A Semi-Markov Storage Model

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1. INTRODUCTION. In recent years a great variety of models in storage theory have been studied from one point of view or another. Unfortunately, there have been relatively few of these where both inputs and releases are considered random. Such models have arisen in the past in economics and business administration. A review of these models has been given by Gani [13], [14]. Arrow, Karlin and Scarf [2] have studied such models along with certain optimization problems. In the present paper we consider a storage model where both inputs and releases, while being random variables, proceed according to an underlying semi-Markov process. Such a structure appears to be lacking in most of the storage models considered thus far in the literature. In the present case the authors were led to such a model by a somewhat different motivation which is discussed briefly below.

In [30] the authors presented a fresh approach to the classical theory of quantal response assays as developed by Finney ([11], [12]) and Bliss [4], among others. The classical theory is based essentially on the hypothesis of existence of a tolerance limit (threshold) for each subject (see Finney [12]). A new non-threshold type model was

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developed in [30] by the authors following the lines of Puri ([27], [28]). The new model is based upon a mechanism by which the test subject releases the drug according to a random process, following an initial fixed single dose, the only input. However, situations such as exposure to natural radiation or to specific chemicals encountered as part of certain occupational hazards, call for a model where not only releases are random but so also are the inputs. The present storage model is a modest attempt in this direction.

A class of continuous time storage models which in some respects resembles the one introduced here was considered by Gani and Pyke [15]. This class of models is such that, for every $t > 0$, the net input in the interval $(0,t)$ is representable as a difference of two independent nonnegative infinitely divisible processes. More specifically these authors considered a separable, centered infinitely divisible process $\{W_t;\ t \geq 0\}$, and defined constructively the level Z_t of the dam with net inputs described by the process $\{W_t\}$. Gani and Pyke studied the distributions of the total time in $(0,t)$ during which the dam is nonempty and empty.

Several other models of interest to the problem of quantal response assays and its extension were found in the vast literature of the theory of storage systems. One early contribution to this theory was a paper on finite dams by Moran [22]. The inputs flowing into these dams during consecutive intervals were assumed to form a sequence of independent identically distributed random variables. Our model retains this characteristic.

In a recent paper [26] Prabhu studied a storage model in which the input $X(t)$ to a dam with infinite capacity during the interval $(0,t)$ is a stochastic process with stationary and independent increments. The release from the dam is continuous and is at a unit rate except when the dam is empty. The net input, or input minus the amount demanded, is given by $Y(t)=X(t)-t$. Prabhu finds, under the conditions $E(X(t)) = \rho t$, $0 < \rho < \infty$, and $\mathrm{Var}\, X(t) < \infty$, that the net input process $Y(t)$ has an asymptotically normal distribution. Other limit distributions he derives are related to the normal in much the same way as the limit results of Section 6.

Brief mention could be made of the models of Hasofer ([16], [17]) and of Karlin and Fabens [18]. The former model, along with that of Gani and Pyke served as a starting point for the present model. The latter paper describes a discrete time inventory model in which only releases proceed according to an underlying semi-Markov structure. However in both of these models the level of the process takes values in the interval $(-\infty,h]$, and therefore their applicability to the quantal response assay problem is questionable. With this we turn to the semi-Markovian model.

2. THE MODEL.

In order to set forth a constructive definition of the process $\{Z(t)\}$ for our model, we first introduce some auxiliary processes. Consider a double sequence of random variables $\{(J_n,T_n),\ n = 0,1,\ldots\}$ taking values in the state space $\mathcal{X} \times [0,\infty)$ with $\mathcal{X} = \{1,2\}$. The process $\{(J_n,T_n)\}$ is defined on a complete probability space $(\Omega,\mathcal{G},P)$ such that

$T_0 = 0$ a.s., $P(J_0 = i) = a_i$, $i = 1,2$; $a_1 + a_2 = 1$, and

$$P(J_n = j,\ T_n \le x | T_0, J_0, T_1, J_1, \ldots, T_{n-1}, J_{n-1} = i) = P(J_n = j,\ T_n \le x | J_{n-1} = i)$$
$$= Q_{ij}(x) = p_{ij}\, H_i(x)$$

for $i,j = 1,2$, $x \in (-\infty,\infty)$ and $n = 1,2,\ldots$. The $Q_{ij}(\cdot)$ are nondecreasing and right continuous mass functions satisfying (i) $Q_{ij}(x) = 0$ for $x \le 0$, (ii) $\sum_j p_{ij} = 1$, where $p_{ij} = Q_{ij}(+\infty)$. Thus $H_i(t) = \sum_j Q_{ij}(t)$.

It is assumed that $H_i(0+) < 1$, $i = 1,2$. and that

$$E_{H_j} \equiv \int_0^\infty (1-H_j(u))du < \infty,\ j = 1,2.$$

It is assumed that $0 < p_{ij} < 1$, $i,j = 1,2$. The marginal sequence $\{J_n, n \ge 0\}$ is a two state Markov chain with $P(J_n=j | J_{n-1}=i) = p_{ij}$. Given the chain $\{J_n\}$ the random variables T_n are conditionally independent. The matrix $\underset{\sim}{P} = (p_{ij})$ is a stochastic matrix and for this reason we shall hereafter drop the subscripts on the p's and set $1-p = p_{11}$, $q = p_{21}$. We assume further that p and q are independent of time. Let $\tau_n = \sum_{i=0}^{n} T_i$, $n = 0,1,2,\ldots$. We define the integer-valued stochastic processes $\{N(t); t \ge 0\}$ and $\{V_t; t \ge 0\}$ as $N(t) = \sup(n: n \ge 0, \tau_n \le t)$, $N_j(t)$ = number of times $J_k = j$ for $0 < k < N(t) + 1$, and $V_t = J_{N(t)}$. The process $\{V_t\}$ is the ordinary semi-Markov process of Pyke [31].

Let state 1 denote the input state, and state 2 the release state. Now we introduce the independent identically distributed (I.I.D.) nonnegative random variables $X_1, X_2, \ldots$ and, independent of the X's, the I.I.D. nonnegative random variables $Y_1, Y_2, \ldots$. The random variable X_i represents the amount of an instantaneous input to the reservoir, the random variable Y_i an instantaneous release from the reservoir. Let B(x) and D(y) denote the common distribution functions of the X_i's and of the Y_i's respectively. We assume B and D are such that $EX_1^2 < \infty$, $EY_1^2 < \infty$.

We define the process $\{Z(t)\}$ constructively as follows. For $0 \leq t < \tau_1$, $Z(t) = Z(0)$ and for $k \geq 1$, and $\tau_k \leq t < \tau_{k+1}$

$$Z(t) = \begin{cases} Z(\tau_k -)+X_k, & \text{if } V_{\tau_k} = 1 \\ \max(0, Z(\tau_k -)-Y_k), & \text{if } V_{\tau_k} = 2. \end{cases}$$

Here it is assumed that the sample paths of the process (J,T) are right continuous. Thus the process Z(t) is almost surely continuous from the right. The process $\{V_t\}$ and hence $\{Z(t)\}$ is separable because of the constructive way each is defined. If at any time the random amount Y_k is greater than the amount actually available, only the available amount is released and the level remains at zero until the next input. The distribution of the waiting time T_{n+1} depends only on the value of J_n.

From Pyke [31] Lemma 3.1 we know that the two dimensional process (J,τ) is a Markov process, and the J process is a Markov chain. Then since $\mathcal{X}$ is finite, it follows (see Pyke [32], Lemma 4.1) that

$$P[N(t) < \infty, \quad \text{for all } t \geq 0] = 1.$$

REMARK In terms of the model described above net input in the interval (0,t) may be expressed as

$$W(t) = \sum_{j=1}^{N_1(t)} X_j - \sum_{j=1}^{N_2(t)} Y_j \ .$$

It can be shown that when $H_1 = H_2 \equiv H$ is a negative exponential distribution with parameter $\eta > 0$, and when $p_{11} = p_{22}$ the present model becomes a special case of that considered by Gani and Pyke [15]. However, the model of Gani and Pyke does not cover the present more general case in which the sequence of inputs and releases is structured through a semi-Markov process.

We introduce the following notation.

$$R_i(t,z,x) = P(Z(t) \leq x | Z(0)=z, J_0=i), \ i = 1,2,$$

for $t \geq 0$, $z \geq 0$, $x \in [0,\infty)$, and the corresponding Laplace transforms

$$\Phi_i(\theta,z,x) = \int_0^\infty \exp(-\theta t) R_i(t,z,x)dt, \ i = 1,2$$

and

$$H_i^*(\theta) = \int_0^\infty \exp(-\theta t)dH_i(t) \qquad , \ i = 1,2$$

where $Re(\theta) > 0$. We put $U(w) = 1$ if $w \geq 0$ and 0 otherwise.

Considering the first jump of the process, if there is one, during $(0,t]$, and whether it is an input or an output, the following backward Kolmogorov integral equations can be easily established for $R_i(t,z,x)$.

$$\text{(1a)} \ R_1(t,z,x) = U(x-z)(1-H_1(t)) + (1-p)\int_0^t \int_0^\infty R_1(t-\tau,z+y,x)dB(y)dH_1(\tau)$$

$$+ \ p\int_0^t [\int_0^z R_2(t-\tau,z-y,x)dD(y) + R_2(t-\tau,0,x)(1-D(z))]dH_1(\tau)$$

(1b) $$R_2(t,z,x)=U(x-z)(1-H_2(t))+q\int_0^t\int_0^\infty R_1(t-\tau,z+y,x)dB(y)dH_2(\tau) +$$

$$+(1-q)\int_0^t[\int_0^z R_2(t-\tau,z-y,x)dD(y) +$$

$$+R_2(t-\tau,0,x)(1-D(z))]dH_2(\tau)$$

(1c) $$R_2(t,0,x)=(1-H_2(t))+q\int_0^t\int_0^\infty R_1(t-\tau,y,x)dB(y)dH_2(\tau) +$$

$$+(1-q)\int_0^t R_2(t-\tau,0,x)dH_2(\tau)$$

for $t \geq 0$, $z > 0$, $x \in [0,\infty)$.

In this generality these integral equations are difficult to solve explicitly. As such we shall attempt, in Sections 3 and 4, to solve equations (1), respectively for the following two tractable special cases:

(A) B and D are both negative exponential, while H_1 and H_2 remain arbitrary.

(B) H_1, H_2 and B are negative exponential, while D remains arbitrary.

In Section 5, for these special cases, we also study the distribution of time to first emptiness. Later in Section 6, we shall consider the asymptotic behavior of the process Z(t), without any restrictions on the form of the distribution functions H_1, H_2, B and D.

3. THE CASE (A) WITH EXPONENTIAL INPUT AND RELEASE DISTRIBUTIONS.

In this section we treat the case where both H_1 and H_2 remain arbitrary and B is a negative exponential distribution with parameter $\alpha > 0$ and D is a negative exponential distribution with parameter $\beta > 0$.

The Laplace transforms of equations (1) with B and D in this form are given below for $z \geq 0$, $\mathrm{Re}(\theta) > 0$.

(2a) $$\Phi_1(\theta,z,x)=U(x-z)(1-H_1^*(\theta))\theta^{-1} + $$
$$+ \alpha(1-p)\exp(\alpha z)H_1^*(\theta)\int_z^\infty \exp(-\alpha v)\Phi_1(\theta,v,x)dv +$$
$$+ \exp(-\beta z)pH_1^*(\theta)[\Phi_2(\theta,0,x)+\beta\int_0^z \exp(\beta v)\Phi_2(\theta,v,x)dv]$$

(2b) $$\Phi_2(\theta,z,x)=U(x-z)(1-H_2^*(\theta))\theta^{-1} +$$
$$+\alpha q\exp(\alpha z)H_2^*(\theta)\int_z^\infty \exp(-\alpha v)\Phi_1(\theta,v,x)dv+$$
$$+\exp(-\beta z)(1-q)H_2^*(\theta)[\Phi_2(\theta,0,x)+\beta\int_0^z \exp(\beta v)\Phi_2(\theta,v,x)dv]$$

(2c) $$\Phi_2(\theta,0,x)=(1-H_2^*(\theta))\theta^{-1}+\alpha qH_2^*(\theta)\int_0^\infty \exp(-\alpha y)\Phi_1(\theta,y,x)dy +$$
$$+ (1-q)H_2^*(\theta)\Phi_2(\theta,0,x).$$

From now on we shall suppress the argument θ of both Laplace transforms and other functions of θ except where its presence is deemed necessary.

