THE DISTRIBUTION OF THE CHARACTERISTIC ROOTS OF $\underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$ UNDER VIOLATIONS*

by

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Lemma 5. Let $S(p \times p)$ be a matrix and $T(p \times p)$ be a complex matrix whose real part is positive definite. Then for $\gamma > -1$.

(2.10) $$L_\kappa^\delta(S) = \Gamma_p(\gamma+m,\kappa)\cdot \frac{2^{\frac{1}{2}p(p-1)}}{(2\pi i)^{\frac{1}{2}p(p+1)}} \int_{Re(T)>0} e^{tr\, T}|T|^{-\gamma-m}\, C_\kappa(I-S\, T^{-1})dT ,$$

(Constantine [2]),

where $m = \frac{1}{2}(p+1)$ and $L_\kappa^\gamma(S)$ is the generalized Laguerre polynomials defined by Constantine [2] as

(2.11) $$e^{-tr\, S}\, L_\kappa^\gamma(S) = \int_{R>0} e^{-tr\, R}|R|^\gamma\, C_\kappa(R)\, A_\gamma(R\, S)d\, R ,$$

where $A_\gamma(R)$ is the Bessel function of matrix argument defined by:

(2.12) $$A_\gamma(R) = \frac{2^{\frac{1}{2}p(p-1)}}{(2\pi i)^{\frac{1}{2}p(p+1)}} \int_{Re(T)>0} e^{tr\, T}\, e^{-tr\, R\, T^{-1}}\, |T|^{-\gamma-m}\, d\, T .$$

3. The distribution of latent roots $S_1\, S_2^{-1}$. The distribution of the latent roots of $S_1\, S_2^{-1}$ will be derived in this section when $S_1(p \times p)$ is distributed $W(p, n_1\, \Sigma_1, \Omega)$ i.e. non-central Wishart distribution on n_1 d.f. with non-centrality Ω and covariance matrix Σ_1 and $S_2(p \times p)$ central Wishart $W(p, n_2, \Sigma_2, 0)$, where $n_1, n_2 \geq p$. Note that for $n_1 \geq p$ we will have p non-zero latent roots of $S_1\, S_2^{-1}$. When $n_1 < p$, $S_1\, S_2^{-1}$ has only n_1 non-zero roots and the density function of the roots of $S_1\, S_2^{-1}$ can be obtained from that for $n_1 \geq p$ if in the latter case the following changes are made:

(3.1) $$(n_1,n_2,p) \rightarrow (p,\, n_1+n_2-p,\, n_1) .$$

Therefore, here we only consider the case where $n_1 \geq p$.

where

$$\Lambda = \Sigma_1^{\frac{1}{2}} \Sigma_2^{-1} \Sigma_1^{\frac{1}{2}} \quad \text{and} \quad C_1(p,n_1,n_2) = \{\Gamma_p(\tfrac{n_1}{2})\, \Gamma_p(\tfrac{n_2}{2})\}^{-1} \quad .$$

Now transform $A_1 = A_2^{\frac{1}{2}} R A_2^{\frac{1}{2}}$ and $A_2 = A_2$. (The same notation is used to denote the matrix R both before and after diagonalization). The Jacobian is $|A_2|^{\frac{1}{2}(p+1)}$, and hence the joint density of R and A_2 is

$$C_1(p,n_1,n_2)\; e^{-\mathrm{tr}\Omega} |\Lambda|^{\frac{1}{2}n_2}\; e^{-\mathrm{tr}\, R A_2} |R|^{\frac{1}{2}(n_1-p-1)}$$
$$\cdot\; e^{-\mathrm{tr}\Lambda A_2} |A_2|^{\frac{1}{2}(n_1+n_2-p-1)}\; {}_0F_1(\tfrac{1}{2}n_1; \Omega A_2^{\frac{1}{2}} R A_2^{\frac{1}{2}}) \quad .$$

