# Exact and Asymptotic Distributions of Some Statistics in Multivariate Analysis

by

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(2.5) $$C(p,n,\Lambda) = \Gamma_p(n/2)[\Gamma_p(n_1/2)\Gamma_p(n_2/2)]^{-1}|\Lambda|^{-n_1/2}.$$

Now using the density (2.3) we get

(2.6) $$E(Y^h) = C(p,n,\Lambda)\pi^{p(p-1)/4} \sum_{k=o}^{\infty} \sum_{\kappa} \frac{(n/2)_\kappa C_\kappa(M)}{k!} \cdot \frac{\prod_{i=1}^{p} \Gamma[n_1/2+ah+k_i-(i-1)/2] \prod_{i=1}^{p} \Gamma[n_2/2+bh-(i-1)/2]}{\prod_{i=1}^{p} \Gamma[n/2+(a+b)h+k_i-(i-1)/2]}.$$

Making use of the inverse Mellin transform, we have the density of Y as

(2.7) $$f(Y) = C(p,n,\Lambda)\pi^{p(p-1)/4} \sum_{k=o}^{\infty} \sum_{\kappa} \frac{(n/2)_\kappa C_\kappa(M)}{k!} Y^{-1} \cdot \frac{1}{2\pi i} \int_C Y^{-h} \frac{\prod_{i=1}^{p} \Gamma[n_1/2+ah+k_i-(i-1)/2] \prod_{i=1}^{p} \Gamma[n_2/2+bh-(i-1)/2]}{\prod_{i=1}^{p} \Gamma[n/2+(a+b)h+k_i-(i-1)/2]} dh.$$

Noting that the integral on the R.H.S. of (2.7) is in the form of the H-function, the non-central density of Y for test (1) can be put in a single general form for different sets of values of a and b as follows:

(2.8) $$f(Y) = C(p,n,\Lambda)\, \alpha \sum_{k=o}^{\infty} \sum_{\kappa} \frac{(n/2)_\kappa \delta C_\kappa(M)}{k!} Y^{-1} H^{r,s}_{t,u}\left(Y \middle| \begin{matrix} (a_i,\alpha_i) i=1,\dots,t \\ (b_i,\beta_i) i=1,\dots,u \end{matrix}\right)$$

where $C(p,n,\Lambda)$ is as in (2.5) and the constants are given in the following table.

$$(2.14)\qquad f(Y)=C_2(p,n,q,\underset{\sim}{P}^2)\alpha\sum_{k=o}^{\infty}\sum_{\kappa}\frac{(n/2)_\kappa(n/2)_\kappa\delta C_\kappa(\underset{\sim}{P}^2)}{(q/2)_\kappa k!}$$

$$\cdot\, H^{r,s}_{t,u}\left(Y\middle|\begin{matrix}(a_i,\alpha_i)i=1,\dots,t\\(b_i,\beta_i)i=1,\dots,u\end{matrix}\right),$$

where the constants $\alpha$, $\delta$, r, s, t, u, $(a_i,\alpha_i)$ and $(b_i,\beta_i)$ are as in Table 1, in which $n_1$ is to be replaced by q throughout.

## 3. SPECIAL CASES

(i) Wilks' $\Lambda$ criterion. Taking a = 0 and b = 1 in (2.8) and using the relation between the H-function and the G-function, we find that the non-central density of Wilks' $\Lambda = \prod_{i=1}^{p}(1-\theta_i)$ is as obtained by Pillai, Al-Ani, Jouris in the three cases [25]

(ii) Wilks-Lawley U-criterion. If a = 1 and b = 0 in (2.8), we obtain the non-central density of Wilks-Lawley U-statistic, $U = \prod_{i=1}^{p}\theta_i$ for test (1), in the form

$$(3.1)\qquad f(u) = C(p,n,\underset{\sim}{\Lambda})\sum_{k=o}^{\infty}\sum_{\kappa}\frac{(n/2)_\kappa C_\kappa(\underset{\sim}{M})}{k!}\Gamma_p(n_2/2)$$

$$\cdot Y^{-1}\, H^{p,o}_{p,p}\left(Y\middle|\begin{matrix}(a_i,\alpha_i)i=1,\dots,p\\(b_i,\beta_i)i=1,\dots,p\end{matrix}\right),$$

where $C(p,n,\underset{\sim}{\Lambda})$ is as in (2.4), $(a_i,\alpha_i) = (n/2+k_i-(i-1)/2,1)$ and $(b_i,\beta_i)=(n_1/2+k_i-(i-1)/2,1)$ $i=1,\dots,p$. (3.1) can also be expressed in terms of the G-function. The density of U for the Manova and Canonical correlation cases can be written down using the substitution (2.10) and (2.13) respectively.

(iii) Taking $a = n_1/2$ and $b = n_2/2$ in (2.8) we obtain the non-central density of the modified likelihood ratio criterion for testing $\Sigma_1 = \Sigma_2$ i.e. of the statistic

$$\lambda = \prod_{i=1}^{p} \theta_i^{n_1/2} (1-\theta_i)^{n_2/2} = |S_1|^{n_1/2} |S|^{-n/2} |S_2|^{n_2/2} \text{ where } S=S_1+S_2,$$

in the form

$$f(\lambda) = C(p,n,\Lambda)\pi^{p(p-1)/4} \sum_{k=o}^{\infty} \frac{(n/2)_\kappa C_\kappa(M)}{k!} \lambda^{-1} H^{2p,o}_{p,2p}\left(\lambda \,\middle|\, {(a_i,\alpha_i) i=1,\ldots,p \atop (b_i,\beta_i) i=1,\ldots,p}\right),$$

where $(a_i,\alpha_i) = (n/2+k_i-(i-1)/2, n/2)$ and $(b_i,\beta_i) =$

$\{(n_1/2+k_i-(i-1)/2, n_1/2), (n_2/2-(i-1)/2, n_2/2)\ i = 1,\ldots,p\}$.

The densities in the other two cases can be written down using (2.10) and (2.13).

(iv) Taking $a = 1$ and $b = -1$ in (2.8) we obtain the noncentral density of the statistic $W = \prod_{i=1}^{p} \theta_i (1-\theta_i)^{-1} = |S_1 S_2^{-1}|$ for test (1) in the form

$$(3.2) \quad f(Y)=C(p,n,\Lambda)\pi^{p(p-1)/2} \sum_{k=o}^{\infty} \sum_{\kappa} \frac{(n/2)_\kappa C_\kappa(M)}{k!\Gamma_p(n/2,\kappa)} Y^{-1}$$

$$H^{p,p}_{p,p}\left(Y \, {(a_i,1)\ i = 1,\ldots,p \atop (b_i,1)\ i = 1,\ldots,p}\right) .$$

where $a_i = 1-n_2/2+(i-1)/2$ and $b_i=n_1/2+k_i-(i-1)/2$. The density in (3.2) can be easily written down in terms of the G-function. The non-central densities of W for the Manova and Canonical correlation cases can be written down using (2.10) and (2.13).

## 4. NON-CENTRAL DISTRIBUTION OF Y IN THE COMPLEX CASE

The non-central density of Y in the complex case can be obtained in a similar manner and is noted below. (a') The general form of the density of Y for test (1) can be written down from (2.8) by making the following substitutions.

$$(4.1)\qquad (\pi^{d},\ n_1/2,\ n_2/2,\ n/2,\ (i-1)/2,\ \Gamma_p(\cdot),\ \Gamma_p(\cdot,\kappa),\ C_\kappa(\cdot),(\cdot)_\kappa)$$

$$\rightarrow (\pi^{2d},\ n_1,\ n_2,\ n,\ (i-1),\ \tilde{\Gamma}_p(\cdot),\ \tilde{\Gamma}_p(\cdot,\kappa),\ \tilde{C}_\kappa(\cdot),\ [\cdot]_\kappa)$$

where $\tilde{\Gamma}_p(\cdot)$, $\tilde{\Gamma}_p(\cdot,\kappa)$, $\tilde{C}_\kappa(\cdot)$ and $[\cdot]_\kappa$ are as defined in James [12].

(b') For the Manova case the general form of the density of Y is obtained from (2.11) by making the substitutions as in (4.1).

(c') In the case of Canonical correlation also, the general form of the density of Y can be written down from (2.14) using (4.1).

## 5. ASYMPTOTIC EXPANSION OF THE DISTRIBUTION OF Y, $a=n_1/2$ AND $b=n_2/2$

First we give some preliminaries.

(a) Preliminaries. For this case, putting $a = n_1/2$ and $b = n_2/2$ in (2.6) we have,

$$(5.1)\qquad E(Y^h) = \frac{\Gamma_p(n/2)}{\Gamma_p(n_1/2)\Gamma_p(n_2/2)}\,|\Lambda|^{-n_1/2}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(n/2)_\kappa\Gamma_p\{\frac{n_1(1+h)}{2},\kappa\}}{k!\,\Gamma_p\{\frac{n(1+h)}{2},\kappa\}}$$

$$\cdot\ \Gamma_p\{n_2(1+h)/2\}\ C_\kappa(M)\ .$$

This can be easily written in the form

$$(5.2)\quad E(Y^h) = \{\Gamma_p(n/2)\Gamma_p[n_1(1+h)/2]\Gamma_p[n_2(1+h)/2]/\Gamma_p[n(1+h)/2]\,\Gamma_p(n_1/2)\Gamma_p(n_2/2)\}|\underset{\sim}{A}|^{-n_1/2}\cdot {}_2F_1(n/2,n_1(1+h)/2;n(1+h)/2,\underset{\sim}{M}).$$

We shall assume that

$$(5.3)\qquad n_i = \tau_i n,\ (i = 1,2) \text{ where } \tau_1 + \tau_2 = 1.$$

The asymptotic expansion of the distribution of Y will be derived in terms of n increasing and also in terms of $m = \rho n$ increasing where $0 < \rho < 1$ and is defined later, with $\tau_1$ and $\tau_2$ fixed. (See Anderson [1], p. 254). The $h^{th}$ moment of

$$(5.4)\qquad W = [n^{(1/2)pn}/n_1^{(1/2)pn_1}\,n_2^{(1/2)pn_2}]\cdot Y$$

is given by

$$(5.5)\quad E(W^h) = n^{(1/2)pnh}\,n_1^{-(1/2)pn_1h}\,n_2^{-(1/2)pn_2h}\{\Gamma_p(n/2)\Gamma_p[n_1(1+h)/2]/\Gamma_p[n(1+h)/2]\Gamma_p(n_1/2)\}\{\Gamma_p[n_2(1+h)/2]/\Gamma_p(n_2/2)\}\,|\underset{\sim}{A}|^{-n_1/2}\cdot {}_2F_1(n/2,n_1(1+h)/2;n(1+h)/2,\underset{\sim}{M}).$$

We shall obtain the asymptotic expansion for (i) $-2\log W$ in terms of n increasing and assuming $\underset{\sim}{M}$ to be of the form $\underset{\sim}{M} = (2/n)\,\underset{\sim}{P}$ where $\underset{\sim}{P}$ is a fixed matrix and (ii) $-2\rho\log W$ in terms of $m = \rho n$ increasing instead of n and assuming $\underset{\sim}{M} = (2/m)\,\underset{\sim}{P}$ where $\underset{\sim}{P}$ is a fixed matrix and the correction factor $\rho$ is given by (see Anderson [1], p. 255)

(5.6) $m = \rho n = n - 2\alpha$ where $\alpha = (\tau_1^{-1}+\tau_2^{-1}-1)(2p^2+3p-1)/12(p+1)$.

We will need the following lemmas proved in [28].

Lemma 5.1. Let $C_\kappa(\underset{\sim}{Z})$ be a zonal polynomial corresponding to the partition $\kappa=\{k_1,k_2,\ldots,k_p\}$ with $k_1+k_2+\ldots+k_p=k$ and $k_1 \geq k_2 \geq k_3\ldots \geq k_p \geq 0$. Putting

$$(5.7)\quad a_1(\kappa) = \sum_{i=1}^{p} k_i(k_i-i),\ a_2(\kappa) = \sum_{i=1}^{p} k_i(4k_i^2-6ik_i+3i^2)$$

Then the following equalities hold:

$$(5.8)\quad \sum_{k=0}^{\infty}\sum_{\kappa} x^k\, C_\kappa(\underset{\sim}{Z})a_1(\kappa)/k! = (x^2 \mathrm{tr}\ \underset{\sim}{Z}^2)e^{\mathrm{tr}(x\underset{\sim}{Z})},$$

$$(5.9)\quad \sum_{k=1}^{\infty}\sum_{\kappa} x^k\, C_\kappa(\underset{\sim}{Z})a_1(\kappa)/(k-1)! = (2x^2\ \mathrm{tr}\ \underset{\sim}{Z}^2+x^3\ \mathrm{tr}\ \underset{\sim}{Z}^2\ \mathrm{tr}\ \underset{\sim}{Z})e^{\mathrm{tr}(x\underset{\sim}{Z})},$$

$$(5.10)\quad \sum_{k=0}^{\infty}\sum_{\kappa} x^k(a_1(\kappa))^2C_\kappa(\underset{\sim}{Z})/k! = \{x^4(\mathrm{tr}\ \underset{\sim}{Z}^2)^2+4x^3\ \mathrm{tr}\ \underset{\sim}{Z}^3+x^2\ \mathrm{tr}\ \underset{\sim}{Z}^2 +x^2(\mathrm{tr}\ \underset{\sim}{Z})^2\}\ e^{\mathrm{tr}(x\underset{\sim}{Z})},$$

$$(5.11)\quad \sum_{k=0}^{\infty}\sum_{\kappa} x^kC_\kappa(\underset{\sim}{Z})a_2(\kappa)/k! = \{4x^3\mathrm{tr}\ \underset{\sim}{Z}^3+3x^2\ \mathrm{tr}\ \underset{\sim}{Z}^2+3x^2(\mathrm{tr}\ \underset{\sim}{Z})^2 +x\ \mathrm{tr}\ \underset{\sim}{Z}\}e^{\mathrm{tr}(x\ \underset{\sim}{Z})},$$

$$(5.12)\quad \sum_{k=2}^{\infty}\sum_{\kappa} C_\kappa(\underset{\sim}{Z})/(k-1)! = (\mathrm{tr}\ \underset{\sim}{Z})e^{\mathrm{tr}\ \underset{\sim}{Z}},$$

and

$$(5.13)\quad \sum_{k=2}^{\infty}\sum_{\kappa} C_\kappa(\underset{\sim}{Z})/(k-2)! = (\mathrm{tr}\ \underset{\sim}{Z})^2\ e^{\mathrm{tr}\ \underset{\sim}{Z}}.$$

Lemma 5.2. With the notations of the lemma 5.1, for large n

$$(5.14)\quad (n/2)_\kappa=(n/2)^k[1+n^{-1}a_1(\kappa)+(1/6n^2)\{k-a_2(\kappa)+3(a_1(\kappa))^2\}+0(n^{-3})],$$

and

$$(5.15)\quad (n/2+a)_\kappa=(n/2)^k[1+(1/2n)\{4ak+2a_1(\kappa)\}+(1/24n^2)\{4k +48a^2k(k-1)+48a(k-1)a_1(\kappa)-4a_2(\kappa)+12(a_1(\kappa))^2\}+0(n^{-3})].$$

(b) Derivation of Asymptotic Expansions. We consider below asymptotic expansions of the distributions of (i) and (ii) above.

(i) Asymptotic expansion of the distribution of -2 log W. Let $\phi(t)$ be the characteristic function of -2 log W. Then from (5.2) we have

$$(5.16)\quad \phi(t) = E(e^{-2it\log W}) = E(W^{-2it}) = C_1(t)C_2(t)C_3(t)|\Lambda|^{-n_1/2}$$

where

$$(5.17)\quad C_1(t) = n^{-itpn}\, n_1^{itpn_1}\, n_2^{itpn_2},$$

$$(5.18)\quad C_2(t) = \Gamma_p(n/2)\Gamma_p(n_1g/2)\Gamma_p(n_2g/2)[\Gamma_p(ng/2)\Gamma_p(n_1/2)\Gamma(n_2/2)]^{-1}$$

and

$$(5.19)\quad g = (1-2it),\ C_3(t) = {}_2F_1(n/2,\ n_1g/2;\ ng/2, M).$$

We shall use the following asymptotic formula for the gamma function as in Anderson ([1], p. 204)

$$(5.20)\quad \log\Gamma(x+h)=\log\sqrt{2\pi}+(x+h-\tfrac{1}{2})\log x-x-\sum_{r=1}^{m}\frac{(-1)^rB_{r+1}(h)}{r(r+1)x^r} + 0(|x|^{-m-1}),$$

which holds for large $|x|$ and fixed h. The Bernoulli polynomial $B_r(h)$ of degree r is given by $(t\, e^{ht})/(e^t-1) = \sum_{r=0}^{\infty} (\frac{t^r}{r!}) B_r(h)$. Some of these which we shall need in the sequel, are listed below.

$$B_1(h) = h-1/2,\ B_2(h) = h^2-h+1/6,$$

(5.21)

$$B_3(h) = h^3-3h^2/2+h/2 \text{ and } B_4(h)=h^4-2h^3+h^2-1/30.$$

Applying the formula (5.20) to each gamma function in $C_2(t)$, we have

(5.22) $$\log C_2(t) = it\ pn \log(n/2)-it\ pn_2\log(n_2/2)-it\ pn_1\log(n_1/2)$$
$$- f \log(g)/2+(r/n)(g^{-1}-1)+(s/n^2)(1-g^{-2})+0(n^{-3}),$$

where

(5.23) $$f = p(p+1)/2,\ r = p(2p^2+3p-1)(\tau_1^{-1}+\tau_2^{-2}-1)/24$$

and

$$s = p(p+1)(2-p^2-p)(\tau_1^{-2}+\tau_2^{-2}-1)/48.$$

It therefore follows that

(5.24) $$C_1(t)C_2(t) = g^{-f/2}\exp[(r/n)(g^{-1}-1)+(s/n^2)(1-g^{-2})+0(n^{-3})]$$
$$= g^{-f/2}[1+(r/n)(g^{-1}-1)+n^{-2}\{s(1-g^{-2})+(r^2/2)(g^{-1}-1)^2\}$$
$$+ 0(n^{-3})]\ .$$

Let $\underset{\sim}{M} = [I - \underset{\sim}{\Lambda}^{-1}] = (2/n)\, \underset{\sim}{P}$ where $\underset{\sim}{P}$ is a fixed matrix. Then

$$(5.25) \qquad |\underset{\sim}{A}|^{-n_1/2} = |\underset{\sim}{I} - \frac{2}{n}\underset{\sim}{P}|^{\tau_1 n/2} .$$

Now using the expansion

$$(5.26) \quad \log|\underset{\sim}{I} - \frac{2}{n}\underset{\sim}{P}| = -(2/n)\mathrm{tr}\,\underset{\sim}{P} - (2/n^2)\mathrm{tr}(\underset{\sim}{P})^2 - (8/3n^3)\mathrm{tr}(\underset{\sim}{P}^3) + 0(n^{-4})$$

we obtain

$$(5.27) \quad |\underset{\sim}{I} - \frac{2}{n}\underset{\sim}{P}|^{\tau_1 n/2} = \exp[(\tau_1 n/2)\log|\underset{\sim}{I} - \frac{2}{n}\underset{\sim}{P}|]$$

$$= e^{(\tau_1 n/2)[-(2/n)\mathrm{tr}\,\underset{\sim}{P} - (2/n^2)\mathrm{tr}(\underset{\sim}{P}^2) - (8/3n^3)\mathrm{tr}(\underset{\sim}{P}^3) + 0(n^{-4})]}$$

$$= e^{-\tau_1 \mathrm{tr}\,\underset{\sim}{P}}[1 - n^{-1}A_1 - n^{-2}A_2 + 0(n^{-3})]$$

where

$$(5.28) \quad A_1 = \tau_1\ \mathrm{tr}(\underset{\sim}{P}^2) \text{ and } A_2 = (4/3)\tau_1 \mathrm{tr}\ (\underset{\sim}{P}^3) - \tau_1^2(\mathrm{tr}\ \underset{\sim}{P}^2)^2/2 .$$

Applying asymptotic formula (5.14) to $(n/2)\kappa$, $(n_1 g/2)\kappa$ and $(ng/2)_\kappa$ we have after some algebraic simplication,

$$(5.29) \quad (n/2)\kappa\ (n_1 g/2)\kappa/(ng/2)\kappa$$

$$= (n\tau_1/2)^k[1 + n^{-1}a_1(\kappa)B(t) + (1/6n^2)\{((k - a_2(\kappa) + 3(a_1(\kappa))^2)$$

$$A(t) - g^{-2}(k - a_2(\kappa) - 3(a_1(\kappa))^2) - D(t)(a_1(\kappa))^2\} + 0(n^{-3})]$$

where

$$(5.30) \quad A(t) = 1 + (\tau_1 g)^{-2},\ B(t) = 1 + (\tau_1^{-1} - 1)/g \text{ and}$$

$$D(t) = 6[g^{-1} + (\tau_1 g^2)^{-1} - (\tau_1 g)^{-1}] .$$

Using (5.29) and the lemma 5.1, we have on simplification,

$$(5.31)\quad C_3(t) = \sum_{k=0}^{\infty}\sum_{\kappa} \frac{(n/2)_\kappa (n_1 g/2)_\kappa C_\kappa(\frac{2}{n} \underset{\sim}{R})}{(ng/2)_\kappa \quad k!}$$

$$= e^{\tau_1 \mathrm{tr}\, \underset{\sim}{R}}[1+(K/n)B(t)+(1/6n^2)\{LA(t)-Mg^{-2}-ND(t)\}+0(n^{-3})]$$

where

$$K = \tau_1^2 \ \mathrm{tr}\ \underset{\sim}{R}^2,\ L = 8\tau_1^3 \ \mathrm{tr}\ \underset{\sim}{R}^3 + 3\tau_1^4(\mathrm{tr}\ \underset{\sim}{R}^2)^2,$$

$$(5.32)\quad M = -3\tau_1^4(\mathrm{tr}\ \underset{\sim}{R}^2)^2-16\tau_1^3\mathrm{tr}\ \underset{\sim}{R}^3-6\tau_1^2\{\mathrm{tr}\ \underset{\sim}{R}^2+(\mathrm{tr}\ \underset{\sim}{R})^2\},$$

and

$$N = \tau_1^4(\mathrm{tr}\ \underset{\sim}{R}^2)^2+4\tau_1^3\mathrm{tr}\ \underset{\sim}{R}^3+\tau_1^2\{\mathrm{tr}\ \underset{\sim}{R}^2+(\mathrm{tr}\ \underset{\sim}{R})^2\}.$$

From (5.16), (5.24), (5.27) and (5.31), we have

$$(5.33)\quad \phi(t) = g^{-f/2}[1+n^{-1}\{\alpha_0+g^{-1}\alpha_1\}+n^{-2}\{\alpha_2+g^{-1}\alpha_3+g^{-2}\alpha_4\}+0(n^{-3})],$$

where the coefficients $\alpha_i$'s are given by

$$\alpha_0 = K - A_1 - r,\ \alpha_1 = K(\tau_1^{-1}-1) + r,$$

$$\alpha_2 = L/6 - A_2 - KA_1 + s + r^2/2 - Kr + A_1 r,$$

$$(5.34)\quad \alpha_3 = r(K-A_1) - r^2 + (\tau_1^{-1}-1)(N-rK-A_1K),$$

and $$\alpha_4 = (L\ \tau_1^{-2}-M)/6 - N\ \tau_1^{-1}-s + r^2/2 + rK(\tau_1^{-1}-1)\ .$$

By inverting the characteristic function in (5.33), using the fact that $(g)^{-f/2}$ is the characteristic function of $\chi_f^2$, a chi square variable with f degrees of freedom, we obtain the following asymptotic expansion for the distribution of -2 log W.

$$(5.35)\quad P(-2\log W \le z) = P(\chi_f^2 \le z) + n^{-1}\{\alpha_0 P(\chi_f^2 \le z) + \alpha_1 P(\chi_{f+2}^2 \le z)\} + n^{-2}\{\alpha_2 P(\chi_f^2 \le z) + \alpha_3 P(\chi_{f+2}^2 \le z) + \alpha_4 P(\chi_{f+4}^2 \le z)\} + O(n^{-3}),$$

where $\alpha_i$'s are defined in (5.33).

(ii) Asymptotic expansion of the distribution of $-2\rho \log W$. Here we shall derive the asymptotic expansion for $-2\rho \log W$ where $\rho$ is given by (5.6). Put $m = \rho n$ and let $m$ tend to infinity instead of $n$. From (5.2), the characteristic function $f(t)$ of $-2\rho \log W$ can be written as

$$(5.36)\quad f(t) = E(e^{-2\rho\, it \log W}) = C_4(t)C_5(t),$$

where $C_4(t)$ and $C_5(t)$ are given by

$$(5.37)\quad C_4(t) = \frac{n^{-pnit\rho}}{n_1^{-pn_1it\rho}\, n_2^{-pn_2it\rho}} \frac{\Gamma_p(n/2)\Gamma_p[\frac{n_1(1-2\rho it)}{2}]\Gamma_p[\frac{n_2(1-2it\rho)}{2}]}{\Gamma_p[\frac{n(1-2\rho it)}{2}]\Gamma_p(n_1/2)\Gamma_p(n_2/2)}$$

and

$$(5.38)\quad C_5(t) = |\underset{\sim}{\Lambda}|^{-\frac{\tau_1}{2}(m+2\alpha)}\, {}_2F_1(\frac{m}{2} + \alpha,\ \frac{mg\tau_1}{2} + \alpha\tau_1;\ \frac{mg}{2} + \alpha,\ \underset{\sim}{M}),$$

$g$ and $\alpha$ being as defined in (5.19) and (5.6) respectively. Now the first factor $C_4(t)$ in (5.36) can be expanded asymptotically (See Anderson [1], p. 255) as follows.

$$(5.39)\quad C_4(t) = g^{-f/2}[1 + (A/m^2)\,(g^{-2}-1) + O(m^{-3})]$$

where

(5.40) $f = p(p+1)/2$, $A = [p(p+1)/48][(p-1)(p+2)(\tau_1^{-2}+\tau_2^{-2}-1)-6\gamma]$

and $\gamma = (\tau_1^{-1}+\tau_2^{-1}-1)^2\ (2p^2+3p-1)^2/36(p+1)^2 = 4\alpha^2$.

