ON THE EXACT NULL DISTRIBUTION OF HOTELLING'S TRACE*

by

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FOREWORD

This is an interim report of the work done under contract F33615-72-C-1400 of the Aerospace Research Laboratories, Air Force Systems Command, United States Air Force. The work reported herein is partly accomplished on Project 7071, "Research in Applied Mathematics", and was technically monitored by P. R. Krishnaiah of the Aerospace Research Laboratories. The work in part was also supported by a Grant from the Ford Foundation Program No. 36390.

ABSTRACT

The paper deals with the density and distribution function of tr $\underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$ where $\underset{\sim}{S}_1$ is distributed central Wishart $W(p,n_1,\underset{\sim}{\Sigma})$ independently of $\underset{\sim}{S}_2$, $W(p,n_2,\underset{\sim}{\Sigma})$. The approach is an improvement on that of Pillai and Young [8]. The density and c.d.f. are obtained explicitly for p=3 and 4 for all non-negative integral $m = (n_1-p-1)/2$, unlike in Pillai and Young where m is restricted to very small integral values.

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2. Preliminaries

Let us consider the integral of the type:

$$(2.1)\qquad \int_{\beta}\cdots\int \exp\left[-t\sum_{i=1}^{k} x_i^{-1}\right] \prod_{i=1}^{k} x_i^{n}(1-x_i)^{m} \prod_{i>j}(x_i-x_j)\prod_{i=1}^{k} dx_i,$$

where n,m are > -1, $t > 0$ independent of the x's and $\beta=\{(x_1,\ldots,x_k)\,|\,0 \le x_1 <\ldots< x_k \le x\}$. Expressing the product $\prod_{i>j}^{k}(x_i-x_j)$ by the Vandermonde's determinant and applying the properties of the determinant, (2.1) can be written as

$$(2.2)\qquad U(x:\ r_k,m;\ldots;r_1,m;t) =$$

$$= \begin{vmatrix} \int_0^x e^{-t/x_k} x_k^{r_k}(1-x_k)^m dx_k & \cdots & \int_0^x e^{-t/x_k} x_k^{r_1}(1-x_k)^m dx_k \\ \vdots & & \vdots \\ \int_0^{x_2} e^{-t/x_1} x_1^{r_k}(1-x_1)^m dx_k & \cdots & \int_0^{x_2} e^{-t/x_1} x_1^{r_1}(1-x_1)^m dx_1 \end{vmatrix}$$

where $r_i=n+i-1$, $i=1,2,\ldots,k$.

Comparing the determinant in (2.2) and that in (3.3) of Pillai [6], the difference is that he has e^{tx} instead of $e^{-t/x}$ in the integrands. Therefore analogous results can be obtained and here will be stated only the result that will be needed later.

<u>Theorem 2.1</u>: The reduction formula to evaluate the determinant (2.2) is given by

$$(2.3)\qquad U(x:\ r_k,m;\ldots;r_1,m;t)=(r_k+m+1)^{-1}[A^{(k)}+B^{(k)}-tC^{(k)}+mD^{(k)}],$$

where

$$A^{(k)}=y^{r_k+1}(1-y)^m e^{-t/y}\Big|_{y=0}^{y=x} \cdot U(x:r_{k-1},m;\ldots;r_1,m;t),$$

$$B^{(k)}=2\sum_{j=1}^{k-1}(-1)^{k+j} I(x:r_k+r_j+1,2m;2t)$$

$$\cdot U(x:r_{k-1},m;\ldots;r_{j+1},m;r_{j-1},m;\ldots;r_1,m;t),$$

$$C^{(k)}=U(x:r_k-1,m;r_{k-1},m;\ldots;r_1,m;t),$$

$$D^{(k)}=U(x:r_k,m-1;r_{k-1},m;\ldots;r_1,m;t)$$

and

$$I(x:r,m;t)=\int_0^x e^{-t/y}y^r(1-y)^m dy.$$

For m=0, (2.3) gives theorem 1 in [8].

