A PROBABILISTIC PROOF OF THE NORMAL

CONVERGENCE CRITERION

by

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Abstract

By embedding partial sum processes into Brownian motion,it is well known that the deMoivre-Laplace central limit theorem is a consequence of the strong law of large numbers. It is the purpose here to show that the embedding technique can be used to establish both the degenerate convergence criterion and the normal convergence criterion for triangular arrays of uniformly asymptotically negligible random variables.

Let X_{nk}, $k = 1, \ldots, k(n)$, $n = 1, 2, \ldots$ be a triangular array of independent random variables.

Theorem: $\mathcal{L}(\sum_k X_{nk}) \to \mathcal{L}(\alpha)$, degenerate law at α, and for every $\varepsilon > 0$, $\max_k P[|X_{nk}| > \varepsilon] \to 0$ as $n \to \infty$ if and only if for every $\tau > 0$ and $\varepsilon > 0$

(i) $\sum_k P[|X_{nk}| > \varepsilon] \to 0$ as $n \to \infty$

(ii) $\sum_k \sigma^2_{nk}(\tau) \to 0$

(iii) $\sum_k a_{nk}(\tau) \to \alpha$

where $X^{\tau}_{nk} = X_{nk} I_{[|X_{nk}| < \tau]}$ and $a_{nk}(\tau) = E(X^{\tau}_{nk})$ and $\sigma^2_{nk}(\tau) = \sigma^2(X^{\tau}_{nk})$.

Proof: Assume $\mathcal{L}(\sum_k X_{nk}) \to \mathcal{L}(\alpha)$ and for every $\varepsilon > 0$, $\max_k P[|X_{nk}| > \varepsilon] \to 0$ as $n \to \infty$. Let X'_{nk} be random variables such that $\mathcal{L}(X'_{nk}) = \mathcal{L}(X_{nk})$ and the X'_{nk} and the X_{nk} are independent. Let $Y_{nk} = X_{nk} - X'_{nk}$ then it is clear that $\mathcal{L}(\sum_k Y_{nk}) \to \mathcal{L}(0)$ and, for every $\varepsilon > 0$, $\max_k P[|Y_{nk}| > \varepsilon] \to 0$ as $n \to \infty$. From the symmetrization inequalities [see 1, p. 245] and the fact that $\mu(X_{nk}) \to 0$ to establish (i) it is sufficient to show that $\sum_k P[|Y_{nk}| > \varepsilon] \to 0$ an $n \to 0$ for every $\varepsilon > 0$. By Levy's inequality [see 1, p. 247],

$$P[\max_k |Y_{nk}| > 2\varepsilon] \le P[\max_{\ell} |\sum_{k=1}^{\ell} Y_{nk}| \ge \varepsilon] \le 2P[|\sum_k Y_{nk}| \ge \varepsilon] \to 0$$

as $n \to \infty$. Thus $\sum_k P[|Y_{nk}| > 2\varepsilon] \to 0$ and (i) is established. To establish (ii) first note that $\sigma^2(Y^{2\tau}_{nk}) \ge 2\sigma^2(X^{\tau}_{nk}) P(|X_{nk}| \le \varepsilon)$ so it is sufficient to show that for every $\tau > 0$, $\sum_k \sigma^2(Y^{\tau}_{nk}) \to 0$ as $n \to \infty$. By (i) we can assume without loss of generality that $P[|Y_{nk}| < \tau_n]=1$ for all n and k, and $\tau_n \downarrow 0$ as $n \to \infty$. We proceed by embedding into Brownian motion.

Let $\Omega = \{\omega: [0,\infty) \to (-\infty,+\infty) | \omega$ is continuous and $\omega(0) = 0\}$, and define $\ell_t(\omega) = \omega(t)$ and let $\mathcal{B}_t$ = the σ-field defined on Ω by $\{\ell_s: s \leq t\}$ and let P be a probability measure such that $(\Omega, \ell_t, \mathcal{B}_t, P)$ is standard Brownian motion. Define stopping times $T(n,k,\cdot)$ so that $\mathcal{L}(\ell_{T(n,k,\cdot)}) = \mathcal{L}(Y_{nk})$ [see 2 for definition and properties of $T(n,k)$]. Then $E(T_{nk}) = \sigma^2(Y_{nk})$ and there is a constant C such that $E(T^2_{nk}) \leq C\, E(Y^4_{nk})$. If $\omega \in \Omega$ define $\omega_t \in \Omega$ by $\omega_t(s) = \omega(t+s) - \omega(t)$, and if S, T are two stopping times define $(S + T)(\omega) = T(\omega) + S(\omega_T)$. Setting $S(n) = \sum_k T(n,k) \mathcal{L}(\ell_{S(n)}) \to \mathcal{L}(0)$. If $\sum_k \sigma^2(Y_{nk}) \not\to 0$ as $n \to \infty$ then by passing to a subsequence if necessary we can assume $\sum_k \sigma^2(Y_{nk}) \to \lambda$ where $0 < \eta < \lambda \leq \infty$ for some $\eta > 0$. We can then choose integers $r(n) \leq k(n)$ such that $\sum_{k=1}^{r(n)} \sigma^2(Y_{kn}) \to \eta$ as $n \to \infty$. Then we have