The existence and uniqueness of a bounded solution to equations (2) can be established by standard techniques. All the necessary details are given in [34] to which the interested reader is referred.

<u>THEOREM</u> 1. <u>The unique bounded solution of the equations</u> (2) <u>is given</u>, <u>for</u> $\mathrm{Re}(\theta) > 0$, <u>by</u>

(3a) $$\Phi_1(\theta,z,x)=\exp(r_2 z)(\alpha-r_2)[\alpha\beta\theta qH_2^*(r_2-r_1)]^{-1}.$$
$$\cdot\{\exp(-r_2 x)[(r_2-r_1)\beta qH_2^*-\{r_1+\beta(1-[1-q]H_2^*)\}\{r_2+\beta(1-H_2^*)\}]$$

$$+ \exp(-r_1x)\{r_2+\beta(1-H_2^*)\}[r_2+\beta(1-[1-q]H_2^*)]r_1/r_2\}, \qquad z > x,$$

(3b) $$\Phi_1(\theta,z,x)=\theta^{-1}-\exp(-r_1x)\{r_2+\beta(1-H_2^*)\}[\alpha\beta\theta qH_2^*(r_2-r_1)]^{-1}\cdot$$

$$\cdot\{\exp(r_1z)(\alpha-r_1)[r_1+\beta(1-[1-q]H_2^*)]$$

$$-\exp(r_2z)[r_2+\beta(1-[1-q]H_2^*)]r_1/r_2\}, \qquad 0 \le z \le x,$$

(3c) $$\Phi_2(\theta,z,x)=\exp(r_2z)(\beta+r_2)\{\beta\theta[r_2+\beta(1-[1-q]H_2^*)](r_2-r_1)\}^{-1}\cdot$$

$$\cdot\{\exp(-r_2x)[(r_2-r_1)\beta qH_2^*-\{r_1+\beta(1-[1-q]H_2^*)\}\{r_2+\beta(1-H_2^*)\}]$$

$$+\exp(-r_1x)\{r_2+\beta(1-H_2^*)\}[r_2+\beta(1-[1-q]H_2^*)]r_1/r_2\}, \qquad z > x,$$

(3d) $$\Phi_2(\theta,z,x)=\theta^{-1}-\exp(-r_1x)\{r_2+\beta(1-H_2^*)\}\{\beta\theta(r_2-r_1)\}^{-1}\cdot$$

$$\cdot\{\exp(r_1z)(\beta+r_1)-\exp(r_2z)(\beta+r_2)r_1/r_2\}, \qquad 0 \le z \le x,$$

where r_1 *and* r_2 *are given with plus and minus respectively, by*

(4) $$r_1(\theta),\ r_2(\theta) = -\tfrac{1}{2}A(\theta) \pm \tfrac{1}{2}[(A(\theta))^2-4B(\theta)]^{\frac{1}{2}},$$

with

(5)
$$A(\theta) = \beta-\alpha+\alpha(1-p)H_1^*-\beta(1-q)H_2^*$$

$$B(\theta) = \alpha\beta\{-(1-H_2^*)(1+H_1^*[q+p-1])+q(H_1^*-H_2^*)\}.$$

REMARK $B(\theta)$ may be expressed in another way as

(6) $$B(\theta) = -\alpha\beta\{(1-H_1^*)(1-H_2^*+qH_2^*)+pH_1^*(1-H_2^*)\},$$

from which it follows that $\mathrm{Re}(A^2-4B(\theta)) \geq 0$. It follows in turn from this that $\mathrm{Re}(r_1) > 0$ and $\mathrm{Re}(r_2) < 0$. It can be shown in addition, that $\mathrm{Re}(r_1) < \alpha$, and that $-\mathrm{Re}(r_2) < \beta$. Two limit properties of r_2 which will be used later on are, for $\beta q < \alpha p$,

$$(7) \quad \lim_{\theta\to 0} (r_2(\theta))^{-1}\{r_2(\theta)+\beta(1-H_2^*(\theta))\}=q(\alpha E_{H_1}+\beta E_{H_2})\alpha^{-1}[E_{H_2}p+qE_{H_1}]^{-1},$$

and

$$(8) \quad \lim_{\theta\to 0} r_2'(\theta)=\alpha\beta[E_{H_2}(p+q)+q(E_{H_1}-E_{H_2})][\beta q-\alpha p].$$

<u>Proof of THEOREM 1.</u>

It can be seen that $\Phi_1(\theta,z,x)$ and $\Phi_2(\theta,z,x)$ are differentiable with respect to z. Differentiating each of (2a) and (2b) twice with respect to z, and collecting terms we obtain the following two second order differential equations

$$(9) \quad \Phi_2''+\Phi_2'[\beta-\alpha-\beta(1-q)H_2^*]+\Phi_2[\alpha\beta(1q)H_2^*-\alpha\beta]$$
$$=-\alpha qH_2^*[\Phi_1'+\beta\Phi_1]-\alpha\beta\theta^{-1}(1-H_2^*)U(x-z),$$

$$(10) \quad \Phi_1''+\Phi_1'[\beta-\alpha+\alpha(1-p)H_1^*]+\Phi_1[\alpha\beta((1-p)H_1^*-1)]$$
$$=\beta pH_1^*[\Phi_2'-\alpha\Phi_2]-\alpha\beta\theta^{-1}(1-H_1^*)U(x-z).$$

Here Φ_i', Φ_i'' and Φ_i''' are corresponding first, second and third order partial derivatives with respect to z. On eliminating Φ_2 and its derivatives from (9) and (10) in a standard manner, one obtains

$$(11) \quad \Phi_1'''+\Phi_1''[2\beta-\alpha+\alpha(1-p)H_1^*-\beta(1-q)H_2^*] +$$

$$+\Phi_1'\beta[\alpha\{-(1-H_2^*)(1+H_1^*(q+p-1))+q(H_1^*-H_2^*)\} +$$

$$+\beta-\alpha+\alpha(1-p)H_1^*-\beta(1-q)H_2^*] -$$

$$-\alpha\beta^2\Phi_1[(1-H_2^*)(1+H_1^*(q+p-1))-q(H_1^*-H_2^*)]$$

$$=-\theta^{-1}\alpha\beta^2U(x-z)[(1-H_2^*)pH_1^*+91-H_1^*)(1-(1-q)H_2^*)].$$

Equation (11) is a third order differential equation with constant coefficients. For $x \geq z$ $\Phi_1 = \theta^{-1}$ is a particular solution of (11). The homogeneous equation associated with equation (11) has auxiliary equation whose roots are $-\beta$, r_1, r_2; the last two are defined by (4). Similarly eliminating Φ_1 from (9) and (10) we obtain the equation

(12) $$\Phi_2''' + \Phi_2''[\beta-2\alpha+\alpha(1-p)H_1^*-\beta(1-q)H_2^*] +$$

$$+\alpha\Phi_2'[\beta(-(1-H_2^*)(1+H_1^*(q+p-1)+q(H_1^*-H_2^*)) -$$

$$-\beta+\alpha-\alpha(1-p)H_1^*+\beta(1-q)H_2^*] -$$

$$-\alpha^2\beta[-(1-H_2^*)(1+H_1^*(q+p-1)+q(H_1^*-H_2^*)]\Phi_2$$

$$=\alpha^2\beta[(1-H_1^*)qH_2-(1-H_2^*)((1-p)H_1^*-1)]U(x-z)\theta^{-1}.$$

Taking into consideration the signs of $Re(r_1)$ and $Re(r_2)$ and the range of z in each case we can express the general solution to equations (2) as

(13a) $$\Phi_1(\theta,z,x)=C_0\exp(-\beta z)+C_1\exp(r_2 z) \quad , z > x$$

(13b) $$\Phi_1(\theta,z,x)=C_2\exp(-\beta z)+C_3\exp(r_1 z)+C_8\exp(r_2 z)+\theta^{-1} \quad , 0 \leq z \leq x,$$

and

(13c) $\Phi_2(\theta,z,x)=C_4\exp(r_2z)$, $z > x$

(13d) $\Phi_2(\theta,z,x)=C_5\exp(\alpha z)+C_6\exp(r_1z)+C_7\exp(r_2z)+\theta^{-1}$, $0 < z \leq x$,

while $\Phi_2(\theta,0,x)$ is described by equation (2c). In order to determine the desired constants C_i, $i = 0,1,\ldots, 8$ we proceed as follows. From (2a), when $z > x$,

$$(14)\quad \int_0^z \Phi_2(\theta,v,x)\exp(\beta v)dv=\exp(\beta z)(\beta pH_1^*)^{-1}[\Phi_1(\theta,z,x)-\Phi_2(\theta,0,x)\beta^{-1} - \\ -\alpha(1-p)H_1^*\exp(\alpha z)\int_z^\infty \exp(-\alpha v)\Phi_1(\theta,v,x)dv]$$

and when $0 \leq z \leq x$

$$(15)\quad \int_0^z \exp(\beta v)\Phi_2(\theta,v,x)dv=\exp(\beta z)(\beta pH_1^*)^{-1}[\Phi_1(\theta,z,x) - \\ -\Phi_2(\theta,0,x)\beta^{-1}-(1-H_1^*)\theta^{-1} - \\ -\alpha(1-p)H_1^*\exp(\alpha z)\int_z^\infty \exp(-\alpha v)\Phi_1(\theta,v,x)dv].$$

Substituting the appropriate forms from (13) for Φ_1 in the right hand sides of both (14) and (15); differentiating each of the resulting equations with respect to z; and collecting terms, we find for $z > x$,

$$(16)\quad \Phi_2(\theta,z,x)=C_1(\beta+r_2)(\beta pH_1^*)^{-1}\exp(r_2z)[1-\alpha(1-p)H_1^*/(\alpha-r_2)],$$

and for $0 < z \leq x$,

$$(17)\quad \Phi_2(\theta,z,x)=\exp(\alpha z)(\alpha+\beta)\alpha(1-p)(\beta p)^{-1}[\exp\{-(\alpha+\beta)x\}(C_2-C_0)(\alpha+\beta)^{-1} - \\ -\exp\{(r_1-\alpha)x\}C_3(r_1-\alpha)^{-1}+\exp\{(r_2-\alpha)x\}(C_1-C_8)(r_2-\alpha)^{-1} +$$

$$+\exp(-\alpha x)(\alpha\theta)^{-1}] +$$

$$+\exp(r_1 z)C_3(\beta+r_1)(\beta p H_1^*)^{-1}[1-\alpha(1-p)H_1^*(\alpha-r_1)^{-1}] +$$

$$+\exp(r_2 z)C_8(\beta+r_2)(\beta p H_1^*)^{-1}[1-\alpha(1-p)H_1^*(\alpha-r_2)^{-1}] + \theta^{-1} .$$

Comparing the coefficients of $\exp(r_1 z)$, $\exp(r_2 z)$, $\exp(\alpha z)$, and the constant term in (16) with those of (13c) and in (17) with those of (13d), it follows that

$$(18)\quad \begin{cases} C_4=C_1(\beta+r_2)(\beta p H_1^*)^{-1}[1-\alpha(1-p)H_1^*(\alpha-r_2)^{-1}] \\ C_5=(\alpha+\beta)\alpha(1-p)(\beta p)^{-1}[\exp\{-(\alpha+\beta)x\}(C_2-C_0)(\alpha+\beta)^{-1} \\ \qquad +\exp(-\alpha x)(\alpha\theta)^{-1}-C_3\exp\{(r_1-\alpha)x\}(r_1-\alpha)^{-1} + \\ \qquad +(C_1-C_8)\exp\{(r_2-\alpha)x\}(r_2-\alpha^{-1}] \\ C_6=C_3(\beta+r_1)(\beta p H_1^*)^{-1}[1-\alpha(1-p)H_1^*(\alpha-r_1)^{-1}] \\ C_7=C_8(\beta+r_2)(\beta p H_1^*)^{-1}[1-\alpha(1-p)H_1^*(\alpha-r_2)^{-1}]. \end{cases}$$

The expression for $\Phi_2(\theta,z,x)$ follows directly once we have found the solution for $\Phi_1(\theta,z,x)$. That is, we have only to determine the constants C_0, C_1, C_2, C_3 and C_8 now. To do this we substitute the general form of Φ_1 and Φ_2 as given in (13) into equations (2). Solving the resulting system with the help of equations (18) we obtain equations (3). The algebra, although tedious, is straightforward and hence is omitted. //

We proceed now to obtain transforms of the (first two) moments of the process $Z(t)$. We shall need the following lemma, which follows itself from a well known result in renewal theory (see Prabhu [24], p. 155).