Substituting x=1, k=p and $i-1=q_i$, i=1,2,...,p, the determinant (2.2) reduces to

(2.4)
$$D(n:q_p,m;\ldots;q_1,m;t) = $$

$$= \begin{vmatrix} \int_0^1 e^{-t/x_p} x_p^{n+q_p}(1-x_p)^m dx_p & \ldots & \int_0^1 e^{-t/x_p} x_p^{n+q_1}(1-x_p)^m dx_p \\ \vdots & & \vdots \\ \int_0^{x_2} e^{-t/x_1} x_1^{n+q_p}(1-x_1)^m dx_1 & \ldots & \int_0^{x_2} e^{-t/x_1} x_1^{n+q_1}(1-x_1)^m dx_1 \end{vmatrix}$$

and theorem (2.1) reduces to the following

<u>Corollary</u>: The reduction formula for the determinant (2.4) is given by

(2.5) $$D(n;\ q_p,m;\ldots;q_1,m;t)=(n+m+q_p+1)^{-1}[E^{(p)}+F^{(p)}-tG^{(p)}+mH^{(p)}]$$

where

$$E^{(p)}=x^{n+q_p+1}(1-x)^m e^{-t/x}\Big|_{x=0}^{x=1}\cdot D(n:q_{p-1},m;\ldots;q_1,m;t),$$

$$F^{(p)}=e^{-2t}\sum_{j=1}^{p-1}(-1)^{p+j}g(m,n:q_p+q_j+3,2;t)$$

$$\cdot D(n:q_{p-1},m;\ldots;q_{j+1},m;q_{j-1},m;\ldots;q_1,m;t),$$

and where

(2.6) $$g(m,n:\ a,b;t)=\int_0^\infty \frac{e^{-tx}(x/b)^{bm}}{(1+x/b)^{b(m+n)+a}}\,dx,$$

$$G^{(p)}=D(n:q_p-1,m;q_{p-1},m;\ldots;q_1,m;t),$$

and

$$H^{(p)}=D(n:q_p,m-1;q_{p-1},m;\ldots;q_1,m;t).$$

The following properties of the determinant D and function g in (2.6) are useful:

1. If any two of q_j's are equal, then the value of the determinant D is zero.

2. If $q_j=j-1$ where j is a positive integer, then we have

(2.7) $$D(n:q_p,m-1;q_{p-1},m;\ldots;q_1,m;t) = \\ = D(n:q_p,m-1;q_{p-1},m-1;\ldots;q_1,m-1;t).$$

3. For any non-negative integer k

(2.8) $$D(n:k,m;t)=I(1:n+k,m;t)=e^{-t}g(m,n:k+2,1;t).$$

4. For any non-negative integer c, we have

(2.9) $$g(m,n+c:a,b;t)=g(m,n:a+bc,b;t)$$

5. By integration by parts, we obtain

(2.10) $$g(0,n:a,b;t)=\frac{1}{t}\left[1-\frac{bn+a}{b}\,g(0,n:a+1,b;t)\right],$$

and for $m \neq 0$ and $b=2$

(2.11) $$g(m,n:a,2;t)=\frac{1}{t}\Big[mg(m-1,n:a+1,2;t)$$
$$-mg(m-1,n:a+2,2;t)-\frac{1}{2}\{2(m+n)+a\}g(m,n:a+1,2;t)\Big].$$

The following results on integrations will be needed later:

a). For any non-negative integer q and $2 \leq r - q$, with $c > 0$ and $0 < u < \infty$ we have:

(2.12) $$I_1(u:q,r;c)=\int_0^u \frac{x^q}{(c+x)^r}\,dx$$
$$=(r-1)^{-1}\left[\frac{Q(q,r;q)}{c^{r-q-1}}-\sum_{i=0}^{q} Q(q,r;i)\,\frac{u^{q-i}}{(c+u)^{r-i-1}}\right]$$

where

(2.13) $$Q(q,r;i)=\prod_{j=1}^{i}\left(\frac{q+1-j}{r-1-j}\right) \text{ and } Q(q,r;0)=1.$$

b). For non-negative integers q and r and for $c > 0$, $d > 0$, $0 < u < \infty$, we have [8]:

(2.14) $$I_2(u:q,r;c,d)=\int_0^u \frac{dx}{(c+x)^q(d-x)^r}$$

$$=A_1\ \ln(1+u/c)-B_1\ \ln(1-u/d)$$

$$- \sum_{i=2}^{q} \frac{A_i}{i-1} \{(c+u)^{1-i} - c^{1-i}\}$$

$$+ \sum_{j=2}^{r} \frac{B_j}{j-1} \{(d-u)^{1-j} - d^{1-j}\},$$

where

(2.15) $$A_i = \{\prod_{k=1}^{q-i} (r+k-1)\}/\{(q-i)!(c+d)^{q+r-i}\}$$

and

(2.16) $$B_j = \{\prod_{k=1}^{r-j} (q+k-1)\}/\{(r-j)!(c+d)^{q+r-j}\},$$

where the empty product is to be interpreted as unity:

c). Finally, let us consider the integral

(2.17) $$h(u:m,m';n,n';a,a';b,b')$$

$$= \int_0^u \frac{(t/b)^{bm}\{(u-t)/b'\}^{b'm'} dt}{(1+t/b)^{b(m+n)+a}\{1+(u-t)/b'\}^{b'(m'+n')+a'}}.$$

Then we have, for non-negative integers bm and b'm',

(2.18) $$h(u:m,m';n,n';a,a';b,b') = b^{bn+a} b'^{b'n'+a'}$$

$$\cdot \sum_{i=0}^{bm} \sum_{j=0}^{b'm'} (-b)^i (-b')^j \binom{bm}{i} \binom{b'm'}{j} I_2(u:bn+a+i, b'n'+a'+j; b, b'+u)$$

where $I_2(u:q,r;c,d)$ is as in (2.14), and for non-negative integers bm, bn, b'm', b'n', a and a'

(2.19) $$J(u:m,m';n,n';a,a';b,b') = \int_0^u h(x:m,m';n,n';a,a';b,b')dx$$

$$= \frac{b^{bn+a}}{b'(m'+n')+a'-1} [b'Q(b'm')I_1(u:bm, b(m+n)+a; b)$$

$$-b'^{b'n'+a'}\sum_{i=0}^{b'm'}\sum_{j=0}^{bm}\sum_{k=0}^{b'm'-i} Q(i)(-b)^j(-b')^k\binom{bm}{j}\binom{b'm'-i}{k}$$

$$.I_2(u:bn+a+j,b'n'+a'+k-1;b,b'+u)],$$

where I_1 and I_2 are as in (2.12) and (2.14) respectively and $Q(i)$ is as defined in (2.13) with $q=b'm'$ and $r=b'(m'+n')+a'$.

3. Some Values of the Determinant D

Let us first use the reduction formula (2.5) for $p=2$. Applying properties 1 and 2 of the determinant D and making use of (2.5) repeatedly m times, we obtain the general formula

$$(3.1)\qquad D(n:1,m;0,m;t)=e^{-2t}(n+m+2)^{-1}[Q(m)g(0,n:2,1;t)$$
$$-\sum_{i=0}^{m} Q(i)g(m-i,n:4,2;t)],$$

where $g(m,n:a,b;t)$ is as in (2.6) and $Q(i)=Q(m,q;i)$ is as defined in (2.13) with $q=m+n+3$.

