ON A MATHEMATICAL THEORY OF QUANTAL RESPONSE ASSAYS AND A NEW MODEL IN DAM THEORY

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CHAPTER I

ON A MATHEMATICAL THEORY OF QUANTAL RESPONSE ASSAYS

1. INTRODUCTION. Consider the following biological phenomenon. At time $t = 0$, each member of a group of hosts such as animals is injected with a dose of a specified virulent organism such as viruses or bacteria, which elicit a characteristic response from the host during the course of time. This response may be death, development of a tumor or some other detectable symptom. If $n(t)$ denotes the number of hosts not responding by time t, the plot against t of either $n(t)$ itself or of the proportion $n(t)/n(0)$ is known as the time dependent response curve. These response curves differ with the dose and with the type of the organism. However, generally speaking, the larger the injected dose, the sooner the host responds. About seven years ago Professor Puri sought to explain these observed response curves through a suitable stochastic model. Upon a search of the existing literature at the time, it was found that most of the models considered until then, were based on the hypothesis of existence of a fixed threshold. This hypothesis postulates that while the organisms are undergoing a certain growth process within the host, as soon as their number touches a fixed threshold N, the host responds. In [39], this hypothesis was abandoned; first because this hypothesis is not strictly correct,

second, it is not clear what value one ought to assign to N in a given situation, and third because the threshold hypothesis necessitates consideration of first passage time problems, thus making the algebra unnecessarily intractable. Instead an alternative hypothesis originally suggested by Professor LeCam was adopted. Here, unlike in the threshold hypothesis, the connection between the number $Z(t)$ of organisms in a host at time t and the host's response is indeterministic in character. More exactly, it is assumed that the value of $Z(t)$ (or possibly of a random variable whose distribution is dependent on the process $\{Z(t)\}$) determines not the presence or the absence of response, but only the probability of response of the host. Mathematically, this amounts to postulating the existence of a nonnegative risk function $f(x,t)$ such that

$$P(\text{host responds during } (t,t+\tau) \mid \text{not responded until } t \text{ and } Z(t) = x) = \delta f(x,t)\tau + o(\tau) \ ,$$

where $\delta \geq 0$ and f satisfies certain mild regularity conditions. Stochastic models based on this more appropriate alternative hypothesis have been explored with a reasonable amount of success in a paper which appeared in the Fifth Berkeley Symposium [39] and again in a later paper connected with bacteriophage reproduction (see Puri [40]). In fact in [39] , it is assumed that the risk function f depends not only on $Z(t)$ but also on the integral $\int_0^t Z(\tau)d\tau$. The latter integral is interpreted as a measure of the amount of toxin produced by the live bacteria during the interval $(0,t)$ assuming, of course, that the toxin excretion rate is constant per bacterium per unit time.

The above models (see [39], [40]) apply to the situations where the response causing agents are self-reproducing such as viruses, bacteria etc. A natural question which arises is how a similar model based on the alternative hypothesis would behave in situations where the agent is not self-reproducing. Such would be the case where, for instance, the agent is a chemical poison, insecticide or drug. This type of situation is encountered in what is commonly described as Quantal Response Assays. The classical theory of quantal response assays has been developed by Finney [15], [16]), Bliss [8], and others. One of the purposes of this theory is to estimate the relative potency of one drug against another by using measures such as E.D.50, the dose which is just about enough to cause response among on the average about 50 per cent of the subjects. Here, typically the experimenter chooses a set of doses of each drug and tests each dose on a batch of subjects. At the end of the test, the experimenter records how many of the subjects responded. In order to analyze the data so obtained, it has been customary to make the following assumptions:

(i) For each subject there exists a tolerance limit or a threshold level T. This limit for a subject is the dose which will be just sufficient to produce the response, so that the subject will respond if $z \geq T$ and will not respond if $z < T$, where z is the dose injected.

(ii) The threshold level T is assumed to be a random variable varying over the population of subjects, with a common distribution. Thus the probability that a randomly chosen subject responds after receiving a dose z, is given by

$$P(z) = P(T \leq z) .$$

It is a common practice with the experimenters to use log dose or $x = \log z$, known as the dose metameter of z. Now if $g(y)$ is a probability density function so that $\int_{-\infty}^{\infty} g(y)\, dy = 1$, the form of the distribution of log T typically can be represented by the density

$$dQ(x) = \eta g(\gamma+\eta x)\, dx ,$$

where γ and η are the usual location and scale parameters respectively. With this, one easily obtains

$$P(z) = \int_{-\infty}^{\gamma+\eta \log z} g(y)\, dy . \tag{1.1}$$

In practice, the choice of $g(y)$ and hence of the distribution of the tolerance limit T is rather arbitrary. Some of the choices of $g(y)$ that have been used in the literature are given below.

$$g(y) = (2\pi)^{-1/2} \exp(-\tfrac{1}{2} y^2); \quad -\infty < y < \infty \tag{1.2}$$

$$g(y) = \begin{cases} \sin 2y & 0 \le y \le \pi/2 \\ 0 & \text{otherwise} , \end{cases}$$

$$g(y) = \frac{1}{2} \operatorname{sech}^2 y, \qquad -\infty < y < \infty. \tag{1.3}$$

The last one has been used by Berkson in his well known work in this area (see [7]), and leads to the following form of P.

$$P(z) = \exp\{2(\gamma+\eta\log z)\}[1+\exp\{2(\gamma+\eta\log z)\}]^{-1} . \tag{1.4}$$

Although the above classical theory has been found useful and is still being used, there are certain unattractive features in it that make one feel like giving it another look. Some of these are as follows: First, there is the same objection of assuming the existence of a tolerance limit or a threshold level for each subject even though the random element is introduced only through allowing this limit to vary randomly from subject to subject. Second, the model as it stands does not lend itself to the consideration of any biological mechanism going on within the host leading to its response. And third, it does not allow the consideration of the time when the response actually occurs if it does; all it considers is whether the response does or does not occur within a fixed length of time. These same features also underlie the more recent work of Ashford [2], Ashford and Smith [3] and Plackett and Hewlett [34], [35], in the case of mixture of drugs.

In this chapter we give the classical theory of quantal response assays a fresh look and construct new stochastic models which attempt to eliminate the above objections. This has been achieved by adopting the alternative approach of the nonthreshold type as discussed above. For the biological phenomenon under consideration, a typical stochastic model of the present type would involve the consideration of the following three main components.

(A) <u>THE INPUT PROCESS</u>. This describes the manner in which the drug is introduced into the subject. We call it the 'Input Process'. One could visualize, depending upon the situation in question, several possibilities of inputs such as a continuous time deterministic input, discrete time deterministic input or a random input according to some random mechanism.

(B) THE RELEASE PROCESS. This describes the manner in which the subject attempts to reduce the level of the drug within its body. This may be carried out either through the process of direct elimination of the drug through natural means or by changing the composition of the drug itself through biochemical processes. We shall call this the 'Release Process'. In principle, this would involve the mechanism going on within the body of the subject which takes into account the manner in which the subject copes with the drug. In experimental situations, the input process is generally controlled by the experimenter. The release process on the other hand is much more involved, and a detailed understanding of it requires a great deal of experience and knowledge of the biological system on the part of the experimenter. This in turn involves, in general, a considerable amount of experimentation probing into the nature of the release mechanism. There has been, in fact, much work done in the past in an attempt to describe this mechanism for certain situations. For instance, the compartment models of, among others, Sapirstein et al [49], Bellman [4], [5], are attempts towards a better understanding of functioning of specific organs and of various biological systems. Unfortunately not too many of these models are stochastic in nature. Again the models in dam theory (see Moran [32], Gani [17], and Prabhu [37], to cite only three references from this vast literature) could be found suitable for combining the aspects of both the input and the release processes.

(C) THE RISK FUNCTION. The most important aspect which appears not to have been considered before in the context of the classical quantal response assays is the consideration of a risk function which ties up the input and the release processes of the drug to the causation of the subject's response. Whether, in any given situation, the risk function depends only on the level Z(t) of the drug at time t, or on some other factors characterizing the biological mechanism going on within the body of the subject, would entail a considerable knowledge of the biological system.

In the next few sections, we shall attempt to incorporate the three aspects listed above into a stochastic model. Although, this has been done here under rather simplified assumptions, the results do indicate that there is something to be gained by approaching this problem from a structural point of view. In this context the reader may also find, among others, the work of Neyman and Scott [33] of great interest. Here the response causing agent is Urethane, while the response is the appearance of a tumor in the lungs of mice.

2. A STOCHASTIC MODEL BASED ON A QUANTAL RESPONSE PROCESS.

2.1. ASSUMPTIONS AND NOTATION. As a first attempt, we consider here a simple stochastic model along the lines discussed above. Following the lines of classical quantal response assays, we assume that for each subject the experiment starts with the administration (input) of a single dose Z(0) = z at time t = 0, with no other inputs thereafter. Thus if Z(t) denotes the amount of drug present at time t in the body of the subject, it is evident that with probability one Z(t) is non-increasing with t. The release process is assumed to have two components. The first one determines how often and at what times the releases occur, while the second one associates with each such occurrence a nonnegative random variable Y denoting the amount of the drug to be released if available. More specifically, if $N(t)$ denotes the number of releases occurring during $(0,t]$, we assume, for simplicity, that $N(t)$ is a Poisson process with parameter $\mu > 0$. Also, given $N(t)$, let $Y_1, Y_2, \ldots, Y_{N(t)}$ denote the random amounts to be released if available, at the release time points as determined by the Poisson process. In particular, it is assumed that conditionally given $N(t)$, the random variables $Y_1, Y_2, \ldots, Y_{N(t)}$ are independently distributed with a common distribution having the probability density fucntion

$$h(y) = \begin{cases} \beta \exp(-\beta y), & y > 0, \\ 0 & \text{elsewhere}, \end{cases} \tag{1.5}$$

where $\beta > 0$. Of course, if at any time, the random amount Y_i is greater than the amount actually available, all the available amount is then released. From the above construction, it follows that

$$Z(t) \equiv \max\,(0, Z(0) - \sum_{j=0}^{N(t)} Y_j); \quad t \geq 0 \tag{1.6}$$

where, by convention, $Y_0 = 0$. Under the Poisson process assumption, it is clear that how often and at what times the releases occur is not influenced by the changes over time in the amount of the drug actually present. This however may not be realistic in certain situations. In Section 5 we shall briefly consider a more general model incorporating this dependence in an appropriate manner. Finally, we consider the risk function. Now $Z(t)$ as defined constructively above is a continuous time stochastic process defined, say, on the probability space $(\Omega,\mathcal{G},P)$ with $\mathcal{X}$ as its state space. Let $f(x,t)$ be a nonnegative bounded function defined and continuous almost everywhere on the product set $\mathcal{X} \times [0,\infty)$. We require this function to be such that for the given process $Z(t)$, the existence and finiteness of the integral $\int_0^t f(Z(\tau,\omega),\tau)\, d\tau$ is guaranteed for every $t > 0$ and for almost all realizations ω of the process $Z(t)$. For a given sample path ω of the process $Z(t)$ we denote the state of the process $Z(t)$ at time τ by $Z(\tau,\omega)$. The function f is the risk function in (C) above. This function relates $Z(t)$ to the process $\{X(t);\ t \geq 0\}$, the <u>Quantal Response Process</u>, which we now introduce.

$X(t)$ is defined as

$$\chi(t)= \begin{cases} 1, & \text{if the subject does not respond until } t \\ 0, & \text{otherwise ,} \end{cases}$$

where $\chi(0) = 1$. Also it is assumed that

$$(1.7)\quad P(\chi(t+\tau) = 0 \mid \chi(t) = 1,\ Z(t) = x) = \delta f(x,t)\tau + o(\tau) \text{ ,}$$

where $\delta > 0$. Using a standard argument it is easy to show that

$$(1.8)\quad P(\chi(t) = 1 \mid \omega) = \exp\{-\delta\int_0^t f(Z(\tau,\omega),\tau)\, d\tau\} \text{ ,}$$

for a given realization ω of the process $Z(t)$. From (1.8) we obtain the transform

$$(1.9) \qquad E(\chi(t)\cdot \exp(-s\, Z(t))) = E[\exp\{-s\, Z(t) - \delta \int_0^t f(Z(\tau),\tau)d\tau\}],$$

where $Re(s) \geq 0$. In particular, this yields

$$(1.10) \qquad P(L > t) = P(\chi(t) = 1) = E[\chi(t)] = E[\exp\{-\delta \int_0^t f(Z(\tau),\tau)d\tau\}],$$

where L is the length of time the subject takes to respond. Taking δ in (1.10) as a dummy variable, it follows that the response time distribution can equivalently be studied by obtaining the distribution of the integral $\int_0^t f(Z(\tau),\tau)d\tau$. The reader may find this particular connection explored in detail elsewhere (see Puri [41], [42], [43], [44]). Now L may not be a proper random variable, since the desired response may never occur. In that case

$$P(L=\infty)=P(\text{no response})=\lim_{t\to\infty} E(X(t))=E(\exp\{-\delta\int_0^\infty f(Z(\tau),\tau)d\tau\}).$$

Inasmuch as the classical theory relies so heavily on the threshold hypothesis we make the following remark in passing. Let Z_L denote the effective level of drug remaining at the time of response, given there is a response. Then by the analysis in [44] the transform of the distribution of Z_L, given a response occurs, is given by

$$(1.11) \qquad E[\exp(-sZ_L)|L < \infty] =$$
$$\delta[P(L < \infty)]^{-1}\int_0^\infty E\{\exp[-sZ(t)-\int_0^t f(Z(\tau),\tau)d\tau]f(Z(t),t)\}dt \ .$$

Unfortunately this expression cannot be applied easily, even in the special case of Section 4.

At this point, we introduce the following notation:

$$W_1(t,z,x) = P(Z(t) \leq x,\ \chi(t) = 1 | Z(0) = z,\ \chi(0) = 1),$$

$$W_1(t,z) \quad = P(\chi(t) = 1 | Z(0) = z,\ \chi(0) = 1),$$

$$W\ (t,z,x) = P(Z(t) \leq x | Z(0) = z)\ ,$$

$$\phi_1(\theta,z,x) = \int_0^\infty \exp(-\theta t)\ W_1(t,z,x)\ dt\ ,$$

$$\phi_1(\theta,z) \quad = \int_0^\infty \exp(-\theta t)\ W_1(t,z)\ dt\ ,$$

$$\phi\ (\theta,z,x) = \int_0^\infty \exp(-\theta t)\ W\ (t,z,x)\ dt\ ,$$

where $Re(\theta) > 0$, $0 \leq x \leq z$, and

$$W(t,0,x) = W(t,z,z) = 1, \quad \text{for } x \geq 0\ ,$$

$$W(t,z,x) = W_1(t,z,x) = 0, \text{ for } x < 0\ .$$

Here the last line follows from the fact that zero is an absorption state for the process $Z(t)$.

In the next subsection, we shall attempt to obtain expressions for the quantities defined above, through setting up the usual Kolmogorov backward integral equations involving these quantities.

2.2. CERTAIN INTEGRAL EQUATIONS AND THEIR SOLUTIONS.

Unless mentioned to the contrary, we assume henceforth that the risk function f does not explicitly depend on time t and depends only on the level $Z(t)$. Moreover, it is assumed that $f(x)$ is differentiable for all $x \geq 0$.

By considering the moment of the first release during $(0,t)$ and the amount to be released, it is easy to establish the following Kolmogorov backward integral equation for the probability $W_1(t,z,x)$ for $x < z$.

$$(1.12)\quad W_1(t,z,x) = \mu\beta \int_0^t \exp\{-(\mu+\delta f(z))u\}\Big[\int_0^{z-x} W_1(t-u,z-y,x)\exp(-\beta y)\, dy + \\ + \int_{z-x}^{z} W_1(t-u,z-y)\exp(-\beta y)\, dy + \\ + \exp(-\delta f(0)(t-u))\int_z^{\infty} \exp(-\beta y)dy\Big]\, du.$$

Taking Laplace transform of both sides of (1.12) we have for $\mathrm{Re}\ \theta > 0$,

$$(1.13)\quad (\mu+\theta+\delta f(z))\phi_1(\theta,z,x) = \mu\beta \int_0^{z-x} \phi_1(\theta,z-y,x)\exp(-\beta y)\, dy + \\ + \mu\beta \int_{z-x}^{z} \phi_1(\theta,z-y)\exp(-\beta y)\, dy + \\ + \exp(-\beta z)[\mu/(\theta+\delta f(0))]\ .$$

Similarly, we have the corresponding equation for $W_1(t,z)$ given by

$$W_1(t,z) = \exp\{-(\mu+\delta f(z))t\}+\mu \exp(-\beta z)\int_0^t \exp\{-(\mu+\delta f(z))u-\delta f(0)(t-u)\}\, du + \\ (1.14)\quad + \mu\beta \int_0^t \exp\{-(\mu+\delta f(z))u\}\int_0^z W_1(t-u,z-y)\exp(-\beta y)\, dy\, du,$$

or equivalently in terms of its Laplace transform, by

$$(1.15)\quad (\mu+\theta+\delta f(z))\phi_1(\theta,z) = 1 + \exp(-\beta z)[\mu/(\theta+\delta f(0))] + \\ + \mu\beta \exp(-\beta z)\int_0^z \exp(\beta v)\phi_1(\theta,v)\, dv\ .$$

Equation (1.15) can easily be converted into the differential equation

$$(1.16) \qquad \phi_1' + \phi_1[\beta+(\delta f'-\mu\beta)(\mu+\theta+\delta f)^{-1}] = \beta(\mu+\theta+\delta f)^{-1} ,$$

where ϕ_1' and f' are the corresponding derivatives with respect to z. Solving (1.16) subject to the initial condition

$$\phi_1(\theta,0) = (\theta+\delta f(0))^{-1} ,$$

we obtain

$$(1.17) \qquad \phi_1(\theta,z) = (\theta+\mu+\delta f(z))^{-1} \exp\{-\beta\int_0^z A(u)\,du\} \cdot \left[(A(0))^{-1} + \beta\int_0^z \exp\{\beta \int_0^v A(u)\,du\}\,dv\right] ,$$

where

$$A(u) = [\theta+\delta f(u)][\theta+\mu+\delta f(u)]^{-1}, \quad u \geq 0 .$$

Substituting (1.17) in (1.13) and solving (1.13) in an analogous manner we have the solution for (1.13) given by

$$(1.18) \qquad \phi_1(\theta,z,x) = (\theta+\mu+\delta f(z))^{-1} \exp\{-\beta \int_0^z A(u)\,du\} \cdot \left[(A(0))^{-1}-\exp\{\beta\int_0^x A(u)du\}+\beta\int_0^x \exp\{\beta\int_0^v A(u)du\}dv\right] ,$$

where $x < z$. As a check, letting $x \to z$ in (1.18) and subtracting the result from (1.17) we obtain, as expected,

$$\int_0^\infty P(Z(t) = z,\ \chi(t) = 1 | Z(0) = z, \chi(0) = 1) \exp(-\theta t)\, dt$$

$$= \phi_1(\theta,z) - \phi_1(\theta,z,z-) = (\theta+\mu+\delta f(z))^{-1} .$$

The expressions for the transforms as given by (1.17) and (1.18), in principle, are sufficient for determining the joint distribution of $\chi(t)$ and $Z(t)$. Unfortunately, to carry out the inversion of these transforms in this generality is rather cumbersome. Later on, we shall carry out their inversion for a special case. Again, if $f(0) > 0$, using a Tauberian argument it follows from (1.17) that

$$\text{(1.19)} \qquad \psi(z) \equiv P(L = \infty | Z(0) = z) = \lim_{\theta\to 0} \theta\ \phi_1(\theta,z) = 0 ,$$

so that L is a proper random variable. In fact, using the relation

$$\text{(1.20)} \qquad \phi_1(\theta,z) = \int_0^\infty \exp(-\theta t)\ P(L > t)\ dt = \frac{1}{\theta}(1 - E[(\exp(-\theta L)])$$

and (1.17) we have

$$\text{(1.21)} \quad E[\exp(-\theta L)] = 1 - (\theta+\mu+\delta f(z))^{-1} \exp\{-\beta \int_0^z A(u)\, du\} .$$

$$\cdot [\theta(A(0))^{-1} + \beta\theta \int_0^z \exp\{\beta \int_0^v A(u)du\}\, dv] .$$

One could now easily obtain moments of L from (1.21). In particular, it follows from (1.20) that

$$\text{(1.22)} \qquad E(L) = \lim_{\theta\to 0} \phi_1(\theta,z) = \phi_1(0,z) .$$

2.3. PROBABILITY OF NO RESPONSE FOR THE CASE WITH $f(0) = 0$.

If $f(0) > 0$, this would mean that the response could be caused even without the presence of the drug. However, in most practical situations this appears unrealistic, except when the response is the death of the subject. Even in the latter case, one could define response as the death caused by the drug and not by other causes; or as an approximation to the actual situation one could ignore the other causes, in which case $f(0) = 0$ would be a reasonable requirement. A more realistic model of this latter situation would be the one which incorporates other causes besides the one due to the drug, since, in principle, all these causes simultaneously compete against each other for the life of the subject. However, at present we shall not venture into this refinement and instead assume $f(0) = 0$ in what follows. With this assumption the random variable L is no longer a proper random variable, since the probability that the subject never responds will be positive. Again, in quantal response assays, where the actual response times are often not reported, one is typically interested only in the probability that the subject never responds. This is valid only as an approximation assuming, of course, that the subject has been under observation for a sufficient length of time. Using (1.17) with $f(0) = 0$, this probability, denoted by $\psi(z)$, is given by

$$\psi(z) = P(\text{subject never responds} \mid Z(0) = z)$$

$$= \lim_{\theta \to 0} \theta \, \Phi_1(\theta, z) \ ,$$

and satisfies the integral equation

$$(1.23)\quad \exp(\beta z)[\mu+\delta f(z)]\psi(z) = \mu + \mu\beta\int_0^z \exp(\beta v)\psi(v)\,dv\ .$$

Differentiation of both sides of (1.23) with respect to z leads to the standard differential equation

$$(1.24)\qquad \frac{\partial\psi(z)}{\partial z} = -\delta\psi(z)[\beta f(z) + f'(z)].$$

Subject to the initial condition $\psi(0) = 1$, equation (1.24) has a unique solution given by

$$(1.25)\quad \psi(z) = \mu(\mu+\delta f(z))^{-1}\exp\{-\beta\int_0^z \delta f(u)[\mu+\delta f(u)]^{-1}\,du\}.$$

From (1.17) we may derive the distribution of the length of time until the response is observed given that the response occurs. For instance, from (1.19) and (1.20) it follows that

$$(1.26)\quad E(L|L<\infty) = \frac{E(LI_{[L<\infty]})}{P(L<\infty)} = \left.\frac{\dfrac{-\partial\{\theta\Phi_1(\theta,z)\}}{\partial\theta}}{P(L<\infty)}\right|_{\theta=0},$$

where in the event the left side is infinite then the limit on the right hand side as $\theta \to 0$ is also infinite. The expression for (1.26) derived using (1.17) is complicated. In Section 4 we shall give a simplified expression for (1.26) for the special case there.