Now as stated before, let

$$\underset{\sim}{I} - \underset{\sim}{\Lambda}^{-1} = (2/m)\ \underset{\sim}{P},$$

where $\underset{\sim}{P}$ is a fixed matrix. We then have

$$(5.41)\quad C_5(t) = |\underset{\sim}{I} - \tfrac{2}{m}\underset{\sim}{P}|^{\frac{\tau_1(m+2\alpha)}{2}}\ {}_2F_1(\tfrac{m}{2} + \alpha,\ \tfrac{m\tau_1 g}{2} + \alpha\tau_1;\ \tfrac{mg}{2} + \alpha, \tfrac{2}{m}\underset{\sim}{P}).$$

Using the asymptotic expansion (5.15) to $(\frac{m}{2} + \alpha)_\kappa$, $(m\tau_1 g/2 + \alpha\tau_1)_\kappa$ and $(mg/2 + \alpha)_\kappa$, we have

$$(5.42)\quad (m/2 + \alpha)_\kappa(m\tau_1 g/2 + \alpha\tau_1)\kappa/(mg/2 + \alpha)\kappa$$

$$= (m\tau_1/2)^k[1+m^{-1}\{2\alpha k+\delta_1 a_1(\kappa)\}+m^{-2}\{k\delta_2+k^2\delta_3+\delta_4 a_1(\kappa)$$

$$+ \delta_5 k a_1(\kappa) + \delta_6 a_2(\kappa)+\delta_7(a_1(\kappa))^2\} + 0(m^{-3})],$$

where

$$\delta_1 = 1 + (\tau_1^{-1}-1)g^{-1}, \delta_2 = -2\alpha^2+[1+(\tau_1^{-2}-1)g^{-2}]/6,$$

$$\delta_3 = 2\alpha^2,$$

$$(5.43)\quad \delta_4 = -2\alpha + 2\alpha g^{-2}(1-\tau_1^{-1}),\ \delta_5 = 2\alpha + 2\alpha(\tau_1^{-1}-1)g^{-1},$$

$$\delta_6 = \{(1-\tau_1^{-2})g^{-2}-1\}/6 \text{ and } \delta_7 = \{1 + (\tau_1^{-1}-1)g^{-1}\}^2/2.$$

From (5.42) and the lemma 5.1, it then easily follows that,

$$(5.44)\quad {}_2F_1\left(\frac{m}{2}+\alpha,\ \frac{m\tau_1 g}{2}+\alpha\tau_1;\ \frac{mg}{2}+\alpha,\ \frac{2}{m}\underset{\sim}{P}\right)$$

$$= e^{\tau_1 \operatorname{tr} \underset{\sim}{P}}[1+m^{-1}(2a\alpha+b\delta_1)+m^{-2}(a\delta_2+a(a+1)\delta_3+b\delta_4+c\delta_5+d\delta_6+e\delta_7)+0(m^{-3})],$$

where the constants a, b, c, d and e are given by

$$a = \tau_1 \operatorname{tr} \underset{\sim}{P};\ b = \tau_1^2 \operatorname{tr} \underset{\sim}{P}^2;\ c = 2\tau_1^2 \operatorname{tr} \underset{\sim}{P}^2 + \tau_1^3 \operatorname{tr} \underset{\sim}{P}^2 \operatorname{tr} \underset{\sim}{P},$$

$$(5.45)\quad d = 4\,\tau_1^3 \operatorname{tr} \underset{\sim}{P}^3 + 3\,\tau_1^2 \operatorname{tr} \underset{\sim}{P}^2 + 3\,\tau_1^2(\operatorname{tr} \underset{\sim}{P})^2 + \tau_1 \operatorname{tr} \underset{\sim}{P}$$

and $$e = \tau_1^4(\operatorname{tr} \underset{\sim}{P}^2)^2 + 4\,\tau_1^3 \operatorname{tr} \underset{\sim}{P}^3 + \tau_1^2[\operatorname{tr} \underset{\sim}{P}^2 + (\operatorname{tr} \underset{\sim}{P})^2].$$

Also

$$(5.46)\quad |\underset{\sim}{I} - \frac{2}{m}\underset{\sim}{P}|^{\frac{\tau_1 m}{2}+\alpha\tau_1} = |\underset{\sim}{I} - \frac{2}{m}\underset{\sim}{P}|^{\tau_1 m/2}\,|\underset{\sim}{I} - \frac{2}{m}\underset{\sim}{P}|^{\alpha\tau_1}$$

and using (5.26) and (5.27) to the factor on the right hand side of (5.46), it can be easily checked that

$$(5.47)\quad |\underset{\sim}{I} - \frac{2}{m}\underset{\sim}{P}|^{\frac{\tau_1 m}{2}+\alpha\tau_1} = e^{-\tau_1 \operatorname{tr} \underset{\sim}{P}}[1-m^{-1}\delta_8+m^{-2}\delta_9+0(m^{-3})],$$

where

$$\delta_8 = \tau_1\{\operatorname{tr} \underset{\sim}{P}^2+2\alpha \operatorname{tr} \underset{\sim}{P}\}$$

and $$\delta_9 = \delta_8^2/2-\tau_1\{4 \operatorname{tr} \underset{\sim}{P}^3/3+2\alpha \operatorname{tr} \underset{\sim}{P}^2\}.$$

We therefore have

$$(5.48)\quad C_5(t) = [1+m^{-1}(2a\alpha+b\delta_1-\delta_8)+m^{-2}(a\delta_2+a(a+1)\delta_3+b\delta_4+c\delta_5$$
$$+d\delta_6+e\delta_7+\delta_9-2a\alpha\delta_8-b\delta_1\delta_8)+0(m^{-3})]$$

and finally from (5.36), (5.39) and (5.48) we have

(5.49) $$f(t)=g^{-f/2}[1+m^{-1}\{\alpha_0+\alpha_1 g^{-1}\}+m^{-2}\{\beta_0+\beta_1 g^{-1}+\beta_2 g^{-2}\}+0(m^{-3})],$$

where

$$\alpha_0 = 2a\alpha+b-\delta_8,\ \alpha_1 = b(\tau_1^{-1}-1),$$

$$\beta_0 = -A+2\alpha^2a^2+(a-d)/6-2\alpha(b-c+a\delta_8)+e/2+\delta_9-b\delta_8,$$

(5.50) $$\beta_1 = (\tau_1^{-1}-1)\ (2c\alpha+e-b\delta_8)$$

and $$\beta_2 = A+(\tau_1^{-2}-1)\ (a-d)/6+2b\alpha(1-\tau_1^{-1})+e(\tau_1^{-1}-1)^2/2.$$

Inverting the characteristic function in (5.49), we have the asymptotic expansion of the distribution of $-2\rho \log W$ in the form,

(5.51) $$P(-2\rho \log W \leq z) = P_r(\chi_f^2 \leq z) + m^{-1}\{\alpha_0 P(\chi_f^2 \leq z) + \alpha_1 P(\chi_{f+2}^2 \leq z)\}+m^{-2}\{\beta_0 P(\chi_f^2 \leq z)+\beta_1 P(\chi_{f+2}^2 \leq z) + \beta_2 P(\chi_{f+4}^2 \leq z)\} + 0(m^{-3}).$$

## 6. ASYMPTOTIC EXPANSION OF THE DISTRIBUTION OF Y, a=1 AND b=0

For this case putting a = 1 and b = 0 in (2.6) we can easily see that

(6.1) $$E(Y^h) = \frac{\Gamma_p(n/2)\Gamma_p(n_1/2+h)}{\Gamma_p(n/2+h)\Gamma_p(n_1/2)} |\underset{\sim}{\Lambda}|^{-n_1/2}\ {}_2F_1(n/2,n_1/2+h;n/2+h,\underset{\sim}{M}).$$

We assume (5.3) and obtain the asymptotic expansion of $L_1$ where

(6.2) $$L_1 = \sqrt{n} \log (Y/\tau_1^p)$$

in terms of n increasing with $\tau_1$ and $\tau_2$ fixed assuming that $\underset{\sim}{M} = [\underset{\sim}{I} - \underset{\sim}{\Lambda}^{-1}] = (2/n)\, \underset{\sim}{P}$ where $\underset{\sim}{P}$ is a fixed matrix. Let $\chi(t)$ be the characteristic function of $L_1$. Then

$$\chi(t) = E(e^{it\, L_1}) = C_6(t)C_7(t), \tag{6.3}$$

where

$$C_6(t) = (1/\tau_1)^{it\sqrt{n}p}\Gamma_p(n/2)\Gamma_p(n_1/2+it\sqrt{n})[\Gamma_p(n/2+it\sqrt{n})\Gamma_p(n_1/2)]^{-1} \tag{6.4}$$

and

$$C_7(t) = |\underset{\sim}{\Lambda}|^{-n_1/2}\ {}_2F_1(n/2, n_1/2+it\sqrt{n};\ n/2+it\sqrt{n},\ \underset{\sim}{M}). \tag{6.5}$$

Using the formula (5.18) to each gamma function on the right hand side of (6.4), we have

$$\begin{aligned} C_6(t) = e^{-pT_1t^2}[1-n^{-1/2}\{fT_1(it)+2pT_2(it)^3/3\} \\ + n^{-1}\{(fT_2+f^2T_1^2/2)\ (it)^2 \\ + 2p(T_3+fT_1T_2)\ (it)^4/3+2p^2T_2^2\ (it)^6/9\}+0(n^{-3/2})], \end{aligned} \tag{6.6}$$

where

$$T_1 = (\tau_1^{-1}-1),\ T_2 = \tau_1^{-2}-1,\ T_3 = \tau_1^{-3}-1 \text{ and } f = p(p+1)/2. \tag{6.7}$$

Now using lemma 5.2 to $(n/2)\kappa$, $(n/2 + it\sqrt{n})\kappa$ and $(n_1/2 + it\sqrt{n})\kappa$ we have

$$\begin{aligned} &(n/2)_\kappa(n_1/2+it\sqrt{n})_\kappa/(n/2 + it\sqrt{n})_\kappa \\ &= (n_1/2)^k[1+n^{-1/2}2\, it\, k\, T_1+n^{-1}\{\tau_1^{-1}a_1(\kappa)+2(it)^2(k^2T_1^2-k\, T_2)\} \\ &\qquad + 0(n^{-3/2})]. \end{aligned} \tag{6.8}$$

As before let $\underset{\sim}{M} = (2/n)\underset{\sim}{P}$ where $\underset{\sim}{P}$ is a fixed matrix. Then from (6.8) and the lemma 5.1, we have after a little simplification,

$$(6.9) \quad {}_2F_1(n/2, n_1/2 + it\sqrt{n};\ n/2 + it\sqrt{n},\ 2/n\ \underset{\sim}{P})$$

$$= e^{\tau_1 \mathrm{tr}\ \underset{\sim}{P}}[1-n^{-1/2}(it)\ A_1+n^{-1}\{(it)^2\ A_2+q\}+0(n^{-3/2})],$$

where

$$(6.10) \quad A_1 = -2T_1\ \mathrm{tr}(\tau_1\ \underset{\sim}{P}),\ q = \tau_1\ \mathrm{tr}\ \underset{\sim}{P}^2$$

and $\quad A_2 = 2\{T_1^2[(\mathrm{tr}\ \tau_1\ \underset{\sim}{P})^2+\mathrm{tr}(\tau_1\ \underset{\sim}{P})]-T_2(\tau_1\ \mathrm{tr}\ \underset{\sim}{P})\}.$

Also from (5.27) we have

$$(6.11) \quad |\underset{\sim}{A}|^{-n_1/2} = |\underset{\sim}{I} - 2/n\ \underset{\sim}{P}|^{\tau_1 n/2} = e^{-\tau_1\ \mathrm{tr}\ \underset{\sim}{P}}[1-n^{-1}q+0(n^{-3/2})]$$

and thus

$$(6.12) \quad C_7(t) = [1-n^{-1/2}(it)\ A_1+n^{-1}(it)^2A_2+0(n^{-3/2})].$$

From (6.3), (6.6) and (6.12), we obtain the following asymptotic expansion for $\chi(t)$.

$$(6.13) \quad \chi(t/\sqrt{2pT_1}) = e^{-t^2/2}\ [1-n^{-1/2}\ D_1+n^{-1}\ D_2+0(n^{-3/2})],$$

where the coefficients $D_1$ and $D_2$ are given by

$$D_1 = (2pT_1)^{-\frac{1}{2}}[(it)(A_1+fT_1) + (3T_1)^{-1}\ T_2(it)^3] \text{ and}$$

$$(6.14) \quad D_2 = (2pT_1)^{-3}\ 4p^2[(it)^2(fT_2+f^2T_1^2/2+fT_1A_1 + A_2)T_1^2$$

$$+ (it)^4\ T_1(T_3+fT_1T_2+A_1T_2)/3+(it)^6\ T_2^2/18].$$

(7.5) $C_9(t) = |\Lambda|^{-n_1/2} \ {}_2F_1(n/2, n_1/2; \frac{n}{2} + it\sqrt{n}, M)$.

Using the formula (5.17) to each gamma function on the right-hand side of (7.4) we get

$$(7.6)\quad C_8(t)=e^{-pR_1t^2}[1-n^{-\frac{1}{2}}\{fR_1(it)+2pR_2(it)^3/3\}$$

$$+ n^{-1}\{(fR_2+f^2R_1^2/2)(it)^2$$

$$+ 2p(R_3+fR_1R_2)(it)^4/3+2p^2R_2^2(it)^6/9\}+0(n^{-3/2})],$$

where coefficients $R_1$, $R_2$, $R_3$ and $R_4$ are given by

$$(7.7)\quad R_1 = \tau_2^{-1}-1,\ R_2 = \tau_2^{-2}-1,\ R_3 = \tau_2^{-3}-1 \text{ and } f = p(p+1)/2.$$

Using lemma 5.1 and 5.2, we have proceeding as in section 6

$$(7.8)\quad {}_2F_1(n/2,\ n_1/2;\ \frac{n}{2} + it\sqrt{n},\ \frac{2}{n} P)$$

$$= e^{\tau_1 \operatorname{tr} P}[1-n^{-\frac{1}{2}}(it)B_1+n^{-1}\{B_2+(it)^2B_3\}+0(n^{-3/2})],$$

where $B_1 = 2(\operatorname{tr} \tau_1 P)$, $B_2 = \tau_1 \operatorname{tr} P^2$ and $B_3 = B_1(4 + B_1)/2$.

Using (7.8) and (6.11), we can write $C_9(t)$ as

$$(7.9)\quad C_9(t) = [1-n^{-\frac{1}{2}}(it)B_1+n^{-1}(it)^2 B_3+0(n^{-3/2})]$$

and thus we have the following asymptotic expansion for $H(t)$.

$$(7.10)\quad H(t/\sqrt{2R_1p}) = e^{-t^2/2}[1-n^{-\frac{1}{2}} \beta_1+n^{-1} \beta_2+0(n^{-3/2})],$$

# CHAPTER II

# ON THE DISTRIBUTION OF THE SPHERICITY TEST CRITERION IN CLASSICAL AND COMPLEX NORMAL POPULATIONS HAVING UNKNOWN COVARIANCE MATRICES

## 1. INTRODUCTION AND SUMMARY

Let $\underset{\sim}{x}$: px1 be distributed $N(\underset{\sim}{\mu},\underset{\sim}{\Sigma})$ where $\underset{\sim}{\mu}$ and $\underset{\sim}{\Sigma}$ are both unknown. Let $\underset{\sim}{S}$ be the sum of product matrix of a sample of size N. To test the hypothesis of sphericity, namely, $H_0$: $\underset{\sim}{\Sigma} = \sigma^2 \underset{\sim}{I}_p$, where $\sigma^2 > 0$ is unknown, against $H_1$: $\underset{\sim}{\Sigma} \neq \sigma^2 \underset{\sim}{I}_p$, Mauchly [20] obtained the likelihood ratio test criterion for $H_0$ in the form $W = |\underset{\sim}{S}|/[(\text{tr}\ \underset{\sim}{S})/p]^p$. Thus the criterion W is a power of the ratio of the geometric mean and the arithmetic mean of the roots $\theta_1, \theta_2,\ldots,\theta_p$ of $|\underset{\sim}{S}-\theta\ \underset{\sim}{I}| = 0$ (see Anderson [1]). For p = 2, Mauchly [20] showed that the density of W is

$$(1.1) \qquad f(w) = \frac{1}{2}(n-1)w^{\frac{1}{2}(n-3)}, \qquad 0 \leq w \leq 1,$$

where n = N-1. The exact distribution in the null case was obtained by Consul [5], [6], in the form

$$(1.2) \quad f(w) = k(p,n)w^{\frac{1}{2}(n-p-1)} G^{p,0}_{p,p}\left(w\,\middle|\,\begin{matrix}\frac{1}{2}(p-1)+(p-1)/p,\ldots,\frac{1}{2}(p-1)\\ 0,\frac{1}{2},\ 1,\ldots,\ \frac{1}{2}(p-1)\end{matrix}\right),$$

where

$$k(p,n) = (2\pi)^{\frac{1}{2}(p-1)} p^{\frac{1}{2} - \frac{1}{2}pn} \Gamma(\tfrac{1}{2}pn) / \prod_{j=1}^{p} \Gamma[\tfrac{1}{2}(n-j-1)],$$

and $G^{m,n}_{p,q}\left(x \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right)$ is the G-function defined in the next section.

In this chapter we have obtained the general moments of W both in real and complex cases for arbitrary covariance matrices and also the corresponding distributions of W in terms of G-function.

## 2. SOME DEFINITIONS AND RESULTS

In this section we give a few definitions and some lemmas which are needed in the sequel. First we define Meijer's G-function by [21]

$$(2.1)\quad G^{m,n}_{p,q}\left(x\middle|\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right) = (2\pi i)^{-1} \int_C \frac{\prod_{j=1}^{m} \Gamma(b_j-s) \prod_{j=1}^{n} \Gamma(1-a_j+s)}{\prod_{j=m+1}^{q} \Gamma(1-b_j+s) \prod_{j=n+1}^{p} \Gamma(a_j-s)} x^s ds,$$

where an empty product is interpreted as unity and C is a curve separating the singularities of $\prod_{j=1}^{m} \Gamma(b_j-s)$ from those of $\prod_{j=1}^{n} \Gamma(1-a_j+s)$, $q \geq 1$, $0 \leq n \leq p \leq q$, $0 \leq m \leq q$; $x \neq 0$ and $|x| < 1$ if $q = p$; $x \neq 0$ if $q > p$.

The G-function of (2.1) can be expressed as a finite number of generalized hypergeometric functions (see Pillai, Al-Ani and Jouris [25] and Luke [18]) and in particular we have

$$(2.3)\quad G^{2,0}_{2,2}\left(x\,\middle|\,{a_1,a_2 \atop b_1,b_2}\right) =$$

$$\frac{x^{b_1}(1-x)^{a_1+a_2-b_1-b_2-1}}{\Gamma(a_1+a_2-b_1-b_2)}\ {}_2F_1(a_2-b_2,a_1-b_2,a_1+a_2-b_1-b_2,1-x)$$

$$0 < x < 1$$

where ${}_2F_1$ here is the Gauss hypergeometric function.

Now we state the Gauss and Legendre's multiplication formula for gamma functions as

$$(2.4)\quad \prod_{r=1}^{n} \Gamma[z+(r-1)/n] = (2\pi)^{\frac{1}{2}(n-1)}\, n^{\frac{1}{2}-nz}\, \Gamma(nz).$$

Further, the hypergeometric function of matrix variates is defined by

$${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;\underset{\sim}{S},\underset{\sim}{T}) = \sum_{k=0}^{\infty}\sum_{\kappa} \frac{(a_1)_\kappa\cdots(a_p)_\kappa C_\kappa(\underset{\sim}{S})C_\kappa(\underset{\sim}{T})}{(b_1)_\kappa\cdots(b_q)_\kappa C_\kappa(\underset{\sim}{I}_m)k!}$$

where the zonal polynomials, $C_\kappa(\cdot)$, and $(\cdot)_\kappa$ are defined in [12].

<u>Lemma 2.1.</u> Let $\underset{\sim}{Z}$: mxm be a complex symmetric matrix whose real part is p.d. and let $\underset{\sim}{T}$: mxm be an arbitrary complex symmetric matrix. Let $\underset{\sim}{S}$: mxm be a real p.d. matrix. Then

$$(2.5)\quad \int_{\underset{\sim}{S}>0} \exp(-\mathrm{tr}\,\underset{\sim}{Z}\,\underset{\sim}{S})|\underset{\sim}{S}|^{t-\frac{1}{2}(m+1)} C_\kappa(\underset{\sim}{T}\,\underset{\sim}{S})d\,\underset{\sim}{S} = \Gamma_m(t,\kappa)|\underset{\sim}{Z}|^{-t}C_\kappa(\underset{\sim}{T}\,\underset{\sim}{Z}^{-1})$$

where $\Gamma_m(t,\kappa)$ is defined in (15) of Constantine [4] and $R(t) > \frac{1}{2}(m-1)$. (See Constantine [4]).

Lemma 2.2. Let $\underset{\sim}{T}$ be as in lemma 2.1. Then

$$(2.6)\quad \int_{\underset{\sim}{S}>0} \exp(-\tfrac{1}{2}\,\mathrm{tr}\,\underset{\sim}{S})|\underset{\sim}{S}|^{t-\frac{1}{2}(m+1)}(\mathrm{tr}\,\underset{\sim}{S})^q\, C_\kappa(\underset{\sim}{T}\,\underset{\sim}{S})d\,\underset{\sim}{S}$$

$$= \Gamma_m(t,\kappa)2^{tm+k+q}\,\Gamma(mt+k+q)C_\kappa(T)/\Gamma(mt+k).$$

Proof. We shall consider the cases when (1) $q \geq 0$ and (2) $q < 0$.

(1) $q \geq 0$. From (2.5), for $u > 0$ we have

$$(2.7)\quad \int_{\underset{\sim}{S}>0} \exp(-\tfrac{1}{2}\,u\,\mathrm{tr}\,\underset{\sim}{S})|\underset{\sim}{S}|^{t-\frac{1}{2}(m+1)}\, C_\kappa(\underset{\sim}{T}\,\underset{\sim}{S})d\,\underset{\sim}{S}$$

$$= 2^{tm+k}\,u^{-tm-k}\,\Gamma_m(t,\kappa)C_\kappa(\underset{\sim}{T}).$$

To prove this case we differentiate (2.7) q times w.r.t. u under the integral sign and let $u = 1$ to obtain

$$(2.8)\quad \int_{\underset{\sim}{S}>0} \exp(-\tfrac{1}{2}\,\mathrm{tr}\,\underset{\sim}{S})|\underset{\sim}{S}|^{t-\frac{1}{2}(m+1)}(\mathrm{tr}\,\underset{\sim}{S})^q\, C_\kappa(\underset{\sim}{T}\,\underset{\sim}{S})d\,\underset{\sim}{S}$$

$$= 2^{tm+k+q}\,\Gamma_m(t,\kappa)\Gamma(tm+k+q)C_\kappa(\underset{\sim}{T})/\Gamma(mt+k).$$

which is also (19) of Khatri [15].

(2) $q < 0$. To prove this case, we integrate (2.7) successively r times w.r.t. u, change the order of integration and let $u = 1$, yielding

$$(2.9)\quad \int_{\underset{\sim}{S}>0} \exp(-\tfrac{1}{2}\,\mathrm{tr}\,\underset{\sim}{S})|\underset{\sim}{S}|^{t-\frac{1}{2}(m+1)}(\mathrm{tr}\,\underset{\sim}{S})^{-r}\, C_\kappa(\underset{\sim}{T}\,\underset{\sim}{S})d\,\underset{\sim}{S}$$

$$= 2^{tm+k-r}\,\Gamma(tm+k-r)\Gamma_m(t,\kappa)C_\kappa(\underset{\sim}{T})/\Gamma(tm+k).$$

Since $\Gamma(tm+k-r)/\Gamma(tm+k) = \prod_{j=1}^{n} (tm+k-j)^{-1}$, (2.9) holds if $tm+k-r > 0$. This proves the lemma.

Lemma 2.3. Let $\underset{\sim}{Z}$: mxm be a complex symmetric matrix whose real part is p.d. and let $\underset{\sim}{T}$: mxm be an arbitrary complex symmetric matrix and $\underset{\sim}{S}$: mxm be a Hermitian matrix. Then

$$(2.10) \quad \int_{\bar{\underset{\sim}{S}}'=\underset{\sim}{S}>0} \exp(-\mathrm{tr}\, \underset{\sim}{Z}\, \underset{\sim}{S}) |\underset{\sim}{S}|^{a-m} \tilde{C}_\kappa(\underset{\sim}{T}\, \underset{\sim}{S}) d\, \underset{\sim}{S} = \tilde{\Gamma}_m(a,\kappa) |\underset{\sim}{Z}|^{-a} \tilde{C}_\kappa(\underset{\sim}{T}),$$

where $\tilde{\Gamma}_m(a,\kappa)$ is defined in [12].

Lemma 2.4. Let $\underset{\sim}{T}$ and $\underset{\sim}{S}$ be as in lemma 2.3. Then

$$(2.11) \quad \int_{\bar{\underset{\sim}{S}}'=\underset{\sim}{S}>0} \exp(-\mathrm{tr}\, \underset{\sim}{S}) |\underset{\sim}{S}|^{a-m} (\mathrm{tr}\, \underset{\sim}{S})^j \tilde{C}_\kappa(\underset{\sim}{T}\, \underset{\sim}{S}) d\, \underset{\sim}{S} = \tilde{\Gamma}_m(a,\kappa)\Gamma(am+k+j)\tilde{C}_\kappa(\underset{\sim}{T})/\Gamma(am+k).$$

Proof. The proof is exactly similar to the proof of lemma 2.2 and hence omitted.

Lemma 2.5. If s is any complex variate and f(x) is a function of a real variable x, such that

$$F(s) = \int_0^\infty x^{s-1} f(x) dx$$

exists, then under certain regularity conditions

$$f(x) = (2\pi i)^{-1} \int_{C-i\infty}^{C+i\infty} x^{-s} F(s) ds.$$

F(s) is called the Mellin transform of f(x), and f(x) is the inverse Mellin transform of F(s). (See Titchmarsh [29])

## 3. DISTRIBUTION OF W IN THE REAL CASE

Let $S$: pxp be distributed as Wishart (n, p, $\Sigma$). Then the distribution of the latent roots $g_1, g_2, \ldots, g_p$ of $S$ has been shown by James [12] to depend only on the latent roots of $\Sigma$ and is given by

$$(3.1) \quad k(p,n,\Sigma)|\Sigma|^{-\frac{1}{2}n} {}_0F_0(-\tfrac{1}{2}\Sigma^{-1}, G)|G|^{\frac{1}{2}(n-p-1)} \prod_{i<j}(g_i-g_j) \prod_{i=1}^{p} dg_i ,$$

where

$$k(p,n,\Sigma) = |\Sigma|^{-\frac{1}{2}n} \pi^{\frac{1}{2}p^2} / \{2^{\frac{1}{2}pn} \Gamma_p(\tfrac{1}{2}n)\Gamma(\tfrac{1}{2}p)\},$$

$$G = \text{diag}(g_1, g_2, \ldots, g_p), \ \infty > g_1 \geq g_2 \geq \ldots \geq g_p > 0 .$$

The distribution (3.1) is not convenient for further development and the convergence of the series is slow. But the convergence may be improved by writing (3.1) in the form suggested by Pillai, Al-Ani and Jouris [25]),

$$(3.2) \quad k(p,n,\Sigma)|G|^{\frac{1}{2}(n-p-1)} \exp(-\tfrac{1}{2}\text{tr } G) \prod_{i<j}(g_i-g_j)\, {}_0F_0(M, G)$$

where

$$M = \tfrac{1}{2}(I - \Sigma^{-1}) .$$

Theorem 3.1. Let $\underset{\sim}{G}$ be distributed as in (3.2) and let $W = |\underset{\sim}{G}|/\{(\text{tr } \underset{\sim}{G})/p\}^p$ be the sphericity criterion. Then the h-th moment of W is given by

$$(3.3) \quad E(W^h) = \frac{p^{ph}|\underset{\sim}{\Sigma}|^{-\frac{1}{2}n}}{2^{\frac{1}{2}pn}\,\Gamma_p(\frac{1}{2}n)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{C_\kappa(\underset{\sim}{M})}{k!} 2^k \frac{\Gamma_p(\frac{1}{2}n+h,k)\Gamma(\frac{1}{2}pn+k)}{\Gamma(\frac{1}{2}pn+ph+k)} .$$

Proof. To find $E(W^h)$ we multiply (3.2) by $|\underset{\sim}{G}|/[(\text{tr } \underset{\sim}{G})/p]^p$, transform $\underset{\sim}{G} \to \underset{\sim}{H}\, \underset{\sim}{V}\, \underset{\sim}{H}'$ where $\underset{\sim}{H}$ is an orthogonal and $\underset{\sim}{V}$ a symmetric matrix, integrate out $\underset{\sim}{H}$ and $\underset{\sim}{V}$ using (44) and (22) of Constantine [4]. We get

$$(3.4) \quad E(W^h) = p^{ph} k(p,n,\underset{\sim}{\Sigma})\Gamma_p(\tfrac{1}{2}p)\pi^{-\frac{1}{2}p^2}$$

$$\cdot \sum_{k=0}^{\infty} \sum_{\kappa} [C_\kappa(\underset{\sim}{M})/C_\kappa(\underset{\sim}{I}_p)k!]$$

$$\int_{\underset{\sim}{V}>0} \exp(-\tfrac{1}{2}\text{ tr } \underset{\sim}{V})|\underset{\sim}{V}|^{(\frac{1}{2}n+h)-\frac{1}{2}(p+1)} (\text{tr } \underset{\sim}{V})^{-ph} C_\kappa(\underset{\sim}{V})d\underset{\sim}{V}.$$

Applying lemma (2.2) to the integral on the R.H.S. of (3.4) we get (3.3).

Theorem 3.2. For any finite p, the p.d.f. of W is

$$(3.5) \quad f(w) = C(p,n,\underset{\sim}{\Sigma}) \sum_{k=0}^{\infty} \sum_{\kappa} \frac{2^k C_\kappa(\underset{\sim}{M})}{k!} p^{\frac{1}{2}-\frac{1}{2}pn-k} \Gamma(\tfrac{1}{2}pn+k)$$

$$\cdot\, w^{\frac{1}{2}(n-p-1)} G^{p,0}_{p,p}\left(w\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_p\end{matrix}\right) ,$$

where

$$C(p,n,\Sigma) = \pi^{\frac{1}{4}p(p-1)} |\Sigma|^{-\frac{1}{2}n} (2\pi)^{\frac{1}{2}(p-1)} / \Gamma_p(\tfrac{1}{2}n),$$

$$a_j = (k+j-1)/p+\tfrac{1}{2}(p-1);\ b_j = k_j+\tfrac{1}{2}(p-j).$$

For p = 2, (3.5) reduces to

$$\text{(3.6)}\quad f(w) = \frac{|\Sigma|^{-\frac{1}{2}n}}{2\Gamma(n-1)} w^{\frac{1}{2}(n-3)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\Gamma(n+k)}{k!} C_\kappa(M) w^{k_1+\frac{1}{2}}$$

$$\cdot\ {}_2F_1(a_2-b_2, a_1-b_2;\ a_1+a_2-b_1-b_2, 1-w).$$

Proof. Applying (2.4) on $\Gamma[p(\frac{1}{2}n+h+k/p)]$ we have from (3.3)

$$E(W^h) = C(p,n,\Sigma) \sum_{k=0}^{\infty} \sum_{\kappa} [\{2^k C_\kappa(M) p^{\frac{1}{2}-\frac{1}{2}pn-k} \Gamma(\tfrac{1}{2}pn+k)\}/k!]$$

$$\prod_{j=1}^{p} [\Gamma\ \tfrac{1}{2}n+h+k_j-\tfrac{1}{2}(i-j)\}/\Gamma\{\tfrac{1}{2}n+((k+j-1)/p)+h\}].$$

Using Lemma 2.5, the density of W has the form

$$\text{(3.7)}\quad f(w) = C(p,n,\Sigma) \sum_{k=0}^{\infty} \sum_{\kappa} \frac{2^k C_\kappa(M)}{k!} p^{\frac{1}{2}-\frac{1}{2}pn-k} \Gamma(\tfrac{1}{2}pn+k) w^{\frac{1}{2}(n-p-1)}$$

$$\cdot\ (2\pi i)^{-1} \int_{C-i\infty}^{C+i\infty} w^{-r} \frac{\prod_{i=1}^{p} \Gamma(r+b_i)}{\prod_{i=1}^{p} \Gamma(r+a_i)}\, dr,$$

where

$$r = \tfrac{1}{2}n+h-\tfrac{1}{2}(p-1),\ b_j = k_j+\tfrac{1}{2}(p-j), a_j = (k+j-1)/p+\tfrac{1}{2}(p-1).$$

Noting that the integral in (3.7) is in the form of Meijer's G-function, we can write the density of W as in (3.5). (3.6) can be obtained easily from (3.5) by putting $p = 2$ in (3.5) and using (2.3).

Remark. Putting $\Sigma = \sigma^2 I$ in (3.5) and (3.6), we can easily deduce the result of Consul in (1.2) [5], [6], amd Mauchly in (1.1), [20].

## 4. DISTRIBUTION OF W IN THE COMPLEX CASE

Let $\tilde{S}$: pxp be distributed as a Complex Wishart $(n,p,\tilde{\Sigma})$ (see Goodman [11]). Then the distribution of the latent roots $g_1, g_2, \ldots, g_p$ of $\tilde{S}$ is (James [12])

$$(4.1)\quad k(p,n,\tilde{\Sigma})\, {}_0\tilde{F}_0(-\tilde{\Sigma}^{-1}, G)\, |G|^{n-p} \prod_{i<j} (g_i - g_j)^2 \prod_{i=1}^{p} dg_i$$

where

$$k(p,n,\tilde{\Sigma}) = \frac{|\tilde{\Sigma}|^{-n} \pi^{p(p-1)}}{\tilde{\Gamma}_p(n)\tilde{\Gamma}_p(p)}\ ; \ \tilde{\Gamma}_p(n) \text{ and}$$

${}_p\tilde{F}_q(a_1, \ldots, a_p;\ b_1, \ldots, b_q;\ S,\ T)$ are defined in (83) and (88) of James [12].

As in the real case, the convergence of (4.1) may be improved by writing it in the form

$$(4.2)\quad k(p,n,\tilde{\Sigma})\, {}_0\tilde{F}_0(\tilde{M}_1, G) \exp(-\mathrm{tr}\, G)\, |G|^{n-p} \prod_{i<j} (g_i - g_j)^2 \prod_{i=1}^{p} dg_i,$$

where $\tilde{M}_1 = I_p - \tilde{\Sigma}^{-1}$.