The distribution is a symmetric function of Ω (Constantine [2]). Hence we can transform $\Omega \to H \Omega H'$ where $H \in O(p)$, and integrating over $O(p)$ using Lemma 1, all the factors in the above expression remain the same except the hypergeometric function which now becomes ${}_0F_1(\frac{1}{2}n_1; \Omega, R A_2)$. Further, expand $e^{-\mathrm{tr}\, R A_2} = {}_0F_0(-R A_2)$ and ${}_0F_1(\frac{1}{2}n_1; \Omega, R A_2)$ in zonal polynomials using (2.2) and apply Lemma 2. The joint density of R and A_2 becomes

$$C_1(p,n_1,n_2) e^{-\mathrm{tr}\Omega} |\Lambda|^{\frac{1}{2}n_2} |R|^{\frac{1}{2}(n_1-p-1)}\; e^{-\mathrm{tr}\Lambda A_2} |A_2|^{\frac{1}{2}(n_1+n_2-p-1)}$$

$$\cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{C_\kappa(\Omega)}{(\frac{1}{2}n_1)_\kappa\, C_\kappa(I_p) k!} \sum_{n=0}^{\infty} \sum_{\nu,\delta} \frac{(-1)^n g_{\kappa,\nu}^{\delta}\, C_\delta(R A_2)}{n!} \quad .$$

Now integrate out A_2 using Lemma 3 we have the density of R of the form and making use of (2.4) and (2.8) we have the result as stated in the theorem. Q.E.D.

For $\Omega = 0$, since $g^{\delta}_{0,\nu} = 1$ and $\delta = \nu$, the expression (3.2) gives the result stated by James [4] page 484 as a special case.

It seems that, the expression (3.2) is not convenient for further development and may not converge for all values of $\underset{\sim}{R}$, $\underset{\sim}{\Lambda}$ and $\underset{\sim}{\Omega}$. For further development we shall prove the following theorem.

<u>Theorem 2</u>. Let $\underset{\sim}{S}_1$, $\underset{\sim}{S}_2$, $\underset{\sim}{R}$ and r_i's be as stated in Theorem 1. Then the joint density function of $r_1, \ldots, r_p$ is given by

$$(3.7) \qquad C(p,n_1,n_2) e^{-\mathrm{tr}\underset{\sim}{\Omega}} |\underset{\sim}{\Lambda}|^{-\frac{1}{2}n_1} |\underset{\sim}{R}|^{\frac{1}{2}(n_1-p-1)} |\underset{\sim}{I}+\lambda\underset{\sim}{R}|^{-\frac{1}{2}(n_1+n_2)} \prod_{i>j} (r_i-r_j)$$

$$\cdot \sum_{k=0}^{\infty} \sum_{\kappa} \left(\frac{n_1+n_2}{2}\right)_{\kappa} \frac{C_{\kappa}(\lambda\underset{\sim}{R}(\underset{\sim}{I}+\lambda\underset{\sim}{R})^{-1})}{k!}$$

$$\sum_{d=0}^{k} \sum_{\delta} \frac{a_{\kappa,\delta}\, C_{\delta}(-\lambda^{-1}\underset{\sim}{\Lambda}^{-1})\, L_{\delta}^{\frac{1}{2}(n_1-p-1)}(\underset{\sim}{\Omega})}{\left(\frac{n_1}{2}\right)_{\delta} C_{\delta}(\underset{\sim}{I})\, C_{\delta}(\underset{\sim}{I})} ,$$

where $\underset{\sim}{\Lambda} = \underset{\sim}{\Sigma}_1^{\frac{1}{2}} \underset{\sim}{\Sigma}_2^{-1} \underset{\sim}{\Sigma}_1^{\frac{1}{2}}$, λ, a positive real number, $C(p,n_1,n_2)$ is as defined in (3.3), $L_{\delta}^{\gamma}(\underset{\sim}{S})$ is as in (2.11) and $a_{\kappa,\delta}$ are constants (Constantine [2], Pillai and Jouris [7]).