LEMMA 1. In order that $E[Z(t)]$ and $E[Z^2(t)]$ exist and be finite it is sufficient that both $H_1(0+) < 1$ and $H_2(0+) < 1$, and both $E(X) < \infty$ and $E(X^2) < \infty$.

Applying Lemma 1, p. 148 of Feller [10], to the process $\{Z(t)\}$ for $\nu = 1$ or 2 we find

$$E[Z^\nu(t)] = \nu\int_0^\infty x^{\nu-1}[1-P(Z(t) \leq x)]\,dx. \tag{19}$$

By Lemma 1 the right hand side integral is finite. Now, taking Laplace transforms, we get

$$\int_0^\infty \exp(-\theta t)E[Z^\nu(t)]dt=\nu\int_0^\infty \exp(-\theta t)\int_0^\infty x^{\nu-1}P(Z(t) > x)dxdt.$$

The integrand on the right hand side above is positive. By Fubini's theorem then

$$\int_0^\infty \exp(-\theta t)E[Z^\nu(t)]dt=\nu\int_0^\infty x^{\nu-1}\int_0^\infty \exp(-\theta t)[1-P(Z(t) \leq x)]dtdx$$

$$=\nu\int_0^\infty x^{\nu-1}[\theta^{-1}-\int_0^\infty \exp(-\theta t)P(Z(t) \leq x)dt]dx .$$

By the above reasoning, the desired Laplace transforms

$$g_i(\theta)=\int_0^\infty \exp(-\theta t)E[Z(t)\,|\,Z(0)=z,\ J_0=i]dt,\ i = 1,2$$

$$h_i(\theta)=\int_0^\infty \exp(-\theta t)E[Z^2(t)\,|\,Z(0)=z,\ J_0=i]dt, i = 1,2$$

defined for $Re(\theta) > 0$, can be obtained to yield

(20) $g_i(\theta)=\theta^{-1}[r_2(\theta)+\beta\{1-H_2^*(\theta)\}][\alpha\beta q H_2^*(\theta)r_1(\theta)\{r_2(\theta)-r_1(\theta)\}]^{-1}\cdot$

$$\cdot\{\exp\{r_1(\theta)z\}(\alpha-r_1(\theta))[r_1(\theta)+\beta\{1-(1-q)H_2^*(\theta)\}]$$

$$-\exp\{r_2(\theta)z\}(\alpha-r_2(\theta))[r_2(\theta)+\beta\{1-(1-q)H_2^*(\theta)\}]r_1(\theta)(r_2(\theta))^{-1}\}$$

and

(21) $h_i(\theta)=\theta^{-1}[r_2(\theta)+\beta\{1-H_2^*(\theta)\}][\alpha\beta q H_2^*(\theta)(r_1(\theta))^2\{r_2(\theta)-r_1(\theta)\}]^{-1}\cdot$

$$\cdot\{\exp\{r_1(\theta)z\}(\alpha-r_1(\theta))[r_1(\theta)+\beta\{1-(1-q)H_2^*(\theta)\}]$$

$$-\exp\{r_2(\theta)z\}(\alpha-r_2(\theta))[r_2(\theta)+\beta\{1-(1-q)H_2^*(\theta)\}]r_1(\theta)(r_2(\theta))^{-1}\}$$

Unfortunately, the inversion of transforms (3), (18) and (19) appears quite cumbersome. This is due, primarily, to the rather complicated dependence of r_1 and r_2 on θ, the parameter of the transform. Even in the simplest special cases the inversion is algebraically involved. Instead, we proceed now to examine the limiting behavior of the process $\{Z(t)\}$ and the asymptotic behavior of its moments. The following theorem can be established by application of a standard Tauberian theorem (Widder [38], p.192) to (3).

THEOREM 2. $\lim_{t\to\infty} P(Z(t)\le x|Z(0)=z,\ J_0=i) = \Psi(x)$, *with the limiting distribution* Ψ *given for* $x\ge 0$, *by*

$$(22)\quad \Psi(x)=\begin{cases}0, & q(p\alpha)^{-1}-\beta^{-1}>0\\ 1-\exp\{[\beta q-\alpha p]x\}(\alpha E_{H_1}+\beta E_{H_2})q\alpha^{-1}[E_{H_2}p+qE_{H_1}]^{-1}, & q(p\alpha)^{-1}-\beta^{-1}\le 0\end{cases}$$

independent of the value of J_0.

In the case, $q(p\alpha)^{-1} < \beta^{-1}$, the limiting distribution function is a negative exponential with a positive probability mass at zero. The interpretation of the criterion for the existence of a proper limiting distribution for $Z(t)$ is straightforward. When average inputs per unit time are less than average releases per unit time, $Z(t)$ has a nondegenerate limiting distribution. When average inputs are equal to or greater than average releases $Z(t)$ tends to infinity. It seems reasonable to distinguish formally between three cases. Borrowing terminology from branching processes, we shall talk of subcritical, criticial and supercritical cases according as $q(p\alpha)^{-1}$ is less than, equal to, or greater than β^{-1}. Let $a = q(p\alpha)^{-1} - \beta^{-1}$.

We shall study now the asymptotic behavior of the first two moments in the critical and supercritical cases.

THEOREM 3. *If* $a > 0$, *then*

(23a) $$E[Z(t)|Z(0)=z,J_0=i] \sim k_1 t \qquad (t \to \infty)$$

(23b) $$E[Z^2(t)|Z(0)=z,J_0=i] \sim 2^{-1}k_1^2t^2 \qquad (t \to \infty)$$

and if $a=0$, *then, for* $i = 1,2$,

(24a) $$E[Z(t)|Z(0)=z,J_0=i] \sim 2(\pi)^{-\frac{1}{2}}k_2^{\frac{1}{2}}t^{\frac{1}{2}} \qquad (t \to \infty)$$

(24b) $$E[Z^2(t)|Z(0)=z,J_0=i] \sim k_2 t \qquad (t \to \infty),$$

where

(25) $$k_1=(\beta q-\alpha p)(\alpha\beta)^{-1}\{E_{H_2}p+qE_{H_1}\}^{-1}$$

(26) $$k_2=\alpha\beta\{E_{H_2}p+qE_{H_1}\}.$$

<u>Proof</u>: Consider first the supercritical case, $a > 0$. From (20) it follows that

(27) $$\lim_{\theta\to 0} \theta^2 g_i(\theta)=k_1.$$

For the second moment, from (21) we have

(28) $$\lim_{\theta\to 0} \theta^3 h_i(\theta)=k_1^2.$$

The result for the supercritical case follows upon application of a Tauberian theorem to both (27) and (28). In the critical case, since

(29) $$\lim_{\theta\to 0} \theta^{3/2} g_i(\theta)=k_2^{\frac{1}{2}}$$

and

(30) $$\lim_{\theta\to 0} \theta^2 h_i(\theta)=k_2,$$

the same theorem yields the desired result for $i = 1,2$. //

In the special case where $H_1=H_2 \equiv H$ and $1-p=q$ the underlying structure is that of a renewal process. Moreover, at each renewal point with the same probability p an input is observed and with probability (1-p) a release is observed. Results are known for discrete versions of this problem, as reported in Takacs [37], for example, and for the continuous time case as examined by Erdos and Kac [8]. For this special case the solution to the integral equation (2) takes the relatively simple form (cf. (3))

$$\Phi(\theta,z,x)=\exp(r_2 z)r_1(r_2+\beta)[\alpha\beta\theta(r_1-r_2)]^{-1}.$$

$$\cdot\{\exp(-r_2 x)(\alpha-r_2)-\exp(-r_1 x)(\alpha-r_1)\}\quad ,\qquad z > x$$

$$\phi(\theta,z,x)=\theta^{-1}-\exp(-r_1x)(\alpha-r_1)[\alpha\beta\theta(r_2-r_1)]^{-1}\cdot$$

$$\cdot\{\exp(r_1z)r_2(r_1+\beta)-\exp(r_2z)r_1(r_2+\beta)\}, \qquad 0 \le z \le x,$$

where $Re(\theta) > 0$ and

$$r_1(\theta)=-2^{-1}c+2^{-1}[c^2-4d]^{\frac{1}{2}}, \quad r_2(\theta)=-2^{-1}c-2^{-1}[c^2-4d]^{\frac{1}{2}},$$

and

$$c=\beta(1-H^*(\theta)p)+\alpha\{(1-p)H^*(\theta)-1\}, \quad d=-\alpha\beta(1-H^*(\theta)).$$

4. THE CASE (B) WITH GENERAL RELEASE AND EXPONENTIAL INPUT DISTRIBUTIONS.

It is natural to ask what generalization can be made of the model treated in Section 3. Specifically, we are interested in freeing that model from the restriction of exponential distributions for both inputs and releases. In this section we examine an initial attempt to consider more general distributions $B(x)$ and $D(y)$. As mentioned in Section 2 with both B and D general a solution, if it exists, is very difficult to generate, by use of transforms or other means. The price of generalizing even one of those distributions is the sacrifice of the semi-Markovian structure of the underlying process. Therefore in this section we abandon the underlying semi-Markovian structure and set $H_1=H_2 \equiv H$, a negative exponential distribution function with parameter $\lambda+\mu$, where $1-p=\lambda/(\lambda+\mu)$ and $q=\mu/(\lambda+\mu)$, and $\lambda,\mu > 0$. Thus the underlying process is Markovian, a fact which allows us to consider not only the backward Kolmogorov integral equations but also the forward integral equations. In the present case equations (1) reduce to a single equation. A solution of the backward equation, the minimal solution, can be constructed using successive approximations. This minimal solution satisfies also the forward equation and is minimal for the latter. Since $N(t) < \infty$, almost surely, here the forward version of equation (1) is valid and can be established by considering the nature of the last jump of the process $Z(t)$ before time t. Considering the last jump of the process $(0,t]$, if there is one, and whether it is an input or a release the following forward Kolmogorov integral equation can be established for

$$R(t,z,x)=P(Z(t)\leq x|Z(0)=z) \text{ for the case } z\geq 0.$$

$$(31)\qquad R(t,z,x)=\exp\{-(\lambda+\mu)t\}U(x-z) + \lambda\int_0^t \exp\{-(\lambda+\mu)(t-\tau)\}d\tau\int_0^x R(\tau,z,x-y)dB(y) + \mu\int_0^t \exp\{-(\lambda+\mu)(t-\tau)\}d\tau\int_0^\infty R(\tau,z,x+y)dD(y),$$

where now z is fixed. The Laplace transform of (31) for $Re(\theta) > 0$ is

$$(32)\qquad \Phi(\theta,z,x)(\lambda+\mu+\theta)=U(x-z)+\lambda\int_0^x \Phi(\theta,z,x-y)dB(y)+\mu\int_0^\infty \Phi(\theta,z,x+y)dD(y).$$

The uniqueness of the solution of equation (32) is proved by use of the principle of contraction mappings, in exactly the same manner as employed by Hasofer in [16]. A tractable solution to (32), presented in Theorems 4 and 5 below, is possible for the case in which B has negative exponential distribution with parameter β while the distribution D remains arbitrary subject to the conditions $D(0)=0$ and that its first moment $E_D=\int_0^\infty y dD(y) < \infty$.