Theorem 5.1. Let $\underset{\sim}{G}$ be distributed as in (4.2) and let $W = |\underset{\sim}{G}|/[(\mathrm{tr}\ \underset{\sim}{G})/p]^p$. Then h-th moment of W is

$$(4.3)\quad \frac{p^{ph}}{\tilde{\Gamma}_p(n)} |\Sigma|^{-n} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{C}_\kappa(\tilde{\underset{\sim}{M}}_1)}{k!} \Gamma(np+k)\tilde{\Gamma}_p(n+h,\kappa)/\Gamma(np+k+ph).$$

Proof: Multiplying (4.2) by $|\underset{\sim}{G}|/[(\mathrm{tr}\ \underset{\sim}{G})/p)]^p$, using the transformation $\underset{\sim}{G} \to \underset{\sim}{U}\ \underset{\sim}{V}\ \bar{\underset{\sim}{U}}'$ where $\underset{\sim}{U}$ is unitary and $\underset{\sim}{V}$ is hermitian p.d. we have on integrating out $\underset{\sim}{U}$ and using the results (see Khatri [14]) that the Jacobian of transformation is

$$J(\underset{\sim}{G},\ \underset{\sim}{U}\ \underset{\sim}{V}) = \prod_{i<j} (g_i - g_j)^2 h_2(\underset{\sim}{U})$$

and that $\displaystyle\int_{\underset{\sim}{U}\ \bar{\underset{\sim}{U}}'=I} h_2(\underset{\sim}{U}) = \frac{\pi^{p(p-1)}}{\tilde{\Gamma}_p(p)}$, we have

$$E(W^h) = \frac{p^{ph}|\tilde{\underset{\sim}{\Sigma}}|^{-n}}{\tilde{\Gamma}_p(n)} \sum_{k=0} \sum_{\kappa} \frac{\tilde{C}_\kappa(\tilde{\underset{\sim}{M}}_1)}{\tilde{C}_\kappa(I_p)k!} \int_{V>0} \exp(-\mathrm{tr}\ \underset{\sim}{V})|\underset{\sim}{V}|^{n+h-p} (\mathrm{tr}\ \underset{\sim}{V})^{-ph} \tilde{C}_\kappa(\underset{\sim}{V})\ d\underset{\sim}{V}$$

Using lemma (2.4) to the integral on the right, we get (4.3).

Theorem 5.2. The density of W is

$$f(w) = \frac{\pi^{\frac{1}{2}p(p-1)} |\tilde{\underset{\sim}{\Sigma}}|^{-n} (2\pi)^{\frac{1}{2}(p-1)}}{\tilde{\Gamma}_p(n)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{C}_\kappa(\tilde{\underset{\sim}{M}}_1)}{k!} \Gamma(pn+k)$$

$$\cdot\ p^{\frac{1}{2}-pn-k}\ w^{n-p}\ G^{p,0}_{p,p}\left(w\,\middle|\,{a_1 a_2,\ldots,a_p \atop b_1,\ldots,b_p}\right)$$

where $a_j = (k/p) + (j-1)/p + (p-1)$, and $b_j = k_j - j + p$.

Proof. The proof is exactly similar to that of theorem 3.2 and hence is omitted.

# CHAPTER III

# THE DISTRIBUTION OF THE SPHERICITY TEST CRITERION UNDER THE NULL HYPOTHESIS

## 1. INTRODUCTION

1 Let $\mathbf{x}$: $p \times 1$ be distributed $N(\boldsymbol{\mu},\boldsymbol{\Sigma})$ where $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are both unknown. Let $\mathbf{S}$ be the sum of product matrix of a sample of size N. To test the hypothesis of sphericity, namely, $H_0$: $\boldsymbol{\Sigma} = \sigma^2 \mathbf{I}_p$, where $\sigma^2 > 0$ is unknown, against $\boldsymbol{\Sigma} \neq \sigma^2 \mathbf{I}_p$, Mauchly [20] obtained the likelihood ratio test criterion for $H_0$ in the form $W = |\mathbf{S}|/[(\mathrm{tr}\ \mathbf{S})/p]^p$. For $p = 2$, Mauchly [20], showed that the density of W is

$$f(w) = \frac{1}{2}(n-1)w^{\frac{1}{2}(n-3)}, \quad 0 \leq w \leq 1, \tag{1.1}$$

where $n = N-1$. The exact distribution in the null case was given by Consul in [5] for some special values of p and in the closed form in [6] in terms of Meijer's G-function, while its non-null distribution is obtained in Chapter II. However the forms of the distribution of W obtained by Consul and later by Mathai and Rathie [17] are not quite suited for computational purposes. No systematic attempt seems to have been made so far to compute the exact percentage points of W. The approximate percentage points for

$p = 3$ have been obtained by Mauchly [20] by fitting a Pearson curve of the form

$$y = k\, x^{p-1}\, (1-x)^{q-1} \tag{1.2}$$

and more recently by Davis [8] for $p = 3, 6, 10$ and $n = 4(1)8, 10, 12, 15, 20$ using a Cornish-Fisher inversion of Box's Series.

The object of the present paper is to develop methods similar to the ones used by Box [2] and U. S. Nair [23], [24], in order to obtain the exact distribution of W in series form and to compute exact percentage points of W. In particular, methods have been given which yield facility in computation for the cases when the sample size is small as well as when sample size is large. Tabulations of percentage points for $p = 2(1)10$ for various significance levels are given and comparisons made with approximate values using (1) Box's series, [Anderson [1], page 263], (2) Mauchly [20], (3) Tukey and Wilks [30], and (4) Davis [8].

## 2. EXACT DISTRIBUTION OF THE SPHERICITY CRITERION W.

For ease in computation, it is desirable to give methods which are particularly suited for extremely small values of N, the sample size, and those which are suited for larger values of N. Thus exact percentage points of W can then be computed for all values of N. In parts (a) and (b) of this section, we shall consider two methods which will achieve the former objective while in part (c) we shall give a method which will achieve the latter objective. The first method which we shall now consider makes use of Mellin transform and Contour integration.

(a) Exact distribution of W through Contour Integration

It has been shown by Mauchly [20], that the h-th moment of the sphericity criterion $W = |\underset{\sim}{S}|/[(\text{tr } \underset{\sim}{S})/p]^p$ is given by

$$(2.1) \quad E(W^h) = p^{ph} \prod_{i=1}^{p} \left[ \frac{\Gamma\{\frac{1}{2}(N-i)+h\}}{\Gamma\{\frac{1}{2}(N-i)\}} \; \frac{\Gamma\{\frac{1}{2}\, p(N-1)\}}{\Gamma\{\frac{1}{2}\, p(N-1)+ph\}} \right].$$

Using Mellin's transform, the density of W is given by

$$(2.2) \quad f(w) = K(p,n)\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{w^{-h-1} p^{ph} \prod_{i=1}^{p} \Gamma\{\frac{1}{2}(N-i)+h\}}{\Gamma\{\frac{1}{2}\, pn+ph\}}\, dh,$$

where $n = N-1$ and $K(p,n) = \Gamma(\frac{1}{2}\, pn)/ \prod_{i=1}^{p} \Gamma(N-i)/2.$ Putting $\frac{1}{2}(N-p)+h=s$ in (2.2), we have

$$(2.3) \quad f(w) = K(p,n)p^{-\frac{1}{2}\, p(N-p)} w^{\frac{1}{2}(N-p)-1} \cdot p(w),$$

where

$$(2.4) \quad p(w) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (w/p^p)^{-s} [\prod_{i=1}^{p} \Gamma(s + \tfrac{p-i}{2})/\Gamma p(s + \tfrac{p-1}{2})]ds,$$

and $c = \frac{1}{2}(N-p)$.

<u>Special</u> <u>Cases</u>. (i) $p = 2$. For $p = 2$, we have

$$(2.5) \quad p(w) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} [(w/4)^{-s}\Gamma(s)\Gamma(s + \tfrac{1}{2})/\Gamma(2s+1)]ds.$$

Using the following duplicating formula for the gamma function

$$(2.6)\qquad \Gamma(s)\Gamma(s+\tfrac{1}{2}) = (\pi)^{\frac{1}{2}}\,\Gamma(2s)/2^{2s-1},$$

in (2.5) we have

$$(2.7)\qquad p(w) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(\pi)^{\frac{1}{2}}[w^{-s}/s]ds\ .$$

The pole of the integrand is at $s = 0$ and the corresponding residue is then $(\pi)^{1/2}$. Hence from (2.3) we have for $p = 2$, the density of W as in (1.1) obtained by Mauchly.

(ii) $p = 3$. In this case, we have from (2.4)

$$(2.8)\qquad p(w) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(w/27)^{-s}[\Gamma(s+1)\Gamma(s+\tfrac{1}{2})\Gamma(s)/\Gamma(3s+3)]ds.$$

Using the duplication formula (2.6), we have

$$(2.9)\qquad p(w) = 2(\pi)^{1/2}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(4w/27)^{-s}[\Gamma(2s)\Gamma(s+1)/\Gamma(3s+3)]ds.$$

We shall again evaluate the integral in (2.9) by Contour integration. The poles of the integrand are at the points

$$(2.10)\qquad s = -m/2,\quad m = 0, -1, -2, \ldots,$$

and the residue at these points can be found by putting $s = t-\frac{1}{2}m$ in the integrand and taking the residue of the integrand at $t = 0$. Thus substituting $s = t-\frac{m}{2}$, the integrand in (2.9) becomes

$$(2.11)\qquad (4w/27)^{-t+\frac{m}{2}}\Gamma(2t-m)\Gamma(t+1-\tfrac{m}{2})/\Gamma\{3t-\tfrac{3}{2}m+3\}\ .$$

To evaluate the integral we need to consider separately the cases when (i) m is even and (ii) m is odd.

First let m be even and say m = 2r. Then the integrand reduces to

$$(2.12) \qquad (4w/27)^{-t+r}\Gamma(2t-2r)\Gamma(t-r+1)/\Gamma(3t-3r+3),$$

which by expanding the gamma functions becomes

$$(2.13) \qquad \frac{3}{2}(4w/27)^{-t+r}\,\frac{\Gamma(2t+1)\Gamma(t+1)[\prod_{i=1}^{3r-3}(3t-i)]}{t\Gamma(3t+1)[\prod_{i=1}^{2r}(2t-i)\prod_{i=1}^{r-1}(t-i)]},$$

valid for r > 1 and the cases when r = 0 and r = 1 have to be considered separately. The expression in (2.13) has a simple pole of first order at t = 0 and its residue at this point is clearly given by

$$(2.14) \qquad \frac{3}{2}(4w/27)^{r}\ (3r-3)!/(2r)!(r-1)!\ .$$

For r = 0, the integrand reduces to

$$(4w/27)^{-t}\Gamma(2t)\Gamma(t+1)/\Gamma(3t+3),$$

which can be written as

$$(2.15) \qquad (4w/27)^{-t}\Gamma(2t+1)\Gamma(t+1)/(2t)\Gamma(3t+3)$$

and has a simple pole at t = 0, the residue at this point being $\frac{1}{4}$ .

For r = 1, the integrand after a little simplification becomes,

$$(2.16)\qquad \tfrac{3}{2}(4w/27)^{-t+1}\Gamma(2t+1)\Gamma(t+1)/(t)(2t-1)(t-1)\Gamma(3t+1)$$

and this has a simple pole at t = 0, the corresponding residue being equal to $\frac{3}{2}(4w/27)$.

Now if m is odd say equal to 2q+1, where q is an integer or zero, then as before, the integrand (2.11) can be easily written down as in (2.13) in the form

$$(2.17)\qquad (4w/27)^{-t+q+\frac{1}{2}}\frac{\Gamma(2t+1)\Gamma(t+\frac{1}{2})\prod_{i=1}^{3q-1}(3t+\frac{1}{2}-i)}{(2t)[\prod_{i=1}^{2q+1}(2t-i)\prod_{i=1}^{q}(t+\frac{1}{2}-i)]\Gamma(3t+\frac{1}{2})},$$

which clearly holds for q > 0; the case q = 0 has to be treated separately.

For q > 0, (2.17) has a pole of first order at t = 0 and its residue at this point is

$$(4w/27)^{q+\frac{1}{2}}\prod_{i=1}^{3q-1}(\tfrac{1}{2}-i)/[2\prod_{i=1}^{2q+1}(-i)\prod_{i=1}^{q}(\tfrac{1}{2}-i)],$$

which can be written in an alternate form as

$$(2.18)\qquad (4w/27)^{q+\frac{1}{2}}\Gamma(3q-\tfrac{1}{2})/2\Gamma(2q+2)\Gamma(q+\tfrac{1}{2})\ .$$

For q = 0, the integrand reduces after a little simplification to

$$(2.19)\qquad (4w/27)^{-t+\frac{1}{2}}\ \Gamma(2t+1)\Gamma(t+\tfrac{1}{2})/(2t)(2t-1)\Gamma(3t+\tfrac{3}{2}),$$

and this has a pole of first order at $t = 0$, the corresponding residue being equal to $-(4w/27)^{1/2}$. Hence finally using Cauchy's Residue Theorem, the integral in (2.8) is seen to be equal to

$$(2.20) \quad p(w) = 2(\pi)^{1/2}\left[\frac{1}{4} - (4w/27)^{1/2} + \sum_{r=1}^{\infty} \frac{\frac{3}{2}(4w/27)^{r}\Gamma(3r-2)}{\Gamma(2r+1)\Gamma(r)} + \sum_{q=1}^{\infty} \frac{(4w/27)^{\frac{1}{2}(2q+1)}\Gamma(3q-\frac{1}{2})}{2\Gamma(2q+2)\Gamma(q+\frac{1}{2})}\right].$$

From (2.3), the density of W for $p = 3$ is therefore

$$(2.21) \quad f(w) = [\Gamma\{3(N-1)/2\}/\prod_{i=1}^{3}\Gamma(\tfrac{N-i}{2})]w^{\frac{1}{2}(N-3)-1}p(w)/3^{\frac{3}{2}(N-3)}$$

where $p(w)$ is as in (2.20). John [13] has recently given an explicit form for the density of W for $p = 3$ but not in a very convenient form for use.

(iii) $p \geq 4$. The cases for values of $p \geq 4$ can be treated in almost a similar way but the method involves psi functions and their derivatives and makes use of the following lemma due to Nair [23].

Lemma 2.1. Let $(a_i)$ be a sequence of numbers, finite or infinite and let

$$(2.22) \quad F(x;t;a_2,a_3,\ldots) \equiv e^{xt+a_2\frac{t^2}{2!}+a_3\frac{t^3}{3!}+\ldots\infty}.$$

Then the nth derivative of $F(x;t;a_2,a_3,\ldots)$ at $t = 0$ is

$$(2.23)\quad D_n(x,a)=\begin{vmatrix} x & -1 & 0 & 0 & 0 & \dots 0 \\ a_2 & x & -1 & 0 & \dots & \dots 0 \\ a_3 & 2a_2 & x & -1 & \dots & \dots 0 \\ a_4 & 3a_3 & 3a_2 & x & -1 & \dots 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_n & \binom{n-1}{1}a_{n-1} & \binom{n-1}{2}a_{n-2} & \dots & \dots & \dots x \end{vmatrix} .$$

Case (1). $p$ odd. First let $p = 2k+1$, $(k > 1)$ be odd and let us denote the integrand in (2.4) by $G(s)$. Then it is easy to see that

$$(2.24)\qquad G(s) = A(W_1)^{-s}\Gamma(s+k)\Gamma(2s)\prod_{i=1}^{k-1}\Gamma(2s+2i)/\Gamma\{p(s+k)\},$$

where

$$(2.25)\qquad A = (\sqrt{\pi})^k 2^{-k(k-2)} \text{ and } W_1 = 2^{2k}w/p^p.$$

The poles of $G(s)$ are at the points given in (2.10) and the residue at these points is equal to the residue of $G(t-\frac{m}{2})$ at $t = 0$. Now

$$(2.26)\qquad G(t-\tfrac{m}{2}) = A(W_1)^{-t+\frac{m}{2}}\Gamma(t-\tfrac{m}{2}+k)\Gamma(2t-m)\prod_{i=1}^{k-1}\Gamma(2t-m+2i)$$

$$\cdot\ [\Gamma\{p(t-\tfrac{m}{2}) + kp\}]^{-1} .$$

We have to consider the cases when (1) $m$ is even and (2) $m$ is odd. First let $m = 2r$ be even. Then we have

$$(2.27)\qquad G(t-r) = ApW_1^r \cdot C(t)/(2t)^k$$

where

$$(2.28)\quad C(t) = \frac{(W_1)^{-t}\Gamma(t+1)[\Gamma(2t+1)]^k \prod_{i=1}^{p(r-k)} (j-pt)/\prod_{j=1}^{2r} (j-2t)}{\prod_{i=1}^{r-k} (j-t)\Gamma(pt+1) \prod_{i=1}^{k-1} \prod_{j=1}^{2r-2i} (j-2t)} .$$

Thus for $r > k$, the pole of $G(t-r)$ is of order $k$ and the residue $R_r$ at $t = 0$ is

$$(2.29)\quad R_r = [Ap\ W_1^r\ 2^{-k}/\Gamma(k)](\tfrac{d}{dr})^{k-1}_{t=0}\ e^{\log C(t)} .$$

Using the formula (Erdélyi; [10])

$$(2.30)\quad \log\Gamma(x+a) = \log\Gamma(a) + x\,\psi(a) + \frac{x^2}{2!}\psi_1(a) + \frac{x^3}{3!}\psi_2(a) + \ldots$$

where

$$(2.31)\quad \psi(a) = \frac{d}{dx}\log\Gamma(x)\Big|_{x=a} \quad\text{and}\quad \psi_j(q) = (\tfrac{d}{dx})^j\ \psi(x)\Big|_{x=a},$$

$\log C(t)$ can be written as

$$(2.32)\quad \log C(t) = b_0 + b_1 t + b_2 \frac{t^2}{2!} + \ldots$$

where

$$(2.33)\quad b_0 = \log [p(r-k)!/\{(r-k)! \prod_{i=0}^{k-1} (2r-2i)!\}],$$

$$b_1 = (1+2k-p)\psi(1) + p \sum_{j=1}^{p(r-k)} (1/j) - \sum_{j=1}^{r-k} (1/j)$$

$$- 2 \sum_{i=0}^{k-1} \sum_{j=1}^{2r-2i} (1/j) - \log W_1$$

and

$$b_q = (1+k2^q-p^q)\psi_{q-1}(1)+\Gamma(q)[\sum_{j=1}^{p(r-k)} (p/j)^q-\sum_{i=0}^{k-1}\sum_{j=1}^{2r-2i}(2/j)^q - \sum_{j=1}^{r-k}(1/j)^q],\ q = 2,3,\ldots\ .$$

Using (2.32) in (2.29) and then applying lemma 2.1, we have

$$(2.34)\qquad R_r = \frac{Ap(W_1^r)[p(r-k)!]}{2^k\Gamma(k)(r-k)!\prod_{i=0}^{k-1}(2r-2i)!} D_{k-1}(W_1,b),\quad (r > k)$$

where

$$(2.35)\qquad D_{k-1}(W_1,b)=\begin{vmatrix} b_1 & -1 & 0 & 0 & 0 & 0 \\ b_2 & b_1 & -1 & 0 & \ldots\ \ldots & 0 \\ b_3 & 2b_2 & b_1 & -1 & \ldots\ \ldots & 0 \\ b_4 & 3b_3 & 3b_2 & b_1 & -1\ \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots\ \ldots & \\ b_{k-1} & \binom{k-2}{1}b_{k-2} & \binom{k-2}{2}b_{k-3} & \ldots & \ldots\ \ldots & b_1 \end{vmatrix}$$

and $b_q$'s are as in (2.33).

If r = 0, then it can be shown that G(t) has a simple pole at t = 0 and the residue R at this point is

$$(2.36)\qquad R = A\Gamma(k)\prod_{i=1}^{k-1}\Gamma(2i)/\{2\Gamma(kp)\}\ .$$

For $r = \ell$ where $\ell = 1,2,\ldots,k-1$, we have from (2.27)

$$(2.37)\qquad G(t-\ell) = \frac{AW_1^{-t+\ell}\Gamma\{t+(k-\ell)\}[\Gamma(2t+1)]^{(\ell+1)}\prod_{i=\ell+1}^{k-1}\Gamma\{2t+(2i-2\ell)\}}{(2t)^{\ell+1}\prod_{j=1}^{2\ell}(2t-j)\Gamma\{pt+p(k-\ell)\}\prod_{i=1}^{\ell-1}\prod_{j=1}^{2\ell-2i}(2t-j)}$$

where $\prod_{i=n}^{m} (\cdot)$ is interpreted as unity if $n > m$.

Thus for $r = \ell$, $\ell = 1,2,\ldots,k-1$, $G(t-\ell)$ has a pole of order $(\ell+1)$ at $t = 0$ and its residue $B_\ell$, on using the lemma and proceeding as above is easily seen to be

$$(2.38) \quad B_\ell = \frac{A(W_1)^\ell \Gamma(k-\ell) \prod_{i=\ell+1}^{k-1} \Gamma(2i-2\ell)}{2^{\ell+1}\Gamma(\ell+1)\Gamma\{p(k-\ell)\}[\prod_{i=0}^{\ell-1} (2\ell-2i)!]} D_\ell(W_1,c), \ell = 1,2,\ldots,k-1,$$

where $D_\ell(W_1,c)$ is equal to the determinant on the right hand side of (2.23) with $x$ replaced by $c_1$, $n$ by $\ell$ and $a_q$'s by $c_q$'s, $q = 2,3,\ldots,\ell$. The coefficients $c_q$'s are given by

$$(2.39) \quad c_1 = \psi(k-\ell)+2[(\ell+1)\psi(1)+\sum_{i=\ell+1}^{k-1} \psi(2i-2\ell)-\sum_{i=0}^{\ell-1} \sum_{j=1}^{2\ell-2i} (1/j)]$$

$$-p\psi(pk-p\ell)-\log W_1$$

and

$$c_q = \psi_{q-1}(k-\ell)+2^q[(\ell+1)\psi_{q-1}(1)+\sum_{i=\ell+1}^{k-1} \psi_{q-1}(2i-2\ell)$$

$$-\sum_{i=0}^{\ell-1} \sum_{j=1}^{2\ell-2i} (\Gamma(q)/j^q)]-p^q\psi_{q-1}(pk-p\ell), \ q = 2,3,\ldots,$$

where $\sum_{i=n}^{m} (\cdot)$ is interpreted as zero if $n > m$.

Similarly for $r = k$, $G(t-k)$ has a pole of order $k$ at $t = 0$ and the residue $R_k$ is

$$(2.40) \quad R_k = \frac{Ap(W_1/2)^k}{\Gamma(k) \prod_{i=0}^{k-1} (2k-2i)!} D_{k-1}(W_1,d)$$

where $D_{k-1}(W_1,d)$ can be obtained from (2.35) by replacing $b_q$'s by $d_q$'s where $d_q$'s are given by

$$d_1=(1+2k-p)\psi(1)-\log W_1-\sum_{i=0}^{k-1}\sum_{j=1}^{2k+2i}(2/j) \quad \text{and}$$

(2.41)

$$d_q=(1+2^q k-p^q)\psi_{q-1}(1)-\sum_{i=0}^{k-1}\sum_{j=1}^{2k+2i}(2^q\Gamma(q)/j^q), q=2,3,\ldots .$$

Now let $m = 2q+1$ be odd. Then it can be easily checked that for $q > 0$, $G(t-q-\frac{1}{2})$ has a pole of order k at $t = 0$ and its residue $G_q$ is given by

$$(2.42) \quad G_q = \frac{(-1)^k A W_1^{q+\frac{1}{2}}\Gamma(k-\frac{1}{2})\prod_{j=1}^{pq}(j+\frac{p}{2}-pk)}{2^k\Gamma(k)\Gamma(kp-\frac{p}{2})\prod_{j=1}^{q}(j-k+\frac{1}{2})} D_{k-1}(W_1,f), \quad q > 0,$$

where $D_{k-1}(W_1,f)$ is the determinant equal to the right hand side of (2.35) with $b_n$'s replaced by $f_n$'s where $f_n$'s are given by

$$f_1 = -\log W_1+\psi(k-\tfrac{1}{2}) + 2k\psi(1)+\sum_{j=1}^{pq}\{p/(j+\tfrac{p}{2}-pk)\}$$

$$-p\psi(kp-\tfrac{p}{2})-\sum_{\ell=0}^{k-1}\sum_{j=1}^{2q-2i+1}(2/j), \text{ and}$$

(2.43)

$$f_n = \psi_{n-1}(k-\tfrac{1}{2})+2k\psi_{n-1}(1)-p^n\psi_{n-1}(pk-\tfrac{k}{2})+\sum_{j=1}^{pq}[p^n\Gamma(n)/(j+\tfrac{p}{2}-pk)^n]$$

$$-\sum_{i=0}^{k-1}\sum_{j=1}^{2q-2i+1}(\frac{2^n\Gamma(n)}{j^n}), \quad n = 2,3,\ldots .$$

Similarly if $q = 0$, then $G(t-\frac{1}{2})$ has a simple pole at $t = 0$ and its residue B is

$$(2.44) \qquad B = -A(W_1)^{1/2}\Gamma(k-\tfrac{1}{2})\prod_{i=1}^{k-1}(2i-1)!/\{2\Gamma p(k-\tfrac{1}{2})\} .$$

Thus if p is odd, we have from (2.3) and Cauchy's Residue theorem that the density of W is

$$(2.45) \qquad f(w) = k(p,n)p^{-\frac{1}{2}p(N-p)}w^{\frac{1}{2}(N-p)-1}[\sum_{r=k+1}^{\infty} R_r + R + \sum_{\ell=1}^{k-1} B_\ell + R_k + \sum_{q=1}^{\infty} G_q + B] .$$

Case (2). p is even. Let $p = 2k(k > 1)$ be even. Then the integrand $H(s)$ in (2.4) can be written as

$$(2.46) \qquad H(s) = A(W_1)^{-s}\Gamma(2s)\prod_{i=1}^{k-1}\Gamma(2s+2i)/\Gamma(ps+pk-k) ,$$

where A and $W_1$ are as in (2.25). The poles of H(s) are at the points given in (2.10) and the residue of H(s) at these points is equal to the residue of $H(t-\frac{m}{2})$ at $t = 0$. Now

$$(2.47) \qquad H(t-\tfrac{m}{2}) = A(W_1)^{-t+\frac{m}{2}}\Gamma(2t-m)\prod_{i=1}^{k-1}\Gamma(2t-m+2i)/\Gamma\{pt-\tfrac{pm}{2} + k(p-1)\}.$$

We have to consider separately the cases when m is even and m is odd. When $m = 2r$, then proceeding as before it is seen that for $r \geq k$, $H(t-r)$ has a pole of order k-1 and the residue $D_r$ at $t = 0$ is given by

$$(2.48)\quad D_r = \frac{(-1)^{k(p-1)} Ap(W_1)^r \Gamma\{pr-k(p-1)+1\}}{2^k \Gamma(k-1) \prod_{i=0}^{k-1} (2r-2i)!} V_{k-2}(W_1,g),\ r \geq k,$$

when the determinant $V_{k-2}(W_1,g)$ is similar to the determinant on the right hand side of (2.35) having (k-2) rows and the elements $b_q$'s being replaced by $g_q$'s where $g_q$'s are given by

$$g_1 = (2k-p)\psi(1)-p \sum_{j=1}^{pr-k(p-1)} (1/j) - \sum_{r=0}^{k-1} \sum_{j=1}^{2r-2i} (2/j) - \log W_1$$

(2.49)

$$g_q = (k2^q - p^q)\psi_{q-1}(1) + \Gamma(q)[\sum_{j=1}^{pr-k(p-1)} (p/j)^q - \sum_{i=0}^{k-1} \sum_{j=1}^{2r-2i} (2/j)^q],$$

$$q = 2, 3, \ldots, \quad .$$

For r = 0, H(t) has a simple pole at t = 0 and the residue D at this point is

$$(2.50)\qquad D = A \prod_{i=1}^{k-1} \Gamma(2i)/2\Gamma k(p-1)\ .$$

For $r = \ell$ where $\ell = 1,2,\ldots,k-1$, $H(t-\ell)$ has a pole of order $\ell+1$ at t = 0 and the residue $E_\ell$ is given by

$$(2.51)\quad E_\ell = \frac{A(W_1)^\ell \prod_{i=\ell+1}^{k-1} \Gamma(2i-2\ell) / \prod_{i=0}^{\ell-1} (2\ell-2i)!}{2^{\ell+1}\Gamma(\ell+1)\Gamma\{p(k-\ell)-k\}} V_\ell(W_1,h) \quad \ell=1,2,\ldots,k-1,$$

where $V_\ell(W_1,h)$ is the determinant of $\ell$th order similar to that in (2.35) with $b_q$'s being replaced by $h_q$'s where $h_q$'s are given by

$$(2.52)\qquad h_1 = -\log W_1+2[(\ell+1)\psi(1)+\sum_{i=\ell+1}^{k-1}\psi(2i-2\ell)-\sum_{i=0}^{\ell-1}\sum_{j=1}^{2\ell-2i}(1/j)]$$

$$-p\psi(pk-p\ell-k)$$

and

$$h_q = 2^q[(\ell+1)\psi_{q-1}(1)\sum_{i=\ell+1}^{k-1}\psi_q(2i-2\ell)-\sum_{i=0}^{\ell-1}\sum_{j=1}^{2\ell-2i}(\Gamma(q)/j^q)]$$

$$-\ p^q\psi_{q-1}(pk-p\ell-k),\ q = 2,3,\ldots\ .$$

Now let $m = 2q + 1$ be odd. Then for $q \geq k-1$, $H(t-q-\frac{1}{2})$ has a pole of order $(k-1)$ at $t = 0$ and the residue $F_q$ is

$$(2.53)\qquad F_q = \frac{Ap(W_1)^{q+\frac{1}{2}}[p(q-k+1)!]}{2^k\Gamma(k-1)\sum_{i=0}^{k-1}(2q+1-2i)!}V_{k-2}(W_1,K),\ q \geq k-1,$$

where $V_{k-2}(W_1,K)$ is the determinant of order k-2 similar to that in (2.35) with the elements given by

$$(2.54)\qquad k_1 = -\log W_1+(2k-p)\psi(1)+\sum_{j=1}^{p(q-k+1)}(p.j)-\sum_{i=0}^{k-1}\sum_{j=1}^{2q+1-2i}(2/j),$$

and

$$k_n = (k2^n-p^n)\psi_{n-1}(1)+\Gamma(n)[\sum_{j=1}^{p(q-k+1)}(p/j)^n-\sum_{i=0}^{k-1}\sum_{j=1}^{2q+1-2i}(2/j)^n],$$

$$n = 2,\ 3,\ldots\ .$$

where

$\sum_{j=n}^{m} (\cdot)$ is interpreted as zero if $m < n$.