<u>Proof</u>. We start from the expression (3.4). Now apply Lemma 4 to ${}_0F_1(\frac{1}{2}n_1; \underset{\sim}{\Omega}^{\frac{1}{2}} \underset{\sim}{A}_1 \underset{\sim}{\Omega}^{\frac{1}{2}})$ to get the joint density of $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$ as

$$(3.8)\qquad C_3(p,n_1,n_2)e^{-\operatorname{tr}\Omega}|\Lambda|^{\frac{1}{2}n_2}|R|^{\frac{1}{2}(n_1-p-1)}$$

$$\cdot \int_{\operatorname{Re}(T)>0} e^{\operatorname{tr}T}|T|^{-\frac{1}{2}n_1}|I - W|^{-\frac{1}{2}(n_1+n_2)}$$

$$|R+(I - W)^{-\frac{1}{2}}\Lambda(I - W)^{-\frac{1}{2}}|^{-\frac{1}{2}(n_1+n_2)}(dT),$$

where

$$C_3(p,n_1,n_2) = 2^{\frac{1}{2}p(p-1)}\Gamma_p(\frac{n_1+n_2}{2}) \,/\, [(2\pi i)^{\frac{1}{2}p(p+1)}\Gamma_p(\frac{n_2}{2})] .$$

Write

$$|R+(I - W)^{-\frac{1}{2}}\Lambda(I - W)^{-\frac{1}{2}}| =$$

$$= |\Lambda|\;|I - W|^{-1}\;|I+R(I - W)^{\frac{1}{2}}\Lambda^{-1}(I - W)^{\frac{1}{2}}| ,$$

then (3.8) becomes

$$(3.9)\qquad C_3(p,n_1,n_2)e^{-\operatorname{tr}\Omega}|\Lambda|^{-\frac{1}{2}n_1}|R|^{\frac{1}{2}(n_1-p-1)}$$

$$\cdot\int_{\operatorname{Re}(T)>0} e^{\operatorname{tr}T}|T|^{-\frac{1}{2}n_1}|I+R(I - W)^{\frac{1}{2}}\Lambda^{-1}(I - W)^{\frac{1}{2}}|^{-\frac{1}{2}(n_1+n_2)}(dT) .$$

Since R is symmetric we can diagonalize by an orthogonal transformation H and can use the same technique as before. After substitutions and making use of (3.5) and integrating over H, we will get the joint density of roots $r_1,\ldots,r_p$ of R in the form

$$(3.12)\quad \int_{Re(T)>0} etr\,T\,|T|^{-\frac{1}{2}n_1} \frac{C_\kappa(I-\lambda^{-1}(I-W)^{\frac{1}{2}}\Lambda^{-1}(I-W)^{\frac{1}{2}})}{C_\kappa(I)}\,dT \quad .$$

Let us use the relation (Constantine [2])

$$(3.13)\quad \frac{C_\kappa(I-S)}{C_\kappa(I)} = \sum_{n=0}^{k} \sum_{\nu} (-1)^n a_{\kappa,\nu} \frac{C_\nu(S)}{C_\nu(I)} \quad ,$$

where $a_{\kappa,\nu}$ are constants discussed earlier. Then (3.12) becomes

$$(3.14)\quad \sum_{n=0}^{k} \sum_{\nu} \frac{(-\lambda^{-1})^n a_{\kappa,\nu}}{C_\nu(I)} \int_{Re(T)>0} etr\,T\,|T|^{-\frac{1}{2}n_1} C_\nu(\Lambda^{-1}(I-W))\,dT \quad .$$

