A Restricted Subset Selection Approach to Ranking and Selection Problems

by

Thomas J. Santner

Purdue University

Department of Statistics
Division of Mathematical Sciences

Mimeograph Series #318

May 1973

*This research was supported by the Office of Naval Research Contract N00014-67-A-0226-00014 at Purdue University.

TABLE OF CONTENTS

TABLE OF CONTENTS

LIST OF TABLES

INTRODUCTION

Let $\pi_1,\ldots,\pi_k$ be k populations with π_i having unknown cdf $F_i \in \mathcal{F}$, a known family of distributions. Each π_i is characterized by an unknown $\lambda_i = \lambda_i(F_i) \in \Lambda$, a specified interval on the real line, and $\pi_{(i)}$ is that population having the ith largest λ_i. The experimenter's goal is to make some inference about $\pi_{(k)}$ (or equivalently $\pi_{(1)}$).

Early statistical work on this problem consisted mainly in deriving tests of hypotheses for the equality of the λ_i's. Bahadur [2], Mosteller [42] and Paulson [47] were among the first research workers to recognize the inadequacy of such tests for homogeneity and to reformulate the problem as a multiple decision problem concerned with the ranking and selection of k populations.

In the two decades since these early papers, ranking and selection problems have become an active area of statistical research and the theory in this field has undergone a somewhat dichotomous development arising from the detailed formulation of a reasonable experimental goal to pursue. One approach pioneered by Bechhofer [8] has been to allow the experimenter to select one population which is guaranteed to be of interest to him with fixed probability P* whenever the unknown parameters lie outside some subset, or zone of indifference, of the entire parameter space. This has been termed the indifference zone approach.

Other contributors whose work falls along these lines are Alam and Rizvi [1] on procedures for multivariate normal populations; Bechhofer and Sobel [9], Sobel and Huyett [56], Barr and Rizvi [5] and Rizvi [5] on single sample procedures for various univariate parametric populations; Sobel [57] on single sample non parametric procedures for univariate populations; Mahamunulu [39] and Desu and Sobel [15, 16] on procedures for selecting fixed size subsets containing the t best populations and Bechhofer, Dunnet and Sobel [10], Bechhofer [15, 16],Chambers and Jaratt [13], Paulson [49, 50, 51] and Bechhofer, Kiefer and Sobel [12] on multi stage and sequential procedures.

In contrast to the indifference zone approach, Gupta [20, 26] proposed a formulation in which the experimenter obtains a subset of the k populations for which there is fixed minimum probability P* over the entire parameter space that the population of interest is included. The name of this approach is the subset selection approach and is derived from the type of procedure employed.

Some recent contributors in the category of subset selection work include Gnanadesikan [18], Gnanadesikan and Gupta [19], Gupta [27], Gupta and Studden [31] and Gupta and Panchapakesan [29] on procedures for multivariate normal distributions; Nagel [43], Gupta and Nagel [32] and Gupta and Panchapakesan [33] on single stage procedures for univariate populations; Patterson [46], Rizvi and Sobel [52], Barlow and Gupta [3], Barlow, Gupta and Panchapakesan [4], McDonald [40, 41] and Gupta and McDonald [30] on single stage non parametric and partially non parametric procedures for univariate populations; and Barron [6], Barron and Gupta [7] and Huang [5] on sequential procedures.

Subset selection procedures are often thought of as screening procedures which enable the experimenter to select a subset of the populations under study which contains the best one ($\pi_{(k)}$) so that the selected subset can be further studied in more intensive fashion. The goal in this thesis is to study fixed sample size procedures which give more flexibility to the experimenter than does the usual subset selection procedure by allowing him to specify an upper bound, m, on the number of populations included in the selected subset. Should the data clearly indicate that a particular population is best, this type of rule still retains that advantage of the subset selection procedure in allowing selection of fewer than m populations. On the other hand if the data make the choice of the best population less obvious, this rule selects a larger subset for further study but guarantees that no more than m populations are selected. Such procedures will be called restricted subset selection procedures.

Formally this thesis studies a generalization of the subset selection and indifference zone goals. Furthermore the procedure proposed by Bechhofer [8] and Gupta [20, 26] are also special cases of the present rules. Both analytic results and numerical values needed to implement the proposed rules are given.

The statistician must base his selection rule on the independent random variables $\{X_{ij}\}$ from π_i which have common cdf F_i.

It is assumed that a consistent sequence of estimators for λ_i, $\{T_n(\cdot)\}$, is available i.e.

$$T_n(X_{i1},\ldots,X_{in}) \overset{p}{\to} \lambda_i .$$

The following goal is studied:

<u>G</u>: Given P^*, $m(1 \leq m \leq k)$ and also possibly n and/or $p(\lambda)$ where $p(\cdot)$ is a function on Λ satisfying $p(\lambda) \leq \lambda$, define a procedure $R(n)$ based on a sample of common size n from each population which selects a subset of populations less than or equal to m in size, contains $\pi_{(k)}$ and satisfies

$$P_{\underset{\sim}{\lambda}}[CS|R(n)] \geq P^* \forall\, \underset{\sim}{\lambda} \varepsilon \Omega(p) = \{\underset{\sim}{\lambda} | \lambda_{[k-1]} \leq p(\lambda_{[k]})\}.$$

The event $[CS|R(n)]$ occurs iff the selected subset contains $\pi_{(k)}$. The following rule is proposed.

<u>R(n)</u>: Select $\pi_i \Leftrightarrow T_{in} \geq \max\{T_{[k-m+1]n}, h_n^{-1}(T_{[k]n})\}$

where

(1) $T_{in} = T_n(X_{i1},\ldots,X_{in})$

(2) $T_{[1]n} \leq \ldots \leq T_{[k]n}$ are the ordered observations

(3) $\{h_n(\cdot)\}$ is a sequence of real valued functions satisfying

(a) For every x and n, $h_n(x) > x$

(b) For every n, $h_n(x)$ is continuous and strictly increasing in x

(c) For every x, $h_n(x) \to x$ as $n \to \infty$

The choices $m=1$ with any $p(\lambda) < \lambda$ and $m=k$ with $p(\lambda) = \lambda$ yield the indifference zone approach and subset selection approach respectively.

In Chapter I a general expression is derived for the probability of a correct selection for an arbitrary underlying true vector of λ_i's. Then under the assumption that for each n the estimators form a stochastically increasing family it is shown that the infimum of the probability of correct selection over $\Omega(p)$ occurs at a point for which the parameters are as close together as possible and yet still in $\Omega(p)$. This amounts to reducing the calculation of a k dimensional infimum to a one dimensional infimum. Sufficient conditions are given under which this last infimum can be evaluated. In Section 1.3 the monotonicity and other properties of R(n) are discussed. In particular, it is proved that the infimum over $\Omega(p)$ of the probability of a correct selection approaches one as $n \to \infty$ so that any P* level is attainable by simply taking n sufficiently large. In Section 1.4 the number of non best populations selected, the total number of populations selected and their expectations are studied both asymptotically and for fixed n. The final two sections of Chapter I discuss some modifications of the previous results to allow selection for $\pi_{(1)}$ and some alternate formulations of the basic problem.

Chapter II discusses applications of the general theory to some of the problems which have been previously studied in the literature using fixed size subset rules and subset selection (m=k) rules. These problems include selection from normal populations for means and variances and from non central χ^2 and non central $\mathcal{F}$ populations for non centrality parameters. An example is given of selecting from uniform populations which illustrates the so called "non regular" case. Three different types of indifference zones are employed and also some work is done to establish criteria for choosing a particular rule R(n). Tables are constructed to allow implementation of the proposed procedure.

In Chapter III a generalization is studied in which the experimenter's goal is to determine one of the t best populations. In the case $t \leq (k-m)$ the indifference zone approach is employed and all the results of the general theory parallel those in the special case t=1. In particular the infimum of the probability of a correct selection occurs at a point when all the λ_i's are as close together as possible and yet $\underset{\sim}{\lambda}$ is still in $\Omega^t(p) = \{\underset{\sim}{\lambda} | \lambda_{[k-t]} \leq p(\lambda_{[k-t+1]})\}$, the preference zone for this problem. Tables are provided to allow implementation of these selection rules. In the case $t > (k-m)$ the subset selection approach can be used to yield results generalizing those of Gupta and Panchapakesan [33].

CHAPTER I

SOME RESTRICTED SUBSET SELECTION PROCEDURES

1.1 Formulation of the Problem

Let $(\mathcal{X},\mathcal{B},P_i)$, $i = 1,\ldots,k$ be k probability spaces hereafter referred to as populations and denoted as π_i, $i = 1,\ldots,k$. Specifically it is assumed that $\mathcal{X}$ is a finite dimensional Euclidean space and $\mathcal{B}$ is the associated Borel σ field. The P_i are unknown; however it is assumed they belong to some specified family $\mathcal{P}$ of probability measures on $(\mathcal{X},\mathcal{B})$. Finally let

(i) $F_i(x) = P_i(-\infty,x]$, $x\varepsilon\mathcal{X}$ be the cdf associated with P_i,

(ii) $\mathcal{F} = \{F(x) = P(-\infty,x] | P\varepsilon\mathcal{P}\}$ be the family of all possible cdf's for π_i and

(iii) $\Omega(\mathcal{F}) = \mathcal{F}^k$ is the set of all possible underlying vectors of cdf's.

Each π_i is characterized by a scalar $\lambda_i = \lambda_i(P_i) \varepsilon \Lambda \subset R$ where Λ is a known interval on the real line and the λ_i's are unknown. Let $\lambda_{[1]} \leq \cdots \leq \lambda_{[k]}$ be the ordered λ_i's, $\Omega = \{\underset{\sim}{\lambda}=(\lambda_1,\ldots,\lambda_k) | \lambda_i\varepsilon\Lambda\forall_i\}$ the space of all possible λ_i's and $\pi_{(i)}$ the (unknown) population with parameter $\lambda_{[i]}$. It is assumed there is no a priori knowledge of the correct pairing of the elements in $\{\pi_i\}$ and $\{\pi_{(i)}\}$. The goal is to define a procedure R to select the "best" population where for sake of definiteness $\pi_{(k)}$ is taken as the best population. Note that in some cases $\pi_{(1)}$ might be the best population. Of course if j $(2 \leq j \leq k)$ populations all have $\lambda_i = \lambda_{[k]}$, the selection of any of these tied populations accomplishes the goal. However in the subset selection formulation, ordinarily the infimum of the probability of correct selection is attained when all populations are identical and an arbitrary one is tagged as the "best" one.

The statistician must base his selection rule on the independent random variables $\{X_{ij}\}$ from π_i which have common cdf F_i. Now it is assumed, as will usually be the case in practice, that a consistent sequence of estimators of λ_i, $\{T_n(\cdot)\}$, is available i.e. $T_n(\cdot)$ is a Borel measurable function on $\mathcal{X}^n$ such that:

$$T_n(X_{i1},\ldots,X_{in}) = T_{in} \xrightarrow{p} \lambda_i \text{ when } \{X_{ij}\}_{j=1}^{n} \text{ are iid from } \pi_i.$$

More generally it suffices to assume $\{T_{in}\}$ converges in probability to a monotone function of λ_i, $\nu(\lambda_i)$, so that selection in terms of the $\nu(\lambda_i)$ is equivalent to that in terms of the λ_i.

At times, for notational ease, it will be convenient to denote T_{in} as T_i suppressing the dependence on n. In addition, when writing the cdf of a random variable say, for example, $G(y/\lambda)$ the slanted line will indicate the value of G at y given the true parameter is λ. When used anywhere else a slanted line will indicate division. Finally the following notation and assumption will be needed.

(1.1.1)

(i) The distribution of T_{in} depends only on λ_i and is absolutely continuous with respect to Lesbesgue measure

(ii) T_{in} has cdf $G_n(y/\lambda_i)$, support $E_n^{\lambda_i}$ and pdf $g_n(y/\lambda_i)$

(iii) For every n, $\{G_n(\cdot/\lambda) | \lambda \varepsilon \Lambda\}$ forms a stochastically increasing family

(iv) $G_n^{(i)}(y) = G_n(y/\lambda_{[i]})$ is the cdf of T_{in} when $\{X_{ij}\}_{j=1}^{n}$ are iid from $\pi_{(i)}$

Remark 1.1.1. The requirements in (1.1.1) are not needed in the proofs of all the results to follow even though they are listed here for convenience sake. For example assumption (iii) alone suffices to prove Theorem 1.2.2.

The following definitions will serve to distinguish between the several types of rules proposed in the literature for the above general problem. Let R be the given procedure based on $T_1,\ldots,T_k$ and S the number of populations it selects.

Def. 1.1.1. R is a fixed size subset selection rule means $\exists s (1 \leq s < k)$ such that $P_{\underset{\sim}{F}}[S=s] = 1 \forall \underset{\sim}{F} \varepsilon \Omega(\mathcal{F})$

Rules for which s=1 were introduced by Bechhofer [8] for selecting the normal population with largest mean. In the general case, fixed size subset rules were introduced by Mahamunulu [39].

Def. 1.1.2. R is a restricted subset selection rule means $\exists s$ $(1 < s < k)$ such that $P_{\underset{\sim}{F}}[1 \leq S \leq s] = 1 \forall \underset{\sim}{F} \varepsilon \Omega(\mathcal{F})$ and R is not a fixed size subset rule.

A class of restricted subset selection procedures for parametric problems is studied in this thesis. Analogs for many of the previously studied procedures are developed using this approach.

Def. 1.1.3. R is a subset selection rule means $P_{\underset{\sim}{F}}[1 \leq S \leq k] = 1$ $\forall \underset{\sim}{F} \in \Omega(\mathcal{F})$ and R is neither a restricted subset selection procedure nor a fixed size subset selection procedure.

Subset selection procedures were proposed by Gupta [20,26] in an application to the normal means problem. Some aspects of the general theory are studied in Gupta and Panchapakesan [33].

The current problem will be solved using a restricted subset selection procedure under the indifference zone formulation. In general, only subset selection procedures can attain all P* levels over the entire Ω space. The motivation for using such procedures is that they can be used as screening devices which allow the experimenter to specify an upper bound, m, on the number of populations included in the selected subset. Should the data clearly indicate that a particular population is best, this type of rule still retains that advantage of the subset selection rule in allowing selection of fewer than m populations. On the other hand, if the data make the choice of the best population less obvious, this rule selects a larger subset for further study but guarantees that no more than m populations are selected.

An indifference zone will be defined in $\Omega(\mathcal{F})$ by means of a function $p: \Lambda \to R$ such that

$$(1.1.2) \begin{cases} \text{(i)} & p(\cdot) \text{ is continuous and non decreasing on } \Lambda \\ \text{(ii)} & p: \Lambda' \overset{\text{onto}}{\to} \Lambda \text{ where} \\ & \Lambda' = \{\lambda \varepsilon \Lambda | p(\lambda) \varepsilon \Lambda\} \\ \text{(iii)} & p(\lambda) < \lambda \quad \forall \lambda \varepsilon \Lambda \end{cases}$$

Remark 1.1.2. Formally both the indifference zone approach and the subset selection approach become special cases of (1.1.5) by allowing $p(\lambda) \leq \lambda$ in (1.1.2) and choosing ($m=1$, $p(\lambda) < \lambda$) and ($m=k$, $p(\lambda) = \lambda$) respectively. The case of importance for this thesis will be $1 < m < k$ and hence the restruction $p(\lambda) < \lambda$. However at times it will be pointed out how the general theory reduces to the previously obtained results for these important subcases and for this discussion the weaker $p(\lambda) \leq \lambda$ will be tacitly assumed.

Example 1.1.1.

(a) $\Lambda = (0,\infty)$ and $p(\lambda) = \delta\lambda (0 < \delta < 1) \Rightarrow \Lambda' = (0,\infty)$

(b) $\Lambda = (-\infty,\infty)$ and $p(\lambda) = \lambda - \delta (\delta > 0) \Rightarrow \Lambda' = (-\infty,\infty)$

(c) $\Lambda = [0,\infty)$ and $p(\lambda) = \lambda - \delta (\delta > 0) \Rightarrow \Lambda' = [\delta,\infty)$

For each $\underset{\sim}{F} \varepsilon \Omega(\mathcal{F})$ let $\underset{\sim}{\lambda}(\underset{\sim}{F}) = (\lambda_1,\ldots,\lambda_k)$ be the vector of λ_i's associated with $\underset{\sim}{F}$. Define

$$\Omega(p) = \{\underset{\sim}{\lambda} \varepsilon \Omega | \lambda_{[k-1]} \leq p(\lambda_{[k]})\}$$
$$\Omega^o(p) = \{\underset{\sim}{\lambda} \varepsilon \Omega | \lambda_{[1]} = \lambda_{[k-1]} = p(\lambda_{[k]})\}$$

and then

$\Omega'(p) = \{\underset{\sim}{F} \varepsilon \Omega(\mathcal{F}) | \underset{\sim}{\lambda}(\underset{\sim}{F}) \varepsilon \Omega(p)\}$ specifies a preference zone in $\Omega(\mathcal{F})$.

Example 1.1.2.

(a) Let Λ and $p(\cdot)$ be as in (a) above $\Rightarrow \Omega'(p) =$

$\{\underset{\sim}{F} | \delta^{-1}\lambda_{[k-1]} \leq \lambda_{[k]}\}$

(b) Let Λ and $p(\cdot)$ be as in (b) above $\Rightarrow \Omega'(p) =$

$\{\underset{\sim}{F} | \lambda_{[k]} - \lambda_{[k-1]} \geq \delta\}$

Finally, a general procedure for selecting a restricted subset of the k populations will be defined. Let $\{h_n(\cdot)\}$ be a sequence of functions for which each $h_n(\cdot)$ is defined on a portion of the line containing $\bigcup_{\lambda\varepsilon\Lambda} E_n^{\lambda}$ and satisfies

(1.1.3)
- (i) For each n and x, $h_n(x) > x$
- (ii) For each n, $h_n(x)$ is continuous and strictly increasing in x
- (iii) For each x, $h_n(x) \to x$ as $n \to \infty$

Define the procedure:

(1.1.4) <u>R(n)</u>: Select $\pi_i \Leftrightarrow T_{in} \geq \max\{T_{[k-m+1]n}, h_n^{-1}(T_{[k]n})\}$ where $T_{[1]n} \leq T_{[2]n} \leq \cdots \leq T_{[k]n}$ are the ordered estimators.

Example 1.1.3.

(a) Take $d > 0$ and $h_n(x) = x + d/\sqrt{n}$

<u>R(n)</u>: Select $\pi_i \Leftrightarrow T_{in} \geq \max\{T_{[k-m+1]n}, T_{[k]n} - d/\sqrt{n}\}$

(b) Let $d > 0$ and $h_n(x) = (1+d/n)x$ and suppose $\bigcup_{\lambda\varepsilon\Lambda} E_{\lambda}^{n} \subset (0,\infty)$

<u>R(n)</u>: Select $\pi_i \Leftrightarrow T_{in} \geq \max\{T_{[k-m+1]n}, (\frac{n}{n+d})\, T_{[k]n}\}$

<u>Goal</u>: Given P^*, $p(\cdot)$ and the sequence $\{R(n)\}$ find the common sample size n necessary to achieve

(1.1.5) $\quad P_{\underset{\sim}{F}}[CS|R(n)] \geq P^* \; \forall \; \underset{\sim}{F} \varepsilon \Omega'(p)$

Here $[CS|R(n)]$ stands for the selection of any restricted size subset containing $\pi_{(k)}$.