In the next section we exhibit a comparison of the classical quantal response model with the present one through the use of the expression (1.25).

3. A COMPARISON OF THE PRESENT MODEL WITH THE CLASSICAL ONE.

It appears rather natural at this stage to look for some kind of direct comparison between the present theory and the classical one. Unfortunately there does not appear to be any simple way of making such a comparison, mainly because the two theories are based on entirely different points of view. However, if we insist on making one, the only way which appears reasonable is to equate the end result common to both the theories. More specifically, by equating the probability of no response under the classical theory, namely

$$1 - P(z) = P(T > z) = \int_{\gamma+\eta \log z}^{\infty} g(y)\, dy ,$$

to the probability of no response under the present model, namely $\psi(z)$, we ask what risk function $f(\cdot)$ of the present model would correspond to a given density function $g(y)$ used in the classical theory. To this end, one can easily solve (~~1.25~~ 1.23) for $f(z)$ in terms of ψ yielding

$$(1.27) \qquad \delta f(z) = [\psi(z)]^{-1}\exp(-\beta z)[\mu\{1-\psi(z)\exp(\beta z)\}+\mu\beta\int_0^z \exp(\beta v)\psi(v)\, dv].$$

Now by replacing $\psi(z)$ with $1 - P(z)$, one obtains the desired risk function f corresponding to a given density g of the classical model. For instance, the risk function corresponding to the normal density (1.2) is given by

$$(1.28) \qquad \delta f(z) = \mu \exp(-\beta z)[1 - H(\gamma+\eta \log z)]^{-1}[1-\exp(\beta z)\cdot$$
$$\cdot(1-H(\gamma+\eta \log z))+\beta\int_0^z \exp(\beta v)(1-H(\gamma+\eta \log v))dv],$$

where $H(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} \exp(-\frac{1}{2}\tau^2)\, d\tau$. Similarly for the density function of (1.3), we have

$$\delta f(z) = \mu \exp(-\beta z)[1+\exp\{2(\gamma+\eta \log z)\}] \cdot \{1 - \exp(\beta z)[1+\exp\{2(\gamma+\eta \log z)\}]^{-1} + \beta\int_0^z \exp(\beta v)[1+\exp\{2(\gamma+\eta \log v)\}]^{-1} dv\}. \tag{1.29}$$

As expected, since $P(0) = 0$, we have $f(0) = 0$ in the above formulas. Similar expressions can be obtained for f that correspond to other densities often used in the classical theory. Unfortunately, as is evident, all such expressions will usually be complicated, so that there appears to be no rationale for choosing one or the other form of the risk function in practice. In the next section, we consider the simplest form of the risk function, namely the linear function $\delta f(x) = \delta x$, which appears reasonable at least as a first approximation. The results obtained by using this simple risk function are then applied to some observed data.

4. AN APPLICATION OF THE MODEL TO OBSERVED DATA.

We shall now restrict ourselves to the case of a linear risk function with $\delta f(x) = \delta x$. For this, we have from (1.18),

$$\phi_1(\theta,z,x) = (\theta+\mu+\delta z)^{-1}[C(z)]^{-1}\exp(-\beta z) \cdot [(1+\mu/\theta) - C(x)\exp(\beta x) + \beta\int_0^x C(v)\exp(\beta v)\, dv], \tag{1.30}$$

where $x < z$ and

$$C(v) = [(\theta+\mu)/(\theta+\mu+\delta v)]^{\beta\mu/\delta},\ v \geq 0\ .$$

Also the expression (1.17) now takes the form

$$(1.31)\qquad \phi_1(\theta,z) = [(\theta+\mu)/(\theta+\mu+\delta z)][C(z)]^{-1}\exp(-\beta z)\cdot$$

$$\cdot\ [1/\theta + \{\beta/(\mu+\theta)\}\int_0^z C(v)\exp(\beta v)\ dv]\ .$$

The transforms (1.30) and (1.31) can be easily inverted to produce the expressions for $W_1(t,z,x)$ and $W_1(t,z)$ respectively; for instance, when $\beta\mu/\delta$ is not an integer,

$$(1.32)\quad W_1(t,z)=\sum_{k=0}^{\infty}\frac{1}{k!}\,[A_k(\frac{\beta\mu}{\delta}-1)\cdot\{\delta z/(\mu+\delta z)\}^k\exp(-\beta z)\mathcal{I}_{k,\mu+\delta z}(t)$$

$$+\ \bar{A}_k(\beta\mu/\delta)\cdot(\delta/\beta)^k\ \mathcal{I}_{k+1,\beta}(z)\cdot t^k\exp\{-(\mu+\delta z)t\}]\ ,$$

where

$$A_k(x) = x(x+1)(x+2)\ \ldots\ (x+k-1),\ k \geq 1;\ A_0(x) \equiv 1,$$

and for $\alpha > 0$,

$$\mathcal{I}_{k,\alpha}(x) = \int_0^x \frac{\alpha^k y^{k-1}}{\Gamma(k)}\exp(-\alpha y)dy;\ k \geq 1;\ \mathcal{I}_{0,\alpha}(x) \equiv 1.$$

It is easy to verify that by letting δ tend to zero in (1.30) one obtains

$$(1.33)\quad \lim_{\delta\to 0}\phi_1(\theta,z,x)=\phi(\theta,z,x)=\mu[\theta(\theta+\mu)]^{-1}\exp\{-\beta\theta(\theta+\mu)^{-1}(z-x)\};\ x < z,$$

a result for the process $Z(t)$ alone without the consideration of the quantal response process $\chi(t)$. Again, for the present case with

$f(x) = x$, we have from (1.25) the expression for the probability of no response, as given by

$$(1.34) \qquad \psi(z) = \exp(-\beta z)[1 + \frac{\delta}{\mu} z]^{-1+\beta\mu/\delta}.$$

Given that a response does occur, we have in this case, from (1.26) and (1.31)

$$(1.35)\ E(L|L < \infty] = \mu^{-1}[\beta\int_0^z [1+v(\delta/\mu)]^{-\beta\mu/\delta}\exp(\beta v)dv-\{z(\beta\mu-\delta)(\mu+\delta z)^{-1}\}]\cdot$$
$$\cdot[\exp(\beta z)\{1+z(\delta/\mu)\}^{1-\beta\mu/\delta}-1]^{-1} .$$

Let Z_0(E.D.50) denote that dosage level which will produce a response with probability one-half. From (1.34) we see that Z_0 satisfies the relation

$$(1.36) \qquad \ln \frac{1}{2} = -\beta Z_0 + (\beta\frac{\mu}{\delta} - 1) \log[1+ \frac{\delta}{\mu} Z_0].$$

The no-response probability (1.34) and equation (1.36) are relevant to the data usually available from a quantal response assay. The expression (1.34) contains essentially two parameters, since μ and δ always appear as μ/δ. However, for fitting the above formula to suitable data, it was found convenient to introduce the reparameterization

$$\rho = \delta/\mu,\ \lambda = \rho(\beta-\rho) ,$$

so that

$$(1.37) \qquad \psi(z) = (1+\rho z)^{\lambda/\rho^2} \exp\{-(\rho+\lambda/\rho)z\} .$$

From (1.37), it follows that Z_0 satisfies the relation

$$(1.38) \qquad \ln \frac{1}{2} = -\left(\rho + \frac{\lambda}{\rho}\right) Z_0 + \frac{\lambda}{\rho^2} \ln (1+\rho Z_0) .$$

The formula (1.37) was fitted to the data based on a study of the toxicity of an insecticide known as Deguelin. The data are due to Martin [29] and have also been used by Berkson [7] in an attempt to fit the classical model of the quantal response assays. In the study proper, concentrations at different dose levels z_i of Deguelin were prepared in an alcohol medium. These were then sprayed on groups of respective sizes n_i of the test insects (Adult Apterous Female) Aphis Rumicis. These sprayings were performed in a carefully controlled way using a special atomizer. After spraying, the insects without further handling were placed in tubes with a small amount of bean foliage. They were checked after about 20 hours for the number r_i of deaths in the ith group. These data are given in Table 1.

TABLE 1: MARTIN'S DATA ON TOXIC EFFECT OF DEGUELIN

Concentration mg/litre (z_i)	10.1	20.2	30.3	40.4	50.5
Total number (n_i)	48	48	49	50	48
Number of Deaths (r_i)	18	34	47	47	48

The formula (1.37) was fitted to the above data by using the standard method of minimum chi-square. The fit appears quite satisfactory, since the observed value of the chi-square is 3.69 (3 degrees of freedom), which is not significant at the 5 percent level where the table value is 7.81. Also the method of maximum likelihood led to the estimates for the parameters ρ and μ, along with their standard errors, as given below. Using these and the relation (1.38), an estimate $\hat{Z}_0$ of E.D.50 was obtained by using a computer search procedure for finding the appropriate root $\hat{Z}_0$ of (1.38). The various estimates of the standard errors, as given here, are based on the standard large sample formulas valid for the maximum likelihood estimates.

$$\hat{\lambda} = 0.00526, \qquad \hat{\rho} = 0.02428, \qquad \hat{Z}_0 = 13.117$$

$$\text{S.E}(\hat{\lambda}) = 0.00102, \qquad \text{S.E}(\hat{\rho}) = 0.0143, \qquad \text{S.E}(\hat{Z}_0) = 1.435.$$

As a passing remark, it may be appropriate to mention here that we also fitted the formula (1.37) to data reported in [7] which pertain to responses to certain bacteria. Here, as expected, the fit was considerably worse. For the four degrees of freedom available in that case, the observed chi-square was 12.9. This being significant indicates the sensitivity of the present model to situations where the response causing agent is self-reproducing. The models appropriate for such situations have already been dealt with elsewhere (see Puri [39], [40]). The present model is, of course, not designed for such situations.

5. A MODEL WITH A GENERALIZATION OF THE RELEASE PROCESS.

In the release process as adopted in the above model, how often and at what times the releases occur is not influenced by the changes over time in the amount of the drug actually present in the subject. In this section we modify the release process of the model in order to take into account the possible effect of the changes over time in the level of the drug on the frequency of the releases. We attempt here to accomplish this through a generalization of the Poisson process of Section 2. Let $\mu(z)$ be a nonnegative bounded function, which may be called the risk function for the release, such that

$$\begin{aligned} &P(\text{a release occurs during } (t,t+\tau)\,|\,Z(t)=z)=\mu(z)\tau+o(\tau), \\ (1.39)\quad & \\ &P(\text{more than one release occur during } (t,t+\tau)\,|\,Z(t)=z)=o(\tau). \end{aligned}$$

The random variables $Y_1, Y_2, \ldots$, denoting the amounts to be released, if available, at the release points governed by (1.39), are as before independently distributed with the common distribution given by (1.5). Clearly, when $\mu(z)$ is a positive constant, we are back to the case of the Poisson release process. All the other assumptions of the model as outlined in Section 2 remain the same with the ~~only~~ exception of (1.39) and that we assume that $f(0) = 0$. It may be remarked here that there is no loss in generality so far as the distribution of the quantal response process $\chi(t)$ is concerned, if we allow $\mu(0)$ to be positive. In the latter case we can still fictitiously talk of the releases, even though the level of the drug may be zero. Thus we assume that $\mu(0) > 0$, for convenience. Let $N(t)$ denote the number of releases occurring during $(0,t]$. Also we introduce the following notation.

$$(1.40)\quad \begin{cases} V_1(k,t,z) = P(\chi(t)=1,\ N(t) = k|Z(0)=z,\ \chi(0)=1),\ k \geq 0, \\ V_1(t,z) \quad = P(\chi(t)=1|Z(0)=z,\ \chi(0)=1), \\ V(k,t,z) \quad = P(N(t)=k|Z(0)=z), k \geq 0. \end{cases}$$

It is not too difficult to show that the random variable $N(t)$ is a proper random variable for every $t \geq 0$, so that the probability of an infinite number of releases occurring during a finite time interval is zero. As such

$$V_1(t,z) = \sum_{k=0}^{\infty} V_1(k,t,z) \ .$$

Again taking into account the first release, if it occurs, it is easy to establish the following system of recurrence relations for the V's.

$$(1.41)\quad V_1(0,t,z)=\exp\{-(\mu(z)+\delta f(z))t\}.$$

$$(1.42)\quad V_1(k,t,z)=\mu(z)\int_0^t \exp\{-(\mu(z)+\delta f(z))u\}\{\int_0^z \beta\exp(-\beta y)V_1(k-1,t-u,z-y)dy$$

$$+\exp(-\beta z)V(k-1,t-u,0)\},\ k \geq 1 \ .$$

Let

$$(1.43)\quad \begin{aligned} V_1^*(k,\theta,z) &= \int_0^\infty \exp(-\theta t)V_1(k,t,z)\ dt \ , \\ V^*(k,\theta,z) &= \int_0^\infty \exp(-\theta t)V\ (k,t,z)\ dt \ , \end{aligned}$$

where Re $\theta > 0$. Then from (1.41) and (1.42) we have

$$(1.44) \qquad V_1^*(0,\theta,z) = [\theta+\mu(z)+\delta f(z)]^{-1}$$

$$(1.45) \qquad V_1^*(k,\theta,z) = \mu(z)[\mu(z)+\delta f(z)+\theta]^{-1}\cdot\{\exp(-\beta z)V^*(k-1,\theta,0) + \beta\int_0^z \exp(-\beta y)V_1^*(k-1,\theta,z-y)dy\},\ k \geq 1 .$$

Clearly

$$(1.46) \qquad V(k,t,0) = \frac{[\mu(0)t]^k}{k!}\exp(-\mu(0)t) ,$$

so that

$$(1.47) \qquad V^*(k,\theta,0) = [\mu(0)]^k[\mu(0)+\theta]^{-k-1} .$$

Using this, one can solve the system (1.44)-(1.45) recursively. However, our aim is to obtain $\psi(z)$, the probability of no response. To this end, adding (1.44)-(1.45) over the possible values of k, we obtain

$$(1.48) \qquad V_1^*(\theta,z) = [\theta+\mu(z)+\delta f(z)]^{-1}\cdot[1+\mu(z)\ \exp(-\beta z)\{\theta^{-1}+ + \beta\int_0^z \exp(\beta v)V_1^*(\theta,v)\ dv\}] .$$

We assume now, for simplicity, that besides $f(z)$, the risk function $\mu(z)$ is also differentiable for $z \geq 0$. With this (1.48) can be easily transformed into the differential equation,

$$(1.49) \qquad \partial V_1^*/\partial z+[\mu(\delta f'+\beta\theta+\beta\delta f)-\mu'(\theta+\delta f)][\mu(\theta+\mu+\delta f)]^{-1}V_1^* = [\beta-(\mu'/\mu)][\theta+\mu+\delta f]^{-1},$$

where f' and μ' denote, respectively, the derivatives of f and μ. Here we have suppressed, for convenience, the arguments of all the functions such as f,μ, etc. Equation (1.49) can be easily solved subject to the initial condition $V_1^*(\theta,0) = 1/\theta$, yielding

$$(1.50) \qquad V_1^*(\theta,z) = \frac{\mu(z)(\theta+\mu_0)}{\mu_0(\theta+\mu(z)+\delta f(z))} \exp(-\beta B(z))\left[\frac{1}{\theta} + \frac{\mu_0}{\theta+\mu_0} \cdot \int_0^z \{\beta\mu(s)-\mu'(s)\}\{\mu(s)\}^{-2}\exp(\beta B(s))ds\right],$$

where $\mu_0 = \mu(0)$ and

$$B(s) = \int_0^s \frac{\theta+\delta f(v)}{\theta+\mu(v)+\delta f(v)}\, dv .$$

Finally, since $\psi(z) = \lim_{\theta\to 0} \theta\, V_1^*(\theta,z)$, it follows from (1.50), that

$$(1.51) \qquad \psi(z) = \mu(z)[\mu(z)+\delta f(z)]^{-1}\exp\{-\beta\int_0^z \delta f(v)[\mu(v)+\delta f(v)]^{-1}dv\}.$$

This then is the generalization of the formula (1.25) where $\mu(z)$ was assumed to be a positive constant. ~~Finally~~ For the special case with $f(x) = x$ and $\mu(x) = \mu_0 + \nu x$ such that $\mu_0 + \nu x > 0$ for $0 \le x \le z$, we have,

$$(1.52) \quad \psi(z) = \exp\{-\frac{\beta\delta}{\nu+\delta} z\}\,[1 + \frac{\nu z}{\mu_0}][1 + \frac{\delta+\nu}{\mu_0} z]^{-1+\beta\delta\mu_0(\delta+\nu)^{-2}}, z \ge 0 .$$

6. DISCUSSION. The present chapter is inspired by the need of giving a fresh look at the classical theory of quantal response assays (see Finney [16]), which appears to have certain unappealing features. Although most of the mathematical models of random phenomena incorporate assumptions which tend to simplify the real situation, by now it is

evident that there are certain fundamental differences in the approach adopted here from the one classically used. For instance, the present approach permits the consideration of the response time, while the classical one does not. Unlike the classical approach, the present one is based on a nonthreshold hypothesis which appears more appealing. Most importantly, however, the present model allows ample room for the consideration of the mechanism of the causation of the response, while the classical theory does not. The mechanism incorporated in the model studied here may be oversimplified for certain situations. However, this, in general, can easily be rectified by incorporating more complicated yet realistic mechanisms into the present theory, usually, of course, at the cost of making the algebra more involved.

In the present model the only input allowed is at the start of the experiment. However, this can easily be extended to cover the general case, where the input pattern over time is controlled and determined ahead of time by the experimenter (see Neyman and Scott [33]). Also situations such as exposure to natural radiation, or to specific chemicals as part of certain occupational hazards, involve perhaps a random mechanism for the input process. Such models involving more elaborate input and release processes supply the inspiration for the work in subsequent chapters of the thesis.

The classical theory of quantal response assays has been extended to the case of multiple responses to one or several drugs (see Ashford [2]) or to the case of a single response to mixture of drugs (see Ashford and Smith [3]). It appears worthwhile to examine and extend the present approach to cover these cases. Also, deeper models along the present lines, while incorporating the role of the defense mechanism utilized by the subject in order to cope with the drug, are very much needed. This mechanism, of course, may vary considerably from one situation to another. In several situations, to gain knowledge of this mechanism itself would need a considerable amount of further experimentation.

Again, in many situations it may appear realistic to consider the risk function f not only dependent on the level Z(t) of the drug but also on some other relevant functionals of the process Z(t). (See for instance, Puri [39], [40], and the work done at the Statistical Laboratory, University of California, Berkeley, to appear in the Proceedings of the Sixth Berkeley Symposium).

In the present model, a special form (1.5) of the common distribution of $Y_1, Y_2, \ldots$, the amounts released, was assumed. This can be generalized to the case with an arbitrary distribution function, say H(y), for the random variables Y's. One can easily set the integral equations analogous to (1.13) and (1.15) for this case. For instance, the equation (1.15) now takes the form

$$(1.53)\quad (\mu+\theta+\delta f(z))\phi_1(\theta,z)=1+\mu(1-H(z))(\theta+\delta f(0))^{-1}+\mu\int_0^z \phi_1(\theta,z-y)\,dH(y).$$

Unfortunately, however, the solution of these equations becomes relatively cumbersome.

Finally, it is hoped that, in due course, the approach adopted here will find its proper place in its usefulness in comparison to the classical approach. This will emerge even more when the experimenter wishes to use the data on response times of the subjects for an appropriate analysis, rather than only on whether the subject does or does not respond in a given period of time.

CHAPTER II

A NEW MODEL IN DAM THEORY

1. INTRODUCTION.

The process $\{Z(t)\}$ of Chapter I may be thought of as the level of a reservoir or dam. The initial injection serves as an input and the subsequent reductions in drug level serve as random releases or demands on the reservoir supply. In this chapter we consider a dam process $\{Z(t)\}$ in which both inputs and releases proceed according to an underlying semi-Markov process. A semi Markovian structure underlying the sequence of both inputs and releases appears to be lacking in most of the reservoir and storage models that have thus far been dealt with in the literature. Also, in the context of the problem studied in Chapter I, it is most natural to build such a structure into the models for quantal response assays.

A class of continuous time storage models which in some respects resembles the one introduced in this chapter was considered by Gani and Pyke [19]. This class of models is such that, for every $t > 0$, the net input in the interval $(0,t)$ is representable as a difference of two independent nonnegative infinitely divisible processes. More specifically those authors considered a separable, centered infinitely divisible process $\{W(t); t \geq 0\}$, and defined constructively the level Z_t of the dam with net inputs described by the process $\{W_t\}$. This they did in the following manner.

Set $Z_0 = z \geq 0$. Define the random variables

$$\tau_1 = \inf\{t \geq 0 : W_t + z \leq 0\},$$

$$T_1 = \sup\{t \geq \tau_1 : W_t \text{ is nonincreasing in } (\tau_1, t)\}.$$

Set

$$Z_t = \begin{cases} W_t + z & \text{if } 0 \leq t < \tau \\ 0 & \text{if } \tau \leq t < T_1 \\ W_{T_1+} - W_{T_1-} & \text{if } t = T_1 \;. \end{cases}$$

Now define recursively for each $k > 1$, the random variables

$$\tau_k = \inf\{t \geq T_{k-1} : \; W_t + Z_{T_{k-1}} \leq 0\},$$

$$T_k = \sup\{u \geq \tau_k \; : \; W_t \text{ is nonincreasing in } (\tau_k, t)\},$$

and set

$$Z_t = \begin{cases} W_t - W_{T_{k-1}-} & \text{if } T_{k-1} < t < \tau_k \\ 0 & \text{if } \tau_k \leq t < T_k \\ W_{T_k+} - W_{T_k-} & \text{if } t = T_k \;. \end{cases}$$

Gani and Pyke studied only the distributions of the total time in (0,t) during which the dam is nonempty and empty.

Several other models of interest to the problem of Chapter I and its extension were found in the vast literature of the theory of storage systems. One of early contributions to this theory was a paper on finite dams by Moran [30]. The inputs flowing into these dams during consecutive time intervals were assumed to form a sequence of independent identically distributed random variables. Models in the literature up to 1963 (see Prabhu [37] and Moran [31]) generally retained, as does ours, the assumption of mutual independence of the inputs. An initial attempt to consider correlated inputs was made by Lloyd and Odoom [26]. In their paper a sequence of inputs during consecutive discrete time intervals constitute a Markov chain with a finite number of states. The levels Z_t of a finite capacity dam are observed at times $t = 0,1,2,\ldots$. During the interval $(t,t+1)$ an inflow X_t is observed. The distribution of these inputs is assumed to have a stable limit distribution. An inflow may cause the level to exceed the capacity of the dam and result in an instantaneous overflow. At the end of each interval, m units of water are instantaneously released, if there are present at least m units. These authors studied only the stationary solution for the model as $t \to \infty$. They point out that the joint process $\{(Z_t,X_t)\}$ is also Markovian. From this the marginal limiting distribution of levels is derived. It is claimed that withdrawal policies of a random nature may be easily incorporated into the model, but not much was achieved in this direction. Additional work was done by Lloyd and Odoom in [27] on the stationarity for the probabilities of dam contents.

