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The Markov Renewal Branching Process

by

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Abstract

We consider a finite Markov renewal process with an associated sequence of nonnegative random variables, having properties similar to the sizes of successive generations in a branching process. This process is called a Markov renewal branching process; it arises in the study of the busy period in several queueing models.

This paper contains a general definition and a discussion of the most important properties of the Markov renewal branching process.

1. Introduction

On a complete probability space $(\Omega,\mathcal{B},P)$, consider a sequence $\{(J_n,X_n,Y_n);\ n \geq 0\}$ of triples of random variables with the following properties:

a. For (almost) all $\omega\in\Omega$, $J_n(\omega)\in\{1,\dots,m\}$, $X_n(\omega)\in[0,\infty)$, and $Y_n(\omega)\in\{0,1,2,\dots\}$, where $1 \leq m < \infty$, $n \geq 0$.

b. $X_0=0$, a.s. and $Y_0 = k \geq 1$, a.s.

c. The bivariate sequence $\{(J_n,X_n);\ n \geq 0\}$ is a regular, irreducible Markov renewal sequence with transition probability matrix $\underline{A}(x) = \{A_{ij}(x)\}$, where

(1) $$A_{ij}(x) = P\{J_{n+1} = j,\ X_{n+1} \leq x | X_n = i\},$$

for all $1 \leq i,\ j \leq m$ and $x \geq 0$. Recall that the Markov renewal sequence is irreducible if and only if the stochastic matrix $\underline{A}(+\infty)$ is irreducible.

d. For every $n \geq 1$, the random pairs $(X_0,Y_0),(X_1,Y_1),\dots,(X_n,Y_n)$, are mutually conditionally independent, given the random variables $J_0,J_1,\dots,J_n$. Furthermore the conditional probabilities

(2) $$A_{ij}(x;\nu) = P\{J_{n+1} = j,\ X_{n+1} \leq x,\ Y_{n+1} = \nu | J_n = i\} =$$

$$P\{J_{n+1} = j,\ X_{n+1} \leq x,\ Y_{n+1} = \nu | J_n = i,\ X_n,\ Y_n,\ J_{n-1},\ X_{n-1},\ Y_{n-1},\dots,\ J_0,\ X_0,\ Y_0\}, \quad \text{for } n \geq 0,$$

are independent of n. We set

(3) $$\sum_{\nu=0}^{\infty} A_{ij}(x;\nu) = A_{ij}(x),$$

for all $x \geq 0$, $1 \leq i, j \leq m$.

For use in the sequel, we define the following:

1. the matrices $\underline{A}(x;\nu) = \{A_{ij}(x;\nu)\}$

2. the means $\beta_1,\ldots,\beta_m$ of the lattice distributions with densities $B_i(\nu) = \sum_{j=1}^{m} A_{ij}(+\infty;\nu)$, $i=1,\ldots,m$. Throughout this paper the means $\beta_1,\ldots,\beta_m$ are assumed to be finite.

3. the row-vector $\underline{\pi} = (\pi_1,\ldots,\pi_m)$ of stationary probabilities of the irreducible stochastic matrix $\underline{A}(+\infty)$.

4. the column-vector $\underline{e} = (1,1,\ldots,1)'$ with m components.

5. the matrix $\underline{A}(\xi,z) = \{\underline{A}_{ij}(\xi,z)\}$, where

(4) $$\underline{A}_{ij}(\xi,z) = \sum_{\nu=0}^{\infty} \int_0^{\infty} e^{-\xi x} d\, A_{ij}(x;\nu) z^{\nu},$$

for all ξ and z satisfying Re $\xi \geq 0$, $|z| \leq 1$.

In order to avoid trivialities we assume that $A_{ij}(+\infty;0) > 0$ for some pairs (i,j). This excludes the uninteresting case where all $Y_n=0$, a.s. for $n \geq 1$.

A number of elementary consequences of these definitions are listed for future reference:

(5) a. $\underline{A}\,(0+,1-) = \underline{A}\,(0,1-) = \underline{A}\,(0+,1) = \underline{A}\,(+\infty)$.

b. $\underline{\pi}\,\underline{A}\,(0+,1-) = \underline{\pi}$, $\quad \underline{\pi}\, e = 1$.

c. $$\left[\frac{\partial}{\partial z}\underline{A}(\xi,z)\right]_{\substack{z=1-\\ \xi=0+}} \underline{e} = \underline{\beta} = (\beta_1,\ldots,\beta_m)'.$$

The Irreducibility Condition

The matrix $\underline{A}(\xi,z)$ may be written as the series expansion

(6) $$\underline{A}(\xi,z) = \sum_{\nu=0}^{\infty} \underline{A}^{(\nu)}(\xi) z^{\nu},$$

for Re $\xi \geq 0$, $|z| \leq 1$. The functional iterates

(7) $$\underline{A}_{[n+1]}(\xi,z) = \sum_{\nu=0}^{\infty} \underline{A}_{[n]}^{(\nu)}(\xi)\ \underline{A}_{[n]}^{\nu}(\xi,z),$$

and

(8) $$_{[n+1]}\underline{A}(\xi,z) = \sum_{\nu=0}^{\infty} \underline{A}^{(\nu)}(\xi)\,_{[n]}\underline{A}^{\nu}(\xi,z),$$

are well-defined for $n \geq 1$, Re $\xi \geq 0$, and $|z| \leq 1$. We shall assume that there exists an integer M, such that the matrices $\underline{A}_{[n]}(0,1)$ and $_{[n]}\underline{A}(0,1)$ are _irreducible_ nonnegative matrices for all $n \geq M$. This irreducibility condition is satisfied under very mild conditions on the matrix $\underline{A}(\xi,z)$ and holds in all current applications of the present general model.

Description of the Model

The stochastic model of interest arises in the discussion of the busy period structure of several queueing problems, having substantially different qualitative descriptions. In order to give a unified discussion of the underlying formal structure, we consider the following urn model.

An urn contains initially $Y_0 = Z_0 = k \geq 1$, identical items. The content of the urn changes only at the transition epochs $S_n = X_1+\ldots+X_n$, $n \geq 1$, of the Markov renewal process with transition probability matrix $\underline{A}(x)$. At the n-th transition, one item is removed and Y_n items are added, and this is done for all $n \geq 1$, until the urn becomes empty.