Remark 1.1.3. It is not obvious that (1.1.5) can ever be achieved for any n for an arbitrary P* and p(·). Section 1.3 will provide a proof for this claim.

1.2 Probability of a Correct Selection and its Infimum

Theorem 1.2.1. For any $\underset{\sim}{F} \varepsilon \Omega(\mathcal{F})$

$$(1.2.1) \quad P_{\underset{\sim}{F}}[CS|R(n)] = \sum_{i=k-m}^{k-1} \sum_{\nu=1}^{\binom{k-1}{i}} \int_{\infty}^{\infty} \prod_{j \varepsilon \mathcal{J}_\nu^i(k)} G_n^{(j)}(y) \prod_{j \varepsilon \bar{\mathcal{J}}_\nu^i(k)} \{G_n^{(j)}(h_n(y)) - G_n^{(j)}(y)\} dG_n^{(k)}(y)$$

where

$$\begin{cases} \{\mathcal{J}_\nu^\ell(k) \mid \nu = 1, \ldots, \binom{k-1}{\ell}\} \text{ is the collection of all subsets of size } \ell \text{ from } \{1, \ldots, k-1\} \\ \bar{\mathcal{J}}_\nu^\ell(k) = \{1, \ldots, k-1\} - \mathcal{J}_\nu^\ell(k) \end{cases}$$

Proof. Let $\{\mathcal{J}_\nu^i(k)\}$ and $\{\bar{\mathcal{J}}_\nu^i(k)\}$ be defined as above and also let

$A_\nu^i = [T_{(k)} > T_{(j)} \forall j \varepsilon \mathcal{J}_\nu^i(k)$ and $T_{(k)} < T_{(j)} \forall j \varepsilon \bar{\mathcal{J}}_\nu^i(k)]$ where

$T_{(\ell)}$ is the random variable corresponding to $\pi_{(\ell)}$

$$\Rightarrow P_{\underset{\sim}{F}}[CS|R(n)] = P_{\underset{\sim}{F}}[h_n(T_{(k)}) \geq T_{[k]} \text{ and } T_{(k)} \geq T_{[k-m+1]}]$$

$$= P_{\underset{\sim}{F}}[h_n(T_{(k)}) \geq T_{[k]} \text{ and } T_{(k)} > \text{at least } (k-m) \; T_{(j)}\text{'s w/} j < k]$$

$$= P_{\underset{\sim}{F}}[h_n(T_{(k)}) \geq T_{[k]} \text{ and } \bigcup_{i=k-m}^{k-1} \bigcup_{\nu=1}^{\binom{k-1}{i}} A_\nu^i]$$

$$= \sum_{i=k-m}^{k-1} \sum_{\nu=1}^{\binom{k-1}{i}} P_{\underset{\sim}{F}}[\mathcal{D}_\nu^i] \text{ where } \mathcal{D}_\nu^i = [h_n(T_{(k)}) \geq T_{(j)} \forall j<k, A_\nu^i]$$

Fix i and ν and then since $h_n(T_{(j)}) > T_{(j)}$ by assumption

$$\Rightarrow [h_n(T_{(k)}) \geq h_n(T_{(j)})] \subset [h_n(T_{(k)}) \geq T_{(j)}]$$

$$\Rightarrow P_{\underset{\sim}{F}}[\mathcal{D}_\nu^i] = P_{\underset{\sim}{F}}\ [h_n(T_{(k)}) \geq T_{(j)} \forall j<k,\ T_{(k)} \geq T_{(j)} \forall j \varepsilon \mathcal{A}_\nu^i(k),$$

$$h_n(T_{(k)}) < h_n(T_{(j)}) \forall j \varepsilon \mathcal{J}_\nu^j(k)]$$

$$= P_{\underset{\sim}{F}}[T_{(k)} > T_{(j)} \forall j \varepsilon \mathcal{A}_\nu^i(k),\ T_{(j)} < h_n(T_{(k)}) < h_n(T_{(j)}) \forall j \varepsilon \mathcal{J}^i(k)]$$

$$= P_{\underset{\sim}{F}}[T_{(k)} > T_{(j)} \forall j \varepsilon \mathcal{A}_\nu^i(k),\ T_{(k)} < T_{(j)} < h_n(T_{(k)}) \forall j \varepsilon \mathcal{J}_\nu^j(k)]$$

$$= \int_{-\infty}^{\infty} \prod_{j \varepsilon \mathcal{A}_\nu^i(k)} G_n^{(j)}(y) \prod_{j \varepsilon \mathcal{J}_\nu^i(k)} \{G_n^{(j)}(h_n(y)) - G_n^{(j)}(y)\} dG_n^{(k)}(y)$$

where $G_n^{(j)}(y)$ is defined to be the cdf of $T_{(j)n}$. This completes the proof.

The point to be emphasized regarding this result is that the $P_{\underset{\sim}{F}}[CS|R(n)]$ only depends on $\underset{\sim}{F}$ through $\underset{\sim}{\lambda}(\underset{\sim}{F})$ which is obvious since the distributions of the T_{in} depend only on the λ_i. Hence the more accurate notation, $P_{\underset{\sim}{\lambda}}[CS|R(n)]$ will be used hereafter to denote the probability of a correct selection computed under the assumption the X_{ij}'s have cdf's $\underset{\sim}{F}$ with parameters $\underset{\sim}{\lambda}$.

As mentioned in Section 1.1 the interest in this thesis is in the case $1 < m < k$. However note that for the cases m=1 and m=k the goal and rule of Section 1.1 and Theorem 1.2.1 reduce to the following

(a) m=1, $p(\lambda) < \lambda$ (Bechhofer type goal and rule)

Goal: Select the population $\pi_{(k)}$ so that $P_{\underset{\sim}{\lambda}}[CS] \geq P^*$ $\underset{\sim}{\lambda} \varepsilon \Omega(p)$

Rule: Choose the population corresponding to $T_{[k]n}$

$$P_{\underset{\sim}{\lambda}}[CS] = \int_{-\infty}^{\infty} \prod_{j=1}^{k-1} G_n^{(j)}(y) dG_n^{(k)}(y)$$

(b) m=k, $p(\lambda) = \lambda$ (Gupta type goal and rule)

Goal: Select a subset of $\{\pi_1,\ldots,\pi_k\}$ which contains $\pi_{(k)}$ such that $P_{\underset{\sim}{\lambda}}[CS] \geq P^* \; \forall \underset{\sim}{\lambda} \varepsilon \Omega$

Rule: Select $\pi_i \Leftrightarrow h_n(T_i) \geq T_{[k]}$.

$$P_{\underset{\sim}{\lambda}}[CS] = \int_{-\infty}^{\infty} \prod_{j=1}^{k-1} G_n^{(j)}(h_n(y)) dG_n^{(k)}(y)$$

The next problem is to determine the infimum of the probability of a correct selection over $\Omega(p)$ for a given R(n). The following lemma due to Mahamunulu [39] and Alam and Rizvi [1] will be needed.

Lemma 1.2.1. Let $\underset{\sim}{X} = (X_1,\ldots,X_k)$ have $k \geq 1$ independent components such that for every i, X_i has cdf $H(\cdot|\theta_i)$. Suppose $\{H(\cdot|\theta)\}$ forms a stochastically increasing family. If $\phi(\underset{\sim}{X})$ is a monotone function of X_i when all other components of $\underset{\sim}{X}$ are held fixed then $E_{\underset{\sim}{\theta}}[\phi(\underset{\sim}{X})]$ is monotone in θ_i in the same direction.

The following notation will be used throughout to denote the incomplete beta function with parameters a and b.

$$I(y;a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^y w^{a-1}(1-w)^{b-1}\, dw$$

Theorem 1.2.2.

$$\inf_{\Omega(p)} P_{\underset{\sim}{\lambda}}[CS|R(n)] = \inf_{\Omega^{o}(p)} P_{\underset{\sim}{\lambda}}[CS|R(n)]$$

$$(1.2.2) \qquad = \inf_{\lambda\varepsilon\Lambda'} \int_{-\infty}^{\infty} \{G_n(h_n(y)/p(\lambda))\}^{k-1} I(\frac{G_n(y/p(\lambda))}{G_n(h_n(y)/p(\lambda))};k-m,m)dG_n(y/\lambda))$$

Proof. The proof is an application of Lemma 1.2.1. Take

$$\phi(\underset{\sim}{T}) = \begin{cases} 1, & T_{(k)} \geq \max\{T_{[k-m+1]}, h_n^{-1}(T_{[k]})\} \\ 0, & \text{otherwise} \end{cases}$$

Recall that in $\underset{\sim}{T} = (T_1,\ldots,T_k)$, $T_{[i]}$ is the ith order statistic and $T_{(i)}$ is the statistic corresponding to $\pi_{(i)}$. Now for $\ell < k$ pick $T' = (T_1',\ldots,T_k')$ satisfying $T_{(j)}' = T_{(j)} \; \forall \; j \neq \ell$ and $T_{(\ell)}' > T_{(\ell)}$. It suffices to prove $\phi(\underset{\sim}{T}) = 0 \Rightarrow \phi(\underset{\sim}{T}') = 0$.

Now $\phi(\underset{\sim}{T}) = 0$

$\Leftrightarrow T_{(k)} < \max\{T_{[k-m+1]}, h_n^{-1}(T_{[k]})\}$

$\Leftrightarrow$ (a) $h_n(T_{(k)}) < T_{[k]}$

or

(b) $T_{(k)} < T_{[k-m+1]}$

If (a) holds $\Rightarrow$ either

(1) $T_{(\ell)}' \leq T_{[k]} \Rightarrow T_{[k]}' = T_{[k]} \Rightarrow h_n(T_{(k)}') = h_n(T_{(k)}) < T_{[k]} = T_{[k]}'$

or

(2) $T_{(\ell)}' > T_{[k]} \Rightarrow T_{[k]}' = T_{(\ell)}' > T_{[k]}$

$$\Rightarrow h_n(T_{(k)}') < T_{[k]} < T_{(\ell)}' = T_{[k]}'$$

So in either case

$$h_n(T'_{(k)}) < T'_{[k]} \leq \max\{T'_{[k]}, h_n(T'_{[k-m+1]})\}$$

$\Rightarrow \phi(\underset{\sim}{T}') = 0$

If (b) holds $\Rightarrow$ either

(1) $T'_{(\ell)} \leq T_{[k-m+1]} \Rightarrow T'_{[k-m+1]} = T_{[k-m+1]}$

$\Rightarrow T'_{(k)} = T_{(k)} < T_{[k-m+1]} = T'_{[k-m+1]}$

or

(2) $T_{(\ell)} > T_{[k-m+1]} \Rightarrow T'_{[k-m+1]} = T_{[k-m+1]} \Rightarrow T'_{(k)} < T'_{[k-m+1]}$

or

(3) $T_{(\ell)} \leq T_{[k-m+1]} < T'_{(\ell)} \Rightarrow T'_{[k-m+1]} = \min\{T_{[k-m+2]}, T'_{(\ell)}\} > T_{[k-m+1]}$

$\Rightarrow T'_{(k)} < T'_{[k-m+1]}$

So in either case

$$h_n(T'_{(k)}) < h_n(T'_{[k-m+1]}) \leq \max\{h_n(T'_{[k-m+1]}), T'_{[k]}\}$$

$\Rightarrow \phi(\underset{\sim}{T}') = 0$ thus ending the proof.

Let

$$\psi(\lambda,n) = \int_{-\infty}^{\infty} \{G_n(h_n(y)/p(\lambda))\}^{k-1} I\left(\frac{G_n(y/p(\lambda))}{G_n(h_n(y)/p(\lambda))}; k-m, m\right) dG_n(y/\lambda) \tag{1.2.3}$$

and then the significance of Theorem 1.2.2 is that it has reduced the calculation of a k dimensional infimum to a one dimensional infimum over Λ'. Furthermore the following lower bound for $\psi(\lambda,n)$ can be easily obtained by noting $h_n(x) > x$ and $\psi(\lambda,n)$ is non decreasing in $h_n(x)$:

$$\psi(\lambda,n) \geq \int_{-\infty}^{\infty} \{G_n(y/p(\lambda))\}^{k-1} dG_n(y/\lambda)$$

Note that the right hand side is actually the one dimensional infimum obtained using the above theorem in the case m=1.

Remark 1.2.1. Gupta and Nagel [32] have introduced the concept of a just rule. A rule R with individual selection probabilities $p_i(x_1,\ldots,x_k)$, $i = 1,\ldots,k$ is said to be just if $x_i > y_i$ and $x_j \le y_j$ for $j \neq i$ implies that $p_i(x_1,\ldots,x_k) \geq p_i(y_1,\ldots,y_k)$. Since $\phi(\underset{\sim}{T})$ is the probability of selecting $\pi_{(k)}$ based on $\underset{\sim}{T}$ for the rule R(n), the proof of Theorem 1.2.2 shows it suffices that R(n) satisfy a related but weaker condition than being just. In fact, it can easily be shown that R(n) is a just rule.

In many cases it is possible to prove that $\psi(\lambda,n)$ is monotone (increasing say) in λ. If this is the case and there exists a smallest $\lambda_0 \in \Lambda'$ then the k dimensional infimum will be completely evaluated as

$$\inf_{\Omega(p)} P_{\underset{\sim}{\lambda}}[CS|R(n)] = \psi(\lambda_0,n).$$

The following two lemmas will be used in the proof of such a result. The first lemma is due to Panchapakesan [45].

Lemma 1.2.2. Let $F(\cdot|\lambda)|\lambda\epsilon\Lambda\}$ be a family of absolutely continuous distributions on the real line with continuous densities $f(\cdot|\lambda)$ and $\phi(x,\lambda)$ a bounded real valued function possessing first partial derivatives ϕ_x and ϕ_λ wrt x and λ respectively and satisfying regularity conditions (1.2.5). Then $E_\lambda[\phi(x,\lambda)]$ is non decreasing in λ provided for all $\lambda\epsilon\Lambda$

$$(1.2.4) \qquad f(x/\lambda)\,\frac{\partial\phi(x,\lambda)}{\partial\lambda} - \frac{\partial F(x/\lambda)}{\partial\lambda}\,\frac{\partial\phi(x,\lambda)}{\partial x} \geq 0 \quad \text{for a.e.x.}$$

Further if (1.2.4) is strictly positive on a set of positive Lesbesgue measure than $E_\lambda[\phi(X,\lambda)]$ is strictly increasing in λ.

One possible set of regularity conditions is:

(1.2.5) (i) For all $\lambda \varepsilon \Lambda$, $\frac{\partial \phi(X,\lambda)}{\partial x}$ is Lesbesgue integrable on R.

(ii) For every $[\lambda_1,\lambda_2] \subset \Lambda$ and $\lambda_3 \varepsilon \Lambda$

$$\left| \frac{\partial \phi(x,\lambda)}{\partial \lambda} f(x/\lambda_3) - \frac{\partial F(x/\lambda)}{\partial \lambda} \frac{\partial \phi(x,\lambda_3)}{\partial x} \right| \leq h(x) \quad \lambda \varepsilon [\lambda_1,\lambda_2]$$

where $h(x)$ depends on λ_i, $i=1,2,3$ and is Lesbesgue integrable on R.

(1.2.5) is needed to justify several of the technical details in the proof of Lemma 1.2.2.

<u>Lemma 1.2.3</u>. For any $1 \leq \ell < n$ and $0 \leq a < c \leq 1$

$$ncI(a/c;\ell,n-\ell+1) \geq ab(a/c) \text{ where } b(y) = I'(y;\ell,n-\ell+1).$$

<u>Proof</u>.
$$ncI(a/c;\ell,n-\ell+1) = nc \sum_{i=\ell}^{n} \binom{n}{i} (a/c)^i (1-a/c)^{n-i}$$
$$\geq nc\binom{n}{\ell} (a/c)^\ell (1-a/c)^{n-\ell}$$
$$= an\binom{n}{\ell} (a/c)^{\ell-1} (1-a/c)^{n-\ell} \geq a \cdot b(a/c)$$
which proves the lemma.

<u>Remark 1.2.2</u>. The following assumptions are essentially needed to assure that (1.2.5) holds. For any $[\lambda_1,\lambda_2] \subset \Lambda'$ and any $\lambda_3 \varepsilon \Lambda'$ there exist $e_1(y)$ and $e_2(y)$ such that

$$(1.2.7)\quad \begin{cases} \text{(i)} \quad \left|\dfrac{\partial G_n(y/p(\lambda))}{\partial\lambda}\right| \leq e_1(y)\ \forall\ \lambda\varepsilon[\lambda_1,\lambda_2] \text{ where} \\[2ex] \qquad (\int e_1(y)dG_n(y/\lambda_3))\ (\int e_1(h_n(y))dG_n(y/\lambda_3)) < \infty \\[2ex] \text{(ii)} \quad \left|\dfrac{\partial G_n(y/\lambda)}{\partial\lambda}\right| \leq e_2(y)\ \forall\ \lambda\varepsilon[\lambda_1,\lambda_2] \text{ where} \end{cases}$$

$$(\int e_2(y)dG_n(h_n(y)/\lambda_3))(\int e_2(y)dG_n(y/\lambda_3)) < \infty$$

<u>Theorem 1.2.3</u>. If $E_n^{\lambda} = E_n\ \forall\lambda\varepsilon\Lambda'$, $G_n(y/\lambda)$ is continuously differentiable and satisfies (1.2.7) and all derivatives in (1.2.8) and (1.2.9) exist and satisfy $\forall\lambda\varepsilon\Lambda'$

$$(1.2.8)\quad g_n(y/\lambda)\ \frac{\partial G_n(h_n(y)/p(\lambda))}{\partial\lambda} - h_n'(y)g_n(h_n(y)/p(\lambda))\ \frac{\partial G_n(y/\lambda)}{\partial\lambda} \geq 0$$

ae in E_n

$$(1.2.9)\quad g_n(y/\lambda)\ \frac{\partial G_n(y/p(\lambda))}{\partial\lambda} - g_n(y/p(\lambda))\ \frac{\partial G_n(y/\lambda)}{\partial\lambda} \geq 0 \text{ ae in } E_n$$

then $\psi(\lambda,n)$ is nondecreasing in λ.