Λ being symmetric we can transform $\Lambda \to H\Lambda H'$ by an orthogonal transformation H and integrate over H using (3.6). Then (3.14) becomes

$$(3.15)\quad \sum_{n=0}^{k} \sum_{\nu} \frac{(-\lambda^{-1})^n a_{\kappa,\nu} C_\nu(\Lambda^{-1})}{C_\nu(I)\;C_\nu(I)} \int_{Re(T)>0} etr\,T\,|T|^{-\frac{1}{2}n_1} C_\nu(I-T^{-1}\Omega)\,dT \quad .$$

Applying Lemma 5 to (3.15) gives

$$(3.16)\quad \frac{(2\pi i)^{\frac{1}{2}p(p+1)}}{2^{\frac{1}{2}p(p-1)}} \sum_{n=0}^{k} \sum_{\nu} \frac{a_{\kappa,\nu} C_\nu(-\lambda^{-1}\Lambda^{-1})\; L_\nu^{\frac{1}{2}(n_1-p-1)}(\Omega)}{\Gamma_p(\frac{n_1}{2},\nu)\; C_\nu(I)\; C_\nu(I)} \quad .$$

Combining (3.11) - (3.16) and making use of (2.8) we have the result as stated in the theorem. Q.E.D.

Formula (3.7) will give special cases:

a) For $\Omega = 0$, it is seen from (2.12) that $A_\gamma(0) = \{\Gamma_p(\gamma+\frac{p+1}{2})\}^{-1}$ (see also Herz [3] page 487) and from (2.11) and (2.7) we have $L_\kappa^\delta(0) = (\gamma+\frac{p+1}{2})_\kappa\, C_\kappa(I)$. Substituting $L_\delta^{\frac{1}{2}(n_1-p-1)}(0)$ into (3.7) and making use of (3.13), we have the result of Khatri [5].

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13. ABSTRACT

The paper deals with the density of the characteristic roots of $\underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$ where $\underset{\sim}{S}_1$ is distributed non-central Wishart $W(p,n_1,\underset{\sim}{\Sigma}_1,\underset{\sim}{\Omega})$ and $\underset{\sim}{S}_2$, $W(p,n_2,\underset{\sim}{\Sigma}_2,\underset{\sim}{0})$. This is basic for an exact study of robustness of tests of at least two multivariate hypotheses.

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1. Introduction. Consider the test of the following two hypotheses: 1) equality of covariance matrices in two p-variate normal populations and 2) equality of p-dimensional mean vectors in ℓ p-variate normal populations having a common covariance matrix. In order to carry out some exact investigations of robustness of tests of 1) when the assumption of normality is violated and of 2) when that of a common covariance matrix is disturbed, a distribution problem is studied in this paper, namely, that of the density of the characteristic roots of $\underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$, where $\underset{\sim}{S}_1$ is distributed $W(p,n_1,\underset{\sim}{\Sigma}_1,\underset{\sim}{\Omega})$ and $\underset{\sim}{S}_2$, $W(p,n_2,\underset{\sim}{\Sigma}_2,\underset{\sim}{0})$, (See Section 3 for definitions). The results of the robustness studies will be reported in a second report.

2. Preliminaries. In this section we state some results which will be needed in the sequal.

Lemma 1. Let $\underset{\sim}{S}(p \times p)$ and $\underset{\sim}{T}(p \times p)$ be positive definite symmetric matrices. Then

$$(2.1)\qquad \int_{O(p)} {}_qF_r(a_1,\ldots,a_q;\ b_1,\ldots,b_r;\ \underset{\sim}{S}\,\underset{\sim}{H}\,\underset{\sim}{T}\,\underset{\sim}{H}')(d\underset{\sim}{H}) = {}_qF_r(a_1,\ldots,a_q;\ b_1,\ldots,b_r;\ \underset{\sim}{S},\underset{\sim}{T}) \qquad \text{(James [4])}$$

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(2.6) $$C_\kappa(\underset{\sim}{S})\ C_\sigma(\underset{\sim}{S}) = \sum_\delta g^\delta_{\kappa,\sigma}\ C_\sigma(\underset{\sim}{S})$$

where $\delta = (d_1 \geq d_2 \geq \ldots \geq d_p \geq 0)$ such that $\sum_{i=1}^{p} d_i = k + s$ and $g^\delta_{\kappa,\sigma}$ are constants. (Constantine [2], (Pillai and Sugiyama [8]).