So, provided $Z_1(\omega) \neq 0,\ldots,Z_{n-1}(\omega) \neq 0$, the content $Z_n(\omega)$ of the urn at the time $X_1+\ldots+X_n+$ is recursively defined by

$$Z_n(\omega) = [Z_{n-1}(\omega) + Y_n(\omega) - 1]^+. \tag{9}$$

The random variable $K(\omega)$ is defined by

$$K(\omega) = \min\{n: \ Z_n(\omega) = 0\}. \tag{10}$$

Clearly $K(\omega) \geq k$ for every ω. K is the index of the transition at which the urn becomes empty.

Furthermore let $I(\omega) = J_{K(\omega)}$, whenever $K(\omega)$ is finite; $I(\omega)$ is undefined whenever $K(\omega) = +\infty$. $I(\omega)$ is the state of the Markov renewal process immediately following the time at which the urn becomes empty.

Finally the random variable $U(\omega)$ is defined by $U(\omega) = X_1(\omega)+\ldots+X_{K(\omega)}(\omega)$, provided $K(\omega)$ is finite and is infinite otherwise; it measures the time until the urn becomes empty for the first time.

The object of this paper is to study the probabilities

$$G_{ij}^{(k)}(x;n) = P\{I = j, \ U \leq x, \ K=n | J_0 = i, \ Z_0 = k\}, \tag{11}$$

for $1 \leq i,j \leq m$, $n \geq 0$, $x \geq 0$, and in particular to derive a necessary and sufficient condition for the equality

$$\sum_{j=1}^{m} \sum_{n=0}^{\infty} G_{ij}^{(k)}(+\infty;n) = 1, \tag{12}$$

to hold for all $i=1,\ldots,m$.

There appear to be at least three different approaches to the study of the probabilities $G_{ij}^{(k)}(x;n)$. Each of these are generalizations of arguments first given for the classical $M|G|1$ queue. It is interesting to note however that the difficulty of the proof of any given theorem or lemma depends substantially on which of these approaches is used.

The three approaches are respectively: (a) the use of recurrence relations for appropriate taboo probabilities, (b) the Markov renewal branching model (c) the nonlinear matrix integral equation.

2. Recursion Formulas

Let ${}_0G_{ij}^{(k)}(x;\nu,n)$ be the conditional probability that at least n transitions occur in the Markov renewal process before the urn becomes empty, that immediately following the n-th transition there are ν items in the urn, that the n-th transition occurs no later than time x and that $J_n = j$, given that $J_0 = i$, $Z_0 = k$. Formally

$$(13)\qquad {}_0G_{ij}^{(k)}(x;\nu,n) = P\{J_n = j,\ Z_n = \nu,\ X_1+\ldots+X_n \le x,\ Z_r \ne 0, \text{ for } 1 \le r \le n-1 | J_0 = i,\ Z_0 = k\}.$$

Clearly

$$(14)\qquad {}_0G_{ij}^{(k)}(x;\nu,0) = \delta_{ij}\delta_{\nu k}U(x),$$

where $U(\cdot)$ is the distribution degenerate at zero. Also for $n \ge 0$,

$$(15)\qquad {}_0G_{ij}^{(k)}(x;\nu,n+1) = \sum_{h=1}^{m}\sum_{\nu'=1}^{\nu+1}\int_0^x {}_0G_{ih}^{(k)}(x-y;\nu',n)\,d\,A_{hj}(y;\nu-\nu'+1),$$

by an application of the law of total probability.

Introducing the generating functions

$$(16)\qquad W_{ij}^{(k)}(\xi,z,w) =$$

$$\sum_{\nu=0}^{\infty} z^{\nu} \sum_{n=0}^{\infty} w^{n} \int_0^{\infty} e^{-\xi x} d\ {}_0G_{ij}^{(k)}(x;\nu;n)$$

and the matrix $\underline{W}^{(k)}(\xi,z,w)$ for Re $\xi \geq 0$, $|z| \leq 1$, $|w| < 1$ we obtain successively that

$$(17) \qquad W_{ij}^{(k)}(\xi,z,w) = z^{k}\delta_{ij} + \sum_{n=0}^{\infty} w^{n+1} \sum_{\nu=0}^{\infty} z^{\nu} \sum_{\nu'=1}^{\nu+1} \sum_{h=1}^{m} \int_0^{\infty} e^{-\xi x} \int_0^{x} d_0G_{ih}^{(k)}(x-y;\nu',n)\ d\,A_{hj}(y;\nu-\nu'+1)$$

$$= z^{k}\delta_{ij} + \sum_{h=1}^{m} \frac{w}{z} \{W_{ih}^{(k)}(\xi,z,w) - W_{ih}^{(k)}(\xi,0,w)\}\ A_{hj}(\xi,z),$$

and in matrix notation

$$(18) \qquad \underline{W}^{(k)}(\xi,z,w)[zI - w\underline{A}(\xi,z)] = z^{k+1}I - w\underline{W}^{(k)}(\xi,0,w)\underline{A}(\xi,z).$$

We note that

$$(19) \qquad \underline{W}_{ij}^{(k)}(\xi,0,w) = \sum_{n=0}^{\infty} w^{n} \int_0^{\infty} e^{-\xi x} d\ G_{ij}^{(k)}(x;n),$$

where $G_{ij}^{(k)}(x;n)$ is as defined in Formula (11). For notational simplicity we write $\underline{Y}^{(k)}(\xi,w)$ for the matrix $\underline{W}^{(k)}(\xi,0,w)$.

The matrix $\underline{Y}^{(k)}(\xi,w)$ may in principle be determined by using the observation that $\underline{W}^{(k)}(\xi,z,w)$ is a matrix with analytic entries in ξ,z,w for Re $\xi > 0$, $|z| < 1$, $|w| < 1$, which is suitably continuous on the boundary; therefore the matrix

$$(20) \qquad [z^{k+1}I - w\ \underline{Y}^{(k)}(\xi,w)\underline{A}(\xi,z)][zI - w\underline{A}(\xi,z)]^{-1}$$

can only have entries with removable singularities inside this region.