<u>Proof</u>. As indicated, the proof is an application of Lemma 1.2.2. Note that

$$\psi(\lambda,n) = \int_{E_n} \phi(y,\lambda)dG_n(y/\lambda) \text{ for the choice}$$

$$\phi(y,\lambda) = \{G_n(h_n(y)/p(\lambda))\}^{k-1}\ I(\frac{G_n(y/p(\lambda))}{G_n(h_n(y)/p(\lambda))};k-m,m). \quad \text{Hence}$$

$$\frac{\partial\phi(y,\lambda)}{\partial y} = (k-1)\{G_n(h_n(y)/p(\lambda))\}^{k-2}g_n(h_n(y)/p(\lambda))h_n'(y)$$
$$I(K_n(y,\lambda);k-m,m)$$
$$+ \{G_n(h_n(y)/p(\lambda))\}^{k-3}b(K_n(y,\lambda))\{G_n(h_n(y)/p(\lambda))g_n(y/p(\lambda))$$
$$- G_n(y/p(\lambda))h_n'(y)\cdot g_n(h_n(y)/p(\lambda))\}$$

$$\frac{\partial\phi(y,\lambda)}{\partial\lambda} = (k-1)\{G_n(h_n(y)/p(\lambda))\}^{k-2}\frac{\partial G_n(h_n(y)/p(\lambda))}{\partial\lambda} I(K_n(y,\lambda);k-m,m)+$$

$$\{G_n(h_n(y)/p(\lambda))\}^{k-3}b(K_n(y,\lambda)).\{G_n(h_n(y)/p(\lambda))\frac{\partial G_n(y/p(\lambda))}{\partial\lambda} -$$

$$G_n(y/p(\lambda))\frac{\partial G_n(h_n(y)/p(\lambda))}{\partial\lambda}\}$$

where $K_n(y,\lambda) = \frac{G_n(y/p(\lambda))}{G_n(h_n(y)/p(\lambda))}$

So (1.2.4) becomes $\forall\lambda\varepsilon\Lambda'$

$$(1.2.10) \quad g_n(y/\lambda)[(k-1)\frac{G_n(h_n(y)/p(\lambda))}{\partial\lambda}\{G_n(h_n(y)/p(\lambda))\}$$

$$I(K_n(y,\lambda);k-m,m) + b(K_n(y,\lambda))$$

$$\{G_n(h_n(y)/p(\lambda))\frac{\partial G_n(y/p(\lambda))}{\partial\lambda} - G_n(y/p(\lambda))\frac{\partial G_n(h_n(y)/p(\lambda))}{\partial\lambda}\}]$$

$$- \frac{\partial G_n(y/\lambda)}{\partial\lambda}[(k-1)G_n(h_n(y)/p(\lambda))g_n(h_n(y)/p(\lambda))h_n'(y)$$

$$I(K_n(y,\lambda);k-m,m) + b(K_n(y,\lambda))$$

$$\{G_n(h_n(y)/p(\lambda))g_n(y/p(\lambda))-h_n'(y)G_n(y/p(\lambda))g_n(h_n(y)/p(\lambda))\}]$$

$$\geq 0 \text{ ae in } E_n$$

Combining terms (1.2.10) can be seen to hold if $\forall\lambda\varepsilon\Lambda'$

$$(1.2.11) \quad \{g_n(y/\lambda)\frac{\partial G_n(y/p(\lambda))}{\partial\lambda} - \frac{\partial G_n(y/\lambda)}{\partial\lambda} g_n(y/p(\lambda))\} \geq 0 \text{ ae in } E_n$$

and

$$(1.2.12) \quad \{g_n(y/\lambda)\frac{\partial G_n(h_n(y)/p(\lambda))}{\partial\lambda} - h_n'(y)g_n(h_n(y)/p(\lambda))\frac{\partial G_n(y/\lambda)}{\partial\lambda}\} \times$$

$$\{(k-1)I(K_n(y,\lambda);k-m,m)G_n(h_n(y)/p(\lambda))-b(K_n(y,\lambda))G_n(y)p(\lambda))\}$$

$$\geq 0 \text{ ae in } E_n$$

But by Lemma 1.2.3 the second factor in (1.2.12) is non negative since $\forall y \varepsilon E^n$, $\lambda \varepsilon \Lambda' \Rightarrow 0 \leq G_n(y/p(\lambda)) \leq G_n(h_n(y)/p(\lambda)) \leq 1$. Hence (1.2.12) and (1.2.11) reduce to (1.2.8) and (1.2.9). Similar arguments show that (1.2.7) imply the regularity conditions needed for Lemma 1.2.2 and hence completes the proof.

Remark 1.2.3. If (1.2.8) and (1.2.9) are identically zero then $\psi(\lambda,n)$ is independent of λ.

Special Cases

(a) $m=1$

$\Rightarrow \phi(y,\lambda) = \{G_n(y/p(\lambda))\}^{k-1}$ and a short calculation shows (1.2.4) is satisfied if (1.2.9) alone holds

(b) $m=k$

$\Rightarrow \phi(y,\lambda) = \{G_n(h_n(y)/p(\lambda))\}^{k-1}$ and again some calculation shows (1.2.4) holds if (1.2.8) alone holds. Note that any P^* level can be attained even when there is no indifference zone and in this case (1.2.8) reduces to the result of Theorem 2.2 of Gupta-Panchapakesan [33].

Example 1.2.1. Location Parameter Families

The following are assumed to hold

(1.2.13)

(1) The cdf of T_{in} is $G_n(y/\lambda_i) = G_n(y-\lambda_i)$ with support $E_n^{\lambda}=R^1$

(2) $\Lambda = (-\infty,\infty)$ and $p(\lambda) = \lambda-\delta$, $(\delta > 0)$

(3) $\sup_{y \varepsilon R^1} g_n(y) < \infty$

(4) For every n, $h_n'(y) \geq 1$ ae

Then (1.2.8) becomes

$$g_n(y/\lambda)[-g_n(h_n(y)/p(\lambda))] - h_n'(y)g_n(h_n(y)/p(\lambda))(-1)g_n(y/\lambda)$$

$\geq (1)g_n(h_n(y)/p(\lambda))g_n(y/\lambda)-g_n(h_n(y)/p(\lambda))g_n(y/\lambda)=0$ ae by (4) of (1.2.13) and (1.2.9) becomes

$$g_n(y/\lambda)[-g_n(y/p(\lambda))] - g_n(y)p(\lambda))[-g_n(y/\lambda)] = 0 \quad \forall y$$

Hence $\psi(\lambda,n)$ is non decreasing in λ

For the usual choice $h_n(x) = x+d_n$ where $d_n \downarrow 0$ both (1.2.8) and (1.2.9) are identically zero and hence $\psi(\lambda,n)$ is independent of λ i.e.

$$\inf_{\lambda\varepsilon\Lambda'} \psi(\lambda,n) = \psi(0,n)$$

<u>Example 1.2.2.</u> Scale Parameter Families

It is assumed the following hold

$$(1.2.14)\begin{cases} (1) & T_{in} \text{ has cdf } G_n(y/\lambda) = G_n(\frac{y}{\lambda}) \text{ with support } E_n^\lambda = [0,\infty) \\ (2) & \sup_{y\geq 0} g_n(y) < \infty \text{ and } \int_0^\infty y dG_n(y) < \infty \\ (3) & \Lambda = (0,\infty) \text{ and } p(\lambda) = \delta\lambda \ (0 < \delta < 1) \\ (4) & yh'_n(y) \geq h_n(y) \geq 0 \quad \text{ae on } [0,\infty) \end{cases}$$

Then (1.2.8) becomes

$$\frac{yh_n'(y)}{\lambda} g_n(h_n(y)/\delta\lambda)g_n(y/\lambda) - \frac{h_n(y)}{\lambda} g_n(h_n(y)/\delta\lambda)g_n(y/\lambda)$$

$= \dfrac{g_n(h_n(y)/\delta\lambda)g_n(y/\delta\lambda)}{\lambda}[yh_n'(y)-h_n(y)] \geq 0 \forall \lambda>0$ and ae by (4) of (1.2.14) and (1.2.9) becomes

$$\frac{y}{\lambda} g_n(y/\delta\lambda)g_n(y/\lambda) - \frac{y}{\lambda} g_n(y/\delta\lambda)g_n(y/\lambda) = 0 \forall y \geq 0 \text{ and } \lambda > 0.$$

Hence $\psi(\lambda,n)$ is non decreasing in λ.

For the usual choice $h_n(x) = d_n x$ where $d_n \downarrow 1$ both (1.2.8) and (1.2.9) are identically zero and hence $\psi(\lambda,n)$ is independent of Λ i.e.

$$\inf_{\lambda \varepsilon \Lambda'} \psi(\lambda,n) = \psi(1,n) .$$

1.3 Properties of {R(n)}

This section will study both the properties of the sequence {R(n)} and the individual rules R(n). For $\underset{\sim}{\lambda} \varepsilon \Omega$ define

$$p^n_{\underset{\sim}{\lambda}}(i) = P_{\underset{\sim}{\lambda}}[R(n) \text{ selects } \pi_{(i)}] \tag{1.3.1}$$

and recall the following two definitions.

Def. 1.3.1. R(n) is a monotone procedure means for every $\underset{\sim}{\lambda} \varepsilon \Omega$ and $1 \leq i < j \leq k$

$$p^n_{\underset{\sim}{\lambda}}(i) \leq p^n_{\underset{\sim}{\lambda}}(j) .$$

Def. 1.3.2. R(n) is an unbiased procedure means for every $\underset{\sim}{\lambda} \varepsilon \Omega$ and $1 \leq j < k$

$$P_{\underset{\sim}{\lambda}}[R(n) \text{ does not select } \pi_{(j)}] \geq P_{\underset{\sim}{\lambda}}[R(n) \text{ does not select } \pi_{(k)}].$$

Of course if R(n) is monotone it is also unbiased. Other optimal properties that will be studied are:

Def. 1.3.3. The sequence of rules {R(n)} is consistent wrt Ω' means

$$\inf_{\Omega'} P[CS|R(n)] \to 1 \text{ as } n \to \infty .$$

Def. 1.3.4. The rule R(n) is strongly monotone in $\pi_{(i)}$ means

$$p^n_{\underset{\sim}{\lambda}}(i) \text{ is } \begin{cases} \uparrow \text{ in } \lambda_{[i]} \text{ when all other components of } \underset{\sim}{\lambda} \text{ are fixed} \\ \downarrow \text{ in } \lambda_{[j]}\ (j \neq i) \text{ when all other components of } \underset{\sim}{\lambda} \text{ are fixed.} \end{cases}$$

<u>Remark 1.3.1</u>. If a rule R(n) is strongly monotone in $\pi_{(k)}$, then

$$\inf_{\Omega(p)} P_{\underset{\sim}{\lambda}}[CS|R(n)] = \inf_{\Omega^{o}(p)} P_{\underset{\sim}{\lambda}}[CS|R(n)].$$

<u>Theorem 1.3.1</u>. If there exists a $\lambda_0 \varepsilon \Lambda'$ and $N \geq 1$ such that $\forall n \geq N$, $\inf_{\lambda \varepsilon \Lambda'} \psi(\lambda,n) = \psi(\lambda_0,n)$, then any sequence $\{R(n)\}$ defined by (1.1.4) is consistent wrt $\Omega(p)$.

<u>Proof</u>. From the hypothesis of the theorem and the result of Theorem 1.2.2 there exists N such that for all $n \geq N$

$$\Rightarrow \inf_{\Omega(p)} P_{\underset{\sim}{\lambda}}[CS|R(n)] = \int_{-\infty}^{\infty} \{G_n(h_n(y)/p(\lambda_0))\}^{k-1} I\left(\frac{G_n(y/p(\lambda_0))}{G_n(h_n(y)/p(\lambda_0))};k-m,m\right) dG_n(y/\lambda_0)$$

Also $T_{in} \overset{p}{\to} \lambda_i \Rightarrow P[T_{in} \leq y] = G_n(y/\lambda_i) \to \begin{cases} 0, & y < \lambda_i \\ 1, & y > \lambda_i \end{cases}$.

Since $\inf_{\Omega(p)} P_{\underset{\sim}{\lambda}}[CS|R(n)] \leq 1$ we need only show that given $\epsilon > 0$ $\exists N \ni \forall n \geq N \Rightarrow \int_{-\infty}^{\infty} V(y,\lambda_0)dG_n(y/\lambda_0) \geq 1 - \epsilon$ where

$$V(y,\lambda_0) = \{G_n(h_n(y)/p(\lambda_0))\}^{k-1} I\left(\frac{G_n(y/p(\lambda_0))}{G_n(h_n(y)/p(\lambda_0))};k-m,m\right)$$

Since $p(\lambda_0) < \lambda_0$ $\exists\alpha \ni p(\lambda_0) < \alpha < \lambda_0 \Rightarrow$ given $\epsilon > 0$ $\exists M$ ($\geq N$) such that $\forall n \geq M$

(a) $G_n(\alpha/\lambda_0) < \epsilon$ (since $\alpha < \lambda_0$)

(b) $\{G_n(h_n(\alpha)/p(\lambda_0))\}^{k-1} I(G_n(\alpha)p(\lambda_0));k-m,m) > 1 - \epsilon$

(since $h_n(\alpha) > \alpha > p(\lambda_0)$)

Now take $A = [\alpha,\infty)$

$\Rightarrow$ $y \varepsilon A$

(a) $1 \geq G_n(h_n(y)/p(\lambda_0)) \geq G_n(h_n(\alpha)/p(\lambda_0))$

(b) $G_n(y/p(\lambda_0)) \geq G_n(\alpha/p(\lambda_0))$

$\Rightarrow \forall y \varepsilon A$

$$V(y,\lambda_0) \geq \{G_n(h_n(\alpha)/p(\lambda_0))\}^{k-1} I(\frac{G_n(\alpha/p(\lambda_0))}{G_n(h_n(y)/p(\lambda_0))};k-m,m)$$

$$\geq \{G_n(h_n(\alpha)/p(\lambda_0))\}^{k-1} I(G_n(\alpha/p(\lambda_0));k-m,m) \geq 1-\epsilon$$

So $\forall n \geq M$

$$\Rightarrow \int_{-\infty}^{\infty} V(y,\lambda_0)dG_n(y/\lambda_0) \geq \int_{\alpha}^{\infty} V(y,\lambda_0)dG_n(y/\lambda_0)$$

$$\geq (1-\epsilon)\int_{\alpha}^{\infty} dG_n(y/\lambda_0) \geq (1-\epsilon)^2$$ and the proof is completed.

Remark 1.3.2. Theorem 1.3.1 shows that any P* requirement (1.1.5) can be met by choosing a sufficiently large common sample size n. In particular if the conditions of Theorem 1.2.3 hold and there exists a least element $\lambda_0 \varepsilon \Lambda' \Rightarrow \inf_{\lambda \varepsilon \Lambda'} \psi(\lambda,n) = \psi(\lambda_0,n)$ for all n and hence Theorem 1.3.1 applies.

Example 1.3.1. The Location Parameter Case

Assume $\pi_1,\ldots,\pi_k$ are populations with cdf's $F(y/\lambda_i)=F(y-\lambda_i)$ and also that

(a) F is known, has bounded density and $\int_{-\infty}^{\infty} |y|dF(y) < \infty$

(b) $\Lambda = R'$ and $p(\lambda) = \lambda-\delta$, ($\delta>0$ is specified by the experimenter)

(c) For all n, $h_n'(x) = 1$ ae

Let $T_{in} = T_n(X_{i1},\ldots,X_{in}) = \frac{1}{n}\sum_{j=1}^{n} X_{ij}$ and then it can be easily seen

(a) $T_{in} \overset{p}{\to} \lambda_i + \mu$ where $\mu = \int_{-\infty}^{\infty} ydF(y)$

(b) $G_n(y/\lambda_i) = G_n(y-\lambda_i)$ is again a location parameter family.

From Example 1.2.1 it follows that $\inf_{\lambda\varepsilon\Lambda'} \psi(\lambda,n) = \psi(0,n)\ \forall n$.

So finally an application of Theorem 1.3.1 gives the result

$\inf_{\Omega(p)} P_{\underset{\sim}{\lambda}}[CS|R(n)] \to 1$ as $n \to \infty$.

Example 1.3.2. The Scale Parameter Case

Analogous to the above situation let π_i have cdf $F(y/\lambda_i) = F(\frac{y}{\lambda_i})$ on $(0,\infty)$ and suppose that

(a) F is known with bounded density and $\int_0^{\infty} ydF(y) < \infty$

(b) For all n, $h_n(y)$ is defined on $[0,\infty)$ and satisfies

$yh_n'(y) = h_n(y) \geq 0$ ae

(c) $\Lambda = (0,\infty)$ and $p(\lambda) = \delta\lambda$, $(0 < \delta < 1)$

Again letting $T_{in} = \frac{1}{n}\sum_{j=1}^{n} X_{ij}$ it can be shown

(a) $T_{in} \overset{p}{\to} \lambda_i\mu$ where $\mu = \int_0^{\infty} ydF(y)$

(b) For every n, $G_n(y/\lambda_i) = G_n(\frac{y}{\lambda_i})$

So from Example 1.2.2 it is seen that $\inf_{\lambda\varepsilon\Lambda'} \psi(\lambda,n) = \psi(1,n)$ for all n and hence $\inf_{\Omega(p)} P_{\underset{\sim}{\lambda}}[CS|R(n)] \to 1$ as $n \to \infty$.

Some standard properties of the individual rules R(n) are now studied.

Theorem 1.3.2. For any $i = 1,\ldots,k$ and every rule $R(n)$ of form (1.1.4), $R(n)$ is strongly monotone in $\pi_{(i)}$.

Proof. Since $p^n_{\underset{\sim}{\lambda}}(i) = E_{\underset{\sim}{\lambda}}[\eta_i(\underset{\sim}{T})]$ where

$$\eta_i(\underset{\sim}{T}) = \begin{cases} 1 & , h_n(T_{(i)} \geq \max\{h_n(T_{[k-m+1]}), T_{[k]}\} \\ 0 & , \text{otherwise} \end{cases}$$

the result of Lemma 1.2.1 can again be used to show the desired monotonicity. It suffices to show

(A) $\eta_i(\underset{\sim}{T})$ is non increasing in $T_{(j)}$ $(j \neq i)$ when all other components are fixed.

(B) $\eta_i(\underset{\sim}{T})$ is non decreasing in $T_{(i)}$ when all other components are fixed.

Part (A) has essentially the same proof as Theorem 1.2.2.

To prove part (B) it is only required to show $\eta_i(T) = 1$ implies $\eta_i(\underset{\sim}{T}') = 1$ for any $\underset{\sim}{T} = (T_1,\ldots,T_k)$ and $\underset{\sim}{T}' = (T_1',\ldots,T_k')$ satisfying $T'_{(i)} > T_{(i)}$ and $T'_{(j)} = T_{(j)}$ $j \neq i$.

$$\begin{aligned} &\eta(\underset{\sim}{T}) = 1 \\ \Leftrightarrow\ & h_n(T_{(i)}) \geq \max\{h_n(T_{[k-m+1]}), T_{[k]}\} \\ \Leftrightarrow\ & \begin{cases} \text{(a)} & T_{(i)} \geq T_{[k-m+1]} \\ & \text{and} \\ \text{(b)} & h_n(T_{(i)}) \geq T_{[k]} \end{cases} \end{aligned}$$

Now (a) $\Rightarrow$ either

$$\begin{cases} (1) & T_{(i)} = T_{[k-m+1]} \Rightarrow T'_{[k-m+1]} = \min\{T'_{(i)}, T_{[k-m+2]}\} \leq T'_{(i)} \\ & \text{or} \\ (2) & T_{(i)} > T_{[k-m+1]} \Rightarrow T'_{[k-m+1]} = T_{[k-m+1]} \leq T_{(i)} < T'_{(i)}. \end{cases}$$

So (a) $\Rightarrow T'_{(i)} \geq T'_{[k-m+1]}$

Similarly in case (b) one of two possibilities occur

$$\begin{cases} (1) & T'_{(i)} \leq T_{[k]} \Rightarrow T'_{[k]} = T_{[k]} \leq h_n(T_{(i)}) \leq h_n(T'_{(i)}) \\ & \text{or} \\ (2) & T'_{(i)} > T_{[k]} \Rightarrow T'_{[k]} = T'_{(i)} \leq h_n(T'_{(i)}) \end{cases}$$

Again (b) $\Rightarrow h_n(T'_{(i)}) \geq T'_{[k]}$. Finally we get $h_n(T'_{(i)}) \geq \max\{T'_{[k]}, h_n(T'_{[k-m+1]})\}$ which implies $\eta(\underset{\sim}{T}') = 1$ and ends the proof.

