# A Generalization of a Formula of Pollaczek and Spitzer as Applied to a Storage Model

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1. Introduction. Let $\xi_1, \xi_2, \ldots$ be a sequence of independent and identically distributed (I.I.D.) real random variables (r.v.). Let

(1) $$S_0 = \xi_0 = 0,\ S_n = \sum_{k=0}^{n} \xi_k,\ \delta_n = \max(0, S_1, \ldots, S_n),\ n \geq 1.$$

In 1952 Pollaczek [3] and in 1956 Spitzer [7] obtained, using different methods, the result

(2) $$\sum_{n=0}^{\infty} \rho^n E\{\exp(-s\delta_n)\} = \exp\{\sum_{k=1}^{\infty} \frac{\rho^k}{k} E[\exp(-sS_k^+)]\},$$

valid for $|\rho| < 1$ and $\mathrm{Re}(s) \geq 0$, where $S_k^+ = \max(0, S_k)$. Earlier in 1948, Wald [10] had suggested a method for studying the distribution of $\delta_n$ based on the following observation:

Define a sequence of r.v.'s, $\eta_0, \eta_1, \eta_2, \ldots$, recursively with

(3) $$\eta_0 = \xi_0,\ \eta_{n+1} = [\eta_n + \xi_{n+1}]^+,\ n \geq 0.$$

It is easy to see that

(4) $$\eta_n = \max(0, \xi_n, \xi_{n-1} + \xi_n, \ldots, \xi_2 + \ldots + \xi_n, \xi_0 + \xi_1 + \ldots + \xi_n),$$

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for n=1, 2,... On the other hand when $\xi_0 = 0$, since $\xi_1, \xi_2, \ldots$ are I.I.D., the distributions of $\delta_n$ and $\eta_n$ are identical. Thus this paves a way of finding the distributions of $\delta_n$ via $\eta_n$.

The sequence $\{\eta_n\}$ is important by itself in that it arises in stochastic models pertaining to several live situations. One such situation will be described briefly little later. Recently Takács ([8], [9]) has studied this sequence for the case where $\xi_0$ is a nonnegative random variable independent of the other $\xi_n$'s, which are I.I.D. for $n \geq 1$. The purpose of this paper is to study the sequence $\{\eta_n\}$, when $\xi_n$'s are not necessarily I.I.D. and instead their distributions vary with the states of a finite state Markov chain (M.C.). To this end let $\{J_n, n=0, 1, 2, \ldots\}$ be a K-state M.C. with stationary one-step transition probabilities matrix $\underset{\sim}{P} = (p_{ij})$, i, j=1, 2,..., K. Let $F_i(\cdot)$, i=1, 2,...,K, be K arbitrary but given cumulative distribution functions (C.D.F.). We then define for $n \geq 0$,

$$(5) \qquad \eta_{n+1} = [\eta_n + X_{n+1}^{(j)}]^+, \text{ when } J_{n+1} = j,\ j=1, 2, \ldots, K,$$

where $\eta_0$ is an arbitrary nonnegative r.v. Here all the X's are mutually independent and are independent of $\eta_0$. Also, the C.D.F. of an X with superscript j is given by $F_j(\cdot)$. The problem then is to obtain an expression in closed form for the transform

$$(6) \qquad \underset{\sim}{V}(s,\rho) \equiv \sum_{n=0}^{\infty} \rho^n E[\exp\{-s\eta_n\}],$$

valid for $|\rho| < 1$ and $Re(s) \geq 0$.

A continuous time analogue of the special case with K=2 has been recently studied by Senturia and Puri [6]. This case, although originally arose in the context of stochastic models on quantal response assays in biology (see Puri and Senturia [4]), also serves as a natural model in the theory of storage. We shall describe it briefly here for a later reference. Let $\{W(t);\ t \geq 0\}$ be an ordinary two-state semi-Markov process defined in the sense of Pyke [5], through a bivariate sequence of r.v.'s $\{(J_n, T_n), n=0, 1, 2,...\}$ with $T_0=0$, where

$$(7) \qquad P(J_n = j,\ T_n \leq x | J_{n-1} = i) = p_{ij}\ H_i(x);$$

$0 < p_{ij} < 1$, i, j=1, 2; and $H_i(\cdot)$ denotes the C.D.F. of the nonnegative waiting time in state i, satisfying $H_i(0) < 1$. The state '1' can be visualized here as a 'release' state, while state '2' as an 'input' state for the storage, in the following sense. Each time the process $W(t)$ moves to an input state, a random nonnegative amount X of the commodity is added to the storage. On the other hand, when it moves to a release state, a random nonnegative amount Y is released, if at least that much is available at the time; if less is available, all of it is released. Let $\eta(t)$ denote the amount available in the storage at time t. $\eta(t)$ then is the continuous time analogue of $\eta_n$ defined earlier.

Senturia and Puri [6] obtained several limit results for the process $\eta(t)$, while putting practically no restrictions on $H_i(\cdot)$, i=1, 2, or on the distributions of X and Y. As such, here we shall be only concerned with the problem of obtaining explicit transform of the distribution of $\eta(t)$. The authors of [6] obtained this only for the following two cases:

(i) X and Y's are both exponentially distributed, while $H_1$ and $H_2$ are arbitrary.

(ii) X is exponentially distributed; distribution of Y is arbitrary and $H_i(x) = 1-\exp(-\alpha_i x)$, $\alpha_i > 0$, i=1, 2.

The general case where X and Y are both arbitrarily distributed was found too difficult for this purpose while using the methods of [6]. Also the more general case for $K > 2$, appears quite intractable. However for the above case with K=2, we shall present a complete solution for the discrete time case in section 3 and for the continuous time case in section 6. But first in the next section, we introduce the approach adopted here, which is due to Takács ([8], [9]).