$$\sum_{k=1}^{r(n)} \sigma^2(T(n,k)) \leq \sum_{k=1}^{r(n)} E(T^2(n,k) \leq C \sum_{k=1}^{r(n)} E(Y^4_{nk}) \to 0$$

as $n \to \infty$. Hence by Chebyshev's inequality $\sum_{k=1}^{r(n)} T(n,k) \xrightarrow{P} \eta$ as $n \to \infty$ and, by continuity of the Brownian paths, $\mathcal{L}(\sum_{k=1}^{r(n)} Y_{nk}) \to N(0,\eta)$. Since $P[\sum_{k=r(n)+1}^{k(n)} Y_{nk} \geq 0] = \frac{1}{2}$, $\mathcal{L}(\sum_{k=1}^{k(n)} Y_{nk}) \not\to \mathcal{L}(0)$. Hence $\sum_{k=1}^{k(n)} \sigma^2(Y_{nk}) \to 0$ and (ii) is established. (iii) now follows easily, because by (i) and by (ii) and Chebyshev's inequality $P[|\sum(X^\tau_{nk} - a_{nk}(\tau))| > \varepsilon] \to 0$, hence $\sum a_{nk}(\tau) \to \alpha$. Necessity is proved and the proof of sufficiency is merely an application of truncation and Chebyshev's inequality and will be omitted.

Theorem 2. If X_{nk} are independent summands then for every $\varepsilon > 0$

$$\mathcal{L}(\sum_k X_{nk}) \to N(\alpha,\sigma^2) \text{ and } \max_k P[|X_{nk}| > \varepsilon] \to 0$$

as $n \to \infty$, if and only if for every $\delta > 0$ and every $\tau > 0$,

(i) $\sum_k P[|X_{nk}| \geq \delta] \to 0$ as $n \to \infty$

(ii) $\sum_k \sigma^2(X^\tau_{nk}) \to \sigma^2$ as $n \to \infty$

(iii) $\sum_k E(X^\tau_{nk}) \to \alpha$ as $n \to \infty$.

Proof: First we prove sufficiency. By (i) we assume without loss of generality $P[|X_{nk}| < \varepsilon_n]=1$ where $\varepsilon_n \downarrow 0$ for all n and k. As in Theorem 1 let $T(n,k)$ be stopping times defined on Brownian motion such that $\mathcal{L}(\ell_{T(n,k)}) = \mathcal{L}(X_{nk}-E(X_{nk}))$ and $E(T^2(n,k)) \leq C\ E((X_{nk}-E(X_{nk}))^4)$. Clearly $(E((X_{nk} - E(X_{nk}))^4))/(\sigma^2(X_{nk})) \to 0$ so that

$$\sum_k \sigma^2(T(n,k)) \leq \sum_k E(T^2(n,k)) \leq C \sum E((X_{nk}-E(X_{nk}))^4) \to 0.$$

So by Chebyshev's inequality $\mathcal{L}(\sum_k T(n,k)) \to \mathcal{L}(\sigma^2)$. By continuity of Brownian paths, $\mathcal{L}(\sum(X_{nk} - E(X_{nk})) = \mathcal{L}(\ell_{S(n)}) \to N(0,\sigma^2)$ where $S(n) = \sum T(n,k)$. Hence by (iii) $\mathcal{L}(\sum_k X_{nk}) \to N(\alpha,\sigma^2)$. Sufficiency is established. To prove necessity let X_{nk}, $k = 1,\ldots,k(n)$, $n = 1,2,\ldots$ be a triangular array such that $\mathcal{L}(\sum_k X_{nk}) \to N(0,1)$ and $\max_k P[|X_{nk}| > \varepsilon] \to 0$ for every $\varepsilon > 0$. First we show (i). Let $Y_{nk} = 2^{-\frac{1}{2}}(X_{nk} - X'_{nk})$ where X_{nk} and X'_{nk} are independent and $\mathcal{L}(X_{nk}) = \mathcal{L}(X'_{nk})$. Clearly $\mathcal{L}(\sum_k Y_{nk}) \to N(0,1)$ and as in the previous theorem it is sufficient to show that for any $a > 0$, $\sum_k P[|Y_{nk}| > a] \to 0$. Define M_n to be the number of Y_{nk}, $k = 1,\ldots,k(n)$ such that $|Y_{nk}| > a$. Let

$C_n = \{|\sum_k Y_{nk}| < \frac{1}{2} a\}$, $P[C_n] \to \frac{1}{\sqrt{2\pi}} \int_{-a/2}^{a/2} e^{-x^2/2} dx = 2(\Phi(a/2) - \frac{1}{2}) > 0$.