Corollary 1.3.1. All rules of form (1.1.4) are monotone and hence unbiased.

Proof. It suffices to show that $p_{\underset{\sim}{\lambda}}(i) \leq p_{\underset{\sim}{\lambda}}(i+1)$ for any $\underset{\sim}{\lambda} \varepsilon \Omega$ and $i = 1,\ldots,k-1$. Assume wlog that $\lambda_\ell = \lambda_{[\ell]}$ for notational ease then

$$p^n_{\underset{\sim}{\lambda}}(i) = p^n_{(\lambda_1,\ldots,\lambda_k)}(i)$$

$\leq p^n_{(\lambda_1,\ldots,\lambda_{i-1},\lambda_{i+1},\lambda_{i+1},\lambda_{i+2},\ldots,\lambda_k)}(i)$ since $p^n(i)$ is $\uparrow$ in $\lambda_{[i]}$

$= p^n_{(\lambda_1,\ldots,\lambda_{i-1},\lambda_{i+1},\lambda_{i+1},\lambda_{i+2},\ldots,\lambda_k)}(i+1)$ since both $\pi_{(i)}$ and $\pi_{(i+1)}$ have the same cdf.

$\leq p^n_{(\lambda_1,\ldots,\lambda_{i-1},\lambda_i,\lambda_{i+1},\ldots,\lambda_k)}(i+1)$ since $p^n_{\underset{\sim}{F}}(i+1) \downarrow$ in $\lambda_{[i]}$

$= p^n_{\underset{\sim}{\lambda}}(i+1)$. This completes the proof of the corollary.

1.4 Expected Number of Selected Populations

If $S(n)$ is the number of populations selected by $R(n)$, $T(n)$ is the number of non best populations selected by $R(n)$ and $p^n_{\underset{\sim}{\lambda}}(i)$ is defined as before by (1.3.1) and if

$$W_i(n) = \begin{cases} 0 & , \quad T_{(i)} \geq \max\{T_{[k-m+1]}, h_n^{-1}(T_{[k]})\} \\ 1 & , \quad \text{otherwise} \end{cases}$$

then the following representations hold

(1.4.1) $$S(n) = \sum_{i=1}^{k} W_i(n)$$

(1.4.2) $$T(n) = \sum_{i=1}^{k-1} W_i(n)$$

(1.4.3) $$p^n_{\underset{\sim}{\lambda}}(i) = E_{\underset{\sim}{\lambda}}[W_i(n)].$$

Theorem 1.4.1. For any $\underset{\sim}{F} \varepsilon \Omega(\mathscr{F})$

$$E_{\underset{\sim}{F}}[S(n)] = \sum_{i=1}^{k} \sum_{p=k-m}^{k-1} \sum_{\nu=1}^{\binom{k-1}{p}} \int_{-\infty}^{\infty} \prod_{j \varepsilon \mathscr{P}_\nu^p(i)} G_n^{(j)}(y) \prod_{j \varepsilon \bar{\mathscr{P}}_\nu^p(i)} \{G_n^{(j)}(h_n(y)) - G_n^{(j)}(y)\}\, dG_n^{(i)}(y)$$

where

$$\begin{cases} G_n^{(i)}(y) = G_n(y/\lambda_{[i]}) \\ \{\mathscr{P}_\nu^p(i) \mid \nu = 1,\ldots,\binom{k-1}{p}\} \text{ is the collection of all subsets of size } p \text{ from } \mathcal{U}(i) = \{1,\ldots,k\} - \{i\} \\ \bar{\mathscr{P}}_\nu^p(i) = \mathcal{U}_{(i)} - \mathscr{P}_\nu^p(i) \end{cases}$$

Proof. From the representation (1.4.1) it can be seen that it suffices to calculate $p^n_{\underset{\sim}{\lambda}}(i)$ for $i = 1,\ldots,k$.

$$p^n_{\underset{\sim}{\lambda}}(i) = P_{\underset{\sim}{\lambda}}[T_{(i)} \geq T_{[k-m+1]},\ h_n(T_{(i)}) \geq T_{(j)} \quad j \neq i]$$

$$= P_{\underset{\sim}{\lambda}}[h_n(T_{(i)}) \geq T_{(j)}\ j \neq i, T_{(i)} > \text{at least } (k-m)\ T_{(\ell)}\text{'s w/}\ell \neq i]$$

Analogous to the argument of Theorem 1.2.1 let

$$B^p_\nu = [T_{(i)} > T_{(j)}\ \forall j \varepsilon \mathscr{P}_\nu(i),\ T_{(i)} < T_{(j)}\ \forall j \varepsilon \bar{\mathscr{P}}_\nu(i)]$$

$$\Rightarrow p^n_{\underset{\sim}{\lambda}}(i) = \sum_{p=k-m}^{k-1} \sum_{\nu=1}^{\binom{k-1}{p}} P_{\underset{\sim}{\lambda}}[h_n(T_{(i)}) \geq T_{(j)} \quad j \neq i,\ B^p_\nu]$$

$$= \sum_{p=k-m}^{k-1} \sum_{\nu=1}^{\binom{k-1}{p}} P_{\underset{\sim}{\lambda}}[T_{(i)} > T_{(j)} \forall j \varepsilon \mathscr{P}^p_\nu(i), T_{(i)} < T_{(j)} < h_n(T_{(i)}) \forall j \varepsilon \bar{\mathscr{P}}^p_\nu(i)]$$

$$= \sum_{p=k-m}^{k-1} \sum_{\nu=1}^{\binom{k-1}{p}} \int_{-\infty}^{\infty} \prod_{j \varepsilon \mathscr{P}_\nu(i)} G^{(j)}_n(y) \prod_{j \varepsilon \bar{\mathscr{P}}_\nu(i)} \{G^{(j)}_n(h_n(y)) - G^{(j)}_n(y)\}\ dG^{(i)}_n(y) \text{ and}$$

completes the argument.

Remark 1.4.1. Again note that the expected number of selected populations depends on $\underset{\sim}{F}$ only through $\underset{\sim}{\lambda}(\underset{\sim}{F})$ and hence the notation $E_{\underset{\sim}{\lambda}}[S(n)]$ will be used rather than $E_{\underset{\sim}{F}}[S(n)]$. The expected value of $T(n)$ can be easily derived in a manner similar to the above using (1.4.2).

Example 1.4.1. Slippage Configuration

Suppose $\underset{\sim}{\lambda} = (\lambda_1, \ldots, \lambda_k)$ has the form $\lambda_{[1]} = \lambda_{[k-1]} = p(\lambda_{[k]}) = \lambda_0$ and $\lambda_{[k]} = \lambda^*$, then

$$E_{\underset{\sim}{\lambda}}[S(n)] = \sum_{p=k-m}^{k-1} \binom{k-1}{p} \int_{-\infty}^{\infty} \{G_n(y/\lambda_0)\}^p \{G_n(h_n(y)/\lambda_0) - G_n(y/\lambda_0)\}^{k-1-p}\ dG_n(y/\lambda^*)$$

$$+(k-1)\sum_{p=k-m}^{k-1}\binom{k-2}{p-1}\int_{-\infty}^{\infty}\{G_n(y/\lambda_0)\}^{p-1}\{G_n(y/\lambda^*)\}\{G_n(h_n(y)/\lambda_0)-$$

$$G_n(y/\lambda_0)\}^{k-1-p}\,dG_n(y/\lambda_0)$$

$$+(k-1)\sum_{p=k-m}^{k-2}\binom{k-2}{p}\int_{-\infty}^{\infty}\{G_n(y/\lambda_0)\}^{p}\{G_n(h_n(y)/\lambda_0)-G_n(y/\lambda_0)\}^{k-2-p}$$

$$\cdot\{G_n(h_n(y)/\lambda^*)-G_n(y/\lambda^*)\}dG_n(y/\lambda_0)$$

In the remainder of the section three topics will be studied:

(a) Asymptotic properties of the sequence $\{S(n)\}$

(b) The supremum of $E_{\underset{\sim}{\lambda}}[S(n)]$ over Ω for fixed n

(c) Both (a) and (b) for $T(n)$.

The results for (a) begin with the following

<u>Theorem 1.4.2</u>. For any $\underset{\sim}{\lambda}\varepsilon\Lambda$ such that $\lambda_{[k]} > \lambda_{[k-1]}$

$$(1.4.4)\qquad p^n_{\underset{\sim}{\lambda}}(i) \to \begin{cases} 1 & , \; i = k \\ 0 & , \; 1 \le i < k \end{cases} \quad \text{as } n \to \infty.$$

<u>Proof</u>. Since $T_{(i)n} \overset{P}{\to} \lambda_{[i]}$ and $G^{(i)}_n(y) = P[T_{(i)n} \le y]$ it follows

$$(1.4.5)\quad \text{that}\quad G^{(i)}_n(y) \to \begin{cases} 1 & , \; y > \lambda_{[i]} \\ 0 & , \; y < \lambda_{[i]}. \end{cases}$$

Also $\lambda_{[k-1]} < \lambda_{[k]} \Rightarrow \exists\alpha\ni\ \lambda_{[k-1]} < \alpha < \lambda_{[k]}$. Let $I(\alpha) = [\alpha,\infty)$

<u>Case A: i = k</u>

$$p^n_{\underset{\sim}{\lambda}}(k) = \sum_{p=k-m}^{k-1}\ \sum_{\nu=1}^{\binom{k-1}{p}}\int_{-\infty}^{\infty} f^{p,\nu}_n(y)\,dG^{(k)}_n(y)$$

where $f^{p,\nu}_n(y) = \underset{j\varepsilon\mathscr{P}^p_\nu(k)}{\pi} G^{(j)}(y) \underset{j\varepsilon\bar{\mathscr{P}}^p_\nu(k)}{\pi}\{G^{(j)}_n(h_n(y))-G^{(j)}_n(y)\}$

Subcase (1): For $k-m \leq p \leq k-2$ and $1 \leq \nu \leq \binom{k-1}{p}$

$$\Rightarrow \bar{\mathcal{J}}^{p}_{\nu}(k) \neq \phi$$

$$\Rightarrow f_n^{p,\nu}(y) \leq \prod_{j \in \bar{\mathcal{J}}^{p}_{\nu}(k)} \{G_n^{(j)}(h_n(y))-G_n^{(j)}(y)\} \leq \prod_{j \in \bar{\mathcal{J}}^{p}_{\nu}(k)} \{1-G_n^{(j)}(y)\}$$

Now given $\epsilon > 0$ pick $N \ni \forall\ n \geq N$

$$\begin{cases} G_n^{(k)}(\alpha) < \epsilon/2 \\ \text{and} \\ \prod_{j \in \bar{\mathcal{J}}^{p}_{\nu}(k)} \{1-G_n^{(j)}(\alpha)\} < \epsilon/2 \end{cases}$$

which is possible since $\lambda_{[k-1]} < \alpha < \lambda_{[k]}$ and (1.4.5) holds.

So $\forall_n \geq N$

$$0 \leq \int_{-\infty}^{\infty} f_n^{p,\nu}(y)dG_n^{(k)}(y) = \int_{-\infty}^{\alpha} f_n^{p,\nu}(y)dG_n^{(k)}(y) + \int_{\alpha}^{\infty} f_n^{p,\nu}(y)dG_n^{(k)}(y)$$

$$\leq \int_{-\infty}^{\alpha} 1G_n^{(k)}(y) + \int_{\alpha}^{\infty} \epsilon/2\ dG_n^{(k)}(y)$$

(since $\forall y \in I(\alpha)$, $f_n^{p,\nu}(y) \leq \prod_{j \in \bar{\mathcal{J}}^{p}_{\nu}(k)} \{1-G_n^{(j)}(y)\} \leq \prod_{j \in \bar{\mathcal{J}}^{p}_{\nu}(k)} \{1-G_n^{(j)}(\alpha)\}$

$\leq \epsilon/2)$

$$\leq G_n^{(k)}(\alpha) + \epsilon/2 \cdot 1 \leq \epsilon$$

i.e. $\int_{-\infty}^{\infty} f_n^{p,\nu}(y)dG_n^{(k)}(y) \to 0$ as $n \to \infty$ for any such p and ν.

Subcase (2): $p=k-1$ and $\nu=1$

Since $\int f_n^{k-1,1}(y)dG_n^{(k)}(y) \leq 1$ it suffices to prove $\forall \epsilon > 0\ \exists N \ni \forall n \geq N$ $\Rightarrow f_n^{k-1,1}(y)dG_n^{(k)}(y) \geq 1-\epsilon$ in order to show $\int f_n^{k-1,1}(y)dG_n^{(k)}(y) \to 1$ as $n \to \infty$. Now $p=k-1$ implies

$$\begin{cases} (1) \quad \mathcal{S}_1^{k-1}(k) = \{1,\ldots,k-1\} \text{ and } \mathcal{S}_1^{k-1}(k) = \phi \\ \quad\quad \text{and} \\ (2) \quad \forall y \varepsilon I(\alpha),\ f_n^{k-1,1}(y) = \prod_{j=1}^{k-1} G_n^{(j)}(y) \geq \prod_{j=1}^{k-1} G_n^{(j)}(\alpha) \end{cases}$$

Given $1 > \epsilon > 0$ let $\epsilon' = 1 - \sqrt{1-\epsilon}$ and choose N such that $\forall n \geq N$

$$\begin{cases} G_n^{(k)}(\alpha) < \epsilon' \\ \text{and} \\ \prod_{j=1}^{k-1} G_n^{(j)}(\alpha) \geq 1-\epsilon' \end{cases}$$

which is possible since $\forall j \leq k-1 \Rightarrow \lambda_{[j]} < \alpha < \lambda_{[k]}$. So $\forall n \geq N$

$$\begin{aligned} \int f_n^{k-1,1}(y)dG_n^{(k)}(y) &\geq \int_\alpha^\infty f_n^{k-1,1}(y)dG_n^{(k)}(y) \\ &\geq \prod_{j=1}^{k-1} G_n^{(j)}(\alpha)\int_\alpha^\infty dG_n^{(k)}(y) \\ &\geq (1-\epsilon')(1-G_n^{(k)}(\alpha)) \geq (1-\epsilon')^2 = 1-\epsilon. \end{aligned}$$

<u>Case B: $1 \leq i \leq k-1$</u>

Fix i in the above range and let $h_n^{p,\nu}(y) = \prod_{j \in \mathcal{S}_\nu^p(i)} G_n^{(j)}(y)$

$$\prod_{j\in\bar{\mathcal{S}}_\nu^p(i)} \{G_n^{(j)}(h_n(y)) - G_n^{(j)}(y)\}$$

Then

$$p_{\underset{\sim}{\lambda}}^n(i) = \sum_{p=k-m}^{k-1} \sum_{\nu=1}^{\binom{k-1}{p}} \int_{-\infty}^{\infty} h_n^{p,\nu}(y)dG_n^{(i)}(y).$$ Since $\int h_n^{p,\nu}(y)dG_n^{(i)}(y) \geq 0$

it suffices to show that given $\epsilon > 0\ \exists N \ni \forall\ n \geq N$, $\int h_n^{p,\nu}(y)dG_n^{(i)}(y) \leq \epsilon$ in order to prove $\int h_n^{p,\nu}(y)dG_n^{(i)}(y) \to 0$ as $n \to \infty$.

<u>Subcase (1)</u>: For p,ν such that $k \in \mathcal{P}_\nu(i)$

Given $\epsilon > 0$ pick N so that $\forall n \geq N$ $\begin{cases} G_n^{(k)}(\alpha) < \epsilon/2 \\ G_n^{(i)}(\alpha) > 1 - \epsilon/2 \end{cases}$

where α is as above. Hence $\forall y < \alpha$, $h_n^{p,\nu}(y) \leq G_n^{(k)}(y) \leq G_n^{(k)}(\alpha)$.

So finally we obtain $\forall n \geq N$,

$$\int_{-\infty}^{\infty} h_n^{p,\nu}(y)\,dG_n^{(i)}(y) = \int_{-\infty}^{\alpha} h_n^{p,\nu}(y)\,dG_n^{(i)}(y) + \int_{\alpha}^{\infty} h_n^{p,\nu}(y)\,dG_n^{(i)}(y)$$

$$\leq \int_{-\infty}^{\alpha} G_n^{(k)}(y)\,dG_n^{(i)}(y) + \int_{\alpha}^{\infty} 1\,dG_n^{(i)}(y)$$

$$\leq \frac{\epsilon}{2} \cdot 1 + (1 - G_n^{(i)}(\alpha)) \leq \epsilon.$$

<u>Subcase (2)</u>: For p,ν such that $k \in \bar{\mathcal{P}}_\nu(i)$

Pick α' such that $\alpha < \alpha' < \lambda_{[k]}$. Now since $h_n(\alpha) \to \alpha$ and $\alpha' < \lambda_{[k]}$ $\exists$ N such that $\forall n \geq N$

$$\begin{cases} G_n^{(k)}(\alpha') < \epsilon/2 \\ h_n(\alpha) < \alpha' \\ G_n^{(i)}(\alpha) > 1 - \epsilon/2 \end{cases}$$

$\Rightarrow$ $\forall n \geq N$ and $y < \alpha$

$$h_n^{p,\nu}(y) \leq \{G_n^{(k)}(h_n(y)) - G_n^{(k)}(y)\}$$

$$\leq G_n^{(k)}(h_n(y))$$

$$\leq G_n^{(k)}(h_n(\alpha)) \leq G_n^{(k)}(\alpha') < \epsilon/2$$

So once again by arguments similar to the above

$\int_{-\infty}^{\infty} h_n^{p,\nu}(y)dG_n^{(i)}(y) < \epsilon \ \forall\ n \geq N$. This completes the proof of the theorem.