**2. PRELIMINARIES.** For the present situation one could easily adopt the approach due to Kingman [2], which involves considering the space of signed measures on the Borel sets of the real lines. However, instead we find it more convenient to adopt a somewhat analogous approach due to Takács ([8], [9]) as described below. Let $\mathcal{A}$ be the space of functions $\phi(s)$, defined for Re(s)=0 on the complex plane, with the property that

$$\phi(s) = E\{\xi \exp(-s\eta)\} \tag{8}$$

for some complex (or real) random variable $\xi$ with $E|\xi| < \infty$, and a real r.v. $\eta$. The joint distribution of $\xi$ and $\eta$ uniquely determines $\phi(s)$, although there are infinitely many possible distributions yielding the same $\phi(s)$. We define the norm of $\phi(s)$ by

$$||\phi|| = \inf_{\xi} E|\xi| \tag{9}$$

where the infimum is taken over all those $\xi$ for which (8) holds. Now, since $|\phi(s)| \leq E|\xi|$, we have $|\phi(s)| \leq ||\phi||$ for Re(s)=0. Again, it is easy to establish that $\mathcal{A}$ is a linear space and with the above norm, it is also complete. Hence $\mathcal{A}$ is a Banach space. On this we define an operator A in the following manner. For every $\phi \epsilon \mathcal{A}$ as defined by (8), let

$$(10) \qquad A\phi(s) = \phi^+(s) = E\{\xi \exp(-s\eta^+)\},$$

for Re(s)=0, where $\eta^+ = \max(0,\eta)$. It is easy to see that the function $\phi^+(s)$ is independent of the particular representation (8) of $\phi$. Also $\phi^+(s)$ is a regular function of s in the domain Re(s) > 0 and continuous for Re(s) $\geq$ 0.

Furthermore, since A is linear and $|A\phi(s)| \leq ||\phi||$ for Re(s) $\geq$ 0, we have $||A|| = 1$, so that A is an operator. Also since for every $\phi \epsilon \mathcal{A}$, $\phi^+ \epsilon \mathcal{A}$ and $A^2=A$, it is a projection. In the following lemma we state some of the properties of the operator A, that we shall find useful later.

**LEMMA 1.**

(i) *If* $\phi_1, \phi_2 \epsilon \mathcal{A}$, *and* $A\phi_1 = \phi_1$ *and* $A\phi_2 = \phi_2$, *then* $A(\phi_1\phi_2) = \phi_1\phi_2$.

(ii) *If* $\phi_1, \phi_2 \epsilon \mathcal{A}$ *and* $A\phi_1 = C_1$ *and* $A\phi_2 = C_2$, *where* $C_1$ *and* $C_2$ *are complex (or real) constants, then* $A(\phi_1\phi_2) = C_1C_2$.

(iii) *For* $\phi_1 \epsilon \mathcal{A}$, *if* $A\phi_1 = C$, *then* $A(\phi_1 A(\phi_2)) = A(\phi_1\phi_2)$ *for all* $\phi_2 \epsilon \mathcal{A}$, *where* C *is a complex (or real) constant.*

(iv) $A\{\exp[\phi(s) - A\phi(s)]\} \equiv 1$, *for all* $\phi \epsilon \mathcal{A}$.

(v) *For every* $\phi \epsilon \mathcal{A}$, *there exist unique* $\phi^+ \epsilon \mathcal{A}$ *and* $\phi^- \epsilon \mathcal{A}$, *such that* $\phi = \phi^+ + \phi^-$, $A\phi^+ = \phi^+$ *and* $A\phi^- = 0$.

The proof of this lemma is omitted, since most of its parts can be easily established, and also some of these can be found in Takács [9].

We now return to the process $\{\eta_n\}$ defined in (5) on a K-state M.C. $J_n$. Let for n=0, 1, 2,...; i=1, 2,..., K,

$$\text{(11)} \qquad \begin{aligned} \gamma_i(s) &= E\{\exp(-sX^{(i)})\}, \ \text{Re}(s)=0, \\ \Gamma_n^{(i)}(s) &= E\{I_n^{(i)}\exp(-s\eta_n)\}, \ \text{Re}(s) \geq 0, \end{aligned}$$

where the C.D.F. of $X^{(i)}$ is given by $F_i(\cdot)$ and $I_n^{(i)} = 1$ if $J_n=i$ and is equal to zero otherwise. An argument based on the forward Kolmogorov equations easily leads to

$$\text{(12)} \qquad \Gamma_{n+1}^{(j)}(s) = A\{\sum_{i=1}^{K} p_{ij}\Gamma_n^{(i)}(s)\gamma_j(s)\}, \ n=0, 1, 2,...; \ j=1, 2,...,K,$$

where $\Gamma_0^{(i)}$, i=1, 2,..., K are given. Let $\underset{\sim}{\Gamma}_n(s) = (\Gamma_n^{(1)}(s),..., \Gamma_n^{(K)}(s))'$ and $\underset{\sim}{D}(\gamma) = (\delta_{ij}\gamma_i(s))$, where $\delta_{ij}$ is the Kronecker delta. Then the system (12) can be equivalently expressed as

$$\text{(13)} \qquad \underset{\sim}{\Gamma}_{n+1}(s) = A\{\underset{\sim}{D}(\gamma)\underset{\sim}{P}'\underset{\sim}{\Gamma}_n(s)\}, \ n=0, 1, 2,...$$

Let

$$\text{(14)} \qquad U^{(i)}(s,\rho) = \sum_{n=0}^{\infty} \rho^n \Gamma_n^{(i)}(s); \ |\rho| < 1, \ \text{Re}(s) \geq 0, \ i=1, 2,..., K,$$

and $\underset{\sim}{U}(s,\rho) = (U^{(1)}(s,\rho),...,U^{(K)}(s,\rho))'$. Then we have

$$(15) \qquad V(s,\rho) = \sum_{i=1}^{K} U^{(i)}(s,\rho),$$

where $V(s,\rho)$ is as defined in (6). It easily follows from (13) that for $|\rho| < 1$ and $\mathrm{Re}(s)=0$,

$$(16) \qquad \underset{\sim}{U}(s,\rho) = \underset{\sim}{\Gamma}_0(s) + \rho A\{\underset{\sim}{D}(\gamma)\underset{\sim}{P}'\underset{\sim}{U}(s,\rho)\},$$

or equivalently

$$(17) \qquad A\{(\underset{\sim}{I}-\rho\underset{\sim}{D}(\gamma)\underset{\sim}{P}')\underset{\sim}{U}\} = \underset{\sim}{\Gamma}_0,$$

bearing in mind that $A\underset{\sim}{U} = \underset{\sim}{U}$ and, since $\eta_0$ is a nonnegative r.v., $A\underset{\sim}{\Gamma}_0 = \underset{\sim}{\Gamma}_0$. Thus in order to obtain an expression for (15), one needs to solve for $\underset{\sim}{U}$ the system of equations (17) involving the operator A for given $\underset{\sim}{D}$, $\underset{\sim}{P}$ and $\underset{\sim}{\Gamma}_0$. Unfortunately in this generality it appears quite intractable to obtain a solution in a closed form. As such we specialize this in the next section to the case of the storage model introduced earlier with K=2.