For $q = \ell$, $\ell = 1,2,\ldots,k-2$, $H(t-\ell-\frac{1}{2})$ has a pole of order $(\ell+1)$ at $t = 0$ and the corresponding residue $G_\ell$ is

$$(2.55)\qquad G_\ell = \frac{A(W_1)^{\ell+\frac{1}{2}} \prod_{i=\ell+1}^{k-1} \Gamma(2i-1+2\ell)}{2^{\ell+1}\Gamma(\ell+1) \prod_{i=0}^{\ell} (2\ell+1-2i)!\,\Gamma p(k-1-\ell)} V_\ell(W_1,m),$$

where as before $V_\ell(W,m)$ is the determinant of order $\ell$ similar to the one in (2.35), with elements given by

$$(2.56)\qquad m_1 = -\log W_1 + 2(\ell+1)\psi(1) + 2\sum_{i=\ell+1}^{k-1} \psi(2i-1-2\ell)$$

$$- \sum_{i=0}^{\ell} \sum_{j=1}^{2\ell+1-2i} (2/j) - p\psi(pk-p-p\ell)$$

and

$$m_n = 2^n[(\ell+1)\psi_{n-1}(1) + \sum_{i=\ell+1}^{k-1} \psi_{n-1}(2i-1-2\ell) - \sum_{i=0}^{\ell} \sum_{j=1}^{2\ell+1-2i} (\Gamma(n)/j^n 0]$$

$$- p^n\psi_{n-1}(pk-p-p\ell) \quad n = 2, 3, \ldots .$$

Finally for $q = 0$, $H(t-\frac{1}{2})$ has a simple pole at $t = 0$ and the residue C is

$$(2.57)\qquad C = -A(W_1)^{\frac{1}{2}} \prod_{i=1}^{k-1} \Gamma(2i-1)/2\Gamma p(k-1).$$

Thus when p is even, the density of W is given by

(2.58) $$f(w) = k(p,n)p^{-\frac{1}{2}p(N-p)} w^{\frac{1}{2}(N-p)-1} [D+C + \sum_{r=k}^{\infty} D_r + \sum_{\ell=1}^{k-1} E_\ell + \sum_{q=k-1}^{\infty} F_q + \sum_{\ell=1}^{k-2} G_\ell] .$$

(b) Distribution of W as a gamma series

We shall now obtain the distribution of W in gamma series form. For this let

(2.59) $$L = W^{\frac{n}{2}} .$$

Then from (2.1) we have,

(2.60) $$E(L^h) = K(p,n)p^{\frac{nph}{2}} \prod_{\alpha=1}^{p} \Gamma\{\tfrac{n}{2}(1+h)+\tfrac{1-\alpha}{2}\}/\Gamma\{\tfrac{1}{2}\, pn(1+h)\} .$$

Now let $\lambda = -2q \log L$ where q is an adjustable constant which can be chosen so as to govern the rate of convergence of the resulting gamma series and $0 < q < \infty$. If $\phi(t)$ is the characteristic function $\lambda$ then

(2.61) $$\phi(t) = K(p,n) \cdot C(t)$$

where

(2.62) $$C(t) = p^{-np\ it\ q} \prod_{\alpha=1}^{p} \Gamma\{\tfrac{n}{2}(1-2q\ it)+\tfrac{1-\alpha}{2}\}/\Gamma\tfrac{pn}{2}(1-2q\ it)$$

and therefore

(2.63) $$\log \phi(t) = \log\{K(p,n)\}-np\ it\ q \log p-\log \Gamma\tfrac{pn}{2}(1-2q\ it) + \sum_{\alpha=1}^{p} \log\Gamma\{\tfrac{n}{2}(1-2q\ it)+\tfrac{1-\alpha}{2}\} .$$

The expansion of log $\phi(t)$ will be based on the following expansion for the gamma function:

$$\text{(2.64)} \qquad \log\Gamma(x+h) = \frac{1}{2}\log(2\pi) + (x+h-\frac{1}{2})\log x - x - \sum_{r=1}^{m} \frac{(-1)^r B_{r+1}(h)}{r(r+1)x^r} + R_{m+1}(x) ,$$

where $R_m(x)$ is the remainder such that $|R_m(x)| \leq \theta/|x^m|$, $\theta$ a constant independent of x, and $B_r(h)$ the Bernoulli polynomial of degree r and order one defined by

$$\frac{\tau e^{h\tau}}{e^{\tau}-1} = \sum_{r=0}^{\infty} \frac{\tau^r}{r!} B_r(h) .$$

Explicitly the polynomials are

$$B_0(h) = 1,\ B_1(h) = h-\frac{1}{2},\ B_2(h) = h^2-h+\frac{1}{6}$$

and in general

$$B_r(h) = h^r - \frac{1}{2} r h^{r-1} + {}_rC_2 B_1 h^{r-2} - {}_rC_4 B_2 h^{r-4} + \ldots$$

(last term is x or a constant)

$$= h^r - \frac{1}{2} r h^{r-1} + \sum_{m=1}^{\leq r/2} (-1)^{m-1} {}_rC_{2m} B_m h^{r-2m} .$$

where ${}_nC_m = n!/(n-m)!m!$, $B_m$ are the Bernoulli numbers and have been tabulated extensively. Using (2.64) we obtain

$$(2.65) \quad \log\phi(t) = \log\{K(p,n)\} + (\tfrac{p-1}{2})\log(2\pi)$$

$$- (\tfrac{p^2+p-2}{4}) \log \{n(1-2q\ it)/2\}$$

$$- (\tfrac{pn-1}{2}) \log p + \sum_{r=1}^{m} (Q_r/n^r)(\tfrac{1-2q\ it}{2})^{-r}$$

$$+ R'_{m+1}(n,t) \quad ,$$

where the coefficients $Q_r$'s are given by

$$Q_r = (-1)^{r-1}[(\sum_{\alpha=1}^{p} B_{r+1}(\tfrac{1-\alpha}{2})) - B_{r+1}(0)/p^r]/r(r+1) \ .$$

The characteristic function of L can then be obtained from (2.65) as

$$(2.66) \quad \phi(t) = K_1(p,n)[n(1-2q\ it)/2]^{-v}(\sum_{j=0}^{\infty} (B_j/n^j)(\tfrac{1-2q\ it}{2})^{-j})$$

$$+ R''_{m+1}(n,t),$$

where

$$K_1(p,n) = K(p,n)(2\pi)^{(\frac{p-1}{2})} p^{-(pn-1)/2} \ ,$$

$$v = (p^2 + p-2)/4 \ ,$$

and the coefficients $B_j$'s which we need in our computations are listed below:

$$B_0 = 1, \qquad B_1 = Q_1, \qquad B_2 = Q_1^2/2 + Q_2 \; ,$$

$$B_3 = Q_1Q_2+Q_3+Q_1^3/6 \; ,$$

$$B_4 = Q_2Q_1^2/2+Q_1^4/24+Q_4+Q_3Q_1+Q_2^2/2 \; ,$$

$$B_5 = Q_1^3Q_2/6+Q_2^2Q_1/2+Q_1^5/120+Q_3Q_1^2/2+Q_5+Q_3Q_2+Q_4Q_1 \; ,$$

$$B_6 = Q_1Q_2Q_3+Q_1^6/720+Q_4Q_1^2/2+Q_1^4Q_2/24+Q_3^2/2+Q_1^2Q_2^2/4+Q_4Q_2$$

$$+Q_1^3Q_3/6+Q_5Q_1+Q_2^3/6+Q_6 \; , \tag{2.67}$$

$$B_7 = Q_1^4Q_3/24+Q_3^2Q_1/2+Q_4Q_2Q_1+Q_1Q_2^3/6+Q_1^2Q_2Q_3/2+Q_5Q_1^2/2$$

$$+Q_1^3Q_4/6+Q_1^7/5040+Q_5Q_2+Q_2^2Q_3/2+Q_1^3Q_2^2/12+Q_7$$

$$+Q_6Q_1+Q_1^5Q_2/120+Q_4Q_3 \; ,$$

$$B_8 = Q_8+3Q_3B_5/8+Q_6Q_2/4+Q_1B_7/8+Q_5Q_1Q_2/4+Q_4B_4/2+Q_4Q_2^2/4$$

$$+5Q_5B_3/8+Q_2Q_3^2/8+3Q_6B_2/4+Q_4Q_2Q_1^2/8+Q_1Q_2^2Q_3/4+Q_2^4/24$$

$$+Q_1^3Q_3Q_2/24+Q_1^2Q_2^3/16+Q_1^4Q_2^2/96+7Q_7Q_1/8+Q_1^6/720 \; ,$$

$$B_9 = Q_9+7Q_7B_2/9+Q_1Q_2Q_3^2/3+Q_1^6Q_3/2160+24Q_1^2Q_3/6+8Q_8Q_1/9$$

$$+2Q_6B_3/3+Q_1^4Q_2Q_3/72+Q_3^3/6+5Q_5B_4/9+Q_1^2Q_2^2Q_3/12$$

$$+Q_4Q_2Q_3/3+Q_1^3Q_3^2/18+4Q_4B_5/9+Q_5Q_1Q_3/3+Q_1B_8/9$$

$$+Q_2^3Q_3/18+2Q_2B_7/9+Q_6Q_3/3 \; ,$$

and

$$B_{10} = Q_{10}+7Q_7Q_1Q_2/10+3Q_6B_4/5+7Q_7Q_3/10+4Q_8B_2/5+9Q_9Q_1/10$$

$$+7Q_7Q_1^3/60+3Q_3B_7/10+Q_5Q_1^3Q_2/12+Q_2^2Q_5Q_1/4+Q_1^5Q_5/240$$

$$+Q_2B_8/5+Q_3Q_1^2Q_5/4+Q_1B_9/10+Q_5^2/2+Q_1Q_4Q_5/2+Q_2Q_3Q_5/2 \; .$$

The rest of the coefficients are given in appendix A.

(c) Distribution of W as a Beta Series

In the sequel we shall need the following theorem, restated from Nair [24].

Theorem 2.1. Let

$$\phi(t) = \int x^t \, p(x) \, dx$$

be the moment function of a random variable x with distribution law p(x). If

$$\phi(t) = 0(t^{-k})$$

with real part of t tending to ∞, then $\phi(t)$ can be expanded as a factorial series of the form

$$\phi(t) = \sum_{n=0}^{\infty} a_n \, \Gamma(t+a)/\Gamma(t+k+n+a)$$

a being an arbitrary non-negative constant.

We shall now obtain the distribution of W. Putting $\frac{1}{2}(N-\lambda)+h=s$ in (2.2), where $\lambda$ is an adjustable constant which can be chosen to govern the rate of convergence, we have

$$f(w) = k(p,N,w,\lambda)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} (w/p^p)^{-s} \frac{\prod_{i=1}^{p} \Gamma(s+\frac{\lambda-i}{2})}{\Gamma p(s+\frac{\lambda-1}{2})} ds , \quad (2.72)$$

where

$$k(p,N,w,\lambda) = \frac{\Gamma\{\frac{1}{2} p(n-1)\}}{\prod_{i=1}^{p} \Gamma(\frac{N-i}{2})} p^{-\frac{1}{2} p(N-\lambda)} w^{\frac{1}{2}(N-\lambda)-1} . \quad (2.73)$$

Using the expansion (2.64) to each gamma functions involved in the integrand of the right hand side of (2.72), we have after some simplification

$$(2.74)\qquad \log\left[\prod_{i=1}^{p}\Gamma(s+\tfrac{\lambda-i}{2})/\Gamma_p(s+\tfrac{\lambda-1}{2})\right]$$

$$= \log\left[(2\pi)^{(p-1)/2}\, s^{-v}/p^{ps+(p\lambda-p-1)/2}\right]+\sum_{r=1}^{\infty}(q_r/s^r)$$

where the coefficients $q_i$'s are given by

$$(2.75)\qquad q_r = (-1)^{r-1}\left[\sum_{i=1}^{p} B_{r+1}(\tfrac{\lambda-i}{2})-p^{-r}B_{r+1}(p(\lambda-1)/2)\right]/r(r+1)$$

and

$$v = (p^2 + p-2)/4 \ .$$

From (2.74), we deduce that

$$(2.76)\qquad \prod_{i=1}^{p}\Gamma(s+\tfrac{\lambda-i}{2})/\Gamma_p(s+\tfrac{\lambda-1}{2})$$

$$= (2\pi)^{(p-1)/2}s^{-v}\left[1+\sum_{r=1}^{\infty}(B_r/s^r)\right]/p^{ps+(p\lambda-p-1)/2},$$

where the coefficients $B_r$'s are as given in (2.67) with $Q_r$'s on the right hand side of (2.67) being replaced by $q_r$'s. Now from (2.72) and (2.76) we have the density of W as

$$(2.77)\qquad f(w) = K_1(p,n)w^{\frac{1}{2}(N-\lambda)-1}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} w^{-s}s^{-v}\left[1+\sum_{r=1}^{\infty}B_r/s^r\right]ds$$

where $K_1(p,n)$ is as given in (2.66).

The integral on the right hand side of (2.77) can be easily computed if v is an integer and its value is by Cauchy's theorem of Residues, the residue of $w^{-s}s^{-v}[1+\sum_{r=1}^{\infty} B_r/s^r]$ at $s = 0$. This is easily seen to be equal to

$$\sum_{r=0}^{\infty} [(-\log w)^{v+r-1} B_r/\Gamma(v+r)],\ B_0 = 1 \tag{2.78}$$

and thus from (2.77), the density of W is

$$f(w) = K_1(p,n) \sum_{r=0}^{\infty} (B_r) w^{\frac{1}{2}(N-\lambda)-1} (-\log w)^{v+r-1}/\Gamma(v+r)\ . \tag{2.79}$$

The probability that W is less than any value, say $x_0$, is

$$P(W \leq x_0) = K_1(p,n) \sum_{r=0}^{\infty} B_r \int_0^{x_0} w^{\frac{1}{2}(N-\lambda)-1} (-\log w)^{v+r-1} dw/\Gamma(v+r). \tag{2.80}$$

For computational purposes, we let

$$I_{v+r-1,u} = \int_0^{x_0} w^u(-\log w)^{v+r-1}\ dw/\Gamma(v+r), \tag{2.81}$$

where $u = \frac{1}{2}(N-\lambda)-1$. Then integrating by parts the R.H.S. of (2.81) it can be easily checked that the following extremely useful recurrence relation holds:

$$I_{v+r-1,u}(x_0) = [x_0^{(u+1)}(-\log x_0)^{v+r-1}/\Gamma(v+r)+I_{v+r-2,u}]/(u+1) \tag{2.82}$$

and

$$I_{0,u}(x_0) = x_0^{(u+1)}/(u+1)\ .$$

With this notation, (2.80) can be rewritten as

$$(2.83) \qquad P(W \leq x_0) = K_1(p,n) \sum_{r=0}^{\infty} B_r \, I_{v+r-1,u}(x_0)$$

where $I_{v+r-1,u}(x_0)$ satisfies the recurrence relation (2.82). It is to be noted that (2.83) holds only if v is an integer. However if v is not an integer, we can appeal to Theorem 2.1, since in this case

$$(2.84) \qquad \phi(s) = s^{-v}[1 + \sum_{r=1}^{\infty} B_r/s^r] = O(s^{-v}) \ .$$

Thus according to Theorem 2.1, we can expend $\phi(s)$ in the factorial series as

$$(2.85) \qquad s^{-v}[1+ \sum_{r=1}^{\infty} B_r/s^r] = \sum_{i=0}^{\infty} R_i \, \Gamma(s)/\Gamma(s+v+i)$$

where the coefficients $R_i$'s can be determined explicitly as is done below.

Using the formula (2.64) to each gamma function on the right hand side of (2.85) we have

$$(2.86) \qquad \log \{\Gamma(s)/\Gamma(s+v+i)\} = [-(v+i)\log s] + \sum_{j=1}^{\infty} (C_{ij}/s^j) \ .$$

where the first few coefficients $C_{ij}$'s which we need for our computations are given below. Let $t_i = v + i$.

$$C_{i1} = -\frac{1}{2}(v+i)(v+i-1) = -t_i(t_i-1)/2$$

$$C_{i2} = (t_i^3 - 3t_i^2/2 + t_i/2)/6$$

$$C_{i3} = -(t_i^4 - 2t_i^3 + t_i^2)/12$$

$$C_{i4} = (t_i^5 - 5t_i^4/2 + 5t_i^3/3 - t_i/6)/20$$

(2.87) $$C_{i5} = -(t_i^6 - 3t_i^5 + 5t_i^4/2 - t_i^2/2)/30$$

$$C_{i6} = (t_i^7 - 7t_i^6/2 + 7t_i^5/2 - 7t_i^3/6 + t_i/6)/42$$

$$C_{i7} = -(t_i^8 - 4t_i^7 + 14t_i^6/3 - 7t_i^4/3 + 2t_i^2/3)/56$$

$$C_{i8} = (t_i^9 - 9t_i^8/2 + 6t_i^7 - 21t_i^5/5 + 2t_i^3 - 3t_i/10)/72$$

$$C_{i9} = -(t_i^{10} - 5t_i^9 + 15_i^8/2 - 7t_i^6 + 5t_i^4 - 3t_i^2/2)/90$$

and

$$C_{i10} = (t_i^{11}-11t_i^{10}/2+55t_i^9/6-11t_i^7+11t_i^5-11t_i^3/2+5t_i/6)/110 .$$

The remaining coefficients are given in appendix A.

Thus from (2.86) we have

(2.88) $$\Gamma(s)/\Gamma(s+v+i) = s^{-(v+i)}(1+\sum_{j=1}^{\infty}(d_{ij}/s^j)] ,$$

where the coefficients $d_{ij}$'s can be obtained in terms of $C_{ij}$ from (2.67) where we replace $B_j$ by $d_{ij}$ and $Q_j$ by $C_{ij}$. Using (2.88) on the right hand side of (2.85) we have finally

(2.89) $$s^{-v}[1+\sum_{r=1}^{\infty}(B_r/s^r)] = \sum_{i=0}^{\infty}R_i s^{-(v+i)}[1+\sum_{j=1}^{\infty}d_{ij}/s^j] .$$

Equating the coefficients of s on both sides of (2.89), it is easy to check that we have the following explicit relations to determine the coefficients $R_i$'s.

$$R_0 = 1, \quad R_1+R_0d_{01} = B_1 ,$$
$$R_2 + R_1d_{11}+R_0d_{02} = B_2 ,$$
$$R_3 + R_2d_{21}+R_1d_{12}+R_0d_{03} = B_3 ,$$
$$R_4 + R_3d_{31}+R_2d_{22}+R_1d_{13}+R_0d_{04} = B_4 ,$$
$$R_5 + R_4d_{41}+R_3d_{32}+R_2d_{23}+R_1d_{14}+R_0d_{05} = R_5,$$
$$R_6 + R_5d_{51}+R_4d_{42}+R_3d_{33}+R_2d_{24}+R_1d_{15}+R_0d_{06} = B_6,$$
$$(2.90) \qquad R_7 + R_6d_{61}+R_5d_{52}+R_4d_{43}+R_3d_{34}+R_2d_{25}+R_1d_{16}+R_0d_{07} = B_7,$$
$$R_8 + R_7d_{71}+R_6d_{62}+R_5d_{53}+R_4d_{44}+R_3d_{35}+R_2d_{26}+R_1d_{17} +R_0d_{08} = B_8,$$
$$R_9 + R_8d_{81}+R_7d_{72}+R_6d_{63}+R_5d_{54}+R_4d_{45}+R_3d_{36}+R_2d_{27} +R_1d_{18}+R_0d_{09} = B_9,$$

and

$$R_{10} = R_9d_{91}+R_8d_{82}+R_7d_{73}+R_6d_{64}+R_5d_{55}+R_4d_{46}+R_3d_{37} +R_2d_{28}+R_1d_{19}+R_0d_{010} = B_{10} .$$

Now using (2.85) in (2.77) and noting that term by term integration is valid since a factorial series is uniformly convergent in a half-plane (see Doetch [9]) we have the density of W in the case that $\underline{v}$ is not an integer in the form

$$(2.91) \qquad f(w) = K_1(p,n) \sum_{i=0}^{\infty} R_i \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} w^{-s}[\Gamma(s)/\Gamma(s+v+i)] \, ds$$

and on using the well known integral (Titchmarsh [29])

$$(2.92) \qquad \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\{\Gamma(s)/\Gamma(s+v)\}ds = (1-x)^{v-1}/\Gamma(v) \quad 0 \le x \le 1$$

$$c > 0$$

the exact distribution of f(w) is in the form of a beta series given by

$$(2.93) \qquad f(w) = K_1(p,n)\sum_{i=0}^{\infty} R_i\, w^{\frac{1}{2}(N-\lambda)-1}(1-w)^{v+i-1}/\Gamma(v+i) \ ,$$

where the coefficients $R_i$'s are as given in (2.90).

The distribution of W is then given by

$$(2.94) \qquad P(W \le x_0) = K_1(p,n)\sum_{i=0}^{\infty} R_i\, I_{x_0}(\tfrac{1}{2}(N-\lambda),\ v+i)/\Gamma(v+i),$$

where $I_{x_0}(p,q)$ is the incomplete beta function $\int_0^{x_0} x^{p-1}(1-x)^{q-1}dx$.

## 3. APPROXIMATIONS TO THE DISTRIBUTIONS OF W

This section is devoted to approximations.

Approximation using the method of Wilks and Tukey.

Let $(a)_h = a(a+1)(a+2)\ldots(a+h-1)$. Then we have

$$\Gamma(a+h) = (a)_h\Gamma(a),$$

and

$$\Gamma(a+rh) = (a)_{rh}\Gamma(a) = \Gamma(a)r^{rh}\prod_{i=1}^{r}(\tfrac{a+i-1}{r})h \ ,$$

where r is a positive integer. With these notations, (2.1) can be written as

$$(3.1) \qquad E(W^h) = \prod_{i=1}^{p}(\tfrac{n}{2}+\tfrac{i-p}{2})_h/(\tfrac{n}{2}+\tfrac{i-1}{p})_h$$

and this can be put in the form (see eq. (6) of Wilks and Tukey [30])

$$(3.2) \qquad E(W^h) = \prod_{i=1}^{p} \left(\frac{1}{x} - A_i + 1\right)_h \Big/ \left(\frac{1}{x} - B_i + 1\right)_h$$

where $x = \frac{2}{n}$, $A_i = 1 - \frac{i-p}{2}$ and $B_i = 1 - (i-1)/p$. We can therefore apply the method of Wilks and Tukey [30], to find a fractional power of the test criterion which is approximately distributed according to an incomplete beta distribution function (Pearson Type I)

$$(3.3) \qquad dF(u) = \Gamma(\alpha+\beta)[\Gamma(\alpha)\Gamma(\beta)]^{-1}\, u^{\alpha-1}(1-u)^{\beta-1}du \; .$$

According to this method, the appropriate values of $\alpha$, $\beta$ and the exponent $r$ of the criterion are given by the solutions of the following equations:

$$(3.4) \qquad (\alpha+\beta)/r = \frac{1}{x} = \frac{n}{2},\; C_1 = \beta \quad \text{and } r = \beta(\beta+1)/(C_2-\beta) \; ,$$

where
$$C_m = \sum_{i=1}^{p} (A_i^m - B_i^m) \; .$$

Using the values of $A_i$ and $B_i$ from (3.2), we have

$$C_1 = (p^2+p-2)/4 \text{ and } C_2 = (2p^3+9p^2+5p-12)/24-1/6p \; .$$

We have the following table of values of $\alpha$, $\beta$ and $r$ for various values of $p$ ($\alpha$ being calculated from rounded values of $r$):

Table 2. Values of $\alpha$, $\beta$ and $\gamma$ for various values of p

| p | r | r (rounded) | $\alpha$ | $\beta$ |
|---|---|---|---|---|
| 2 | 2 | 2 | n-1 | 1 |
| 3 | 2.74 | 3 | (3n-5)/2 | 2.5 |
| 4 | 3.47 | 3 | (3n-9)/2 | 4.5 |
| 5 | 4.21 | 4 | 2n-7 | 7 |
| 6 | 4.95 | 5 | 5(n-4)/2 | 10 |
| 7 | 5.69 | 6 | (6(n-27)/2 | 13.5 |
| 8 | 6.43 | 6 | (6n-35)/2 | 17.5 |
| 9 | 7.17 | 7 | (7n-44)/2 | 22 |
| 10 | 7.92 | 8 | 4n-27 | 27 |

## Approximation using Box's Series

Following the procedure of Box [2], we have (see Anderson [1], page 263) the following asymptotic expansion for W:

(3.5) $$\Pr\{-n\rho \log W \le z\} = \Pr(x_f^2 \le z) + w_2(\Pr\{x_{f+4}^2 < z\} - \Pr\{x_f^2 \le z\}) + 0(n^{-3})$$

where $$\rho = 1-(2p^2+p+2)/(6pn)$$

and $$w_2 = (p+2)(p-1)(p-2)(2p^3+6p^2+3p+2)/(288p^2n^2\rho^2) .$$

### Mauchly's Approximation

Mauchly [20] has computed approximate percentage points of W, only for p = 3 by fitting a Pearson curve of the type

$$y = K\, x^{p-1}\, (1-x)^{q-1}$$

by adjusting p and q so as to obtain agreement with the first two moments of the actual distribution.

### Davis Approximation

Davis [8] has obtained percentile approximations for $-n\rho \log W$, then $\rho$ is the same as in (3.5), by means of a Cornish-Fisher Inversion of Box's series, expressing the percentage points of the distribution in terms of chi-squared percentiles.

Comparisons of the accuracy of these four approximations are carried out in the next section.

## 4. COMPUTATIONS USING SERIES FORMS AND THE APPROXIMATIONS

Some of the cases studied are summarized in Tables 3 - 5. Table 3 gives 5% and 1% points for the exact distribution of W, together with the percentage points as approximated by Mauchly's, Box's series and Wilks-Tukey's approximations for various values of p and N. Table 4 gives the .005, .01, .025, .05, .1 and .25 significance points of the exact distribution of W for p = 3(1)10 and various N. Table 5 gives comparison with Davis' approximations.

Table 3 reveals that even for moderate sample size N, the approximations given by Mauchly for $p = 3$ is extremely poor. Box's series approximation is reasonably good for small values of p and even moderate values of N. Davis' results are generally correct to the decimal he has given but his table is incomplete in regard to small values of N for the values of p he has considered, i.e. $p = 3$, 6 and 10.