For $p=3$, the use of formula (2.5), with the help of the properties of the determinant D listed in section 2, gives the expression

$$(3.2)\qquad D(n:2,m;1,m;0,m;t)$$
$$=e^{-3t}(n+m+3)^{-1}[(n+2)^{-1}Q(m)\{g(0,n:2,1;t)-g(0,n:4,2;t)\}$$
$$+\sum_{i=0}^{m} Q(i)\{g(m-i,n:5,2;t)g(m-i,n:3,1;t)-g(m-i,n:6,2;t)g(m-i,n:2,1;t)\}]$$

where $Q(i)$, as before, is defined by (2.13) but now $q=m+n+4$.

Finally, for $p=4$, we first state the following values of the determinants:

(3.3) $$D(n:2,m;1,m;t)=D(n+1:1,m;0,m;t)$$

$$=e^{-2t}(n+m+3)^{-1}[Q(m,q+1,m)g(0,n:3,1;t)$$

$$-\sum_{i=0}^{m} Q(m,q+1;i)g(m-i,n:6,2;t)],$$

which is obtained using (3.1) and (2.9). Note that, here $q=m+n+3$ for the definition of $Q(i)$ in (2.13). Next we need to obtain the value of the determinant $D(n:2,m;0,m;t)$. For this, we have to eliminate t using (2.10) and (2.11) in the expression $tD(n:1,m;0,m;t)$. The result is

(3.4) $$D(n:2,m;0,m;t)=e^{-2t}(n+m+2)^{-1}(n+m+3)^{-1}$$

$$\cdot[(n+m+2)\{g(m,n:4,2;t)-g(m,n:5,2;t)\}$$

$$+(n+2)Q(m,q;m)\{g(0,n;2,1;t)+g(0,n:3,1;t)\}$$

$$-(n+2)\sum_{i=0}^{m} Q(m,q;i)g(m-i,n:4,2;t)$$

$$-\sum_{i=0}^{m} (n+m-i+2)Q(m,q;i)g(m-i,n:5,2;t)$$

$$+\sum_{i=0}^{m} (m-i)Q(m,q;i)\{g(m-1-i,n:5,2;t)-g(m-1-i,n:6,2;t)\}],$$

where, again, $Q(m,q;i)$ is as in (2.13) with $q=m+n+3$.

From the above and repeating the use of formula (2.5) we finally obtain the general expression for the determinant D for $p=4$ as

(3.5) $$D(n:3,m;2,m;1,m;0,m;t)$$

$$=e^{-4t}(n+m+4)^{-1}[Q(m,q+2;m)e^{3t}D(n:2,0;1,0;0,0;t)$$

$$-\sum_{i=0}^{m} Q(m,q+2;i)\{g(m-i,n:6,2;t)e^{2t}D(n:2,m-i;1,m-i;t)$$

$$-g(m-i,n:7,2;t)e^{2t}D(n:2,m-i;0,m-i;t)$$

$$+g(m-i,n:8,2;t)e^{2t}D(n:1,m-i;0,m-i;t)\}]$$

where the D determinants on the right hand side are defined as in (3.2) for m=0, (3.3), (3.4) and (3.1) respectively.

4. The Exact Null Density Function of $U^{(p)}$

Let $\underset{\sim}{S}_1(p\times p)$ be distributed $W(p,n_1,\underset{\sim}{\Sigma},\underset{\sim}{\Omega})$, i.e. non-central Wishart distribution on n_1 d.f. with non-centrality parameter $\underset{\sim}{\Omega}$ and covariance matrix $\underset{\sim}{\Sigma}$, independently of $\underset{\sim}{S}_2(p\times p)$, central Wishart $W(p,n_2,\underset{\sim}{\Sigma},\underset{\sim}{0})$. If $c_1,\ldots,c_p$ are the latent roots of $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$, then the joint density of $c_1,\ldots,c_p$ when $\underset{\sim}{\Omega}=0$ is given by

$$f(c_1,\ldots,c_p)=C(p,m,n)\{\prod_{i=1}^{p} c_i^m/(1+c_i)^q\}\prod_{i>j}(c_i-c_j), \tag{4.1}$$