<u>THEOREM</u> 4. <u>For</u> z=0, <u>equation</u> (32) <u>has the unique bounded solution</u>

(33) $\Phi(\theta,0,x)=\theta^{-1}-(\alpha-\gamma(\theta))(\alpha\theta)^{-1}\exp(-\gamma(\theta)x),\ x \geq 0,\ \mathrm{Re}\theta > 0$

<u>with</u>

(34) $\int_0^\infty \exp(-\theta t)P(Z(t)=0|Z(0)=0)dt=\gamma(\theta)(\alpha\theta)^{-1},$

<u>where</u> $\gamma(\theta)$ <u>is the unique root of</u>

(35) $\lambda+\mu+\theta-\alpha\lambda(\alpha-r)^{-1}-\mu\int_0^\infty \exp(-ry)dD(y)=0,$

<u>with</u> $0 < \mathrm{Re}(r(\theta)) < \alpha$.

We need first the following lemma, which follows by an application of Rouché's theorem.

<u>LEMMA</u> 2. <u>For</u> α, λ, μ <u>all positive</u>, $\mathrm{Re}(\theta) > 0$, <u>equation</u> (35) <u>has a unique root</u> $r(\theta)=\gamma(\theta)$ <u>in</u> $0 < \mathrm{Re}(r(\theta)) < \alpha-\delta$, <u>where</u> $\delta > 0$ <u>is small</u>. <u>Moreover</u> $0 < |\gamma(\theta)| \leq \alpha|(\mu+\theta)/(\lambda+\mu+\theta)|$.

<u>Proof of THEOREM</u> 4. We show that the solution of (32) is of the form

(36) $\Phi(\theta,0,x)=\theta^{-1}+C(\theta)\exp(-rx).$

We know $\Phi(\theta,0,x) \to \theta^{-1}$, as $x \to \infty$. Now this is possible only if $\mathrm{Re}(r(\theta)) > 0$. Furthermore, from (34) we require that $|\gamma(\theta)| < \alpha$. So the only values of $r(\theta)$ in which we are interested are those for which $0 < \mathrm{Re}(r(\theta)) < \alpha$. Substitution of (36) into (32) yields an identity in x. Comparing the coefficients of $\exp(rx)$ and of $\exp(\alpha x)$ on both sides of this identity, we obtain (35) and the relation

(37) $\theta^{-1}+C(\theta)(\alpha-r)^{-1} = 0.$

By Lemma 2 equation (35) has a unique root $r=\gamma(\theta)$ in $0 < |r(\theta)| < \alpha\left|\frac{\mu+\theta}{\lambda+\mu+\theta}\right| < \alpha$, $\mathrm{Re}(\theta) > 0$. Once $\gamma(\theta)$ has been determined (37) then yields the term $C(\theta)$, which in turn yields (33). The uniqueness of the solution of (34) guarantees that (33) is the only such solution. Finally, (34) follows since

$$\int_0^\infty \exp(-\theta t)P(Z(t)=0|Z(0)=0)dt=\lim_{x\to 0+}\Phi(\theta,0,x). \qquad //$$

Let H be an arbitrary function, defined on the nonnegative half of the real line, which is integrable in every finite subinterval of that half line and which can be expressed as the difference of two monotone nondecreasing functions.

Let also

$$(38) \qquad K(s) = \sum_{k=0}^{\infty} H^{(k)}(s),$$

where

$$H^{(0)}(s)\equiv 1,\ H^{(k)}(s) = \int_0^s H^{(k-1)}(s-u)dH(u),\ 0 \le s \le z,\ k = 1,2,\ldots$$

<u>THEOREM</u> 5. <u>Suppose</u> $D(s)/s \le A < \infty$ <u>for</u> $0 < s \le \epsilon$ <u>for some</u> $\epsilon > 0$, $A > 0$. <u>Then for</u> $z > 0$, <u>equation</u> (32) <u>has the unique solution</u>

$$(39) \qquad \Phi(\theta,z,x)=\theta^{-1}-\exp(-\gamma(\theta)x)(\alpha-\gamma(\theta)) \cdot[(\alpha\theta)^{-1}+(\lambda+\mu+\theta)^{-1}\int_0^z \exp(\alpha u)K(z-u)du],\ x\ge z$$

<u>and for</u> $0 \le x < z$,

$$(40) \qquad \Phi(\theta,z,x)=\theta^{-1}-(\lambda+\mu+\theta)^{-1} K(z-x)-(\alpha-\gamma(\theta))\exp\{-\gamma(\theta)x\}\cdot[(\alpha\theta)^{-1}+(\lambda+\mu+\theta)^{-1}\int_0^z \exp(\alpha u)K(z-u)du],$$

where K is given in (38) and

(41) $$H(s)=[\mu D(s)+\lambda(1-\exp(\alpha s))](\lambda+\mu+\theta)^{-1}, \quad 0 \leq s \leq z.$$

Proof: Let

$$\Phi(\theta,z,x) = \Phi_1(\theta,z,x) \quad \text{for } 0 \leq x < z$$

$$\Phi(\theta,z,x) = \Phi_2(\theta,z,x) \quad \text{for } z \leq x.$$

Equation (32) may then be broken into the two parts: for $0 \leq x \leq z$,

(42) $$\Phi_1(\theta,z,x)(\lambda+\mu+\theta)=\exp(-\alpha x)\lambda\alpha\int_0^x \Phi_1(\theta,z,v)\exp(\alpha v)dv + \mu[\int_0^{z-x}\Phi_1(\theta,z,x+y)dD(y)+\int_{z-x}^{\infty}\Phi_2(\theta,z,x+y)dD(y)],$$

and for $x \geq z$,

(43) $$\Phi_2(\theta,z,x)(\lambda+\mu+\theta)=1+\mu\int_0^{\infty}\Phi_2(\theta,z,x+y)dD(y)+ +\exp(-\alpha x)\alpha\lambda[\int_0^z \exp(\alpha v)\Phi_1(\theta,z,v)dv + \int_z^x \Phi_2(\theta,z,v)\exp(\alpha v)dv].$$

Now when z=0 we saw that Φ_2 had the form

(44) $$\Phi_2(\theta,z,x)=\theta^{-1}+W\exp(-rx), \quad x \geq z \quad Re(r) > 0.$$

We shall construct a solution of (32) this time by putting Φ_2 as in (44) and setting

(45) $$\Phi_1(\theta,z,x)=\theta^{-1}+W\exp(-rx)+g(\theta,z,x), \quad 0 \leq x < z,$$

where $g(\theta,z,x)$ is a function to be determined. Substitution of (44) into (43) produces an identity in x. Comparing the coefficients of exp(-rx) and exp(-αx) on both sides of this identity we obtain the relations

$$(46)\qquad \lambda+\mu+\theta-\alpha\lambda(\alpha-r)^{-1}-\mu\int_0^\infty \exp(-ry)\,dD(y)=0,\ \ r\neq\alpha,$$

and

$$(47)\qquad \exp(\alpha z)(\alpha\theta)^{-1}-J(\theta,z)+W\ \exp\{(\alpha-r)z\}(\alpha-r)^{-1}=0,\ \ \alpha\neq r,$$

where

$$(48)\qquad J(\theta,z)=\int_0^z \exp(\alpha v)\Phi_1(\theta,z,v)\,dv.$$

By Lemma 2 we can determine a unique root, $r=\gamma(\theta)$, from (46) which satisfies $0 < \mathrm{Re}(\gamma(\theta)) < \alpha$ with $\mathrm{Re}(\theta) > 0$. Straightforward substitution of (44) and (45) into (42) leads to the following integral equation.

$$(49)\qquad g(\theta,z,z-s)(\lambda+\mu+\theta)=-1+\mu\int_0^s g(\theta,z,z-s+y)\,dD(y)-$$
$$-\alpha\lambda\int_0^s g(\theta,z,\xi+z-s)\exp(\alpha\xi)\,d\xi.$$

Here we have made the change of variable $s=z-x$, $0 \leq s \leq z$. Setting $\Gamma(\theta,s)=g(\theta,z,z-s)$, we see that equation (49) becomes

$$(50)\qquad \Gamma(\theta,s)=-(\lambda+\mu+\theta)^{-1}+\int_0^s \Gamma(\theta,s-y)\,dH(y),$$

where H is defined in (41). Equation (50) is a Volterra type equation. It is shown in the appendix that in this case this equation has the solution

$$\Gamma(\theta,s)=-(\lambda+\mu+\theta)^{-1}K(s).$$

Converting from Γ back to g we find $\Phi_1(\theta,z,x)$ in terms of W by means of (45). Then using (47) and (48) we obtain

$$(51)\quad J(\theta,z)=\exp\{(\alpha-\gamma)z\}[(\exp(\alpha z)-1)(\alpha\theta)^{-1}-(\lambda+\mu+\theta)^{-1}\int_0^z \exp(\alpha v)U(z-v)dv].$$

In turn, therefore, from (47)

$$(52)\quad W=\exp\{-(\alpha-\gamma)z\}(\alpha-\gamma)[-\exp(\alpha z)(\alpha\theta)^{-1}+(\exp(\alpha z)-1)(\alpha\theta)^{-1} - -(\lambda+\mu+\theta)^{-1}\int_0^z \exp(\alpha v)U(z-v)dv].$$

From (44) and (45), equations (39) and (40) now follow. //

We shall now investigate the limit behavior of Z(t) as $t \to \infty$. To this end we must investigate the behavior of $\gamma(\theta)$ as $\theta \to 0$. By an application of Rouche's Theorem we saw that equation (35) has, for fixed θ, a unique root $r(\theta)=\gamma(\theta)$ such that $0 < |\gamma(\theta)| \le \alpha|(\mu+\theta)/(\lambda+\mu+\theta)|$. We now state the following lemma which can be proved by exactly the same technique as employed by Benes in [3].

<u>LEMMA</u> 3. <u>For</u> $\theta > 0$,

$$\lim_{\theta\downarrow 0}\gamma(\theta) = \begin{cases} 0 & \text{if } \alpha\mu\lambda^{-1}E_D \le 1 \\ \alpha[1-\lambda\{\lambda+\mu(1-\zeta)\}^{-1}] & \text{if } \alpha\mu\lambda^{-1}E_D > 1, \end{cases}$$

<u>where</u> ζ <u>is the least nonnegative root of the equation</u> $\xi=\Phi(\xi)$, $0 \le \xi < 1$,

$$\Phi(\xi)=D^*(\alpha[1-\lambda\{\lambda+\mu(1-\xi)\}^{-1}]),\quad D^*(r)=\int_0^\infty \exp(-ry)dD(y),$$

<u>and</u> $E_D=\int_0^\infty y\,dD(y)$.

REMARK. Here the root ζ exists, and the reader may find its properties discussed on p. 274 of Feller [9].

Lemma 3 and a standard Tauberian argument (Widder [38], p.192) can now be used to obtain the following theorem.

THEOREM 6. *Under the conditions of Theorem 5*

$$\lim_{t\to\infty} P(Z(t)\le x \mid Z(0)=z) = \Psi(x),\ x \ge 0,$$

independent of the value of z, *where the distribution* Ψ *is given for* $x \ge 0$ *by*

$$\Psi(x) = \begin{cases} 1-(\alpha-\gamma^*)\alpha^{-1}\exp(-\gamma^* x), & E_D > \lambda(\alpha\mu)^{-1} \\ 0, & E_D \le \lambda(\alpha\mu)^{-1}, \end{cases}$$

where $\gamma^*=\alpha[1-\lambda\{\lambda+\mu(1-\zeta)\}^{-1}]$ *and* ζ *is as in Lemma 3.*

Once again we see that if average releases per unit time exceed average inputs per unit time, then Z(t) has a nondegenerate limiting distribution--an exponential with positive mass at zero.

From Theorems 4 and 5 we can derive the moments of the process Z(t). Since Z(t) is a nonnegative random variable Laplace transforms of the first two moments of Z(t) are readily obtained. The complicated expressions for these transforms are given in full detail in [34].

5. DISTRIBUTION OF TIME TO FIRST EMPTINESS.

It is reasonable to ask, "how long does it take before the dam becomes empty for the first time, given that Z(0)=z is positive?"

In this section we shall attempt to answer this question for the two special cases (A) and (B) of the model considered in Sections 3 and 4.

