(A_n^{(i)},R(n,k,i)\geq\delta)+\sum_{i=1}^{k} P(A_n^{(i)},T(n,k,i)\geq\delta).$$

We shall restrict attention to the first term on the right side; the second term can be similarly treated. Now for $n \geq n_2$, we have

$$(39)\quad \sum_{i=1}^{k} P(A_n^{(i)},R(n,k,i)\geq\delta)$$

$$\leq \sum_{i=1}^{k} P(A_n^{(i)},(\eta_{\nu(n)}-\eta_{N_{i-1}})\geq\delta(N_{i-1})^{1/2})$$

$$\leq \sum_{i=1}^{k} P(A_n^{(i)},\max_{N_{i-1}<\ell\leq N_i}(\eta_\ell-\eta_{N_{i-1}})\geq\delta(N_{i-1})^{1/2})$$

$$\leq \varepsilon/2+\sum_{i=1}^{k} P(A^{(i)},(\eta_{N_i}-\eta_{N_{i-1}})\geq\delta(N_{i-1})^{1/2})$$

$$\leq \varepsilon/2+\sum_{i=1}^{k} P(A^{(i)},\max(S_{N_{i-1}+1},\ldots,S_{N_i})\geq\eta_{N_{i-1}}+\delta(N_{i-1})^{1/2})$$

$$\leq \varepsilon/2+\sum_{i=1}^{k} P(A^{(i)},\max_{N_{i-1}<\ell\leq N_i}(\sum_{j=N_{i-1}+1}^{\ell} X_j)\geq\eta_{N_{i-1}}-S_{N_{i-1}}+\delta(N_{i-1})^{1/2})$$

$$\leq \varepsilon/2+\sum_{i=1}^{k} P(A^{(i)},\max_{N_{i-1}<\ell\leq N_i}\left|\sum_{j=N_{i-1}+1}^{\ell} X_j/(N_i-N_{i-1})^{1/2}\right| \geq \delta[N_{i-1}/(N_i-N_{i-1})]^{1/2})$$

$$\leq \varepsilon/2+3\sum_{i=1}^{k} \frac{N_i-N_{i-1}}{\delta^2\, N_{i-1}}\,(P(A^{(i)}))^{1/2} \ .$$

Here we have used (30) in the top third inequality, while at the end we have used theorem 5.

Now since the set of continuity points of the distribution of ν is everywhere dense, we can choose the points of subdivision of $(\alpha,\beta]$ such that for any $\theta > 0$,

$$a_i - a_{i-1} = (\beta-\alpha)(1+\varepsilon_i\theta)/k,\ i = 1,2,\ldots,k, \tag{40}$$

holds for some ε_i with $|\varepsilon_i| \le 1$. Taking this subdivision of $(\alpha,\beta]$, we obtain

$$\begin{aligned}\sum_{i=1}^{k} \frac{N_i - N_{i-1}}{\delta^2 N_{i-1}} (P(A^{(i)}))^{1/2} &\le \sum_{i=1}^{k} \frac{(\beta-\alpha)f(n)}{\delta^2[\alpha f(n)]} \left(\frac{1+\theta}{k}\right)(P(A^{(i)}))^{1/2} \\ &\le \frac{(\beta-\alpha)f(n)}{\delta^2[\alpha f(n)]} \left(\frac{1+\theta}{k}\right)\left(\sum_{i=1}^{k} P(A^{(i)})\right)^{1/2}\cdot(k)^{1/2} \\ &\le \frac{(\beta-\alpha)f(n)}{\delta^2[\alpha f(n)]}(1+\theta)k^{-1/2}.\end{aligned} \tag{41}$$

Here the last but one inequality follows from the Cauchy inequality. Now for $0 < \delta < \alpha$, choose $n_3 (n_3 \ge n_2)$ such that for $n \ge n_3$

$$\frac{f(n)}{[\alpha f(n)]} \le \frac{1}{\alpha-\delta},$$

so that using this and (41), it follows from (39) that for $n \ge n_3$,

$$\sum_{i=1}^{k} P(A_n^{(i)}, R(n,k,i) \ge \delta) \le \varepsilon + \frac{3(\beta-\alpha)(1+\theta)}{\delta^2(\alpha-\delta)}(k)^{-1/2}. \tag{42}$$

Finally taking k large enough, the last term can be made arbitrarily small. Hence the theorem follows.