Table 3. Percentage Points for W from the Exact Distribution and Approximations: Mauchly, Wilks & Tukey and Box Series

p = 3

| | N = 8 | | N = 10 | | N = 12 | |
|---|---|---|---|---|---|---|
| | 5% | 1% | 5% | 1% | 5% | 1% |
| Mauchly | .172 | .083 | .278 | .165 | .366 | .243 |
| Wilks & Tukey | .14780 | .07355 | .24194 | .14434 | .32344 | .21458 |
| Box Series | .14050 | .06843 | .23576 | .13912 | .31842 | .20989 |
| Exact | .14026 | .06815 | .23564 | .13898 | .31836 | .20981 |

p = 4

| | N = 6 | | N = 10 | | N = 15 | |
|---|---|---|---|---|---|---|
| Wilks & Tukey | $.0^{2}274$ | $.0^{3}46$ | .09040 | .04563 | .24680 | .16636 |
| Box Series | $.0^{2}471$ | $.0^{2}102$ | .09789 | .05061 | .25365 | .17229 |
| Exact | $.0^{2}38662$ | $.0^{3}69040$ | .097393 | .050095 | .25352 | .17211 |

p = 5

| | N = 7 | | N = 10 | | N = 14 | |
|---|---|---|---|---|---|---|
| Wilks & Tukey | $.0^{2}159$ | $.0^{3}33$ | .03103 | .01373 | .11354 | .06852 |
| Box Series | $.0^{2}186$ | $.0^{3}44$ | .03192 | .01428 | .11497 | .06957 |
| Exact | $.0^{2}12621$ | $.0^{3}21839$ | .031104 | .013613 | .11460 | .069151 |

p = 6

| | N = 8 | | N = 12 | | N = 14 | |
|---|---|---|---|---|---|---|
| Wilks & Tukey | $.0^{3}86$ | $.0^{3}20$ | .02610 | .01264 | .05238 | .02922 |
| Box Series | $.0^{3}77$ | $.0^{3}19$ | .02502 | .01206 | .05106 | .02835 |
| Exact | $.0^{3}42669$ | $.0^{3}71870$ | .024325 | .011478 | .050510 | .027821 |

p = 7

| | N = 9 | | N = 11 | | N = 15 | |
|---|---|---|---|---|---|---|
| Wilks & Tukey | $.0^{3}45$ | $.0^{3}11$ | $.0^{2}415$ | $.0^{2}159$ | .02972 | .01620 |
| Box Series | $.0^{3}33$ | $.0^{4}9$ | $.0^{2}345$ | $.0^{2}132$ | .02765 | .01491 |
| Exact | $.0^{3}14730$ | $.0^{4}24239$ | $.0^{2}29501$ | $.0^{2}10165$ | .027115 | .014444 |

p = 8

| | N = 10 | | N = 12 | | N = 14 | |
|---|---|---|---|---|---|---|
| Wilks & Tukey | $.0^{4}9$ | $.0^{4}2$ | $.0^{2}129$ | $.0^{3}45$ | $.0^{2}551$ | $.0^{2}246$ |
| Box Series | $.0^{3}15$ | $.0^{4}4$ | $.0^{2}155$ | $.0^{3}59$ | $.0^{2}606$ | $.0^{2}279$ |
| Exact | $.0^{4}51489$ | $.0^{5}83064$ | $.0^{2}12329$ | $.0^{3}41204$ | $.0^{2}56126$ | $.0^{2}24756$ |

p = 9

| | N = 12 | | N = 14 | | N = 16 | |
|---|---|---|---|---|---|---|
| Wilks & Tukey | $.0^{3}22$ | $.0^{4}7$ | $.0^{2}151$ | $.0^{3}61$ | $.0^{2}493$ | $.0^{2}236$ |
| Box Series | $.0^{3}25$ | $.0^{4}8$ | $.0^{2}155$ | $.0^{3}65$ | $.0^{2}496$ | $.0^{2}241$ |
| Exact | $.0^{3}13971$ | $.0^{4}35438$ | $.0^{2}12945$ | $.0^{3}49428$ | $.0^{2}46163$ | $.0^{2}21595$ |

p = 10

| | N = 12 | | N = 14 | | N = 16 | |
|---|---|---|---|---|---|---|
| Wilks & Tukey | $.0^{4}28$ | $.0^{5}7$ | $.0^{3}34$ | $.0^{3}12$ | $.0^{2}151$ | $.0^{3}67$ |
| Box Series | $.0^{4}29$ | $.0^{5}9$ | $.0^{3}31$ | $.0^{3}12$ | $.0^{2}141$ | $.0^{3}63$ |
| Exact | $.0^{5}64552$ | $.0^{5}10036$ | $.0^{3}21066$ | $.0^{4}66814$ | $.0^{2}12140$ | $.0^{3}50647$ |

Table 4. Percentage Points of Sphericity Criterion W

p = 2

| N \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 3 | $.0^{4}25000$ | $.0^{3}10000$ | $.0^{3}62500$ | $.0^{2}25000$ | .010000 | .062500 |
| 4 | $.0^{2}50000$ | .010000 | .025000 | .050000 | .10000 | .25000 |
| 5 | .029240 | .046416 | .08550 | .13572 | .21544 | .39685 |
| 6 | .070711 | .10000 | .15811 | .22361 | .31623 | .50000 |
| 7 | .12011 | .15849 | .22865 | .30171 | .39811 | .57435 |
| 8 | .17100 | .21544 | .29240 | .36840 | .46416 | .62996 |
| 9 | .22007 | .26827 | .34855 | .42489 | .51795 | .67295 |
| 10 | .26591 | .31623 | .39764 | .47287 | .56234 | .70711 |
| 11 | .30808 | .35938 | .44054 | .51390 | .59948 | .73487 |
| 12 | .34657 | .39811 | .47818 | .54928 | .63096 | .75786 |
| 13 | .38162 | .43288 | .51135 | .58003 | .65793 | .77720 |
| 14 | .41352 | .46416 | .54074 | .60696 | .68129 | .79370 |
| 15 | .44258 | .49239 | .56693 | .63073 | .70170 | .80793 |
| 16 | .46912 | .51795 | .59038 | .65184 | .71969 | .82034 |
| 17 | .49340 | .54117 | .61149 | .67070 | .73564 | .83124 |
| 18 | .51567 | .56234 | .63058 | .68766 | .74989 | .84090 |
| 19 | .53616 | .58171 | .64792 | .70297 | .76270 | .84951 |
| 20 | .55505 | .59945 | .66373 | .71687 | .77526 | .85724 |
| 22 | .58870 | .63096 | .69150 | .74113 | .79433 | .87055 |
| 24 | .61775 | .65793 | .71509 | .76160 | .81113 | .88159 |
| 26 | .64305 | .68129 | .73535 | .77908 | .82540 | .89090 |
| 28 | .66527 | .70170 | .75295 | .79418 | .83768 | .89885 |
| 30 | .68492 | .71969 | .76836 | .80736 | .84834 | .90572 |
| 34 | .71810 | .74989 | .79409 | .82925 | .86596 | .91700 |
| 38 | .74501 | .77426 | .81470 | .84668 | .87992 | .92587 |
| 42 | .76727 | .79433 | .83157 | .86089 | .89125 | .93303 |
| 46 | .78597 | .81113 | .84563 | .87269 | .90063 | .93893 |
| 50 | .80191 | .82540 | .85753 | .88265 | .90852 | .94387 |
| 60 | .83302 | .85317 | .88056 | .90186 | .92367 | .95332 |
| 70 | .85570 | .87333 | .89718 | .91566 | .93452 | .96005 |
| 80 | .87297 | .88862 | .90975 | .92606 | .94267 | .96508 |
| 90 | .88655 | .90063 | .91958 | .93418 | .94901 | .96898 |
| 100 | .89751 | .91030 | .92748 | .94069 | .95410 | .97210 |
| 120 | .91411 | .92491 | .93939 | .95049 | .96172 | .97678 |
| 140 | .92609 | .93544 | .94794 | .95751 | .96718 | .98011 |
| 160 | .93513 | .94337 | .95438 | .96279 | .97127 | .98261 |
| 180 | .94221 | .94957 | .95940 | .96690 | .97446 | .98454 |
| 200 | .94789 | .95455 | .96342 | .97019 | .97701 | .98609 |
| 250 | .95817 | .96354 | .97069 | .97613 | .98160 | .98888 |
| 300 | .96507 | .96957 | .97555 | .98010 | .98467 | .99074 |

Table 4 (Continued)

p = 3

| N \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 4 | $.0^{5}39305$ | $.0^{4}15228$ | $.0^{4}99478$ | $.0^{3}40104$ | $.0^{2}16700$ | .011603 |
| 5 | $.0^{2}11700$ | $.0^{2}23667$ | $.0^{2}61070$ | .012679 | .026853 | .076732 |
| 6 | $.0^{2}88748$ | .014398 | .027585 | .045683 | .076928 | .16044 |
| 7 | .025882 | .037466 | .061687 | .090921 | .13590 | .24004 |
| 8 | .050467 | .068151 | .10225 | .14026 | .19471 | .31002 |
| 9 | .079827 | .10285 | .14486 | .18921 | .24970 | .37019 |
| 10 | .11161 | .13898 | .18696 | .23564 | .29971 | .42176 |
| 11 | .14418 | .17494 | .22726 | .27876 | .34471 | .46613 |
| 12 | .17647 | .20981 | .26516 | .31836 | .38503 | .50453 |
| 13 | .20786 | .24391 | .30048 | .35457 | .42118 | .53800 |
| 14 | .23799 | .27457 | .33321 | .38762 | .45365 | .56738 |
| 15 | .26666 | .30417 | .36350 | .41779 | .48290 | .59335 |
| 16 | .29383 | .33192 | .39149 | .44538 | .50934 | .61645 |
| 17 | .31948 | .35789 | .41737 | .47065 | .53332 | .63712 |
| 18 | .34366 | .38219 | .44133 | .49386 | .55516 | .65571 |
| 19 | .36644 | .40492 | .46355 | .51522 | .57511 | .67251 |
| 20 | .38789 | .42619 | .48417 | .53493 | .59340 | .68778 |
| 22 | .42713 | .46482 | .52124 | .57006 | .62573 | .71444 |
| 24 | .46203 | .49889 | .55354 | .60040 | .65338 | .73694 |
| 26 | .49319 | .52908 | .58190 | .62684 | .67729 | .75618 |
| 28 | .52111 | .55598 | .60696 | .65006 | .69816 | .77281 |
| 30 | .54624 | .58007 | .62926 | .67060 | .71651 | .78732 |
| 34 | .58958 | .62136 | .66715 | .70529 | .74730 | .81144 |
| 38 | .62556 | .65540 | .69811 | .73343 | .77210 | .83066 |
| 42 | .65584 | .68391 | .72386 | .75670 | .79248 | .84634 |
| 46 | .68166 | .70811 | .74559 | .77626 | .80953 | .85736 |
| 50 | .70393 | .72891 | .76417 | .79293 | .82400 | .87035 |
| 60 | .74809 | .76997 | .80064 | .82546 | .85211 | .89155 |
| 70 | .78086 | .80028 | .82737 | .84918 | .87249 | .90679 |
| 80 | .80612 | .82356 | .84779 | .86723 | .88794 | .91828 |
| 90 | .82617 | .84199 | .86390 | .88143 | .90006 | .92725 |
| 100 | .84247 | .85694 | .87693 | .89289 | .90981 | .93444 |
| 120 | .46737 | .87972 | .89672 | .91024 | .92454 | .94527 |
| 140 | .88548 | .89624 | .91103 | .92276 | .93513 | .95303 |
| 160 | .89925 | .90878 | .92186 | .93221 | .94312 | .95886 |
| 180 | .91006 | .91861 | .93034 | .93961 | .94936 | .96340 |
| 200 | .91877 | .92654 | .93716 | .94555 | .95436 | .96704 |
| 250 | .93462 | .94092 | .94952 | .95629 | .96340 | .97361 |
| 300 | .94529 | .95059 | .95781 | .96350 | .96945 | .97799 |

Table 4 (Continued)

p = 4

| N \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 5 | $.0^{6}91162$ | $.0^{5}36645$ | $.0^{4}23265$ | $.0^{4}95283$ | $.0^{3}40030$ | $.0^{2}29305$ |
| 6 | $.0^{3}33678$ | $.0^{3}69040$ | $.0^{2}18194$ | $.0^{2}38662$ | $.0^{2}84730$ | .026147 |
| 7 | $.0^{2}30556$ | $.0^{2}50312$ | $.0^{2}99040$ | .016868 | .029512 | .066529 |
| 8 | .010209 | .015033 | .025485 | .038664 | .060019 | .11410 |
| 9 | .022162 | .030463 | .047058 | .066398 | .095554 | .16287 |
| 10 | .038208 | .050095 | .072584 | .097393 | .1396 | .20994 |
| 11 | .057311 | .072583 | .10033 | .12972 | .17030 | .25404 |
| 12 | .078477 | .096785 | .12902 | .16211 | .20651 | .29477 |
| 13 | .10089 | .12183 | .15780 | .19381 | .24102 | .33213 |
| 14 | .12391 | .14708 | .18610 | .22435 | .27358 | .36631 |
| 15 | .14708 | .17211 | .21356 | .25352 | .30412 | .39756 |
| 16 | .17006 | .19663 | .23999 | .28119 | .33269 | .42615 |
| 17 | .19263 | .22044 | .26528 | .30736 | .35936 | .45236 |
| 18 | .21462 | .24343 | .28938 | .33205 | .38425 | .47643 |
| 19 | .23595 | .26553 | .31230 | .35332 | .40749 | .49860 |
| 20 | .25655 | .28673 | .33406 | .37723 | .42920 | .51905 |
| 22 | .29546 | .32641 | .37429 | .41734 | .46850 | .55550 |
| 24 | .33132 | .36261 | .41046 | .45301 | .50304 | .58698 |
| 26 | .36428 | .39559 | .44305 | .48484 | .53356 | .61440 |
| 28 | .39455 | .42567 | .47247 | .51337 | .56068 | .63847 |
| 30 | .42235 | .45313 | .49912 | .53903 | .58492 | .65977 |
| 34 | .47149 | .50132 | .54542 | .58326 | .62635 | .69571 |
| 38 | .51337 | .54207 | .58415 | .61995 | .66039 | .72486 |
| 42 | .54938 | .57689 | .61695 | .65082 | .68883 | .74894 |
| 46 | .58059 | .60692 | .64506 | .67712 | .71293 | .76918 |
| 50 | .60788 | .63307 | .66939 | .69978 | .73359 | .78641 |
| 60 | .66298 | .68558 | .71790 | .74471 | .77429 | .82004 |
| 70 | .70468 | .72509 | .75409 | .77801 | .80425 | .84454 |
| 80 | .73729 | .75584 | .78211 | .80366 | .82721 | .86318 |
| 90 | .76346 | .78045 | .80441 | .82401 | .84536 | .87784 |
| 100 | .78491 | .80057 | .82259 | .84055 | .86007 | .88966 |
| 120 | .81798 | .83149 | .85042 | .86580 | .88244 | .90756 |
| 140 | .84225 | .85413 | .87072 | .88415 | .89865 | .92046 |
| 160 | .86083 | .87141 | .88617 | .89809 | .91094 | .93021 |
| 180 | .87550 | .88504 | .89832 | .90904 | .92057 | .93782 |
| 200 | .88737 | .89606 | .90814 | .91787 | .92832 | .94394 |
| 250 | .90906 | .91615 | .92599 | .93390 | .94238 | .95501 |
| 300 | .92375 | .92974 | .93804 | .94470 | .95183 | .96243 |

Table 4 (Continued)

p = 5

| N \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 6 | $.0^{6}24579$ | $.0^{6}98368$ | $.0^{5}72524$ | $.0^{4}25776$ | $.0^{3}10959$ | $.0^{3}83762$ |
| 7 | $.0^{3}10563$ | $.0^{3}21839$ | $.0^{3}58374$ | $.0^{2}12621$ | $.0^{2}28373$ | $.0^{2}92522$ |
| 8 | $.0^{2}10968$ | $.0^{2}18281$ | $.0^{2}36768$ | $.0^{2}64001$ | .011530 | .027554 |
| 9 | $.0^{2}40994$ | $.0^{2}61227$ | .010628 | .016501 | .026388 | .053105 |
| 10 | $.0^{2}97579$ | .013613 | .021543 | .031104 | .046080 | .082916 |
| 11 | .018156 | .024161 | .035852 | .049192 | .069047 | .11473 |
| 12 | .0290262 | .037303 | .052770 | .069704 | .093963 | .14705 |
| 13 | .041953 | .052479 | .071536 | .091741 | .11983 | .17893 |
| 14 | .056485 | .069151 | .091503 | .11460 | .14594 | .20983 |
| 15 | .072206 | .086848 | .11215 | .13775 | .17180 | .23944 |
| 16 | .088751 | .10518 | .13309 | .16082 | .19710 | .26760 |
| 17 | .10582 | .12385 | .15402 | .18354 | .22163 | .29428 |
| 18 | .12317 | .14261 | .17473 | .20575 | .24527 | .31947 |
| 19 | .14061 | .16129 | .19507 | .22731 | .26797 | .34324 |
| 20 | .15799 | .17974 | .21492 | .24817 | .28969 | .36563 |
| 22 | .19215 | .21560 | .25292 | .28761 | .33025 | .40663 |
| 24 | .22503 | .24971 | .28847 | .32400 | .36713 | .44311 |
| 26 | .25634 | .28186 | .32151 | .35746 | .40063 | .47566 |
| 28 | .28593 | .31200 | .35214 | .38818 | .43110 | .50482 |
| 30 | .31379 | .34018 | .38049 | .41641 | .45885 | .53106 |
| 34 | .36449 | .39103 | .43108 | .46628 | .50740 | .57625 |
| 38 | .40909 | .43536 | .47461 | .50878 | .54831 | .61370 |
| 42 | .44838 | .47413 | .51231 | .54529 | .58315 | .64520 |
| 46 | .48312 | .50821 | .54519 | .57692 | .61314 | .67203 |
| 50 | .51397 | .53834 | .57407 | .60456 | .63919 | .69513 |
| 60 | .57761 | .60010 | .63275 | .66033 | .69137 | .74089 |
| 70 | .62690 | .64760 | .67745 | .70250 | .73049 | .77478 |
| 80 | .66607 | .68517 | .71257 | .73544 | .76088 | .80086 |
| 90 | .69790 | .71558 | .74086 | .76186 | .78514 | .82155 |
| 100 | .72425 | .74069 | .76411 | .78351 | .80495 | .83836 |
| 120 | .76529 | .77966 | .80005 | .81686 | .83535 | .86399 |
| 140 | .79575 | .80850 | .82652 | .84133 | .85757 | .88262 |
| 160 | .81924 | .83068 | .84682 | .86004 | .87451 | .89676 |
| 180 | .83790 | .84827 | .86287 | .87481 | .88786 | .90787 |
| 200 | .85307 | .86255 | .87588 | .88677 | .89864 | .91682 |
| 250 | .88095 | .8875 | .89969 | .90860 | .91828 | .93307 |
| 300 | .89995 | .90656 | .91584 | .92337 | .93155 | .94401 |

Table 4 (Continued)

p = 6

| N \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 7 | $.0^{7}70557$ | $.0^{6}29697$ | $.0^{5}18030$ | $.0^{5}74790$ | $.0^{4}31547$ | $.0^{3}24844$ |
| 8 | $.0^{4}34541$ | $.0^{4}71870$ | $.0^{3}19456$ | $.0^{3}42669$ | $.0^{3}97879$ | $.0^{2}33335$ |
| 9 | $.0^{3}40126$ | $.0^{3}67578$ | $.0^{2}13837$ | $.0^{2}25527$ | $.0^{2}45255$ | .011336 |
| 10 | $.0^{2}16522$ | $.0^{2}24979$ | $.0^{2}44243$ | $.0^{2}70038$ | .011482 | .024216 |
| 11 | $.0^{2}42686$ | $.0^{2}60326$ | $.0^{2}97479$ | .014353 | .021791 | .041033 |
| 12 | $.0^{2}85127$ | .011478 | .017390 | .024325 | .034966 | .060679 |
| 13 | .014444 | .018800 | .027141 | .036529 | .050383 | .082172 |
| 14 | .021960 | .027821 | .038682 | .050510 | .067439 | .10473 |
| 15 | .030903 | .038302 | .051661 | .065830 | .085610 | .12776 |
| 16 | .041061 | .049980 | .065743 | .082100 | .104468 | .15085 |
| 17 | .052219 | .062606 | .080634 | .098998 | .12368 | .17368 |
| 18 | .064174 | .075950 | .096078 | .11626 | .14298 | .19605 |
| 19 | .076743 | .089816 | .11187 | .13367 | .16217 | .21782 |
| 20 | .089763 | .10403 | .12783 | .15107 | .18112 | .23890 |
| 22 | .11662 | .13299 | .15975 | .18538 | .21788 | .27885 |
| 24 | .14388 | .16196 | .19107 | .21850 | .25277 | .31577 |
| 26 | .17096 | .19041 | .22132 | .25008 | .28556 | .34973 |
| 28 | .19745 | .21798 | .25027 | .27997 | .31624 | .38093 |
| 30 | .22313 | .24449 | .27778 | .30812 | .34485 | .40960 |
| 34 | .27153 | .29397 | .32844 | .35939 | .39633 | .46025 |
| 38 | .31571 | .33866 | .37355 | .40450 | .44105 | .50339 |
| 42 | .35576 | .37885 | .41364 | .44424 | .48005 | .54044 |
| 46 | .39199 | .41497 | .44935 | .47936 | .51425 | .57254 |
| 50 | .42477 | .44748 | .48125 | .51055 | .54441 | .60056 |
| 60 | .49406 | .51571 | .54754 | .57485 | .60606 | .65708 |
| 70 | .54917 | .56955 | .59930 | .62460 | .65331 | .69978 |
| 80 | .59381 | .61292 | .64067 | .66413 | .69059 | .73311 |
| 90 | .63060 | .64852 | .67442 | .69622 | .72071 | .75983 |
| 100 | .66139 | .67822 | .70246 | .72278 | .74553 | .78171 |
| 120 | .70994 | .72488 | .74628 | .76412 | .78400 | .81540 |
| 140 | .74642 | .75981 | .77891 | .79479 | .81241 | .84011 |
| 160 | .77480 | .78691 | .80415 | .81843 | .83423 | .85899 |
| 180 | .79750 | .80854 | .82423 | .83720 | .85152 | .87389 |
| 200 | .81605 | .82620 | .84058 | .85246 | .86555 | .88595 |
| 250 | .85037 | .85879 | .87069 | .88048 | .89124 | .90796 |
| 300 | .87391 | .88110 | .89124 | .89957 | .90870 | .92285 |

Table 4 (Continued)

p = 7

| N \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 8 | $.0^{7}21289$ | $.0^{7}86044$ | $.0^{6}55120$ | $.0^{5}22835$ | $.0^{5}95942$ | $.0^{4}72580$ |
| 9 | $.0^{4}115809$ | $.0^{4}24239$ | $.0^{4}66388$ | $.0^{3}14730$ | $.0^{3}34311$ | $.0^{2}12149$ |
| 10 | $.0^{3}14825$ | $.0^{3}25197$ | $.0^{3}52402$ | $.0^{3}94336$ | $.0^{2}17761$ | $.0^{2}46290$ |
| 11 | $.0^{3}66517$ | $.0^{2}10165$ | $.0^{2}18324$ | $.0^{2}29501$ | $.0^{2}49404$ | .010845 |
| 12 | $.0^{2}18510$ | $.0^{2}26462$ | $.0^{2}43552$ | $.0^{2}65237$ | .010119 | .019815 |
| 13 | $.0^{2}39356$ | $.0^{2}53692$ | $.0^{2}82864$ | .011790 | .017307 | .031195 |
| 14 | $.0^{2}70567$ | $.0^{2}92955$ | .013667 | .018704 | .026327 | .044531 |
| 15 | .011263 | .014435 | .020431 | .027115 | .036919 | .059370 |
| 16 | .016537 | .020729 | .028448 | .036821 | .048798 | .075298 |
| 17 | .022812 | .028074 | .037553 | .047610 | .061690 | .091967 |
| 18 | .029994 | .036345 | .047578 | .059270 | .075347 | .10909 |
| 19 | .037977 | .045412 | .058355 | .071609 | .089554 | .12644 |
| 20 | .046648 | .055143 | .069730 | .084457 | .10413 | .14384 |
| 22 | .065631 | .076124 | .093740 | .11111 | .13380 | .17825 |
| 24 | .086164 | .098448 | .11870 | .13831 | .16346 | .21158 |
| 26 | .10761 | .12146 | .14396 | .16541 | .19254 | .24343 |
| 28 | .12949 | .14467 | .16905 | .19200 | .22067 | .27360 |
| 30 | .15142 | .16774 | .19367 | .21781 | .24767 | .30205 |
| 34 | .19449 | .21253 | .24073 | .26653 | .29794 | .35389 |
| 38 | .23555 | .25472 | .28433 | .31106 | .34320 | .39952 |
| 42 | .27402 | .29390 | .32429 | .35146 | .38380 | .43972 |
| 46 | .30974 | .33002 | .36076 | .38801 | .42019 | .47524 |
| 50 | .34277 | .36321 | .39399 | .42108 | .45287 | .50678 |
| 60 | .41458 | .43479 | .46486 | .49099 | .52126 | .57177 |
| 70 | .47344 | .49296 | .52177 | .54656 | .57505 | .62203 |
| 80 | .52215 | .54082 | .56817 | .59156 | .61826 | .66192 |
| 90 | .56296 | .58071 | .60661 | .62863 | .65366 | .69431 |
| 100 | .59755 | .61441 | .63890 | .65965 | .68314 | .72109 |
| 120 | .65283 | .66805 | .69003 | .70854 | .72937 | .76277 |
| 140 | .69496 | .70876 | .72861 | .74526 | .76391 | .79366 |
| 160 | .72806 | .74066 | .75872 | .77381 | .79067 | .81747 |
| 180 | .75474 | .76630 | .78284 | .79664 | .81201 | .83636 |
| 200 | .77668 | .78735 | .80260 | .81529 | .82941 | .85172 |
| 250 | .81754 | .82649 | .83923 | .84979 | .86149 | .87991 |
| 300 | .84580 | .85349 | .86441 | .87344 | .88343 | .89910 |

Table 4 (Continued)

p = 8

| N \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 9 | $.0^{8}71788$ | $.0^{7}27598$ | $.0^{6}17155$ | $.0^{6}72189$ | $.0^{5}31412$ | $.0^{4}20202$ |
| 10 | $.0^{5}39473$ | $.0^{5}83064$ | $.0^{4}22961$ | $.0^{4}51489$ | $.0^{3}12180$ | $.0^{3}44134$ |
| 11 | $.0^{4}55082$ | $.0^{4}94377$ | $.0^{3}19900$ | $.0^{3}36314$ | $.0^{3}69598$ | $.0^{2}18771$ |
| 12 | $.0^{3}26703$ | $.0^{3}41204$ | $.0^{3}75447$ | $.0^{2}12329$ | $.0^{2}21036$ | $.0^{2}47822$ |
| 13 | $.0^{3}79550$ | $.0^{2}11491$ | $.0^{2}19226$ | $.0^{2}29243$ | $.0^{2}46224$ | $.0^{2}93706$ |
| 14 | $.0^{2}17954$ | $.0^{2}24756$ | $.0^{2}38847$ | $.0^{2}56126$ | $.0^{2}83944$ | .015649 |
| 15 | $.0^{2}33920$ | $.0^{2}45162$ | $.0^{2}67510$ | $.0^{2}93791$ | .013445 | .023497 |
| 16 | $.0^{2}56696$ | $.0^{2}73433$ | .010564 | .014227 | .019719 | .032724 |
| 17 | $.0^{2}86711$ | .010982 | .015313 | .020106 | .027108 | .043115 |
| 18 | .012404 | .015421 | .020950 | .026931 | .035479 | .054455 |
| 19 | .016850 | .020620 | .027401 | .034597 | .044691 | .066544 |
| 20 | .021969 | .026523 | .034584 | .042993 | .054605 | .079202 |
| 22 | .034018 | .040171 | .050778 | .061544 | .076025 | .10563 |
| 24 | .048080 | .055801 | .068835 | .081781 | .098838 | .13274 |
| 26 | .063675 | .072873 | .088141 | .10304 | .12235 | .15986 |
| 28 | .080369 | .090921 | .10820 | .12482 | .14605 | .18654 |
| 30 | .097792 | .10956 | .12861 | .14671 | .16957 | .21248 |
| 34 | .13368 | .14749 | .16941 | .18984 | .21517 | .26153 |
| 38 | .16963 | .18497 | .20900 | .23105 | .25801 | .30642 |
| 42 | .20463 | .22109 | .24659 | .26972 | .29769 | .34717 |
| 46 | .23812 | .25537 | .28186 | .30567 | .33419 | .38404 |
| 50 | .26985 | .28764 | .31474 | .33892 | .36766 | .41741 |
| 60 | .34107 | .35942 | .38699 | .41123 | .43966 | .48794 |
| 70 | .40149 | .41973 | .44689 | .47052 | .49797 | .54398 |
| 80 | .45272 | .47053 | .49686 | .51959 | .54581 | .58935 |
| 90 | .49642 | .51364 | .53895 | .56068 | .58561 | .62671 |
| 100 | .53399 | .55054 | .57479 | .59551 | .61917 | .65795 |
| 120 | .59497 | .61020 | .63235 | .65116 | .67250 | .70718 |
| 140 | .64217 | .65616 | .67642 | .69354 | .71289 | .74412 |
| 160 | .67967 | .79256 | .71117 | .72684 | .74448 | .77285 |
| 180 | .71015 | .72207 | .73925 | .75367 | .76986 | .79580 |
| 200 | .73538 | .74646 | .76238 | .77572 | .79067 | .81456 |
| 250 | .78277 | .79216 | .80559 | .81680 | .82932 | .84922 |
| 300 | .81582 | .82394 | .83554 | .84520 | .85595 | .87299 |

Table 4 (Continued)

p = 9

| N \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 10 | $.0^{8}25350$ | $.0^{8}92163$ | $.0^{7}56095$ | $.0^{6}23259$ | $.0^{5}10164$ | $.0^{5}61014$ |
| 11 | $.0^{5}13612$ | $.0^{5}28789$ | $.0^{5}80284$ | $.0^{4}18169$ | $.0^{4}42570$ | $.0^{3}16718$ |
| 12 | $.0^{4}20532$ | $.0^{4}35438$ | $.0^{4}75647$ | $.0^{3}13971$ | $.0^{3}27192$ | $.0^{3}75487$ |
| 13 | $.0^{3}10683$ | $.0^{3}16629$ | $.0^{3}30880$ | $.0^{3}51137$ | $.0^{3}88727$ | $.0^{2}20826$ |
| 14 | $.0^{3}33897$ | $.0^{3}49428$ | $.0^{3}83940$ | $.0^{2}12945$ | $.0^{2}20813$ | $.0^{2}43517$ |
| 15 | $.0^{3}80903$ | $.0^{2}11264$ | $.0^{2}17946$ | $.0^{2}26291$ | $.0^{2}399930$ | $.0^{2}76857$ |
| 16 | $.0^{2}16061$ | $.0^{2}21595$ | $.0^{2}32774$ | $.0^{2}46163$ | $.0^{2}67287$ | .012112 |
| 17 | $.0^{2}28055$ | $.0^{2}36693$ | $.0^{2}53582$ | $.0^{2}73143$ | .010304 | .017596 |
| 18 | $.0^{2}44630$ | $.0^{2}57070$ | $.0^{2}80753$ | .010744 | .014715 | .024063 |
| 19 | $.0^{2}66139$ | $.0^{2}8300$ | .011439 | .014894 | .019925 | .031413 |
| 20 | $.0^{2}92748$ | .011455 | .015437 | .019734 | .025874 | .039535 |
| 22 | .016116 | .019398 | .025210 | .031285 | .039702 | .057658 |
| 24 | .024859 | .029330 | .037060 | .044940 | .055599 | .077608 |
| 26 | .035265 | .040948 | .050587 | .060218 | .072995 | .098694 |
| 28 | .047052 | .053924 | .065397 | .076672 | .091395 | .12038 |
| 30 | .059939 | .067945 | .081137 | .093923 | .11040 | .14225 |
| 34 | .088001 | .098046 | .11426 | .12963 | .14902 | .18545 |
| 38 | .11775 | .12949 | .14914 | .16553 | .18713 | .22683 |
| 42 | .14799 | .16109 | .18165 | .20057 | .22377 | .26572 |
| 46 | .17794 | .19211 | .21411 | .23416 | .25850 | .30189 |
| 50 | .20710 | .22208 | .24517 | .26601 | .29111 | .33536 |
| 60 | .27498 | .29117 | .31575 | .33759 | .36349 | .40820 |
| 70 | .33480 | .35143 | .37641 | .39836 | .42410 | .46791 |
| 80 | .38693 | .40335 | .42832 | .44991 | .47503 | .51732 |
| 90 | .43229 | .44866 | .47289 | .49388 | .51815 | .55869 |
| 100 | .47190 | .48787 | .51140 | .53167 | .55501 | .59374 |
| 120 | .53733 | .55232 | .57426 | .59302 | .61448 | .64973 |
| 140 | .58885 | .60282 | .62318 | .64050 | .66020 | .69235 |
| 160 | .63029 | .64331 | .66220 | .67821 | .69636 | .72583 |
| 180 | .66429 | .67643 | .69401 | .70886 | .72564 | .75279 |
| 200 | .69264 | .70400 | .72040 | .73422 | .74981 | .77495 |
| 250 | .74638 | .75611 | .77011 | .78186 | .79505 | .81621 |
| 300 | .78422 | .79271 | .80488 | .81507 | .82648 | .84472 |