Ali Khan and Gani [1] studied the time dependent solution of the Lloyd-Odoom model. They considered a dam of infinite capacity with initial content $Z_0 = u > 0$ at $t = 0$. In the interval $(t,t+1)$ an input X_t flows into the dam where $t = 0,1,2,\ldots$, and

$X_t = 0,1,2,\ldots, m < \infty$, such that

$$P(X_{t+1}=j \mid X_t=i) = p_{ij} \quad (i,j = 0,1,\ldots,m) \; .$$

At each (t-0) there is a release of water from the dam such that this is

$$\begin{cases} 1 & \text{if } Z_{t-1}+X_{t-1} > 1 \\ Z_{t-1}+X_{t-1} & \text{if } Z_{t-1}+X_{t-1} \leq 1 \; . \end{cases}$$

According to this release rule the dam content Z_t at time t is given by

$$Z_t = Z_{t-1}+X_{t-1} - \min(Z_{t-1}+X_{t-1}, 1), \quad t = 1,2,\ldots \; .$$

Ali Khan and Gani studied the transition probabilities

$$P(k,j \mid u,i;t) = P(Z_t=k, X_{t-1}=j \mid Z_0=u, X_{-1}=i)$$

where $i,j = 0,1,\ldots,m; u > 0; k \geq 0$ and X_{-1} is the initial input. From this they derived expressions for the transition probabilities of the dam content at time $t \geq 0$.

In a recent paper [38] Prabhu studied a storage model in which the input $X(t)$ to a dam with infinite capacity during the interval $(0,t)$ is a stochastic process with stationary and independent increments. The release from the dam is continuous and is at a unit rate except when the dam is empty. The net input, or input minus the amount demanded, is given by $Y(t)=X(t)-t$. Prabhu finds that, under the conditions $E(X(t)) = \rho t$, $0 < \rho < \infty$, and $\mathrm{Var}\, X(t) < \infty$, the net input process Y(t) has an asymptotically normal distribution. Other limit distributions he derives are related to the normal in much the same way as the limit results of Chapter IV.

As is evident from the above discussion and a review of the literature, most of the models that have been considered up to this time have either a random input process or a random release process. There does not appear to be a model with both random input and random release structured as in the model we introduce in this chapter.

Brief mention could be made of the models of Hasofer ([20], [21]) and of Karlin and Fabens [23]. The former model, along with that of Gani and Pyke served as a starting point for the present model. The latter paper describes a discrete time inventory model in which only releases proceed according to an underlying semi-Markov structure. However in both of these models the level of the process takes values in the interval $(-\infty,h]$, and therefore their applicability to the problem of Chapter I is questionable. With this we turn to the semi-Markovian model.

1.1. THE MODEL.

In order to set forth a constructive definition of the process $\{Z(t)\}$ for our model, we first introduce some auxiliary processes. Consider a double sequence of random variables $\{(J_n,T_n),\ n = 0,1,\ldots\}$ taking values in the state space $\mathcal{X}\times[0,\infty)$ with $\mathcal{X} = [1,2]$. $\{(J_n,T_n)\}$ is defined on a complete probability space $(\Omega,\mathcal{G},P)$ such that $T_0 = 0$ a.s.,

$$P(J_0=i) = a_i,\ i = 1,2; \quad a_1 + a_2 = 1,$$

and

$$P(J_n=j,\ T_n \le x | T_0, J_0, T_1, J_1, \dots, T_{n-1}, J_{n-1}=i)$$

$$= P(J_n=j,\ T_n \le x | J_{n-1}=i)$$
$$= Q_{ij}(x) = p_{ij}H_i(x).$$

for $i,j = 1,2$, $x\varepsilon(-\infty,\infty)$ and $n=1,2,\dots$. The $Q_{ij}(\cdot)$ are nondecreasing and right continuous mass functions satisfying

$$\text{(2.2)} \quad \begin{aligned} &\text{(i)} \quad Q_{ij}(x) = 0 \qquad \text{for } x \le 0 \\ &\text{(ii)} \quad \sum_j p_{ij} = 1, \quad \text{where } p_{ij} = Q_{ij}(+\infty) \ . \end{aligned}$$

Thus $H_i(t) = \sum_j Q_{ij}(t)$. It is assumed that $H_i(0+) < 1$, $i = 1,2.$ and that

$$E_{H_j} \equiv \int_0^\infty (1-H_j(u))du < \infty,\ j = 1,2.$$

It is assumed that $0 < p_{ij} < 1$, $i,j = 1,2$. The matrix $\underset{\sim}{P} = (p_{ij})$ is a stochastic matrix and for this reason we shall hereafter drop the subscripts on the p's and set $1-p = p_{11}$, $q = p_{21}$. We assume further that p and q are independent of time.

The marginal sequence $\{J_n,\ n \ge 0\}$ is a two state Markov chain with $P(J_n=j|J_{n-1}=i) = p_{ij}$. Given the chain $\{J_n\}$ the random variables T_n are conditionally independent in the sense that

$$P(T_1 \le x_1, \dots, T_n \le x_n | J_0, J_1, \dots, J_{n-1}) = \prod_{i=1}^n P(T_i \le x_i | J_{i-1}).$$

Let $\tau_n = \sum_{i=0}^{n} T_i$, $n = 0,1,2,\ldots$. We define the integer-valued stochastic processes $\{N(t); t \geq 0\}$, $\{N_j(t); t \geq 0\}$ and $\{V_t; t \geq 0\}$ as

$$N(t) = \sup(n: n \geq 0,\ \tau_n \leq t)$$

$$N_j(t) = \text{number of times } J_k = j \text{ for } 0 < k < N(t) + 1,$$

$$V_t = J_{N(t)} .$$

The process $\{V_t\}$ is the ordinary semi-Markov process of Pyke [46].

Let state 1 denote the input state, and state 2 the release state. Now we introduce the independent identically distributed nonnegative random variables $X_1, X_2, \ldots$ and, independent of the X's, the independent identically distributed nonnegative random variables $Y_1, Y_2, \ldots$. The random variable X_i represents the amount of an instantaneous input to the reservoir, the random variable Y_i an instantaneous release from the reservoir. Let $B(x)$ and $D(y)$ denote the common distribution functions of the X_i and of the Y_i respectively.

We define the process $\{Z(t)\}$ constructively as follows. For $0 \leq t < \tau_1$, $Z(t) = Z(0)$ and for $k \geq 1$, and $\tau_k \leq t < \tau_{k+1}$

$$Z(t) = \begin{cases} Z(\tau_k -) + X_k, & \text{if } V_{\tau_k} = 1 \\ \\ \max(0, Z(\tau_k -) - Y_k), & \text{if } V_{\tau_k} = 2. \end{cases}$$

Here it is assumed that the sample paths of the process (J,T) are right continuous. Thus the process $Z(t)$ is almost surely continuous from the right. The process $\{V_t\}$ and hence $\{Z(t)\}$ is separable because of the constructive way each is defined.

In the constructive definition above we may regard X_k as the input corresponding to the kth visit of the process V_t to state 1, and similarly for Y_k. If at any time the random amount Y_k is greater than the amount actually available, only the available amount is released and the level remains at zero until the next input. The distribution of the waiting time T_{n+1} depends only on the value of J_n.

From Pyke [46] Lemma 3.1 we know that the two dimensional process (J,τ) is a Markov process, and the J process is a Markov chain. ~~Also,~~ Then since $\mathcal{X}$ is finite, it follows (see Pyke [~~47~~ 46], Lemma 4.1) that

$$P[N(t) < \infty, \quad \text{for all } t \geq 0 \mid J_0 = i] = 1,$$

for $i=1,2$.

<u>REMARK</u> In terms of the model described above net input in the interval $(0,t)$ may be expressed as

$$W(t) = \sum_{j=1}^{N_1(t)} X_j - \sum_{j=1}^{N_2(t)} Y_j \; .$$

It can be shown that when $H_1=H_2 \equiv H$ is of the form

$$H(t)=\begin{cases} 1 - \exp(-\eta t) & , \quad t \geq 0 \\ \\ 0 & , \quad t < 0 \end{cases}$$

where $\eta > 0$, and when $p_{11} = p_{22}$ the present model becomes a special case of that considered by Gani and Pyke [19]. However, the model of Gani and Pyke does not cover the present more general case in which the sequence of inputs and releases is structured through a semi-Markov process.

We introduce the following notation.

$$R_i(t,z,x) = P(Z(t) \leq x|Z(0)=z,\ J_0=i),\ i = 1,2,$$

for $t \geq 0$, $z \geq 0$, $x \in [0,\infty)$, and the corresponding Laplace transforms

$$\Phi_i(\theta,z,x) = \int_0^\infty \exp(-\theta t)R_i(t,z,x)dt,\ i = 1,2$$

and

$$H_i^*(\theta) = \int_0^\infty \exp(-\theta t)dH_i(t) \qquad ,\ i = 1,2$$

where $\text{Re}(\theta) > 0$. We put

$$U(w) = \begin{cases} 1 & \text{if } w \geq 0 \\ & \\ 0 & \text{if } w < 0 . \end{cases}$$

Considering the first jump of the process, if there is one, during $(0,t]$, and whether it is an input or an output, the following backward Kolmogorov integral equations can be easily established for $R_i(t,z,x)$.

(2.4a) $$R_1(t,z,x)=U(x-z)(1-H_1(t))+(1-p)\int_0^t\int_0^\infty R_1(t-\tau,z+y,x)dB(y)dH_1(\tau)+$$
$$+p\int_0^t[\int_0^z R_2(t-\tau,z-y,x)dD(y)+R_2(t-\tau,0,x)(1-D(z))]dH_1(\tau)$$

(2.4b) $$R_2(t,z,x)=U(x-z)(1-H_2(t))+q\int_0^t\int_0^\infty R_1(t-\tau,z+y,x)dB(y)dH_2(\tau)+$$
$$+(1-q)\int_0^t\Big[\int_0^z R_2(t-\tau,z-y,x)dD(y)+$$
$$+R_2(t-\tau,0,x)(1-D(z))\Big]dH_2(\tau)$$

(2.4c) $$R_2(t,0,x)=(1-H_2(t))+\alpha q\int_0^t\int_0^\infty R_1(t-\tau,y,x)dB(y)dH_2(\tau)+$$
$$+(1-q)\int_0^t R_2(t-\tau,0,x)dH_2(\tau).$$

for $t \geq 0$, $z > 0$, $x \in [0,\infty)$.

In this generality these integral equations are difficult to solve explicitly. As such we shall attempt, in the next few sections, to solve equations (2.4), in some tractable special cases. In Section 2 we examine the special case where both B and D are negative exponential distribution functions. In Section 3, it is assumed that $H_1=H_2 \equiv H$ is a negative exponential distribution function, and B is assumed to be negative exponential while D remains arbitrary.

2. THE MODEL WITH EXPONENTIAL DISTRIBUTION FOR EACH INPUT AND EACH RELEASE.

In this section we treat the case where both H_1 and H_2 remain arbitrary, and

(2.5) $$B(y) = \begin{cases} 1 - \exp(-\alpha y), & y \geq 0 \\ 0, & y < 0 \end{cases}$$

$$(2.6)\quad D(y)=\begin{cases} 1-\exp(-\beta y)\ , & y\geq 0\\ 0 & ,\ y<0\ , \end{cases}$$

where α , $\beta > 0$.

The Laplace transforms of equations (2.4) under the assumptions in (2.5) and (2.6) are given below for $z \geq 0$, $\mathrm{Re}(\theta) > 0$.

$$(2.7a)\quad \Phi_1(\theta,z,x)=U(x-z)(1-H_1^*(\theta))\theta^{-1}+$$
$$+\alpha(1-p)\exp(\alpha z)H_1^*(\theta)\int_z^\infty \exp(-\alpha v)\Phi_1(\theta,v,x)dv+$$
$$+\exp(-\beta z)pH_1^*(\theta)[\Phi_2(\theta,0,x)+\beta\int_0^z \exp(\beta v)\Phi_2(\theta,v,x)dv]$$

$$(2.7b)\quad \Phi_2(\theta,z,x)=U(x-z)(1-H_2^*(\theta))\theta^{-1}+$$
$$+\alpha q\exp(\alpha z)H_2^*(\theta)\int_z^\infty \exp(-\alpha v)\Phi_1(\theta,v,x)dv+$$
$$+\exp(-\beta z)(1-q)H_2^*(\theta)[\Phi_2(\theta,0,x)+\beta\int_0^z \exp(\beta v)\Phi_2(\theta,v,x)dv]$$

$$(2.7c)\quad \Phi_2(\theta,0,x)=(1-H_2^*(\theta))\theta^{-1}+\alpha qH_2^*(\theta)\int_0^\infty \exp(-\alpha y)\Phi_1(\theta,y,x)dy+$$
$$+(1-q)H_2^*(\theta)\Phi_2(\theta,0,x).$$

From now on we shall suppress the argument θ of both Laplace transforms and other functions of θ except where its presence is deemed necessary.

With regard to the existence and uniqueness of solutions to equations (2.7) we shall construct a solution by converting equations (2.7) into two third order differential equations with constant coefficients (see (2.22) and (2.24) below). A solution to each of these equations exists (cf. Ince, [22], p. 73). The uniqueness of a bounded solution of the equations (2.7) may be shown in the following way. Let (Φ_1,Φ_2), $(\tilde{\Phi}_1,\tilde{\Phi}_2)$ be two bounded solutions of (2.7). The difference pair $V_1 = \Phi_1-\tilde{\Phi}_1$, $V_2 = \Phi_2-\tilde{\Phi}_2$ satisfies the two equations

$$V_1 = \alpha(1-p)\exp(\alpha z)H_1^*(\theta)\int_z^\infty \exp(-\alpha v)V_1(v)dv$$

$$+ \beta p \exp(-\beta z)H_1^*(\theta)\int_0^z \exp(\beta v)V_2(v)dv$$

$$V_2 = \alpha q H_2^*(\theta)\int_z^\infty \exp(-\alpha v)V_1(v)dv$$

$$+ \beta(1-q)\exp(-\beta z)H_2^*(\theta)\int_0^z V_2(v)\exp(\beta v)dv ,$$

where $Re(\theta) > 0$.

Let $\sup_{z\geq 0}|V_1(z)| = M_1$, $\sup_{z\geq 0}|V_2(z)| = M_2$.

It then follows that

$$|V_1(z)| \leq |H_1^*(\theta)|[(1-p)M_1+pM_2],$$

$$|V_2(z)| \leq |H_2^*(\theta)|[qM_1+(1-q)M_2],$$

for every $z \geq 0$. Hence

(2.8) $\quad M_1 \leq |H_1^*(\theta)|M_2$

(2.9) $\quad M_2 \leq |H_2^*(\theta)|M_1$.

But (2.8) and (2.9) together imply

(2.10) $\quad M_1 \leq |H_1^*(\theta)||H_2^*(\theta)|M_1$.

Since $\mathrm{Re}(\theta) > 0$ implies $|H_1^*(\theta)| < 1$ and $|H_2^*(\theta)| < 1$, the only way (2.10) can be satisfied is if $M_1 \equiv 0$. This in turn implies by (2.9) that $M_2 \equiv 0$ and hence $(\Phi_1(z), \Phi_2(z))$ and $(\tilde{\Phi}_1(z), \tilde{\Phi}_2(z))$ coincide for every $z \geq 0$. The solution to equation (2.7) is thus unique.

THEOREM 2.1. *The unique bounded solution of the equations* (2.7) *is given, for* $\mathrm{Re}(\theta) > 0$, *by*

$$\begin{aligned}(2.11\text{a})\quad \Phi_1(\theta,z,x) &= \exp(r_2z)(\alpha-r_2)[\alpha\beta\theta qH_2^*(r_2-r_1)]^{-1}\cdot\\ &\cdot\{\exp(-r_2x)[(r_2-r_1)\beta qH_2^*-\{r_1+\beta(1-[1-q]H_2^*)\}\{r_2+\beta(1-H_2^*)\}]\\ &+\exp(-r_1x)\{r_2+\beta(1-H_2^*)\}[r_2+\beta(1-[1-q]H_2^*)]r_1/r_2\},\ z > x\end{aligned}$$

$$\begin{aligned}(2.11\text{b})\quad \Phi_1(\theta,z,x) &= \theta^{-1}-\exp(-r,x)\{r_2+\beta(1-H_2^*)\}[\alpha\beta\theta qH_2^*(r_2-r_1)]^{-1}\cdot\\ &\cdot\{\exp(r_1z)(\alpha-r_1)[r_1+\beta(1-[1-q]H_2^*)]\\ &-\exp(r_2z)[r_2+\beta(1-[1-q]H_2^*)]r_1/r_2\},\ 0 \leq z \leq x\end{aligned}$$

$$\begin{aligned}(2.11\text{c})\quad \Phi_2(\theta,z,x) &= \exp(r_2z)(\beta+r_2)\{\beta\theta[r_2+\beta(1-[1-q]H_2^*)](r_2-r_1)\}^{-1}\cdot\\ &\cdot\{\exp(-r_2x)[(r_2-r_1)\beta qH_2^*-\{r_1+\beta(1-[1-q]H_2^*)\}\{r_2+\beta(1-H_2^*)\}]\\ &+\exp(-r_1x)\{r_2+\beta(1-H_2^*)\}[r_2+\beta(1-[1-q]H_2^*)]r_1/r_2\},\ z > x\end{aligned}$$

$$\begin{aligned}(2.11\text{d})\quad \Phi_2(\theta,z,x) &= \theta^{-1}-\exp(-r_1x)\{r_2+\beta(1-H_2^*)\}\{\beta\theta(r_2-r_1)\}^{-1}\cdot\\ &\cdot\{\exp(r_1z)(\beta+r_1)-\exp(r_2z)(\beta+r_2)r_1/r_2\},\ 0 \leq z \leq x,\end{aligned}$$

where

$$(2.12)\qquad r_1(\theta) = -\frac{1}{2}A(\theta) + \frac{1}{2}[(A(\theta))^2-4\,B(\theta)]^{1/2},$$

$$(2.13)\qquad r_2(\theta) = -\frac{1}{2}A(\theta) - \frac{1}{2}[(A(\theta))^2-4B(\theta)]^{1/2},$$

<u>and</u>

$$A(\theta) = \beta-\alpha+\alpha(1-p)H_1^*(\theta)-\beta(1-q)H_2^*(\theta)$$

$$B(\theta) = \alpha\beta\{-(1-H_2^*)(1+H_1^*[q+p-1])+q(H_1^*-H_2^*)\}.$$

<u>REMARK</u> $B(\theta)$ may be expressed in another way as

$$(2.14)\qquad B(\theta) = -\alpha\beta\{(1-H_1^*)(1-H_2^*+qH_2^*)+pH_1^*(1-H_2^*)\},$$

from which it follows that $\mathrm{Re}(A^2-4B(\theta)) \geq 0$. It follows in turn from this that $\mathrm{Re}(r_1) > 0$ and $\mathrm{Re}(r_2) < 0$. It can be shown in addition, that $\mathrm{Re}(r_1) < \alpha$, and that $-\mathrm{Re}(r_2) < \beta$. Two limit properties of r_2 which will be used later on are, for $\beta q < \alpha p$,

$$(2.15)\qquad \lim_{\theta\to 0}(r_2(\theta))^{-1}\{r_2(\theta)+\beta(1-H_2^*(\theta))\}=q(\alpha E_{H_1}+\beta E_{H_2})\alpha^{-1}[E_{H_2}p+qE_{H_1}]^{-1},$$

and

$$(2.16)\qquad \lim_{\theta\to 0} r_2'(\theta)=\alpha\beta[E_{H_2}(p+q)+q(E_{H_1}-E_{H_2})][\beta q-\alpha p]^{-1}.$$

<u>Proof of THEOREM 2.1.</u>

It can be seen that $\Phi_1(\theta,z,x)$ and $\Phi_2(\theta,z,x)$ are differentiable with respect to z. Differentiating each of (~~2.11a~~ 2.7a) and (~~2.11b~~ 2.7b) twice with respect to z, and collecting terms we obtain the following two second order differential equations

$$(2.17)\quad \Phi_2''+\Phi_2'[\beta-\alpha-\beta(1-q)H_2^*]+\Phi_2[\alpha\beta(1-q)H_2^*-\alpha\beta]$$

$$=-\alpha qH_2^*[\Phi_1'+\beta\Phi_1]-\alpha\beta\theta^{-1}(1-H_2^*)U(x-z),$$

$$(2.18)\quad \Phi_1''+\Phi_1'[\beta-\alpha+\alpha(1-p)H_1^*]+\Phi_1[\alpha\beta((1-p)H_1^*-1)]$$

$$=\beta pH_1^*[\Phi_2'-\alpha\Phi_2]-\alpha\beta\theta^{-1}(1-H_1^*)U(x-z).$$

Here Φ_i', Φ_i'' and Φ_i''' are corresponding first, second and third order partial derivatives with respect to z. From (2.18) it follows that

$$(2.19)\quad \Phi_2'-\alpha\Phi_2=(\beta pH_1^*)^{-1}[\Phi_1''+\Phi_1'\{\beta-\alpha+\alpha(1-p)H_1^*\}+\Phi_1\alpha\beta((1-p)H_1^*-1)+$$

$$+\alpha\beta\theta^{-1}(1-H_1^*)U(x-z)]\ .$$

Differentiating (2.19) with respect to z we obtain

$$(2.20)\quad \Phi_2''-\alpha\Phi_2'=(\beta pH_1^*)^{-1}[\Phi_1'''+\Phi_1''\{\beta-\alpha+\alpha(1-p)H_1^*\}+\Phi_1'\alpha\beta((1-p)H_1^*-1)],$$

and differentiating (2.18) with respect to z we get

$$(2.21)\quad \Phi_1'''+\Phi_1''\{\beta-\alpha+\alpha(1-p)H_1^*\}+\Phi_1'\alpha\beta\{(1-p)H_1^*-1\}=\beta pH_1^*[\Phi_2'-\alpha\Phi_2].$$

Direct substitution, of (2.20) for $\Phi_2''-\alpha\Phi_2'$ and then of (2.21) for $\Phi_2'-\alpha\Phi_2$, into the left hand side of (2.17) yields, after collecting terms,

$$(2.22)\quad \Phi_1'''+\Phi_1''[2\beta-\alpha+\alpha(1-p)H_1^*-\beta(1-q)H_2^*]+$$

$$+\Phi_1'\beta[\alpha\{-(1-H_2^*)(1+H_1^*(q+p-1))+q(H_1^*-H_2^*)\}+$$

(continued)

$$+\beta-\alpha+\alpha(1-p)H_1^*-\beta(1-q)H_2^*]-$$

$$-\alpha\beta^2\Phi_1[(1-H_2^*)(1+H_1^*(q+p-1))-q(H_1^*-H_2^*)]$$

$$=-\theta^{-1}\alpha\beta^2U(x-z)[(1-H_2^*)pH_1^*+(1-H_1^*)(1-(1-q)H_2^*)].$$

Equation (2.22) is a third order differential equation with constant coefficients. For $x \geq z$ $\Phi_1 = \theta^{-1}$ is a particular solution of (2.22). The homogeneous equation associated with equation (2.22) has auxiliary equation given by

$$(2.23)\qquad r^3+[2\beta-\alpha+\alpha(1-p)H_1^*-\beta(1-q)H_2^*]r^2+$$

$$+r\beta\{\alpha[-(1-H_2^*)(1+H_1^*(q+p-1))+q(H_1^*-H_2^*)]+$$

$$+\beta-\alpha+\alpha(1-p)H_1^*-\beta(1-q)H_2^*\}-$$

$$-\alpha\beta^2[(1-H_2^*)(1+H_1^*(q+p-1))-q(H_1^*-H_2^*)]=0.$$

The roots of (2.23) are $-\beta$, r_1, r_2; the last two are defined by (2.12) and (2.13).