Tables of the coefficients $g^\delta_{\kappa,\sigma}$ for various values of k and s can be found, for instance, in [6].

Lemma 3. Let $\underset{\sim}{R}(p \times p)$ be a complex symmetric matrix whose real part is positive definite, and let $\underset{\sim}{T}(p \times p)$ be an arbitrary complex symmetric matrix. Then

(2.7) $$\int_{\underset{\sim}{S} > 0} e^{-\mathrm{tr}\ \underset{\sim}{R}\ \underset{\sim}{S}} |\underset{\sim}{S}|^{t - \frac{1}{2}(p+1)}\ C_\kappa(\underset{\sim}{S}\ \underset{\sim}{T})d\underset{\sim}{S}$$
$$= \Gamma_p(t,\kappa)\ |\underset{\sim}{R}|^{-t}\ C_\kappa(\underset{\sim}{T}\ \underset{\sim}{R}^{-1}), \qquad \text{(Constantine [1])}$$

the integration being over the space of positive definite $p \times p$ matrices, and valid for all complex numbers t such that $\mathrm{Re}(t) > \frac{1}{2}(p-1)$. The constant $\Gamma_p(t,\kappa)$ is given by

(2.8) $$\Gamma_p(t,\kappa) = \pi^{\frac{1}{4}p(p-1)} \prod_{i=1}^{p} \Gamma(t+k_i - \tfrac{1}{2}(i-1)) = (t)_\kappa\ \Gamma_p(t) \quad .$$

Lemma 4. Let $\underset{\sim}{S}(p \times p)$ be a positive definite symmetric matrix and $\underset{\sim}{T}(p \times p)$ a complex matrix whose real part is positive definite symmetric. Then

(2.9) $$\frac{2^{\frac{1}{2}p(p-1)}\ \Gamma_p(b)}{(2\pi i)^{\frac{1}{2}p(p+1)}} \int_{\mathrm{Re}(\underset{\sim}{T}) = \underset{\sim}{X}_0 > 0} e^{\mathrm{tr}\ \underset{\sim}{T}} |\underset{\sim}{T}|^{-b}\ {}_qF_r(a_1,\ldots,a_q;b_1,\ldots,b_r;\underset{\sim}{T}^{-1}\underset{\sim}{S})d\underset{\sim}{T}$$
$$= {}_qF_{r+1}\ (a_1,\ldots,a_q;\ b_1,\ldots,b_r,b;\ \underset{\sim}{S})\ , \qquad \text{(James [4])}$$

__Theorem 1__. Let the $p \times p$ matrices $\underset{\sim}{S}_1$ and $\underset{\sim}{S}_2$ be independently distributed, $\underset{\sim}{S}_1$ having $W(p, n_1, \underset{\sim}{\Sigma}_1, \Omega)$ and $\underset{\sim}{S}_2$ having $W(p, n_2, \underset{\sim}{\Sigma}_2, \underset{\sim}{0})$. Let $\underset{\sim}{R} = \text{diag}(r_1,\ldots,r_p)$, where $r_1,\ldots,r_p$ are the roots of $\underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$ such that $0 < r_1 < r_2 < \ldots < r_p < \infty$. Then the joint density function of $r_1,\ldots,r_p$ is given by

$$(3.2) \qquad C(p,n_1,n_2)e^{-\text{tr}\underset{\sim}{\Omega}}|\underset{\sim}{\Lambda}|^{-\frac{1}{2}n_1} |\underset{\sim}{R}|^{\frac{1}{2}(n_1-p-1)} \prod_{i>j}(r_i-r_j)$$