Table 4 (Continued)

p = 10

| N \ α | .005 | .010 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 11 | $.0^{8}11246$ | $.0^{8}35733$ | $.0^{7}19324$ | $.0^{7}77218$ | $.0^{6}33526$ | $.0^{5}21386$ |
| 12 | $.0^{6}47154$ | $.0^{5}10036$ | $.0^{5}28256$ | $.0^{5}64552$ | $.0^{4}15891$ | $.0^{4}60952$ |
| 13 | $.0^{5}76669$ | $.0^{4}13324$ | $.0^{4}28760$ | $.0^{4}53699$ | $.0^{3}10610$ | $.0^{3}30376$ |
| 14 | $.0^{4}42583$ | $.0^{4}66814$ | $.0^{3}12568$ | $.0^{3}21066$ | $.0^{3}37102$ | $.0^{3}89399$ |
| 15 | $.0^{3}14331$ | $.0^{3}21078$ | $.0^{3}36286$ | $.0^{3}56666$ | $.0^{3}92523$ | $.0^{2}19892$ |
| 16 | $.0^{3}36052$ | $.0^{3}50647$ | $.0^{3}81824$ | $.0^{2}12140$ | $.0^{2}18755$ | $.0^{2}37055$ |
| 17 | $.0^{3}75024$ | $.0^{2}10179$ | $.0^{2}15665$ | $.0^{2}22346$ | $.0^{2}33072$ | $.0^{2}61162$ |
| 18 | $.0^{2}13670$ | $.0^{2}18040$ | $.0^{2}26712$ | $.0^{2}36920$ | $.0^{2}52683$ | $.0^{2}92661$ |
| 19 | $.0^{2}22587$ | $.0^{2}29142$ | $.0^{2}41802$ | $.0^{2}56300$ | $.0^{2}78244$ | .013126 |
| 20 | $.0^{2}34640$ | $.0^{2}43855$ | $.0^{2}61253$ | $.0^{2}80714$ | .010952 | .017701 |
| 22 | $.0^{2}69174$ | $.0^{2}84980$ | .011376 | .014478 | .018907 | .028781 |
| 24 | .011834 | .014208 | .018414 | .022817 | .028932 | .042056 |
| 26 | .018193 | .021449 | .027092 | .032865 | .040709 | .057042 |
| 28 | .025886 | .030071 | .037194 | .044348 | .053894 | .073283 |
| 30 | .034761 | .039886 | .048484 | .056983 | .068156 | .090387 |
| 34 | .055364 | .062312 | .073700 | .084683 | .098783 | .12595 |
| 38 | .078662 | .087261 | .10111 | .11422 | .13076 | .16184 |
| 42 | .10353 | .11356 | .12949 | .14436 | .16285 | .19695 |
| 46 | .12913 | .14036 | .15800 | .17428 | .19429 | .23066 |
| 50 | .15483 | .16705 | .18607 | .20346 | .22464 | .26265 |
| 60 | .21714 | .23106 | .25238 | .27151 | .29443 | .33462 |
| 70 | .27444 | .28925 | .31169 | .33157 | .35512 | .39574 |
| 80 | .32589 | .34109 | .36390 | .38395 | .40748 | .44759 |
| 90 | .37168 | .38694 | .40970 | .42955 | .45270 | .49182 |
| 100 | .41235 | .42747 | .44991 | .46937 | .49195 | .52983 |
| 120 | .48083 | .49536 | .51674 | .53515 | .55634 | .59151 |
| 140 | .53578 | .54954 | .56969 | .58695 | .60669 | .63923 |
| 160 | .58059 | .59356 | .61249 | .62861 | .64700 | .67713 |
| 180 | .61773 | .62994 | .64770 | .66279 | .67994 | .70791 |
| 200 | .64895 | .66046 | .67715 | .69129 | .70732 | .73339 |
| 250 | .70872 | .71871 | .73313 | .74529 | .75903 | .78122 |
| 300 | .75125 | .76003 | .77268 | .78332 | .79529 | .81455 |

Table 5. Correction factors for Mauchly's sphericity test $-2\rho \log W$.

| N | p = 3 | | | | p = 6 | | | | p = 10 | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 5% | | 1% | | 5% | | 1% | | 5% | | 1% | |
| | Exact | Davis | Exact | Davis | Exact | Davis | Exact | Davis | Exact | Davis | Exact | Davis |
| 5 | 1.07404 | 1.074 | 1.09101 | 1.091 | | | | | | | | |
| 6 | 1.03761 | 1.0376 | 1.04629 | 1.0463 | | | | | | | | |
| 7 | 1.02279 | 1.0228 | 1.02804 | 1.0280 | 0.80918 | — | 0.86151 | — | | | | |
| 8 | 1.01529 | 1.0153 | 1.01881 | 1.0188 | 0.74526 | — | 0.76618 | — | | | | |
| 9 | 1.01097 | 1.0110 | 1.01349 | 1.0135 | 1.10560 | 1.105 | 1.12270 | 1.12 | | | | |
| 11 | 1.00644 | 1.0064 | 1.00790 | 1.0079 | 1.05084 | 1.0508 | 1.05810 | 1.058 | 1.46774 | — | 1.55147 | — |
| 13 | 1.00423 | 1.0042 | 1.00519 | 1.0052 | 1.03026 | 1.0303 | 1.03434 | 1.0343 | 1.15373 | 1.15 | 1.17243 | — |
| 16 | 1.00257 | 1.0026 | 1.00315 | 1.0031 | 1.01693 | 1.0169 | 1.01910 | 1.0191 | 1.06697 | 1.067 | 1.07328 | 1.073 |
| 21 | 1.00137 | 1.0014 | 1.00167 | 1.0017 | 1.00844 | 1.0084 | 1.00949 | 1.0095 | 1.02875 | 1.0287 | 1.03112 | 1.0311 |

# CHAPTER IV

# DISTRIBUTION OF THE LIKELIHOOD RATIO CRITERION FOR TESTING $\Sigma = \Sigma_0$

## 1. INTRODUCTION

Let p x 1 vectors $x_1, x_2, \ldots, x_N$ be a random sample from a p-variate normal distribution with unknown mean vector $\mu$ and positive definite covariance matrix $\Sigma$. The likelihood ratio criterion for testing the hypothesis $H_0$: $\Sigma = \Sigma_0$ against the alternatives $H_1$: $\Sigma \neq \Sigma_0$, for some given positive definite matrix $\Sigma_0$, is given by (Anderson [1]),

$$(1.1) \qquad \lambda = (e/N)^{\frac{Np}{2}} |S \Sigma_0^{-1}|^{N/2} e^{-\frac{1}{2} \operatorname{tr} \Sigma_0^{-1} S}$$

where $S = \sum_{i=1}^{N} (x_i - \bar{x})(x_i - \bar{x})'$ and $\bar{x} = \sum_{i=1}^{N} x_i/N$.

This likelihood ratio is not unbiased. However, if the criterion is modified by reducing the sample size N to the degrees of freedom n = N-1, then Sugiura and Nagao [27] have shown that the test is unbiased. The monotonicity of the power function with respect to each of the p characteristic roots of $\Sigma \Sigma_0^{-1}$ is established by Nagao [22] and Das Gupta [7].

Let

$$(1.2) \qquad \lambda_1 = (e/n)^{\frac{np}{2}} |\underset{\sim}{S}\, \underset{\sim}{\Sigma}_0^{-1}|^{\frac{n}{2}}\, e^{-\frac{1}{2}\, \mathrm{tr}\, \underset{\sim}{\Sigma}_0^{-1} \underset{\sim}{S}}$$

be the modified likelihood ratio statistic. Korin [17] expressed the null distribution of $-2 \log \lambda_1$ in the form of an asymptotic series of central chi-square distributions and computed its percentage points but his tables are incomplete in regard to small values of n for the values of $p = 3(1)10$. Recently Davis [8] expressed the percentage points of $-2 \log \lambda_1$ in terms of chisquared percentiles using a Cornish-Fisher inversion of Box's series but his tables are also incomplete in regard to small values of n for the values of p he has considered i.e. $p = 6$ and 10.

The object of the present chapter is to develop a method similar to the one used in Chapter III, in order to obtain the exact distribution of $L = \lambda_1^{2/n}$ in a series form and to compute percentage points of L to any degree of accuracy even for small sample sizes. Tables of percentage points for $p = 2(1)10$ for various significance levels are given and comparisons made with the results of Korin [17] and Davis [8].

## 2. DERIVATION OF THE DISTRIBUTION OF $L = \lambda_1^{2/n}$

The h-th moment of $\lambda_1$ under the null hypothesis is given by

$$(2.1) \qquad E(\lambda_1^h) = (2e/n)^{\frac{nhp}{2}} [\Gamma_p\{n(1+h)/2\}/\Gamma_p(n/2)] \cdot (1+h)^{-np(1+h)/2} ,$$

where $\Gamma_p(x) = \pi^{p(p-1)/4} \prod_{i=1}^{p} \Gamma\{x - (i-1)/2\}$ .

Let

$$L = \lambda_1^{2/n} .$$

Then

$$(2.2) \qquad E(L^h) = E(\lambda_1^{2h/n}) = \frac{(2e/n)^{ph}}{\prod_{\alpha=1}^{p} \Gamma(\frac{n+1-\alpha}{2})} \frac{\prod_{\alpha=1}^{p} \Gamma\{\frac{n}{2} + h + \frac{1-\alpha}{2}\}}{(1+2h/n)^{np(1+2h/n)/2}} .$$

Using inverse Mellins' transform, the density of L is given by

$$(2.3) \quad f(L) = [\prod_{\alpha=1}^{p} \Gamma(\frac{n+1-\alpha}{2})]^{-1} \cdot \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{L^{-h-1}(2e/n)^{ph} \prod_{\alpha=1}^{p} \Gamma\{\frac{n}{2} + h + \frac{1-\alpha}{2}\}}{(1+2h/n)^{np(1+2h/n)/2}} dh .$$

Putting $\frac{n}{2}$ +h=t in (2.3), we have

$$(2.4) \quad f(L) = K(p,n) \cdot L^{\frac{n}{2} - 1} \cdot \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} L^{-t} C(t) dt$$

where $c = \frac{n}{2}$ ,

$$(2.5) \qquad C(t) = (e/t)^{pt} \prod_{\alpha=1}^{p} \Gamma\{t + \frac{1-\alpha}{2}\} ,$$

and

$$(2.6) \qquad K(p,n) = (2e/n)^{-pn/2} [\prod_{\alpha=1}^{p} \Gamma(\frac{n+1-\alpha}{2})]^{-1} .$$

Using the expansion (2.64) of Chapter III to each gamma function in (2.5), we have

(2.7) $$\log C(t) = \frac{p}{2} \log (2\pi) - \frac{p(p+1)}{4} \log t + [A_1/t + A_2/t^2 + \ldots + A_\gamma/t^\gamma + \ldots]$$

where the coefficients $A_\gamma$'s are given by

(2.8) $$A_\gamma = (-1)^{\gamma-1} \left[ \sum_{\alpha=1}^{p} B_{\gamma+1} \left(\frac{1-\alpha}{2}\right)\right] / \gamma(\gamma+1)$$

where $B_\gamma(x)$ is the Bernoulli polynomial of degree $\gamma$ and order unity. Thus

(2.9) $$C(t) = (2\pi)^{p/2}\, t^{-p(p+1)/4}\, [1 + B_1/t + B_2/t^2 + \ldots]$$

where the coefficients $B_i$'s can be obtained from (2.67) of Chapter III.

Using (2.9) in (2.4), we have the density of L as

(2.10) $$f(L) = K(p,n) \cdot L^{\frac{n}{2}-1} \cdot (2\pi)^{p/2} \cdot \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} L^{-t} t^{-v} [1 + \sum_{\gamma=1}^{\infty} B_\gamma / t^\gamma] dt$$

where $K(p,n)$ is as given in (2.6) and $v = p(p+1)/4$.

The integral on the right hand side of (2.10) can be easily computed if $v$ is an integer and its value is by Cauchy's theorem of residues, the residue of $L^{-t}\, t^{-v}\, [1 + \sum_{\gamma=1}^{\infty} B_\gamma / t^\gamma]$ at $t = 0$. This is easily seen to be equal to

(2.11) $$\sum_{\gamma=0}^{\infty} [(-\log L)^{v+\gamma-1} B_\gamma / \Gamma(v+\gamma)], \quad B_0 = 1$$

and thus from (2.10), the density of L is

(2.12) $$f(L) = K(p,n)\, (2\pi)^{p/2}\, L^{\frac{n}{2}-1} \sum_{\gamma=0}^{\infty} (B_\gamma)(-\log L)^{v+\gamma-1} / \Gamma(v+\gamma).$$

The probability that L is less than any value, say $x_0$, is

$$(2.13)\quad P(L \leq x_0) = K(p,n)(2\pi)^{p/2} \sum_{\gamma=0}^{\infty} (B_\gamma) \int_0^{x_0} L^{\frac{n}{2}-1} (-\log L)^{v+\gamma-1}\, dL/\Gamma(v+\gamma) .$$

For computational purposes, we let

$$(2.14)\quad I_{v+\gamma-1,u} = \int_0^{x_0} L^u (-\log L)^{v+\gamma-1}\, dL/\Gamma(v+\gamma)$$

where $u = \frac{n}{2} - 1$. Then integrating by parts the right hand side of (2.14), we have the following recurrence relation:

$$(2.15)\quad I_{v+\gamma-1,u}(x_0) = [x_0^{u+1}(-\log x_0)^{v+\gamma-1}/\Gamma(v+\gamma) + I_{v+\gamma-2,u}]/(u+1)$$

and

$$(2.16)\quad I_{0,u}(x_0) = x_0^{(u+1)}/(u+1).$$

With this notation (2.14) can be written as

$$(2.17)\quad P(L \leq x_0) = K(p,n)(2\pi)^{p/2} \sum_{\gamma=0}^{\infty} (B_\gamma) I_{v+\gamma-1,u}(x_0)$$

where $I_{v+\gamma-1,u}(x_0)$ satisfies the recurrence relations (2.15) and (2.16). It is to be noted that (2.17) holds only if $v = p(p+1)/4$ is an integer. Otherwise we can appeal to Theorem 2.1 of Chapter III, since in this case

$$(2.18)\quad \phi(t) = t^{-v}[1 + \sum_{\gamma=1}^{\infty} B_\gamma/t^\gamma] = O(t^{-v}) .$$

Thus according to the theorem, we can expand $\phi(t)$ in the factorial series as

$$(2.19)\qquad \phi(t) = t^{-v}\,[1+\sum_{\gamma=1}^{\infty} B_\gamma/t^\gamma]$$

$$= \sum_{i=0}^{\infty} R_i\Gamma(t+\lambda)/\Gamma(t+v+i+\lambda)$$

where $\lambda$ is an arbitrary positive constant and can be chosen to govern the rate of convergence of the resulting series. The coefficients $R_i$'s can be determined explicitly as is done below.

Using the formula (2.6) of Chapter III to each gamma function on the right hand side of (2.19), we have

$$(2.20)\quad \log\{\Gamma(t+\lambda)/\Gamma(t+v+i+\lambda)\} = [-(v+i)\log t + \sum_{j=1}^{\infty} A_{ij}/t_j]$$

where the first few coefficients $A_{ij}$'s which are needed in our computations are listed below.

$$A_{i1} = -[t_i^2-\lambda^2-v]/2$$

$$A_{i2} = [t_i^3-\lambda^3-3(t_i^2-\lambda^2)/2+(t_i-\lambda)/2]/6$$

$$A_{i3} = -[t_i^4-\lambda^4-2(t_i^3-\lambda^3)+(t_i^2-\lambda^2)]/12$$

$$A_{i4} = [t_i^5-\lambda^5-5(t_i^4-\lambda^4)/2+5(t_i^3-\lambda^3)/3-(t_i-\lambda)/6]/20$$

$$(2.21)\quad A_{i5} = -[t_i^6-\lambda^6-3(t_i^5-\lambda^5)/2+5(t_i^4-\lambda^4)/2-(t_i^2-\lambda^2)/2]/30$$

$$A_{i6} = [t_i^7-\lambda^7-7(t_i^6-\lambda^6)/2+7(t_i^5-\lambda^5)/2$$
$$-7(t_i^3-\lambda^3)/6+(t_i-\lambda)/6]/42$$

$$A_{i7} = -[t_i^8-\lambda^8-4(t_i^7-\lambda^7)+14(t_i^6-\lambda^6)/3$$
$$-7(t_i^4-\lambda^4)/3+2(t_i^2-\lambda^2)/3]/56$$

$$A_{i8} = [t_i^9-\lambda^9-9(t_i^8-\lambda^8)/2+6(t_i^7-\lambda^7)-21(t_i^5-\lambda^5)/5$$
$$+2(t_i^3-\lambda^3)-3(t_i-\lambda)/10]/72$$

$$A_{i9} = -[t_i^{10}-\lambda^{10}-5(t_i^9-\lambda^9)+15(t_i^8-\lambda^8)/2$$
$$-7(t_i^6-\lambda^6)+5(t_i^4-\lambda^4)-3(t_i^2-\lambda^2)/2]/90$$

$$A_{i10} = [t_i^{11}-\lambda^{11}-11(t_i^{10}-\lambda^{10})/2+55(t_i^9-\lambda^9)/6$$
$$-11(t_i^7-\lambda^7)+11(t_i^5-\lambda^5)-11(t_i^3-\lambda^3)/2]/110$$

where $t_i = \lambda+v+i$. The remaining coefficients are listed in Appendix B. Thus from (2.20) we have

$$\Gamma(t+\lambda)/\Gamma(t+\lambda+v+i) = t^{-(v+i)}[1+\sum_{j=1}^{\infty}(Q_{ij}/t^j)], \tag{2.22}$$

where the coefficients $Q_{ij}$'s can be obtained in terms of $A_{ij}$ as in Chapter III. Using (2.22) on the right hand side of (2.19) we have

$$t^{-v}[1+\sum_{\gamma=1}^{\infty}(B_\gamma/t^\gamma)] = \sum_{i=0}^{\infty} R_i t^{-(v+i)}[1+\sum_{j=1}^{\infty}(Q_{ij}/t^j)]. \tag{2.23}$$

Equating the coefficients of t on both sides of (2.23), it can be seen that $R_i$'s can be determined using (2.90) of Chapter III. Now using (2.19) in (2.10) and noting that the term by term integration is valid since a factorial series is uniformly convergent in a half-plane (see Doetch [9]) we have the density of L in the case that v is not an integer in the form

$$(2.24)\quad f(L) = K(p,n)\, L^{\frac{n}{2}-1} (2\pi)^{p/2} \sum_{i=0}^{\infty} R_i \frac{1}{2\pi i} \cdot \int_{c-i\infty}^{c+i\infty} L^{-t}[\Gamma(t+\lambda)/\Gamma(t+\lambda+v+i)]\, dt$$

$$= K(p,n)(2\pi)^{p/2} \sum_{i=0}^{\infty} R_i\, L^{\frac{n}{2}+\lambda-1} (1-L)^{v+i-1}/\Gamma(v+i) .$$

The distribution of L is then given by

$$(2.25)\quad P(L \le x_0) = K(p,n)(2\pi)^{p/2} \sum_{i=0}^{\infty} R_i\, I_{x_0}(\tfrac{n}{2}+\lambda, v+i)/\Gamma(v+i)$$

where $I_{x_0}(p,q)$ is the incomplete beta function $\int_0^{x_0} x^{p-1}(1-x)^{q-1}dx$.

## 3. COMPUTATIONS OF PERCENTAGE POINTS

.005, .01, .025, .05, .1 and .25 significance points of $L = \lambda_1^{2/n}$ were computed for p = 2(1)10 and various n using (2.17) and (2.25) and these are presented in table 6 . The computation was carried out on CDC 6500 using double precision arithmetic. Table 7 compares the exact values with those obtained by Korin [17] and Davis [8].

Table 6. Percentage Points of $L = \lambda_1^2/n$ for Testing $\Sigma = \Sigma_0$

p = 2

| n \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 2 | $.0^{4}11569$ | $.0^{4}46327$ | $.0^{3}29074$ | $.0^{2}11716$ | $.0^{2}047596$ | .031333 |
| 3 | $.0^{2}27788$ | $.0^{2}55954$ | .014187 | .028861 | .059260 | .15763 |
| 4 | .018257 | .029250 | .054811 | .088639 | .14444 | .28133 |
| 5 | .047798 | .068263 | .10982 | .15809 | .22903 | .38018 |
| 6 | .085961 | .11453 | .16797 | .22533 | .30386 | .45748 |
| 7 | .12773 | .16242 | .22387 | .28636 | .36795 | .51863 |
| 8 | .16994 | .20897 | .27544 | .34047 | .42251 | .56786 |
| 9 | .21086 | .25281 | .32217 | .38807 | .46909 | .60820 |
| 10 | .24963 | .29344 | .36422 | .42995 | .50913 | .64180 |
| 11 | .28591 | .33080 | .40199 | .46689 | .54380 | .67017 |
| 12 | .31962 | .36503 | .43594 | .49961 | .57407 | .69443 |
| 13 | .35085 | .39636 | .46653 | .52874 | .60067 | .71541 |
| 14 | .37974 | .42505 | .49417 | .55479 | .62422 | .73371 |
| 15 | .40645 | .45136 | .51923 | .57821 | .64521 | .74982 |
| 16 | .43118 | .47554 | .54203 | .59935 | .66401 | .76410 |
| 17 | .45410 | .49779 | .56285 | .61852 | .68094 | .77685 |
| 18 | .47536 | .51834 | .58191 | .63599 | .69627 | .78829 |
| 19 | .49513 | .53734 | .59943 | .65195 | .71022 | .79863 |
| 20 | .51355 | .55496 | .61557 | .66659 | .72295 | .80800 |
| 22 | .54678 | .58658 | .64433 | .69252 | .74535 | .82437 |
| 24 | .57592 | .61411 | .66914 | .71474 | .76442 | .83817 |
| 26 | .60165 | .63828 | .69077 | .73400 | .78085 | .84996 |
| 28 | .62450 | .65966 | .70978 | .75084 | .79515 | .86016 |
| 30 | .64492 | .67869 | .72660 | .76569 | .80770 | .86906 |
| 32 | .66328 | .69573 | .74160 | .77888 | .81881 | .87690 |
| 34 | .67985 | .71107 | .75505 | .79067 | .82871 | .88385 |
| 36 | .69490 | .72495 | .76718 | .80127 | .83759 | .89006 |
| 38 | .70861 | .73758 | .77817 | .81086 | .84559 | .89564 |
| 40 | .72115 | .74910 | .78817 | .81956 | .85285 | .90068 |
| 45 | .74828 | .77395 | .80965 | .83819 | .86832 | .91138 |
| 50 | .77062 | .79434 | .82719 | .85334 | .88085 | .92001 |
| 55 | .78935 | .81137 | .84177 | .86590 | .89121 | .92710 |
| 60 | .80526 | .82580 | .85409 | .87648 | .89991 | .93304 |
| 65 | .81894 | .83819 | .86464 | .88552 | .90732 | .93808 |
| 70 | .83083 | .84894 | .87376 | .89332 | .913712 | .94242 |
| 75 | .84127 | .84835 | .88174 | .90013 | .91928 | .94619 |
| 80 | .85049 | .86666 | .88876 | .90613 | .92417 | .94950 |
| 85 | .85870 | .87405 | .89500 | .91144 | .92851 | .95242 |
| 90 | .86606 | .88067 | .90058 | .91619 | .93237 | .95502 |
| 95 | .87269 | .88662 | .90560 | .92045 | .93584 | .95736 |
| 100 | .87869 | .89201 | .91013 | .92430 | .93897 | .95946 |

Table 6 (continued)

p = 3

| n \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 3 | $.0^{5}2506$ | $.0^{4}10160$ | $.0^{4}64805$ | $.0^{3}26498$ | $.0^{2}11027$ | $.0^{2}78035$ |
| 4 | $.0^{3}81873$ | $.0^{3}16678$ | $.0^{2}43296$ | $.0^{2}90548$ | .019385 | .056755 |
| 5 | $.0^{2}65792$ | .010719 | .020707 | .034585 | .058931 | .12591 |
| 6 | .019979 | .029067 | .048280 | .071783 | .10854 | .19607 |
| 7 | .040201 | .054573 | .082592 | .11426 | .16034 | .26062 |
| 8 | .065194 | .084426 | .11991 | .15788 | .21047 | .31801 |
| 9 | .093006 | .11638 | .15782 | .20042 | .25732 | .36846 |
| 10 | .12215 | .14891 | .19492 | .24079 | .30037 | .41275 |
| 11 | .15160 | .18105 | .23046 | .27854 | .33964 | .45172 |
| 12 | .18067 | .21220 | .26407 | .31356 | .37534 | .48614 |
| 13 | .20894 | .24205 | .29564 | .34594 | .40781 | .51671 |
| 14 | .23617 | .27044 | .32517 | .37583 | .43736 | .54398 |
| 15 | .26223 | .29732 | .35272 | .40341 | .46431 | .56844 |
| 16 | .28705 | .32270 | .37842 | .42888 | .48896 | .59048 |
| 17 | .31063 | .34663 | .40239 | .45243 | .51155 | .61044 |
| 18 | .33300 | .36916 | .42475 | .47426 | .53231 | .62858 |
| 19 | .35419 | .39039 | .44564 | .49450 | .55145 | .64513 |
| 20 | .37427 | .41038 | .46517 | .51333 | .56914 | .66030 |
| 22 | .41130 | .44701 | .50062 | .54724 | .60075 | .68710 |
| 24 | .44458 | .47965 | .53187 | .57688 | .62813 | .71002 |
| 26 | .47455 | .50887 | .55958 | .60298 | .65207 | .72984 |
| 28 | .50164 | .53512 | .58430 | .62612 | .67315 | .74714 |
| 30 | .52661 | .55882 | .60647 | .64676 | .69186 | .76238 |
| 32 | .54857 | .58029 | .62644 | .66528 | .70857 | .77589 |
| 34 | .56899 | .59983 | .64452 | .68198 | .72357 | .78796 |
| 36 | .58769 | .61768 | .66096 | .69711 | .73712 | .79880 |
| 38 | .60488 | .63403 | .67597 | .71089 | .74941 | .80859 |
| 40 | .62073 | .64907 | .68973 | .72348 | .76061 | .81747 |
| 45 | .65540 | .68184 | .71954 | .75065 | .78470 | .83645 |
| 50 | .68434 | .70907 | .74418 | .77300 | .80440 | .85187 |
| 55 | .70886 | .73206 | .76486 | .79169 | .82081 | .86464 |
| 60 | .72987 | .75170 | .78247 | .80754 | .83468 | .87538 |
| 65 | .74808 | .76868 | .79763 | .82117 | .84657 | .88454 |
| 70 | .76401 | .78350 | .81083 | .83299 | .85685 | .89245 |
| 75 | .77805 | .79654 | .82242 | .84335 | .86587 | .89935 |
| 80 | .79052 | .80811 | .83267 | .85251 | .87381 | .90541 |
| 85 | .80168 | .81843 | .84181 | .86065 | .88086 | .91079 |
| 90 | .81171 | .82771 | .85000 | .86795 | .88717 | .91559 |
| 95 | .82077 | .83609 | .85739 | .87452 | .89284 | .91990 |
| 100 | .82901 | .84369 | .86408 | .88047 | .89797 | .92379 |

Table 6 (Continued)

p = 4

| n \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 4 | $.0^{6}68258$ | $.0^{5}27562$ | $.0^{4}17377$ | $.0^{4}71482$ | $.0^{3}30192$ | $.0^{2}22422$ |
| 5 | $.0^{3}26081$ | $.0^{3}53590$ | $.0^{2}14157$ | $.0^{2}30195$ | $.0^{2}66607$ | .020829 |
| 6 | $.0^{2}24414$ | $.0^{2}40310$ | $.0^{2}79751$ | .013653 | .024052 | .054981 |
| 7 | $.0^{2}83707$ | .012366 | .021080 | .032154 | .050263 | .096845 |
| 8 | .018557 | .025595 | .039760 | .056400 | .081716 | .14106 |
| 9 | .032548 | .042820 | .062385 | .084139 | .11560 | .18472 |
| 10 | .049524 | .062930 | .087449 | .11362 | .15008 | .22637 |
| 11 | .068633 | .084918 | .11378 | .14363 | .18402 | .26541 |
| 12 | .089139 | .10798 | .14054 | .17338 | .21678 | .30166 |
| 13 | .11045 | .13149 | .16715 | .20236 | .24803 | .33516 |
| 14 | .13210 | .15503 | .19322 | .23030 | .27762 | .36606 |
| 15 | .15377 | .17829 | .21852 | .25702 | .30550 | .39454 |
| 16 | .17521 | .20104 | .24291 | .28248 | .33172 | .42082 |
| 17 | .19625 | .22315 | .26631 | .30664 | .35634 | .44510 |
| 18 | .21677 | .24455 | .28869 | .32955 | .37945 | .46756 |
| 19 | .23670 | .26517 | .31004 | .35123 | .40114 | .48838 |
| 20 | .25599 | .28500 | .33040 | .37175 | .42151 | .50772 |
| 22 | .29256 | .32229 | .36823 | .40954 | .45867 | .54249 |
| 24 | .32645 | .35652 | .40252 | .44344 | .49164 | .57283 |
| 26 | .35777 | .38791 | .43362 | .47392 | .52101 | .59951 |
| 28 | .38669 | .41670 | .46188 | .50143 | .54731 | .62312 |
| 30 | .41339 | .44314 | .48764 | .52634 | .57097 | .64416 |
| 32 | .43808 | .46746 | .51117 | .54898 | .59235 | .66301 |
| 34 | .46092 | .48987 | .53273 | .56962 | .61175 | .67998 |
| 36 | .48210 | .51057 | .55255 | .58851 | .62943 | .69536 |
| 38 | .50178 | .52973 | .57081 | .60586 | .64560 | .70933 |
| 40 | .52009 | .54751 | .58768 | .62184 | .66044 | .72210 |
| 45 | .56068 | .58677 | .62470 | .65673 | .69269 | .74962 |
| 50 | .59514 | .61992 | .65574 | .68582 | .71940 | .77222 |
| 55 | .62471 | .64824 | .68211 | .71042 | .74188 | .79109 |
| 60 | .65034 | .67270 | .70478 | .73147 | .76105 | .80708 |
| 65 | .67274 | .69403 | .72446 | .74970 | .77758 | .82081 |
| 70 | .69249 | .71277 | .74170 | .76562 | .79198 | .83272 |
| 75 | .71001 | .72937 | .75692 | .77965 | .80464 | .84315 |
| 80 | .72566 | .74418 | .77046 | .79211 | .81585 | .85236 |
| 85 | .73972 | .75746 | .78258 | .80323 | .82585 | .86055 |
| 90 | .75243 | .76943 | .79349 | .81324 | .83482 | .86788 |
| 95 | .76396 | .78029 | .80336 | .82227 | .84292 | .87447 |
| 100 | .77446 | .79017 | .81234 | .83048 | .85026 | .88045 |