for $0 < c_1 <\ldots< c_p < \infty$, where $q=m+n+p+1$, $m=\frac{1}{2}(n_1-p-1)$, $n=\frac{1}{2}(n_2-p-1)$ and

$$C(p,m,n)=\pi^{\frac{1}{2}p}\prod_{i=1}^{p}\frac{\Gamma(\frac{1}{2}(2m+2n+p+i+2)}{\Gamma(\frac{1}{2}(2m+i+1))\Gamma(\frac{1}{2}(2n+i+1))\Gamma(\frac{1}{2}i)}.$$

To find the density of $U^{(p)}=\sum_{i=1}^{p} c_i=\mathrm{tr}\,\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$, we will use the Laplace transform of $U^{(p)}$ with respect to $f(c_1,\ldots,c_p)$. It is given by (after making transformation $x_i=(1+c_{p-i+1})^{-1}$, $i=1,2,\ldots,p$)

$$L(t:p,m,n)=C(p,m,n)e^{pt}\int\underset{\mathcal{B}}{\cdots}\int\exp(-t\sum_{i=1}^{p}x_i^{-1})$$

$$\prod_{i=1}^{p}x_i^n(1-x_i)^m\prod_{i>j}(xi-xj)\prod_{i=1}^{p}dx_i,$$

where $\mathcal{B}=\{(x_1,\ldots,x_p)\,|\,0 < x_1 <\ldots< x_p < 1\}$.

Applying the results in section 2, the above Laplace transform can be written as:

$$(4.2)\qquad L(t:p,m,n)=C(p,m,n)e^{pt}D(n:q_p,m;\ldots;q_1,m;t),$$

where $D(n:q_p,m;\ldots;q_1,m;t)$ is exactly as in (2.4) with $q_j=j-1, j=1,2,\ldots,p$. By the uniqueness of the Laplace transform, (4.2) will give the density of $U^{(p)}$ if we take its inverse. Therefore, let us denote by $D^*(n:q_p,m;\ldots;q_1,m;t)$ the inverse Laplace transform of $e^{pt}D(n:q_p,m;\ldots;q_1,m;t)$. Then, the density of $U^{(p)}$ can be written in the form

$$(4.3)\qquad f(U^{(p)})=C(p,m,n)D^*(n:q_p,m;\ldots;q_1,m;U^{(p)}).$$

Let now $g^*(u:m,n;a,b)$ be the inverse Laplace transform of $g(m,n:a,b;t)$ in (2.6), then clearly

$$(4.4)\qquad g^*(u:m,n;a,b)=(u/b)^{bm}/(1+u/b)^{b(m+n)+a},$$

and if $h(x:m,m';n,n';a,a';b,b')$ the inverse Laplace transform of $g(m,n:a,b;t)$ $g(m',n':a',b';t)$, then $h(x:m,m';n,n';a,a';b,b')$ is as described in (2.17).

Now, for p=2, we can see from (3.1) that

$$(4.5)\qquad D^*(n:1,m;0,m;u)=(n+m+2)^{-1}[Q(m,q;m)g^*(u:0,n:2,1) - \sum_{i=0}^{m} Q(m,q;i)g^*(u:m-i,n;4,2)]$$

so that the density function of $U^{(2)}$, after using (4.4), is given by:

$$(4.6)\qquad f(U^{(2)})=\frac{C(2,m,n)}{n+m+2}[Q(m,q;m)(1+U^{(2)})^{-n-2}$$

$$-\sum_{i=0}^{m}\{Q(m,q;i)(\tfrac{1}{2}U^{(2)})^{2(m-i)}/(1+\tfrac{1}{2}U^{(2)})^{2(m-i+n)+4}\}],$$

for $0 < U^{(2)} < \infty$ and zero otherwise, where $Q(m,q;i)$ is as defined in (2.13) with $q=m+n+3$.