**3. THE CASE OF STORAGE MODEL.** Let us consider the case with K=2, where for convenience, we write $p=p_{12}=1-p_{11}$ and $q=p_{21}=1-p_{22}$. The case with $p + q=1$ becomes equivalent to the case of I.I.D. r.v.'s with common C.D.F. given by $qF_1 + pF_2$. Since the results for this case are already known, we assume that $p + q\neq 1$. For the same reason we also assume that $0 < p < 1$ and $0 < q < 1$. The equation (17) can be rewritten for this case as

$$U^{(1)} - \rho(1-p)A(\gamma_1 U^{(1)}) - \rho q A(\gamma_1 U^{(2)}) = \Gamma_0^{(1)}$$

(18)

$$U^{(2)} - \rho p A(\gamma_2 U^{(1)}) - \rho(1-q)A(\gamma_2 U^{(2)}) = \Gamma_0^{(2)}.$$

We shall attempt to solve (18) assuming that $A\gamma_2 = \gamma_2$, while $\gamma_1$ remaining arbitrary. The assumption $A\gamma_2 = \gamma_2$ is consistent with the fact that the state '2' of our M.C. is the 'input' state corresponding to our storage model, so that $F_2(\cdot)$ is concentrated only on the nonnegative half of the real line. We shall need the following theorem which is a generalization of a result due to Takács [8].

THEOREM 1. *Let for* $|\rho| < 1$, $\text{Re}(s) \geq 0$, $W(s,\rho) \equiv \sum_{n=0}^{\infty} \rho^n W_n(s)$, *where* $W_n \in \mathscr{A}$, $AW_n = W_n$ *for* $n=0,1,2,\ldots,$ *and* $\sum_{n=0}^{\infty} |W_n||\rho|^n < \infty$. *Let* $W(s,\rho)$ *satisfy the equation*

(19) $$A\{(1-\rho\,\phi(s,\rho))W(s,\rho)\} = A(R(s,\rho)), \quad |\rho| < 1, \ \text{Re}(s) = 0,$$

*for given* $\phi(s,\rho)$ *and* $R(s,\rho)$ *satisfying*

(20) $$\phi(s,\rho) = \sum_{n=0}^{\infty} \rho^n \phi_n(s), \quad R(s,\rho) = \sum_{n=0}^{\infty} \rho^n R_n(s), \quad |\rho| < 1, \ \text{Re}(s) = 0,$$

*with* $\phi_n \in \mathscr{A}$, $R_n \in \mathscr{A}$, $n=0, 1, 2, \ldots,$ *and*

(21) $$\sum_{n=0}^{\infty} |\rho|^n |\phi_n| < 1, \quad \sum_{n=0}^{\infty} |\rho|^n |R_n| < \infty.$$

*Then we have*

$$W(s,\rho) = \exp\{-A \log (1-\rho\phi(s,\rho))\}$$
(22)
$$\times A[R(s,\rho) \exp\{-\log (1-\rho\phi(s,\rho) + A \log (1-\rho\phi(s,\rho)\}].$$

PROOF. Substituting the series expressions for $W,\phi$ and $R$ in (19) we obtain on comparing the coefficients of $\rho^m$ on both sides

$$W_0(s) = A[R_0(s)]$$
(23)
$$W_{m+1}(s) = A[R_{m+1}(s) + \sum_{i=0}^{m} \phi_{m-i}(s)W_i(s)],\ m \geq 0.$$

Thus we can solve (23) recursively for $W_n(s)$, n=0, 1, 2,... Consequently the solution of (19) exists and is unique. To complete the proof all we need to show is that (22) satisfies (23) or equivalently (19). For this we let

(24) $$h(s,\rho) = \exp\{\log(1-\rho\,\phi(s,\rho)) - A \log(1-\rho\,\phi(s,\rho))\}.$$

Then (22) can be rewritten as

(25) $$(1-\rho\,\phi(s,\rho))W(s,\rho) = h(s,\rho)A\{R(s,\rho)[h(s,\rho)]^{-1}\}.$$

We note from Lemma 1 (iv) that $A[h(s,\rho)] = 1$. Consequently, on applying the operator A on both sides of (25) and using Lemma 1 (iii) we have

(26) $$A\{1-\rho\,\phi(s,\rho)W(s,\rho)\} = A\{h(s,\rho)A[R(s,\rho)(h(s,\rho))^{-1}]\}$$
$$= A(R(s,\rho)),$$

which coincides with (19). This complete the proof.

The following theorem now provides the solution to (18).