If $M_n = r$ then of the 2^r possible ways of assigning positive signs to the $|Y_{nk}| > a$ at most $\binom{r}{\ell}$, $\ell = [r/2]$ (this is the maximum number of incomparable subsets of a set of size r) allow $|\sum_k Y_{nk}| < a/2$. Hence

$P[C_n] \leq P[M_n=0] + \sum_{r=1}^{\infty} b_r\, P[M_n=r]$ where $b_r = \binom{r}{\ell} 2^{-r}$. If $\limsup_n \sum_k P[|Y_{nk}| > a] = \eta > 0$, then we can, by passing to a subsequence, assume $\lim_n \sum P[|Y_{nk}| > a] = \eta$. First $\eta \neq +\infty$, because in that case $P[M_n=r] \to 0$ for $r = 0,1,2,\ldots$ which implies $P[C_n] \to 0$, which is a contradiction. Next we show that $\eta = 0$. Suppose not. Let $I(n) = \{k: |Y_{nk}| < a\}$, and $J(n)$ be the complement of $I(n)$. Then

$$
\begin{aligned}
P[\sum_k Y_{nk} > Na] &\geq P[M_n=N\ ]P[\sum_{k\varepsilon I(n)} Y_{nk} \geq 0]P[\sum_{k\varepsilon J(n)} Y_{nk} > Na] \\
&\geq P[M_n=N]\tfrac{1}{2}\, P[Y_{nk} > 0 \text{ for all } k \ \varepsilon\ J(n)] \\
&\to \frac{e^{-\eta}(\eta)^N}{(N)!} \frac{1}{2} \left(\tfrac{1}{2}\right)^N = p(\eta,N);
\end{aligned}
$$

also $\lim_{n\to\infty} P[\sum_k Y_{nk} > Na] \leq \frac{1}{\sqrt{2\pi}} \int_{Na}^{\infty} e^{-x^2/2}\, dx \leq \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{N^2a^2}{2}}}{Na} = q(N)$.

Hence $\lim_{N\to\infty} \frac{p(\eta,N)}{q(N)} = \infty$ which implies $\eta = 0$.

Thus (i) is established. To show (ii) we may assume without loss of generality that $P[|X_{nk}| \leq \tau_n]=1$ for all n and k where $\tau_n \downarrow 0$. As in the previous theorem, let $(\Omega, \ell_t, \mathcal{B}_t, P)$ be standard Brownian motion and let $T(n,k)$ be a stopping time such that $\mathcal{L}(\ell_{T(n,k)}) = \mathcal{L}(X_{nk} - E(X_{nk}))$ and $E(T(n,k)) = \sigma^2(X_{nk})$, and $E(T^2(n,k)) \leq C\, E((X_{nk} - E(X_{nk}))^4)$. By passing to a subsequence, if necessary, we can assume that $\lim \sum_k \sigma^2(X_{nk}) \to \lambda$, $0 \leq \lambda \leq +\infty$. Suppose $\lambda > 1$. We can now choose integers $r(n)$ such that $\sum_{k=1}^{r(n)} \sigma^2(X_{nk}) \to 1 + \varepsilon$ for some $\varepsilon > 0$.

Then we have

$$\sum_{k=1}^{r(n)} \sigma^2(T(n,k)) \leq \sum_{k=1}^{r(n)} E(T^2(n,k)) \leq C \sum_{k=1}^{r(n)} E((X_{nk}-E(X_{nk}))^4) \to 0,$$

because (i) implies that $(\sigma^2(X_{nk}))^{-1} E((X_{nk}-E(X_{nk}))^4) \to 0$ as $n \to \infty$. So by Chebyshev's inequality $\mathcal{L}(\sum_{k=1}^{r(n)} T(n,k)) \to \mathcal{L}(1+\varepsilon)$ hence by the continuity of Brownian paths $\mathcal{L}(\sum_{k=1}^{r(n)} (X_{nk}-E(X_{nk}))) \to N(0,1+\varepsilon)$. Thus

$$\lim_{n\to\infty} \max_x P[a-x < \sum_{k=1}^{k(n)} X_{nk} < a+x] \leq (2\pi(1+\varepsilon))^{-\frac{1}{2}} \int_{-a}^{a} e^{-\frac{x^2}{2(1+\varepsilon)}} dx$$

and hence $\mathcal{L}(\sum_k X_{nk}) \not\to N(0,1)$. By a similar argument we show $\lambda < 1$ implies $\mathcal{L}(\sum_k (X_{nk} - E(X_{nk}))) \to N(0,\lambda)$, which is impossible. Thus (ii) is established. From (ii) we have that $\mathcal{L}(\sum_k T(n,k)) \to \mathcal{L}(1)$ and thus $\mathcal{L}(\sum_k (X_{nk}-E(X_{nk}))) \to N(0,1)$ and hence $\sum_k E(X_{nk}) \to 0$. The theorem is proved.

References

[1] LOÉVE, M. (1955). *Probability Theory*. Van Nostrand, New York.

[2] SKOROKHOD, A. (1965). *Studies in the Theory of Random Processes* Addison-Wesley, Reading.

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