Proceeding in the case of Φ_2 exactly as for Φ_1 we arrive at the following differential equation.

$$(2.24)\quad \Phi_2'''+\Phi_2''[\beta-2\alpha+\alpha(1-p)H_1^*-\beta(1-q)H_2^*]+$$

$$+\alpha\Phi_2'[\beta(-(1-H_2^*)(1+H_1^*(q+p-1)+q(H_1^*-H_2^*))-$$

$$-\beta+\alpha-\alpha(1-p)H_1^*+\beta(1-q)H_2^*]-$$

$$-\alpha^2\beta[-(1-H_2^*)(1+H_1^*(q+p-1)+q(H_1^*-H_2^*)]\Phi_2$$

$$=\alpha^2\beta[(1-H_1^*)qH_2-(1-H_2^*)((1-p)H_1^*-1)]U(x-z)\theta^{-1}.$$

For $x \geq z$ $\Phi_2 = \theta^{-1}$ is a particular solution of (2.24). The corresponding auxiliary equation is

$$(2.25) \quad r^3+r^2[\beta-2\alpha+\alpha(1-p)H_1^*-\beta(1-q)H_2^*]+$$
$$+\alpha r[\beta(-(1-H_2^*)(1+H_1^*(q+p-1))+q(H_1^*-H_2^*)-$$
$$-\beta+\alpha-\alpha(1-p)H_1^*+\beta(1-q)H_2^*]-$$
$$-\alpha^2\beta[-(1-H_2^*)(1+H_1^*(q+p-1))+q(H_1^*-H_2^*)]=0.$$

The three roots of (2.25) are α, r_1 and r_2. Taking into consideration the signs of $\mathrm{Re}(r_1)$ and $\mathrm{Re}(r_2)$ and the range of z in each case we can express the general solution to equations (2.7) as

$$(2.26a) \quad \Phi_1(\theta,z,x)=C_0\exp(-\beta z)+C_1\exp(r_2 z) \qquad , z > x$$

$$(2.26b) \quad \Phi_1(\theta,z,x)=C_2\exp(-\beta z)+C_3\exp(r_1 z)+C_8\exp(r_2 z)+\theta^{-1} \qquad , 0 \leq z \leq x,$$

and

$$(2.26c) \quad \Phi_2(\theta,z,x)=C_4\exp(r_2 z) \qquad , z > x$$

$$(2.26d) \quad \Phi_2(\theta,z,x)=C_5\exp(\alpha z)+C_6\exp(r_1 z)+C_7\exp(r_2 z)+\theta^{-1} \qquad , 0 < z \leq x,$$

while $\Phi_2(\theta,0,x)$ is described by equation (2.7c). In order to determine the desired constants C_i, $i = 0,1,\ldots,8$ we proceed as follows. From (2.7a), when $z > x$,

$$(2.27)\quad \int_0^z \Phi_2(\theta,v,x)\exp(\beta v)dv=\exp(\beta z)(\beta p H_1^*)^{-1}[\Phi_1(\theta,z,x)-\Phi_2(\theta,0,x)\beta^{-1}-$$

$$-\alpha(1-p)H_1^*\exp(\alpha z)\int_z^\infty \exp(-\alpha v)\Phi_1(\theta,v,x)dv]$$

and when $0 \le z \le x$

$$(2.28)\quad \int_0^z \exp(\beta v)\Phi_2(\theta,v,x)dv=\exp(\beta z)(\beta p H_1^*)^{-1}[\Phi_1(\theta,z,x)-$$

$$-\Phi_2(\theta,0,x)\beta^{-1}-(1-H_1^*)\theta^{-1}-$$

$$-\alpha(1-p)H_1^*\exp(\alpha z)\int_z^\infty \exp(-\alpha v)\Phi_1(\theta,v,x)dv].$$

Substituting the appropriate forms from (2.26) for Φ_1 in the right hand sides of both (2.27) and (2.28); differentiating each of the resulting equations with respect to z; and collecting terms, we find

$$(2.29)\quad \Phi_2(\theta,z,x)=C_1(\beta+r_2)(\beta p H_1^*)^{-1}\exp(r_2 z)[1-\alpha(1-p)H_1^*/(\alpha-r_2)],\quad z > x$$

and

$$(2.30\quad \Phi_2(\theta,z,x)=\exp(\alpha z)(\alpha+\beta)\alpha(1-p)(\beta p)^{-1}[\exp\{-(\alpha+\beta)x\}(C_2-C_0)(\alpha+\beta)^{-1}-$$

$$-\exp\{(r_1-\alpha)x\}C_3(r_1-\alpha)^{-1}+\exp\{(r_2-\alpha)x\}(C_1-C_8)(r_2-\alpha)^{-1}+$$

$$+\exp(-\alpha x)(\alpha\theta)^{-1}]+$$

$$+\exp(r_1 z)C_3(\beta+r_1)(\beta p H_1^*)^{-1}[1-\alpha(1-p)H_1^*(\alpha-r_1)^{-1}]+$$

$$+\exp(r_2 z)C_8(\beta+r_2)(\beta p H_1^*)^{-1}[1-\alpha(1-p)H_1^*(\alpha-r_2)^{-1}]+$$

$$+\theta^{-1}\qquad\qquad ,\ 0 < z \le x.$$

Comparing the coefficients of $\exp(r_1 z)$, $\exp(r_2 z)$, $\exp(\alpha z)$, and the constant term in (2.29) with those of (2.26c) and in (2.30) with those of (2.26d), it follows that

$$(2.31)\begin{cases} C_4 = C_1(\beta+r_2)(\beta p H_1^*)^{-1}[1-\alpha(1-p)H_1^*(\alpha-r_2)^{-1}] \\ C_5 = (\alpha+\beta)\alpha(1-p)(\beta p)^{-1}[\exp\{-(\alpha+\beta)x\}(C_2-C_0)(\alpha+\beta)^{-1} \\ \qquad +\exp(-\alpha x)(\alpha\theta)^{-1}-C_3\exp\{(r_1-\alpha)x\}(r_1-\alpha)^{-1}+ \\ \qquad +(C_1-C_8)\exp\{(r_2-\alpha)x\}(r_2-\alpha)^{-1}] \\ C_6 = C_3(\beta+r_1)(\beta p H_1^*)^{-1}[1-\alpha(1-p)H_1^*(\alpha-r_1)^{-1}] \\ C_7 = C_8(\beta+r_2)(\beta p H_1^*)^{-1}[1-\alpha(1-p)H_1^*(\alpha-r_2)^{-1}]. \end{cases}$$

The expression for $\Phi_2(\theta,z,x)$ follows directly once we have found the solution for $\Phi_1(\theta,z,x)$. That is, we have only to determine the constants C_0, C_1, C_2, C_3 and C_8 now. To do this we substitute the general form of Φ_1 and Φ_2 as given in (2.26) into equations (2.7). Comparing the coefficients of $\exp(\alpha z)$, $\exp(\beta z)$, $\exp(r_1 z)$, $\exp(r_2 z)$, and the constant term in each of these equations, we arrive at the relations,

$$(2.32)\begin{cases} C_0[1-\alpha(1-p)H_1^*\cdot(\alpha+\beta)^{-1}]-pAH_1^* = 0 \\ C_2[1-\alpha(1-p)H_1^*\cdot(\alpha+\beta)^{-1}]-pBH_1^* = 0 \\ \alpha q H_2^* C_0(\alpha+\beta)^{-1}+(1-q)H_2^*A = 0 \\ C_5[1-(1-q)\beta H_2^*\cdot(\alpha+\beta)^{-1}]-\alpha q H_2^* E = 0 \\ \alpha q H_2 C_2\cdot(\alpha+\beta)^{-1}+(1-q)H_2^*B = 0, \end{cases}$$

$$\Phi_2(\theta,0,x)=[1-(1-q)H_2^*]^{-1}[(1-H_2^*)\theta^{-1}+\alpha qH_2^*\{(\alpha\theta)^{-1}+E+C_2(\alpha+\beta)^{-1}+$$
$$+C_3(\alpha-r_1)^{-1}+D(\alpha-r_2)^{-1}\}],$$

where

$$A=\Phi_2(\theta,0,x)+\beta\{C_5(\exp\{(\alpha+\beta)x\}-1)(\alpha+\beta)^{-1}+C_6(\exp\{(r_1+\beta)x\}-1)(r_1+\beta)^{-1}+$$
$$+[C_7(\exp\{(r_2+\beta)x\}-1)-C_4\exp\{(r_2+\beta)x\}](r_2+\beta)^{-1}+$$
$$+(\exp(\beta x)-1)(\beta\theta)^{-1}\},$$

$$B=\Phi_2(\theta,0,x)-\beta\{(\beta\theta)^{-1}+C_5(\alpha+\beta)^{-1}+C_6(r_1+\beta)^{-1}+C_7(r_2+\beta)^{-1}\},$$

and

$$E=(C_0-C_2)\exp\{-(\alpha+\beta)x\}(\alpha+\beta)^{-1}-C_3\exp\{-(\alpha-r_1)x\}(\alpha-r_1)^{-1}+$$
$$+(C_1-C_8)\exp\{-(\alpha-r_2)x\}(\alpha-r_2)^{-1}-\exp(-\alpha x)(\alpha\theta)^{-1}.$$

Solving the system (2.32) with the help of equations (2.31) we obtain (2.11a) and (2.11b). The algebra, although tedious, is straightforward and hence we omit it. Equations (2.31) together with the solution to (2.32) for Φ_1 yield (2.11c) and (2.11d) for Φ_2. //

We proceed now to obtain transforms of the (first two) moments of the process Z(t). We shall need the following lemma, which follows itself from a well known result in renewal theory given as Lemma 2 in the Appendix.

LEMMA 2.1. *In order that* $E[Z(t)]$ *and* $E[Z^2(t)]$ *exist and be finite it is sufficient that both* $H_1(0+) < 1$ *and* $H_2(0+) < 1$, *and both* $E(X) < \infty$ *and* $E(X^2) < \infty$.

Applying Lemma 3 of the Appendix to the process $\{Z(t)\}$ for $\nu = 1$ or 2 we find

$$E[Z^{\nu}(t)] = \nu\int_0^{\infty} x^{\nu-1}[1-P(Z(t) \leq x)]dx.$$

By Lemma 2.1 the right hand side integral is finite. Now, taking Laplace transforms, we get

$$\int_0^{\infty} \exp(-\theta t)E[Z^{\nu}(t)]dt=\nu\int_0^{\infty} \exp(-\theta t)\int_0^{\infty} x^{\nu-1}P(Z(t) > x)dxdt.$$

The integrand on the right hand side above is positive. By Fubini's theorem then

$$\int_0^{\infty} \exp(-\theta t)E[Z^{\nu}(t)]dt=\nu\int_0^{\infty} x^{\nu-1}\int_0^{\infty} [1-P(Z(t) \leq x)]dtdx$$

$$=\nu\int_0^{\infty} x^{\nu-1}[\theta^{-1}-\int_0^{\infty} \exp(-\theta t)P(Z(t) \leq x)dt]dx .$$

By the above reasoning, the desired Laplace transforms

$$g_i(\theta)=\int_0^{\infty} \exp(-\theta t)E[Z(t)|Z(0)=z,\ J_0=i]dt,\ i = 1,2$$

$$h_i(\theta)=\int_0^{\infty} \exp(-\theta t)E[Z^2(t)|Z(0)=z,\ J_0=i]dt, i = 1,2$$

defined for $Re(\theta) > 0$, can be obtained to yield

$$(2.33)\quad g_i(\theta)=\theta^{-1}[r_2(\theta)+\beta\{1-H_2^*(\theta)\}][\alpha\beta qH_2^*(\theta)r_1(\theta)\{r_2(\theta)-r_1(\theta)\}]^{-1}\cdot$$

$$\cdot\{\exp\{r_1(\theta)z\}(\alpha-r_1(\theta))[r_1(\theta)+\beta\{1-(1-q)H_2^*(\theta)\}]$$

$$-\exp\{r_2(\theta)z\}(\alpha-r_2(\theta))[r_2(\theta)+\beta\{1-(1-q)H_2^*(\theta)\}]r_1(\theta)(r_2(\theta))^{-1}\}$$

and

$$(2.34)\quad h_i(\theta)=\theta^{-1}[r_2(\theta)+\beta\{1-H_2^*(\theta)\}][\alpha\beta q H_2^*(\theta)(r_1(\theta))^2\{r_2(\theta)-r_1(\theta)\}]^{-1}\cdot$$

$$\cdot\{\exp\{r_1(\theta)z\}(\alpha-r_1(\theta))[r_1(\theta)+\beta\{1-(1-q)H_2^*(\theta)\}]$$

$$-\exp\{r_2(\theta)z\}(\alpha-r_2(\theta))[r_2(\theta)+\beta\{1-(1-q)H_2^*(\theta)\ \}]r_1(\theta)(r_2(\theta))^{-1}\}$$

for $i = 1$ or 2. The inversion of transforms (2.11), (2.33) and (2.34) would yield the desired results. Unfortunately, the inversion process appears quite cumbersome. This is due, primarily, to the rather complicated dependence of r_1 and r_2 on θ, the parameter of the transform. Even in the simplest special cases the inversion is algebraically involved. Instead, we proceed now to examine the limiting behavior of the process $\{Z(t)\}$ and the asymptotic behavior of its moments.

<u>THEOREM</u> 2.2. $\quad Z(t) \xrightarrow{\mathcal{L}} Z$, <u>as</u> $t \to \infty$,

<u>where</u> $P(Z = \infty) = 1$ <u>if</u> $\beta q \geq \alpha p$

$\qquad P(Z < \infty) = 1$ <u>if</u> $\beta q < \alpha p$,

<u>with the limiting distribution given by</u>

$$(2.35)\quad \psi_i(z) = \begin{cases} 0, & \beta q \geq \alpha p \\ 1-\exp\{[\beta q-\alpha p]x\}(\alpha E_{H_1}+\beta E_{H_2})q\alpha^{-1}[E_{H_2}p+qE_{H_1}]^{-1}, & \beta q<\alpha p, \end{cases}$$

for $i = 1,2$ and $x \geq 0$.

<u>Proof</u>: Let $\psi_i(x) = \lim\limits_{\theta\to 0} \theta\, \Phi_i(\theta,z,x)$, $i = 1,2$. By the standard Tauberian Theorem 1 of the Appendix $\psi_i(x) \equiv \lim\limits_{t\to\infty} R_i(t,z,x)$. The result follows upon application of this theorem to (2.11). //

In the subcritical case, $\beta q < \alpha p$, the limiting distribution function is a negative exponential with a positive probability mass at zero. The interpretation of the criterion for the existence of a proper limiting distribution for $Z(t)$ is straightforward. When average inputs per unit time are less than average releases per unit time, $Z(t)$ has a nondegenerate limiting distribution. When average inputs are equal to or greater than average releases $Z(t)$ tends to infinity. This same condition will appear again in Chapter III in the study of first emptiness properties. It seems reasonable therefore to distinguish formally between the following three cases. Borrowing terminology from branching processes, we shall talk of subcritical, critical and supercritical cases according as $q\alpha^{-1}$ is less than, equal to, or greater than $p\beta^{-1}$.

We shall study now the asymptotic behavior of the first two moments in the critical and supercritical cases.

<u>THEOREM</u> 2.3. <u>If</u> $q\alpha^{-1} > p\beta^{-1}$, <u>then</u>

(2.36a) $\quad E[Z(t)|Z(0)=z,J_0=i] \sim k_1 t \qquad (t \to \infty)$

(2.36b) $\quad E[Z^2(t)|Z(0)=z,J_0=i] \sim 2^{-1}k_1^2 t^2 \qquad (t \to \infty)$

<u>and</u> <u>if</u> $q\alpha^{-1} = p\beta^{-1}$, <u>then</u>, <u>for</u> $i = 1,2$,

(2.37a) $\quad E[Z(t)|Z(0)=z,J_0=i] \sim 2(\pi)^{-1/2}k_2^{1/2}t^{1/2} \qquad (t \to \infty)$

(2.37b) $\quad E[Z^2(t)|Z(0)=z,J_0=i] \sim k_2 t \qquad (t \to \infty)$,

where

(2.38) $$k_1=(\beta q-\alpha p)(\alpha\beta)^{-1}\{E_{H_2}p+qE_{H_1}\}^{-1}$$

(2.39) $$k_2=\alpha\beta\{E_{H_2}p+qE_{H_1}\}.$$

<u>Proof</u>: Consider first the supercritical case, $\beta q > \alpha p$. From (2.33) it follows that

(2.40) $$\lim_{\theta\to 0} \theta^2 g_i(\theta)=k_1.$$

For the second moment, from (2.34) we have

(2.41) $$\lim_{\theta\to 0} \theta^3 h_i(\theta)=k_1^2.$$

The result for the supercritical case follows upon application of Tauberian Theorem 3 of the Appendix to both (2.40) and (2.41). In the critical case, since

(2.42) $$\lim_{\theta\to 0} \theta^{3/2} g_i(\theta)=k_2^{1/2}$$

and

(2.43) $$\lim_{\theta\to 0} \theta^2 h_i(\theta)=k_2,$$

the same theorem yields the desired result for $i = 1,2$. //

In the special case where $H_1=H_2 \equiv H$ and $1-p=q$ the underlying structure is that of a renewal process. Moreover, at each renewal point with the same probability p an input is observed and with probability (1-p) a release is observed. Results are known for discrete versions of this problem, as reported in Takacs [52], for example, and for the continuous time case as examined by Erdos and Kac[12]. For this special case the solution to the integral equation (2.7) takes the relatively simple form (cf. (2.11)).

$$\Phi(\theta,z,x)=\exp(r_2 z)r_1(r_2+\beta)[\alpha\beta\theta(r_1-r_2)]^{-1}\cdot$$

$$\cdot\{\exp(-r_2 x)(\alpha-r_2)-\exp(-r_1 x)(\alpha-r_1)\}\quad, \qquad z > x$$

$$\Phi(\theta,z,x)=\theta^{-1}-\exp(-r_1 x)(\alpha-r_1)[\alpha\beta\theta(r_2-r_1)]^{-1}\cdot$$

$$\cdot\{\exp(r_1 z)r_2(r_1+\beta)-\exp(r_2 z)r_1(r_2+\beta)\},\quad 0 \leq z \leq x,$$

where $Re(\theta) > 0$ and

$$r_1(\theta)=-2^{-1}c+2^{-1}[c^2-4d]^{1/2},$$

$$r_2(\theta)=-2^{-1}c-2^{-1}[c^2-4d]^{1/2},$$

and

$$c=\beta(1-H^*(\theta)p)+\alpha\{(1-p)H^*(\theta)-1\}$$

$$d=-\alpha\beta(1-H^*(\theta)).$$

3. THE MODEL WITH GENERAL DISTRIBUTION FOR EACH RELEASE AND EXPONENTIAL DISTRIBUTION FOR EACH INPUT.

It is natural to ask what generalization can be made of the model treated in Section 2. Specifically, we are interested in freeing that model from the restriction of exponential distributions for both inputs and releases. In this section we examine an initial attempt to consider more general distributions $B(x)$ and $D(y)$. As mentioned in Section 2 with both B and D general a solution, if it exists, is very difficult to generate, by use of transforms or other means. The price of generalizing even one of those distributions is the sacrifice of the semi-Markovian structure of the underlying process. Therefore in this section we abandon the underlying semi-Markovian structure by setting $H_1=H_2 \equiv H$, and by assuming that

$$H(t)=\begin{cases} 1-\exp\{-(\lambda+\mu)t\} & , t \geq 0 \\ 0 & , t < 0, \end{cases}$$

where $1-p=\lambda/(\lambda+\mu)$ and $q=\mu/(\lambda+\mu)$, and $\lambda,\mu > 0$. Thus the underlying process is Markovian, a fact which allows us to consider not only the backward Kolmogorov integral equations but also the forward integral equations. In the present case equations (2.4) reduce to a single equation. A solution of the backward equation, the minimal solution, can be constructed using successive approximations. This minimal solution satisfies also the forward equation and is minimal for the latter. Since $N(t) < \infty$, almost surely, here the forward version of equation (2.4) is valid and can be established by considering the nature of the last jump of the process $Z(t)$ before time t. Considering the last jump of the process $(0,t]$, if there is one, and whether it is an input or a release the following forward Kolmogorov integral equation can be established for

$R(t,z,x)=P(Z(t)\leq x|Z(0)=z)$ for the case $z \geq 0$.

$$(2.44)\quad R(t,z,x)=\exp\{-(\lambda+\mu)t\}U(x-z)+$$

$$+\lambda\int_0^t \exp\{-(\lambda+\mu)(t-\tau)\}d\tau\int_0^x R(\tau,z,x-y)dB(y)$$

$$+\mu\int_0^t \exp\{-(\lambda+\mu)(t-\tau)\}d\tau\int_0^\infty R(\tau,z,x+y)dD(y),$$

where now z is fixed. The Laplace transform of (2.44) for $Re(\theta) > 0$ is

$$(2.45)\quad \Phi(\theta,z,x)(\lambda+\mu+\theta)=U(x-z)+\lambda\int_0^x \Phi(\theta,z,x-y)dB(y)+$$

$$+\mu\int_0^\infty \Phi(\theta,z,x+y)dD(y).$$

The uniqueness of the solution of equation (2.45) is proved as follows by use of the principle of contraction mappings. Let M be the metric space of all bounded complex-valued functions defined on $0 \leq x < \infty$ and integrable in any finite subinterval of $0 \leq x < \infty$. Take as the metric for this space

$$\rho(f,g)=\sup_{x\geq 0}|g(x)-f(x)| \quad , \; f,g\epsilon M.$$

Now define a mapping, $A: M \to M$, of M into itself, by the equation

$$A\Phi(x)=U(x-z)(\lambda+\mu+\theta)^{-1}+\lambda(\lambda+\mu+\theta)^{-1}\int_0^x \Phi(x-y)dB(y)+$$

$$+\mu(\lambda+\mu+\theta)^{-1}\int_0^\infty \Phi(x+y)dD(y).$$

Then for all $x \geq 0$

$$|A\Phi_2(x)-A\Phi_1(x)|\leq|\lambda+\mu+\theta|^{-1}\rho(\Phi_1,\Phi_2)(\lambda+\mu).$$

Hence we have

$$\rho(A\Phi_1, A\Phi_2) \leq |\lambda+\mu+\theta|^{-1}(\lambda+\mu)\rho(\Phi_1,\Phi_2).$$

Since $\mathrm{Re}(\theta) > 0$, A is a contraction mapping. By Theorem 4 of the Appendix A has a unique fixed point which is the unique solution of equation (2.45).