THEOREM 2. *Let for* K=2, $A\gamma_2 = \gamma_2$, *while* $\gamma_1$ *remaining arbitrary*. *Then we have for* $|\rho| < 1$, $Re(s) \geq 0$,

(27) $$\sum_{n=0}^{\infty} \rho^n E\{\exp(-s\eta_n)\} = \frac{\Gamma_0^{(2)} + (1-\rho(1-p-q)\gamma_2)U^{(1)}}{1-\rho(1-q)\gamma_2},$$

*where*

(28) $$U^{(1)}(s,\rho) = \exp\{-A \log (1-\rho B_1(s,\rho))\} \times A\{B_2(s,\rho)\exp[- \log (1-\rho B_1(s,\rho)) + A \log (1-\rho B_1(s,\rho))]\},$$

(29) $$B_1(s,\rho) = [(1-p) + \frac{\rho\ q\gamma_2}{1-\rho(1-q)\gamma_2}]\gamma_1$$

*and*

(30) $$B_2(s,\rho) = [\Gamma_0^{(1)} + \frac{\rho q\gamma_1}{1-\rho(1-q)\gamma_2}\Gamma_0^{(2)}].$$

PROOF. Since $A\gamma_2 = \gamma_2$ and $AU^{(i)} = U^{(i)}$, i=1, 2, by using Lemma 1 (i), equation (18) becomes

(31) $$U^{(1)}-\rho(1-p)A(\gamma_1 U^{(1)}) - \rho q\ A(\gamma_1 U^{(2)}) = \Gamma_0^{(1)}$$

(32) $$U^{(2)}-\rho p\gamma_2 U^{(1)} - \rho(1-q)\gamma_2 U^{(2)} = \Gamma_0^{(2)}.$$

Solving (32) for $U^{(2)}$, we have

(33) $$U^{(2)} = (\Gamma_0^{(2)} + \rho p\gamma_2 U^{(1)})(1-\rho(1-q)\gamma_2)^{-1}.$$

Substituting it in (31), we obtain after some manipulation,

$$(34)\qquad A\{(1-\rho B_1(s,\rho))U^{(1)}(s,\rho)\} = A(B_2(s,\rho)).$$

Solving this for $U^{(1)}(s,\rho)$ with the help of theorem 1, one obtains (28). Finally using (33) and the fact that $V(s,\rho) = U^{(1)}(s,\rho) + U^{(2)}(s,\rho)$, one obtains (27). This completes the proof.

For the special case, where $P(\eta_0=0,\ I_0^{(1)}=1) = 1$ or equivalently $\Gamma_0^{(1)}(s)=1$ and $\Gamma_0^{(2)}(s)=0$, on using Lemma 1 (iv) the expression (25) simplifies considerably yielding

$$(35)\qquad \sum_{n=0}^{\infty} \rho^n E(\exp[-s\eta_n]) = B_3(s,\rho)\exp\{-A\log[1-\rho B_1(s,\rho)]\},$$

valid for $|\rho| < 1$, $\mathrm{Re}(s) \geq 0$, where

$$(36)\qquad B_3(s,\rho) = [1-\rho(1-p-q)\gamma_2][1-\rho(1-q)\gamma_2]^{-1}.$$

In the next section we present an alternative but more direct approach for obtaining (35), although it can with an equal ease be applied to obtain (27). This approach appears more revealing, in that it throws some light on the two factors on the right side of (35), particularly the factor $B_3$.

4 AN ALTERNATIVE APPROACH. Let us consider again the special case of the last section with $K=2$, $\Gamma_0^{(1)}(s)=1$, $\Gamma_0^{(2)}(s)=0$ and $\gamma_2$ satisfying $A\gamma_2 = \gamma_2$. Here $P(\eta_0=0) = 1$ and at the step zero the M.C. $J_n$ starts with the release state '1'. With $N(0)=0$, define the r.v.'s $N(n)$, $n \geq 1$, recursively as

$$(37)\qquad N(n) = \inf\{m:\ m > N(n-1),\ J_m = 1\},$$

so that $N(n+1) - N(n) - 1$ is the number of inputs that occurs between the nth and (n+1)th release, the zeroth release being at step zero. Let for n=0, 1, 2,..., the amounts of these inputs be denoted by the mutually independent r.v.'s $X_{i,n}^{(2)}$, i=1, 2,..., $N(n+1) - N(n) - 1$ (with common C.D.F. $F_2(\cdot)$), whenever $N(n+1) - N(n) \geq 2$ and by $X_{0,n}^{(2)} = 0$ whenever $N(n+1) - N(n) = 1$. Using these the following algebraic steps follow quite easily, while taking $|\rho| < 1$, and $\mathrm{Re}(s) \geq 0$.

(38)
$$\sum_{m=0}^{\infty} \rho^m E\{\exp(-s\eta_m)\}$$
$$= E\{\sum_{m=0}^{\infty} \rho^m \exp(-s\eta_m)\}$$
$$= E\{\sum_{n=0}^{\infty} \rho^{N(n)} \exp(-s\eta_{N(n)}) \sum_{r=0}^{N(n+1)-N(n)-1} \rho^r \exp(-s \sum_{i=0}^{r} X_{in}^{(2)})\}$$
$$= E\{\sum_{n=0}^{\infty} \rho^{N(n)} \exp(-s\eta_{N(n)}) \sum_{r=0}^{N(n+1)-N(n)-1} (\rho\gamma_2)^r\}$$
$$= \sum_{n=0}^{\infty} E\{\rho^{N(n)} \exp(-s\eta_{N(n)}) E[\frac{1 - (\rho\gamma_2)^{N(n+1)-N(n)}}{1 - \rho\gamma_2} |N(n)]\}.$$

On the other hand the process $N(n)$, n=0, 1, 2,..., is known to have independent increments, so that $N(n+1)-N(n)$ is independent of $N(n)$. Furthermore it can be easily established that

(39)
$$P(N(n+1)-N(n)=1) = 1-p$$
$$P(N(n+1)-N(n)=k) = p(1-q)^{k-1}q, \quad k \geq 1,$$

which after some algebra yields

(40)
$$E\{\frac{1 - (\rho\gamma_2)^{N(n+1)-N(n)}}{1 - \rho\gamma_2} |N(n)\} = \frac{1 - E(\rho\gamma_2)^{N(n+1)-N(n)}}{1 - \rho\gamma_2} = B_3(s,\rho).$$

Substituting this in (38), we obtain

(41) $$\sum_{m=0}^{\infty} \rho^m E[\exp(-s\eta_m)] = B_3(s,\rho) \sum_{n=0}^{\infty} E\{\rho^{N(n)} \exp(-s\eta_{N(n)})\}.$$