$$\cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{C_\kappa(\underset{\sim}{\Omega})}{(\frac{1}{2}n_1)_\kappa C_\kappa(\underset{\sim}{I}_p)k!} \sum_{n=0}^{\infty} \sum_{\nu,\delta} \frac{(-1)^n g^{\delta}_{\kappa,\nu}(\frac{n_1+n_2}{2})_\delta C_\delta(\underset{\sim}{\Lambda}^{-1}) C_\delta(\underset{\sim}{R})}{C_\delta(\underset{\sim}{I}_p)n!} ,$$

where

$$(3.3) \qquad C(p,n_1,n_2) = \pi^{\frac{1}{2}p^2} \Gamma_p(\frac{n_1+n_2}{2})]/[\Gamma_p(\frac{n_1}{2}) \Gamma_p(\frac{n_2}{2}) \Gamma_p(\frac{p}{2})] .$$

__Proof__. Joint density of $\underset{\sim}{S}_1$ and $\underset{\sim}{S}_2$ is

$$\{\Gamma_p(\frac{n_1}{2}) \Gamma_p(\frac{n_2}{2}) |2\underset{\sim}{\Sigma}_1|^{\frac{1}{2}n_1}|2\underset{\sim}{\Sigma}_2|^{\frac{1}{2}n_2}\}^{-1} e^{-\text{tr}\underset{\sim}{\Omega}} e^{-\frac{1}{2}\text{tr}\, \underset{\sim}{\Sigma}_1^{-1}\underset{\sim}{S}_1} |\underset{\sim}{S}_1|^{\frac{1}{2}(n_1-p-1)}$$

$$e^{-\frac{1}{2}\text{tr}\, \underset{\sim}{\Sigma}_2^{-1}\underset{\sim}{S}_2} |\underset{\sim}{S}_2|^{\frac{1}{2}(n_2-p-1)} {}_0F_1(\tfrac{1}{2}n_1; \tfrac{1}{2}\underset{\sim}{\Sigma}_1^{-1} \underset{\sim}{\Omega} \underset{\sim}{S}_1) .$$

Since the roots are invariant under the simultaneous transformations $\underset{\sim}{A}_1 = \frac{1}{2}\underset{\sim}{\Sigma}_1^{-\frac{1}{2}} \underset{\sim}{S}_1\underset{\sim}{\Sigma}_1^{-\frac{1}{2}}$ and $\underset{\sim}{A}_2 = \frac{1}{2}\underset{\sim}{\Sigma}_1^{-\frac{1}{2}} \underset{\sim}{S}_2 \underset{\sim}{\Sigma}_1^{-\frac{1}{2}}$, we have now the joint density of $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$:

$$(3.4) \qquad C_1(p,n_1,n_2)e^{-\text{tr}\underset{\sim}{\Omega}}|\underset{\sim}{\Lambda}|^{\frac{1}{2}n_2} e^{-\text{tr}\underset{\sim}{A}_1}|\underset{\sim}{A}_1|^{\frac{1}{2}(n_1-p-1)} e^{-\text{tr}\underset{\sim}{\Lambda}\underset{\sim}{A}_2}|\underset{\sim}{A}_2|^{\frac{1}{2}(n_2-p-1)} {}_0F_1(\tfrac{1}{2}n_1;\underset{\sim}{\Omega}\underset{\sim}{A}_1),$$

$$C_1(p,n_1,n_2)\ e^{-\mathrm{tr}\Omega}|\Lambda|^{-\frac{1}{2}n_1}\ |R|^{\frac{1}{2}(n_1-p-1)}$$

$$\cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{C_\kappa(\Omega)}{(\frac{1}{2}n_1)_\kappa C_\kappa(I_p)k!} \sum_{n=0}^{\infty} \sum_{\nu,\delta} \frac{(-1)^n\ g^{\delta}_{\kappa,\nu}\Gamma_p(\frac{n_1+n_2}{2},\delta)\ C_\delta(R\ \Lambda^{-1})}{n!} .$$