A tractable solution is possible for the case in which

$$B(y) = \begin{cases} 1-\exp(-\alpha y) & , y \geq 0 \\ 0 & , y < 0, \end{cases}$$

where $\alpha > 0$, while the distribution D remains arbitrary subject to the conditions $D(0)=0$ and that its first moment $E_D = \int_0^\infty y dD(y) < \infty$.

From now on we assume that B has the above exponential form and present the solution, first in the case where $z=0$.

<u>THEOREM</u> 2.4. <u>For</u> $z=0$, <u>equation</u> (2.45) <u>has the unique bounded solution</u>

$$(2.46)\quad \Phi(\theta,0,x)=\theta^{-1}-(\alpha-\gamma(\theta))(\alpha\theta)^{-1}\exp(-\gamma(\theta)x),\ x \geq 0,\ \mathrm{Re}\theta > 0$$

<u>with</u>

$$(2.47)\quad \int_0^\infty \exp(-\theta t)P(Z(t)=0|Z(0)=0)dt=\gamma(\theta)(\alpha\theta)^{-1},$$

<u>where</u> $\gamma(\theta)$ <u>is the unique root of</u>

$$(2.48)\quad \lambda+\mu+\theta-\alpha\lambda(\alpha-r)^{-1}-\mu\int_0^\infty \exp(-ry)dD(y)=0,$$

<u>with</u> $0 < \mathrm{Re}(r(\theta)) < \alpha$.

We need first

LEMMA 2.2. *For* α, λ, μ *all positive*, $\mathrm{Re}(\theta) > 0$, *equation* (2.48) *has a unique root* $r(\theta)=\gamma(\theta)$ *in* $0 < \mathrm{Re}(r(\theta)) < \alpha-\delta$, *where* $\delta > 0$ *is small*. *Moreover* $0 < |\gamma(\theta)| \leq \alpha|(\mu+\theta)/(\lambda+\mu+\theta)|$.

We prove Lemma 2.2 at the end of this chapter.

Proof of THEOREM 2.4. We show that the solution of (2.45) is of the form

$$\Phi(\theta,0,x)=\theta^{-1}+C(\theta)\exp(-rx). \tag{2.49}$$

We know $\Phi(\theta,0,x) \to \theta^{-1}$, as $x \to \infty$. Now this is possible only if $\mathrm{Re}(r(\theta)) > 0$. Furthermore, from (2.47) we require that $|\gamma(\theta)| < \alpha$. So the only values of $r(\theta)$ in which we are interested are those for which $0 < \mathrm{Re}(r(\theta)) < \alpha$. Substitution of (2.49) into (2.45) yields an identity in x. Comparing the coefficients of $\exp(rx)$ and of $\exp(\alpha x)$ on both sides of this identity, we obtain (2.48) and the ~~relations~~ relation

$$\theta^{-1}+C(\theta)(\alpha-r)^{-1} = 0. \tag{2.50}$$

By Lemma ~~2.4~~ 2.2 equation (2.48) has a unique root $r=\gamma(\theta)$ in $0 < |r(\theta)| < \alpha\left|\frac{\mu+\theta}{\lambda+\mu+\theta}\right| < \alpha$, $\mathrm{Re}(\theta) > 0$. Once $\gamma(\theta)$ has been determined (2.50) then yields the term $C(\theta)$, which in turn yields (2.46). The uniqueness of the solution of (2.47) guarantees that (2.46) is the only such solution. Finally, since

$$\int_0^\infty \exp(-\theta t)P(Z(t)=0|Z(0)=0)dt=\lim_{x\to 0+} \Phi(\theta,0,x)$$

(2.47) follows. //

We shall next prove a lemma which is essential in the case $z > 0$. Let H be an arbitrary function, defined on the nonnegative half of the real line, which is integrable in every finite subinterval of that half line and which can be expressed as the difference of two monotone nondecreasing functions.

Let also

$$K(s) = \sum_{k=0}^{\infty} H^{(k)}(s), \tag{2.51}$$

where

$$H^{(k)}(s)=\int_0^s H^{(k-1)}(s-u)dH(u),\quad 0 \le s \le z,\ k = 1,2,\ldots$$

$$H^{(0)}(s)=1 \qquad\qquad ,\ 0 \le s \le z.$$

<u>LEMMA</u> 2.3. (i) <u>The Volterra equation</u>

$$F(\xi) = a+\int_0^\xi F(\xi-y)dH(y),\quad 0 \le \xi \le b < \infty \tag{2.52}$$

where F is the unknown function, H has the properties listed above, and a is a constant, has solution given by

$$(2.53)\qquad F(\xi) = a[1+H(\xi)+H*H(\xi)+H*H*H(\xi)+\dots]$$

$$= a\ K(\xi),$$

and this solution is unique, provided $|K(\xi)|$ *converges uniformly in* $0 \leq \xi \leq b < \infty$.

Here (*) *denotes the convolution operation.*

(ii) *Let D be such that* $D(s)/s \leq A < \infty$ *for* $0 < s \leq \varepsilon$, *where* $\varepsilon > 0$ *is small. Then for*

$$(2.54)\qquad H(s) = [\mu D(s)+\lambda(1-\exp(\alpha s))](\lambda+\mu+\theta)^{-1},\ 0 \leq s \leq z,$$

where z *is fixed but otherwise arbitrary,* K(s) *exists and is finite for* $0 \leq s \leq z$.

Proof: (i) The assertion follows first by a substitution of (2.53) into (2.52). Then the interchange of summation and integration operations is justified since the series $|K(\xi)|$ converges uniformly in $0 \leq \xi \leq b < \infty$. Also since $|K| < \infty$, for a fixed $\varepsilon > 0$, there is an $n_0=n_0(\varepsilon,s)$ such that $|H^{(n)}(s)| < \varepsilon$ for $n \geq n_0$. Now consider the difference, V, of two solutions of equation (2.52). V satisfies $V=H*V$, and hence

$$V=H^{(n)}*V \quad \text{for all } n.$$

But the remark above indicates that $H^{(n)}(s) \to 0$ for all s as $n \to \infty$, and hence $V(s)=0$. The solution of (2.52) is thus unique.

(ii) Let

$$M=|\lambda+\mu+\theta|^{-1}[\mu A+\alpha\lambda\exp(\alpha z)]. \tag{2.55}$$

It can be shown that $D(s)/s \leq A < \infty$ for $0 < s \leq \varepsilon$, $\varepsilon > 0$ implies $D(s)/s \leq A < \infty$ for $0 < s \leq z$ where z is fixed but otherwise arbitrary, and A is used in a generic sense here.

Thus M in (2.55) is finite. The assertion of the lemma now follows from the fact that $|K(s)| \leq \exp(Ms)$, which is proved below by using an induction argument.

$$H^{(0)}(s) \equiv 1$$

$$|H^{(1)}(s)| \leq |\lambda+\mu+\theta|^{-1}[\mu As+\lambda(\exp(\alpha s)-1)]$$

$$\leq |\lambda+\mu+\theta|^{-1}[\mu A+\alpha\lambda\exp(\alpha z)]s=Ms,$$

where in the first inequality we made use of the fact that $D(s)/s \leq A < \infty$ for $0 \leq s \leq z$. Suppose that $|H^{(k)}(s)| \leq M^{k-1}s^{k-1}/(k-1)!$ We have shown this to be true for $k = 2$. Then

$$(2.56)\quad |H^{(k)}(s)| \le |\lambda+\mu+\theta|^{-1}\{\mu\int_0^s |H^{(k-1)}(s-u)|dD(u)+$$

$$+\lambda\int_0^s |H^{(k-1)}(s-u)|d\{\exp(\alpha u)-1\}\}$$

$$\le |\lambda+\mu+\theta|^{-1}M^{k-1}[(k-1)!]^{-1}[\mu\int_0^s (s-u)^{k-1}dD(u)+$$

$$+\alpha\lambda\int_0^s (s-u)^{k-1}\exp(\alpha u)du] \;.$$

Two successive integrations by parts of the first integral on the right hand side of (2.56) yields

$$|H^{(k)}(s)| \le M^{k-1}[(k-1)!]^{-1}|\lambda+\mu+\theta|^{-1}[\mu A+\alpha\lambda\exp(\alpha z)]s^k k^{-1}$$

$$= M^k s^k/k!$$

It follows by induction therefore that

$$(2.57)\quad |H^{(k)}(s)| \le M^k s^k/k! \;,\; k = 0,1,2,\ldots;\; 0 \le s \le z \;.$$

Thus $|\sum_{k=0}^{\infty} H^{(k)}(s)| \le \sum_{k=0}^{\infty}|H^{(k)}(s)| \le \sum_{k=0}^{\infty} M^k s^k/k! = \exp(Ms).$ //

<u>REMARK</u>. In the case D has a density the condition, $D(s)/s \le A < \infty$ for $0 \le s \le \varepsilon$, is satisfied and the lemma holds. The condition (2.57) guarantees, by the Weierstrass M-test, that $K(\xi)$ converges uniformly.

<u>THEOREM</u> 2.5. *Suppose* $D(s)/s \le A < \infty$ *for* $0 \le s \le \varepsilon$ *for* $\varepsilon > 0$. *Then for* $z > 0$, *equation* (2.45) *has the unique solution*

$$(2.58)\quad \phi(\theta,z,x)=\theta^{-1}-\exp(-\gamma(\theta)x)(\alpha-\gamma(\theta))$$

$$\cdot[(\alpha\theta)^{-1}+(\lambda+\mu+\theta)^{-1}\int_0^z \exp(\alpha u)K(z-u)du], x\ge z$$

and

$$(2.59)\quad \Phi(\theta,z,x)=\theta^{-1}-(\lambda+\mu+\theta)^{-1}K(z-x)-$$

$$-\exp(-\gamma(\theta)x)(\alpha-\gamma(\theta))\cdot$$

$$\cdot[(\alpha\theta)^{-1}+(\lambda+\mu+\theta)^{-1}\int_0^z \exp(\alpha u)K(z-u)du],\ 0 \leq x < z.$$

where K is given in (2.51) and H by (2.54).

Proof: Let

$$\Phi(\theta,z,x) = \Phi_1(\theta,z,x) \quad \text{for } 0 \leq x < z$$

$$\Phi(\theta,z,x) = \Phi_2(\theta,z,x) \quad \text{for } z \leq x .$$

Equation (2.45) may then be broken into the two parts

$$(2.60)\quad \Phi_1(\theta,z,x)(\lambda+\mu+\theta)=\exp(-\alpha x)\lambda\alpha\int_0^x \Phi_1(\theta,z,v)\exp(\alpha v)dv$$
$$+\mu[\int_0^{z-x} \Phi_1(\theta,z,x+y)dD(y)+\int_{z-x}^{\infty} \Phi_2(\theta,z,x+y)dD(y)],$$
$$0 \leq x < z$$

and

$$(2.61)\quad \Phi_2(\theta,z,x)(\lambda+\mu+\theta)=1+\exp(-\alpha x)\alpha\lambda[\int_0^z \exp(\alpha v)\Phi_1(\theta,z,v)dv$$
$$+\int_z^x \Phi_2(\theta,z,v)\exp(\alpha v)dv]$$
$$+\mu\int_0^{\infty} \Phi_2(\theta,z,x+y)dD(y) \quad , \quad x \geq z .$$

Now when z=0 we saw that Φ_2 had the form

(2.62) $\Phi_2(\theta,z,x)=\theta^{-1}+W\exp(-rx)$, $x \geq z$ $\mathrm{Re}(r) > 0$.

We shall construct a solution of (2.45) this time by putting Φ_2 as in (2.62) and setting

(2.63) $\Phi_1(\theta,z,x)=\theta^{-1}+W\exp(-rx)+g(\theta,z,x)$, $0 \leq x < z$,

where $g(\theta,z,x)$ is a function to be determined. Substitution of (2.62) into (2.61) produces an identity in x. Comparing the coefficients of $\exp(-rx)$ and $\exp(-\alpha x)$ on both sides of this identity we obtain the relations

(2.64) $$\lambda+\mu+\theta-\alpha\lambda(\alpha-r)^{-1}-\mu\int_0^\infty \exp(-ry)dD(y)=0,\ r \neq \alpha,$$

and

(2.65) $$\exp(\alpha z)(\alpha\theta)^{-1}-J(\theta,z)+W\exp\{(\alpha-r)z\}(\alpha-r)^{-1}=0,\ \alpha \neq r,$$

where

(2.66) $$J(\theta,z)=\int_0^z \exp(\alpha v)\Phi_1(\theta,z,v)dv.$$

By Lemma 2.2 we can determine a unique root, $r=\gamma(\theta)$, from (2.64) which satisfies $0 < \mathrm{Re}(\gamma(\theta)) < \alpha$ with $\mathrm{Re}(\theta) > 0$. Straightforward substitution of (2.62) and (2.63) into (2.60) leads to the following integral equation.

(2.67) $$g(\theta,z,z-s)(\lambda+\mu+\theta)=-1+\mu\int_0^s g(\theta,z,z-s+y)dD(y)-$$
$$-\alpha\lambda\int_0^s g(\theta,z,\xi+z-s)\exp(\alpha\xi)d\xi.$$

Here we have made the change of variable $s=z-x$, $0 \leq s \leq z$. Setting $\Gamma(\theta,s)=g(\theta,z,z-s)$, we see that equation (2.67) becomes

$$(2.68) \quad \Gamma(\theta,s)=-(\lambda+\mu+\theta)^{-1}+\int_0^s \Gamma(\theta,s-y)dH(y),$$

where H is defined in (2.54). Equation (2.68) is a Volterra equation, so that from Lemma 2.3 it follows that

$$\Gamma(\theta,s)=-(\lambda+\mu+\theta)^{-1}K(s).$$

Converting from Γ back to g we find $\Phi_1(\theta,z,x)$ in terms of W by means of (2.63). Then using (2.65) and (2.66) we obtain

$$(2.69) \quad J(\theta,z)=\exp\{(\alpha-\gamma)z\}[(\exp(\alpha z)-1)(\alpha\theta)^{-1}-(\lambda+\mu+\theta)^{-1}\int_0^z \exp(\alpha v)U(z-v)dv].$$

In turn, therefore, from (2.65)

$$(2.70) \quad W=\exp\{-(\alpha-\gamma)z\}(\alpha-\gamma)[-\exp(\alpha z)(\alpha\theta)^{-1}+(\exp(\alpha z)-1)(\alpha\theta)^{-1}-$$
$$-(\lambda+\mu+\theta)^{-1}\int_0^z \exp(\alpha v)U(z-v)dv].$$

From (2.62) and (2.63), equations (2.58) and (2.59) now easily follow.//

We shall now investigate the limit behavior of $Z(t)$ as $t \to \infty$. To this end we must investigate the behavior of $\gamma(\theta)$ as $\theta \to 0$. Knowledge of this behavior will then allow us to apply the Tauberian Theorem 1 of the Appendix to $\Phi(\theta,z,x)$. By an application of Rouche's Theorem we saw that equation (2.48) has, for fixed θ, a unique root $r(\theta)=\gamma(\theta)$ such that $0 < |\gamma(\theta)| \leq \alpha|(\mu+\theta)/(\lambda+\mu+\theta)|$. We now prove the following lemma.

<u>LEMMA</u> 2.4. <u>For</u> $\theta > 0$,

$$\lim_{\theta\downarrow 0} \gamma(\theta) = \begin{cases} 0 & \text{if } \alpha\mu\lambda^{-1}E_D \leq 1 \\ \alpha[1-\lambda\{\lambda+\mu(1-\zeta)\}^{-1}] & \text{if } \alpha\mu\lambda^{-1}E_D > 1 \end{cases},$$

<u>where</u> ζ <u>is the least nonnegative root of the equation</u> $\xi=\Phi(\xi)$, $0 \leq \xi < 1$ <u>and</u> Φ <u>is such that</u> $\Phi'(\xi) > 0$, $\Phi''(\xi) > 0$. <u>Here</u> $E_D=\int_0^\infty y dD(y)$.

<u>REMARK</u>. The root ζ exists and its properties are discussed on p. 274 of Feller [13].

<u>Proof of LEMMA</u> ~~2.6~~ 2.4. Consider, instead of γ, the function ξ of γ defined by $\gamma=\alpha[1-\lambda\{\lambda+\mu(1-\xi)\}^{-1}]$, whence $\xi=\{\mu\alpha-(\mu+\lambda)\gamma\}\{\mu(\alpha-\gamma)\}^{-1}$ and since $0 < \mathrm{Re}\gamma(\theta) < \alpha$, $0 \leq \mathrm{Re}(\xi) < 1$. Set $\Phi(\xi)=D^*(\alpha[1-\lambda\{\lambda+\mu(1-\xi)\}^{-1}])$, where $D^*(r)=\int_0^\infty \exp(-ry)dD(y)$. The function $\Phi(\xi)$ generates a discrete probability distribution with mean $\alpha\mu\lambda^{-1}E_D$, and satisfies the conditions of the theorem. Equation (2.48) may then be rewritten as

$$\theta\mu^{-1}+\xi=\Phi(\xi) .$$

The proof from this point on is due solely to Benes [6]. This suggests that as $\theta \to 0$ along the real axis, ξ approaches a root of the familiar equation from branching processes, $\xi=\Phi(\xi)$. We now show that $\xi(\theta) \to \zeta$ as $\theta \to 0$ along the real axis. If θ is real then so is ξ. Also if $\theta > 0$, then $\xi < \zeta$, because $\theta > 0 \Rightarrow \Phi(\xi) > \xi$, and in $0 \leq \xi < 1$ this is possible only if $\xi < \zeta$ since $\Phi(0) > 0$, and $\xi=\Phi(\xi)$ has at most two roots in (0,1) one of them being 1. To show that $0 < \theta < \theta' \Rightarrow \xi(\theta) > \xi(\theta')$

write $\xi=\xi(\theta)$, $\xi'=\xi(\theta')$. Then $0 < \theta < \theta'$ and $\xi < \zeta$, $\xi' < \zeta$ imply $\Phi(\xi)-\Phi(\xi') < \xi-\xi'$. Now $\Phi'(y)$ is steadily increasing in $0 < y < 1$ ($\Phi''(y) > 0$, $0 < y < 1$); so if for some u we have both $u < \zeta$ and $\Phi'(u) > 1$, then $\Phi(u) > u$ and $\Phi(1) > 1$, a contradiction to the fact that $\Phi(1)=1$. So $\Phi'(y) \leq 1$ for $y < \zeta$. Now if $\xi \leq \xi'$, this would imply $\Phi(\xi')-\Phi(\xi) \leq \xi'-\xi < \Phi(\xi')-\Phi(\xi)$, which is impossible. Hence $\xi > \xi'$.

It remains to show that, given $u < \zeta$, there exists a $\theta > 0$ such that $\xi(\theta) > u$. The equation

$$x = \lambda[\Phi(u)-u] \ , \text{ i.e. } x\lambda^{-1}+u=\Phi(u)$$

uniquely determines an $x > 0$, and for this x we must have $\xi(x)=u$ or else equation (2.48) does not have a unique root (contradiction). If now $0 < \theta < x$, then $\xi(\theta) > \xi(x)=u$, as was to be proved. It follows that as $\theta \to 0$ along the real axis

$$\xi \to \begin{cases} 1 & \text{if } \Phi'(1) \leq 1 \\ \zeta & \text{if } \Phi'(1) > 1 \ . \end{cases}$$

From this the desired result follows. //

We now use Lemma 2.4 to prove the following theorem.

THEOREM 2.6. Under the conditions of Theorem 2.5

$$Z(t) \xrightarrow{\mathcal{L}} Z \quad \text{as} \quad t \to \infty,$$

where

$$P(Z=\infty)=1 \quad \underline{\text{if}} \quad E_D \overset{\leq}{\not>} \lambda(\alpha\mu)^{-1}$$

$$P(Z<\infty)=1 \quad \underline{\text{if}} \quad E_D \overset{>}{\not\leq} \lambda(\alpha\mu)^{-1}.$$

<u>independent of the value of z.</u>

<u>The distribution of</u> Z <u>is given for</u> $x \geq 0$ <u>by</u>

$$(2.75) \quad P(Z \leq x)=\begin{cases} 1-(\alpha-\gamma^*)\alpha^{-1}\exp(-\gamma^* x), & E_D > \lambda(\alpha\mu)^{-1} \\ 0 & , E_D \leq \lambda(\alpha\mu)^{-1}, \end{cases}$$

<u>where</u> $\gamma^*=\alpha[1-\lambda\{\lambda+\mu(1-\zeta)\}^{-1}]$ <u>and</u> ζ <u>is as in Lemma 2.4</u>.

<u>Proof</u>: By the Tauberian Theorem 1 of the Appendix

$$P(Z \leq x)=\lim_{\theta\downarrow 0} \theta\Phi(\theta,z,x), \text{ for } x \geq 0.$$

Applying this argument to Φ first in (2.46) and then in (2.58) we arrive at (2.75) with the aid of Lemma 2.6. Thus the limit is independent of the initial condition $Z(0)=z$. //

Once again we see that if average releases per unit time exceed average inputs per unit time, then $Z(t)$ has a nondegenerate limiting distribution -- an exponential with positive mass at zero.