5. SUPERCRITICAL CASE. We shall need the following theorem in order to prove the main result of this section. This theorem, by itself, appears to be some interest.

THEOREM 8. Let $a > 0$. Then

(43) $$\lim_{n\to\infty} P(\eta_n - S_n \leq x) = W(x),$$

where $W(\cdot)$ is a distribution function of a nonnegative proper random variable. Furthermore, this distribution is same as that of the limit of a subcritical process obtained by replacing X_n by $-X_n$, for all n. The proof of this theorem is omitted as it follows along the lines of the proof of theorem 1 and in particular from (7). Finally, we have the following theorem as the analogue of theorem 1.

THEOREM 9. Let $a > 0$ and Var $X_n = 1$. Let $\nu(n)$ be as defined in theorem 6. Then

(44) $$\lim_{n\to\infty} P\left(\frac{\eta_{\nu(n)} - a\nu(n)}{\sqrt{\nu(n)}} \leq x\right) = \Phi(x) .$$

PROOF. The proof of this theorem follows the same lines as those of theorem 7. However, we shall mention here briefly the points of difference, while skipping the details. Analogous to (24) we write

(45) $$\frac{\eta_{\nu(n)} - a\nu(n)}{\sqrt{\nu(n)}} = \frac{\eta_{N_{i-1}} - aN_{i-1}}{\sqrt{N_{i-1}}} + R_1(n,k,i) - T_1(n,k,i),$$

where

$$R_1(n,k,i) = \left(\frac{N_{i-1}}{\nu(n)}\right)^{1/2}\left\{\frac{\eta_{\nu(n)} - \eta_{N_{i-1}} - a(\nu(n) - N_{i-1})}{\sqrt{N_{i-1}}}\right\},$$

and

$$T_1(n,k,i) = \left\{\frac{\eta_{N_{i-1}} - aN_{i-1}}{\sqrt{N_{i-1}}}\right\}\left(1 - \left(\frac{N_{i-1}}{\nu(n)}\right)^{1/2}\right).$$

Also, analogous to (26) define the events

(46) $$C_1(n,k,i,\delta) = \{|R_1(n,k,i) - T_1(n,k,i)| < \delta\},\ i = 1,2,\ldots,k.$$

Now proceed exactly along the same lines of argument as those of theorem 7, during the course of which, we would need the mixing property of the sequence $(\eta_n - na)/n^{1/2}$, as exhibited in theorem 4, until finally we are required to show that for any $\varepsilon > 0$, there exists n_3 such that for $n \geq n_3$,

$$(47) \qquad \sum_{i=1}^{k} P(A_n^{(i)},\ \bar{C}_1(n,k,i,\delta)) \leq \varepsilon \ ,$$

the analogue of (37). Again, it is easily seen that

$$(48) \quad \sum_{i=1}^{k} P(A_n^{(i)}, \bar{C}_1(n,k,i,\delta)) \leq \sum_{i=1}^{k} P(A_n^{(i)}, |R_1(n,k,i)| \geq \delta/2) + \sum_{i=1}^{k} P(A_n^{(i)}, |T_1(n,k,i)| \geq \delta/2).$$

As before we restrict to the first summation on the right side, while the last can be analogously treated. Now it can be easily shown that for $n \geq n_2$, we have

$$(49) \quad \sum_{i=1}^{k} P(A_n^{(i)}, |R_1(n,k,i)| \geq \delta/2)$$
$$\leq \sum_{i=1}^{k} P\Big(A_n^{(i)}, \Big| \frac{\eta_{\nu(n)} - \eta_{N_{i-1}} - a(\nu(n) - N_{i-1})}{\sqrt{N_{i-1}}} \Big| \geq \delta/2\Big)$$
$$\leq \sum_{i=1}^{k} P\Big(A_n^{(i)}, \max_{N_{i-1} < \ell \leq N_i} \Big| \frac{\eta_\ell - \eta_{N_{i-1}} - a(\ell - N_{i-1})}{\sqrt{N_{i-1}}} \Big| \geq \delta/2\Big)$$
$$\leq \varepsilon/2 + \sum_{i=1}^{k} P\Big(A^{(i)}, \max_{N_{i-1} < \ell \leq N_i} \Big| \frac{\eta_\ell - \eta_{N_{i-1}} - a(\ell - N_{i-1})}{\sqrt{N_{i-1}}} \Big| \geq \delta/2\Big)$$
$$\leq \varepsilon/2 + \sum_{i=1}^{k} P\Big(A^{(i)}, \max_{N_{i-1} < \ell \leq N_i} \Big| \frac{\eta_\ell - S_{N_{i-1}} - a(\ell - N_{i-1})}{\sqrt{N_{i-1}}} \Big| + \Big| \frac{\eta_{N_{i-1}} - S_{N_{i-1}}}{\sqrt{N_{i-1}}} \Big| \geq \delta/2\Big).$$