Corollary 1.4.1. For any $\underset{\sim}{\lambda}\epsilon\Omega$ such that $\lambda_{[k]} > \lambda_{[k-1]}$

$$W_i(n) \overset{q.m.}{\rightarrow} \begin{bmatrix} 1 & , & i = k \\ 0 & , & i \leq i < k \end{bmatrix} \text{ as } n \rightarrow \infty \tag{1.4.6}$$

Proof. When $i = k$

$$E_{\underset{\sim}{\lambda}}[(W_k(n)-1)^2] = 0 \cdot P_{\underset{\sim}{\lambda}}[W_k(n)=1] + 1 \cdot P_{\underset{\sim}{\lambda}}[W_k(n)=0]$$

$$= 1-p^n_{\underset{\sim}{\lambda}}(k) \rightarrow 0 \text{ as } n \rightarrow \infty \quad \text{by (1.4.4)}$$

For $i < k$, $E_{\underset{\sim}{\lambda}}[\{W_k(n)\}^2] = 0 \cdot P_{\underset{\sim}{\lambda}}[W_i(n)=0]+1 \cdot P_{\underset{\sim}{\lambda}}[W_i(n)=1]$

$$= p^n_{\underset{\sim}{\lambda}}(i) \rightarrow 0 \text{ as } n \rightarrow \infty \text{ by (1.4.4).}$$

Remark 1.4.2. The following types of asymptotic behavior also hold for any $\underset{\sim}{\lambda}\epsilon\Omega$ satisfying $\lambda_{[k]} > \lambda_{[k-1]}$,

(1.4.7) $W_i(n) \overset{p}{\rightarrow} \begin{bmatrix} 1 & , & i = k \\ 0 & , & i < k \end{bmatrix}$ as $n \rightarrow \infty$ since convergence in L^2 implies convergence in probability.

(1.4.8) $S(n) \overset{p}{\rightarrow} 1$ as $n \rightarrow \infty$ since $S(n) = \sum_{i=1}^{k} W_i(n)$

(1.4.9) $S(n) \overset{q.m.}{\rightarrow} 1$ as $n \rightarrow \infty$ since $|S(n)-1| \leq m-1$

(1.4.10) $E_{\underset{\sim}{\lambda}}[S(n)] \rightarrow 1$ as $n \rightarrow \infty$ since $(S(n)-1) \leq (S(n)-1)^2$

The next results will study some properties of S(n) when n is fixed. In particular, conditions will be given which guarantee that the supremum of $E_{\underset{\sim}{\lambda}}[S(n)]$ in Ω occurs at some point $\underset{\sim}{\lambda}=(\lambda_1,\ldots,\lambda_k)$ for which $\lambda_{[1]} = \lambda_{[k]}$.

It will be assumed that the regularity conditions (1.4.11) hold in some of the theorems which follow.

(1.4.11)
(i) $E_n^\lambda = E_n$ for every $\lambda\varepsilon\Lambda$

(ii) For any $[\lambda_1,\lambda_2] \subset \Lambda$ there exists $r^*(y)$ depending on λ_1, $\lambda_2 \ni \left|\frac{dG_n(y/\lambda)}{d\lambda}\right| \leq r^*(y)\ \forall y\varepsilon R^1$ and $\forall\lambda\varepsilon[\lambda_1,\lambda_2]$ where $r^*(y)$ satisfies

$$\begin{cases} \int r^*(h_n(y))dG_n(y/\lambda') < \infty\ \forall\ \lambda' \geq \lambda_2 \text{ and} \\ \int r^*(y)dG_n(h_n(y)/\lambda') < \infty \quad \forall\lambda' \geq \lambda_2 \end{cases}$$

Theorem 1.4.3. If the condition (1.4.11) is satisfied and $\forall\lambda_1$, λ_2 in Λ with $\lambda_1 \leq \lambda_2$

$$(1.4.12)\quad \frac{\partial G_n(h_n(y)/\lambda_1)}{\partial\lambda_1} g_n(y/\lambda_2) - \frac{\partial G_n(y/\lambda_1)}{\partial\lambda_1} g_n(h_n(y)/\lambda_2)h_n'(y) \geq 0$$

ae in E_n

then $E_{\underset{\sim}{\lambda}}[S(n)]$ is non decreasing in $\lambda_{[1]}$ on $\Lambda(\lambda_{[2]}) = \{\lambda\varepsilon\Lambda|\lambda\leq\lambda_{[2]}\}$ for any fixed $(\lambda_{[2]},\ldots,\lambda_{[k]})$.

Proof. Fix $\lambda_{[2]} \leq\ldots\leq \lambda_{[k]}$ for the following argument and then

$E_{\underset{\sim}{\lambda}}[S(n)] = T_1(\underset{\sim}{\lambda}) + T_2(\underset{\sim}{\lambda})$ where

$$T_1(\underset{\sim}{\lambda}) = \sum_{p=k-m}^{k-1} \sum_{\nu=1}^{\binom{k-1}{p}} \int_{E_n} e_1^{p,\nu}(y)dG_n^{(1)}(y)$$

$$T_2(\underset{\sim}{\lambda}) = \sum_{r=2}^{k} \sum_{p=k-m}^{k-1} \sum_{\nu=1}^{\binom{k-1}{p}} \int_{E_n} e_r^{p,\nu}(y)dG_n^{(r)}(y) \text{ where}$$

$$e_i^{p,\nu}(y) = \prod_{j\in\mathscr{P}_\nu^p(i)} G_n^{(j)}(y) \prod_{j\in\bar{\mathscr{P}}_\nu^p(i)} \{G_n^{(j)}(h_n(y)) - G_n^{(j)}(y)\}.$$

Now $T_2(\underset{\sim}{\lambda})$ can be rewritten as

$$T_2(\underset{\sim}{\lambda}) = \sum_{p=k-m}^{k-1} \sum_{\nu=1}^{\binom{k-1}{p}} \sum_{r\ni 1\in\mathscr{P}_\nu^p(r)} \int_{E_n} e_r^{p,\nu}(y)\,dG_n^{(r)}(y)$$
$$+ \sum_{p=k-m}^{k-1} \sum_{\nu=1}^{\binom{k-1}{p}} \sum_{r\ni 1\in\bar{\mathscr{P}}_\nu^p(r)} \int_{E_n} e_r^{p,\nu}(y)\,dG_n^{(r)}(y).$$

For any $A \subset \{1,\ldots,k\}$ of size s, let $\{\mathscr{P}_\nu^p(A)\,|\,\nu=1,\ldots,\binom{k-s}{p}\}$ be the collection of all subsets of size p from $\{1,\ldots,k\} - A$. Note that for any fixed $p=k-m,\ldots,k-1$ and $r=2,\ldots,k$

$$\{\mathscr{P}_\nu^p(r)\,|\,1\in\mathscr{P}_\nu^p(r)\} = \{\mathscr{P}_\nu^{p-1}(1,r)\cup\{1\}\,|\,\nu=1,\ldots,\binom{k-2}{p-1}\} \tag{1.4.13}$$

while for any $p=k-m,\ldots,k-2$ and $r=2,\ldots,k$

$$\{\mathscr{P}_\nu^p(r)\,|\,1\notin\mathscr{P}_\nu^p(r)\} = \{\mathscr{P}_\nu^p(1,r)\,|\,\nu=1,\ldots,\binom{k-2}{p}\}. \tag{1.4.14}$$

So

$$T_2(\underset{\sim}{\lambda}) = \sum_{p=k-m}^{k-1} \sum_{r=2}^{k} \sum_{\nu=1}^{\binom{k-2}{p-1}} \int_{E_n} W_r^{p,\nu}(y) G_n^{(1)}(y)\,dG_n^{(r)}(y)$$
$$+ \sum_{p=k-m}^{k-2} \sum_{r=2}^{k} \sum_{\nu=1}^{\binom{k-2}{p}} \int_{E_n} z_r^{p,\nu}(y)\{G_n^{(1)}(h_n(y)) - G_n^{(1)}(y)\}\,dG_n^{(r)}(y)$$

where

(1) $$W_r^{p,\nu}(y) = \prod_{j\in\mathscr{P}_\nu^{p-1}(1,r)} G_n^{(j)}(y) \prod_{j\in\bar{\mathscr{P}}_\nu^{p-1}(1,r)} \{G_n^{(j)}(h_n(y)) - G_n^{(j)}(y)\}$$

(2) $$z_r^{p,\nu}(y) = \prod_{j\in\mathscr{P}_\nu^{p}(1,r)} G_n^{(j)}(y) \prod_{j\in\bar{\mathscr{P}}_\nu^{p}(1,r)} \{G_n^{(j)}(h_n(y)) - G_n^{(j)}(y)\}.$$

$T_1(\lambda)$ can be rewritten as follows. For any $p=k-m,\ldots,k-2$ and $\nu=1,\ldots,\binom{k-1}{p}$, $\int_{E_n} e_1^{p,\nu}(y)dG_n^{(1)}(y) = \int_{E_n} U(y)dV(y)$ for

$$dV(y) = dG_n^{(1)}(y) \text{ and } U(y)= \prod_{j\in\mathscr{P}_\nu^p(i)} G_n^{(j)}(y) \prod_{j\in\bar{\mathscr{P}}_\nu^p(i)} \{G_n^{(j)}(h_n(y))-G_n^{(j)}(y)\}$$

$$\Rightarrow \int_{E_n} e_1^{p,\nu}(y)dG_n^{(1)}(y)$$

$$= - \sum_{r\in\mathscr{P}_\nu^p(1)} \int_{E_n} \prod_{j\in\mathscr{P}_\nu^p(1)-\{r\}} G_n^{(j)}(y) \prod_{j\in\bar{\mathscr{P}}_\nu^p(1)} \{G_n^{(j)}(h_n(y))-G_n^{(j)}(y)\} G_n^{(1)}(y)dG_n^{(r)}(y)$$

$$- \sum_{r\in\bar{\mathscr{P}}_\nu^p(1)} \int_{E_n} \prod_{j\in\mathscr{P}_\nu^p(1)} G_n^{(j)}(y) \prod_{j\in\bar{\mathscr{P}}_\nu^p(1)-\{r\}} \{G_n^{(j)}(h_n(y))-G_n^{(j)}(y)\}G_n^{(1)}(y) \{g_n^{(r)}(h_n(y))h_n'(y)-g_n^{(r)}(y)\}dy$$

For $p=k-1$ and $\nu=1$

$$\int_{E_n} \prod_{j=2}^{k} G_n^{(j)}(y)dG_n^{(1)}(y) = 1- \sum_{r=2}^{k} \int_{E_n} \prod_{\substack{j=1 \\ j\neq r}}^{k} G_n^{(j)}(y)dG_n^{(r)}(y).$$

Now note that for any fixed $p=k-m,\ldots,k-1$ and $r=2,\ldots,k$

(1.4.15) $\{\mathscr{P}_\nu^p(1) \mid r\in\mathscr{P}_\nu^p(1)\} = \{\mathscr{P}_\nu^{p-1}(1,r)\cup\{r\} \mid \nu=1,\ldots,\binom{k-2}{p-1}\}$

while for any $p=k-m,\ldots,k-2$ and $r=2,\ldots,k$

$$\{\mathscr{P}_\nu^p(1) \mid r\in\bar{\mathscr{P}}_\nu^p(1)\} = \{\mathscr{P}_\nu^p(1,r) \mid \nu=1,\ldots,\binom{k-2}{p}\}$$

$$\Rightarrow T_1(\lambda) = 1- \sum_{p=k-m}^{k-1} \sum_{r=2}^{k} \sum_{\nu=1}^{\binom{k-2}{p-1}} \int_{E_n} W_r^{p,\nu}(y)G_n^{(1)}(y)dG_n^{(r)}(y)$$

$$- \sum_{p=k-m}^{k-1} \sum_{r=2}^{k} \sum_{\nu=1}^{\binom{k-2}{p}} \int_{E_n} z_r^{p,\nu}(y) G_n^{(1)}(y) \{ g_n^{(r)}(h_n(y)) h_n'(y) - g_n^{(r)}(y) \} dy$$

So

$$E_{\underset{\sim}{\lambda}}[S(n)] = T_1(\underset{\sim}{\lambda}) + T_2(\lambda)$$

$$= 1 + \sum_{p=k-m}^{k-2} \sum_{r=2}^{k} \sum_{\nu=1}^{\binom{k-2}{p}} \int_{E_n} z_r^{p,\nu}(y) \{ G_n^{(1)}(h_n(y)) g_n^{(r)}(y) - G_n^{(1)}(y) g_n^{(r)}(h_n(y)) h_n'(y) \} dy$$

and finally

$$\frac{dE_{\underset{\sim}{\lambda}}[S(n)]}{d\lambda_{[1]}} = \sum_{p=k-m}^{k-2} \sum_{r=2}^{k} \sum_{\nu=1}^{\binom{k-2}{p}} \int_{E_n} z_r^{p,\nu}(y) \cdot \left\{ \frac{\partial G_n^{(1)}(h_n(y))}{\partial \lambda_{[1]}} g_n^{(r)}(y) - \frac{\partial G_n^{(1)}(y)}{\partial \lambda_{[1]}} g_n^{(r)}(h_n(y)) h_n'(y) \right\} dy. \tag{1.4.16}$$

By (1.4.12) for every $r=2,\ldots,k$

$$\frac{\partial G_n^{(1)}(h_n(y))}{\partial \lambda_{[1]}} g_n^{(r)}(y) - \frac{\partial G_n^{(1)}(y)}{\partial \lambda_{[1]}} g_n^{(r)}(h_n(y)) h_n'(y) \geq 0 \text{ a.e. in } E_n$$

$\Rightarrow$ the derivative in (1.4.16) is non negative and completes the proof.

Remark 1.4.3. If $G_n(\cdot)$ is a location parameter family and $h_n(y) = y+d_n$ $(d_n \downarrow 0)$ then (1.4.12) is simply the requirement of monotone likelihood ratio; i.e.

$$(1.4.12) \Leftrightarrow -g_n(y+d_n-\lambda_1) g_n(y-\lambda_2) + g_n(y-\lambda_1) g_n(y+d_n-\lambda_2) \geq 0 \quad \forall \lambda_1 < \lambda_2,\ y \in R'$$

$$\Leftrightarrow \frac{g_n(y+d_n-\lambda_2)}{g_n(y+d_n-\lambda_1)} \geq \frac{g_n(y-\lambda_2)}{g_n(y-\lambda_1)}$$

$$\Leftrightarrow g_n(y-\lambda) \text{ has MLR.}$$

Similarly if $G_n(\cdot)$ is a scale parameter family on $(0,\infty)$ and $h_n(y) = d_n y(d_n+1)$ then (1.4.12) is equivalent to MLR.

<u>Corollary 1.4.2.</u> If for every fixed $\lambda_{[2]} \leq \cdots \leq \lambda_{[k]}$, $\frac{dE_{\underset{\sim}{\lambda}}[S(n)]}{d\lambda_{[1]}} \geq 0$ for $\lambda_{[1]}$ in $\Lambda(\lambda_{[2]})$, then the $\sup_{\Omega} E_{\underset{\sim}{\lambda}}[S(n)] = \sup_{\lambda\varepsilon\Lambda} \gamma(\lambda,n)$ where

$$(1.4.17)\quad \gamma(\lambda,n) = k\int_{E_n^{\lambda}} \{G_n(h_n(y)/\lambda)\}^{k-1} I\left(\frac{G_n(y/\lambda)}{G_n(h_n(y)/\lambda)};k-m,m\right) dG_n(y/\lambda)$$

Furthermore if the hypotheses of Theorem 1.4.3 hold for $\lambda_1=\lambda_2$ then $\gamma(\lambda,n)$ is non decreasing in λ and hence if there is a greatest element $\lambda_0\varepsilon\Lambda \Rightarrow \sup_{\Omega} E_{\underset{\sim}{\lambda}}[S(n)] = \gamma(\lambda_0,n)$.

<u>Proof.</u> It suffices to prove $\forall\, q < k$ and fixed $\lambda_{[q+1]} \leq \cdots \leq \lambda_{[k]}$ that $E_{\underset{\sim}{\lambda}(q)}[S(n)] \uparrow$ in λ on $\Lambda(\lambda_{[q+1]})$ where the underlying $\underset{\sim}{\lambda}(q) = (\lambda,\ldots,\lambda,\lambda_{[q+1]},\ldots,\lambda_{[k]})$. Let $\underset{\sim}{\lambda}' = (\lambda_{[1]},\ldots,\lambda_{[k]})$ and note from Theorem 1.4.1 that $E_{\underset{\sim}{\lambda}'}[S(n)]$ is invariant under permutations of $\underset{\sim}{\lambda}'$.

$$\Rightarrow \frac{dE_{\underset{\sim}{\lambda}(q)}[S(n)]}{d\lambda} = \sum_{i=1}^{q} \frac{\partial E_{\underset{\sim}{\lambda}'}[S(n)]}{\partial\lambda_{[i]}}\Bigg|_{\underset{\sim}{\lambda}(q)}$$

$$= \frac{q\,\partial E_{\underset{\sim}{\lambda}'}[S(n)]}{\partial\lambda_{[1]}}\Bigg|_{\underset{\sim}{\lambda}(q)}$$

But from the previous proof $\frac{\partial E_{\underset{\sim}{\lambda}'}[S(n)]}{\partial\lambda_{[1]}}\Bigg|_{\underset{\sim}{\lambda}(q)} \geq 0.$

Hence the supremum over Ω of $E[S(n)]$ occurs at some point where all the $\lambda_{[i]}$'s are equal.

Since $\gamma(\lambda,n) = E[\phi(Y,\lambda)]$ for

$$\phi(y,\lambda) = \{G_n(h_n(y)/\lambda)\}^{k-1} I\left(\frac{G_n(y/\lambda)}{G_n(h_n(y)/\lambda)};k-m,m\right)$$

Lemma 1.2.2 can be applied and the sufficient condition (1.2.5) that $\gamma(\lambda,n)$ be non decreasing reduces to

$$\{G_n(h_n(y)/\lambda)\}^{k-3}\{g_n(y/\lambda)\frac{\partial G_n(h_n(y)/\lambda)}{\partial\lambda} - \frac{\partial G_n(y/\lambda)}{\partial\lambda} g_n(h_n(y)/\lambda)h_n'(y)\}\cdot$$

$$\{(k-1)G_n(h_n(y)/\lambda)I\left(\frac{G_n(y/\lambda)}{G_n(h_n(y)/\lambda)};k-m,m\right) - G_n(y/\lambda)b\left(\frac{G_n(y/\lambda)}{G_n(h_n(y)/\lambda)}\right)\} \geq 0$$

$\forall\ \lambda$ and ae y

$$(1.4.18) \quad \Leftrightarrow \{g_n(y/\lambda)\frac{\partial G_n(h_n(y)/\lambda)}{\partial\lambda} - \frac{\partial G_n(y/\lambda)}{\partial\lambda} g_n(h_n(y)/\lambda)h_n'(y)\} \geq 0$$

$\forall\ \lambda$ and ae y

since the third factor is non negative by Lemma 1.2.3. But (1.4.18) is precisely the hypothesis of the Theorem 1.4.3 and hence $\gamma(\lambda,n)$ is nondecreasing in λ. The final part of the result is obvious.

Remark 1.4.3. Theorem 1.4.3 generalizes the result of Gupta-Panchapakesan [33] which shows that (1.4.12) implies that the supremum of $E[S(n)]$ occurs at some point having $\lambda_{[1]} = \lambda_{[k]}$ in the special case m=k.