R being symmetric, can be diagonalized by an orthogonal transformation H such that $H\ R\ H' = \mathrm{diag}\ (r_1,\ldots,r_p)$ where $r_1,\ldots,r_p$ are roots of R. For uniqueness, we assume that the elements in the first row of H are positive and the roots are arranged in the order $0 < r_1 < r_2 < \ldots < r_p < \infty$. The volume element dR becomes (Constantine [1], James [4])

$$dR = \prod_{i>j} (r_i - r_j) \prod_{i=1}^{p} dr_i (d\ H) \quad . \tag{3.5}$$

Substituting R in the above expression and integrating over H, we have the joint density of $r_1,\ldots, r_p$ in the form

$$C_1(p,n_1,n_2)\ e^{-\mathrm{tr}\Omega}|\Lambda|^{-\frac{1}{2}n_1}|R|^{\frac{1}{2}(n_1-p-1)} \prod_{i>j} (r_i-r_j)$$

$$\cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{C_\kappa(\Omega)}{(\frac{1}{2}n_1)_\kappa C_\kappa(I_p)k!} \sum_{k=0}^{\infty} \sum_{\nu,\delta} \frac{(-1)^n\ g^{\delta}_{\kappa,\nu}\Gamma_p(\frac{n_1+n_2}{2},\delta)}{n!}$$

$$2^{-p} \int_{O(p)} C_\delta(HRH'\Lambda^{-1})(dH) .$$

The factor 2^{-p} multiplying the integral arises from the restriction that the elements in the first row of H are positive. Finally we use the property of the zonal polynomials given by James [4] for normalized measure

$$\int_{O(p)} C_\kappa(H\ S\ H'T)(d\ H) = [C_\kappa(S)\ C_\kappa(T)]/C_\kappa(I_p) \tag{3.6}$$

$$C_2(p,n_2)\, e^{-\mathrm{tr}\Omega}|\Lambda|^{\frac{1}{2}n_2}\, e^{-\mathrm{tr}\Lambda A_2}|A_2|^{\frac{1}{2}(n_2-p-1)}\,|A_1|^{\frac{1}{2}(n_1-p-1)}$$

$$\cdot\int_{\mathrm{Re}(T)=X_0>0} e^{\mathrm{tr}T}|T|^{-\frac{1}{2}n_1}\, e^{-\mathrm{tr}(I - W)A_1}\,(d\,T) \quad ,$$

where

$$C_2(p,n_2) = 2^{\frac{1}{2}p(p-1)} \Big/ \left[(2\pi i)^{\frac{1}{2}p(p+1)}\, \Gamma_p\left(\frac{n_2}{2}\right)\right], \quad W = \Omega^{\frac{1}{2}}T^{-1}\Omega^{\frac{1}{2}}$$

and $T = X_0 + i\,Y$ with X_0 positive definite symmetric matrix and Y a non-singular real symmetric matrix such that $(I - W)$ is nonsingular. The roots are invariant under the simultaneous transformations

$$(I - W)^{\frac{1}{2}}\, A_1(I - W)^{\frac{1}{2}} = B_1$$

and

$$(I - W)^{\frac{1}{2}}\, A_2(I - W)^{\frac{1}{2}} = B_2 \quad .$$

Making substitutions in the above density and after taking the Jacobians into consideration, we obtain the joint density of B_1 and B_2 as follows:

$$C_2(p,n_2)e^{-\mathrm{tr}\Omega}|\Lambda|^{\frac{1}{2}n_2}e^{-\mathrm{tr}B_1}|B_1|^{\frac{1}{2}(n_1-p-1)}\,|B_2|^{\frac{1}{2}(n_2-p-1)}$$

$$\cdot\int_{\mathrm{Re}(T)>0} e^{\mathrm{tr}T}|T|^{-\frac{1}{2}n_1}|I - W|^{-\frac{1}{2}(n_1+n_2)}\, e^{-\mathrm{tr}(I - W)^{-\frac{1}{2}}\Lambda(I - W)^{-\frac{1}{2}}B_2}(dT) \quad .$$