This approach, which was used in a number of queueing models discussed by Çinlar [1,2] and Neuts [8,9], appears to require the introduction of a number of technical nondegeneracy assumptions in order to be able to construct the matrix $\underline{Y}^{(k)}(\xi,w)$ at those points (ξ,w), where the eigenvalues of the matrix $w\underline{A}(\xi,z)$ exhibit certain singularities. By using alternate approaches, we shall show that these conditions, which are usually impossible to check explicitly, are in fact not needed.

Theorem 1

If $\eta(\xi,z)$ and $\underline{u}(\xi,z)$ are respectively an eigenvalue and the corresponding right eigenvector of the matrix $\underline{A}(\xi,z)$, then

$$\underline{W}^{(k)}(\xi,z,w)\underline{u}(\xi,z) = [z - w\eta(\xi,z)]^{-1}[z^{k+1}\underline{u}(\xi,z) - w\eta(\xi,z)\underline{Y}^{(k)}(\xi,w)\underline{u}(\xi,z)]. \tag{21}$$

If the quantity $\chi(\xi,w)$ satisfies

$$\chi(\xi,w) = w\eta[\xi,\chi(\xi,w)], \quad 0 < |\chi(\xi,w)| < 1, \tag{22}$$

at a point (ξ,w) and if $\underline{u}(\xi,\chi(\xi,w))$ can be defined so as to be analytic in a neighborhood of the point (ξ,w), then $\chi^{k}(\xi,w)$ is an eigenvalue of $Y^{(k)}(\xi,w)$, with corresponding eigenvector $\underline{u}(\xi,\chi(\xi,w))$.

In particular, for $\xi > 0$, $0 < z < 1$, the matrix $\underline{A}(\xi,z)$ is an irreducible, nonnegative matrix. Its maximal eigenvalue $\eta^{\circ}(\xi,z)$ of algebraic and geometric multiplicity one, so that $\eta^{\circ}(\xi,z)$ is analytic in a neighborhood of every such point (ξ,z). Moreover $\underline{u}^{\circ}(\xi,z)$ can be defined so that all its components are strictly positive and are also analytic in a neighborhood of every such point (ξ,z).

The smallest positive root $\chi^{\circ}(\xi,w)$ of the equation

$$\chi^{\circ}(\xi,w) = w\eta^{\circ}[\xi,\chi^{\circ}(\xi,w)], \tag{23}$$

satisfies $0 < \chi^{\circ}(\xi,w) < 1$, and is <u>the maximal eigenvalue</u> of the matrix $\underline{Y}^{(1)}(\xi,w)$.

Proof

By multiplying the matrices in Equation (18) on the right by $\underline{u}(\xi,z)$ and simplifying, we obtain (21). The stated assumptions imply furthermore that the point $z=\chi(\xi,w)$, which lies in the unit disk must be a removable singularity of the expression on the right of Equation (21). Therefore we have that

(24) $$\chi^{k+1}(\xi,w)\underline{u}[\xi,\chi(\xi,w)] = w\eta[\xi,\chi(\xi,w)]\underline{\gamma}^{(k)}(\xi,w)\underline{u}[\xi,\chi(\xi,w)].$$

By use of Equation (22), we obtain

(25) $$\underline{\gamma}^{(k)}(\xi,w)\underline{u}[\xi,\chi(\xi,w)] = \chi^{k}(\xi,w)\underline{u}[\xi,\chi(\xi,w)].$$

In particular, for the maximal eigenvalue the analyticity conditions are always satisfied for $\xi > 0$, $0 < z < 1$. We shall show below that the equation (23) has a unique solution satisfying $0 < \chi^{\circ}(\xi,w) < 1$. Replacing $\chi(\xi,w)$ by $\chi^{\circ}(\xi,w)$ in Equation (25), we note in addition that the eigenvector $\underline{u}[\xi,\chi^{\circ}(\xi,w)]$ has all its components strictly positive. Since for $\xi > 0$, $w > 0$, the matrix $\underline{\gamma}^{(1)}(\xi,w)$ is irreducible and nonnegative, it follows that $\gamma^{\circ}(\xi,w)$ must be the maximal eigenvalue of $\underline{\gamma}^{(1)}(\xi,w)$. Indeed, the Perron-Frobenius eigenvalue of an irreducible, nonnegative matrix is the only eigenvalue having a real, strictly positive right eigenvector. The irreducibility of $\underline{\gamma}^{(k)}(\xi,w)$, $k \geq 1$, is a ready consequence of the irreducibility assumptions stated in (7) and (8).

3. The Markov Renewal Branching Process

The matrix $\gamma^{(k)}(\xi,w)$ in Formula (20) may also be determined by an argument similar to those used in the theory of branching processes.

The $Y_0 = k$ items initially in the urn guarantee that at least k transitions will occur in the Markov renewal process $\{(J_n, X_n),\ n \geq 0\}$ before the urn becomes empty. We refer to these k items as the <u>first</u> generation of items. Any items added to the urn as a result of the first k transitions make up the <u>second</u> generation of items. There will be $M_1 = Y_1 + \ldots + Y_k'$, such items. If $M_1 = 0$, the urn is empty after exactly k transitions, but if $M_1 > 0$, then M_1 additional transitions at least occur before the first emptiness of the urn. The $M_2 = Y_{k+1} + \ldots + Y_{k+M_1}$, items added during these M_1 transitions make up the third generation of items.

Continuing in this manner, we define recursively the random variables M_r, $r=0,1,\ldots$, until an index r' is reached for which $M_{r'} = 0$. When this occurs, we say that emptiness occurs after exactly r' generations of items. We so obtain a sequence (of random length) of integer-valued random variables $M_0 = k, M_1, \ldots, M_R$, where

(26) $$\{R(\omega) = r'\} = \{\omega:\ M_0 M_1 \ldots M_{r'-1} \neq 0,\ M_{r'} = 0\}.$$

We also introduce the random variables $M_0^* = 0$, $M_1^* = M_0, \ldots, M_R^* = M_0 + M_1 + \ldots + M_{R-1}$, and note that $M_R^* = K$, a.s. The random variable M_r^* counts the total number of items in the first r generations. Furthermore, the random variables $I_0^* = J_0$, $I_1^* = J_{M_1^*}, \ldots, I_R^* = J_{M_R^*}$, describe respectively the initial state of the underlying Markov renewal process and the states of this process after all transitions corresponding to the first, second,...,R-th generations have been completed.