Expressed in terms of g^* and h functions, the density of $U^{(3)}$ can be easily obtained. From (3.2) we can find the $D^*(n:2,m;1,m;0,m;u)$, so that the density function of $U^{(3)}$ is given by

$$f(U^{(3)})=(n+m+3)^{-1}C(3,m,n) \tag{4.7}$$
$$\cdot[(n+2)^{-1}Q(m,q+1;m)\{g^*(U^{(3)}:0,n;2,1)-g^*(U^{(3)}:0,n;4,2)\}$$
$$+\sum_{i=0}^{m}Q(m,q+1;i)\{h(U^{(3)}:m-i,m-i;n,n;5,3;2,1)$$
$$-h(U^{(3)}:m-i,m-i;n,n;6,2;2,1)\}]$$

for $0 < U^{(3)} < \infty$ and zero otherwise.

Finally, from (3.5) we also can obtain the density function of $U^{(4)}$. To shorten the expression we let $D_1^*(u;i)$ be the inverse Laplace transform of $g(m-i,n:6,2;t)e^{2t}D(n:2,m-i;1,m-i;t)$, so that

$$D_1^*(u;i)=(n+m-i+3)^{-1} \tag{4.8}$$
$$\cdot[Q(m-i,q+1;m-i)h(u:m-i,0;n,n;6,3;2,1)$$
$$-\sum_{j=0}^{m-i}Q(m-i,q+1;j)h(u:m-i,m-i-j;n,n;6,6;2,2)],$$

and $D_2^*(U;i)$ be that of

$$g(m-i,n:7,2;t)e^{2t}D(n:2,m-i;0,m-i;t), \text{ so that}$$

$$D_2^*(u;i)=(n+m-i+3)^{-1}(n+m-i+2)^{-1} \tag{4.9}$$
$$\cdot[(n+m-i+2)\{h(u:m-i,m-i;n,n;7,4;2,2)$$
$$-h(u:m-i,m-i:n,n;7,5;2,2)\}$$
$$+(n+2)Q(m-i,q;m-i)\{h(u:m-i,0:n,n;7,2;2,1)$$

$$+h(u:m-i,0;n,n;7,3;2,1)\}$$

$$-(n+2)\sum_{j=0}^{m-i} Q(m-i,q;j)h(u:m-i,m-i-j;n,n;7,4;2,2)$$

$$-\sum_{j=0}^{m-i} (n+m-i-j+2)Q(m-i,q;j)h(u:m-i,m-i-j;n,n;7,5;2,2)$$

$$+\sum_{j=0}^{m-i} (m-i-j)Q(m-i,q;j)\{h(u:m-i,m-1-i-j;n,n;7,5;2,2)$$

$$-h(u:m-i,m-1-i-j;n,n;7,6;2,2)\}].$$

Also we let the inverse Laplace transform of

$$g(m-i,n;8,2;t)e^{2t}D(n:1,m-i;0,m-i;t) \text{ be}$$

(4.10)
$$D_3^*(u;i)=(m-i+n+2)^{-1}$$

$$\cdot[Q(m-i,q;m-i)h(u:m-i,0:n,n;8,2;2,1)$$

$$-\sum_{j=0}^{m-i} Q(m-i,q;j)h(u:m-i,m-i-j;n,n;8,4;2,2)],$$

and that of $e^{3t}D(n:2,0;1,0;0,0;t)$ be

(4.11)
$$D_4^*(u)=(n+3)^{-1}(2n+5)^{-1}[(n+2)^{-1}g^*(u:0,n;2,1)$$

$$-(n+2)^{-1}(2n+5)g^*(u:0,n;4,2)$$

$$+2g^*(u:0,n;5,2)h(u:0,0;n,n;5,3;2,1)].$$

The density function of $U^{(4)}$ therefore can be written as

(4.12)
$$f(U^{(4)})=(n+m+4)^{-1}C(4,m,n)[Q(m,q+2;m)D_4^*(U^{(4)})$$

$$-\sum_{i=0}^{m} Q(m,q+2;i)\{D_1^*(U^{(4)};i)-D_2^*(U^{(4)};i)+D_3^*(U^{(4)};i)\}],$$