This shows that the factor $B_3(s,\rho)$ is nothing but the contribution to $V(s,\rho)$ coming from the 'input' steps that fall between various 'release' steps of the M.C. $J_n$. We shall show while omitting details that, as expected, the sum on the right side of (41) coincides with $\exp\{-A \log (1-\rho B_1)\}$, the term on the right side of (35). Let

(42) $$\Psi_n(s,\rho) = E\{\rho^{N(n)} \exp(-s\eta_{N(n)})\}, \ n \geq 1,$$
$$V_1(s,\rho) = \sum_{n=0}^{\infty} \Psi_n(s,\rho),$$

with $\Psi_0=1$. Calculations analogous to those of (41) and a forward Kolmogorov equation argument easily lead to the recursive relation

(43) $$\Psi_{n+1}(s,\rho) = \rho A\{B_1(s,\rho)\Psi_n(s,\rho)\}, \ n \geq 0.$$

From this and the fact that $\Psi_0=1$, it follows that

(44) $$A\{(1-\rho B_1(s,\rho))V_1(s,\rho)\} = 1.$$

Finally using theorem 1, equation (44) yields

(45) $$V_1(s,\rho) = \sum_{n=0}^{\infty} E\{\rho^{N(n)} \exp(-s\eta_{N(n)})\} = \exp\{-A \log (1-\rho B_1(s,\rho))\}.$$

This complete an alternative derivation of (35).

The above approach at least in part indicates why the solution to the general problem for $K > 2$ with $\gamma_i$'s all arbitrary, is so intractable. In essence, it involves keeping track of the different types of transitions of the M.C. $J_n$, while allowing the operator A to sweep at every transition.

Even with K=2, when both $\gamma_i$'s are arbitrary, the above approach becomes already involved. It may be remarked here that similar difficulties arise when one wishes to study the random variable $\delta_n = \max(0, S_1, \ldots, S_n)$, where the sums $S_n$'s are no longer based on I.I.D. random variables, but on r.v.'s defined on a K-state M.C. Recently Arjas [1] has considered this problem unfortunately without any success.

5. METHOD OF FACTORIZATION. The following theorem gives a method of factorization which helps the evaluations of quantities such as $A \log (1-\rho B_1(s,\rho))$, which arise in the expression for $V(s,\rho)$, such as (28). The theorem (without a trivial modification) is due to Takács [8] and is given here without proof. Later we apply this to an example.

THEOREM 3. *Suppose* $|\rho| < 1$ *and that for* $\mathrm{Re}(s)=0$, *we have*

(46) $$1-\rho B(s,\rho) = \Phi^{+}(s,\rho)\, \Phi^{-}(s,\rho),$$

*where* $\Phi^{+}(s,\rho)$ *is a regular function of* $s$ *in the domain* $\mathrm{Re}(s) > 0$, *continuous and free from zeros in* $\mathrm{Re}(s) \geq 0$, *and* $\lim_{|s| \to \infty} \log \Phi^{+}(s,\rho)/s=0$ $(\mathrm{Re}(s) > 0)$; *furthermore* $\Phi^{-}(s,\rho)$ *is a regular function of* $s$ *in the domain* $\mathrm{Re}(s) < 0$, *continuous and free from zeros in* $\mathrm{Re}(s) \leq 0$ *and* $\lim_{|s| \to \infty} \log \Phi^{-}(s,\rho)/s=0$ $(\mathrm{Re}\, s < 0)$.

*Then*

(47) $$A \log[1-\rho B(s,\rho)] = \log \Phi^{+}(s,\rho) + \log \Phi^{-}(0,\rho),$$

*for* $\mathrm{Re}\, s \geq 0$.

EXAMPLE. For the case with K=2, let $P(\eta_0=0, I_0^{(1)}=1) = 1$, and $A\gamma_2 = \gamma_2$, where $\gamma_2 = E[\exp(-sX^{(2)})]$ is otherwise arbitrary. Also let $P(X^{(1)} \leq x) = \exp(\alpha x)$

for $x \leq 0$, $\alpha > 0$, so that $\gamma_1(s) = \alpha(\alpha-s)^{-1}$ for $Re(s) < \alpha$ and $A\gamma_1 = 1$. Substituting these in (29) we have for $|\rho| < 1$,

$$(48) \qquad 1-\rho B_1(s,\rho) = g_\rho(s)[(s-\alpha)(1-(1-q)\rho\gamma_2(s))]^{-1},$$

where

$$(49) \qquad g_\rho(s) = s-\alpha + (1-q)\rho\alpha\gamma_2 - (1-q)\rho s\gamma_2 + \alpha\rho(1-p) - \rho^2(1-p-q)\alpha\gamma_2.$$

Now by applying Rouché's theorem twice it can be easily shown that as long as $|\rho| < 1$, $g_\rho(s)=0$ has exactly two roots, one in the domain $Re(s) > 0$ and the other in the domain $Re(s) < 0$. Let these roots be denoted by $s_1(\rho)$ and $s_2(\rho)$ respectively. We may now write down

$$(50) \qquad 1-\rho B_1(s,\rho) = \Phi^+(s,\rho)\ \Phi^-(s,\rho),$$

where

$$(51) \qquad \Phi^+(s,\rho) = g_\rho(s)[(s-\alpha)(s-s_2(\rho))]^{-1}$$

and

$$(52) \qquad \Phi^-(s,\rho) = (s-s_2(\rho))[1-(1-q)\rho\gamma_2(s)]^{-1}.$$

It is not too difficult to see that $\Phi^+$ and $\Phi^-$ given by (51) and (52) satisfy the desired conditions of theorem 3. Thus using this theorem we obtain from (35) after some simplification

$$(53) \qquad V(s,\rho) = s_2(\rho)[1-\rho(1-p-q)\gamma_2(s)][1-\rho(1-p-q)]^{-1}[(1-q)(s_2(\rho)-s)]^{-1},$$

valid for $|\rho| < 1$, $Re(s) \geq 0$.