Finally it should be noted that $\gamma(\lambda,n)$ is independent of λ when $G_n(\cdot/\lambda)$ is a location or scale parameter family and hence the evaluation of $\sup_{\Omega} E[S(n)]$ is complete.

Another associated random variable of interest is the number of non best populations selected, $T(n)$. The expected value of the number of non best populations selected can be written

$$E_{\underset{\sim}{\lambda}}[T(n)] = E_{\underset{\sim}{\lambda}}[S(n)] - p^n_{\underset{\sim}{\lambda}}(k).$$

As usual, the dependence of T_{in} on n is suppressed in the notation.

<u>Goal</u>: Given P*, u(·) and the sequence of rules {R'(n)} find the smallest common sample size n necessary to achieve

(1.5.6) $$P_{\underset{\sim}{F}}[CS|R'(n)] \geq P^* \; \forall \underset{\sim}{F}\varepsilon\Omega'(u).$$

The event [CS|R'(n)] stands for the selection of any restricted subset containing $\pi_{(1)}$.

Once again the notation introduced in Section 1.2 will be used to emphasize the dependence of the various quantities of interest on the λ_i's.

<u>Theorem 1.5.1</u>. For any $\underset{\sim}{\lambda}\varepsilon\Omega$

$$P_{\underset{\sim}{\lambda}}[CS|R'(n)] = \sum_{p=k-m}^{k-1} \sum_{\nu=1}^{\binom{k-1}{p}} \int_{-\infty}^{\infty} \prod_{j\varepsilon\mathscr{P}_\nu^p(1)} \{1-G_n^{(j)}(y)\} \prod_{j\varepsilon\bar{\mathscr{P}}_\nu^p(1)} \{G_n^{(j)}(y) - G_n^{(j)}(H_n(y))\} dG_n^{(1)}(y)$$

where

$$\begin{cases} G_n^{(j)} = G_n(y/\lambda_{[j]}) \\ \{\mathscr{P}_\nu^p(1)|\nu=1,\ldots,\binom{k-1}{p}\} \text{ is the collection of all subsets of size p from } \{2,\ldots,k\}. \end{cases}$$

Using the assumption concerning the stochastically increasing nature of the family $\{G_n(y/\lambda)|\lambda\varepsilon\Lambda\}$ and the method of Theorem 1.2.2 the following reduction result can be proved.

<u>Theorem 1.5.2</u>.

(1.5.7) $$\inf_{\underset{\sim}{\lambda}\varepsilon\Omega(u)} P_{\underset{\sim}{\lambda}}[CS|R'(n)] = \inf_{\underset{\sim}{\lambda}\varepsilon\Omega^o(u)} P[CS|R'(n)] = \inf_{\lambda\varepsilon\Lambda'} \eta(\lambda,n)$$

Analagous to the definitions of Section 1.1 let

$$(1.5.2)\quad \begin{cases} \Omega(u) = \{\underset{\sim}{\lambda}\varepsilon\Omega \mid u(\lambda_{[1]}) \le \lambda_{[2]}\} \\ \\ \Omega^{o}(u) = \{\underset{\sim}{\lambda}\varepsilon\Omega \mid u(\lambda_{[1]}) = \lambda_{[2]} = \lambda_{[k]}\} \end{cases}$$

and then

(1.5.3) $\Omega'(u) = \{\underset{\sim}{F}\varepsilon\Omega(\mathscr{F}) \mid \underset{\sim}{\lambda}(\underset{\sim}{F})\varepsilon\Omega(u)\}$ specifies a preference zone in $\Omega(\mathscr{F})$.

<u>Example 1.5.1</u>. Let $\Lambda=[0,\infty)$, $\delta_1 > 0$, $\delta_2 > 1$ and

$$u(\lambda)= \begin{cases} \lambda + \delta_1, & 0 \le \lambda \le \delta_1/(\delta_2-1) \\ \delta_2\lambda, & \lambda \ge \delta_1/(\delta_2-1) \end{cases}$$

then $\Lambda'=[0,\infty)$ and $\Omega'(u) = \{\underset{\sim}{F} \mid \lambda_{[2]} \ge \max\{\lambda_{[1]}+\delta_1, \delta_2\lambda_{[1]}\}\}$

Suppose that $\{H_n(x)\}$ is a sequence of real valued functions defined for each n on a portion of the real line containing $\bigcup_{\lambda\varepsilon\Lambda} E_n^{\lambda}$ and such that

(i) $H_n(x) < x \;\forall x$ and n

(ii) For each n, the function $H_n(x)$ is continuous and strictly increasing

(iii) For every x, $H_n(x) \uparrow x$ as $n \to \infty$

The following selection procedure, $R'(n)$, is proposed for selecting $\pi_{(1)}$,

(1.5.5) <u>R'(n)</u>: Select $\pi_i \Leftrightarrow T_i \le \min\{T_{[m]}, H_n^{-1}(T_{[1]})\}$

Corollary 1.4.3. If the hypotheses of Corollary 1.4.2 hold then

$\sup_{\Omega} E_{\underset{\sim}{\lambda}}[T(n)] = \frac{(k-1)}{k} \sup_{\lambda\varepsilon\Lambda} \gamma(\lambda,n)$ where $\gamma(\lambda,n)$ is defined by

(1.4.17). Furthermore if the hypotheses of Theorem 1.4.3 hold

for $\lambda_1 = \lambda_2$ then $\gamma(\lambda,n)$ is non decreasing in λ and hence if there

is a greatest element $\lambda_0 \varepsilon \Lambda \Rightarrow \sup_{\Omega} E_{\underset{\sim}{\lambda}}[T(n)] = (k-1)\gamma(\lambda_0,n)/k$.

Proof. For any $\underset{\sim}{\lambda} = (\lambda_{[1]},\ldots,\lambda_{[k]})\varepsilon\Omega$ let $\underset{\sim}{\lambda}([k]) = (\lambda_{[k]},\ldots,\lambda_{[k]})$

then $\forall \underset{\sim}{\lambda}\varepsilon\Omega$ the hypotheses imply $E_{\underset{\sim}{\lambda}}[S(n)] \leq E_{\underset{\sim}{\lambda}([k])}[S(n)]$. Also the

strong monotonicity of R(n) implies $p^n_{\underset{\sim}{\lambda}}(k) \geq p^n_{\underset{\sim}{\lambda}([k])}(k)$. So

$$E_{\underset{\sim}{\lambda}}[T(n)] \leq E_{\underset{\sim}{\lambda}([k])}[T(n)] = \frac{(k-1)}{k} \gamma(\lambda_{[k]},n)$$

$\Rightarrow \sup_{\Omega} E_{\underset{\sim}{\lambda}}[T(n)] = \frac{(k-1)}{k} \sup_{\lambda\varepsilon\Lambda} \gamma(\lambda,n)$ and the remaining conclusions follow from previous work.

Remark 1.4.4. For any $\underset{\sim}{\lambda}\varepsilon\Omega$ such that $\lambda_{[k]} > \lambda_{[k-1]}$, $T(n) \overset{p}{\to} 0$ as $n \to \infty$.

1.5 Selection for the Smallest Parameter

In the present modification, k populations $\pi_1,\ldots,\pi_k$ are studied where each population is characterized as in Section 1.1. The experimenter is now interested in determining $\pi_{(1)}$ the population characterized by $\lambda_{[1]}$. The assumptions (1.1.1) are supposed met and a preference zone is specified by a function $u: \Lambda \to R'$ satisfying

$$(1.5.1) \begin{cases} \text{(i)} & u(\cdot) \text{ is continuous and non decreasing on} \\ & \Lambda' = \{\lambda\varepsilon\Lambda | u(\lambda)\varepsilon\Lambda\} \\ \text{(ii)} & u(\lambda) > \lambda \ \forall\ \lambda\varepsilon\Lambda' \end{cases}$$

where

$$(1.5.8)\quad \eta(\lambda,n)=\int_{-\infty}^{\infty}\{1-G_n(H_n(y))\}^{k-1}I\left(\frac{1-G_n(y/u(\lambda))}{1-G_n(H_n(y)/u(\lambda))};k-m,m\right)\,dG_n(y/\lambda)$$

The next goal of interest is to discover conditions which allow the evaluation of the one dimensional infimum of (1.5.7). As in Theorem 1.2.3 regularity conditions of the type (1.2.9) are required. They take the following form in this case: For any $[\lambda_1,\lambda_2] \subset \Lambda'$ and $\lambda_3\varepsilon\Lambda'$ there exist $e_1(y)$ and $e_2(y)$ such that

$$(1.5.9)\quad \begin{cases} \text{(i)} & \left|\dfrac{\partial G_n(y/u(\lambda))}{\partial\lambda}\right| \leq e_1(y)\ \forall\lambda\varepsilon[\lambda_1,\lambda_2] \text{ where} \\ & \left(\int e_1(y)\,dG_n(y/\lambda_3)\right)\left(\int e_1(H_n(y))\,dG_n(y/\lambda_3)\right) < \infty \\ \text{(ii)} & \left|\dfrac{\partial G_n(y/\lambda)}{\partial\lambda}\right| \leq e_2(y)\ \forall\lambda\varepsilon[\lambda_1,\lambda_2] \text{ where} \\ & \left(\int e_2(y)\,dG_n(H_n(y)/\lambda_3)\right)\left(\int e_2(y)\,dG_n(y/\lambda_3)\right) < \infty \end{cases}$$

<u>Theorem 1.5.3</u>. If $E_n^{\lambda} = E_n\ \forall\ \lambda\varepsilon\Lambda'$ and $G_n(y/\lambda)$, $H_n(y)$ and $u(\lambda)$ are sufficiently regular so that both (1.5.9) holds and $\forall\ \lambda\varepsilon\Lambda'$:

$$(1.5.10)\quad H_n'(y)g_n(H_n(y)/u(\lambda))\frac{\partial G_n(y/\lambda)}{\partial\lambda} - g_n(y/\lambda)\frac{\partial G_n(H_n(y)/u(\lambda))}{\partial\lambda} \geq 0$$

for ae y in E_n

and

$$(1.5.11)\quad g_n(y/u(\lambda))\frac{\partial G_n(y/\lambda)}{\partial\lambda} - g_n(y/\lambda)\frac{\partial G_n(y/u(\lambda))}{\partial\lambda} \geq 0 \text{ for ae } y \text{ in } E_n$$

then $\eta(\lambda,n)$ is non decreasing in λ.

The proof of this result is similar to that of Theorem 1.2.3. Note that if (1.5.10) and (1.5.11) are identically zero then $\eta(\lambda,n)$ is independent of λ.

Next it is shown that (1.5.6) can be attained for all P* levels by taking n sufficiently large.

Theorem 1.5.4. If there exists $\lambda_0 \varepsilon \Lambda'$ such that for all n sufficiently large

(1.5.12) $$\inf_{\lambda\varepsilon\Lambda'} \eta(\lambda,n) = \eta(\lambda_0,n)$$

then $\{R'(n)\}$ is consistent wrt $\Omega(u)$.

Formally the definition of consistency is the same as in Section 1.3 but the meaning of a correct selection is different.

Remark 1.5.1. If the conditions (1.5.10) and (1.5.11) hold for every n sufficiently large and there is a smallest $\lambda^* \varepsilon \Lambda'$ then (1.5.12) holds with $\lambda_0 = \lambda^*$.

Some properties of the rules R'(n) are now studied. For $\underset{\sim}{\lambda} \varepsilon \Omega$ let

$$p_{\underset{\sim}{\lambda}}^{'n}(i) = P_{\underset{\sim}{\lambda}}[R'(n) \text{ selects } \pi_{(i)}]$$

Def. 1.5.1. The rule R'(n) is reverse strongly monotone in $\pi_{(i)}$ means $p_{\underset{\sim}{\lambda}}^{'n}(i)$ is

$$\begin{cases} \downarrow \text{ in } \lambda_{[i]} \text{ when all other components are fixed} \\ \uparrow \text{ in } \lambda_{[j]} (j \neq i) \text{ when all other components are fixed.} \end{cases}$$

Def. 1.5.2. The rule R'(n) is reverse monotone means $\forall\ 1 \leq i < j \leq k$ and $\underset{\sim}{\lambda} \varepsilon \Omega$ $p_{\underset{\sim}{\lambda}}^{'n}(i) \geq p_{\underset{\sim}{\lambda}}^{'n}(j)$.

Def. 1.5.3. The rule R'(n) is reverse unbiased means $\forall k \geq i > 1$ and $\underset{\sim}{\lambda} \varepsilon \Omega$

$$P_{\underset{\sim}{\lambda}}[R'(n) \text{ rejects } \pi_{(i)}] \geq P_{\underset{\sim}{\lambda}}[R'(n) \text{ rejects } \pi_{(1)}].$$

Note that reverse monotonicity implies reverse unbiasedness.

Theorem 1.5.3. For any $i=1,\ldots,k$, any procedure R'(n) of form (1.5.5) is reverse strongly monotone in $\pi_{(i)}$.

The result analagous to Corollary 1.3.1 is

Corollary 1.5.1. All rules of form (1.5.5) are reverse monotone and reverse unbiased.

The next area to be studied concerns S'(n), the number of populations R'(n) selects. S'(n) is an integer valued random variable taking values between 1 and m. Using the representation

$$S'(n) = \sum_{i=1}^{k} Z_i(n) \text{ where} \tag{1.5.13}$$

$$Z_i(n) = \begin{cases} 1 & , \quad T_{(i)} \leq \min\{T_{[m]}, H_n^{-1}(T_{[1]})\} \\ 0 & , \quad \text{otherwise} \end{cases} \tag{1.5.14}$$

the following results can be obtained.

Theorem 1.5.6. For any $\underset{\sim}{\lambda} \varepsilon \Omega$

$$E_{\underset{\sim}{\lambda}}[S'(n)] = \sum_{i=1}^{k} \sum_{p=k-m}^{k-1} \sum_{\nu=1}^{\binom{k-1}{p}} \int_{-\infty}^{\infty} \prod_{j \varepsilon \mathcal{P}_\nu(i)} \{1-G_n^{(j)}(y)\} \prod_{j \varepsilon \bar{\mathcal{P}}_\nu(i)} \{G_n^{(j)}(y)-G_n^{(j)}(H_n(y))\} dG_n^{(i)}(y) \tag{1.5.15}$$

The general expression (1.5.15) shows that $E_{\underset{\sim}{\lambda}}[S'(n)]$ depends only on the λ_i's and is symmetric under permutations of the coordinates. If larger and larger samples are drawn from each π_i the number of populations selected decreases to one if there is but one "best" population.

Theorem 1.5.7. For any $\underset{\sim}{\lambda}\varepsilon\Omega$ such that $\lambda_{[1]} < \lambda_{[2]}$

(1.5.16) $$p'^{n}_{\underset{\sim}{\lambda}}(i) \to \begin{cases} 1 & , \ i = 1 \\ 0 & , \ 2 \le i \le k \end{cases} \quad \text{as } n \to \infty$$

The proof follows along the lines of the corresponding result for $\pi_{(k)}$.

Remark 1.5.2. For any $\underset{\sim}{\lambda}\varepsilon\Omega$ such that $\lambda_{[1]} < \lambda_{[2]}$ the result (1.5.16) implies the following asymptotic behavior.

(i) $$Z_i(n) \overset{p}{\to} \begin{cases} 1 & , \ i = 1 \\ 0 & , \ 2 \le i \le k \end{cases} \quad \text{as } n \to \infty$$

(ii) $S'(n) \overset{p}{\to} 1$ as $n \to \infty$

(iii) $E_{\underset{\sim}{\lambda}}[S'(n)] \to 1$ as $n \to \infty$

Similar results hold for $T'(n)$, the number of non best populations selected.

As in Section 1.4 certain regularity conditions will be needed for several of the remaining results. One such set of conditions is:

(1.5.17) (i) $E_n^{\lambda} = E_n$ for every $\lambda\varepsilon\Lambda$

(ii) For any $[\lambda_1, \lambda_2] \subset \Lambda \ \exists \ s^*(y)$ depending only on λ_1 and λ_2 such that

$$\left| \frac{\partial G_n(y/\lambda)}{\partial \lambda} \right| \leq s^*(y) \ \forall \lambda \varepsilon [\lambda_1, \lambda_2] \text{ where}$$

$$\begin{cases} \int_{E_n} s^*(H_n(y)) dG_n(y/\lambda') < \infty \ \forall \lambda' \geq \lambda_2 \text{ and} \\ \int_{E_n} s^*(y) dG_n(H_n(y)/\lambda') < \infty \ \forall \lambda' \geq \lambda_2 \end{cases}$$

Theorem 1.5.8. If the conditions (1.5.17) are satisfied and $\forall \lambda_1, \lambda_2$ in Λ with $\lambda_1 \leq \lambda_2$

$$(1.5.18) \quad H_n'(y) g_n(H_n(y)/\lambda_2) \frac{\partial G_n(y/\lambda_1)}{\partial \lambda_1} - \frac{\partial G_n(H_n(y)/\lambda_1)}{\partial \lambda_1} g_n(y/\lambda_2) \geq 0$$

ae in E_n

then $E_{\underset{\sim}{\lambda}}[S'(n)]$ is non decreasing in $\lambda_{[1]}$ on $\Lambda(\lambda_{[2]}) = \{\lambda \varepsilon \Lambda | \lambda \leq \lambda_{[2]}\}$ for any fixed $(\lambda_{[2]}, \ldots, \lambda_{[k]})$.

Remark 1.5.3. If $G_n(y/\lambda)$ is a location (scale) parameter family and $H_n(y) = y - d_n(y/c_n)$ with $d_n > 0$ $(c_n > 1)$ then (1.5.18) is the condition that $g_n(y/\lambda)$ has monotone likelihood ratio.

Corollary 1.5.2. If for every fixed $(\lambda_{[2]}, \ldots, \lambda_{[k]})$, $\frac{\partial E_{\underset{\sim}{\lambda}}[S'(n)]}{\partial \lambda_{[1]}} \geq 0$ on $\Lambda(\lambda_{[2]})$ then $\sup_{\Omega} E_{\underset{\sim}{\lambda}}[S'(n)] = \sup_{\lambda \varepsilon \Lambda} \gamma'(\lambda, n)$ where

$$\gamma'(\lambda,n) = k \int_{E_n} \{1 - G_n(H_n(y)/\lambda)\}^{k-1} I\left(\frac{1-G_n(y/\lambda)}{1-G_n(H_n(y)/\lambda)}; k-m, m\right) dG_n(y/\lambda)$$

Furthermore if the hypotheses of Theorem 1.5.8 hold for $\lambda_1 = \lambda_2$ then $\gamma'(\lambda,n)$ is non decreasing in λ and hence if there is a greatest element $\lambda_0 \epsilon \Lambda$, $\sup_{\Omega} E_{\underset{\sim}{\lambda}}[S(n)] = \gamma'(\lambda_0,n)$.

Remark 1.5.4. If $G_n(\cdot/\lambda)$ is a location or scale parameter family, $\gamma'(\lambda,n)$ is independent of λ.