Let $Z(0)=z$ be strictly positive. Let T_E be the time until the dam first becomes empty. That is, $T_E=\inf\{t|Z(t)=0,\ t>0\}$. In addition, throughout this section, we shall use the notation

$$\tilde{R}_i(t,z,x)=P(Z(t)\leq x,\ T_E>t|Z(0)=z,J_0=i),\ i=1,2,$$

$$\tilde{R}\ (t,z,x)=P(Z(t)\leq x,\ T_E>t|Z(0)=z),$$

$$1-F\ (t)=P(T_E>t|Z(0)=z),\ 1-F_i(t)=P(T_E>t|Z(0)=z,J_0=i),\ i=1,2,$$

for $t\geq 0$, $x\geq 0$. Let $\tilde{\Phi}_i(\theta,z,x)$, $F_i^*(\theta)$, $i=1,2$, $\tilde{\Phi}(\theta,z,x)$, and $F^*(\theta)$ be, for $\mathrm{Re}(\theta)>0$, the Laplace transforms of $\tilde{R}_i(t,z,x)$, $F_i(t)$, $i=1,2$, $R(t,z,x)$, and $F(t)$ respectively.

5.1. CASE (A).

By considering the first jump, if there is one, of the process $Z(t)$ in the interval $(0,t]$, along with the size and nature of that jump, we can establish in a straightforward manner the following backward Kolmogorov integral equations for $\tilde{R}_i(t,z,x)$ for $z>0$.

$$(53)\quad \tilde{R}_1(t,z,x)=U(x-z)(1-H_1(t))+\alpha(1-p)\exp(\alpha z)\int_0^t\int_z^\infty \exp(-\alpha v)\tilde{R}_1(t-\tau,v,x)dvdH_1(\tau)$$
$$+p\beta\exp(-\beta z)\int_0^t\int_0^z \exp(\beta v)\tilde{R}_2(t-\tau,v,x)dvdH_1(\tau),$$

$$(54)\quad \tilde{R}_2(t,z,x)=U(x-z)(1-H_2(t))+\alpha q\exp(\alpha z)\int_0^t\int_z^\infty \exp(-\alpha v)\tilde{R}_1(t-\tau,v,x)dvdH_2(\tau)$$
$$+(1-q)\beta\exp(-\beta z)\int_0^t\int_0^z \exp(\beta v)\tilde{R}_2(t-\tau,v,x)dvdH_2(\tau).$$

In terms of their Laplace transforms, (53) and (54) take the following form for $\mathrm{Re}(\theta) > 0$.

(55) $$\tilde{\Phi}_1(\theta,z,x)=U(x-z)(1-H_1^*)\theta^{-1}+\alpha(1-p)\exp(\alpha z)H_1^*\int_z^\infty \exp(-\alpha v)\tilde{\Phi}_1(\theta,v,x)dv + +p\beta\exp(-\beta z)H_1^*\int_0^z \exp(\beta v)\tilde{\Phi}_2(\theta,v,x)dv$$

(56) $$\tilde{\Phi}_2(\theta,z,x)=U(x-z)(1-H_2^*)\theta^{-1}+\alpha q\exp(\alpha z)H_2^*\int_z^\infty \exp(-\alpha v)\tilde{\Phi}_1(\theta,v,x)dv + +(1-q)\beta\exp(-\beta z)H_2^*\int_0^z \tilde{\Phi}_2(\theta,v,x)\exp(\beta v)dv.$$

We have the following

THEOREM 7. *The bounded solution of equations* (55) *and* (52) *is unique and is given by*

(57a) $$\tilde{\Phi}_1(\theta,z,x)=\exp(r_2 z)[\alpha\theta(r_1-r_2)]^{-1}[1-\alpha(1-p)H_1^*/(\alpha-r_2)]^{-1}\cdot \cdot\{\exp(-r_1 x)(\alpha-r_1)[1-\alpha(1-p)H_1^*/(\alpha-r_1)][\alpha(1-H_1^*)-r_2] - -\alpha p H_1^*(r_1-r_2)-\exp(-r_2 x)[\alpha(1-H_1^*)-r_1][\alpha-r_2-\alpha(1-p)H_1^*]\}, \quad z>x,$$

(57b) $$\tilde{\Phi}_1(\theta,z,x)=\theta^{-1}+\exp\{(z-x)r_1\}(r_1-\alpha)[\alpha\theta(r_1-r_2)]^{-1}[\alpha(1-H_1^*)-r_2] + +\exp(r_2 z)[\alpha\theta(r_1-r_2)]^{-1}[1-\alpha(1-p)H_1^*/(\alpha-r_2)]^{-1}\cdot \cdot\{\exp(-r_1 x)(\alpha-r_1)[1-\alpha(1-p)H_1^*/(\alpha-r_1)][\alpha(1-H_1^*)-r_2] - -\alpha p H_1^*(r_1-r_2)\}, \quad 0\le z\le x,$$

(58a) $\tilde{\Phi}_2(\theta,z,x)=\exp(r_2z)(\beta+r_2)[\beta p\alpha\theta H_1^*(r_1-r_2)]^{-1}\cdot$

$$\cdot\{\exp(-r_1x)(\alpha-r_1)[1-\alpha(1-p)H_1^*/(\alpha-r_1)][\alpha(1-H_1^*)-r_2] -$$

$$-\alpha p(r_1-r_2)H_1^*-\exp(-r_2x)[\alpha(1-H_1^*)-r_1][\alpha-r_2-\alpha(1-p)H_1^*]\}, \quad z>x,$$

(58b) $\tilde{\Phi}_2(\theta,z,x)=\theta^{-1}+\exp\{(z-x)r_1\}(r_1-\alpha)(\beta+r_1)[\alpha(1-H_1^*)-r_2]\cdot$

$$\cdot[1-\alpha(1-p)H_1/(\alpha-r_1)]^{-1}[\alpha\theta\beta p(r_1-r_2)H_1^*]^{-1} +$$

$$+\exp(r_2z)(\beta+r_2)[\alpha\beta\theta pH_1^*(r_1-r_2)]^{-1}\{\exp(-r_1x)(\alpha-r_1)\cdot$$

$$\cdot[1-\alpha(1-p)H_1^*/(\alpha-r_1)][\alpha(1-H_1^*)-r_2]-\alpha pH_1^*(r_1-r_2)\}, \quad 0\le z\le x,$$

where $\mathrm{Re}(\theta) > 0$ *and* $r_1(\theta)$ *and* $r_2(\theta)$ *are given by* (4). The technique of the proof is exactly the same as that of Theorem 1. The proof therefore is omitted.

The distribution of time to first emptiness is given in terms of its Laplace transform in the theorem below. The proof of this theorem is omitted as it follows from (57b) and (58b) by letting $x \to \infty$, while keeping the signs of $\mathrm{Re}(r_1)$ and $\mathrm{Re}(r_2)$ in mind.

THEOREM 8. *The Laplace transforms of the distribution of time to first emptiness are given by*

$$F_1^*(\theta)=\exp\{r_2(\theta)z\}pH_1^*(\theta)[\theta\{1-\alpha(1-p)H_1^*(\theta)/(\alpha-r_2(\theta))\}]^{-1}, \quad z > 0,$$

$$F_2^*(\theta)=\exp\{r_2(\theta)z\}(\beta+r_2(\theta))(\beta\theta)^{-1}, \quad z > 0,$$

where $r_1(\theta)$ *and* $r_2(\theta)$ *are given in* (4).

We have seen in Section 3 that the behavior of $Z(t)$ and of $E[Z(t)]$, for large t, is different for the three cases, subcritical, critical, and supercritical. A similar behavior is indicated below for the random variable T_E. Let $\psi(z) = P(T_E=\infty|Z(0)=z)$, $\psi_i(z) = P(T_E=\infty|Z(0)=z, J_0=i)$, $i = 1,2$. Application of the relevant Tauberian theorem and that T_E is a nonnegative random variable yields the following theorem.

__THEOREM__ 9. (A) __The limiting distribution, as__ $t \to \infty$, __of__ T_E __is given by__,

$$\psi_1(z) = \begin{cases} 0, & a \leq 0 \\ 1-p\exp\{-(\beta q-\alpha p)z\}[1-\alpha(1-p)\{\alpha+(\beta q-\alpha p)\}^{-1}], & a > 0 \end{cases}$$

__and__

$$\psi_2(z) = \begin{cases} 0, & a \leq 0 \\ 1-\exp\{-(\beta q-\alpha p)z\}\{\beta-(\beta q-\alpha p)\}\beta^{-1}, & a > 0 \end{cases}$$

(B) $E[T_E|Z(0)=z,J_0=i]=\infty$, $i = 1,2$, $z > 0$; if $a=0$, __and if__ $a < 0$ __then__

$$E[T_E|Z(0)=z,J_0=1]=p^{-1}\{E_{H_1}-(\alpha k_1)^{-1}[1-p+\alpha zp]\},$$

and

$$E[T_E|Z(0)=z,J_0=2]= -(1+\beta z)(\beta k_1)^{-1}, \quad (k_1 < 0),$$

__for__ $z > 0$ __where__ k_1 __is given in__ (25).

5.2. CASE (B).

In this sub-section we study the distribution of T_E, the time until first emptiness for the model of Section 4. As in Section 4 attention will be focused on a forward rather than a backward Kolmogorov integral equation. Considering the last jump of the process Z(t) during (0,t], if there is one, and whether it is an input or a release, subject to the additional condition that the dam is not yet emtpy at time t, the following forward Kolmogorov integral equation can be established for $\tilde{R}(t,z,x)$, for $z > 0$.

(59) $$\tilde{R}(t,z,x)=U(x-z)\exp\{-(\lambda+\mu)t\} +$$
$$+\lambda\int_0^t \exp\{-(\lambda+\mu)(t-\tau)\}d\tau\int_0^x R(\tau,z,x-y)dB(y) +$$
$$+\mu\int_0^t \exp\{-(\lambda+\mu)(t-\tau)\}d\tau\int_0^\infty \tilde{R}(\tau,z,x+y)dD(y) -$$
$$-\mu\int_0^t \exp\{-(\lambda+\mu)(t-\tau)\}d\tau\int_0^\infty \tilde{R}(\tau,z,y)dD(y).$$

The corresponding Laplace transform of (59) satisfies, for $Re(\theta) > 0$,

(60) $$\tilde{\Phi}(\theta,z,x)(\lambda+\mu+\theta)=U(x-z)+\lambda\int_0^x\tilde{\Phi}(\theta,z,x-y)dB(y) +$$
$$+\mu[\int_0^\infty\tilde{\Phi}(\theta,z,x+y)dD(y)-\int_0^\infty\tilde{\Phi}(\theta,z,y)dD(y)].$$

It can be shown, by exactly the same argument used in Section 4, that (60) has a unique bounded solution, which unfortunately is difficult to obtain in any closed form in its present generality. However, as before for the case (B) the solution is tractable and is given below.

<u>THEOREM</u> 10. <u>Given</u> $\epsilon > 0$, <u>suppose</u> $D(s)/s \leq M < \infty$, <u>for</u> $0 < s \leq \epsilon$, <u>for some</u> ϵ, $M > 0$. <u>Then the unique bounded solution of equation</u> (60) <u>is given by</u>

(61) $$\tilde{\Phi}(\theta,z,x)=\theta^{-1}+C(\theta,z)+A\exp(-\gamma(\theta)x)-(\lambda+\mu+\theta)^{-1}K(z-x),\quad 0 \leq x < z$$

(62) $$\tilde{\Phi}(\theta,z,x)=\theta^{-1}+C(\theta,z)+A\exp(-\gamma(\theta)x)\qquad ,\ x \geq z,$$

<u>where</u>

$$C(\theta,z)=\theta^{-1}\mu[1+\theta^{-1}\mu\{1-D^*(\gamma)(\alpha-\gamma)\alpha^{-1}\}]^{-1}.$$

$$\cdot[-\theta^{-1}\{1-D^*(\gamma)(\alpha-\gamma)\alpha^{-1}\}+(\lambda+\mu+\theta)^{-1}\{\int_0^z K(z-y)dD(y)+$$

$$+\int_0^z \exp(\alpha v)k(z-v)dv\}],$$

$$A=-\alpha^{-1}(\alpha-\gamma)C(\theta,z)-(\alpha-\gamma)[(\alpha\theta)^{-1}+(\lambda+\mu+\theta)^{-1}\int_0^z \exp(\alpha v)K(z-v)dv],$$

$D^*(\gamma)=\int_0^\infty \exp(-\gamma y)dD(y)$, and $\gamma=\gamma(\theta)$ and K are as in Section 3.