From Theorems 2.4 and 2.5 we can derive the moments of the process $Z(t)$. Since $Z(t)$ is a nonnegative random variable Lemma 3 of the Appendix provides us with Laplace transforms of the first two moments of $Z(t)$. From (2.58) and (2.59) it follows that

(2.71) $$\int_0^\infty \exp(-\theta t)E[Z(t)\,|\,Z(0)=z]dt$$

$$=\frac{(\alpha-\gamma(\theta))}{\gamma(\theta)}\{(\alpha\theta)^{-1}+(\lambda+\mu+\theta)^{-1}\int_0^z K(z-u)\exp(\alpha u)du\}$$

$$-(\lambda+\mu+\theta)^{-1}\int_0^z K(z-x)dx$$

and

(2.72) $$\int_0^\infty \exp(-\theta t)E[Z^2(t)\,|\,Z(0)=z]dt$$

$$=2(\alpha-\gamma(\theta))[\gamma(\theta)]^{-2}\{(\alpha\theta)^{-1}+(\lambda+\mu+\theta)^{-1}\int_0^z u\ \exp(\alpha u)K(z-u)du\}$$

$$-2(\lambda+\mu+\theta)^{-1}\int_0^z x\ K(z-x)dx.$$

Finally, we prove Lemma 2.2 which is stated on page 58.

Proof of LEMMA 2.2. Let $f(r)+g(r)=\lambda+\mu+\theta-\alpha\lambda(\alpha-r)^{-1}-\mu\int_0^\infty \exp(-ry)dD(y)$, where $f(r)=\lambda+\mu+\theta-\alpha\lambda(\alpha-r)^{-1}$, $g(r)=-\mu\int_0^\infty \exp(-ry)dD(y)$. Consider the contour C consisting of the following parts.

C_1: $r=it, -c\le t\le c$; C_2: $r=t+ic$, $0\le t\le\alpha$

C_3: $r=t-ic, 0\le t\le\alpha$; C_4: $r=\alpha+it$, $\delta\le t\le c$

C_5: $r=\alpha+it, -c\le t\le\delta$; C_6: $r=\delta e^{i\phi}$, $\pi/2\le\phi\le 3\pi/2$,

where $c > 0$, $\delta > 0$ and δ is small. Both f and g are analytic inside and on the closed contour C. Now on C we have $|g(r)| \le \mu$, and it is not hard to show that on C_1, C_2 and C_3 $|g(r)| < |f(r)|$ provided $c > \alpha/2$. Consider now C_4 and C_5. We wish to show that $|\lambda+\mu+\theta-\alpha\lambda(\alpha-r)^{-1}| > \mu$ on C_4 and C_5. But we have

$$\mathrm{Re}(\lambda+\mu+\theta-\alpha\lambda(\alpha-r)^{-1}) > \mu$$

$$\Leftrightarrow \mathrm{Re}(\lambda[1-\alpha(\alpha-r)^{-1}]) \geq 0$$

$$\Leftrightarrow \mathrm{Re}(\alpha(\alpha-r)^{-1}) \leq 1$$

$$\Leftrightarrow \mathrm{Re}(\frac{\alpha}{\alpha-r} \cdot \frac{\alpha-\bar{r}}{\alpha-\bar{r}}) \leq 1 .$$

Set $r = x+it$. Then we have

$$\mathrm{Re}(\lambda+\mu+\theta-\alpha\lambda(\alpha-r)^{-1}) > \mu$$

$$\Leftrightarrow \alpha(\alpha-x) \leq |\alpha-r|^2 .$$

But on C_4 and C_5 $x=\alpha$ so that the above condition becomes $0 \leq |t^2|$, which is always true.

Finally, consider C_6: Pick $\delta > 0$ small enough that the point $\tau=\alpha(\mu+\theta)/(\lambda+\mu+\theta)$ is contained in C and

$$|\alpha\lambda/(\alpha-r)| > \mu+|\lambda+\mu+\theta| \text{ for } r=\delta e^{i\phi}, \pi/2 \leq \phi \leq 3\pi/2.$$

Then on C_6

$$|\lambda+\mu+\theta-\alpha\lambda(\alpha-r)^{-1}| \geq ||\lambda+\mu+\theta|-|\alpha\lambda/(\alpha-r)|| > \mu.$$

We have shown that $|g(r)| < |f(r)|$ on C. Hence by Rouche's theorem (listed in the Appendix) $f(r)$ and $f(r)+g(r)$ have the same number of zeros inside C. However $f(r)$ has exactly one zero inside C, namely $r(\theta)=\alpha(\mu+\theta)(\lambda+\mu+\theta)^{-1}$. This proves the uniqueness and that $0 < |\gamma(\theta)| \leq \alpha|(\mu+\theta)/(\lambda+\mu+\theta)|$. //

CHAPTER III

DISTRIBUTION OF TIME TO FIRST EMPTINESS

1. INTRODUCTION AND NOTATION.

It is reasonable to ask, "how long does it take before the dam becomes empty for the first time, given that $Z(0)=z$ is positive?" In the context of the quantal response biological assay problem this question might be rephrased as, "how long does it take the subject to reduce the residual level of drug to zero, and thus, from that point on, escape the risk of death from the drug?" In this chapter we shall attempt to answer such questions for the model presented in Chapter II.

Let $Z(0)=z$ be strictly positive. Let T_E be the time until the dam first becomes empty. That is,

$$T_E=\inf\{t|Z(t)=0,t > 0\}.$$

In addition, throughout this chapter, we shall use the notation

$$\tilde{R}_i(t,z,x)=P(Z(t) \leq x,\ T_E > t|Z(0)=z,J_0=i),\ i = 1,2,$$

$$\tilde{R}\ (t,z,x)=P(Z(t)\leq x,\ T_E > t|Z(0)=z),$$

$$1-F_i(t)=P(T_E > t|Z(0)=z,J_0=i) \qquad , i = 1,2,$$

$$1-F\ (t)=P(T_E > t|Z(0)=z),$$

for $t \geq 0$, $x \geq 0$. Let $\tilde{\Phi}_i(\theta,z,x)$, $F_i^*(\theta)$, $i = 1,2$, $\tilde{\Phi}(\theta,z,x)$, and $F^*(\theta)$ be, for $\mathrm{Re}(\theta) > 0$, the Laplace transforms of $\tilde{R}_i(t,z,x)$, $F_i(t)$, $i = 1,2$, $R(t,z,x)$, and $F(t)$ respectively.

2. THE MODEL WITH EXPONENTIAL DISTRIBUTION FOR EACH INPUT AND EACH RELEASE.

In this section we consider the model of Section 2.2. The next section deals with the model of Section 2.3.

By considering the first jump, if there is one, of the process $Z(t)$ (cf. Section 2.2) in the interval $(0,t]$, along with the size and nature of that jump, we can establish in a straightforward manner the following backward Kolmogorov integral equations for $\tilde{R}_i(t,z,x)$ for $z > 0$.

$$(3.1)\quad \tilde{R}_1(t,z,x)=U(x-z)(1-H_1(t))+\alpha(1-p)\cancel{\exp}\,e^{\alpha(z)}\int_0^t\int_z^\infty \exp(-\alpha v)\tilde{R}_1(t-\tau,v,x)dvdH_1(\tau)$$

$$+p\beta\exp(-\beta z)\int_0^t\int_0^z \exp(\beta v)\tilde{R}_2(t-\tau,v,x)dvdH_1(\tau),$$

$$(3.2)\quad \tilde{R}_2(t,z,x)=U(x-z)(1-H_2(t))+\alpha q\exp(\alpha z)\int_0^t\int_z^\infty \exp(-\alpha v)\tilde{R}_1(t-\tau,v,x)dvdH_2(\tau)$$

$$+(1-q)\beta\exp(-\beta z)\int_0^t\int_0^z \exp(\beta v)\tilde{R}_2(t-\tau,v,x)dvdH_2(\tau).$$

In terms of their Laplace transforms, (3.1) and (3.2) take the following form for $\mathrm{Re}(\theta) > 0$.

$$(3.3)\quad \tilde{\Phi}_1(\theta,z,x)=U(x-z)(1-H_1^*)\theta^{-1}+\alpha(1-p)\exp(\alpha z)H_1^*\int_z^\infty \exp(-\alpha v)\tilde{\Phi}_1(\theta,v,x)dv+$$

$$+p\beta\exp(-\beta z)H_1^*\int_0^z \exp(\beta v)\tilde{\Phi}_2(\theta,v,x)dv$$

$$(3.4)\quad \tilde{\Phi}_2(\theta,z,x)=U(x-z)(1-H_2^*)\theta^{-1}+\alpha q\exp(\alpha z)H_2^*\int_z^\infty \exp(-\alpha v)\tilde{\Phi}_1(\theta,v,x)dv+$$
$$+(1-q)\beta\exp(-\beta z)H_2^*\int_0^z \tilde{\Phi}_2(\theta,v,x)\exp(\beta v)dv.$$

We have the following

<u>THEOREM</u> 3.1. <u>The bounded solution of equations</u> (3.3) <u>and</u> (3.4) <u>is unique and is given by</u>

$$(3.5a)\quad \tilde{\Phi}_1(\theta,z,x)=\exp(r_2z)[\alpha\theta(r_1-r_2)]^{-1}[1-\alpha(1-p)H_1^*/(\alpha-r_2)]^{-1}\cdot$$
$$\cdot\{\exp(-r_1x)(\alpha-r_1)[1-\alpha(1-p)H_1^*/(\alpha-r_1)][\alpha(1-H_1^*)-r_2]-$$
$$-\alpha pH_1^*(r_1-r_2)-\exp(-r_2x)[\alpha(1-H_1^*)-r_1][\alpha-r_2-\alpha(1-p)H_1^*]\},z>x$$

$$(3.5b)\quad \tilde{\Phi}_1(\theta,z,x)=\theta^{-1}+\exp\{(z-x)r_1\}(r_1-\alpha)[\alpha\theta(r_1-r_2)]^{-1}[\alpha(1-H_1^*)-r_2]+$$
$$+\exp(r_2z)[\alpha\theta(r_1-r_2)]^{-1}[1-\alpha(1-p)H_1^*/(\alpha-r_2)]^{-1}\cdot$$
$$\cdot\{\exp(-r_1x)(\alpha-r_1)[1-\alpha(1-p)H_1^*/(\alpha-r_1)][\alpha(1-H_1^*)-r_2]-$$
$$-\alpha pH_1^*(r_1-r_2)\}\ ,\quad 0\le z\le x$$

$$(3.6a)\quad \tilde{\Phi}_2(\theta,z,x)=\exp(r_2z)(\beta+r_2)[\beta p\alpha\theta H_1^*(r_1-r_2)]^{-1}\cdot$$
$$\cdot\{\exp(-r_1x)(\alpha-r_1)[1-\alpha(1-p)H_1^*/(\alpha-r_1)][\alpha(1-H_1^*)-r_2]-$$
$$-\alpha p(r_1-r_2)H_1^*-\exp(-r_2x)[\alpha(1-H_1^*)-r_1][\alpha-r_2-\alpha(1-p)H_1^*]\},z>x$$

$$(3.6b)\quad \tilde{\Phi}_2(\theta,z,x)=\theta^{-1}+\exp\{(z-x)r_1\}(r_1-\alpha)(\beta+r_1)[\alpha(1-H_1^*)-r_2]\cdot$$
$$\cdot[1-\alpha(1-p)H_1/(\alpha-r_1)]^{-1}[\alpha\theta\beta p(r_1-r_2)H_1^*]^{-1}+$$
$$+\exp(r_2z)(\beta+r_2)[\alpha\beta\theta pH_1^*(r_1-r_2)]^{-1}\{\exp(-r_1x)(\alpha-r_1)\cdot$$
$$\cdot[1-\alpha(1-p)H_1^*/(\alpha-r_1)][\alpha(1-H_1^*)-r_2]-\alpha pH_1^*(r_1-r_2)\},0\le z\le x,$$

<u>where</u> $\text{Re}(\theta) > 0$ <u>and</u> $r_1(\theta)$ and $r_2(\theta)$ are given by (2.12) and (2.13).

<u>Proof</u>: The technique of the proof is exactly the same as that of Theorem 2.1. and we therefore sketch only an outline of the proof. Proceeding as in the case of Theorem 2.1, we arrive at the same set of third order differential equations (2.22) and (2.24). The general solution to equations (3.3) and (3.4) has therefore the same form as in (2.26). This is not unexpected, since equations (3.3) and (3.4) differ from (2.7) by the constant term involving $\Phi_i(\theta,0,x)$ in each case. Thus the same relations (2.31) hold between the constants C_4, C_5, C_6, C_7 and C_1, C_2, C_3, C_8 respectively. We proceed now in the same manner as before. A substitution of the general solution (2.26) into (3.3) and (3.4) yields an identity. By comparing coefficients on both sides of this identity we are led to the relations

$$(3.7)\quad \begin{cases} C_0[1-\alpha(1-p)H_1^*/(\alpha+\beta)]-pH_1^*A=0 \\ C_2[1-\alpha(1-p)H_1^*/(\alpha+\beta)]+\beta pH_1^*B_1=0 \\ C_5[1-(1-q)\beta H_2^*/(\alpha+\beta)]-\alpha qH_2^*E=0 \\ \alpha qC_0H_2^*/(\alpha+\beta)+(1-q)H_2^*A=0 \\ \alpha qC_2H_2^*/(\alpha+\beta)-(1-q)H_2^*\beta B_1=0, \end{cases}$$

where

$$A=\beta\{C_5[\exp\{(\alpha+\beta)x\}-1](\alpha+\beta)^{-1}+C_6[\exp\{(r_1+\beta)x\}-1](r_1+\beta)^{-1}+$$

$$+C_7[\exp\{(r_2+\beta)x\}-1](r_2+\beta)^{-1}+C_4\exp\{(r_2+\beta)x\}(r_2+\beta)^{-1}+$$

$$+(\exp(\beta x)-1)(\beta\theta)^{-1}\},$$

$$B=C_5(\alpha+\beta)^{-1}+C_6(r_1+\beta)^{-1}+C_7(r_2+\beta)^{-1}+(\beta\theta)^{-1},$$

and

$$E=(C_0-C_2)\exp\{-(\alpha+\beta)x\}(\alpha+\beta)^{-1}-C_3\exp\{-(\alpha-r_1)x\}(\alpha-r_1)^{-1}+$$

$$+(C_1-C_8)\exp\{-(\alpha-r_2)x\}(\alpha-r_2)^{-1}-\exp(-\alpha x)(\alpha\theta)^{-1}.$$

The solution of the system (3.7) is obtained with the help of equations (2.31). The values of the constants so derived, when put into ~~(2.36)~~ 2.26 yield the solutions (3.5a) and (3.5b). The algebra, although tedious, is straightforward and thus is omitted. As in the proof of Theorem 2.1 the solution $\tilde{\Phi}_2$, as given in (3.6a) and (3.6b), is derived from the solution $\tilde{\Phi}_1$, as given in (3.5a) and (3.5b), by use of the relations (2.31). We have thus exhibited, by construction, a solution of (3.3) and (3.4). The uniqueness is argued in the same way as in Section 2.2.//

The distribution of time to first emptiness is given in terms of its Laplace transform in the theorem below. The proof of this theorem is omitted as it easily follows from (3.5b) and (3.6b) by letting $x \to \infty$, while keeping the signs of $\mathrm{Re}(r_1)$ and $\mathrm{Re}(r_2)$ in mind.

<u>THEOREM</u> 3.2. <u>The Laplace transforms of the distribution of time to first emptiness are given by</u>

(3.8) $F_1^*(\theta)=\exp\{r_2(\theta)z\}pH_1^*(\theta)[\theta\{1-\alpha(1-p)H_1^*(\theta)/(\alpha-r_2(\theta))\}]^{-1}, z > 0,$

(3.9) $F_2^*(\theta)=\exp\{r_2(\theta)z\}(\beta+r_2(\theta))(\beta\theta)^{-1}$, $z > 0$,

where $r_1(\theta)$ *and* $r_2(\theta)$ *are given in* (2.12) *and* (2.13) *respectively*.

We have seen in Chapter II and we shall see again in Chapter IV that the behavior of $Z(t)$ and of $E[Z(t)]$, for large t, is different for the three cases, subcritical, critical, and supercritical. A similar behavior is indicated below for the random variable T_E.

Let $\psi_i(z) = P(T_E=\infty|Z(0)=z,J_0=i)$, $i = 1,2$,

$\psi(z) = P(T_E=\infty|Z(0)=z)$.

THEOREM 3.3. (A) *The limiting distribution, as* $t \to \infty$, *of* T_E *is given by*,

$$(3.10)\quad \psi_1(z)=\begin{cases} 0, & \beta q \le \alpha p \\ 1-p\exp\{-(\beta q-\alpha p)z\}[1-\alpha(1-p)\{\alpha+(\beta q-\alpha p)\}^{-1}], & \beta q > \alpha p \end{cases}$$

and

$$(3.11)\quad \psi_2(z)=\begin{cases} 0, & \beta q \le \alpha p \\ 1-\exp\{-(\beta q-\alpha p)z\}\{\beta-(\beta q-\alpha p)\}\beta^{-1}, & \beta q > \alpha p \end{cases}$$

(B) $E[T_E|Z(0)=z,J_0=i]=\infty$, $i = 1,2$, $z > 0$; if $\beta q = \alpha p$ *and if* $\beta q < \alpha p$, *then*

$$E[T_E|Z(0)=z,J_0=1]=p^{-1}\{E_{H_1}-(\alpha k_1)^{-1}[1-p+\alpha zp]\},$$

and

$$E[T_E|Z(0)=z,J_0=2]= -(1+\beta z)(\beta k_1)^{-1}, \quad (k_1 < 0),$$

for $z > 0$ where k_1 is given in (2.38).

Proof: For proof of part (A) it is easily seen that

$$\psi_i(z)=\lim_{\theta\to 0} \theta\, \tilde{\Phi}_i(\theta,z,\infty),\ i = 1,2 \text{ and } \psi(z)=\lim_{\theta\to 0} \theta\, \tilde{\Phi}(\theta,z,\infty)$$

and that these limits exist (T_E is an increasing function of t). Thus by the usual Tauberian argument

$$\psi_i(z)=\lim_{\theta\to 0} \theta(\theta^{-1}-F_i^*(\theta)),\ i = 1,2, \text{ and } \psi(z) =\lim_{\theta\to 0} \theta(\theta^{-1}-F^*(\theta)).$$

With this, formulas (3.10) and (3.11) follow from (3.8) and (3.9). The results for part (B) follow from (3.8) and (3.9) and Lemma 3 of the Appendix. It is necessary to apply ℓ'Hospital's rule and the limit result (2.16). //

Thus in the case where $\beta q \leq \alpha p$, T_E tends in law, as $t \to \infty$, to a finite random variable, while when $\beta q > \alpha p$ it does not. For the sub-critical case higher moments may be derived in a similar manner.

3. THE MODEL WITH GENERAL DISTRIBUTION FOR EACH RELEASE AND EXPONENTIAL DISTRIBUTION FOR EACH INPUT.

In this section we study the distribution of T_E, the time until first emptiness for the model of Section 2.3. As in Section 2.3. attention will be focused on a forward rather than a backward Kolmogorov integral equation. Considering the last jump of the process Z(t) during (0,t], if there is one, and whether it is an input or a release, subject to the additional condition that the dam is not yet empty at time t, the following forward Kolmogorov equation can be established for $\tilde{R}(t,z,x)$, for $z > 0$.

$$(3.12) \quad \tilde{R}(t,z,x)=U(x-z)\exp\{-(\lambda+\mu)t\}+ +\lambda\int_0^t \exp\{-(\lambda+\mu)(t-\tau)\}d\tau\int_0^x \tilde{R}(\tau,z,x-y)dB(y) +\mu\int_0^t \exp\{-(\lambda+\mu)(t-\tau)\}d\tau\int_0^\infty P(y<Z(\tau)\leq x+y,T_E>\tau|Z(0)=z)dD(y).$$

Since

$$P(y<Z(\tau)\leq x+y,T_E>\tau|Z(0)=z)=\tilde{R}(\tau,z,x+y)-\tilde{R}(\tau,z,y),$$

$$(3.12) \quad \tilde{R}(t,z,x)=U(x-z)\exp\{-(\lambda+\mu)t\}+ +\lambda\int_0^t \exp\{-(\lambda+\mu)(t-\tau)\}d\tau\int_0^x R(\tau,z,x-y)dB(y)+ +\mu\int_0^t \exp\{-(\lambda+\mu)(t-\tau)\}d\tau\int_0^\infty \tilde{R}(\tau,z,x+y)dD(y)- -\mu\int_0^t \exp\{-(\lambda+\mu)(t-\tau)\}d\tau\int_0^\infty \tilde{R}(\tau,z,y)dD(y).$$

The corresponding Laplace transform of (3.12) satisfies, for $Re(\theta) > 0$,

$$(3.13) \quad \tilde{\Phi}(\theta,z,x)(\lambda+\mu+\theta)=U(x-z)+\lambda\int_0^x \tilde{\Phi}(\theta,z,x-y)dB(y)+ +\mu[\int_0^\infty \tilde{\Phi}(\theta,z,x+y)dD(y)-\int_0^\infty \tilde{\Phi}(\theta,z,y)dD(y)].$$

It can be shown, by exactly the same argument used in Section 2.3., that (3.13) has a unique bounded solution, which unfortunately is difficult to obtain in any closed form in its present generality. However, there does exist a tractable special case to which we now turn our attention. From now on we shall assume that

$$(3.14)\qquad B(y) = \begin{cases} 1-\exp(-\beta y), & y \geq 0 \\ 0, & y < 0, \end{cases}$$

$\beta > 0$, while $D(\cdot)$ remains arbitrary. With this we obtain, in the next theorem, a solution of equation (3.13).

THEOREM 3.4. *Given* $\varepsilon > 0$, *suppose* $D(s)/s \leq K_1 < \infty$, *for* $0 < s \leq \varepsilon$. *Then the unique bounded solution of equation* (3.13) *is given by*

$$(3.15)\quad \tilde{\Phi}(\theta,z,x)=\theta^{-1}+C(\theta,z)+A\exp(-\gamma(\theta)x)-(\lambda+\mu+\theta)^{-1}K(z-x),\ 0 \leq x < z$$

$$(3.16)\quad \tilde{\Phi}(\theta,z,x)=\theta^{-1}+C(\theta,z)+A\exp(-\gamma(\theta)x)\qquad ,\ x \geq z,$$

where

$$(3.17)\quad C(\theta,z)=\theta^{-1}\mu[1+\theta^{-1}\mu\{1-D^*(\gamma)(\alpha-\gamma)\alpha^{-1}\}]^{-1}\cdot$$

$$\cdot[-\theta^{-1}\{1-D^*(\gamma)(\alpha-\gamma)\alpha^{-1}\}+(\lambda+\mu+\theta)^{-1}\{\int_0^z K(z-y)dD(y)+$$

$$+\int_0^z \exp(\alpha v)K(z-v)dv\}],$$

$$(3.18)\quad A=-\alpha^{-1}(\alpha-\gamma)C(\theta,z)-(\alpha-\gamma)[(\alpha\theta)^{-1}+(\lambda+\mu+\theta)^{-1}\int_0^z \exp(\alpha v)K(z-v)dv],$$

$D^*(\gamma)=\int_0^\infty \exp(-\gamma y)dD(y)$, *and* $\gamma=\gamma(\theta)$ and K are as in Section 2.3.