6. CONTINUOUS TIME CASE. Consider an arbitrary K-state semi-Markov process $\{W(t);\ t \geq 0\}$ defined through a bivariate sequence of r.v.'s $\{J_n, T_n), n \geq 0\}$ with $T_0=0$, where

(54) $$P(J_n=j,\ T_n \leq x | J_{n-1}=i) = p_{ij}H_i(x);\ i,\ j=1,\ 2,\ldots,\ K,$$

$H_i(\cdot)$ and $(p_{ij})$ are as defined in section 1, with $H_i(0) < 1$, $i=1, 2,\ldots,K$. Let for $n=0, 1, 2,\ldots$, $\tau_n = \sum_{i=0}^{n} T_i$. We define $\eta(t)$ constructively as:

(55) $$\eta(t) = \begin{cases} \eta_0, & 0 \leq t < \tau_1 \\ [\eta(\tau_k-) + X_k^{(j)}]^+, & \tau_k \leq t < \tau_{k+1};\ \text{if } J_k = W(\tau_k)=j, \end{cases}$$

so that $\eta(t)$ is continuous from the right. With this, the $\eta_n$ as defined in (5) is also equal to $\eta(\tau_n)$. We shall study first the joint distribution of $\eta_n$ and $\tau_n$. In particular, we shall obtain an expression for

(56) $$\bar{V}(s_1,\ s_2,\ \rho) = \sum_{n=0}^{\infty} \rho^n E\{\exp(-s_1\eta_n - s_2\tau_n)\},$$

valid for $|\rho| < 1$ and $\mathrm{Re}(s_i) \geq 0$, $i=1, 2$. As before, let

(57) $$\begin{aligned} \gamma_i(s) &= E\{\exp(-sX^{(i)})\},\ \mathrm{Re}(s)=0, \\ \theta_i(s) &= E\{\exp(-sT^{(i)})\},\ \mathrm{Re}(s) \geq 0, \end{aligned}$$

$i=1, 2,\ldots, K$, where the transform $\theta_i$ corresponds to the distribution of the waiting time in state i. Analogous to (11) and (14), let for $|\rho| < 1$, $\mathrm{Re}(s_j) \geq 0$, $j=1, 2$,

(58) $$\Gamma_n^{(i)}(s_1,\ s_2) = E\{I_n^{(i)}\exp(-s_1\eta_n - s_2\tau_n)\},\ n \geq 0,$$

and

(59) $$U^{(i)}(s_1,\ s_2,\ \rho) = \sum_{n=0}^{\infty} \rho^n \Gamma_n^{(i)}(s_1,\ s_2),$$

for i=1, 2,..., K, with $\underset{\sim}{\Gamma}_n$ and $\underset{\sim}{U}$ denoting the corresponding vectors. Also let $\underset{\sim}{D}(\gamma) = (\delta_{ij}\gamma_i(s))$ and $\underset{\sim}{D}(\theta) = (\delta_{ij}\theta_i(s))$.

At this stage we need to reconsider as follows the approach due to Takács with a slight modification. Consider the space $\mathcal{A}_1$ of functions $\phi(s_1,s_2)$ defined for $\mathrm{Re}(s_i)=0$, i=1, 2, on the product space of two complex planes, with the property that

(60) $$\phi(s_1,s_2) = E\{\xi \exp(-s_1\eta-s_2\tau)\},$$

for some complex (or real) r.v. $\xi$ with $E|\xi| < \infty$, and two real r.v.'s $\eta$ and $\tau$. (It is evident that the old space $\mathcal{A}$ is contained in $\mathcal{A}_1$. On the other hand treating the factor $\xi \exp(-s_2\tau)$ of (60) as the r.v. $\xi$ of (8) and with $s_1$ in (60) replaced by s, it follows that $\mathcal{A}_1 \subset \mathcal{A}$, so that the two spaces $\mathcal{A}$ and $\mathcal{A}_1$ are identical.) With the norm as defined before in (9), the space $\mathcal{A}_1$ is a Banach space. Again, as in (10), we define for every $\phi\epsilon\, \mathcal{A}_1$, an operator $A_1$ as

(61) $$A_1\phi(s_1,s_2) = E\{\xi \exp(-s_1\eta^+-s_2\tau)\},$$

for $\mathrm{Re}(s_i)=0$, i=1, 2. Treating the factor $\xi \exp(-s_2\tau)$ as the r.v. $\xi$ of (10), it is evident that the operator A and $A_1$ are essentially the same. As a result the operator $A_1$ has the same properties as those of operator A. In particular, this operator satisfies the properties listed in Lemma 1, with the slight modification that we now allow the various C's to be functions of $s_2$. Consequently, the result parallel to that of theorem 1 holds.

The analogue of (13) for the present case is now given by

(62) $$\underset{\sim}{\Gamma}_{n+1}(s_1,s_2) = A_1\{\underset{\sim}{D}(\gamma)\underset{\sim}{P}'\underset{\sim}{D}(\theta)\underset{\sim}{\Gamma}_n(s_1,s_2)\};\ \mathrm{Re}(s_i)=0,\ i=1,\ 2,$$

n=0, 1, 2,..., where $\Gamma_0^{(i)}$, i=1, 2,..., K are given. From this it follows that

(63) $\quad A_1\{[\underset{\sim}{I}-\rho \underset{\sim}{D}(\gamma)\underset{\sim}{P}'\underset{\sim}{D}(\theta)]\ \underset{\sim}{U}\ \} = \underset{\sim}{\Gamma}_0,$

with $A_1\underset{\sim}{U} = \underset{\sim}{U}$ and $A_1\underset{\sim}{\Gamma}_0=\underset{\sim}{\Gamma}_0$, taking $\eta_0$ to be nonnegative. Finally for the case with K=2, and $A_1\gamma_2=\gamma_2$ (taking $p_{12}=p$, $p_{21}=q$, $0 < p < 1$, $0 < q < 1$, and $p+q\neq 1$) we obtain the result analogous to that of theorem 2, namely for $|\rho| < 1$, $Re(s_i) \geq 0$, i=1, 2,