Table 6 (Continued)

p = 5

| n \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 5 | $.0^{6}209$ | $.0^{6}813$ | $.0^{5}571$ | $.0^{4}211$ | $.0^{4}883$ | $.0^{3}678$ |
| 6 | $.0^{4}8631$ | $.0^{3}1787$ | $.0^{3}4787$ | $.0^{2}1038$ | $.0^{2}2341$ | $.0^{2}7707$ |
| 7 | $.0^{3}91558$ | $.0^{2}15289$ | $.0^{2}30854$ | $.0^{2}53894$ | $.0^{2}97536$ | .023522 |
| 8 | $.0^{2}34846$ | $.0^{2}52164$ | $.0^{2}90895$ | .014165 | .022760 | .046217 |
| 9 | $.0^{2}894223$ | .011779 | .018714 | .027119 | .040364 | .073262 |
| 10 | .015873 | .021176 | .031545 | .043440 | .061249 | .10262 |
| 11 | .025655 | .033050 | .046932 | .062212 | .084226 | .13284 |
| 12 | .037426 | .04693 | .064206 | .082619 | .10836 | .16301 |
| 13 | .050797 | .062331 | .082772 | .10400 | .13296 | .19252 |
| 14 | .065391 | .078826 | .10214 | .12584 | .15754 | .22102 |
| 15 | .080870 | .096049 | .12193 | .14777 | .18176 | .24833 |
| 16 | .096950 | .11370 | .14185 | .16952 | .20539 | .27435 |
| 17 | .11340 | .13155 | .16167 | .19088 | .22830 | .29907 |
| 18 | .13001 | .14942 | .18124 | .21175 | .25040 | .32249 |
| 19 | .14666 | .16715 | .20043 | .23201 | .27165 | .34465 |
| 20 | .16321 | .18465 | .21916 | .25163 | .29203 | .36562 |
| 22 | .19568 | .21865 | .25508 | .28883 | .33022 | .40422 |
| 24 | .22696 | .25104 | .28877 | .32329 | .36513 | .43880 |
| 26 | .25679 | .28165 | .32020 | .35511 | .39702 | .46987 |
| 28 | .28506 | .31043 | .34945 | .38447 | .42617 | .49787 |
| 30 | .31174 | .33742 | .37663 | .41156 | .45285 | .52321 |
| 32 | .33687 | .36270 | .40190 | .43654 | .47733 | .54621 |
| 34 | .36052 | .38637 | .42539 | .45971 | .49983 | .56717 |
| 36 | .38276 | .40854 | .44726 | .48115 | .52057 | .58634 |
| 38 | .40369 | .42932 | .46764 | .50104 | .53974 | .60392 |
| 40 | .42337 | .44880 | .48667 | .51954 | .55748 | .62011 |
| 45 | .46775 | .49249 | .52906 | .56052 | .59655 | .65542 |
| 50 | .50618 | .53010 | .56523 | .59525 | .62943 | .68481 |
| 55 | .53967 | .56272 | .59640 | .62503 | .65745 | .70964 |
| 60 | .56907 | .59125 | .62350 | .65080 | .68158 | .73087 |
| 65 | .59505 | .61636 | .64727 | .67331 | .70258 | .74924 |
| 70 | .61815 | .63864 | .66825 | .69313 | .72101 | .76527 |
| 75 | .63881 | .65851 | .68691 | .71071 | .73731 | .77939 |
| 80 | .65738 | .67635 | .70361 | .72641 | .75182 | .79191 |
| 85 | .67417 | .69244 | .71864 | .74050 | .76482 | .80309 |
| 90 | .68942 | .70702 | .73223 | .75322 | .77654 | .81314 |
| 95 | .70331 | .72030 | .74458 | .76476 | .78715 | .82222 |
| 100 | .71603 | .73243 | .75585 | .77528 | .79680 | .83045 |

Table 6 (Continued)

p = 6

| n \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 6 | $.0^{7}679$ | $.0^{6}256$ | $.0^{5}156$ | $.0^{5}637$ | $.0^{4}272$ | $.0^{3}212$ |
| 7 | $.0^{4}2922$ | $.0^{4}6086$ | $.0^{3}1650$ | $.0^{3}3626$ | $.0^{3}8340$ | $.0^{2}2857$ |
| 8 | $.0^{3}34475$ | $.0^{3}58147$ | $.0^{2}11936$ | $.0^{2}21210$ | $.0^{2}39263$ | $.0^{2}98987$ |
| 9 | $.0^{2}14392$ | $.0^{2}21796$ | $.0^{2}38712$ | $.0^{2}61448$ | .010108 | .021457 |
| 10 | $.0^{2}37618$ | $.0^{2}53261$ | $.0^{2}86316$ | .012744 | .019416 | .036790 |
| 11 | $.0^{2}75778$ | .010237 | .015555 | .021817 | .031466 | .054938 |
| 12 | .012966 | .016912 | .024487 | .033043 | .045723 | .075008 |
| 13 | .019867 | .025215 | .035157 | .04026 | .061643 | .096263 |
| 14 | .028141 | .034941 | .047256 | .060366 | .078740 | .11814 |
| 15 | .037607 | .045856 | .060477 | .07570 | .096606 | .14020 |
| 16 | .048072 | .057730 | .074544 | .091731 | .11491 | .16215 |
| 17 | .059348 | .070352 | .089215 | .10819 | .13341 | .18377 |
| 18 | .071262 | .083532 | .10429 | .12488 | .15189 | .20491 |
| 19 | .083659 | .097107 | .11959 | .14163 | .17021 | .22546 |
| 20 | .096406 | .11094 | .13499 | .15831 | .18826 | .24538 |
| 22 | .12250 | .13894 | .16566 | .19111 | .22232 | .28315 |
| 24 | .14884 | .16682 | .19567 | .22274 | .25644 | .31816 |
| 26 | .17492 | .19416 | .22465 | .25292 | .28772 | .35048 |
| 28 | .20043 | .22065 | .25239 | .28153 | .31705 | .38029 |
| 30 | .22515 | .24614 | .27881 | .30854 | .34448 | .40778 |
| 32 | .24896 | .27053 | .30387 | .33397 | .37011 | .43316 |
| 34 | .27181 | .29381 | .32759 | .35790 | .39405 | .45661 |
| 36 | .29366 | .31597 | .35062 | .38039 | .41643 | .47833 |
| 38 | .31453 | .33703 | .37121 | .40154 | .43736 | .49848 |
| 40 | .33443 | .35704 | .39124 | .42144 | .45695 | .51721 |
| 45 | .38017 | .40277 | .43664 | .46626 | .50078 | .55866 |
| 50 | .42070 | .44301 | .47621 | .50503 | .53838 | .59375 |
| 55 | .45668 | .47854 | .51089 | .53880 | .57090 | .62381 |
| 60 | .48873 | .51006 | .54146 | .56842 | .59927 | .64980 |
| 65 | .51741 | .53816 | .56857 | .59457 | .62421 | .67249 |
| 70 | .54318 | .56332 | .59275 | .61781 | .64628 | .69245 |
| 75 | .56643 | .58597 | .61443 | .63859 | .66595 | .71015 |
| 80 | .58751 | .60645 | .63396 | .65726 | .68358 | .72594 |
| 85 | .60668 | .62504 | .65165 | .67413 | .69946 | .74011 |
| 90 | .62420 | .64199 | .66774 | .68943 | .71384 | .75290 |
| 95 | .64025 | .65750 | .68242 | .70338 | .72692 | .76450 |
| 100 | .65501 | .67175 | .69588 | .71615 | .73886 | .77507 |

Table 6 (Continued)

p = 7

| n \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 7 | $.0^{7}330$ | $.0^{6}105$ | $.0^{6}555$ | $.0^{5}214$ | $.0^{5}891$ | $.0^{4}696$ |
| 8 | $.0^{4}101$ | $.0^{4}210$ | $.0^{4}576$ | $.0^{3}128$ | $.0^{3}299$ | $.0^{2}106$ |
| 9 | $.0^{3}1301$ | $.0^{3}2213$ | $.0^{3}4611$ | $.0^{3}832$ | $.0^{2}1570$ | $.0^{2}4110$ |
| 10 | $.0^{3}5897$ | $.0^{3}9025$ | $.0^{2}1630$ | $.0^{2}2630$ | $.0^{2}4416$ | $.0^{2}9742$ |
| 11 | $.0^{2}16566$ | $.0^{2}23716$ | $.0^{2}39120$ | $.0^{2}58722$ | $.0^{2}91321$ | .017970 |
| 12 | $.0^{2}35507$ | $.0^{2}48512$ | $.0^{2}75041$ | .010700 | .015747 | .028516 |
| 13 | $.0^{2}64117$ | $.0^{2}84584$ | .012465 | .017094 | .02412 | .040984 |
| 14 | .010298 | .013217 | .018749 | .024932 | .034030 | .054961 |
| 15 | .015203 | .019083 | .026246 | .034038 | .045216 | .07006 |
| 16 | .021073 | .025970 | .034812 | .04422 | .057427 | .08595 |
| 17 | .027829 | .033766 | .044293 | .055279 | .070428 | .10236 |
| 18 | .035373 | .042352 | .054532 | .067037 | .084013 | .11906 |
| 19 | .043603 | .051608 | .065385 | .079330 | .098006 | .13586 |
| 20 | .05242 | .061417 | .076720 | .092015 | .11226 | .15264 |
| 22 | .071424 | .082284 | .10038 | .11809 | .14107 | .18569 |
| 24 | .091711 | .10423 | .12476 | .14450 | .16972 | .21763 |
| 26 | .11273 | .12670 | .14929 | .17072 | .19773 | .24813 |
| 28 | .13406 | .14927 | .17359 | .19640 | .22482 | .27707 |
| 30 | .15540 | .17165 | .19740 | .22131 | .25082 | .30440 |
| 32 | .17650 | .19363 | .22055 | .24533 | .27566 | .33013 |
| 34 | .19721 | .21506 | .24292 | .26837 | .29931 | .35433 |
| 36 | .21742 | .23587 | .26447 | .29043 | .32178 | .37708 |
| 38 | .23705 | .25598 | .28516 | .31148 | .34310 | .39847 |
| 40 | .25607 | .27537 | .30498 | .33156 | .36333 | .41859 |
| 45 | .30077 | .32067 | .35089 | .37771 | .40944 | .46388 |
| 50 | .34145 | .36158 | .39190 | .41858 | .44990 | .50303 |
| 55 | .37834 | .39845 | .42855 | .45486 | .48552 | .53711 |
| 60 | .41177 | .43170 | .46137 | .48716 | .51705 | .56698 |
| 65 | .44211 | .46176 | .49087 | .51605 | .54510 | .59333 |
| 70 | .46970 | .48900 | .51747 | .54199 | .57019 | .61674 |
| 75 | .48155 | .51375 | .54154 | .56540 | .59273 | .63766 |
| 80 | .49912 | .53632 | .56342 | .58660 | .61309 | .65645 |
| 85 | .51231 | .55698 | .58337 | .60588 | .63155 | .67342 |
| 90 | .52010 | .57593 | .60162 | .62349 | .64836 | .68882 |
| 95 | .54119 | .59337 | .61838 | .63963 | .66373 | .70284 |
| 100 | .55222 | .60948 | .63382 | .65446 | .67783 | .71567 |

Table 6 (Continued)

p = 8

| n \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 9 | $.0_{5}294$ | $.0_{5}674$ | $.0_{4}196$ | $.0_{4}448$ | $.0_{3}107$ | $.0_{3}393$ |
| 10 | $.0_{4}490$ | $.0_{4}841$ | $.0_{3}177$ | $.0_{3}325$ | $.0_{3}624$ | $.0_{2}169$ |
| 11 | $.0_{3}2400$ | $.0_{3}3707$ | $.0_{3}6799$ | $.0_{2}1113$ | $.0_{2}1903$ | $.0_{2}4342$ |
| 12 | $.0_{3}72041$ | $.0_{2}10418$ | $.0_{2}17462$ | $.0_{2}26605$ | $.0_{2}42133$ | $.0_{2}85765$ |
| 13 | $.0_{2}16368$ | $.0_{2}22596$ | $.0_{2}35523$ | $.0_{2}51410$ | $.0_{2}77049$ | .014418 |
| 14 | $.0_{2}31108$ | $.0_{2}41466$ | $.0_{2}62100$ | $.0_{2}86418$ | .012414 | .021771 |
| 15 | $.0_{2}52269$ | $.0_{2}67778$ | $.0_{2}97682$ | .013177 | .018299 | .030474 |
| 16 | $.0_{2}80314$ | .010184 | .014225 | .018707 | .025270 | .040327 |
| 17 | .011537 | .014359 | .019541 | .025159 | .033206 | .051132 |
| 18 | .015731 | .019272 | .025653 | .032438 | .041978 | .062700 |
| 19 | .020580 | .024873 | .032485 | .040443 | .051455 | .074859 |
| 20 | .026039 | .031101 | .039952 | .049070 | .061512 | .087458 |
| 22 | .038571 | .045177 | .056459 | .067802 | .082923 | .11347 |
| 24 | .052873 | .060971 | .074544 | .087927 | .10544 | .13993 |
| 26 | .068506 | .077999 | .093668 | .10887 | .12846 | .16627 |
| 28 | .085081 | .095847 | .11339 | .13019 | .15157 | .19210 |
| 30 | .10227 | .11418 | .13338 | .15156 | .17444 | .21719 |
| 32 | .11980 | .13273 | .15338 | .17273 | .19686 | .24140 |
| 34 | .13748 | .15129 | .17319 | .19353 | .21870 | .26464 |
| 36 | .15512 | .16971 | .19267 | .21385 | .23986 | .28689 |
| 38 | .17261 | .18787 | .21173 | .23360 | .26029 | .30813 |
| 40 | .18985 | .20568 | .23030 | .25273 | .27996 | .32840 |
| 45 | .23144 | .24835 | .27434 | .29774 | .32580 | .37496 |
| 50 | .27046 | .28804 | .31482 | .33870 | .36708 | .41618 |
| 55 | .30670 | .32466 | .35182 | .37585 | .40420 | .45276 |
| 60 | .34020 | .35832 | .38557 | .40952 | .43760 | .48532 |
| 65 | .37109 | .38923 | .416350 | .44006 | .46773 | .51442 |
| 70 | .39956 | .41760 | .44446 | .46784 | .49500 | .54055 |
| 75 | .42583 | .44369 | .47018 | .49315 | .51974 | .56412 |
| 80 | .45008 | .46771 | .49377 | .51629 | .54228 | .58546 |
| 85 | .47251 | .48987 | .51546 | .53751 | .56287 | .60486 |
| 90 | .49329 | .51035 | .53545 | .55701 | .58175 | .62257 |
| 95 | .51259 | .52934 | .55392 | .57499 | .59911 | .63879 |
| 100 | .53053 | .54696 | .57103 | .59161 | .61513 | .65370 |

Table 6 (Continued)

p = 9

| n \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 10 | $.0^{5}124$ | $.0^{5}260$ | $.0^{5}722$ | $.0^{4}163$ | $.0^{4}392$ | $.0^{3}147$ |
| 11 | $.0^{4}185$ | $.0^{4}320$ | $.0^{4}684$ | $.0^{3}126$ | $.0^{3}246$ | $.0^{3}687$ |
| 12 | $.0^{4}970$ | $.0^{3}151$ | $.0^{3}281$ | $.0^{3}466$ | $.0^{3}810$ | $.0^{2}191$ |
| 13 | $.0^{3}3099$ | $.0^{3}4523$ | $.0^{3}7692$ | $.0^{2}1188$ | $.0^{2}1913$ | $.0^{2}4012$ |
| 14 | $.0^{3}74384$ | $.0^{2}10366$ | $.0^{2}16541$ | $.0^{2}24265$ | $.0^{2}36974$ | $.0^{2}71266$ |
| 15 | $.0^{2}14841$ | $.0^{2}19973$ | $.0^{2}30359$ | $.0^{2}42820$ | $.0^{2}62518$ | .011287 |
| 16 | $.0^{2}26040$ | $.0^{2}34091$ | $.0^{2}49857$ | $.0^{2}68150$ | $.0^{2}61569$ | .016469 |
| 17 | $.0^{2}41592$ | $.0^{2}53236$ | $.0^{2}75439$ | .010050 | .013787 | .022608 |
| 18 | $.0^{2}61860$ | $.0^{2}77705$ | .010724 | .013982 | .018734 | .029614 |
| 19 | $.0^{2}87034$ | .010759 | .014529 | .018585 | .024404 | .037386 |
| 20 | .011714 | .014283 | .018905 | .023817 | .030732 | .045819 |
| 22 | .019172 | .022853 | .029299 | .035959 | .045084 | .064268 |
| 24 | .028386 | .033239 | .041555 | .049957 | .061224 | .084226 |
| 26 | .039112 | .045140 | .055293 | .065365 | .078637 | .10509 |
| 28 | .051078 | .058248 | .070152 | .081782 | .096887 | .12641 |
| 30 | .064022 | .072275 | .085811 | .098868 | .11562 | .14781 |
| 32 | .077706 | .086969 | .10200 | .11635 | .13456 | .16905 |
| 34 | .091921 | .10211 | .11851 | .13400 | .15348 | .18995 |
| 36 | .10649 | .11753 | .13514 | .15165 | .17224 | .21037 |
| 38 | .12127 | .13307 | .15177 | .16916 | .19072 | .23024 |
| 40 | .13613 | .14861 | .16827 | .18644 | .20882 | .24950 |
| 45 | .17307 | .18694 | .20852 | .22819 | .25210 | .29481 |
| 50 | .20894 | .22382 | .24672 | .26739 | .29226 | .33606 |
| 55 | .24316 | .25875 | .28254 | .30383 | .32923 | .37348 |
| 60 | .27550 | .29155 | .31589 | .33751 | .36314 | .407381 |
| 65 | .30587 | .32219 | .34683 | .36858 | .39422 | .43814 |
| 70 | .33428 | .35076 | .37549 | .39722 | .42271 | .46609 |
| 75 | .36084 | .37735 | .40204 | .42364 | .44887 | .49157 |
| 80 | .38563 | .40210 | .42664 | .44803 | .47293 | .51486 |
| 85 | .40878 | .42515 | .44947 | .47059 | .49510 | .53619 |
| 90 | .43042 | .44664 | .47068 | .49149 | .51557 | .55581 |
| 95 | .45066 | .46671 | .49041 | .51089 | .53453 | .57389 |
| 100 | .46962 | .48545 | .50881 | .52893 | .55211 | .59059 |

Table 6 (Continued)

p = 10

| n \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 11 | $.0^{6}463$ | $.0^{6}962$ | $.0^{5}262$ | $.0^{5}592$ | $.0^{4}143$ | $.0^{4}547$ |
| 12 | $.0^{5}6996$ | $.0^{5}1215$ | $.0^{4}2625$ | $.0^{4}4906$ | $.0^{4}9696$ | $.0^{3}2773$ |
| 13 | $.0^{4}3902$ | $.0^{4}6127$ | $.0^{3}1154$ | $.0^{3}1936$ | $.0^{3}3415$ | $.0^{3}8253$ |
| 14 | $.0^{3}13203$ | $.0^{3}19434$ | $.0^{3}33497$ | $.0^{3}52370$ | $.0^{3}85630$ | $.0^{2}18458$ |
| 15 | $.0^{3}33375$ | $.0^{3}46925$ | $.0^{3}75906$ | $.0^{2}11275$ | $.0^{2}17443$ | $.0^{2}34545$ |
| 16 | $.0^{3}69756$ | $.0^{2}94719$ | $.0^{2}14596$ | $.0^{2}20844$ | $.0^{2}30892$ | $.0^{2}57271$ |
| 17 | $.0^{2}12760$ | $.0^{2}16854$ | $.0^{2}24986$ | $.0^{2}34574$ | $.0^{2}49504$ | $.0^{2}87007$ |
| 18 | $.0^{2}21158$ | $.0^{2}27320$ | $.0^{2}39237$ | $.0^{2}52903$ | $.0^{2}73624$ | .012380 |
| 19 | $.0^{2}32552$ | $.0^{2}41244$ | $.0^{2}57677$ | $.0^{2}76085$ | .010338 | .016745 |
| 20 | $.0^{2}47226$ | $.0^{2}58879$ | $.0^{2}80485$ | .010421 | .013870 | .021758 |
| 22 | $.0^{2}87070$ | .010575 | .013931 | .017497 | .022523 | .033526 |
| 24 | .014116 | .016798 | .021501 | .026372 | .033067 | .047238 |
| 26 | .020893 | .024459 | .030588 | .036806 | .045184 | .062439 |
| 28 | .028915 | .033395 | .040974 | .048531 | .058548 | .078717 |
| 30 | .038025 | .043423 | .052430 | .061282 | .072856 | .095719 |
| 32 | .048057 | .054353 | .064736 | .07482 | .087841 | .11316 |
| 34 | .058846 | .066004 | .077692 | .088920 | .10328 | .13080 |
| 36 | .070239 | .078215 | .091124 | .10341 | .11898 | .14847 |
| 38 | .082098 | .090840 | .10488 | .11813 | .13480 | .16603 |
| 40 | .094302 | .10376 | .11883 | .13297 | .15061 | .18336 |
| 45 | .12567 | .13666 | .15394 | .16990 | .18955 | .22529 |
| 50 | .15732 | .16952 | .18850 | .20583 | .22691 | .26469 |
| 55 | .18845 | .20159 | .22185 | .24016 | .26224 | .30133 |
| 60 | .21860 | .23243 | .25361 | .27261 | .29536 | .33521 |
| 65 | .24748 | .26183 | .28365 | .30311 | .32625 | .36646 |
| 70 | .27497 | .28967 | .31193 | .33165 | .35500 | .39526 |
| 75 | .30102 | .31596 | .33847 | .35833 | .38173 | .42183 |
| 80 | .32565 | .34073 | .36336 | .38325 | .40658 | .44636 |
| 85 | .34891 | .36405 | .38669 | .40652 | .42970 | .46904 |
| 90 | .37084 | .38598 | .40856 | .42826 | .45124 | .49005 |
| 95 | .39153 | .40662 | .42907 | .44860 | .47132 | .50955 |
| 100 | .41105 | .42606 | .44833 | .46765 | .49007 | .52769 |

Table 7. Upper Percentage Points of $-2 \log \lambda_1$ for testing $\Sigma = \Sigma_0$

1%

| | p = 4 | | | p = 6 | | | | p = 10 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| n | Korin | Series (2.17) | n | Korin | Davis | Series (2.25) | n | Korin | Davis | Series (2.25) |
| 4 | - | 51.2067 | 6 | - | - | 91.068 | 11 | - | - | 152.510 |
| 5 | - | 37.65785 | 7 | - | - | 67.9490 | 12 | - | - | 135.813 |
| 6 | - | 33.08249 | 8 | - | - | 59.5996 | 13 | - | - | 126.1024 |
| 7 | 30.8 | 30.74936 | 9 | - | - | 55.1577 | 14 | - | - | 119.6427 |
| 8 | 29.33 | 29.32298 | 10 | - | 52.3 | 52.35128 | 15 | - | 114.9 | 114.9656 |
| 9 | 28.36 | 28.35673 | 12 | - | 48.96 | 48.95695 | 16 | - | - | 111.39217 |
| 10 | 27.66 | 27.65726 | 15 | - | 46.234 | 46.23382 | 18 | - | - | 106.24872 |
| 11 | 27.13 | 27.12675 | 20 | 43.99 | 43.9754 | 43.97543 | 20 | - | 102.70 | 102.69722 |
| 12 | 26.71 | 26.71020 | 25 | 42.80 | 42.7890 | 42.78900 | 25 | - | 97.234 | 97.23448 |
| 14 | 26.10 | 26.09754 | 30 | 42.07 | 42.0559 | 42.05590 | 30 | - | 94.1027 | 94.10278 |

5%

| | p = 4 | | | p = 6 | | | | p = 10 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| n | Korin | Series (2.17) | n | Korin | Davis | Series (2.25) | n | Korin | Davis | Series (2.25) |
| 4 | - | 38.18424 | 6 | - | - | 71.781 | 11 | - | - | 132.403 |
| 5 | - | 29.01339 | 7 | - | - | 55.4554 | 12 | - | - | 119.069 |
| 6 | 25.8 | 25.76297 | 8 | - | - | 49.24692 | 13 | - | - | 111.1458 |
| 7 | 24.06 | 24.06050 | 9 | - | - | 45.8294 | 14 | - | - | 105.7642 |
| 8 | 23.00 | 23.00227 | 10 | - | 43.6 | 43.6268 | 15 | - | 101.8 | 101.8159 |
| 9 | 22.28 | 22.27753 | 12 | - | 40.919 | 40.91920 | 16 | - | - | 98.77239 |
| 10 | 21.75 | 21.74883 | 15 | - | 38.7138 | 38.71385 | 18 | - | - | 94.35388 |
| 11 | 21.35 | 21.34555 | 20 | 36.87 | 36.8638 | 36.86380 | 20 | - | 91.28 | 91.27911 |
| 12 | - | 21.02752 | 25 | 35.89 | 35.8847 | 35.88471 | 25 | - | 86.516 | 86.51565 |
| 13 | 20.77 | 20.77014 | 30 | 35.28 | 35.2774 | 35.27737 | 40 | 80.7 | 80.7067 | 80.70671 |

# CHAPTER V

# DISTRIBUTION OF THE LIKELIHOOD RATIO CRITERION FOR TESTING $\Sigma = \Sigma_0$, $\mu = \mu_0$

## 1. INTRODUCTION

Let a p-variate random sample of size N from the normal distribution with mean $\mu$ and covariance matrix $\Sigma$ be denoted by $x_1, x_2, \ldots, x_N$. The likelihood ratio criterion for testing the hypothesis $H_0$: $\Sigma = \Sigma_0$ and $\mu = \mu_0$ against alternatives $H_1$: $\Sigma \neq \Sigma_0$ or $\mu \neq \mu_0$, where $\Sigma_0$ is a given positive definite matrix and $\mu_0$ a given vector, is expressed as (Anderson [1])

$$(1.1) \qquad L = (e/N)^{\frac{Np}{2}} |S\,\Sigma_0^{-1}|^{\frac{N}{2}} e^{-\frac{1}{2} \operatorname{tr} \Sigma_0^{-1}\{S+N(\bar{x}-\mu_0)(\bar{x}-\mu_0)'\}}$$

where $S = \sum_{\alpha=1}^{N} (x_\alpha-\bar{x})(x_\alpha-\bar{x})'$ and $\bar{x} = \sum_{\alpha=1}^{N} x_\alpha/N$. Although the exact distribution of L is unknown, it has been shown that the asymptotic distribution of -2 log L is a chisquare with $\frac{1}{2}$ p(p+1)+p degrees of freedom. No further information on the distribution of L appears to be available.

In this chapter, the null distribution of L is obtained first using the derivations for the chisquare distribution and then using the derivations for the beta distribution by methods similar to those used in Chapter III. From these percentage points of L can be computed to any degree of accuracy even for small sample sizes. Tabulations of percentage points for p = 2(1)6 for various significance levels are given.

## 2. DISTRIBUTION OF L AS A CHISQUARE SERIES

The h-th moment of L under the null hypothesis is known to be (Anderson [1])

$$(2.1) \qquad E(L^h) = (2e/N)^{Nhp/2}[\Gamma_p\{(n+Nh)/2\}/\Gamma_p(n/2)] \cdot (1+h)^{-\frac{Np}{2}(1+h)}$$

where n = N-1. Let

$$(2.2) \qquad \lambda = -2 \log L.$$

If $\phi(t)$ is the characteristic function of $\lambda$, then

$$(2.3) \qquad \phi(t) = \frac{(2e/N)^{-Npit}}{(1-2it)^{-Np(1-2it)/2}} \frac{\prod_{\alpha=1}^{p} \Gamma\{\frac{N}{2}(1-2it)-\frac{\alpha}{2}\}}{\prod_{\alpha=1}^{p} \Gamma(\frac{N-\alpha}{2})},$$

and therefore

$$(2.4) \qquad \log \phi(t) = -Npit \log(2e/N) + \sum_{\alpha=1}^{p} \log\Gamma\{\frac{N}{2}(1-2it)-\frac{\alpha}{2}\} - \sum_{\alpha=1}^{p} \log\Gamma(\frac{N-\alpha}{2}) - \frac{Np}{2}(1-2it)\log(1-2it).$$

Using the expansion (2.64) of Chapter III to each gamma function in (2.4) we have

$$(2.5) \quad \log \phi(t) = \frac{p}{2} \log(2\pi) + \sum_{\alpha=1}^{p} \log \Gamma\left(\frac{N-\alpha}{2}\right) + \frac{pN}{2} \log(N/2e)$$

$$- \left(\frac{p^3+3p}{4}\right) \log \Gamma\left\{\frac{N}{2}(1-2it)\right\} + \sum_{\gamma=1}^{m} (Q_\gamma/N^\gamma)\left(\frac{1-2it}{2}\right)^{-\gamma}$$

$$+ R'_{m+1}(N,t),$$

where the coefficients $Q_r$'s are given by

$$(2.6) \quad Q_r = (-1)^{r-1} \sum_{\alpha=1}^{p} B_{r+1}(-\alpha/2)/r(r+1).$$

The characteristic function of L can then be obtained from (2.5) as

$$(2.7) \quad \phi(t) = K(p,N)[N(1-2it)/2]^{-v}\left(\sum_{j=0}^{\infty} (B_j/N^j)\left(\frac{1-2it}{2}\right)^{-j}\right)$$

$$+ R''_{m+1}(N,t),$$

where

$$(2.8) \quad K(p,N) = (2\pi)^{p/2} (N/2e)^{\frac{pN}{2}} \left[\prod_{\alpha=1}^{p} \Gamma\left(\frac{N-\alpha}{2}\right)\right]^{-1},$$

and

$$(2.9) \quad v = (p^2 + 3p)/4 .$$

The coefficients $B_j$'s can be expressed in terms of $Q_j$'s using (2.67) of Chapter III.

Since $(1-2it)^{-\frac{n}{2}}$ is the characteristic function of a chisquare density with n degrees of freedom, say $g_n(\chi^2)$, the density of $\lambda$ can be derived from (2.7) in the form

$$(2.10) \qquad f(\lambda) = K(p,N) \sum_{j=0}^{\infty} (B_j)(2/N)^{j+v} g_{2(j+v)}(\chi^2) + R'''_{m+1}(N).$$

The probability that $\lambda$ is larger than any value, say $\lambda_0$, is

$$(2.11) \qquad P(\lambda \geq \lambda_0) = K(p,N) \sum_{j=0}^{\infty} (B_j)(2/N)^{j+v} G_{2(j+v)}(\chi^2) + R_{m+1}(N),$$

where $G_{2(j+v)}(\chi^2) = \int_{\lambda_0}^{\infty} g_{2(j+v)}(\chi^2)\, d\chi^2$

and $R_{m+1}(N) = \frac{1}{2\pi} k(p,N) \int_{\lambda_0}^{\infty} \int_{-\infty}^{\infty} e^{-it\lambda} \sum_{j=0}^{\infty} (B_j)(2/N)^{j+v}(1-2it)^{-(j+v)}$

$$\cdot\ [\exp\{R_{m+1}(N,t)\}-1]\ dt\ d\lambda\ .$$

Thus from (2.11) we have that the distribution of $\lambda$ may be obtained from a series of chisquare distributions.