On the other hand, in view of theorem 8, since for $i = 1,2,\ldots,k$,

$$(\eta_{N_{i-1}} - S_{N_{i-1}})(N_{i-1})^{-1/2} \overset{P}{\to} 0,$$

as $n \to \infty$, for $0 < 4\gamma < \delta$, we can find $n_3 (n_3 \geq n_2)$ such that for $n \geq n_3$, we have

(50) $$\sum_{i=1}^{k} P((\eta_{N_{i-1}} - S_{N_{i-1}}) \geq \gamma(N_{i-1})^{1/2}) \leq \varepsilon/2 .$$

Thus for $n \geq n_3$, we have from (49),

(51) $$\sum_{i=1}^{k} P(A_n^{(i)}, |R_1(n,k,i)| \geq \delta/2)$$

$$\leq \varepsilon + \sum_{i=1}^{k} P(A^{(i)}, (\eta_{N_{i-1}} - S_{N_{i-1}}) \leq \gamma(N_{i-1})^{1/2}, \max_{N_{i-1} < \ell \leq N_i} \left|\frac{\eta_\ell - S_{N_{i-1}} - a(\ell - N_{i-1})}{\sqrt{N_{i-1}}}\right| \geq \frac{\delta}{2} - \gamma)$$

$$\leq \varepsilon + \sum_{i=1}^{k} P(A^{(i)}, (\eta_{N_{i-1}} - S_{N_{i-1}}) \leq \gamma(N_{i-1})^{1/2}, \max_{N_{i-1} < \ell \leq N_i} |\max(\eta_{N_{i-1}} - S_{N_{i-1}}, S_{N_{i-1}+1} - S_{N_{i-i}},$$

$$\ldots, S_\ell - S_{N_{i-1}}) - a(\ell - N_{i-1})| \geq (\frac{\delta}{2} - \gamma)(N_{i-1})^{1/2})$$

$$\leq \varepsilon + \sum_{i=1}^{k} P(A^{(i)}, \max_{N_{i-1} < \ell \leq N_i} |\max(S_{N_{i-1}+1} - S_{N_{i-1}}, \ldots, S_\ell - S_{N_{i-1}}) - a(\ell - N_{i-1})|$$

$$\geq (\frac{\delta}{2} - \gamma)(N_{i-1})^{1/2})$$

$$\leq \varepsilon + \sum_{i=1}^{k} p(A^{(i)}, \max_{N_{i-1} < \ell \leq N_i} |\sum_{j=N_{i-1}+1}^{\ell} (X_j - a)| \geq (\frac{\delta}{2} - \gamma)(N_{i-1})^{1/2})$$

$$\leq \varepsilon + \sum_{i=1}^{k} P(A^{(i)}, \max_{N_{i-1} < \ell \leq N_i} |\sum_{j=N_{i-1}+1}^{\ell} (X_j - a)| (N_i - N_{i-1})^{-1/2} \geq (\frac{\delta}{2} - \gamma)[N_{i-1}/(N_i - N_{i-1})]^{1/2})$$

$$\leq \varepsilon + 3 \sum_{i=1}^{k} \frac{N_i - N_{i-1}}{(\frac{\delta}{2} - \gamma)^2 N_{i-1}} (P(A^{(i)}))^{1/2} .$$

Here for going from second inequality to the third, among others, we have used the fact that $4\gamma < \delta$. The last step of (51), of course, follows from theorem 5. The rest of the argument is same as the one for theorem 7, and is therefore omitted. This completes the proof.

We close with the remark that all the above results can easily be extended to cover the case where $S_0 = X_0$ is a nonnegative random variable. When $\nu(n)$ is nonrandom and is equal to n, this case has recently been considered by Takác [13].

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