(64) $$\bar{V}(s_1,s_2,\rho) = \frac{\Gamma_0^{(2)} + [1-\rho(1-q)\theta_2\gamma_2 + \rho p\gamma_2\theta_1]\ U^{(1)}}{1-\rho(1-q)\gamma_2\theta_2},$$

where

(65) $$U^{(1)} = \exp\{-A_1 \log [1-\rho\bar{B}_1(s_1,s_2,\rho)]\}$$
$$\times A_1\{\bar{B}_2(s_1,s_2,\rho) \exp [- \log (1-\rho\bar{B}_1(s_1,s_2,\rho)) + A_1 \log (1-\rho\bar{B}_1(s_1,s_2,\rho))]\},$$

(66) $$\bar{B}_1(s_1,s_2,\rho) = [(1-p) + \frac{\rho pq\gamma_2\theta_2}{1-\rho(1-q)\gamma_2\theta_2}]\ \gamma_1\theta_1,$$

and

(67) $$\bar{B}_2(s_1,s_2,\rho) = \Gamma_0^{(1)} + \rho q\theta_2\gamma_1\Gamma_0^{(2)}(1-\rho(1-q)\gamma_2\theta_2)^{-1}.$$

As before, with $\Gamma_0^{(1)}=1$, $\Gamma_0^{(2)}=0$, (64) simplifies to

(68) $$\bar{V}(s_1,s_2,\rho) = \bar{B}_3(s_1,s_2,\rho)\exp\{-A_1 \log [1-\rho\bar{B}_1(s_1,s_2,\rho)]\},$$

valid for $|\rho| < 1$, $Re(s_i) \geq 0$, i=1, 2, where

(69) $$\bar{B}_3(s_1,s_2,\rho) = [1-\{\rho(1-q)\theta_2-\rho p\theta_1\}\gamma_2][1-\rho(1-q)\gamma_2\theta_2]^{-1}.$$

In order to obtain the transform for $\eta(t)$, besides $\bar{V}$ we also need the expression for

(70) $$\bar{V}_1(s_1,s_2,\rho) = \sum_{n=0}^{\infty} \rho^n E\{\exp(-s_1\eta_n-s_2\tau_{n+1})\},$$

valid for $|\rho| < 1$ and $Re(s_i) \geq 0$, i=1, 2. For this, let for i=1, 2,..., K,

(71) $$W^{(i)}(s_1,s_2,\rho) = \sum_{n=0}^{\infty} \rho^n L_n^{(i)}(s_1,s_2),$$

where

(72) $$L_n^{(i)}(s_1,s_2,\rho) = E\{I_n^{(i)} \exp(-s_1\eta_n - s_2\tau_{n+1})\},$$

so that $\overline{V}_1 = W^{(1)}+\ldots+W^{(k)}$. A modified version of equation (63) for the vector $\underset{\sim}{W}=(W^{(1)},\ldots,W^{(K)})'$, is given by

(73) $$A_1\{[\underset{\sim}{I}-\rho\underset{\sim}{D}(\theta)\underset{\sim}{D}(\gamma)\underset{\sim}{P}']\underset{\sim}{W}\} = \underset{\sim}{L}_0,$$

where $A_1\underset{\sim}{W}=\underset{\sim}{W}$ and $A_1\underset{\sim}{L}_0=\underset{\sim}{L}_0$. As before when K=2, $A_1\gamma_2=\gamma_2$, the solution for $\overline{V}_1$ turns out to be

(74) $$\overline{V}_1(s_1,s_2,\rho) = [L_0^{(2)} + (1 + \rho\theta_2\gamma_2(1-p-q))W^{(1)}][1-\rho(1-q)\theta_2\gamma_2]^{-1},$$

with $|\rho| < 1$ and $Re(s_i) \geq 0$, i=1, 2. Here $W^{(1)}$ is same as $U^{(1)}$ of (65) with $\overline{B}_2$ replaced by $C(s_1,s_2,\rho)$, where

(75) $$C(s_1,s_2,\rho) = L_0^{(1)} + \rho q\theta_1\gamma_1 L_0^{(2)}[1-\rho(1-q)\gamma_2\theta_2]^{-1}.$$

Also, as is evident from the definition

(76) $$L_0^{(i)}(s_1,s_2) = \theta_i(s_2) \cdot \Gamma_0^{(i)}(s_1,s_2),\ i=1,\ 2.$$

Again, for the case with $\Gamma_0^{(1)}=1$ and $\Gamma_0^{(2)}=0$, expression (74) simplifies to

(77) $$\overline{V}_1(s_1,s_2,\rho) = \frac{[1 + \rho\theta_2\gamma_2(1-p-q)]\theta_1}{1 - \rho(1-q)\theta_2\gamma_2} \cdot \exp\{-A_1 \log[1-\rho\overline{B}_1(s_1,s_2,\rho)]\},$$

with $|\rho| < 1$, $Re(s_i) \geq 0$, i=1, 2.

Finally the following theorem gives the expression for the desired transform for $\eta(t)$, namely

(78) $$G(s_1,s_2) = \int_0^{\infty} \exp(-s_2 t)E[\exp(-s_1\eta(t))]\, dt,$$

valid for $Re(s_1) \geq 0$ and $Re(s_2) > 0$.

THEOREM 4. Let $0 < p < 1$, $0 < q < 1$, $p+q \neq 1$, $H_i(0) < 1$, for $i=1, 2, \ldots, K$, $Re(s_1) \geq 0$, $Re(s_2) > 0$ and $|\rho| < 1$. Then

$$G(s_1,s_2) = \frac{1}{s_2} \lim_{\rho \to 1} [\overline{V}(s_1,s_2,\rho)-\overline{V}_1(s_1,s_2,\rho)], \tag{79}$$

where $\overline{V}$ and $\overline{V}_1$ are defined by (56) and (70) respectively. In particular, when $K=2$, $A_1\gamma_2=\gamma_2$ and $P(I_0^{(1)}=1, \eta_0=0)=1$,

$$G(s_1,s_2) = \exp[-A_1 \log (1-\overline{B}_1(1))] \\ \times [1-(1-q)\gamma_1\theta_2-p\gamma_1\theta_1-\theta_1-(1-p-q)\gamma_2\theta_1\theta_2][s_2\{1-(1-q)\gamma_2\theta_2\}]^{-1}, \tag{80}$$

where $\overline{B}_1(1) = \overline{B}_1(s_1,s_2,1)$.