Proof: We proceed exactly as in the proof of Theorem 2.5. Let

$$\tilde{\Phi}(\theta,z,x)=\tilde{\Phi}_1(\theta,z,x)\qquad \text{for } 0 \leq x < z$$

$$\tilde{\Phi}(\theta,z,x)=\tilde{\Phi}_2(\theta,z,x)\qquad \text{for } x \geq z$$

$$H(\theta,z)\ = \int_0^\infty \tilde{\Phi}(\theta,z,y)dD(y).$$

Then equation (3.13) may be broken into the two parts

$$(3.19)\quad \tilde{\Phi}_1(\theta,z,x)(\lambda+\mu+\theta)=\alpha\lambda\exp(-\alpha x)\int_0^x \exp(\alpha v)\ \tilde{\Phi}_1(\theta,z,v)dv$$
$$+\mu\left[\int_0^{z-x}\tilde{\Phi}_1(\theta,z,x+y)dD(y)+\int_{z-x}^{\infty}\tilde{\Phi}_2(\theta,z,x+y)dD(y)\right]$$
$$-\mu H(\theta,z)\qquad ,\ 0 \le x < z$$

and

$$(3.20)\quad \tilde{\Phi}_2(\theta,z,x)(\lambda+\mu+\theta)=1+\alpha\lambda\exp(-\alpha x)\left[\int_0^z \exp(\alpha v)\tilde{\Phi}_1(\theta,z,v)dv\right.$$
$$\left.+\int_z^x \tilde{\Phi}_2(\theta,z,v)dv\right]+$$
$$+\mu\int_0^{\infty}\tilde{\Phi}_2(\theta,z,x+y)dD(y)-\mu H(\theta,z),\ x \ge z.$$

We shall construct a solution of (3.13) by putting

$$(3.21)\quad \tilde{\Phi}_2=\theta^{-1}+C^*(\theta,z)+A^*\exp(-rx),\ x \ge z,$$

and

$$(3.22)\quad \tilde{\Phi}_1=\theta^{-1}+C^*(\theta,z)+A^*\exp(-rx)+g(\theta,z,x),\ 0 \le x < z,$$

where $g(\theta,z,x)$ is a function to be determined, and $0 < \mathrm{Re}(r(\theta)) < \alpha$. Substitution of (3.21) into ~~(3.13)~~ 3.20 produces an identity in x. Comparing the coefficients of $\exp(rx)$ and $\exp(-\alpha x)$ and the constant terms in that identity we obtain the relations

$$(3.23)\quad \lambda+\mu+\theta - \frac{\alpha\lambda}{\alpha-r} -\mu D^*(r)=0,\quad \alpha\neq r\ ,$$

$$(3.24)\quad J(\theta,z)-\alpha^{-1}\exp(\alpha z)(\theta^{-1}+C^*(\theta,z))-(\alpha-r)^{-1}A^*\exp\{(\alpha-r)z\}=0,$$

and

(3.25) $$\theta C^*(\theta,z)+\mu H(\theta,z)=0,$$

where

(3.26) $$J(\theta,z)=\int_0^z \exp(\alpha v)\tilde{\Phi}_1(\theta,z,v)dv,$$

which is independent of x. By Lemma ~~2.4~~ 2.2 equation (3.23) has a unique root $r=\gamma(\theta)$ in $0 < |r(\theta)| \leq \alpha\left|\frac{\mu+\theta}{\lambda+\mu+\theta}\right|$, $\mathrm{Re}(\theta) > 0$.

Substitution of (3.21) and (3.22) into (~~3.20~~ 3.19) leads us to an integral equation of the form

(3.27) $$g(\theta,z,z-s)(\lambda+\mu+\theta)=-\mu H(\theta,z)-\theta(\theta^{-1}+C(\theta,z))-$$
$$-\alpha\lambda\exp(-\alpha x)\int_0^s \exp(\alpha w)g(\theta,z,w+z-s)dw+$$
$$+\mu\int_0^s g(\theta,z,z-s+y)dD(y),$$

Now let $\Gamma(\theta,s)=g(\theta,z,z-s)$. Equation (3.27) then becomes, taking (3.25) into account,

(3.28) $$\Gamma(\theta,s)=-(\lambda+\mu+\theta)^{-1}+\int_0^s G(s-y)dH(y),$$

where H is given in (2.54).

Equation (3.28) is the same Volterra equation as (~~2.98~~ 2.68) and has, as indicated in Section 2.3, the unique solution

$$\Gamma(\theta,s)=-(\lambda+\mu+\theta)^{-1}K(s),$$

where $K(s) = \sum_{k=0}^{\infty} H^{(k)}(s)$. Converting from Γ back to g we find $\tilde{\Phi}_1(\theta,z,x)$ in terms of A* by means of (3.22). Then using (3.24) and (3.26) we obtain

$$(3.29)\quad A^*=-(\alpha-\gamma)[\alpha^{-1}\{\theta^{-1}+C^*(\theta,z)\}+(\lambda+\mu+\theta)^{-1}\int_0^z \exp(\alpha v)K(z-v)dv].$$

By the definition of $H(\theta,z)$ and (3.21) and (3.22) it follows that $H(\theta,z)$ satisfies

$$(3.30)\quad H(\theta,z)=\theta^{-1}-\mu\theta^{-1}H(\theta,z)+A^*D^*(\gamma)-(\lambda+\mu+\theta)^{-1}\int_0^z K(z-y)dD(y).$$

Finally from (3.29), (3.30) and (3.25) it follows that

$$(3.31)\quad H(\theta,z)=-[1+\mu\theta^{-1}\{1-D^*(\gamma)(\alpha-\gamma)\alpha^{-1}\}]^{-1}[-\theta^{-1}\{1-D^*(\gamma)(\alpha-\gamma)\alpha^{-1}\}+$$
$$+(\lambda+\mu+\theta)^{-1}\{\int_0^z K(z-y)dD(y)+\int_0^z \exp(\alpha v)K(z-v)dv\}].$$

Now we get $C^*(\theta,z)$ from (3.31) and (3.25). C^* has the form of the right hand side of (3.17). And thus A^* in (3.29) has the same form as the right side of (3.18).

The uniqueness of this solution follows in exactly the same way as in Theorem 2.5. This completes the proof. //

We now have the distribution of T_E given in the following theorem which follows from (3.16) by letting $x \to \infty$ and keeping in mind that $Re(\gamma(\theta)) > 0$.

THEOREM 3.5. *The Laplace transform of the distribution of* T_E *is given, for* $Re(\theta) > 0$, *and* $z > 0$, *by*

$$(3.32)\quad F^*(\theta)=[\theta+\mu\{1-D^*(\gamma)(\alpha-\gamma)\alpha^{-1}\}]^{-1}\cdot$$
$$\cdot[1-\mu(\lambda+\mu+\theta)^{-1}\{\int_0^z K(z-y)dD(y)-\int_0^z \exp(\alpha v)K(z-v)dv\}].$$

Analogous to the result in Section 2 we have

<u>THEOREM</u> 3.6. (A) <u>The limiting distribution of</u> T_E, <u>as</u> $t \to \infty$, <u>is given by</u>

$$(3.33) \quad \psi(z) = \begin{cases} 0 & \text{if } (\lambda/\alpha) \le E_D\mu \\ 1-[1+\mu(E_D+\alpha^{-1})(\alpha^{-1}\lambda-\mu E_D)^{-1}]^{-1}\cdot S(z), & \text{if } (\lambda/\alpha) > E_D\mu \end{cases}$$

<u>where</u>

$$S(z)=[(\alpha^{-1}\lambda-\mu E_D)(E_D+\alpha^{-1})-(\mu+\lambda)^{-1}\{\int_0^z K(z-y)dD(y)+ +\alpha\int_0^z \exp(\alpha y)K(z-y)dy\}]\cdot$$

(B) $E[T_E|Z(0)=z] = \infty$, <u>if</u> $(\lambda/\alpha) \ge \mu E_D$

<u>and</u>

$$E[T_E|Z(0)=z]=\mu^{-1}[1-D^*(\gamma^*)(\alpha-\gamma^*)\alpha^{-1}]^{-1}W(z) \quad , \text{ if } (\lambda/\alpha) < \mu E_{D_1}$$

<u>where</u>

$$W(z) = [1-\mu(\mu+\lambda)^{-1}\{\int_0^z K(z-y)dD(y)+\int_0^z \exp(\alpha v)K(z-v)dv\}],$$

<u>and</u> $\gamma^*=\lim_{\theta\to 0} \gamma(\theta)$ (cf. Section 2.3).

We omit the proof as it follows in an analogous manner to the proof of Theorem 2.5. We see that again in the critical and supercritical cases T_E tends in law, as $t \to \infty$, to a proper random variable, while in the ~~sub~~ super critical case it does not.

CHAPTER IV

LIMIT BEHAVIOR OF THE PROCESS $\{Z(t)\}$

1. THE APPROACH AND SOME NOTATION.

In this chapter we study the limit behavior of the process $\{Z(t)\}$, representing the level of the dam for the storage model of Section 2.1. A standard approach to this problem would be to locate first a suitable sequence of points of regeneration of the process such as the time points of first emptiness and subsequent returns to emptiness in the present case. Then, conditioning on the number and location in time of the points in this sequence, partition the time interval under consideration into its component parts and study the corresponding components of the process separately. It turns out, however, that this standard approach, although attractive, is not the most fruitful one in the present case. Instead we find it more convenient to consider the sequence of points of return to the release state, which, in general, are not regeneration points of the process. This consideration leads to the representation (4.5) which expresses $Z(t)$ in terms of an auxiliary process associated with this sequence. It is this representation which, when suitably exploited, leads to the main results of this chapter. In Chapter II, the solution of equations (2.4 ~~2.5~~) for an exact expression of the distribution of $Z(t)$ was obtained for special cases, in which the random variables determining the amounts of either inputs or releases or both are assumed to be exponentially distributed. However, for the limit results presented in this chapter we make no such assumption.

We carry over all the assumptions and notation for the model of Section **2.1**. In particular, there we let $\{(J_n, T_n),\ n=0,1,2,\ldots\}$ be a double sequence of random variables taking values in the state space $\{1,2\} \times [0,\infty)$ where 1 corresponds to an input state and 2 to a release state. We assumed that $T_0 = 0$ a.s. and that $P(J_0 = j) = a_j$ where the a_j satisfy (i) $a_j \geq 0$, (ii) $a_1 + a_2 = 1$, and

$$P(J_n = j,\ T_n \leq x \mid J_0, J_1, T_1, \ldots, T_{n-1}, J_{n-1} = i)$$

$$= P(J_n = j,\ T_n \leq x \mid J_{n-1} = i) = p_{ij} H_i(x) = Q_{ij}(x)$$

for all $x \varepsilon (-\infty, \infty)$ and $n=1,2,\ldots, i,j=1,2$ (cf. also (2.2)). The distribution functions $H_1(x)$, $H_2(x)$ satisfy $H_1(0+) < 1$, $H_2(0+) < 1$ and $H_1(+\infty) = 1$, $H_2(+\infty) = 1$. We set

$$q = Q_{21}(+\infty) = P[J_{n+1} = 1 \mid J_n = 2],\ 0 < q < 1$$

$$p = Q_{12}(+\infty) = P[J_{n+1} = 2 \mid J_n = 1],\ 0 < p < 1.$$

We have then

$$H_j(t) = \sum_{k=1}^{2} Q_{jk}(t) ,$$

and we assume that

$$E_{H_j} = \int_0^\infty (1-H_j(u))du < \infty , \ j=1,2 . \tag{4.1}$$

We set

$$\tau_n = \sum_{i=0}^{n} T_i, \ n=0,1,2,..., \tag{4.2}$$

and define the integer valued stochastic processes $\{N(t); t \geq 0\}$ and $\{N_j(t);t \geq 0\}$ as

$$N(t) = \sup\{n \geq 0: \tau_n \leq t\} ,$$

$$N_j(t) = \text{number of times } J_k = j \text{ for } 0 < k < N(t)+1, \ j=1,2.$$

Thus $N_2(t)$ represents the number of visits to state 2 (release) in the interval $(0,t]$.

The process $Z(t)$ was defined constructively in Section 2.1. We now introduce some additional notation. Let $\rho_1,\rho_2,\rho_3,...,$ be the sequence of lengths of time between successive returns to state 2 after time $t = 0$. These lengths are independent and identically distributed. Set $\rho_0 \equiv 0$, and let $\sigma_n = \sum_{j=0}^{n} \rho_j$, $n=0,1,....$ Clearly

$$N_2(t) = k \Leftrightarrow \sigma_k \leq t < \sigma_{k+1}, \ k = 0,1,2,....$$

Again, $\sigma_{N_2(t)}$ denotes the time of the last release, if there is one, before time t. Let us suppose $J_0 = 2$ and let ν_j denote the number of inputs occurring in the interval (σ_{j-1},σ_j) for j=1,2,... The j-th release must, by definition of ρ_j, occur at σ_j. ν_1 is thus (starting from state 2) the number of visits to state 1 (input) until the first return to state 2 (release).

In Chapter II we defined the independent identically distributed nonnegative random variables $X_1,X_2,\ldots$, the 'inputs', independent of N(t), with common distribution function B(x), $x \geq 0$; also the independent identically distributed nonnegative random variables $Y_1,Y_2,\ldots$, the 'releases', independent of the X's and of N(t), with common distribution function D(y); $y \geq 0$. We put, by convention $X_0 = Y_0 \equiv 0$, and assume that B and D are such that $E(X) < \infty$ and $E(Y) < \infty$. Let the random variables V_j, be defined by

$$V_j = X_1^{(j)}+X_2^{(j)}+\ldots+X_{\nu_j}^{(j)} - Y_j^{(j)}, j=1,2,\ldots, \tag{4.3}$$

while V_0 is an arbitrary nonnegative random variable. The ν_j, j=1,2,... are independent of the X's and Y's, are mutually independent, and follow a common distribution, namely that of ν_1. Consequently, V_j, j=1,2,... is a sequence of independent identically distributed random variables.

Finally, define the following sequence of random variables.

$$\begin{aligned} \eta_0 &= V_0 \\ \eta_{n+1} &= \max(0,\eta_n+V_{n+1}),\ n = 0,1,2,\ldots \end{aligned} \tag{4.4}$$

The sequence η_n has been studied extensively, more recently by Takacs [52].

Now by Lemma 1 of the Appendix $N_2(t)$ is almost surely finite for all $t \geq 0$. This in turn implies there is a last release before time t and that the random variables ν_j, j=1,2,... are also almost surely finite. Thus, by the above structure and the constructive definition of the process Z(t) (cf. Section 2.1) it is evident that

$$Z(t) \equiv \eta_{N_2(t)} + \sum_{j=0}^{I_t} X_j, \quad t \geq 0 , \tag{4.5}$$

holds, almost surely, where I_t denotes the number of inputs occurring during $(\sigma_{N_2(t)}, t]$.

In the next section we examine more closely the components comprising this representation.

2. SOME PRELIMINARY RESULTS.

Unless stated otherwise we take $J_0 = 2$ throughout this section. Then it is easy to establish that

$$\begin{aligned} P(\rho_1 \leq x, \nu_1 = 0) &= (1-q)H_2(x) \\ P(\rho_1 \leq x, \nu_1 = k) &= q(1-p)^{k-1}pH_2 * H_1^{(k)}(x); \; k \geq 1. \end{aligned} \tag{4.6}$$

From this it follows that

$$P(\rho_1 \leq x) = (1-q)H_2(x) + pqH_2 * H_1 * \sum_{k=1}^{\infty} [(1-p)H_1(x)]^{(k-1)}, \tag{4.7}$$

and

(4.8) $$P(\nu_1 = 0) = (1-q),\ P(\nu_1 = k) = pq(1-p)^{k-1},\ k \geq 1 .$$

We have the following two theorems.

<u>THEOREM</u> 4.1. <u>I_t has the following distribution for $t \geq 0$.</u>

(4.9)
$$P(I_t = 0) = (1-H_2)*U(t)$$
$$P(I_t = k) = q(1-p)^{k-1}H_1*H_2^{(k-1)}*(1-H_2)*U(t),\ k \geq 1,$$

<u>where</u> $$U(t) = \sum_{k=0}^{\infty} P(\rho_0 + \rho_1+\ldots+\rho_k \leq t).$$

<u>Proof</u>: Since $P(N_2(t) < \infty) = 1$ we have

$$P(I_t = 0) = P(I_t = 0,\ N_2(t) = 0)+\sum_{k=1}^{\infty} P(I_t = 0,\ N_2(t) = k),$$

or equivalently,

$$P(I_t = 0)=(1-H_2(t))+\sum_{k=1}^{\infty}\int_0^t(1-H_2(t-x))d_x\, P(\rho_0+\rho_1+\ldots+\rho_k \leq x)$$
$$=(1-H_2)*U(t).$$

For $k \geq 1$ we get, by the same argument,

$$P(I_t = k) = \sum_{j=0}^{\infty} P(I_t = k,\ N_2(t) = j)$$
$$= q(1-p)^{k-1}H_2*H_1^{(k-1)}*(1-H_1)(t)+$$
$$+ q(1-p)^{k-1}H_2*H_1^{(k-1)}*\sum_{j=1}^{\infty}\int_0^t(1-H_1(t-x))d_xP(\rho_0+\rho_1+\ldots+\rho_j \leq x)$$
$$= q(1-p)^{k-1}H_2*H_1^{(k-1)}*(1-H_1)*U(t),$$

which is the desired result. //

<u>THEOREM 4.2.</u> $\sum_{j=0}^{I_t} X_j \xrightarrow{\mathcal{L}} T$, <u>as</u> $t \to \infty$,

<u>where T is an almost surely finite (nonnegative) random variable, with Laplace transform given by (4.14).</u>

<u>Proof:</u> The Laplace Stieltjes transform of the random variable $\sum_{j=0}^{I_t} X_j$ is given by

$$(4.10)\quad E[\exp\{-s\sum_{j=0}^{I_t} X_j\}]=(1-H_2)*U(t)+$$

$$+\sum_{k=1}^{\infty}[\gamma_1(s)]^k q(1-p)^{k-1}H_2*H_1^{(k-1)}*(1-H_1)*U(t),$$

where $s > 0$ and $\gamma_1(s) = E[\exp(-sX)]$. From (4.7) and an application of the monotone convergence theorem

$$(4.11)\quad E(\rho_1)=E_{H_2}+E_{H_1}pq\sum_{k=1}^{\infty}k(1-p)^{k-1}=E_{H_2}+pqE_{H_1}\{-\frac{d}{dp}\sum_{k=0}^{\infty}(1-p)^k\}$$

$$=E_{H_2}+p^{-1}q\ E_{H_1} < \infty.$$

Now by the Key Renewal Theorem (cf. Smith [50], p.15), listed also in the Appendix, it follows that

$$(4.12)\qquad \lim_{t\to\infty}(1-H_2)*U(t) = E_{H_2}\mu^{-1},$$

where $\mu = E(\rho_1)$. Also by the same theorem

$$\lim_{t\to\infty} H_2*H_1^{(k-1)}*(1-H_1)*U(t) = \mu^{-1}\int_0^{\infty} H_2*H_1^{(k-1)}*(1-H_1)(u)du\ .$$

It can be shown that

$$(4.13) \qquad \int_0^\infty H_2 * H_1^{(k-1)*}(1-H_1)(u)du = E_{H_1} \;, \quad k \geq 1 \;.$$

Using (4.12) and (4.13) in (4.10) it follows that

$$(4.14) \qquad \lim_{t\to\infty} E[\exp\{-s \sum_{j=0}^{I_t} X_j\}] = \mu^{-1} E_{H_2} + q\gamma_1(s) E_{H_1} \{\mu[1-(1-p)\gamma_1(s)]\}^{-1}$$

$$= h(s), \text{ say.}$$

By Theorem 2, p. 408 of [14] h(s) is the transform of a possibly defective distribution F_T and the convergence in law holds. Since $\gamma(0) = 1$ it follows from (4.14) that

$$h(0) = \mu^{-1}(E_{H_2} + q\, p^{-1} E_{H_1}) \;.$$

Now it follows from (4.11) that $h(0) = 1$. This completes the proof. //

We prove now the following lemma which will be essential for the limit results in Section 3.

LEMMA 4.1. *Let $\{\upsilon_t\}$ be a sequence of random variables defined on appropriate probability spaces $(\Omega_t, \mathcal{A}_t, P)$, T an almost surely finite random variable defined on $(\Omega, \mathcal{A}, P)$ such that $\lim_{t\to\infty} P(\upsilon_t \leq x) = P(T \leq x)$ for every continuity point x of the distribution function of T. Suppose also that g is any function such that $g(t) \to \infty$ as $t \to \infty$. Then*

$$(4.15) \qquad [\upsilon_t / g(t)] \xrightarrow{P} 0, \quad \text{as } t \to \infty.$$

Proof: Let F_{υ_t} and F_T be the distribution functions of υ_t and of T respectively. Fix $\varepsilon > 0$, $\delta > 0$. Since $T < \infty$ a.s. there exists a continuity point A of F_T such that

$$F_T(A) > 1-\delta \text{ and } F_T(-A) < \delta.$$

We have

$$P(|\upsilon_t/g(t)| < \varepsilon) = F_{\upsilon_t}(\varepsilon g(t)) - F_{\upsilon_t}(-\varepsilon g(t)-).$$

There exists $t_1 = t_1(\varepsilon)$ such that for $t > t_1$

$$-\varepsilon \leq F_{\upsilon_t}(A) - F_T(A) \leq \varepsilon$$

and another t_2 such that $\varepsilon g(t) \geq A$ when $t \geq t_2$. Take $t \geq \max(t_1, t_2)$. Then we have

$$\begin{aligned} P(|\upsilon_t/g(t)| < \varepsilon) &\geq F_T(A) - \varepsilon - (F_T(-A-)+\varepsilon) \\ &\geq F_T(A) - F_T(-A-) - 2\varepsilon \\ &\geq 1 - 2(\varepsilon+\delta). \end{aligned}$$

If we let $\varepsilon \to 0$ then $\delta \to 0$, the assertion follows. //

Considering the random variable V_j we observe that $E(V_j)$ exists, is finite and is given by

$$E(V_j) = E(X)\, E(\nu_j) - E(Y), \; j = 1, \ldots \quad . \tag{4.16}$$

This follows from Wald's fundamental identity, since the ν_j's are almost surely finite and from the assumption that the first moments of B and D are finite. It follows from (4.16) and (4.8) that

$$E(V_j) = E(X)q\,p^{-1} - E(Y)\ ,\ j = 1,2,\ldots\ .$$

In a similar manner it can be shown that

$$\mathrm{Var}(V_j) = qp^{-1}E(X^2)+[E(X)]^2\{2(1-p)-q\}qp^{-2} -[E(Y)]^2+E(Y^2), j=1,2,\ldots\ .$$

From now on we set $a = E(X)q\,p^{-1} - E(Y)$, $(=EV_j)$, and

$$b = qp^{-1}\{E(X^2)-2E(XY)\}+2[E(X)]^2q(1-p)p^{-2} + E(Y^2),\ (=EV_j^2)\ .$$

The sequence η_n employed in the representation (4.5) has the following property which will be useful in the sequel.