The random variables $T_0 = 0$, and T_r, $1 \leq r \leq R$, are defined by

(27) $$T_r = \sum_{\nu=1}^{M_r^*} X_\nu,$$

and we note that T_r is the total length of time until all transitions due to the first r generation of items have been completed. Moreover $T_R = U$, a.s.

Setting $\tau_0 = 0$ and $\tau_r = T_r - T_{r-1}$, for $1 \leq r \leq R(\omega)$, it follows that the sequence

$$(28) \qquad \{(I_r^*, M_r, \tau_r), \; 0 \leq r \leq R\}$$

is a Markov renewal sequence (with a random stopping time R), [3].

Let ${}_0Q^{(r)}(i,k;\, j,k';\, n;\, x)$ with Laplace-Stieltjes transform ${}_0q^{(r)}(i,k;\, j,k';\, n;\, \xi)$ be the probability

$$(29) \qquad P\{I_r^* = j,\; M_r = k',\; M_r^* = n,\; T_r \leq x,\; M_\nu \neq 0,\; \nu=1,\ldots,r-1 \,|\, I_0^* = i,\; Y_0 = k\},$$

then we have

$$(30) \qquad {}_0q^{(r+1)}(i,k;j,k';n;\xi) = \sum_{h=1}^{m} \sum_{\nu=1}^{n} {}_0q^{(r)}(i,k;h,\nu;n-\nu;\xi)\, {}_0q^{(1)}(h,\nu;j,k';\xi)$$

for $r \geq 0$, provided we set

$$(31) \qquad {}_0q^{(0)}(i,k;j,k';n;\xi) = \delta_{ij}\, \delta_{kk'}\, \delta_{0n},$$

where the deltas are Kronecker deltas.

Introducing the generating functions

$$(32) \qquad {}_0\Phi_{ij}^{(r)}(\xi,z,w) = \sum_{n=1}^{\infty} \sum_{k'=0}^{\infty} {}_0q^{(r)}(i,k;j,k';n;\xi)\, z^{k'} w^n,$$

the equation (30) implies that

$$(33) \qquad {}_0\Phi_{ij}^{(r+1)}(\xi,z,w) = \sum_{h=1}^{m} \sum_{\nu=1}^{\infty} \sum_{n=1}^{\infty} {}_0q^{(n)}(i,k;h,\nu;n;\xi) w^n [w^\nu \underline{A}^\nu(\xi,z)]_{hj}.$$

We observe that the matrix ${}_0\Phi^{(r+1)}(\xi,z,w)$ may therefore be written as

(34) $${}_0\Phi^{(r+1)}(\xi,z,w) = {}_0\Phi^{(r)}[\xi, w\,\underline{A}(\xi,z), w] - {}_0\Phi^{(r)}(\xi,0,w),$$

where the substitution of the matrix $w\,\underline{A}(\xi,z)$ for the scalar variable z is performed as follows. The matrix ${}_0\Phi^{(r)}(\xi,z,w)$ may be written in the general form

(35) $${}_0\Phi^{(r)}(\xi,z,w) = \sum_{\nu=0}^{\infty} B_\nu^{(r)}(\xi,w)z^\nu,$$

for Re $\xi \geq 0$, $|z| \leq 1$, $|w| < 1$ or Re $\xi > 0$, $|z| \leq 1$, $|w| \leq 1$. Since in this region the norm of the matrix $w\,\underline{A}(\xi,z)$ is at most one, the series defined by

(36) $$\sum_{\nu=0}^{\infty} B_\nu^{(r)}(\xi,w)[w\,\underline{A}(\xi,z)]^\nu,$$

converges. The resulting matrix with analytic entries in ξ, z, and w in the region of interest is denoted by ${}_0\Phi^{(r)}[\xi, w\,\underline{A}(\xi,z), w]$.

Formula (34) can be most conveniently written in terms of the matrix functional iterates $\underline{A}_{[n]}^{(k)}(\xi,z,w)$ defined by

(37) $$\underline{A}_{[0]}^{(k)}(\xi,z,w) = z^k I, \qquad \underline{A}_{[n+1]}^{(k)}(\xi,z,w) = \underline{A}_{[n]}^{(k)}[\xi, w\,\underline{A}(\xi,z), w],$$

for $n \geq 0$, Re $\xi \geq 0$, $|z| \leq 1$, $|w| < 1$, or Re $\xi > 0$, $|z| \leq 1$, $|w| \leq 1$. A direct induction argument shows that

(38) $${}_0\Phi^{(0)}(\xi,z,w) = z^k I, \qquad {}_0\Phi^{(r)}(\xi,z,w) = \underline{A}_{[r]}^{(k)}(\xi,z,w) - \underline{A}_{[r-1]}^{(k)}(\xi,0,w),$$

for $r \geq 1$, $k \geq 1$.

<u>Remark</u>

It is worthwhile to stress the fact that in general $\underline{A}_{[n]}^{(k)}(\xi,z,w)$ is <u>not</u> the k-th power of the matrix $\underline{A}_{[n]}^{(1)}(\xi,z,w)$. Also, it is not in general true that

$$\underline{A}^{(k)}_{[n+1]}(\xi,z,w)$$

is equal to $w^k\underline{A}^k[\xi,\ \underline{A}^{(k)}_{[n]}(\xi,z,w)]$.

These invalid equalities were inadvertently used in Neuts [10,13], thereby invalidating some steps in the discussion of the busy period in both queueing models. Fortunately none of the substantive theorems in these papers are affected by this error. Since the Markov renewal branching process under discussion here, is precisely the abstract formulation of the busy period in both papers, the minor corrections needed in the proofs are obtained by particularizing the arguments in the present paper to the special problems discussed in [10] and [13].

Setting z=0 in Formula (32), we observe that

$$(39)\qquad {}_0\Phi^{(r)}_{ij}(\xi,0,w) = \sum_{n=1}^{\infty} {}_0q^{(r)}(i,k;j,0;n;\xi)w^n,$$

is the transform of the conditional probability that the urn becomes empty no later than time x, in exactly r generations of items, in exactly n transitions of the underlying Markov renewal process and that the final state J_n of the Markov renewal process is j, given that $J_0= i$, $Y_0= k$.