We omit the proof of Theorem 10. It follows along the same lines as that of Theorem 5 and employs the exact same technique except that we start by constructing a solution of (60) by putting

$$\tilde{\Phi}_2=\theta^{-1}+C^*(\theta,z)+A^*\exp(-rx),\quad x \geq z,$$

and

$$\tilde{\Phi}_1=\theta^{-1}+C^*(\theta,z)+A^*\exp(-rx)+g(\theta,z,x),\quad 0 \leq x < z,$$

where $g(\theta,z,x)$ is a function to be determined, and $0 < \mathrm{Re}(r(\theta)) < \alpha$.

We now have the distribution of T_E given in the following theorem which follows from (62) by letting $x \to \infty$ and keeping in mind that $Re(\gamma(\theta)) > 0$.

THEOREM 11. *The Laplace transform of the distribution of* T_E *is given, for* $Re(\theta) > 0$, *and* $z > 0$, *by*

$$F^*(\theta)=[\theta+\mu\{1-D^*(\gamma)(\alpha-\gamma)\alpha^{-1}\}]^{-1}.$$

$$\cdot[1-\mu(\lambda+\mu+\theta)^{-1}\{\int_0^z K(z-y)dD(y)-\int_0^z \exp(\alpha v)K(z-v)dv\}].$$

Analogous to the result in Section 5.1 we give without proof

THEOREM 12. (A) *The limiting distribution of* T_E, *as* $t \to \infty$, *is given by*

$$\psi(Z) = \begin{cases} 0 & , \text{ if } (\lambda/a) \le E_D\mu \\ 1-[1+\mu(E_D+\alpha^{-1})(\alpha^{-1}\lambda-\mu E_D)^{-1}]^{-1}\cdot S(z) & , \text{ if } (\lambda/\alpha) > E_D\mu \end{cases}$$

where

$$S(z)=[(\alpha^{-1}\lambda-\mu E_D)(E_D+\alpha^{-1})-(\mu+\lambda)^{-1}\{\int_0^z K(z-y)dD(y) + +\alpha\int_0^z \exp(\alpha y)K(z-y)dy\}].$$

(B) $E[T_E|Z(0)=z] = \infty$, *if* $(\lambda/\alpha) \ge \mu E_D$

and

$$E[T_E|Z(0)=z]=\mu^{-1}[1-D^*(\gamma^*)(\alpha-\gamma^*)\alpha^{-1}]^{-1}W(z) \quad , \text{ if } (\lambda/\alpha) < \mu E_{D_1}$$

where

$$W(z) = [1-\mu(\mu+\lambda)^{-1}\{\int_0^z K(z-y)dD(y)+\int_0^z \exp(\alpha v)K(z-v)dv\}],$$

and $$\gamma^* = \lim_{\theta \to 0} \gamma(\theta) \text{ (cf. Section 4).}$$

6. LIMIT RESULTS (GENERAL CASE).

6.1. THE APPROACH AND SOME NOTATION.

In order to obtain exact and explicit results we considered, in Section 3 and 4, the two special cases (A) and (B) in which the random variables determining the amounts of inputs, releases, or both were assumed to be exponentially distributed. However in this section we shall study the limit behavior of the process Z(t) without making any assumption about the forms of the input, release or waiting time (sojourn) distributions.

A standard approach to this problem would be to locate first a suitable sequence of points of regeneration of the process such as the time points of first emptiness and subsequent returns to emptiness in the present case. Then, conditioning on the number and location in time of the points in this sequence, partition the time interval under consideration into its component parts and study the corresponding components of the process separately. It turns out, however, that this standard approach, although attractive, is not the most fruitful one in the present case. Instead we find it more convenient to consider the sequence of points of return to the release state, which, in general, are not regeneration points of the process Z(t). This consideration leads to the representation (65) which expresses Z(t) in terms of an auxiliary process associated with this sequence. It is this representation which, when suitably exploited, leads to the main results presented in this section.

We introduce some additional notation. Let $\rho_1, \rho_2, \rho_3, \ldots,$ be the sequence of lengths of time between successive returns to state 2 after time $t = 0$, these lengths being I.I.D. random variables. Let $\rho_0 \equiv 0$, and $\sigma_n = \sum_{j=0}^{n} \rho_j$, $n=0,1,\ldots$. Clearly for $k=0,1,2,\ldots,$ $N_2(t) = k$ if and only if $\sigma_k \leq t < \sigma_{k+1}$. Again, $\sigma_{N_2(t)}$ denotes the time of the last release, if there is one, before time t. Let us suppose $J_0 = 2$ and let ν_j denote the number of inputs occurring in the interval (σ_{j-1}, σ_j) for $j=1,2,\ldots$ The j-th release must, by definition of ρ_j, occur at σ_j. ν_1 is thus (starting from state 2) the number of visits to state 1 (input) until the first return to state 2 (release).

We put, by convention $X_0=Y_0\equiv 0$. Let the random variables V_j, be defined by

$$(63) \qquad V_j = X_1^{(j)}+X_2^{(j)}+\ldots+ X_{\nu_j}^{(j)} - Y_j^{(j)}, \quad j=1,2,\ldots,$$

V_0 being an arbitrary nonnegative random variable. The ν_j, $j=1,2,\ldots$ are independent of the X's and Y's, are mutually independent, and follow a common distribution, namely that of ν_1. Consequently, V_j, $j=1,2,\ldots$ is a sequence of I.I.D. random variables. Finally, define the following sequence of random variables recursively.

$$(64) \qquad \eta_0 = V_0, \qquad \eta_{n+1} = \max(0,\eta_n+V_{n+1}), \quad n = 0,1,2,\ldots$$

This sequence has been studied extensively, more recently by Takacs [37].

Again $N_2(t)$ is almost surely finite for all $t \geq 0$, which in turn implies that there is a last release before time t and that the random variables ν_j, $j=1,2,\ldots$ are also almost surely finite. Thus, by the above structure and the constructive definition of the process Z(t) it is evident that

(65) $$Z(t) \equiv \eta_{N_2(t)} + \sum_{j=0}^{I_t} X_j, \quad t \geq 0,$$

holds, almost surely, where I_t denotes the number of inputs occurring during $(\sigma_{N_2(t)}, t]$.

In the next section we examine more closely the components comprising the representation (65).

6.2. SOME PRELIMINARY RESULTS.

Unless stated otherwise we shall take $J_0 = 2$ throughout this section. It can then be established that

(66)
$$P(\rho_1 \leq x, \ \nu_1 = 0) = (1-q)H_2(x)$$
$$P(\rho_1 \leq x, \ \nu_1 = k) = q(1-p)^{k-1}pH_2^* H_1^{(k)}(x); \ k \geq 1.$$

From this it follows that

(67) $$P(\rho_1 \leq x) = (1-q)H_2(x)pqH_2^*H_1^* \sum_{k=1}^{\infty} [(1-p)H_1(x)]^{(k-1)},$$

and

(68) $$P(\nu_1 = 0) = (1-q), \ P(\nu_1 = k) = pq(1-p)^{k-1}, \ k \geq 1.$$

These, in turn, lead to the following two theorems.

THEOREM 13. I_t has the following distribution for $t \geq 0$.

(69)
$$P(I_t = 0) = (1-H_2)*U(t)$$
$$P(I_t = k) = q(1-p)^{k-1}H_1*H_2^{(k-1)}*(1-H_2)*U(t), \ k \geq 1,$$

<u>where</u> $$U(t) = \sum_{k=0}^{\infty} P(\rho_0 + \rho_1 + \ldots + \rho_k \le t).$$

<u>Proof</u>: Since $P(N_2(t) < \infty) = 1$ we have

$$P(I_t = 0) = P(I_t = 0,\ N_2(t) = 0) + \sum_{k=1}^{\infty} P(I_t = 0,\ N_2(t) = k),$$

or equivalently,

$$P(I_t = 0) = (1-H_2(t)) + \sum_{k=1}^{\infty} \int_0^t (1-H_2(t-x))\, d_x\, P(\rho_0+\rho_1+\ldots+\rho_k \le x)$$

$$= (1-H_2)*U(t).$$

For $k \ge 1$ we get, by the same argument,

$$P(I_t = k) = \sum_{j=0}^{\infty} P(I_t = k,\ N_2(t) = j)$$

$$= q(1-p)^{k-1} H_2 * H_1^{(k-1)} * (1-H_1)(t) +$$

$$+ q(1-p)^{k-1} H_2 * H_1^{(k-1)} * \sum_{j=1}^{\infty} \int_0^t (1-H_1(t-x))\, d_x P(\rho_0+\rho_1+\ldots+\rho_j \le x)$$

$$= q(1-p)^{k-1} H_2 * H_1^{(k-1)} * (1-H_1) * U(t),$$

which is the desired result. //

<u>THEOREM 14</u>. $\lim_{t\to\infty} P\left(\sum_{j=0}^{I_t} X_j \le x\right) = \Psi(x)$, <u>for every continuity point of</u> Ψ, <u>where</u> Ψ <u>is a proper distribution function, with Laplace transform given by (74)</u>.

<u>Proof</u>: The Laplace Stieltjes transform of the random variable $\sum_{j=0}^{I_t} X_j$ is given by

$$(70)\qquad E[\exp\{-s\sum_{j=0}^{I_t} X_j\}] = (1-H_2)*U(t) + $$

$$+ \sum_{k=1}^{\infty} [\gamma_1(s)]^k q(1-p)^{k-1} H_2*H_1^{(k-1)}*(1-H_1)*U(t),$$

where $s > 0$ and $\gamma_1(s) = E[\exp(-sX)]$. From (67) and an application of the monotone covergence theorem

$$(71)\qquad E(\rho_1)=E_{H_2} + E_{H_1} pq \sum_{k=1}^{\infty} k(1-p)^{k-1} = E_{H_2} + p^{-1} q\, E_{H_1} < \infty .$$

Now by the Key Renewal Theorem (cf. Smith [35], p. 15), it follows that, in the nonlattice case,

$$(72)\qquad \lim_{t\to\infty} (1-H_2)*U(t) = E_{H_2} \mu^{-1} ,$$

where $\mu = E(\rho_1)$. Also by the same theorem

$$\lim_{t\to\infty} H_2*H_1^{(k-1)}*(1-H_1)*U(t) = \mu^{-1} \int_0^{\infty} H_2*H_1^{(k-1)}*(1-H_1)(u)du.$$

Analagous results hold in the lattice case. It can be shown that

$$(73)\qquad \int_0^{\infty} H_2*H_1^{(k-1)}*(1-H_1)(u)du = E_{H_1},\ k \geq 1 .$$

Using (72) and (73) in (70) it follows that

$$(74)\qquad \lim_{t\to\infty} E[\exp\{-s \sum_{j=0}^{I_t} X_j\}] = \mu^{-1}E_{H_2} + q\gamma_1(s)E_{H_1} \{\mu[1-(1-p)\gamma_1(s)]\}^{-1}$$

$$= h(s),\ \text{say}.$$

By Theorem 2, p. 408 of [10] h(s) is the transform of a possibly defective distribution F and the convergence in law holds.

Since $\gamma_1(0) = 1$ it follows from (74) that $h(0) = 1$, which establishes the theorem. //

The following lemma, which will be essential for the limit results in Section 6.3, is given here without proof.

LEMMA 4. *Let $\{\nu_t\}$ be a sequence of random variables defined on appropriate probability spaces $(\Omega_t, \mathcal{A}_t, P)$, T an almost surely finite random variable defined on $(\Omega, \mathcal{A}, P)$ such that $\lim_{t\to\infty} P(\nu_t \le x) = P(T \le x)$ for every continuity point x of the distribution function of T. Suppose also that g is any function such that $g(t) \to \infty$ as $t \to \infty$. Then*

$$[\nu_t/g(t)] \xrightarrow{P} 0, \text{ as } t \to \infty. \tag{75}$$

Considering the random variable V_j we observe that $E(V_j)$ exists, is finite and is given by

$$E(V_j) = E(X)\,E(\nu_j) - E(Y), \quad j = 1, \ldots\,. \tag{76}$$

It follows from (76) and (68) that $E(V_j) = E(X)qp^{-1} - E(Y)$, $j = 1, 2, \ldots$. In a similar manner it can be shown that

$$\mathrm{Var}(V_j) = qp^{-1}E(X^2) + [E(X)]^2\{2(1-p)-q\}qp^{-2} - [E(Y)]^2 + E(Y^2), \quad j = 1, 2, \ldots\,.$$

We set $a = E(X)qp^{-1} - E(Y)$, $(=EV_j)$, the same as in previous sections, and $b = qp^{-1}\{E(X^2) - 2E(XY)\} + 2[E(X)]^2q(1-p)p^{-2} + E(Y^2)$, $(=EV_j^2)$.