where $D_1^*(U^{(4)};i)$, $D_2^*(U^{(4)};i)$, $D_3^*(U^{(4)};i)$ and $D_4^*(U^{(4)})$ are as in (4.8), (4.9), (4.10) and (4.11) respectively while the Q's are as in (2.13) with $q=m+n+3$.

5. The Distribution Function of $U^{(p)}$

If $F(U^{(p)})$ denotes the c.d.f. of $U^{(p)}$, then upon integrating (4.3) we obtain the general form of the c.d.f. of $U^{(p)}$ as

$$F(U^{(p)})=C(p,m,n)\int_0^{U^{(p)}} D^*(n:q_p,m;\ldots;q_1,m;x)\,dx \tag{5.1}$$

Now, integrating (4.6) we obtain the c.d.f. of $U^{(2)}$ in the form

$$F(U^{(2)})=(n+m+2)^{-1}C(2,m,n)[Q(m,q;m)B_\nu(1,n+1) \tag{5.2}$$
$$-2\sum_{i=0}^{m} Q(m,q;i)B_w(2m-2i+1,2n+3)],$$

where $\nu=U^{(2)}/(1+U^{(2)})$, $w=U^{(2)}/(2+U^{(2)})$ and $B_z(a,b)$ denotes the incomplete beta function. C.d.f. (5.2) is an alternative form of that obtained by Hotelling [4].

Next, using the fact that

$$\int_0^u g^*(x:0,n;a,b)\,dx=\frac{b}{bn+a-1}\,(1-g^*(u:0,n;a-1,b)), \tag{5.3}$$

and integrating the expression in (4.7), we get the c.d.f. of $U^{(3)}$ as stated in the following theorem:

<u>Theorem 5.1</u>: The exact null distribution function of $U^{(3)}$ is given by

$$F(U^{(3)})=\frac{C(3,m,n)}{n+m+3}\,[\frac{Q(m,q+1;m)}{n+2}\,\{(n+1)^{-1}(2n+3)^{-1} \tag{5.4}$$
$$-(n+1)^{-1}(1+U^{(3)})^{-(n+1)}+2(2n+3)^{-1}(1+\tfrac{1}{2}U^{(3)})^{-(2n+3)}\}$$

$$+\sum_{i=0}^{m} Q(m,q+1;i)\{J(U^{(3)}:m-i,m-i;n,n;5,3;2,1)$$

$$-J(U^{(3)}:m-i,m-i;n,n;6,2;2,1)\}],$$

for $0 < U^{(3)} < \infty$ and zero otherwise, where $Q(m,q;i)$ is as in (2.13) with $q=m+n+3$ and $J(U^{(3)}:m,m';n,n';a,a';b,b')$ is as in (2.17).

Pillai and Young [8] have obtained the c.d.f. of $U^{(3)}$ for small values of m, i.e., for m=0(1)5. Their expression is so complicated that it is very difficult to write it down explicitly, since for their c.d.f. they need tables of constants and of values of the integrals in the determinants separately for each value of m, a and b. The c.d.f. of $U^{(3)}$ which is obtained here, as we can see from (5.4), holds for all non-negative integers m and its expression is much simpler.