By use of Formula (38), we observe that

$$(40)\qquad \sum_{r=0}^{N} {}_0\Phi^{(r)}_{ij}(\xi,0,w) = [\underline{A}^{(k)}_{[N]}(\xi,0,w)]_{ij},$$

so that the quantity on the right is the transform of the conditional probability that the urn becomes empty no later than time x, after at most N generations of items and after exactly n transitions of the underlying Markov renewal process and that $J_n= j$, given that $J_0= i$, $Y_0= k$.

For every $N \geq 1$, the matrix $\underline{A}_{[N]}^{(k)}(\xi,0,w)$ is the transform of a bivariate, substochastic, semi-Markov matrix. Moreover for every $\xi \geq 0$, $0 \leq w \leq 1$, the entries of the matrix $\underline{A}_{[N]}^{(k)}(\xi,0,w)$ are monotone nondecreasing in N by Formula (40). Since the family of all m x m substochastic semi-Markov matrices is weakly compact, it follows that the matrix

$$\lim_{N \to \infty} \underline{A}_{N}^{(k)}(\xi,0,w), \tag{41}$$

exists and is the transform matrix of a (possibly substochastic) bivariate semi-Markov matrix. Also, by the probability interpretation of the matrix in (41), it follows that

$$\gamma^{(k)}(\xi,w) = \lim_{N \to \infty} \underline{A}_{[N]}^{(k)}(\xi,0,w). \tag{42}$$

By analytic continuation, Formula (42) is valid for all (ξ,w) with Re $\xi \geq 0$, $|w| \leq 1$. This result is summarized in the following theorem.

<u>Theorem 2</u>

The matrix $\gamma^{(k)}(\xi,w)$ is the limit of the matrix functional iterates $\underline{A}_{[N]}^{(k)}(\xi,0,w)$, defined in Formula (37).

For every $\xi \geq 0$, $0 \leq w \leq 1$, the convergence in (42) is monotone, nondecreasing in N.

<u>Theorem 3</u>

The matrix $\gamma^{(k)}(\xi,w)$ satisfies

$$\gamma^{(k)}(\xi,w) = [\gamma^{(1)}(\xi,w)]^{k}, \tag{43}$$

for all (ξ,w) with Re $\xi \geq 0$, $|w| \leq 1$, and all $k \geq 1$.

Proof

We recall the probability interpretation of the entries $\gamma_{ij}^{(k)}(\xi,w)$ as given by Formula (19), and consider the event that in the urn model, the first emptiness occurs no later than time x after exactly $n \geq k$ transitions, that $J_n = j$, given the initial conditions $J_0 = i$, $Y_0 = k$.

Let the k random variables $U_1,\ldots,U_k$ denote respectively the first passage times from the urn content k to k-1, k-1 to k-2,...,1 to 0. Let $U_1',\ldots,U_k'$ denote the number of transitions in the underlying finite Markov renewal process, while the urn content decreases from k to k-1 for the first time, from k-1 to k-2 for the first time, and so on. Finally let $V_0 = J_0$, a.s. and let V_ν be the state of the finite Markov renewal process at the time $U_1+\ldots+U_\nu+0$, for $\nu=1,\ldots,k$. A classical property of the first passage times in Markov renewal processes then implies that the k pairs $(U_1,U_1'),\ldots,(U_k,U_k')$, are conditionally independent given the k+1 random variables $V_0,V_1,\ldots,V_k$.

Furthermore the conditional probability $P\{U_\nu \leq x,\ U_\nu' = n,\ V_\nu = j \mid V_{\nu-1} = i\}$, is equal to $G_{ij}^{(1)}(x;n)$, because of the spacial homogeneity of the content process. Its transform with respect to x and n is therefore given by $\gamma_{ij}^{(1)}(\xi,w)$.

Finally, since $I = V_k$, $K = U_1'+\ldots+U_k'$, and $U = U_1+\ldots+U_k$, it follows that

$$(44) \qquad \gamma_{ij}^{(k)}(\xi,w) = \sum_{i_1=1}^{m} \cdots \sum_{i_{k-1}=1}^{m} \gamma_{ii_1}^{(1)}(\xi,w)\cdots\gamma_{i_{k-1}j}^{(1)}(\xi,w),$$

and hence Formula (43) holds.

Remark

Theorem 3 could be anticipated in view of the property of the eigenvalues and right eigenvectors of $\gamma^{(k)}(\xi,w)$, shown in Theorem 1. A direct analytic proof seems to involve a complicated discussion of the singularities of the eigenvectors of the matrix $zI - w\,\underline{A}(\xi,z)$, which is avoided by the probabilistic proof given here.

Theorem 4

The matrix $\gamma^{(1)}(\xi,w) = \gamma(\xi,w)$ is the unique matrix with entries analytic in ξ and w for Re $\xi > 0$, $|w| \le 1$, which satisfies the functional equation

(45) $$Z = w\,\underline{A}[\xi,Z], \qquad \text{with } ||Z|| \le 1,$$

for every Re $\xi \ge 0$, $|w| \le 1$.

Proof

Consider the probabilities $G_{ij}^{(k)}(x;n)$ as defined in Formula (11). By conditioning on the number of items in the urn and on the state of the Markov renewal process at the time of the first transition we obtain

(46) $$G_{ij}^{(1)}(x;n) = \delta_{n1}A_{ij}(x;0) + \sum_{h=1}^{m}\sum_{\nu=1}^{n-1}\int_0^x A_{ih}(x-y;\nu)\,d\,G_{hj}^{(\nu)}(y;n-1),$$

for all $x \ge 0$, $n \ge 1$, $1 \le i$, $j \le m$.

Upon evaluation of the transforms

(47) $$\gamma_{ij}^{(k)}(\xi,w) = \sum_{n=k}^{\infty} w^n \int_0^\infty e^{-\xi x} d\,G_{ij}^{(k)}(x;n),$$

and by use of Theorem 3, we obtain from Formula (46) that

(48) $$\gamma(\xi,w) = w\,\underline{A}[\xi,\ \gamma(\xi,w)].$$

Since $\gamma(\xi,w)$ is the transform of a bivariate (substochastic) semi-Markov matrix, it is clearly of norm not exceeding one for Re $\xi \ge 0$, $|w| \le 1$.