The sequence η_n employed in the representation (65) has the following property which we shall find useful.

(77) $\eta_n = \max(S_n - S_n,\ S_n - S_{n-1}, \ldots,\ S_n - S_1, S_n),\ n = 1,2,\ldots$

where $S_n = \sum_{i=0}^{n} V_i$ (cf. Takacs [37], p. 344). A sequence of random variables closely related to the η_n and also essential in the analysis of the limit results for Z(t) is the following.

(78) $\eta_n^* = \max(0, \zeta_1,\ \zeta_2, \ldots,\ \zeta_n),\ n = 0,1,2\ldots,$

where $\zeta_n = S_n - V_0$. The exact distribution of η_n^* is covered by the well known Spitzer identity [36]. If $\eta_0 = 0$, η_n and η_n^* have the same distribution. If η_0 is an arbitrary nonnegative random variable, the distribution of η_n is covered by Takacs [37]. Let

(79) $$\eta = \sup_{0 \leq n < \infty} (S_n - V_0)$$

The random variable η is nonnegative and possible infinite. We exclude the case $P(V_n=0)=1$ since in that case the independence of the X's and Y's is violated when B and D are nontrivial. Consider then the case $P(V_n=0) < 1$. We summarize some known limiting results for η_n and η_n^*.

(R_1) (Takacs [37],p.350). If $a < 0$, then $P(\eta<\infty)=1$. On the other hand if $a \geq 0$, then $P(\eta=\infty)=1$.

(R_2) (Lindley [19], p. 281; Takacs [37]p.345). If $P(V_n=0) < 1$, then we have

$$\lim_{n\to\infty} P(\eta_n \leq x) = P(\eta \leq x) ,$$

regardless of the distribution of η_0. As a consequence of the proof of Lindley's result we have that η_n and η_n^* have the same limit distribution.

(R_3) (Erdos and Kac [8]). If $a = 0$, $E(V_n^2) = 1$, then

(80) $$\lim_{n\to\infty} P(\eta_n^* n^{-\frac{1}{2}} \leq x) = L(x),$$

where L is given by

(81) $$L(x) = \begin{cases} 2\Phi(x) - 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0, \end{cases}$$

and $\Phi(x)$ is the standard normal distribution function.

(R_4) (Chung [5], p. 1163; also a shorter proof by Puri [29]). If $a > 0$, then $\eta_n^* n^{-1} \xrightarrow{a.s.} a$, as $n \to \infty$. If moreover Var $V_n = 1$, then

(82) $$\lim_{n\to\infty} P((\eta_n^* - an)\, n^{-\frac{1}{2}} \leq x) = \Phi(x).$$

We remark here that, starting with $J_0 = 1$, we could go through exactly the same analysis as we did for $J_0 = 2$ in this section. The limit results for the case $a \geq 0$ are independent of the value of J_0. However, the limit results for the case $a < 0$ will, in general, depend upon the value of J_0. As was observed for the cases (A) and (B) (see Sections 3 and 4), in the subcritical case the process Z(t) tends in law to a proper random variable as $t \to \infty$. While this seems to be true in the present general case as well, we shall, however restrict ourselves in the next section only to the critical and supercritical cases.

6.3 THE LIMIT RESULTS.

First it is essential to state an important limiting property of $N_2(t)$, the number of releases in the interval (0,t]. Since $N_2(t)$ also represents the number of visits of the Markov chain $\{J_n\}$ to the state 2, it could be visualized as a renewal process (cf. Çinlar [7], p. 125). Denote the distribution function which induces this renewal process by F. We have $\mu = E(\rho_1) = \int_0^\infty (1-F(u))du < \infty$ from (71). The following well known result is stated without proof (see Chung [6], p. 127).

THEOREM 15. *Let* $N_2(t)$ *denote the number of renewals in* $(0,t]$ *of the renewal process induced by* F. *Then*

$$N_2(t)\ t^{-1} \xrightarrow{P} \mu^{-1} > 0, \ \text{as}\ t \to \infty. \tag{83}$$

We turn first to the critical case, where a = 0. In proving the following limit theorems we employ the approach used by Renyi in [33] where he studied the asymptotic distribution of the sum of a random number of independent random variables.

THEOREM 16. *Let* $a = 0$, $b < \infty$. *Without loss of generality we take* $b = 1$. *Then*

$$\lim_{t\to\infty} P(Z(t)\ t^{-\frac{1}{2}} \le x) = L(x\mu^{\frac{1}{2}}),$$

where the distribution function L *is given by* (81).

Proof: Using the representation (65) we have

$$Z(t)\ t^{-\frac{1}{2}} = \eta_{N_2(t)} t^{-\frac{1}{2}} + \sum_{j=0}^{I_t} X_j t^{-\frac{1}{2}}. \tag{84}$$

Theorem 14 and Lemma 4 together allow us to focus attention on the first term on the right hand side of (84). In view of Theorem 15, there exists a sequence $\epsilon_t > 0$, with $\epsilon_t \downarrow 0$ as $t \to \infty$, such that

(85) $$P(|N_2(t) - \mu^{-1}t| > \mu^{-1}\,\epsilon_t\, t) \leq \mu^{-1}\,\epsilon_t .$$

Define the set $A_t = \{\omega: |N_2(t)-\mu^{-1}t| \leq \epsilon_t\mu^{-1}t\}$, $t \geq 0$.

We thus have

(86) $$P(\bar{A}_t) \leq \mu^{-1}\epsilon_t$$

where $\bar{A}_t$ denotes the complement of A_t. Let now

$$n_1(t) = [(1-\epsilon_t)\mu^{-1}t] \text{ and } n_2(t) = [(1+\epsilon_t)\mu^{-1}t] ,$$

where [....] denotes the integral part of the number in the brackets. For convenience the arguments of n_1 and n_2 will sometimes be suppressed. Both $n_1(t)$ and $n_2(t) \to \infty$ as $t \to \infty$. Moreover, on A_t, $n_1(t) < N_2(t) \leq n_2(t)$. We can now write

(87) $$\eta_{N_2(t)}t^{-\frac{1}{2}} = \eta_{n_1}n_1^{-\frac{1}{2}}[n_1/t]^{\frac{1}{2}}I_{A_t} + \{\eta_{N_2(t)} - \eta_{n_1}\}n_1^{-\frac{1}{2}}[n_1/t]^{\frac{1}{2}}\, I_{A_t} + \\ + \eta_{N_2(t)}t^{-\frac{1}{2}}\, I_{\bar{A}_t},$$

where I_G is the indicator function of the set G. By (86) it suffices to consider the first two terms on the right hand side of (87). From (83) it follows that $n_1(t)/t \overset{P}{\to} \mu^{-1}$, as $t \to \infty$. Further, it can be shown using

$$P(\eta_n^* n^{-\frac{1}{2}} \leq x, (V_o + \zeta_n)n^{-\frac{1}{2}} \leq x) \leq P(\eta_n n^{-\frac{1}{2}} \leq x) \leq P(\eta_n^* n^{-\frac{1}{2}} \leq x), \text{ all } x \geq 0$$

and a similar argument to that in proof of (R_2) Section 6.2, that $\eta_n n^{-\frac{1}{2}}$ and $\eta_n^* n^{-\frac{1}{2}}$ have the same limit distribution. By (R_3) Section 6.2 the first term on the right side of (87) tends in probability to $\mu^{-\frac{1}{2}}\xi$, and thus it suffices to show that

$$|\eta_{N_2(t)}-\eta_{n_1}|n_1^{-\frac{1}{2}} I_{A_t} \xrightarrow{P} 0, \text{ as } t \to \infty.$$

Now fix an $\epsilon > 0$. Then

$$P(|\eta_{N_2(t)}-\eta_{n_1}|I_{A_t} \geq \epsilon n_1^{\frac{1}{2}}) \leq P(\sup_{n_1<j\leq n_2}|\eta_j-\eta_{n_1}| \geq \epsilon n_1^{\frac{1}{2}}).$$

The form of the limit in (81) allows us, for $\delta > 0$, to choose a point $A = A(\delta)$ of L, so small that $P(\xi \leq A) \leq \delta/8$. Now, from (80) there exists a $t_1(\delta)$ such that for $t > t_1(\delta)$

$$|P(\eta_{n_1} \leq An_1^{\frac{1}{2}}) - P(\xi \leq A)| \leq \delta/8.$$

Hence

(88) $$P(\eta_{n_1} \leq An_1^{\frac{1}{2}}) \leq \delta/4 \text{ for } t > t_1(\delta) .$$

Now

(89) $$P(\sup_{n_1<j\leq n_2}|\eta_j-\eta_{n_1}|>\epsilon n_1^{\frac{1}{2}})=P(\sup_{n_1<j\leq n_2}|\eta_j-\eta_{n_1}|>\epsilon n_1^{\frac{1}{2}},\ \eta_{n_1}>An_1^{\frac{1}{2}})$$
$$+P(\sup_{n_1<j\leq n_2}|\eta_j-\eta_{n_1}|>\epsilon n_1^{\frac{1}{2}},\eta_{n_1}\leq An_1^{\frac{1}{2}}).$$

By (88) the second term in (89) is less than $\delta/4$ for $t \geq t_1(\delta)$. In order to consider the first term on the right hand side of (89) we set $\epsilon'=\min(\epsilon,A)$. Choose $t > t_1(\delta)$. Since

$$\{\sup_{n_1<j\leq n_2}|\eta_j-\eta_{n_1}|>\epsilon' n_1^{\frac{1}{2}},\ \eta_{n_1} > \epsilon' n_1^{\frac{1}{2}}\}$$

$$\subseteq \{\sup_{n_1<j\leq n_2}|\sum_{i=n_1+1}^{j} V_i|>\epsilon' n_1^{\frac{1}{2}}\},$$

it follows by the Kolmogorov inequality (Chung [6], p. 109) that

(90) $$P(\sup_{n_1<j\le n_2} |\eta_j-\eta_{n_1}|>\epsilon n_1^{\frac{1}{2}}, \eta_{n_1} > A_{n_1}^{\frac{1}{2}})$$

$$\le P(\sup_{n_1<j\le n_2} |\sum_{i=n_1+1}^{j} V_i|>\epsilon' n_1^{\frac{1}{2}}) \le (n_2-n_1)\{(\epsilon')^2 n_1\}^{-1}.$$

The right hand side of (90) tends to zero as $t \to \infty$. Letting $\delta \to 0$, we conclude that $|\eta_{N_2(t)}-\eta_{n_1}|n_1^{-\frac{1}{2}}I_{A_t} \xrightarrow{P} 0$, completing the proof. //

Consider now the supercritical case, where $a > 0$. We first prove

THEOREM 17. *Let* $a > 0$, $b-a^2=1$. *Then*

(91) $$\lim_{n\to\infty} P((\eta_n-an)n^{-\frac{1}{2}} \le x) = \Phi(x) \text{ as } n \to \infty.$$

Proof: By the definition of η_n we can write

(92) $$(\eta_n-an)n^{-\frac{1}{2}}=[\max(S_n-S_n,S_n-S_{n-1},\ldots, S_n-S_1,S_n)-an]n^{-\frac{1}{2}}$$

$$=[\max(-S_n,-S_{n-1},\ldots,-S_1,0)\ n^{-\frac{1}{2}}+(S_n-an)n^{-\frac{1}{2}}.$$

The first term on the extreme right of (92) tends to zero in probability by Lemma 4. For we use the fact that $\max(0,-S_1,-S_2,\ldots,-S_n)$ corresponds to a subcritical process by replacing V_n's by $-V_n$'s. It follows from (R_1) and (R_2) of Section 6.2 that $\max(0,-S_1,\ldots,-S_n)$ tends in law to a proper random variable. Finally, by the central limit theorem (Feller [10], p. 187) the second term on the extreme right of (92) converges in distribution in the desired manner. This completes the proof. //