PROOF. Since $H_i(0)< 1$, $i=1, 2,\ldots, K$, $\sum_{n=0}^{\infty} P(\tau_n \leq t < \tau_{n+1}) = 1$, so that

$$\begin{aligned} P(\eta(t) \leq x) &= \sum_{n=0}^{\infty} P(\eta(t) \leq x,\ \tau \leq t < \tau_{n+1}) \\ &= \sum_{n=0}^{\infty} P(\eta_n \leq x,\ \tau_n \leq t < \tau_{n+1}) \\ &= \sum_{n=0}^{\infty} [P(\eta_n \leq x,\ \tau_n \leq t)-P(\eta_n \leq x,\ \tau_{n+1} \leq t)] \\ &= \sum_{n=0}^{\infty} P(\eta_n \leq x,\ \tau_n \leq t) - \sum_{n=0}^{\infty} P(\eta_n \leq x,\ \tau_{n+1} \leq t); \end{aligned} \tag{81}$$

the last step follows from the fact that $\sum_{n=0}^{\infty} P(\tau_n \leq t) < \infty$, which itself is a simple consequence of the assumption that $H_i(0) < 1$, $i=1, 2,\ldots, K$. Thus using (81), we have for $Re(s_i) > 0$, $i=1, 2$,

$$(82)\qquad G(s_1,s_2) = \int_0^\infty \exp(-s_2 t) \int_0^\infty \exp(-s_1 x)\, dP(\eta(t) \le x)\, dt$$

$$= s_1 \int_0^\infty \exp(-s_2 t) \int_0^\infty \exp(-s_1 x) P(\eta(t) \le x)\, dxdt$$

$$= s_1 \sum_{n=0}^{\infty} \int_0^\infty \int_0^\infty \exp(-s_1 x - s_2 t) P(\eta_n \le x,\ \tau_n \le t)\, dtdx$$

$$- s_1 \sum_{n=0}^{\infty} \int_0^\infty \int_0^\infty \exp(-s_1 x - s_2 t) P(\eta_n \le x,\ \tau_{n+1} \le t)\, dtdx$$

$$= \frac{1}{s_2} \sum_{n=0}^{\infty} E\{\exp(-s_1 \eta_n - s_2 \tau_n)\} - \frac{1}{s_2} \sum_{n=0}^{\infty} E\{\exp(-s_1 \eta_n - s_2 \tau_{n+1})\}.$$

Here the interchanges of the summation and the integral signs can be easily justified from the fact that

$$(83)\qquad \sum_{n=0}^{\infty} E\{\exp(-\mathrm{Re}(s_2)\tau_n)\} \le \sum_{n=0}^{\infty} E\{\exp[-\mathrm{Re}(s_2)(\xi_1+\ldots+\xi_n)]\} < \infty,$$

where $\xi_1,\ldots,\xi_n$ are mutually independent with common distribution same as that of $\min(\nu_1,\ldots,\nu_k)$, where $\nu_i$'s are independent with C.D.F. of $\nu_i$ given by $H_i(\cdot)$, i=1, 2,..., K. The last inequality in (83) holds since $H_i(0) < 1$, for all i and $\mathrm{Re}(s_2) > 0$. In (82) we have also used the standard fact that for $\mathrm{Re}(s_i) > 0$, i=1, 2,

$$(84)\qquad E\{\exp(-s_1\eta_n - s_2\tau_n)\} = s_1 s_2 \int_0^\infty \int_0^\infty \exp(-s_1 x - s_2 t) P(\eta_n \le x,\ \tau_n \le t)\, dxdt.$$

The relation (79) now easily follows from (81) and from the definitions of $\bar{V}$ and $\bar{V}_1$. Again (80) is an easy consequence of (79), (64) and (77), keeping in mind that the process of taking limit as $\rho \to 1$ under the operator $A_1$ remains valid as long as $\mathrm{Re}(s_2) > 0$. Finally by continuity it follows that

(79) and (80) also remain valid when $\mathrm{Re}(s_1) = 0$. This completes the proof of the theorem.

We close with the remark that for the case with $K > 2$, it is obvious that the above derivations will go through in an analogous fashion as long as all but one $\gamma_i$'s satisfy $A\gamma_i = \gamma_i$. However the problem of getting a solution of (17) or of (63) in a close form, for the general case with $K > 2$ and arbitrary $\gamma_i$'s, remains open.

## REFERENCES

[1] Arjas, E. (1972) On a fundamental identity in the theory of semi-Markov processes. Adv. Appl. Prob. 4, 258-270.

[2] Kingman, J.F.C. (1966) On the algebra of queues. J. Appl. Prob. 3, 285-326.

[3] Pollaczek, F. (1952) Fonctions caractéristiques de certaines répartitions définies au moyen de la notion d'ordre. Application á la théorie des attentes. C.R. Acad. Sci. 234, 2334-2336.

[4] Puri, P.S. and Senturia, J. (1972) On a mathematical theory of quantal response assays. Proc. Sixth Berkeley Symp. Math. Statist. & Prob. (held June 1971), 4, 231-247.

[5] Pyke, R. (1961) Markov renewal processes. Ann. Math. Statist. 32, 1231-1259.

[6] Senturia, J. and Puri, P.S. (1972) A semi-Markov storage model. To appear in Adv. Appl. Prob.

[7] Spitzer, F. (1956) A combinatorial lemma and its application to probability theory. Trans. Amer. Math. Soc. 82, 323-339.

[8] Takács, L. (1970) On the distribution of the maximum of sums of mutually independent and identically distributed random variables. Adv. Appl. Prob. 2, 344-354.

[9] Takács, L. (1972) On a formula of Pollaczek and Spitzer. Studia Mathematics XLI, 27-34.

[10] Wald, A. (1948) On the distribution of the maximum of successive cumulative sums of independent but not identically distributed chance variables. Bull. Amer. Math. Soc. 54, 422-430.