Apply the transformation $B_1 = B_2^{\frac{1}{2}}\, R\, B_2^{\frac{1}{2}}$ and $B_2 = B_2$ and integrate out B_2 using (2.7) with $T = 0$ and also (2.8), then we have the density of R in the form

$$(3.10)\qquad C_3(p,n_1,n_2)e^{-\mathrm{tr}\Omega}|\Lambda|^{-\frac{1}{2}n_1}|R|^{\frac{1}{2}(n_1-p-1)}\prod_{i>j}(r_i-r_j)$$

$$\cdot\int_{\mathrm{Re}(T)>0} e^{\mathrm{tr}T}|T|^{-\frac{1}{2}n_1}\int_{O(p)} 2^{-p}|I+HRH'(I-W)^{\frac{1}{2}}\Lambda^{-1}(I-W)^{\frac{1}{2}}|^{-\frac{1}{2}(n_1+n_2)}(dH)(dT)\ .$$

Following Khatri [5], we can write

$$|I+H\,R\,H'A| = |I+\lambda R|\ |I-(I-\lambda^{-1}A)\ H\ (\lambda R)\ (I+\lambda R)^{-1}H'|\ ,$$

where λ is a positive real number and in our case $A=(I-W)^{\frac{1}{2}}\Lambda^{-1}(I-W)^{\frac{1}{2}}$.
After making use of James [4]

$$\int_{O(p)}|I-H\,R\,H'A|^{-a}\,(dH) = {}_1F_0(a;\ A,\ R)$$

and the formula (2.4), the expression (3.10) now becomes

$$(3.11)\qquad C_4(p,n_1,n_2)e^{-\mathrm{tr}\Omega}|\Lambda|^{-\frac{1}{2}n_1}|R|^{\frac{1}{2}(n_1-p-1)}|I+\lambda R|^{-\frac{1}{2}(n_1+n_2)}\prod_{i>j}(r_i-r_j)$$

$$\cdot\int_{\mathrm{Re}(T)>0} e^{\mathrm{tr}T}|T|^{-\frac{1}{2}n_1}{}_1F_0\left(\frac{n_1+n_2}{2};\ I-\lambda^{-1}(I-W)^{\frac{1}{2}}\Lambda^{-1}(I-W)^{\frac{1}{2}},\lambda R(I+\lambda R)^{-1}\right)dT,$$

where

$$C_4(p,n_1,n_2) = [2^{\frac{1}{2}p(p-1)}\Gamma_p\left(\frac{n_1+n_2}{2}\right)\pi^{\frac{1}{2}p^2}]/[(2\pi i)^{\frac{1}{2}p(p+1)}\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{p}{2}\right)]\ .$$

Now we can make use of (2.2) to expand ${}_1F_0$ and integrate term by term. The expression (3.11) involves the integral of the form

b) For $\underset{\sim}{\Lambda} = \underset{\sim}{I}$ and $\lambda = 1$ and using the relation (Constantine [1966])

$$\frac{L_{\delta}^{\gamma}(\underset{\sim}{S})}{(\gamma+m)_{\kappa} C_{\kappa}(\underset{\sim}{I})} = \sum_{n=0}^{k} \sum_{\nu} \frac{(-1)^{n} a_{\kappa,\nu} C_{\nu}(\underset{\sim}{S})}{(\gamma+m)_{\nu} C_{\nu}(\underset{\sim}{I})} ,$$

where $m = \frac{1}{2}(p+1)$ whenever $\underset{\sim}{S}$ is $(p \times p)$ matrix and setting $\nu = \delta = \kappa$ we have the result of Constantine [1], James [4].

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14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Distribution						
Characteristic roots						
Covariance matrices						
MANOVA						
Violations						
Robustness						