(4.17) $\eta_n = \max(S_n-S_n,\ S_n-S_{n-1},\ldots,S_n-S_1,S_n),\ n=1,2,\ldots$

where $S_n = \sum_{i=0}^{n} V_i$ (cf. Takacs [52], p. 344). A sequence of random variables closely related to the η_n and also essential in the analysis of the limit results for Z(t) is the following .

(4.18) $\eta_n^* = \max(0,\zeta_1,\ \zeta_2,\ldots,\zeta_n),\ n = 0,1,2,\ldots,$

where $\zeta_n = S_n-V_0$. The exact distribution of η_n^* is covered by the well known Spitzer identity [51]. If $\eta_0=0$, η_n and η_n^* have the same distribution. If η_0 is an arbitrary nonnegative random variable, the distribution of η_n is covered in the paper by Takacs [52]. Let

$$(4.19) \qquad \eta = \sup_{0 \leq n < \infty} (S_n - V_0)$$

The random variable η is nonnegative and possibly infinite. As in Chapter II, we call the process $\{\eta_n\}$ subcritical, critical or supercritical according as $a < 0$, $a = 0$, or $a > 0$. We exclude the case $P(V_n=0)=1$ since in that case the independence of the X's and Y's is violated when B and D are nontrivial, and otherwise we would have $P(\eta=0)=1$.

Consider then the case $P(V_n=0) < 1$. We summarize some known limiting results for η_n and η_n^*.

(R_1) (Takacs [52], p. 350) If $E(|V_n|) < \infty$ and $a < 0$, then $P(\eta<\infty)=1$. On the other hand if $a \geq 0$, then $P(\eta=\infty)=1$.

(R_2) (Lindley [25], p. 281; Takacs [52] p. 345). If $E(|V_n|) < \infty$ and $P(V_n=0) < 1$, then we have

$$\lim_{n\to\infty} P(\eta_n \leq x) = P(\eta \leq x) ,$$

regardless of the distribution of η_0. As a consequence of the proof of Lindley's result we have that η_n and η_n^* have the same limit distribution.

(R_3) (Erdos and Kac [12]). If $a = 0$, $E(V_n^2) = 1$, then

$$(4.20) \qquad \eta_n^* \, n^{-1/2} \xrightarrow{\mathcal{L}} \xi,$$

where ξ is a random variable with distribution function

$$(4.21) \qquad L(x) = \begin{cases} 2\Phi(x) - 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0, \end{cases}$$

and $\Phi(x)$ is the standard normal distribution function.

(R_4) (Chung [9], p. 1163; also a shorter proof by Puri [45]). If $a > 0$, then $\eta_n^* n^{-1} \xrightarrow{a.s.} a$, as $n \to \infty$. If moreover Var $V_n=1$, then

$$(4.22) \qquad \lim_{n\to\infty} P((\eta_n^* - an)\, n^{-1/2} \leq x) = \Phi(x) .$$

We remark here that, starting with $J_0 = 1$, we could go through exactly the same analysis as we did for $J_0 = 2$ in this section. The limit results for the case $a \geq 0$ are independent of the value of J_0. However, the limit results for the case $a < 0$ will, in general, depend upon the value of J_0. Unfortunately, there are other problems associated with this latter case. We reserve comment on this, until the end of the next section.

3. THE LIMIT RESULTS.

First it is essential to state an important limiting property of $N_2(t)$, the number of releases in the interval $(0,t]$. Since $N_2(t)$ also represents the number of visits of the Markov chain $\{J_n\}$ to the state 2, it could be visualized as a renewal process (cf. Çinlar [11], p. 125). Denote the distribution function which induces this renewal process by F. We have $\mu = E(\rho_1) = \int_0^\infty (1-F(u))du < \infty$ from (4.11). The following well known result is stated without proof (see Chung [10], p. 127).

THEOREM 4.3. Let $N_2(t)$ denote the number of renewals in $(0,t]$ of the renewal process induced by F. Then

$$N_2(t)\, t^{-1} \xrightarrow{P} c > 0, \text{ as } t \to \infty, \tag{4.23}$$

where $c = \mu^{-1}$.

We turn first to the critical case, where $a = 0$. In proving the following limit theorem we employ the approach used by Renyi in [48] where he studied the asymptotic distribution of the sum of a random number of independent random variables.

THEOREM 4.4. Let $a = 0$, $b < \infty$. Without loss of generality we take $b = 1$. Then

$$\lim_{t\to\infty} P(Z(t)\, t^{-1/2} \leq x) = L(xc^{-1/2}) ,$$

where the distribution function L is given by (4.21).

Proof: Using the representation (4.5) we have

$$Z(t)\, t^{-1/2} = \eta_{N_2(t)} t^{-1/2} + \sum_{j=0}^{I_t} X_j t^{-1/2} . \tag{4.24}$$

By Theorem 4.2 $\sum_{j=0}^{I_t} X_j$ tends in law, as $t \to \infty$, to an almost surely finite random variable. Lemma 4.1 allows us consequently to focus attention on the first term on the right hand side of (4.24). In view of Theorem 4.3., there exists $\varepsilon_t > 0$, with $\varepsilon_t \downarrow 0$ as $t \to \infty$, such that

$$P(|N_2(t) - ct| > c\, \varepsilon_t\, t) \leq c\, \varepsilon_t . \tag{4.25}$$

Define the set $A_t = \{\omega: \ |N_2(t)-ct| \leq \varepsilon_t ct\}$, $t \geq 0$.
We thus have

$$(4.26) \qquad P(\bar{A}_t) \leq c\ \varepsilon_t$$

where $\bar{A}_t$ denotes the complement of A_t. Let now

$$n_1(t) = [(1-\varepsilon_t)ct] \text{ and } n_2(t) = [(1+\varepsilon_t)ct] ,$$

where [....] denotes the integral part of the number in the brackets. For convenience the arguments of n_1 and n_2 will sometimes be suppressed. Both $n_1(t)$ and $n_2(t) \to \infty$ as $t \to \infty$. Moreover, on A_t, $n_1(t) < N_2(t) \leq n_2(t)$. We can now write

$$(4.27) \quad \eta_{N_2(t)} t^{-1/2} = \eta_{n_1} n_1^{-1/2} [n_1/t]^{1/2} I_{A_t} + \{\eta_{N_2(t)} - \eta_{n_1}\} n_1^{-1/2} [n_1/t]^{1/2} I_{A_t} + \eta_{N_2(t)} t^{-1/2} I_{\bar{A}_t} ,$$

where I_B is the indicator function of the set B. By (4.26) it suffices to consider the first two terms on the right hand side of (4.27). From (4.23) it follows that $n_1(t)/t \xrightarrow{P} c$, as $t \to \infty$. Further, it can be shown using

$$P(\eta_n^* n^{-1/2} \leq x, (V_0+\zeta_n) n^{-1/2} \leq x) \leq P(\eta_n n^{-1/2} \leq x) \leq P(\eta_n^* n^{-1/2} \leq x), \text{ all } x \geq 0$$

and a similar argument to that in proof of (R_2) Section $\underset{\sim}{1}$, that $\eta_n n^{-1/2}$ and $\eta_n^* n^{-1/2}$ have the same limit distribution. By (R_3) Section $\underset{\sim}{1}$ the first term on the right side of (4.27) tends in probability to $c^{1/2}\ \xi$, and thus it suffices to show that

$$|\eta_{N_2(t)}-\eta_{n_1}|n_1^{-1/2}\, I_{A_t} \xrightarrow{P} 0, \text{ as } t \to \infty.$$

Now fix an $\varepsilon > 0$. Then

$$P(|\eta_{N_2(t)}-\eta_{n_1}|I_{A_t} \geq \varepsilon_{n_1}^{1/2}) \leq P(\sup_{n_1<j\leq n_2} |\eta_j-\eta_{n_1}| \geq \varepsilon n_1^{1/2}) .$$

The form of the limit in (4.21) allows us, for $\delta > 0$, to choose a continuity point $A = A(\delta)$ of L, so small that $P(\xi \leq A) \leq \delta/8$. Now, from (4.20) there exists a $t_1(\delta)$ such that for $t > t_1(\delta)$

$$|P(\eta_{n_1} \leq A_{n_1}^{1/2}) - P(\xi \leq A)| \leq \delta/8.$$

Hence

$$P(\eta_{n_1} \leq A_{n_1}^{1/2}) \leq \delta/4 \quad \text{for } t > t_1(\delta) . \tag{4.28}$$

Now

$$P(\sup_{n_1<j\leq n_2} |\eta_j-\eta_{n_1}|>\varepsilon n_1^{1/2})=P(\sup_{n_1<j\leq n_2} |\eta_j-\eta_{n_1}|>\varepsilon n_1^{1/2},\eta_{n_1}>A_{n_1}^{1/2}) \tag{4.29}$$
$$+P(\sup_{n_1<j\leq n_2} |\eta_j-\eta_{n_1}|>\varepsilon n_1^{1/2},\eta_{n_1}\leq A_{n_1}^{1/2}).$$

By (4.28) the second term in (4.29) is less than $\delta/4$ for $t \geq t_1(\delta)$. In order to consider the first term on the right hand side of (4.29) we set $\varepsilon'=\min(\varepsilon,A)$. Choose $t > t_1(\delta)$. Since

$$\{\sup_{n_1<j\le n_2} |\eta_j-\eta_{n_1}|>\varepsilon' n_1^{1/2},\ \eta_{n_1}>\varepsilon' n_1^{1/2}\}$$

$$\subseteq \bigcup_{n_1<j\le n_2} \{|\sum_{i=n_1+1}^{j} V_i|>\varepsilon' n_1^{1/2},\ |\sum_{i=n_1+1}^{j-1} V_i| \le \varepsilon' n_1^{1/2}\}$$

$$\subseteq \{\sup_{n_1<j\le n_2} |\sum_{i=n_1+1}^{j} V_i|>\varepsilon' n_1^{1/2}\}\ ,$$

it follows that

$$(4.30)\quad P(\sup_{n_1<j\le n_2} |\eta_j-\eta_{n_1}|>\varepsilon n_1^{1/2},\ \eta_{n_1} > A_{n_1}^{1/2})$$

$$\le P(\sup_{n_1<j\le n_2} |\eta_j-\eta_{n_1}|>\varepsilon' n_1^{1/2},\ \eta_{n_1}>\varepsilon' n_1^{1/2})$$

$$\le P(\sup_{n_1<j\le n_2} |\sum_{i=n_1+1}^{j} V_i|>\varepsilon' n_1^{1/2})\ .$$

Now by the Kolmogorov inequality (Chung [10], p. 109) it follows that

$$(4.31)\quad P(\sup_{n_1<j\le n_2} |\sum_{i=n_1+1}^{j} V_i|>\varepsilon' n_1^{1/2}) \le (n_2-n_1)\{(\varepsilon')^2 n_1\}^{-1} = 2\varepsilon_t\{(\varepsilon')^2(1-\varepsilon_t)\}^{-1}.$$

The right hand side of (4.31) tends to zero as $t \to \infty$. Letting $\delta \to 0$, we conclude that $|\eta_{N_2(t)}-\eta_{n_1}|n_1^{-1/2} I_{A_t} \xrightarrow{P} 0$, completing the proof. //

Consider now the supercritical case, where $a > 0$. We first prove the

<u>THEOREM</u> 4.5: <u>Let</u> $a > 0$, $b-a^2=1$. <u>Then</u>

$$(4.32)\quad \lim_{n\to\infty} P((\eta_n-an)n^{-1/2} \le x) = \Phi(x) \ \underline{\text{as}}\ n \to \infty.$$

Proof: By the definition of η_n we can write

$$(4.33)\qquad (\eta_n - an)n^{-1/2} = [\max(S_n - S_n, S_n - S_{n-1}, \ldots, S_n - S_1, S_n) - an]n^{-1/2}$$

$$= \max(-S_n, -S_{n-1}, \ldots, -S_1, 0)n^{-1/2} + (S_n - an)n^{-1/2}.$$

The first term on the extreme right of (4.33) tends to zero in probability by Lemma 4.1. For we use the fact that $\max(0, -S_1, -S_2, \ldots, -S_n)$ corresponds to a subcritical process by replacing V_n's by $-V_n$'s. It follows from (R_1) and (R_2) of Section 1 that $\max(0, -S_1, \ldots, -S_n)$ tends in law to a proper random variable. Finally, by the central limit theorem (Feller [14], p. 187) the second term on the extreme right of (4.33) tends in law to a random variable with the desired distribution. This completes the proof. //

Immediately from Theorem 4.5 follows the

COROLLARY 4.1. *Let* $a > 0$, $b - a^2 = 1$. *Then*

$$(4.34)\qquad \eta_n n^{-1} \xrightarrow{P} a, \text{ as } n \to \infty.$$

Proof: Consider $\upsilon_n = (\eta_n - a_n)n^{-1/2}$, $g(n) = n^{1/2}$. The result follows upon application of Lemma 4.1 to $\upsilon_n / g(n)$. //

We now wish to prove

THEOREM 4.6. *Let* $a > 0$ *and* $b - a^2 = 1$. *Let* $N_2(t)$ *be as defined in Section 1. Then*

$$(4.35)\qquad \lim_{t\to\infty} P((Z(t) - aN_2(t))t^{-1/2} \le x) = \Phi(xc^{-1/2}).$$

Proof: We write, using the representation (4.5),

$$(4.36)\qquad (Z(t)-aN_2(t))t^{-1/2}=(\eta_{N_2(t)}-aN_2(t))t^{-1/2}+[\sum_{j=0}^{I_t} X_j]t^{-1/2}.$$

By the same argument set forth in the proof of Theorem 4.4 it suffices to show that the first term on the right hand side of (4.36) tends in law to a random variable $c^{1/2}\zeta$, say, where the distribution function of ζ is Φ. In view of Theorem 4.3 define $\{\varepsilon_t\}$, $n_1(t)$, $n_2(t)$ and A_t in the same way as in the proof of Theorem 4.4. Then we can write

$$(4.37)\quad (\eta_{N_2(t)}-aN_2(t))t^{-1/2}=(\eta_{n_1}-an_1)n_1^{-1/2}(n_1/t)^{1/2}+$$

$$+\{\eta_{N_2(t)}-\eta_{n_1}-a(N_2(t)-n_1)\}n_1^{-1/2}(n_1/t)^{1/2}.$$

By Theorem 4.5 and the definition of $n_1(t)$, the first term on the right of (4.37) tends in law to $c^{1/2}\zeta$. By (4.26) it suffices to show therefore that

$$\{\eta_{N_2(t)}-\eta_{n_1}-a(N_2(t)-n_1)\}n_1^{-1/2}\; I_{A_t} \xrightarrow{P} 0, \text{ as } t\to\infty.$$

Now

$$|\{\eta_{N_2(t)}-\eta_{n_1}-a(N_2(t)-n_1)\}I_{A_t}n_1^{-1/2}|$$

$$\leq \sup_{n_1<j\leq n_2} |\eta_j-\eta_{n_1}-a(j-n_1)|n_1^{-1/2}$$

$$\leq \sup_{n_1<j\leq n_2} \{\max(-S_j,-S_{j-1},\ldots,-S_{n_1},\ldots-S_1,0)-\max(-S_{n_1},\ldots,-S_1,0)+$$

(continued)

$$(4.38) \qquad +|S_j - S_{n_1} - a(j-n_1)|\} n_1^{-1/2}$$

$$\leq \sup_{n_1<j\leq n_2} \{\max(-S_j, -S_{j-1}, \ldots, -S_{n_1}, \ldots -S_1, 0) - \max(-S_{n_1}, \ldots, -S_1, 0)\} n_1^{-1/2} +$$

$$+ \sup_{n_1<j\leq n_2} |S_j - S_{n_1} - a(j-n_1)| n_1^{-1/2}$$

$$\leq \{\max(-S_{n_2}, -S_{n_2-1}, \ldots, -S_1, 0) - \max(-S_{n_1}, \ldots, -S_1, 0)\} n_1^{-1/2} +$$

$$+ \sup_{n_1<j\leq n_2} \left| \sum_{i=n_1+1}^{j} (V_i - a) \right| n_1^{-1/2}.$$

By the same reasoning set forth in proof of Theorem 4.5 both $\max(0, -S_1 - S_2, \ldots, -S_{n_2}) n_1^{-1/2}$ and $\max(0, -S_1, \ldots, -S_{n_1}) n_1^{-1/2}$ can be shown to tend to zero in probability as $t \to \infty$. Moreover, by the Kolmogorov inequality, for any arbitrary constant $\delta > 0$, we have,

$$(4.39) \qquad P\left(\sup_{n_1<j\leq n_2} \left| \sum_{i=n_1+1}^{j} (V_i - a) \right| \geq \delta n_1^{1/2} \right) \leq (n_2 - n_1)(\delta^2 n_1)^{-1}.$$

The right side of (4.39) tends to zero as $t \to \infty$. This completes the proof. //

As a consequence of Theorem 4.6 we have

COROLLARY 4.2. Let $a > 0$, $b - a^2 = 1$. Then, as $t \to \infty$,

$$Z(t) t^{-1} \xrightarrow{P} ac \; .$$

In the subcritical case we know from (2.35) that, as $t \to \infty$, $Z(t)$ tends in law to a finite random variable, in the special case where

$B(y) = 1-\exp(-\alpha y)$ and $D(y) = 1-\exp(-\beta y)$, α, $\beta > 0$; $y \geq 0$. In the more general setting of this chapter it is not easy to deal with the subcritical case. Any analysis of the asymptotic distribution of $Z(t)$ for large t must depend upon the joint distribution of $\eta_{N_2(t)}$ and $\sum_{j=0}^{I_t} X_j$. Moreover, these two random variables are by no means independent and it is not known how their joint limiting distribution behaves. This appears to be an open question.

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APPENDIX

APPENDIX

In this appendix we collect some useful lemmas and theorems referred to in the text. With the notation as in Section 2.1 except as noted, we state the following results.

LEMMA 1. (Pyke [~~47~~ 46], Lemma 4.1). If $m < \infty$, m the order of $\mathcal{X}$, then for all states $i \in \mathcal{X}$

$$P[N(t) < \infty, \text{ for all } t \geq 0 | J_0 = i] = 1.$$

LEMMA 2. (cf. Prabhu [36], p. 155). $N_k(t)$, $k = 1,2$, is a proper random variable with finite moments of all orders, that is, for ~~k = 1,2,~~ $F(0+)<1$, F the c.d.f. inducing the process, and $k=1,2$,

(i) $P(N_k(t) < \infty) = 1$

(ii) $E([N_k(t)]^\nu) < \infty$, $\nu = 1,2,\ldots$.

LEMMA 3. (Feller [14], Lemma 1, p. 148). If F is the distribution function of a nonnegative random variable, then for any $\nu > 0$

$$\int_0^\infty x^\nu \, dF(x) = \nu \int_0^\infty x^{\nu-1}[1-F(x)]dx$$

in the sense that if one side converges so does the other.

We have also the following Tauberian theorem.

THEOREM 1. (Widder [54], p. 187). Let $f(s) = \int_0^\infty \exp(-st)d\alpha(t)$ converge for $s > 0$, and let

$$\lim_{s \to 0+} f(s) = A .$$

Then

$$\lim_{t \to \infty} \alpha(t) = A ,$$

if and only if

$$\beta(t) = \int_0^t u d\alpha(u) = o(t).$$

NOTE: If $\alpha(t) = \int_0^t a(u)du$, then $a(t) = o(t^{-1})$, $(t \to \infty)$ implies $\beta(t) = o(t)$, $(t \to \infty)$.

Another useful Tauberian theorem is the following

THEOREM 2. (Widder [54], p. 192). If $\alpha(t)$ is nondecreasing and such that the integral $f(s) = \int_0^\infty \exp(-st)d\alpha(t)$ converges for $s > 0$, and if for some non-negative number γ and the constant A

$$f(s) \sim As^{-\gamma} , \quad (s \to 0+)$$

then

$$\alpha(t) \sim At^{\gamma}[\Gamma(\gamma+1)], \quad (t \to +\infty) .$$

From Theorem 2 we have

THEOREM 3. *Let* $f^*(s) = \int_0^\infty \exp(-st)\alpha(t)dt$. *Under the conditions of Theorem* 2 *above, if*

$$s^{\gamma+1} f^*(s) \sim A, \quad (s \to 0+)$$

then

$$\alpha(t) \sim At^{\gamma} [\Gamma(\gamma+1)]^{-1}, \ (t \to \infty).$$

Proof: $f(s) = sf^*(s)$ and the result follows upon direct application of Theorem 2 to $f(s)$. //

For the proof of existence and uniqueness of solutions to certain integral equations in Chapter II and III we use the contraction mapping theorem. We state it here without proof.

THEOREM 4. (Kolmogorov and Fomin [24], p. 73). *Every contraction mapping* A *defined on the metric space* $\mathcal{R}$ *has one and only one fixed point* (i.e. *the equation* $Ax = x$ *has one and only one solution*).

KEY RENEWAL THEOREM (Smith [50], p. 15, or Prabhu [36], p. 166). *Let* $Q(t)$ *be a nonnegative, nonincreasing function of* $t \geq 0$, *such that* $\int_0^\infty Q(t)dt < \infty$. *Then*

$$\int_0^t Q(t-\tau)dU(\tau) \to \mu^{-1} \int_0^\infty Q(t)dt, \text{ as } t \to \infty,$$

the limit being interpreted as zero if $\mu = \infty$. *Here* $U(t) = \sum_{k=0}^{\infty} F^{(k)}(t)$, $t \geq 0$, *for the distribution function* F (nonlattice) ~~(of a nonnegative random variable)~~ *with* $\mu \equiv \int_0^\infty [1-F(u)]du$.

<u>ROUCHE'S THEOREM</u>. (Titchmarsh [53], p. 116). *If $f(z)$ and $g(z)$ are analytic inside and on a closed contour C, and $|g(z)|<|f(z)|$ on C, then $f(z)$ and $f(z)+g(z)$ have the same number of zeros inside C.*