The proof of the uniqueness of the solution is identical to that given by Purdue [17] for the case w=1. As in Purdue's Theorem 3.2, it is sufficient to show that there exists a number $\sigma \ge 0$ for which the quantity

(49) $$\Psi(\sigma) = \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{n=1}^{\infty} n \int_0^\infty e^{-\sigma x} d\,A_{ij}(x;n) < 1.$$

Since $\Psi(0+) = \sum_{i=1}^{m} \beta_i < \infty$, and $\Psi(\sigma)$ is decreasing and tends to zero as σ tends to infinity, such a number σ clearly exists. As in Purdue's proof, the result follows by appealing to a contraction mapping theorem and an analytic continuation argument.

Remark

For w=1, the Equation (48) is the transform version of a nonlinear matrix integral equation. The matrix $\underline{G}(x) = \{\underline{G}_{ij}(x)\}$, where

(50) $$G_{ij}(x) = \sum_{n=0}^{\infty} G_{ij}^{(1)}(x;n),$$

is then the unique (substochastic) semi-Markov matrix with transform matrix $\gamma(\xi,1)$, which satisfies the system of integral equations

(51) $$G(x) = \sum_{n=0}^{\infty} A(\cdot;n) * G^{(n)}(x), \qquad x \geq 0,$$

where $G^{(n)}(\cdot)$ is the n-fold matrix convolution of the semi-Markov matrix $G(\cdot)$.

4. The Probability of Eventual Emptiness

Next we examine the probabilities of eventual emptiness of the urn. The quantity

(52) $$\sum_{j=1}^{m} G_{ij}(+\infty), \qquad i=1,\ldots,m,$$

is clearly the probability that the urn becomes empty eventually, given that $J_0 = i$, $Z_0 = 1$. We shall derive a necessary and sufficient condition under which the quantities in (52) are equal to one for all i=1,...,m.

Theorem 5

The matrix $G(+\infty) = \gamma(0+,1)$, is stochastic if and only if

(53) $$\sum_{i=1}^{m} \pi_i \beta_i \leq 1.$$

Proof

For k=1, w=1, the Equation (18) becomes

$$\text{(54)} \qquad \underline{W}^{(1)}(\xi,z,1)[z\ I - \underline{A}(\xi,z)] = z^2 I - \underline{\gamma}(\xi,1)\underline{A}(\xi,z),$$

For $\xi \geq 0$, $0 < z \leq 1$, the matrix $\underline{A}(\xi,z)$ is an irreducible matrix with nonnegative entries. Let $\eta(\xi,z)$ be the maximal eigenvalue of $\underline{A}(\xi,z)$ and let $\underline{u}(\xi,z)$ be the corresponding right eigenvector, which can be chosen so that all components of $\underline{u}(\xi,z)$ are analytic for $\xi > 0$, $0 < z < 1$, continuous at $\xi = 0$, $z = 1$, for $\xi \to 0+$, $z \to 1-$, and so that in addition all components of $\underline{u}(\xi,z)$ are strictly positive for $\xi \geq 0$, $0 < z \leq 1$.

Multiplying both sides of Equation (54) on the right by $\underline{u}(\xi,z)$ we obtain,

$$\text{(55)} \qquad \underline{W}^{(1)}(\xi,z,1)[z-\eta(\xi,z)]\underline{u}(\xi,z) = z^2\underline{u}(\xi,z) - \eta(\xi,z)\gamma(\xi,1)\underline{u}(\xi,z).$$

Since the vector $\underline{W}^{(1)}(\xi,z,1)\underline{u}(\xi,z)$, has analytic components for $\xi > \theta$, $0 < z < 1$, which are suitably continuous as $\xi \to 0+$, $z \to 1-$, the vector on the right must vanish if z is replaced by $\chi(\xi)$, where $\chi(\xi)$ is any root of the equation

$$\text{(56)} \qquad z = \eta(\xi,z), \qquad 0 < z \leq 1,\ \xi > 0.$$

As was shown in [13], by using convexity properties of $\eta(\xi,z)$ established in [12], the Equation (56) has a unique root $\chi(\xi)$ satisfying $0 < \chi(\xi) < 1$, for $\xi > 0$, which tends increasingly as $\xi \to 0+$, to the smallest positive root of the Equation

$$\text{(57)} \qquad z = \eta(0+,z)$$

Replacing z by $\chi(\xi)$ in (55) we obtain

$$\text{(58)} \qquad \chi^2(\xi)\underline{u}[\xi,\chi(\xi)] - \chi(\xi)\gamma(\xi,1)\underline{u}[\xi,\chi(\xi)] = \underline{0}.$$

for $\xi > 0$. Dividing by $\chi(\xi) \neq 0$, we obtain that $\chi(\xi)$ is a positive eigenvalue of the nonnegative matrix $\underline{\gamma}(\xi,1)$, with corresponding right eigenvector $\underline{u}[\xi,\chi(\xi)]$ having all positive components. It follows therefore that the quantity $\chi(\xi)$ is the maximal eigenvalue of the matrix $\gamma(\xi,1)$ for $\xi > 0$, and by continuity that the maximal eigenvalue of $\gamma(0+,1)$ is given by the smallest positive root of the Equation (57).

The smallest positive root $\chi(0+)$ of Equation (57) is equal to one, if and only if the left hand derivative $\eta'(0+,1-)$ of $\eta(0+,z)$ is less than or equal to one. If $\eta'(0+,1-) > 1$, the quantity $\chi(0+)$ satisfies $0 < \chi(0+) < 1$.

Finally, it is well-known [1,12], that $\eta'(0+,1-)$ is given by the quantity $\sum_{i=1}^{m} \pi_i \beta_i$.