Now, in order to obtain the c.d.f. of $U^{(4)}$ let us integrate the expressions in (4.8), (4.9), (4.10) and (4.11), in that order, and using (2.17) and (5.3) we get

(5.5) $$J_1(u;i)=(n+m-i+3)^{-1}$$

$$\cdot[Q(m-i,q+1;m-i)J(u:m-i,0;n,n;6,3;2,1)$$

$$-\sum_{j=0}^{m-i} Q(m-i,q+1;j)J(u:m-i,m-i-j;n,n;6,6,2,2)],$$

(5.6) $$J_2(u;i)=(n+m-i+3)^{-1}(n+m-i+2)^{-1}$$

$$\cdot[(n+m+i+2)\{J(u:m-i,m-i;n,n;7,4;2,2)$$

$$-J(u:m-i,m-i;n,n;7,5;2,2)\}$$

$$+(n+2)Q(m-i,q;m-i)\{J(u:m-i,0;n,n;7,2;2,1)$$

$$+J(u:m-i,0;n,n;7,3;2,1)\}$$

$$-(n+2)\sum_{j=0}^{m-i} Q(m-i,q;j)J(u:m-i,m-i-j;n,n;7,4;2,2)$$

$$-\sum_{j=0}^{m-i} (n+m-i-j+2)Q(m-i,q;j)J(u:m-i,m-i-j;n,n;7,5;2,2)$$

$$+\sum_{j=0}^{m-i} (m-i-j)Q(m-i,q;j)\{J(u:m-i,m-1-i-j;n,n;7,5;2,2)$$

$$-J(u:m-i,m-1-i-j;n,n;7,6;2,2)\}],$$

(5.7) $$J_3(u;i)=(n+m-i+3)^{-1}[Q(m-i,q;m-i)J(u:m-i,0;n,n;8,2;2,1)$$

$$-\sum_{j=0}^{m-i} Q(m-i,q;j)J(u:m-i,m-i-j;n,n;8,4;2,2)]$$

and finally:

(5.8) $$J_4(u)=(n+2)^{-1}(n+3)^{-1}(2n+5)^{-1}[-(2n+1)(n+1)^{-1}(2n+3)^{-1}$$

$$-(n+1)^{-1}g^*(u:0,n;1,1)+2(2n+5)(2n+3)^{-1}g^*(u:0,n;3,2)$$

$$-2g^*(u:0,n;4,2)+(n+2)J(u:0,0;n,n;5,3;2,1)].$$

Using the above, integration of $f(U^{(4)})$ in (4.12) gives the following result:

<u>Theorem 5.2</u>: The exact null distribution function of $U^{(4)}$ is given by

(5.9) $$F(U^{(4)})=(n+m+4)^{-1}C(4,m,n)[Q(m,q+2;m)J_4(U^{(4)})$$

$$-\sum_{i=0}^{m} Q(m,q+2;i)\{J_1(U^{(4)};i)-J_2(U^{(4)};i)+J_3(U^{(4)};i)\}]$$

for $0 < U^{(4)} < \infty$ and zero otherwise, where $J_1(U^{(4)};i)$, $J_2(U^{(4)};i)$, $J_3(U^{(4)};i)$ and $J_4(U^{(4)})$ are as in (5.5), (5.6), (5.7) and (5.8) respectively.

For the values of m=0,1 and 2, the c.d.f. of $U^{(4)}$ has been obtained by Pillai and Young [8] giving the necessary tables of constants and the values of the integrals needed. The result in theorem (5.2) has simpler expression and holds for all non-negative intgers m.

<u>Remark</u>: The density and c.d.f. of $U^{(p)}$ become more complicated as p becomes larger. Even for p=5, convolutions of three independent beta type variables are involved and the expressions become very cumbersome.

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13. ABSTRACT

The paper deals with the density and distribution function of tr $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ where $\underset{\sim}{S}_1$ is distributed central Wishart $W(p,n_1,\underset{\sim}{\Sigma})$ independently of $\underset{\sim}{S}_2$, $W(p,n_2,\underset{\sim}{\Sigma})$. The approach is an improvement on that of Pillai and Young [7]. The density and c.d.f. are obtained explicitly for p=3 and 4 for all non-negative integral $m=(n_1-p-1)/2$, unlike in Pillai and Young where m is restricted to very small integral values.

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