Immediately from Theorem 17 follows the

COROLLARY 1. Let $a > 0$, $b-a^2 = 1$. Then

$$\eta_n n^{-1} \xrightarrow{P} a, \text{ as } n \to \infty.$$

Finally, we have for the supercritical case,

THEOREM 18. Let $a > 0$ and $b-a^2 = 1$. Let $N_2(t)$ be as defined in Section 2. Then

$$\lim_{t\to\infty} P((Z(t) - aN_2(t))t^{-\frac{1}{2}} \leq x) = \Phi(x\mu^{\frac{1}{2}}).$$

Proof: We write, using the representation (65),

$$(93) \qquad (Z(t)-aN_2(t))t^{-\frac{1}{2}} = (\eta_{N_2(t)} - aN_2(t))t^{-\frac{1}{2}} + [\sum_{j=0}^{I_t} X_j]t^{-\frac{1}{2}}.$$

By the same argument set forth in the proof of Theorem 16 it suffices to show that the first term on the right hand side of (93) converges in distribution to $\Phi(x\mu^{\frac{1}{2}})$ for every x. In view of Theorem 15 define $\{\epsilon_t\}$, $n_1(t)$, $n_2(t)$ and A_t in the same way as in the proof of Theorem 16. Then we can write

$$(94) \qquad (\eta_{N_2(t)} - aN_2(t))t^{-\frac{1}{2}} = (\eta_{n_1} - an_1)n_1^{-\frac{1}{2}}(n_1/t)^{\frac{1}{2}} + \\ + \{\eta_{N_2(t)} - \eta_{n_1} - a(N_2(t)-n_1)\}n_1^{-\frac{1}{2}}(n_1/t)^{\frac{1}{2}}.$$

By Theorem 17 and the definition of $n_1(t)$, the first term on the right of (94) converges in distribution to $\Phi(x\mu^{\frac{1}{2}})$ for every x. By (86) it suffices to show therefore that

$$\{\eta_{N_2(t)} - \eta_{n_1} - a(N_2(t)-n_1)\}n_1^{-\frac{1}{2}} I_{A_t} \xrightarrow{P} 0, \text{ as } t \to \infty.$$

Now

$$|\{\eta_{N_2(t)}-\eta_{n_1}-a(N_2(t)-n_1)\}I_{A_t}n_1^{-\frac{1}{2}}| \leq \sup_{n_1<j\leq n_2} |\eta_j-\eta_{n_1}-a(j-n_1)|n_1^{-\frac{1}{2}}$$

$$\leq \{\max(-S_{n_2},-S_{n_2-1},\ldots,-S_1,0)-\max(-S_{n_1},\ldots,-S_1,0)\}n_1^{-\frac{1}{2}} +$$

$$+ \sup_{n_1<j\leq n_2} |\sum_{i=n_1+1}^{j} (V_i-a)|n_1^{-\frac{1}{2}} .$$

By the same reasoning set forth in proof of Theorem 17 both $\max(0,-S_1-S_2,\ldots,-S_{n_2})n_1^{-\frac{1}{2}}$ and $\max(0,-S_1,\ldots,-S_{n_1})n_1^{-\frac{1}{2}}$ can be shown to tend to zero in probability as $t \to \infty$. Moreover, by the Kolmogorov inequality, for any arbitrary constant $\delta > 0$, we have,

$$(95) \qquad P(\sup_{n_1<j<n_2} |\sum_{i=n_1+1}^{j} (V_i-a)| \geq \delta n_1^{\frac{1}{2}}) \leq (n_2-n_1)(\delta^2 n_1)^{-1}.$$

The right side of (95) tends to zero as $t \to \infty$. This completes the proof. //

As a consequence of Theorem 18 we have

COROLLARY 2. Let $a > 0$, $b=a^2=1$. Then, as $t \to \infty$,

$$Z(t)t^{-1} \xrightarrow{P} a\mu^{-1}.$$

7. A FEW CONCLUDING REMARKS.

We end by making the following remarks. Models in the literature up to 1963 (see Prabhu [25] and Moran [23]) generally retained, as does ours, the assumption of mutual independence of the inputs. An initial attempt to consider correlated inputs (but with deterministic release) was made by Lloyd and Odoom [20]. In their paper a sequence of inputs during consecutive discrete time intervals constitute a Markov chain with a finite number of states. The levels Z_t of a finite capacity dam are observed at times $t = 0,1,2\ldots$. During the interval $(t,t+1)$ an inflow X_t is observed. The distribution of these inputs is assumed to have a stable limit distribution. An inflow may cause the level to exceed the capacity of the dam and result in an instantaneous overflow. At the end of each interval, m units of water are instantaneously released, if there are present at least m units. These authors studied the stationary solution for the model as $t \to \infty$. They point out that the joint process $\{(Z_t,X_t)\}$ is also Markovian. From this the marginal limiting distribution of levels is derived. It is claimed that withdrawal policies of a random nature may be easily incorporated into the model, but not much was achieved in this direction. Additional work was done by Lloyd and Odoom in [21] on the stationarity for the probabilities of dam contents. Ali Khan and Gani [1] studied the time dependent solution of the Lloyd-Odoom model.

In the context of the quantal response assay problem mentioned in the beginning of this paper, it would seem desirable to introduce a similar type of dependence, perhaps a Markovian one to begin with, into the sequence of inputs and releases. Moreover, as was done in [30], it would be appropriate to incorporate an element of dependence of the waiting times between inputs and releases on the level of the process $Z(t)$. This is so in view of the fact that many biological mechanisms are known to become active only in response to changes in the level of the stimulus. The analysis of the models incorporating these features will be the subject of a future study.

APPENDIX

We demonstrate here the proof of the lemma necessary for the proof of Theorem 5.

<u>LEMMA</u> (i) <u>The Volterra equation</u>

$$F(\xi) = a + \int_0^\xi F(\xi - y)\,dH(y), \quad 0 \le \xi \le b < \infty \tag{96}$$

<u>in</u> F, <u>where</u> H <u>is given and has the properties stated before Theorem</u> $\underset{\sim}{5}$, <u>and</u> $\underset{\sim}{a}$ <u>is a given constant, has solution given by</u>

$$F(\xi) = a[1+H(\xi)+H*H(\xi)+H*H*H(\xi)+\ldots.] = aK(\xi), \tag{97}$$

<u>and this solution is unique, provided</u> K(ξ) <u>converges uniformly in</u> $0 \le \xi \le b < \infty$.

(ii) <u>Let</u> D <u>be such that</u> $D(s)/s \le A < \infty$ <u>for</u> $0 \le s \le \epsilon$, <u>for some</u> $\epsilon > 0$ <u>and some constant</u> $A > 0$. <u>Then for</u>

$$H(s) = [\mu D(s)+\lambda(1-\exp(\alpha s))](\lambda+\mu+\theta)^{-1}, \; 0 \le s \le z,$$

<u>where</u> z <u>is fixed but otherwise arbitrary</u>, K(s) <u>exists and is finite for</u> $0 \le s \le z$.

<u>Proof</u>: (i) The assertion follows first by a substitution of (97) into (96). Then the interchange of summation and integration operations is justified since the series K(ξ) converges uniformly in $0 \le \xi \le b < \infty$. Also Since $|K| < \infty$, for a fixed $\epsilon > 0$, there is an $n_0 = n_0(\epsilon, s)$ such that $|H^{(n)}(s)| < \epsilon$ for $n \ge n_0$. Now consider the difference, V, of two solutions of equation (96). V satisfies V=H*V, and hence $V = H^{(n)} * V$ for all n.

But the remark above indicates that $H^{(n)}(s) \to 0$ for all s as $n \to \infty$, and hence $V(s)=0$. The solution of (96) is thus unique.

(ii) Let

(98) $$M = |\lambda+\mu+\theta|^{-1}[\mu A+\alpha\lambda\exp(\alpha z)].$$

It can be shown that $D(s)/s \leq A < \infty$ for $0 < s \leq \epsilon$, $\epsilon > 0$ implies $D(s)/s \leq A < \infty$ for $0 < s \leq z$ where z is fixed but otherwise arbitrary, and A is used in a generic sense here.

Thus M in (98) is finite. The assertion of the lemma now follows from the fact that $|K(s)| \leq \exp(Ms)$, which is proved by a straightforward induction argument, i.e., that

(99) $$|H^{(k)}(s)| \leq M^k s^k/k!, \; k=0,1,2,\ldots, \; 0 \leq s \leq z.$$ //

<u>REMARK</u>. In the case D has a density the condition, $D(s)/s \leq A < \infty$ for $0 \leq s \leq \epsilon$, is satisfied and the lemma holds. The condition (99) guarantees, by the Weierstrass M-test, that $K(\xi)$ converges uniformly.

REFERENCES

[1] Ali Khan, M. S. and Gani, J. (1968). Infinite dams with inputs forming a Markov chain. Jour. Appl. Prob. 5, 72-83.

[2] Arrow, Karlin, S. and Scarf. (1958). Studies in the Mathematical Theory of Inventory and Production. Stanford University Press.

[3] Benes, V. E. (1957). On queues with Poisson arrivals. Ann. Math. Stat. 28, 670-677.

[4] Bliss, C. I. (1952). The statistics of bioassay, with special reference to the vitamins. Reprinted, with additions, from Vitamin Methods, Volume II, pages 445-628. Academic Press, New York.

[5] Chung, K. L. (1948). Asymptotic distribution of the maximum cumulative sum of independent random variables. Bull. Amer. Math. Soc. 54, 1162-1170.

[6] Chung, K. L. (1968). A Course in Probability Theory. Harcourt Brace and World, New York.

[7] Çinlar, E. (1969). Markov renewal theory. Adv. in Appl. Prob. 1, 123-187.

[8] Erdös, P. and Kac, M. (1946). On certain limit theorems of the theory of probability. Bull. Amer. Math. Soc. 52, 292-302.

[9] Feller, W. (1957). An Introduction to Probability Theory and Its Applications, Volume I. John Wiley and Sons, Inc., New York.

[10] Feller, W. (1966). An Introduction to Probability Theory and Its Applications, Volume II. John Wiley and Sons, Inc., New York.

[11] Finney, D. J. (1947). Probit Analysis. A Statistical Treatment Of the Sigmoid Response Curve. Cambridge University Press, Cambridge.

[12] Finney, D. J. (1952). Statistical Method in Biological Assay. Charles Griffin and Co., London.

[13] Gani, J. (1955) Some problems in the theory of provisioning and of dams. Biometrika 42, 179-200.

[29] Puri, P. S. (1971). On the asymptotic distribution of the maximum of sums of a random number of I.I.D. random variables. Department of Statistics, Purdue University Mimeograph Series number 265.

[30] Puri, P. S. and Senturia, J. (1972). On a mathematical theory of quantal response assays. To appear in Proc. Sixth Berkeley Symp. on Math. Stat. and Prob. (Biology-Health Section) held in June 1971.

[31] Pyke, R. (1961). Markov renewal processes: definitions and preliminary results. Ann. Math. Stat. **32**, 1231-1242.

[32] Pyke, R. (1961). Markov renewal processes with finitely many states. Ann. Math. Stat. **32**, 1243-1259.

[33] Renyi, A. (1957). On the asymptotic distribution of the sum of a random number of independent random variables. Acta. Mathematica Acad. Sci. Hung. **8**, 193-199.

[34] Senturia, J. (1972). On a mathematical theory of quantal response assays and a new model in dam theory. Ph.D. Thesis. Purdue University.

[35] Smith, W. L. (1955). Regenerative stochastic processes. Proc. Roy. Soc. A, **232**, 6-31.

[36] Spitzer, F. (1956). A combinatorial lemma and its application to probability theory. Trans. Amer. Math. Soc. **82**, 323-339.

[37] Takacs, L. (1970). On the distribution of the maximum of sums of mutually independent and identically distributed random variables. Adv. Appl. Prob. **2**, 344-354.

[38] Widder, D. V. (1941). The Laplace Transform. Princeton University Press, Princeton.