## 3. DISTRIBUTION OF L AS A BETA SERIES

Let

$$(3.1) \qquad \lambda = L^{2/N}\ .$$

Then from (2.1), we have

$$(3.2) \qquad E(\lambda^h) = \frac{(2e/N)^{ph}}{\prod_{\alpha=1}^{p} \Gamma(\frac{N-\alpha}{2})} \frac{\prod_{\alpha=1}^{p} \Gamma\{\frac{N}{2} + h - \frac{\alpha}{2}\}}{(1+2h/N)^{Np(1+2h/N)/2}}\ .$$

Using inverse Mellin's transform, the density of $\lambda$ is given by

$$(3.3) \qquad f(\lambda) = \frac{1}{\prod_{\alpha=1}^{p} \Gamma(\frac{N-\alpha}{2})} \cdot \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\lambda^{-h-1}(2e/N)^{ph} \prod_{\alpha=1}^{p} \Gamma\{\frac{N}{2}+h-\frac{\alpha}{2}\}}{(1+2h/N)^{Np(1+2h/N)/2}}\, dh.$$

Putting $\frac{N}{2} + h = t$ in (3.3), we have

$$(3.4) \qquad f(\lambda) = K_1(p,N)\, \lambda^{\frac{N}{2}-1} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \lambda^{-t}\, C(t)dt$$

where $c = N/2$,

$$(3.5) \qquad C(t) = (e/t)^{pt} \prod_{\alpha=1}^{p} \Gamma(t - \frac{\alpha}{2})$$

and

$$(3.6) \qquad K_1(p,N) = (2e/N)^{-pN/2} \left[\prod_{\alpha=1}^{p} \Gamma(\frac{N-\alpha}{2})\right]^{-1} .$$

Using the expansion (2.64) of Chapter III to each gamma function in (3.6), we have

$$(3.7) \qquad \log C(t) = \frac{p}{2} \log(2\pi) - \frac{(p^2+3p)}{4} \log t$$

$$+ [A_1/t + A_2/t^2 + \ldots + A_r/t^r + \ldots]$$

where the coefficients $A_r$'s are given by

$$(3.8) \qquad A_r = (-1)^{r-1}\left[\sum_{\alpha=1}^{p} B_{r+1}(-\alpha/2)\right]/r(r+1).$$

Thus from (3.7), we have

$$(3.9)\quad C(t) = (2\pi)^{p/2} t^{-(p^2+3p)/4}[1+B_1/t+B_2/t^2+\ldots+B_r/t^r+\ldots]$$

where the coefficients $B_r$'s can be computed using (2.10) of Chapter IV.

Proceeding as in Chapter IV, it can be seen that when $v = (p^2+3p)/4$ is an integer, the probability that $\lambda$ is less than any given value, say $\lambda_0$, is

$$(3.10)\quad P(\lambda \leq \lambda_0) = K_1(p,N)(2\pi)^{p/2} \sum_{r=0}^{\infty} (B_r) I_{v+r-1,u}(\lambda_0)$$

where $u = \frac{N}{2} - 1$ and $I_{v+r-1,u}(\lambda_0)$ satisfies the recurrence relation (2.15) of Chapter IV. If $v = (p^2+3p)/4$ is not an integer. The distribution of $\lambda$ is given by

$$(3.11)\quad P(\lambda \leq \lambda_0) = K_1(p,N)(2\pi)^{p/2} \sum_{i=0}^{\infty} R_i I_{\lambda_0}(\tfrac{N}{2}+\lambda,\ v+i)/\Gamma(v+i)$$

where $I_{\lambda_0}(p,q)$ is the incomplete beta function $\int_0^{\lambda_0} x^{p-1}(1-x)^{q-1}dx$ .

## 4. COMPUTATIONS OF PERCENTAGE POINTS

.005, .01, .025, .05, .1 and .25 significance points of $\lambda = L^{2/N}$ were computed for p = 2(1)8 and various n using (2.11), (3.10) and (3.11) and these are presented in table 7. The computation was carried out on CDC 6500 using double precision arithmetic.

Table 8. Percentage Points of $\lambda = L^{2/N}$ for testing $\underset{\sim}{\Sigma} = \underset{\sim}{\Sigma}_0$, $\underset{\sim}{\mu} = \underset{\sim}{\mu}_0$

p = 2

| N \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 4 | $.0^2 11950$ | $.0^2 24255$ | $.0^2 62516$ | .012967 | .027422 | .078104 |
| 5 | $.0^2 90438$ | .014658 | .028055 | .046404 | .078032 | .16230 |
| 6 | .026253 | .037975 | .062461 | .091968 | .13730 | .24199 |
| 7 | .051029 | .068870 | .10323 | .14149 | .19621 | .31188 |
| 8 | .080540 | .10371 | .14595 | .19050 | .25119 | .37189 |
| 9 | .11243 | .13993 | .18810 | .23694 | .30114 | .42330 |
| 10 | .14505 | .17593 | .22840 | .28001 | .34605 | .46750 |
| 11 | .17738 | .21080 | .26628 | .31956 | .38628 | .50575 |
| 12 | .20877 | .24408 | .30155 | .35570 | .42233 | .53910 |
| 13 | .23889 | .27553 | .33424 | .38867 | .45471 | .56837 |
| 14 | .26754 | .30509 | .36447 | .41878 | .48387 | .59425 |
| 15 | .29468 | .33281 | .39240 | .44630 | .51024 | .61726 |
| 16 | .32031 | .35874 | .41824 | .47151 | .53415 | .63785 |
| 17 | .34446 | .38299 | .44215 | .49466 | .55593 | .65638 |
| 18 | .36720 | .40568 | .46431 | .51597 | .57582 | .67313 |
| 19 | .38862 | .42692 | .48489 | .53563 | .59406 | .68834 |
| 20 | .40879 | .44682 | .50404 | .55381 | .61083 | .70221 |
| 22 | .44572 | .48300 | .53853 | .58634 | .64061 | .72659 |
| 24 | .47862 | .51499 | .56871 | .61457 | .66622 | .74730 |
| 26 | .50804 | .54341 | .59529 | .63926 | .68848 | .76512 |
| 28 | .53448 | .56881 | .61886 | .66104 | .70798 | .78060 |
| 30 | .55832 | .59161 | .63989 | .68037 | .72521 | .79417 |
| 32 | .57992 | .61218 | .65876 | .69764 | .74054 | .80617 |
| 34 | .59956 | .63082 | .67579 | .71316 | .75425 | .81685 |
| 36 | .61749 | .64778 | .69121 | .72718 | .76660 | .82642 |
| 38 | .63392 | .66328 | .70525 | .73990 | .77777 | .83504 |
| 40 | .64902 | .67750 | .71808 | .75150 | .78793 | .84285 |
| 45 | .68191 | .70834 | .74579 | .77645 | .80970 | .85948 |
| 50 | .70923 | .73385 | .76858 | .79687 | .82742 | .87294 |
| 55 | .73227 | .75528 | .78763 | .81388 | .84213 | .88404 |
| 60 | .75195 | .77355 | .80380 | .82827 | .85454 | .89336 |
| 65 | .76896 | .78928 | .81769 | .84060 | .86514 | .90130 |
| 70 | .78379 | .80299 | .82975 | .85129 | .87430 | .90814 |
| 75 | .79684 | .81502 | .84031 | .86063 | .88230 | .91409 |
| 80 | .80842 | .82567 | .84965 | .86887 | .88934 | .91932 |
| 85 | .81875 | .83517 | .85795 | .87619 | .89559 | .92395 |
| 90 | .82803 | .84369 | .86539 | .88274 | .90117 | .92807 |
| 95 | .83640 | .85137 | .87209 | .88863 | .90619 | .93177 |
| 100 | .84400 | .85834 | .87815 | .89396 | .91072 | .93511 |

Table 8 (Continued)

p = 3

| N \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 5 | $.0^{3}34244$ | $.0^{3}70203$ | $.0^{2}18466$ | $.0^{2}39217$ | $.0^{2}85876$ | .026462 |
| 6 | $.0^{2}30928$ | $.0^{2}50905$ | .010015 | .017047 | .029804 | .067094 |
| 7 | .010307 | .015172 | .025706 | .038979 | .060467 | .11482 |
| 8 | .02233 | .030688 | .047382 | .066822 | .096110 | .16366 |
| 9 | .038451 | .050397 | .072989 | .097894 | .13358 | .21074 |
| 10 | .057614 | .072947 | .10079 | .13027 | .17095 | .25483 |
| 11 | .078828 | .097194 | .12952 | .16269 | .20716 | .29553 |
| 12 | .10127 | .12227 | .15832 | .19439 | .24166 | .33285 |
| 13 | .12432 | .14754 | .18662 | .22493 | .27421 | .36699 |
| 14 | .14750 | .17258 | .21408 | .25408 | .30473 | .39819 |
| 15 | .17049 | .19710 | .24051 | .28174 | .33327 | .42675 |
| 16 | .19306 | .22090 | .26578 | .30789 | .35991 | .45292 |
| 17 | .21506 | .24389 | .28987 | .33256 | .38478 | .47695 |
| 18 | .23638 | .26598 | .31278 | .35580 | .40798 | .49908 |
| 19 | .25697 | .28717 | .33452 | .37770 | .42967 | .51950 |
| 20 | .27680 | .30745 | .35515 | .39833 | .44994 | .53839 |
| 22 | .31415 | .34533 | .39326 | .43609 | .48671 | .57217 |
| 24 | .34852 | .37986 | .42755 | .46974 | .51912 | .60148 |
| 26 | .38008 | .41132 | .45847 | .49982 | .54783 | .62711 |
| 28 | .40906 | .44003 | .48644 | .52684 | .57343 | .64970 |
| 30 | .43570 | .46627 | .51180 | .55119 | .59636 | .66975 |
| 32 | .46022 | .49031 | .53489 | .57325 | .61701 | .68766 |
| 34 | .48284 | .51240 | .55598 | .59330 | .63569 | .70375 |
| 36 | .50374 | .53273 | .57530 | .61160 | .65267 | .71828 |
| 38 | .52310 | .55150 | .59306 | .62836 | .66816 | .73146 |
| 40 | .54107 | .56888 | .60943 | .64375 | .68234 | .74347 |
| 45 | .58077 | .60710 | .64523 | .67727 | .71307 | .76930 |
| 50 | .61432 | .63923 | .67511 | .70510 | .73842 | .79043 |
| 55 | .64300 | .66659 | .70041 | .72855 | .75969 | .80802 |
| 60 | .66778 | .69014 | .72209 | .74857 | .77777 | .82290 |
| 65 | .68939 | .71062 | .74087 | .76586 | .79334 | .83565 |
| 70 | .70839 | .72859 | .75729 | .78094 | .80688 | .84668 |
| 75 | .72523 | .74448 | .77177 | .79420 | .81876 | .85634 |
| 80 | .74024 | .75862 | .78463 | .80596 | .82926 | .86485 |
| 85 | .75371 | .77129 | .79612 | .81645 | .83862 | .87241 |
| 90 | .76585 | .78270 | .80645 | .82487 | .84701 | .87917 |
| 95 | .77687 | .79303 | .81579 | .83437 | .85457 | .88525 |
| 100 | .78690 | .80243 | .82427 | .84208 | .86142 | .89074 |

Table 8 (Continued)

p = 4

| N \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 6 | $.0^{3}10678$ | $.0^{3}220703$ | $.0^{3}58981$ | $.0^{2}12748$ | $.0^{2}28695$ | $.0^{2}93374$ |
| 7 | $.0^{2}11061$ | $.0^{2}18433$ | $.0^{2}37061$ | $.0^{2}64488$ | .011612 | .027730 |
| 8 | $.0^{2}41278$ | $.0^{2}61637$ | .010696 | .016600 | .026537 | .053367 |
| 9 | $.0^{2}98136$ | .013688 | .021655 | .031256 | .046289 | .083241 |
| 10 | .018242 | .024272 | .036006 | .049390 | .069303 | .11510 |
| 11 | .029144 | .037447 | .052960 | .069939 | .094254 | .14743 |
| 12 | .042098 | .052653 | .071755 | .092003 | .12014 | .17933 |
| 13 | .056655 | .069348 | .091745 | .11488 | .14626 | .21023 |
| 14 | .072395 | .087064 | .11241 | .13804 | .17213 | .23983 |
| 15 | .088956 | .10541 | .13336 | .16112 | .19743 | .26798 |
| 16 | .10604 | .12409 | .15430 | .18385 | .22196 | .29465 |
| 17 | .12340 | .14286 | .17501 | .20605 | .24560 | .31983 |
| 18 | .14084 | .16154 | .19534 | .22761 | .26829 | .34358 |
| 19 | .15823 | .17999 | .21519 | .24846 | .29004 | .36596 |
| 20 | .17544 | .19812 | .23449 | .26856 | .31075 | .38705 |
| 22 | .20901 | .23314 | .27127 | .30645 | .34941 | .42568 |
| 24 | .24112 | .26627 | .30555 | .34134 | .38454 | .46010 |
| 26 | .27157 | .29741 | .33736 | .37339 | .41647 | .49087 |
| 28 | .30029 | .32655 | .36682 | .40282 | .44552 | .51849 |
| 30 | .32728 | .35376 | .39409 | .42987 | .47201 | .54339 |
| 32 | .35261 | .37916 | .41934 | .45476 | .49624 | .56594 |
| 34 | .37637 | .40287 | .44275 | .47772 | .51845 | .58642 |
| 36 | .39864 | .42501 | .46449 | .49893 | .53887 | .60511 |
| 38 | .41955 | .44570 | .48470 | .51858 | .55769 | .62222 |
| 40 | .43917 | .46506 | .50353 | .53680 | .57508 | .63794 |
| 45 | .48326 | .50835 | .54532 | .57705 | .61326 | .67213 |
| 50 | .52127 | .54544 | .58085 | .61103 | .64528 | .70050 |
| 55 | .55429 | .57752 | .61137 | .64007 | .67248 | .72440 |
| 60 | .58320 | .60549 | .63784 | .66515 | .69585 | .74479 |
| 65 | .60868 | .63007 | .66100 | .68701 | .71615 | .76240 |
| 70 | .63129 | .65182 | .68141 | .70622 | .73393 | .77774 |
| 75 | .65148 | .67119 | .69953 | .72323 | .74963 | .79123 |
| 80 | .66961 | .68855 | .71573 | .73839 | .76360 | .80319 |
| 85 | .68597 | .70419 | .73028 | .75200 | .77609 | .81385 |
| 90 | .70080 | .71835 | .74343 | .76426 | .78734 | .82342 |
| 95 | .71432 | .73123 | .75537 | .77538 | .79752 | .83206 |
| 100 | .72668 | .74300 | .76625 | .78550 | .80677 | .83990 |

Table 8 (Continued)

p = 5

| N \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 7 | $.0^{4}3483$ | $.0^{4}7246$ | $.0^{3}1959$ | $.0^{3}4301$ | $.0^{3}9868$ | $.0^{2}3360$ |
| 8 | $.0^{3}40379$ | $.0^{3}67996$ | $.0^{2}13921$ | $.0^{2}24668$ | $.0^{2}45511$ | .011394 |
| 9 | $.0^{2}16608$ | $.0^{2}25107$ | $.0^{2}44458$ | $.0^{2}70365$ | .011532 | .024312 |
| 10 | $.0^{2}42873$ | $.0^{2}60583$ | $.0^{2}97873$ | .014408 | .021870 | .041164 |
| 11 | $.0^{2}85445$ | .011519 | .017449 | .024403 | .035072 | .060841 |
| 12 | .014486 | .018858 | .027220 | .036629 | .050512 | .082357 |
| 13 | .022021 | .027896 | .038779 | .050629 | .067587 | .10493 |
| 14 | .030978 | .038392 | .051773 | .065965 | .085773 | .12798 |
| 15 | .041149 | .050083 | .065870 | .082248 | .10464 | .15107 |
| 16 | .052319 | .062720 | .080771 | .099155 | .12386 | .17390 |
| 17 | .064284 | .076075 | .096224 | .11642 | .14316 | .19627 |
| 18 | .076861 | .089948 | .11202 | .13384 | .16236 | .21804 |
| 19 | .089887 | .10417 | .12798 | .15125 | .18131 | .23912 |
| 20 | .10322 | .11861 | .14398 | .16852 | .19991 | .25947 |
| 22 | .13098 | .14766 | .17568 | .20229 | .23577 | .29789 |
| 24 | .15761 | .17642 | .20651 | .23467 | .26961 | .33329 |
| 26 | .18444 | .20446 | .23613 | .26541 | .30133 | .36584 |
| 28 | .21055 | .23152 | .26436 | .29442 | .33096 | .39573 |
| 30 | .23575 | .25745 | .29114 | .32171 | .35856 | .42321 |
| 32 | .25996 | .28218 | .31646 | .34732 | .38428 | .44849 |
| 34 | .28310 | .30571 | .34035 | .37135 | .40824 | .47181 |
| 36 | .30519 | .32805 | .36289 | .39389 | .43057 | .49335 |
| 38 | .32622 | .34924 | .38414 | .41503 | .45142 | .51329 |
| 40 | .34624 | .36932 | .40417 | .43488 | .47090 | .53179 |
| 45 | .39209 | .41507 | .44945 | .47946 | .51435 | .57263 |
| 50 | .43257 | .45519 | .48878 | .51789 | .55148 | .60710 |
| 55 | .46839 | .49049 | .52315 | .55126 | .58353 | .63653 |
| 60 | .50021 | .52173 | .55336 | .58046 | .61141 | .66194 |
| 65 | .52863 | .54952 | .58010 | .60619 | .63587 | .68408 |
| 70 | .55411 | .57436 | .60390 | .62901 | .65749 | .70352 |
| 75 | .57707 | .59669 | .62522 | .64939 | .67672 | .72074 |
| 80 | .59785 | .61684 | .64439 | .66768 | .69393 | .73608 |
| 85 | .61674 | .63512 | .66174 | .68418 | .70943 | .74984 |
| 90 | .63396 | .65177 | .67750 | .69914 | .72344 | .76224 |

Table 8 (Continued)

p = 6

| N \ α | .005 | .01 | .025 | .05 | .1 | .25 |
|---|---|---|---|---|---|---|
| 8 | $.0^{4}1175$ | $.0^{4}2449$ | $.0^{4}6667$ | $.0^{3}1493$ | $.0^{3}3467$ | $.0^{2}1228$ |
| 9 | $.0^{3}14899$ | $.0^{3}25325$ | $.0^{3}52651$ | $.0^{3}94699$ | $.0^{2}17839$ | $.0^{2}46618$ |
| 10 | $.0^{3}66790$ | $.0^{2}10206$ | $.0^{2}18395$ | $.0^{2}29611$ | $.0^{2}49579$ | .010881 |
| 11 | $.0^{2}18575$ | $.0^{2}26552$ | $.0^{2}43693$ | $.0^{2}65440$ | .010148 | .019869 |
| 12 | $.0^{2}39475$ | $.0^{2}53849$ | $.0^{2}83094$ | .011822 | .017350 | .031265 |
| 13 | $.0^{2}70751$ | $.0^{2}93190$ | .013700 | .018747 | .026384 | .044617 |
| 14 | .011289 | .014467 | .020474 | .027168 | .036988 | .059468 |
| 15 | .016571 | .020769 | .028500 | .036886 | .048878 | .075407 |
| 16 | .022853 | .028122 | .037615 | .047684 | .061779 | .092085 |
| 17 | .030043 | .036401 | .047647 | .059352 | .075444 | .10922 |
| 18 | .038032 | .045474 | .058431 | .071697 | .089657 | .12657 |
| 19 | .046710 | .055212 | .069812 | .084550 | .10424 | .14397 |
| 20 | .055967 | .065492 | .081652 | .097763 | .11903 | .16128 |
| 22 | .075822 | .087246 | .10624 | .12479 | .14879 | .19522 |
| 24 | .096888 | .10999 | .13143 | .15201 | .17821 | .22783 |
| 26 | .11861 | .13315 | .15665 | .17889 | .20685 | .25885 |
| 28 | .14055 | .15633 | .18154 | .205119 | .23443 | .28816 |
| 30 | .16241 | .17923 | .20583 | .23047 | .26082 | .31575 |
| 32 | .18397 | .20164 | .22937 | .25484 | .28595 | .34165 |
| 34 | .20507 | .22344 | .25206 | .27816 | .30982 | .36596 |
| 36 | .22560 | .24454 | .27386 | .30042 | .33244 | .38875 |
| 38 | .24550 | .26489 | .29475 | .32163 | .35387 | .41014 |
| 40 | .26474 | .28448 | .31473 | .34182 | .37415 | .43022 |
| 45 | .30982 | .33010 | .36084 | .38809 | .42027 | .47532 |
| 50 | .35069 | .37114 | .40190 | .42892 | .46059 | .51418 |
| 55 | .38764 | .40803 | .43848 | .46506 | .49600 | .54791 |
| 60 | .42105 | .44121 | .47117 | .49717 | .52727 | .57742 |
| 65 | .45131 | .47114 | .50049 | .52583 | .55504 | .60341 |
| 70 | .47877 | .49822 | .52688 | .55153 | .57983 | .62647 |
| 75 | .50377 | .52279 | .55073 | .57468 | .60209 | .64704 |
| 80 | .52660 | .54517 | .57238 | .59563 | .62215 | .66550 |

# CHAPTER VI

## SUMMARY AND CONCLUSION

The study of the exact and asymptotic distributions of some statistics including some well-known likelihood ratio test criteria in multivariate analysis has been carried out in this dissertation. The main objective has been to present in convenient forms the distributions of these statistics in order to facilitate further work including numerical computations. This research was carried out for the reason that earlier authors who attempted to study these distributions either gave expressions which were practically not suitable for further use or merely restricted themselves to asymptotic derivations.

In the first chapter, the non-central distributions of statistics of the form $Y = \prod_{i=1}^{p} \theta_i^a (1-\theta_i)^b$, where a and b are real numbers have been obtained in the form of H-functions as a result of employing inverse Mellin transform and then asymptotic expansions of the distribution of Y have been obtained for suitable values of (a, b). It may be possible to express the H-functions in a simple computational form by using the methods of contour integration as done in Chapter III at least in some cases and thus the power study of Y for some sets of values (a, b) may be made.

In Chapter II the moments of the sphericity criteria W were obtained in the null case and then use was made of the Mellin transform and Meijer's G-functions to find its non-central distribution in a closed form. Here the methods of contour integration may be employed to obtain the non-central distribution of W in a more suitable form for further work including numerical computations and thus power study of W can be facilitated.

In Chapter III the exact distribution of the sphericity criterion W was obtained in a form from which the percentage points of W were computed for p = 2(1)10 and various degrees of freedom n. The methods employed are quite general and could be used to obtain the exact distribution of other likelihood ratio criteria. In particular the methods can be applied to obtain the exact distribution of the likelihood ratio test criterion for testing the hypothesis that all off diagonal elements of $\Sigma$ are zero, while the diagonal elements are equal in sets. Obviously the sphericity test considered in Chapter III is a special case of this hypothesis.

In Chapter IV, the density of the likelihood ratio criterion $L = \lambda_1^{2/n}$ for testing $\Sigma = \Sigma_0$ was obtained in a form which could be used to compute percentage points to any degree of accuracy even for small sized samples. The methods employed in this chapter along with the methods in Chapter I may yield an expression for the asymptotic non-central distribution of $\lambda_1$ in terms of central or non-central chisquare distributions.

In Chapter V, the methods of Chapter III and IV were further used to obtain the distribution of $\lambda = L^{2/N}$ for testing $\Sigma = \Sigma_0$, $\mu = \mu_0$ as a chisquare as well as a Beta series from which percentage points for p = 2(1)6 were computed. It may be possible that the methods used in this chapter together with the methods of Chapter I may yield a suitable form for the asymptotic distribution of $\lambda$.

In conclusion, the dissertation embodies the theoretical work solving once for all the distribution problems at least in the null case of some well-known likelihood ratio test criteria and makes available much needed tabulations which are fairly complete. The methods obtained to solve these distribution problems (See Chapter III) are of wide application.

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# REFERENCES


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# APPENDICES

## Appendix A

Coefficients $B_i$'s (i = 11,...,14) are:

$$B_{11} = Q_{11}+8Q_1Q_2Q_8/11+2Q_2B_9/11+8Q_3Q_8/11$$
$$+9Q_9B_2/11+4Q_8Q_1^3/33+Q_1B_{10}/11+7Q_2Q_1^2Q_7/22$$
$$+10Q_{10}Q_1/11+7Q_1^4Q_7/264+3Q_3B_8/11+7Q_4Q_7/11$$
$$+6Q_6B_5/11+7Q_3Q_1Q_7/11+5Q_5B_6 11+7Q_2^2Q_7/22$$

$$B_{12} = Q_{12}+Q_1B_{11}/12+Q_2B_{10}/6+Q_3B_9/4+Q_4B_8/3+5Q_5B_7/12+Q_6B_6/2$$
$$+7Q_7B_5/12+2Q_8B_4/3+3Q_9B_3/4+5Q_{10}Q_1^2/12+5Q_{10}Q_2/6+11Q_{11}B_1/12$$

$$B_{13} = Q_{13}+Q_1B_{12}/13+2Q_2B_{11}/13+3Q_3B_{10}/13+4Q_4B_9/13+5Q_5B_8/13+6Q_6B_7/13$$
$$+7Q_7B_6/13+8Q_8B_5/13+9Q_9B_4/13+10Q_{10}B_3/13+11Q_{11}Q_1^2/26$$
$$+11Q_{11}Q_2/13+12Q_{12}B_1/13$$

$$B_{14} = Q_{14}+Q_1B_{13}/14+3Q_3B_{11}/14+Q_2B_{12}/7+2Q_4B_{10}/7+5Q_5B_9/14+3Q_6B_8/7$$
$$+3Q_6B_8/7+Q_7B_7/2+4Q_8B_6/7+9Q_9B_5/14+5Q_{10}B_4/7+11Q_{11}B_3/14$$
$$+3Q_{12}Q_1^2/7+6Q_{12}Q_2/7+13Q_{13}B_1/14$$

Coefficients $C_{ij}$'s (j = 11,...,14) are:

$$C_{i11} = -(t_i^{12}-6t_i^{11}+11t_i^{10}-33t_i^8/2+22t_i^6-33t_i^4/2+5t_i^2)$$

$$C_{i12} = t_i^{13}-13t_i^{12}/2+13t_i^{11}-143t_i^9/6+288t_i^7/7-429t_i^5/10+65t_i^3/3-691t_i/210$$

$$C_{i13} = -(t_i^{14}-7t_i^{13}+91t_i^{12}/6-1001t_i^{10}/30+143t_i^8/2-1001t_i^6/10+455t_i^4/6$$
$$-691t_i^2/30)$$

Coefficients $R_{is}$ (i = 11,...,14) are:

$$R_{11}+R_{10}d_{10,1}+R_9d_{92}+R_8d_{83}+R_7d_{74}+R_6d_{65}+R_5d_{56}+R_4d_{47}+R_3d_{38}$$
$$+R_2d_{29}+R_1d_{1,10}+R_0d_{0,11} = B_{11}$$

$$R_{12}+R_{11}d_{11,1}+R_{10}d_{10,2}+R_9d_{93}+R_8d_{84}+R_7d_{75}+R_6d_{66}+R_5d_{57}+R_4d_{48}$$
$$+R_3d_{39}+R_2d_{210}+R_1d_{1,11}+R_0d_{0,12} = B_{12}$$

$$R_{13}+R_{12}d_{12,1}+R_{11}d_{11,2}+R_{10}d_{10,3}+R_9d_{94}+R_8d_{85}+R_7d_{76}+R_6d_{67}$$
$$+R_5d_{58}+R_4d_{49}+R_3d_{3,10}+R_2d_{2,11}+R_1d_{1,12}+R_0d_{0,13} = B_{13}$$

$$R_{14}+R_{13}d_{13,1}+R_{12}d_{12,2}+R_{11}d_{11,3}+R_{10}d_{10,4}+R_9d_{95}+R_8d_{86}+R_7d_{77}$$
$$+R_6d_{68}+R_5d_{59}+R_4d_{4,10}+R_3d_{3,11}+R_2d_{2,12}+R_1d_{1,13}$$
$$+R_0d_{0,14} = B_{14}.$$

## Appendix B

Coefficients $A_{ij}$'s (j = 11,...,15) are:

$$A_{i11} = -[(t_i^{12}-\lambda^{12})-6(t_i^{11}-\lambda^{11})+11(t_i^{10}-\lambda_i^{10})-33(t_i^{8}-\lambda^{8})/2$$
$$+22(t_i^{6}-\lambda^{6})-33(t_i^{4}-\lambda^{4})/2+5(t_i^{2}-\lambda^{2})]/132$$

$$A_{i12} = [(t_i^{13}-\lambda^{13})-13(t_i^{12}-\lambda^{12})/2+13(t_i^{11}-\lambda^{11})-143(t_i^{9}-\lambda^{9})/6$$
$$+286(t_i^{7}-\lambda^{7})/7-429(t_i^{5}-\lambda^{5})/10+65(t_i^{3}-\lambda^{3})/3$$
$$-691(t_i-\lambda)/210]/156$$

$$A_{i13} = -[(t_i^{14}-\lambda^{14})-7(t_i^{13}-\lambda^{13})+91(t_i^{12}-\lambda^{12})/6-1001(t_i^{10}-\lambda^{10})/30$$
$$+143(t_i^{8}-\lambda^{8})/2-1001(t_i^{6}-\lambda^{6})/10+455(t_i^{4}-\lambda^{4})/6$$
$$-691(t_i^{2}-\lambda^{2})/30]/182$$

$$A_{i14} = [(t_i^{15}-\lambda^{15})-15(t_i^{14}-\lambda^{14})/2+35(t_i^{13}-\lambda^{13})/2$$
$$-91(t_i^{11}-\lambda^{11})/2+715(t_i^{9}-\lambda^{9})/6-429(t_i^{7}-\lambda^{7})/2$$
$$+455(t_i^{5}-\lambda^{5})/2-691(t_i^{3}-\lambda^{3})/6+35(t_i-\lambda)/2]/210$$

$$A_{i15} = -[(t_i^{16}-\lambda^{16})-8(t_i^{15}-\lambda^{15})+20(t_i^{14}-\lambda^{14})-182(t_i^{12}-\lambda^{12})/3$$
$$+4004(t_i^{10}-\lambda^{10})/21-1287(t_i^{8}-\lambda^{8})/3+$$
$$+20020(t_i^{6}-\lambda^{6})/33-1382(t_i^{4}-\lambda^{4})/3+140(t_i^{2}-\lambda^{2})]/240$$