Example 1.5.2. Location Parameter Family

Let $\pi_1,\ldots,\pi_k$, Λ and T_{in} be as in example 1.3.1. Suppose $u(\lambda) = \lambda+\delta$ $(\delta > 0)$ and $\{H_n(x)\}$ is any sequence satisfying (1.5.4) and the additional assumption $H_n'(x) \leq 1$. (1.5.10) becomes

$$H_n'(y)g_n(H_n(y)/u(\lambda))(-g_n(y/\lambda))-g_n(y/\lambda)(-g_n(H_n(y)/u(\lambda))$$
$$\geq g_n(y/\lambda)g_n(H_n(y)/u(\lambda))-1\cdot g_n(y/\lambda)g_n(H_n(y)/u(\lambda))=0 \;\forall\; y,\lambda \text{ in } R^1$$

and (1.5.11) becomes

$$g_n(y/\lambda)g_n(y/u(\lambda))-g_n(y/u(\lambda))g_n(y/\lambda)=0 \;\forall y,\lambda \text{ in } R^1.$$

Hence $\eta(\lambda,n)$ is non decreasing in λ. For the usual choice $H_n(x) = x-d_n(d_n\uparrow 0)$ both (1.5.10) and (1.5.11) are identically zero and hence

$$\inf_{\underset{\sim}{\lambda}\epsilon\Omega(u)} P_{\underset{\sim}{\lambda}}[CS|R'(n)]=\int_{-\infty}^{\infty}\{1-G_n(y-d_n-\delta)\}^{k-1}I\left(\frac{1-G_n(y-\delta)}{1-G_n(y-\delta-d_n)};k-m,m\right)dG_n(y).$$

Finally if it is known that $g_n(y/\lambda)=g_n(y-\lambda)$ has monotone likelihood ratio then

$$\sup_{\Omega} E_{\underset{\sim}{\lambda}}[S'(n)]=k\int_{-\infty}^{\infty}\{1-G_n(y-d_n)\}^{k-1}I\left(\frac{1-G_n(y)}{1-G_n(y-d_n)};k-m,m\right)dG_n(y).$$

The results for the scale parameter family follow in exactly the same manner.

Conclusions for T'(n) corresponding to Corollary 1.4.3 follow along the same lines as before and will not be explicitly stated here.

1.6 Some Alternate Formulations

Formulation I

The idea of this formulation is to show that by taking larger indifference zones the same rule R based on a fixed number of observations from each population can be made to attain any P* level. Bechhofer [8] provides an example showing that his proposed rule for the normal means problem has this property.

In the general case the machinery of Section 1.1 is assumed and also that $(0,\infty) \subset \Lambda$. P^*, n, m and $h_n(y)$ are fixed. Finally it is assumed there exists a class of indifference zones $\mathcal{P}' = \{p_\xi : \Lambda \to R^1 | \xi \varepsilon I\}$ where I is an interval and if $\xi_0 = \sup\{\xi | \xi \varepsilon I\}$, the following hold

$$(1.6.1) \quad \begin{cases} \text{(i)} & \text{For every } \xi \varepsilon I,\ p_\xi(\lambda) \leq \lambda \\ \text{(ii)} & \text{For every } \xi \varepsilon I,\ p_\xi(\lambda) \text{ is continuous and strictly increasing in } \lambda. \\ \text{(iii)} & p_\xi^{-1}(\lambda) \to \infty \text{ as } \xi \to \xi_0 \text{ for every } \lambda \varepsilon \Lambda \end{cases}$$

The following theorem gives conditions under which (1.1.5) can be met

Theorem 1.6.1. If for every $\xi \varepsilon I$, $\inf_{\lambda \varepsilon \Lambda'_\xi} \psi(\lambda,n) = \psi(\lambda_0^\xi,n)$ where λ_0^ξ satisfies $p_\xi(\lambda_0^\xi) = \lambda_0$ and if $\forall y \varepsilon \bigcup_{\lambda \varepsilon \Lambda} E_n^\lambda$, $G_n(y/\lambda) \to 0$ as $\lambda \to \infty$ then $\inf_{\Omega(p_\xi)} P[CS|R] \to 1$ as $\xi \to \xi_0$

Remark 1.6.1. Note that $\psi(\lambda,n)$ depends on the particular indifference zone used, $p_\xi(\lambda)$, even though the notation suppresses this fact.

Proof. By hypothesis $\forall_\xi \varepsilon I$

$$\inf_{\Omega(p_\xi)} P[CS|R] = \sum_{p=k-m}^{k-1} \binom{k-1}{p} \int \{G_n(y/\lambda_0)\}^p \{G_n(h_n(y)/\lambda_0) - G_n(y/\lambda_0)\}^{k-1-p} \, dG_n(y/u_\xi(\lambda_o))$$

where $u_\xi(\lambda) = p_\xi^{-1}(\lambda)$.

Let $g_p(y) = \{G_n(y/\lambda_0)\}^p \{G_n(h_n(y)/\lambda_0) - G_n(y/\lambda_0)\}^{k-1-p}$ and then two cases will be studied:

Case A $p < k - 1$

$\Rightarrow g_p(\pm \infty) = 0$ and $\forall y$, $G_n(y/u_\xi(\lambda_o)) \to 0$ as $\xi \to \xi_0$

$\Rightarrow \int g_p(y) dG_n(y/u_\xi(\lambda_o)) \to 0$ by the extended Helly Bray Lemma.

Case B $p = k-1$

Integration by parts gives

$$\int g_k(y) dG_n(y/u_\xi(\lambda)) = 1 - k\int G_n(y/u_\xi(\lambda_o)) \{G_n(y/\lambda_o)\}^{k-2} dG_n(y/\lambda_o)$$

$\to 1$ as $\xi \to \xi_0$ by dominated convergence, (1.6.1) and the hypothesis of the theorem. This completes the proof of the theorem.

The main application of Theorem 1.6.1 will be to location and scale parameter families as the examples of Chapter II will show.

Formulation II

The following formulation is commonly specified in the literature when using the subset selection approach and although the approach here will even employ an indifference zone, difficulty still arises as example 1.6.1 will show. It is assumed the experimenter has a priori specified $P^*, m, p(\lambda)$ and n. The statistician must construct a rule attaining the given probability requirement (1.1.5).

Again let us adopt the notation of Section 1.1 where n is fixed and let H denote any class of functions satisfying the requirements of Gupta-Panchapakesan [33]. Specifically these requirements are that if $E = \bigcup_{\lambda \varepsilon \Lambda} E_n^{\lambda}$ then $H = \{h_{c,d}\colon\ E \to R' \mid c \varepsilon [1,\infty), d \varepsilon [0,\infty)\}$ is a class of functions such that

(1.6.2)
- (i) For every pair c and d, $h_{c,d}(x) \geq x \ \forall x \varepsilon E$
- (ii) $h_{1,0}(x) \equiv x$
- (iii) For every x, $h_{c,d}(x)$ is continuous in c and d.
- (iv) For every $x \varepsilon E$ and c, $h_{c,d}(x) \to \infty$ as $d \to \infty$ and/or for every $x \varepsilon E$ and d, $h_{c,d}(x) \to \infty$ as $c \to \infty$.

Again for ease of notation it will be convenient to simply write $h(\cdot)$ rather than $h_{c,d}(\cdot)$. For each $h \varepsilon H$ define

R(h): Select $\pi_i \Leftrightarrow h(T_i) \geq \max\{T_{[k]}, h(T_{[k-m+1]})\}$

Gupta-Panchapakesan [33] prove that (1.6.2) are sufficient to guarantee the existence of a rule in H satisfying any given P^* requirement (1.1.5) in the subset selection case (m=k). The following example shows the typical behavior in the restricted subset selection case.

Example 1.6.1. Suppose each π_i, i=1,...,k is characterized by θ_i and one observation, X_i, is taken from π_i which has cdf $F(y/\theta_i) = F(y-\theta_i)$. Let

$$H = \{h_d: R^1 \to R^1 \mid h_d(x) = x+d, d\varepsilon[0,\infty]\}$$

and suppose $\delta > 0$ is given.

Goal: Given P*,m and δ find a rule R(h), hεH satisfying

$$P_{\underset{\sim}{\theta}}[CS|R(h)] \geq P^* \ \forall\ \underset{\sim}{\theta}\varepsilon\Omega(\delta)=\{\underset{\sim}{\theta}|\theta_{[k]}-\theta_{[k-1]} \geq \delta\}.$$

Now for each hεH, R(h) is defined by

R(h): Select $\pi_i \Leftrightarrow X_i \geq \max\{X_{[k-m+1]}, X_{[k]}-d\}$

It can easily be seen that the maximum P* attainable in the class of rules $\{R(h)|h\varepsilon H\}$, P^*_{max}, can be computed as

$$P^*_{max} = \sup_{d\geq 0}\ \inf_{\Omega(\delta)} P[CS|R(n)]$$

$$= (k-m)\binom{k-1}{k-m}\int_{-\infty}^{\infty}\{1-F(w-\delta)\}\{F(w)\}^{k-m-1}\{1-F(w)\}^{m-1}dF(w)$$

Two difficulties should be noted

(i) If it is desired to meet the P* condition on all of Ω(i.e.δ=0)

$$\Rightarrow P^*_{max}=(k-m)\binom{k-1}{k-m}\int_{-\infty}^{\infty}\{F(w)\}^{k-m-1}\{1-F(w)\}^{m}dF(w)$$

=m/k is the maximum attainable P* level and this is achieved for the fixed size subset rule which chooses the m populations corresponding to the m largest order statistics.

(ii) Even if a preference zone is specified (δ>0), since $1-F(w-\delta) < 1 \forall w$

$$\Rightarrow P^*_{max} < (k-m)\binom{k-1}{k-m}\int_{-\infty}^{\infty}(1)\{F(w)\}^{k-m-1}\{1-F(w)\}^{m-1}dF(w)=1.$$

So even in this case not all probability levels can be achieved over the preference zone.

CHAPTER II

APPLICATIONS OF RESTRICTED SUBSET SELECTION PROCEDURES

2.1 Selection of the Normal Population with Largest Mean

Because of the importance of normal theory in practical applications this problem will be discussed in some detail and a set of tables provided to facilitate the use of the proposed procedure. From a formal point of view the goals in Sections 1.1 and 1.6 can be combined into a general goal for which the subset selection approach and the indifference zone approach become special cases. Also the rules proposed by Bechhofer [8], Gupta [20,26] and Desu and Sobel [15] are special cases of the proposed rule, R(n).

Let $\pi_i \sim N(\mu_i, \sigma^2)$ for i=1,...,k and suppose the common σ^2 is known. Each population is characterized by its mean and if $\mu_{[1]} \leq \cdots \leq \mu_{[k]}$ are the ordered means and $\pi_{(i)}$ is the populations with mean $\mu_{[i]}$ then the best population is $\pi_{(k)}$. For $d > 0$ and $n \geq 1$ define

(2.1.1) R(n): Select $\pi_i \Leftrightarrow \bar{X}_i \geq \max\{\bar{X}_{[k-m+1]}, \bar{X}_{[k]} - d\sigma/\sqrt{n}\}$

where $\bar{X}_i$ is the sample mean from π_i. In the terminology of Section 1.1 $T_{in} = \bar{X}_i \overset{p}{\to} \mu_i = \lambda_i$ and the goal reduces to

Goal G': Given P*, $\delta > 0$ and $\{R(n)\}$ (i.e. m and d) find the smallest n so that

(2.1.2) $$P_{\underset{\sim}{\mu}}[CS|R(n)] \geq P^* \;\forall\; \underset{\sim}{\mu}\varepsilon\Omega(\delta) = \{\underset{\sim}{\mu}|\mu_{[k]}-\mu_{[k-1]} \geq \delta\}.$$

Theorem 2.1.1.

(2.1.3) $$\inf_{\Omega(\delta)} P_{\underset{\sim}{\mu}}[CS|R(n)] = \int_{-\infty}^{\infty} \{\Phi(y+d+\frac{\sqrt{n}\delta}{\sigma})\}^{k-1} I\left(\frac{\Phi(y+\frac{\sqrt{n}\delta}{\sigma})}{\Phi(y+d+\frac{\sqrt{n}\delta}{\sigma})};k-m,m\right) d\Phi(y)$$

From Example 1.3.1 dealing with the location parameter family or directly by an application of the dominated convergence theorem the following equalities can be verified.

(2.1.4) (i) $$\lim_{n\to\infty} \inf_{\Omega(\delta)} P_{\underset{\sim}{\mu}}[CS|R(n)] = 1$$

(2.1.5) (ii) $$\lim_{\delta\to\infty} \inf_{\Omega(\delta)} P_{\underset{\sim}{\mu}}[CS|R(n)] = 1$$

(2.1.6) (iii) $$\lim_{d\to\infty} \inf_{\Omega(\delta)} P_{\underset{\sim}{\mu}}[CS|R(n)] = \sup_{d\geq 0} \inf_{\Omega(\delta)} P_{\underset{\sim}{\mu}}[CS|R(n)]$$

$$= (k-m)\binom{k-1}{k-m}\int_{-\infty}^{\infty} \{1-\Phi(y-\frac{\sqrt{n}\delta}{\sigma})\}\{\Phi(y)\}^{k-m-1}\{1-\Phi(y)\}^{m-1} d\Phi(y)$$

Note that for a fixed sample size, any P* level in (2.1.2) can be met by choosing δ sufficiently large, however not all (P^*,δ) goals can be met by selecting d sufficiently large. In particular, when attempting to meet a subset selection type goal the difficulty of example (1.6.1) arises i.e. $\sup_{d\geq 0} \inf_{\Omega} P_{\underset{\sim}{\mu}}[CS|R(n)] = m/k$.

Since all or some of n,d and δ may be flexible within bounds in a practical problem, the following more general goal may be required in certain situations.

<u>Goal G</u>: Given P^*,m and also possibly n and/or $\delta \geq 0$ define a procedure R(n) based on a sample of common size n from each population which selects a subset of populations containing $\pi_{(k)}$, does not exceed m in size and yet satisfies (2.1.2).

The proposed procedure is (2.1.1) where both n and d are constants to be determined.

<u>Remark 2.1.1</u>. Equation (2.1.6) shows that it may be impossible to attain (2.1.2) for certain combinations of P^*,n and δ. In this case the requirements on n and δ must be relaxed.

<u>Remark 2.1.2</u>. The following previously studied cases are specializations of the goal G and rule R(n).

(i) Given: P^*, m=k, δ=0 and n

⇒ <u>G</u>: Define a procedure R which selects a subset of $\{\pi_1,\ldots,\pi_k\}$ containing $\pi_{(k)}$ and satisfies $P_{\underset{\sim}{\mu}}[CS|R] \geq P^* \; \forall \underset{\sim}{\mu}\varepsilon\Omega$. The proposed rule reduces to

<u>R</u>: Select $\pi_i \Leftrightarrow \bar{X}_i \geq \bar{X}_{[k]} - d\sigma/\sqrt{n}$

This goal and procedure were studied by Gupta [20,26].

(ii) Given: P^*, m=1, $\delta > 0$

⇒ <u>G</u>: Define a sequence of rules {R(n)} each of which chooses a single population and find the smallest n so that $P_{\underset{\sim}{\mu}}[CS|R(n)] \geq P^* \forall \; \underset{\sim}{\mu}\varepsilon\Omega(\delta)$. The rule R(n) reduces to

R(n): Select π_i corresponding to $\bar{X}_{[k]}$.

Bechhofer [8] studied this goal and procedure.

Also note that the rule proposed by Desu and Sobel [15] is a special case of R(n) for the choice $m=s(1 < s < k)$ and $d=+\infty$. All the results obtained for these three rules are special cases of the general results for R(n). In particular (2.1.3) reduces to give the results of:

(i) Bechhofer (m=1)

$$\inf_{\Omega(\delta)} P_{\underset{\sim}{\mu}}[CS|R(n)] = \int_{-\infty}^{\infty} \{\Phi(y + \frac{\sqrt{n}\delta}{\sigma})\}^{k-1} d\Phi(y)$$

(ii) Desu and Sobel ($1 < m < k$, $d=+\infty$)

$$\inf_{\Omega(\delta)} P_{\underset{\sim}{\mu}}[CS|R(n)] = \int_{-\infty}^{\infty} \sum_{p=k-m}^{k-1} \binom{k-1}{p} \{\Phi(y + \frac{\sqrt{n}\delta}{\sigma})\}^{p} \cdot \{1-\Phi(y + \frac{\sqrt{n}\delta}{\sigma})\}^{k-1-p} d\Phi(y)$$

$$= (k-m)\binom{k-1}{k-m} \int_{-\infty}^{\infty} \{1-\Phi(y-\frac{\sqrt{n}\delta}{\sigma})\}\{\Phi(y)\}^{k-m-1}\{1-\Phi(y)\}^{m-1} d\Phi(y)$$

(iii) Gupta (m=k, d > 0)

$$\inf_{\Omega} P_{\underset{\sim}{\mu}}[CS|R] = \int_{-\infty}^{\infty} \{\Phi(y+d)\}^{k-1} \sum_{p=0}^{k-1} \binom{k-1}{p} \{\frac{\Phi(y)}{\Phi(y+d)}\}^{p} \{1-\frac{\Phi(y)}{\Phi(y+d)}\}^{k-1-p} d\Phi(y)$$

$$= \int_{-\infty}^{\infty} \{\Phi(y+d)\}^{k-1} d\Phi(y)$$

Since {R(n)} is of the form (1.1.4) with $h_n(x) = x + d\sigma/\sqrt{n}$ satisfying (1.1.3) and all the other hypotheses of Section 1.1 are satisfied, the general theory gives the following results:

A. For any $i=1,\ldots,k$ and any n, R(n) is strongly monotone in $\pi_{(i)}$.

B. If $p^n_{\underset{\sim}{\mu}}(i) = P_{\underset{\sim}{\mu}}[R(n) \text{ selects } \pi_{(i)}]$ and $S(n)$ is defined as in Section 1.4 then $\forall \underset{\sim}{\mu} \varepsilon \Omega$ such that $\mu_{[k]} > \mu_{[k-1]}$

$$\text{(1)} \quad p^n_{\underset{\sim}{\mu}}(i) \to \begin{cases} 1 , & i = k \\ 0 , & 1 \le i < k \end{cases} \quad \text{as } n \to \infty$$

(2) $S(n) \to 1$ in probability as $n \to \infty$

C. For any $\underset{\sim}{\mu} \varepsilon \Omega$

$$E_{\underset{\sim}{\mu}}[S(n)] = \sum_{i=1}^{k} \sum_{p=k-m}^{k-1} \sum_{\nu=1}^{\binom{k-1}{p}} \int_{-\infty}^{\infty} \prod_{j \varepsilon \mathscr{P}^p_\nu(i)} \Phi(y+\frac{\sqrt{n}}{\sigma}(\mu_{[i]}-\mu_{[j]})) \times$$

$$\prod_{j \varepsilon \bar{\mathscr{P}}^p_\nu(i)} \{\Phi(y+d+\frac{\sqrt{n}}{\sigma}(\mu_{[i]}-\mu_{[j]}))-\Phi(y+\frac{\sqrt{n}}{\sigma}(\mu_{[i]}-\mu_{[j]}))\} d\Phi(y)$$

D. $E_{\underset{\sim}{\mu}}[S(n)]$ is $\uparrow$ in $\mu_{[1]}$ on $(-\infty, \mu_{[2]}]$

E. $\sup_{\Omega} E_{\mu}[S(n)] = k\int_{-\infty}^{\infty} \{\Phi(y+d)\}^{k-1} I(\frac{\Phi(y)}{\Phi(y+d)}; k-m, m) \; d\Phi(y)$

Some short computations also show that

F. $\sup_{\Omega} E_{\mu}[S(n)]$ is non decreasing in d.

G. If $n(d)$ is the sample size needed for the rule $R(n)$ in (2.1.1) to attain a fixed (P^*, δ) requirement, then $n(d)$ is non increasing in d.