The limit matrix $\Gamma = \gamma(0+,1)$, plays an important role in many applications of the Markov renewal branching process. By setting w=1, and ξ=0+, in Equation (48) we see that

$$(59) \qquad \Gamma = \underline{A}[0+,\Gamma] = \sum_{\nu=0}^{\infty} \underline{A}(+\infty;\nu)\Gamma^{\nu}.$$

Furthermore, as shown in the proof of Theorem 5, the spectral radius of Γ is equal to the smallest positive root of the Equation (57). This implies that Γ is strictly substochastic, if $\sum_{i=1}^{m} \pi_i \beta_i > 1$, and is stochastic if and only if $\sum_{i=1}^{m} \pi_i \beta_i \leq 1$.

We shall now examine all solutions to the equation

$$(60) \qquad X = \sum_{\nu=0}^{\infty} \underline{A}(+\infty;\nu)X^{\nu}$$

which belong to the class $\mathcal{X}$ of irreducible, nonnegative and substochastic matrices.

Theorem 6

If $X \in \mathcal{X}$ and satisfies (60), then the spectral radius $\rho(X)$ is a root of Equation (57) in (0,1 .

The matrix Γ is a minimal solution of (60) in $\mathcal{X}$, i.e. every other solution $X \in \mathcal{X}$ satisfies $\Gamma \leq X$, where $\leq$ denotes the entry-wise inequality for matrices. The minimal solution is unique. The Equation (56) always has a stochastic solution. If $\sum_{i=1}^{m} \pi_i \beta_i \leq 1$, then Γ is the unique, irreducible solution of (60) in $\mathcal{X}$.

Proof

Let $X\underline{u} = z\underline{u}$, $\underline{u} > \underline{0}$. Since $X \in \mathcal{X}$, we have $0 \leq z \leq 1$. Equation (60) implies that

$$(61) \qquad z\underline{u} = X\underline{u} = \sum_{\nu=0}^{\infty} \underline{A}(+\infty;\nu) z^{\nu} \underline{u} = \underline{A}(0+,z)\underline{u},$$

in terms of the matrix $\underline{A}(\xi,z)$ at ξ=0+. Let $\underline{v}$ be a left eigenvector of A(0+,z), corresponding to the maximal eigenvalue $\eta(0+,z)$ of the irreducible positive matrix $\underline{A}(0+,z)$. Let $\underline{v}$ be chosen, so that $\underline{v} > \underline{0}$. Equation (61) yields

$$(62) \qquad \underline{v}(z\underline{u}) = z\underline{v}\ \underline{u} = \eta(0+,z)\underline{v}\ \underline{u}.$$

Since $\underline{v}\ \underline{u} > 0$, this implies that z must be a root of Equation (57).

The solution Γ corresponds by Theorem 5, to the smallest positive root of (57).

Next we construct a solution X* to (60), which we shall show to be a minimal solution. We define recursively the sequence of matrices $X_0 = 0$, $X_{n+1} = \sum_{\nu=0}^{\infty} \underline{A}(+\infty;\nu) X_n^{\nu}$, for $n \geq 0$.

The sequence $\{X_n, n \geq 0\}$, is monotone, nondecreasing. This is a particular case of the fact that for all $X \in \mathcal{X}$ and $Y \in \mathcal{X}$

(63) $$X \leq Y, \rightarrow \sum_{\nu=0}^{\infty} \underline{A}(+\infty;\nu)X^{\nu} \leq \sum_{\nu=0}^{\infty} \underline{A}(+\infty;\nu)Y^{\nu}.$$

This implication is proved as follows. Let $X^k \leq Y^k$, then $X^k(X-Y) \leq 0$, and hence $X^{k+1} \leq X^kY$. Also $(X^k-Y^k)Y \leq 0$, and hence $X^kY \leq Y^{k+1}$. Combining both inequalities yields $X^{k+1} \leq Y^{k+1}$. Therefore, by induction, $X \leq Y$, implies that $X^k \leq Y^k$, for all $k \geq 0$. Equation (63) is now an obvious corrolary. Applying (63), to the sequence $\{X_n\}$, proves that $X_n \leq X_{n+1}$, since $0 \leq X_0$.

The sequence $\{X_n, n \geq 0\}$ has a limit in $\mathcal{X}$, which we denote by X^*. The matrix X^* is clearly a solution of Equation (60).

Next we show that every other solution Y of (60) in $\mathcal{X}$ satisfies $X^* \leq Y$. The matrix Y satisfies $Y \geq 0 = X_0$, and therefore

(64) $$Y = \sum_{\nu=0}^{\infty} A(+\infty;\nu)Y^{\nu} \geq X_1,$$

and by repeated iteration we obtain $Y \geq X_n$. Letting n tend to infinity, we obtain $Y \geq X^*$.

Let Y now be any solution to (60), whose maximal eigenvalue is equal to the smallest positive root of the Equation (57). Since $X^* \leq Y$, the maximal eigenvalue $\rho(X^*)$ does not exceed the maximal eigenvalue $\rho(Y)$ of Y. Therefore $\rho(X^*)$ is also equal to the smallest positive root of Equation (57), or $\rho(X^*) = \rho(Y)$. This in turn implies that $X^* = Y$, and in particular that $X^* = \Gamma$.

The Equation (60) always has at least one stochastic solution. This follows by application of Brouwer's fixed point theorem to the set of stochastic matrices.

We can now summarize the results about the solution set of (60) as follows:

a. If $\sum_{i=1}^{m} \pi_i \beta_i \leq 1$, then Γ is irreducible and stochastic. Since every other stochastic solution X must satisfy $\Gamma \leq X$, we obtain $\Gamma = X$ and Γ is therefore the unique solution to (60), since there are no substochastic solutions.

b. If $\sum_{i=1}^{m} \pi_i \beta_i > 1$, then Γ is irreducible and strictly substochastic. $\rho(\Gamma)$ is the smallest positive root of Equation (57). The matrix Γ is the only strictly substochastic solution to Equation (60).

The Equation (60) also has at least one stochastic solution X and $\Gamma \leq X$. We do not know whether the stochastic solution in this case is also unique.

5. Applications

The Markov renewal branching process occurs in the study of the busy period of a large number of queueing models. We list several of these as examples.

a. The M|G|1 Queue [11]

In this case m=1, and the matrix $\underline{A}(\xi,z)$ reduces to a single entry.

(65) $$A(\xi,z) = h(\xi + \lambda - \lambda z),$$

where $h(\cdot)$ is the Laplace-Stieltjes transform of the service time distribution $H(\cdot)$ and λ is the arrival rate.

In the case of group arrivals,

(66) $$\underline{A}(\xi,z) = h[\xi + \lambda - \lambda\varphi(z)],$$

where $\varphi(\cdot)$ is the probability generating function of the group size density.

b. The M|SM|1 Queue

This model was studied by Çinlar [1] and by Neuts [8]. If the service times are governed by the semi-Markov matrix $\underline{Q}(\cdot)$ with matrix $\underline{q}(\xi)$ of Laplace-Stieltjes transforms, then the busy period process is a Markov renewal branching process with

(67) $\underline{A}(\xi,z) = \underline{q}[\xi + \lambda-\lambda z]$.

Many queueing models with group service or with dispatching are particular cases of the M|SM|1 model.

c. Two Servers in Series with a Finite Intermediate Waitingroom

This model was discussed in Neuts [9,10]. The matrix $\underline{A}(\xi,z)$ is complicated and will not be given here. The equilibrium condition for this case is also fairly complex and involves the computation of the stationary probabilities of the matrix $\underline{A}(0,1)$. The resulting expression for the traffic intensity does not have the intuitive simplicity of that in other queues.

d. A Queue with Fluctuating Arrival and Service Rates

Two closely related, but not identical, models of queues subject to fluctuations in arrival and service rates were studied by Naor and Yechiali [7] and by Neuts [13]. See also Purdue [16] and Yechiali [18].

For the model studied in [13], the matrix $\underline{A}(\xi,z)$ is given by

(68) $$A_{ij}(\xi,z) = \int_0^\infty e^{-\xi u} P_{ij}(z,u)\, d\, H_i(u),$$

where $H_i(\cdot)$, $i=1,\ldots,m$, are probability distributions on $[0,\infty)$. The function $P_{ij}(z,u)$ is the (i,j)-th entry of the matrix

(69) $$\underline{P}(z,u) = \exp\{-u[\Lambda(1-z) + \Delta(I-P)]\},$$

where $\Lambda = \text{diag}\ (\lambda_1,\ldots,\lambda_m)$, $\lambda_i \geq 0$, $i=1,\ldots,m$; $\Delta = \text{diag}(\sigma_1,\ldots,\sigma_m)$, $\sigma_i > 0$, $i=1,\ldots,m$; and P is an irreducible stochastic matrix of order m. The busy period process is again a Markov renewal branching process.

e. Discrete Time Versions of the preceding models

The numerical analysis problems related to the preceding models have not been investigated extensively. These problems are very substantial and require further theoretical investigation. The enormous numerical difficulties are somewhat easier to overcome for the discrete versions, than for the continuous parameter models. For an extensive discussion of the value of discrete queues, we refer to Dafermos and Neuts [4], and to the investigation of models of the M|G|1 type in Neuts [14], Klimko and Neuts [5,6] , Neuts and Heimann [15].

It is clear that the equations governing the busy period structure in these cases are the formal analogues of those for the corresponding continuous parameter cases. The discrete time queues correspond to lattice Markov renewal proceses, while the sojourn times for the continuous parameter models have general distributions.

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Bibliography

[1] Çinlar, E. (1967)
Time Dependence of Queues with semi-Markovian Services.
J. Appl. Prob. 4, 356-364.

[2] Çinlar, E. (1967)
Queues with semi-Markovian Arrivals
J. Appl. Prob. 4, 365-379.

[3] Çinlar, E. (1969)
Markov Renewal Theory
Advances Appl. Prob. 1, 123-187.

[4] Dafermos, S. and Neuts, M. F. (1971)
A Single Server Queue in Discrete Time.
Cahiers du Centre de Recherche Opérationnelle, Brussels, 13, 23-40.

[5] Klimko, E. and Neuts, M. F. (1973)
The Single Server Queue in Discrete Time, Numerical Analysis II
Nav. Res. Logistics Quarterly, forthcoming in 1973.

[6] Klimko, E. and Neuts, M. F. (1973)
The Single Server Queue in Discrete Time, Numerical Analysis III
Nav. Res. Logistics Quarterly, forthcoming in 1973.

[7] Naor, P. and Yechiali, U. (1971)
Queueing Problems with Heterogeneous Arrivals and Service
Operations Research 19, 722-734.

[8] Neuts, M. F. (1966)
The Single Server Queue with Poisson Input and semi-Markov Service Times
J. Appl. Prob. 3, 202-230.

[9] Neuts, M. F. (1968)
Two Queues in Series with a Finite, Intermediate Waitingroom
J. Appl. Prob. 5, 123-142.

[10] Neuts, M. F. (1968)
Two Servers in Series, treated in terms of a Markov Renewal Branching Process
Adv. Appl. Prob. 2, 110-149.

[11] Neuts, M. F. (1969)
The Queue with Poisson Input and General Service Times, treated as a Branching Process
Duke Math. J. 36, 215-232.

[12] Neuts, M. F. and Purdue, P. (1971)
Multivariate Semi-Markov Matrices
J. Australian Math. Soc. 13, 107-113.

[13] Neuts, M. F. (1971)
A Queue Subject to Extraneous Phase Changes
Adv. Appl. Prob. 3, 78-119.

[14] Neuts, M. F. (1973)
The Single Server Queue in Discrete Time, Numerical Analysis I
Nav. Res. Logistics Quarterly, forthcoming in 1973

[15] Neuts, M. F. and Heimann, D. (1972)
The Single Server Queue in Discrete Time, Numerical Analysis IV
Purdue Mimeo Series No. 301, Department of Statistics, Purdue University, W. Lafayette, IN. 47907.

[16] Purdue, P. (1972)
On the Use of Analytic Matric Functions in Queueing Theory
Ph.D. Thesis, Department of Statistics, Purdue University, W. Lafayette, IN.

[17] Purdue, P. (1973)
Nonlinear Matrix Integral Equations of Volterra Type in Queueing Theory
Forthcoming in J. Appl. Prob.

[18] Yechiali, U. (1973)
A Queueing Type Birth and Death Process defined on a Continuous time Markov Chain
Operations Research, forthcoming

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