Table I has been prepared for the purpose of implementing $\{R(n)\}$ and it lists the values of $\sqrt{n}\delta/\sigma$ satisfying

$$\text{(2.1.7)} \quad P^* = \int_{-\infty}^{\infty} \{\Phi(y+d+\frac{\sqrt{n}\delta}{\sigma})\}^{k-1} I\left(\frac{\Phi(y+\frac{\sqrt{n}\delta}{\sigma})}{\Phi(y+d+\frac{\sqrt{n}\delta}{\sigma})}; k-m, m\right) \; d\Phi(y)$$

for various P^*, k, m and d values.

Table I

Lists the value of $\frac{\sqrt{n}\delta}{\sigma}$ needed to attain P* levels .75, .90 and .975 for the rules given by d = .4, .8, 1.2 and 1.6 for various k and m.

	d	.4			.8		
		P*			P*		
k	m	.75	.90	.975	.75	.90	.975
3	2	1.070	1.859	2.750	.788	1.561	2.436
4	2	1.335	2.093	2.952	1.098	1.836	2.680
	3	1.287	2.057	2.932	.916	1.678	2.537
5	2	1.510	2.252	3.096	1.297	2.019	2.832
	3	1.454	2.204	3.063	1.106	1.845	2.688
	4	1.447	2.201	3.060	1.055	1.805	2.664
6	2	1.639	2.370	3.198	1.442	2.153	2.958
	3	1.577	2.319	3.163	1.248	1.970	2.798
	4	1.569	2.312	3.155	1.183	1.921	2.765
	5	1.569	2.311	3.155	1.170	1.913	2.756
7	2	1.739	2.461	3.282	1.556	2.257	3.054
	3	1.676	2.406	3.242	1.360	2.074	2.887
	4	1.664	2.398	3.234	1.288	2.015	2.843
	5	1.663	2.397	3.233	1.267	2.002	2.838
8	2	1.822	2.537	3.349	1.647	2.342	3.131
	3	1.756	2.480	3.308	1.452	2.157	2.962
	4	1.742	2.473	3.301	1.375	2.093	2.914
	5	1.741	2.472	3.300	1.348	2.075	2.903
9	2	1.892	2.603	3.408	1.724	2.415	3.196
	3	1.824	2.543	3.363	1.530	2.229	3.026
	4	1.810	2.533	3.361	1.449	2.160	2.972
	5	1.808	2.531	3.359	1.418	2.137	2.957
10	2	1.953	2.656	3.460	1.791	2.478	3.252
	3	1.882	2.597	3.410	1.596	2.290	3.079
	4	1.868	2.587	3.407	1.512	2.219	3.024
	5	1.864	2.583	3.403	1.479	2.193	3.006
15	2	2.169	2.860	3.641	2.027	2.698	3.464
	3	2.094	2.793	3.590	1.837	2.513	3.278
	4	2.074	2.778	3.582	1.744	2.431	3.212
	5	2.069	2.776	3.581	1.699	2.397	3.193
20	2	2.309	2.992	3.766	2.179	2.843	3.593
	3	2.232	2.924	3.705	1.991	2.659	3.417
	4	2.209	2.905	3.702	1.898	2.573	3.347
	5	2.203	2.903	3.699	1.848	2.533	3.314

Table I (cont.)

	d	1.2			1.6		
		P*			P*		
k	m	.75	.90	.975	.75	.90	.975
3	2	.590	1.351	2.211	.462	1.223	2.075
4	2	.954	1.684	2.504	.875	1.602	2.414
	3	.599	1.345	2.189	.359	1.093	1.921
5	2	1.181	1.892	2.696	1.123	1.830	2.627
	3	.841	1.560	2.380	.669	1.376	2.173
	4	.686	1.429	2.272	.375	1.101	1.929
6	2	1.340	2.041	2.830	1.295	1.994	2.775
	3	1.016	1.719	2.516	.881	1.572	2.353
	4	.849	1.572	2.400	.597	1.300	2.097
	5	.783	1.521	2.365	.428	1.155	1.983
7	2	1.464	2.157	2.939	1.426	2.115	2.897
	3	1.153	1.844	2.633	1.039	1.722	2.488
	4	.980	1.687	2.499	.768	1.456	2.237
	5	.896	1.623	2.451	.588	1.295	2.100
8	2	1.565	2.251	3.024	1.530	2.213	2.987
	3	1.261	1.946	2.728	1.164	1.840	2.597
	4	1.087	1.786	2.583	.904	1.584	2.349
	5	.995	1.708	2.520	.722	1.413	2.202
9	2	1.648	2.329	3.095	1.617	2.297	3.062
	3	1.353	2.033	2.799	1.266	1.934	2.692
	4	1.178	1.868	2.649	1.016	1.688	2.446
	5	1.080	1.783	2.588	.836	1.518	2.291
10	2	1.719	2.397	3.163	1.691	2.367	3.132
	3	1.431	2.105	2.870	1.352	2.018	2.768
	4	1.256	1.940	2.721	1.110	1.778	2.528
	5	1.153	1.851	2.647	.933	1.607	2.372
15	2	1.969	2.637	3.387	1.949	2.613	3.363
	3	1.704	2.365	3.107	1.650	2.303	3.037
	4	1.537	2.201	2.951	1.435	2.083	2.818
	5	2.428	2.100	2.866	1.273	1.923	2.666
20	2	2.130	2.788	3.530	2.113	2.769	3.511
	3	1.879	2.529	3.263	1.834	2.478	3.205
	4	1.716	2.368	3.110	1.635	2.276	3.003
	5	1.607	2.265	3.015	1.484	2.122	2.849

All computations were made on a CDC 6500 using Gauss Hermite quadrature based on twenty nodes to perform the numerical integration. Checks on the accuracy of the program for m=1 showed that the algorithm produced values which were off by at most one or two digits in the third place. The author assumes full responsibility for all tables constructed in the thesis.

In general, given P* and δ there will be many choices of d and n satisfying (2.1.7). Since each R(n) will always select at most m populations no matter which d (≥ 0) is used, the following list of possible criteria is proposed for choosing d.

A. A Minimax Approach. Consider the following three loss functions.

(i) $L_1(\underset{\sim}{\mu},d) = E_{\underset{\sim}{\mu}}[S(n)]$ where R(n) is based on d. $L_1(\cdot,\cdot)$ is an appropriate loss function if the cost of taking additonal observations from a population is cheap given that some have already been taken. In this case the loss can be considered solely due to selecting a large number of populations. Since $\sup_{\Omega} L_1(\underset{\sim}{\mu},d) = \sup_{\Omega} E_{\underset{\sim}{\mu}}[S(n)]$ is $\uparrow$ in d,

$\Rightarrow \inf_{0\leq d\leq\infty} \sup_{\Omega} L_1(\underset{\sim}{\mu},d) = 1$ and is achieved at d=0. Hence the minimax rule is:

$R_1(n)$: Select π_i corresponding to $\bar{X}_{[k]}$.

Remark 2.1.3. In some cases it seems more appropriate to let $L_1(\underset{\sim}{\mu},d) = E_{\underset{\sim}{\mu}}[T(n)]$ where T(n) is the number of non best populations selected rather than $L_1(\underset{\sim}{\mu},d) = E_{\underset{\sim}{\mu}}[S(n)]$. It can be shown that the same rules are picked using both loss functions.

(ii) $L_2(\underset{\sim}{\mu},d) = e(P^*,\delta,m,d) = n(d)/n(\infty)$ where $n(d)$ is the solution of (2.1.7). L_2 is the ratio of the sample sizes of the rules using d and ∞ needed to attain the same probability level. It is appropriate if the experimenter only cares to keep the cost of the total number of observations small since he knows $1 \leq S(n) \leq m$ holds. Since

$$\sup_{\Omega} L_2(\underset{\sim}{\mu},d) = n(d)/n(\infty) \text{ is } \downarrow \text{ in } d$$

$$\Rightarrow \inf_{0 \leq d \leq \infty} \sup_{\Omega} L_2(\underset{\sim}{\mu},d) = 1 \text{ and is achieved at } d=\infty.$$

Hence the minimax rule is

$\underline{R_2(n)}$: Select the populations corresponding to $\bar{X}_{[k-m+1]},\ldots,$ $\bar{X}_{[k]}$.

(iii) $L_3(\underset{\sim}{\mu},d) = f(E_{\underset{\sim}{\mu}}[S(n)],\ n(d))$ where $f(x,y)$ is non decreasing in x and y. L_3 penalizes the experimenter for both large sample sizes and large expected subset sizes.

$$(2.1.8) \quad \sup_{\Omega} L_3(\underset{\sim}{\mu},d) = f(k\int_{-\infty}^{\infty} \{\Phi(y+d)\}^{k-1} I(\frac{\Phi(y)}{\Phi(y+d)};k-m,m)\,d\Phi(y),\ n(d))$$

For any given P^*,δ,m and k (2.1.8) can be minimized using numerical techniques to determine the minimax rule.

B. <u>An ϵ Minimax Approach</u>. The objective is to choose rules which are less conservative than the minimax rules of (A) in that they take advantage of the subset selection nature of R(n) for $d \in (0,\infty)$.

(i) Choose d^* so that

$$\sup_{\Omega} L_1(\underset{\sim}{\mu},d^*) = \inf_{0 \leq d \leq \infty} \sup_{\Omega} L_1(\underset{\sim}{\mu},d) + \epsilon = 1 + \epsilon$$

$\Rightarrow$ d* satisfies

$$\int_{-\infty}^{\infty} \{\Phi(y+d^*)\}^{k-1} I\left(\frac{\Phi(y)}{\Phi(y+d^*)};k-m,m\right) d\Phi(y) = (1+\epsilon)/k$$

(ii) Choose d* to satisfy $e(P^*,\delta,m,d^*) = 1+\epsilon$

$\Rightarrow$ choose d* $\ni n(d^*) = n(\infty)[1+\epsilon]$. Table II shows that even for small values of d, $e(P^*,\delta,m,d)$ is close to one indicating only a slight additional cost for using the rule defined by d. On the other hand the savings realized by using R(n) over the fixed subset size rule ($d=+\infty$) is measured by $(m-E_{\underset{\sim}{\mu}}[S(n)])$. This difference, of course, depends on the unknown $\underset{\sim}{\mu}$. To investigate this quantity numerically Tables III and IV have been constructed which list $E_{\underset{\sim}{\mu}}[S(n)]$ under the following configurations:

(a) Equispaced Means $\Rightarrow$ $\mu = (\alpha,\alpha+\delta,\ldots,\alpha+(k-1)\delta)$.

For certain values of P*,d, $\sqrt{n}\delta/\sigma$ and k, Table III lists

$$E_{\underset{\sim}{\mu}}[S(n)] = \sum_{i=1}^{k} \sum_{p=k-m}^{k-1} \sum_{\nu=1}^{\binom{k-1}{p}} \int_{-\infty}^{\infty} \prod_{j\in\mathcal{P}_\nu(i)} \Phi\left(y+\frac{\sqrt{n}}{\sigma}(i-j)\delta\right) \times$$

$$\prod_{j\in\bar{\mathcal{P}}_\nu(i)} \left\{\Phi\left(y+d+\frac{\sqrt{n}}{\sigma}(i-j)\delta\right)-\Phi\left(y+\frac{\sqrt{n}}{\sigma}(i-j)\delta\right)\right\}d\Phi(y).$$

(b) Slippage $\Rightarrow$ $\underset{\sim}{\mu} = (\alpha,\ldots,\alpha,\alpha+\delta)$

Again for certain P*,d, $\sqrt{n}\delta/\sigma$,k and m values Table IV gives

$$E_{\underset{\sim}{\mu}}[S(n)] = \sum_{p=k-m}^{k-1} \binom{k-1}{p} \int_{-\infty}^{\infty} \left\{\Phi\left(y+\frac{\sqrt{n}\delta}{\sigma}\right)\right\}^{p} \left\{\Phi\left(y+d+\frac{\sqrt{n}\delta}{\sigma}\right)-\Phi\left(y+\frac{\sqrt{n}\delta}{\sigma}\right)\right\}^{k-1-p} d\Phi(y)$$

$$+(k-1)\sum_{p=k-m}^{k-1} \binom{k-2}{p-1} \int_{-\infty}^{\infty} \left\{\Phi\left(y-\frac{\sqrt{n}\delta}{\sigma}\right)\right\}\{\Phi(y)\}^{p-1}\{\Phi(y+d)-\Phi(y)\}d\Phi(y)$$

Table II

This table lists $n(d)/n(\infty)$ where $n(a)$ is the sample size necessary for the rule $R(n)$: Select $\pi_i \Leftrightarrow \bar{X}_i \geq \max\{\bar{X}_{[k-m+1]}, \bar{X}_{[k]} - a\sigma/\sqrt{n}\}$ to satisfy $P_{\mu}[CS/R(n)] \geq P^* \forall \underset{\sim}{\mu} \in \Omega(\delta)$

	d	.4		.8		1.2		1.6	
		P*							
h	m	.90	.975	.90	.975	.90	.975	.90	.975
3	2	2.90	1.98	2.04	1.55	1.53	1.28	1.25	1.12
4	2	1.84	1.57	1.42	1.29	1.19	1.13	1.08	1.05
	3	7.59	3.46	5.05	2.59	3.24	1.93	2.14	1.49
5	2	1.58	1.43	1.27	1.20	1.11	1.08	1.04	1.03
	3	3.39	3.30	2.38	1.83	1.70	1.44	1.32	1.20
	4	17.38	5.24	11.69	3.97	7.32	2.89	4.35	2.08
6	2	1.46	1.36	1.20	1.16	1.08	1.06	1.03	1.02
	3	2.55	2.02	1.84	1.58	1.40	1.28	1.17	1.12
	4	5.67	3.28	3.91	2.52	2.62	1.90	1.79	1.45
7	2	1.38	1.31	1.16	1.13	1.06	1.05	1.02	1.02
	3	2.17	1.84	1.61	1.46	1.26	1.21	1.11	1.08
	4	3.81	2.66	2.67	2.05	1.89	1.59	1.40	1.27
8	2	1.34	1.28	1.14	1.12	1.05	1.04	1.02	1.02
	3	1.97	1.72	1.49	1.38	1.21	1.17	1.08	1.06
	4	3.08	2.34	2.20	1.83	1.60	1.43	1.26	1.19
9	2	1.30	1.25	1.12	1.10	1.04	1.03	1.01	1.01
	3	1.83	1.64	1.41	1.33	1.17	1.14	1.06	1.05
	4	2.67	2.16	1.94	1.69	1.45	1.34	1.19	1.14

Table III

Using the rule R(n) and under the configuration (α, $\alpha+\delta$,...,$\alpha+(k-1)\delta$) this table gives in order the triple a) the expected number of selected populations, b) the expected sum of ranks of the selected populations and c) the expected proportion of selected populations ((a) divided by m)

Number of Populations Studied

m	d \ $\frac{\sqrt{n}}{\sigma}\delta$					
		k = 3				
2	.4	1.3111	1.2800	1.2237	1.1649	1.1156
		2.5300	2.1262	1.7606	1.4906	3.3121
		0.6555	0.6400	0.6118	0.5825	0.5578
	.7	1.5039	1.4588	1.3751	1.2839	1.2038
		2.9134	2.4731	2.0451	1.7073	1.4698
		0.7520	0.7294	0.6875	0.6420	0.6019
		k = 4				
2	.4	1.3619	1.3090	1.2316	1.1660	1.1157
		3.2056	2.3924	1.8184	1.4971	1.3124
		0.6810	0.6545	0.6158	0.5830	0.5578
	.7	1.5691	1.4972	1.3862	1.2855	1.2039
		3.7113	2.8056	2.1237	1.7172	1.4704
		0.7845	0.7486	0.6931	0.6427	0.6020
3	.4	1.4391	1.3629	1.2568	1.1750	1.1183
		3.3970	2.5213	1.8765	1.5173	1.3183
		0.4797	0.4543	0.4189	0.3917	0.3728
	.7	1.7789	1.6483	1.4611	1.3139	1.2126
		4.2343	3.1766	2.3037	1.7845	1.4910
		0.5930	0.5494	0.4870	0.4380	0.4042
		k = 5				
2	.4	1.3956	1.3208	1.2326	1.1660	1.1157
		3.8362	2.5299	1.8277	1.4973	1.3124
		0.6978	0.6604	0.6163	0.5830	0.5578
	.7	1.6097	1.5119	1.3875	1.2855	1.2039
		4.4502	2.9794	2.3170	1.7175	1.4704
		0.8048	0.7560	0.6938	0.6428	0.6020
3	.4	1.4995	1.3845	1.2588	1.1751	1.1183
		4.1402	2.6964	1.8893	1.5176	1.3183
		0.4998	0.4615	0.4196	0.3917	0.3728
	.7	1.8785	1.6862	1.4650	1.3140	1.2125
		5.2408	3.4475	2.3276	1.7851	1.4910
		0.6262	0.5621	0.4884	0.4380	0.4042
4	.4	1.5165	1.3920	1.2601	1.1752	1.1183
		4.1910	2.7184	1.8932	1.5180	1.3183
		0.3791	0.3480	0.3150	0.2938	0.2796
	.7	1.9571	1.7230	1.4724	1.3148	1.2126
		5.4774	3.5593	2.3499	1.7875	1.4912
		0.4893	0.4308	0.3681	0.3287	0.3031

Table IV

Using the rule R(n) and under the configuration $(\alpha,\alpha,\ldots,\alpha+\delta)$ the table gives in order the triple a) the expected number of selected populations, b) the expected sum of ranks of the selected populations and c) the expected proportion of selected populations ((a) divided by m)

Number of Populations Studied

m	d \ $\frac{\sqrt{n}}{\sigma}\delta$	.10	.50	.90	1.30	1.70
			k = 3			
2	.4	1.3120	1.2996	1.2702	1.2270	1.1766
		2.5773	2.3611	2.1156	1.8627	1.6259
		0.6560	0.6498	0.6351	0.6135	0.5883
	.7	1.5052	1.4872	1.4437	1.3783	1.3003
		2.9629	2.7352	2.4657	2.1740	1.8861
		0.7526	0.7436	0.7219	0.6892	0.6502
			k = 4			
2	.4	1.3641	1.3529	1.3243	1.2792	1.2233
		3.3491	3.0598	2.7192	2.3554	2.0028
		0.6821	0.6765	0.6622	0.6396	0.6116
	.7	1.5720	1.5568	1.5169	1.4523	1.3696
		3.8654	3.5571	3.1877	2.7804	2.3685
		0.7860	0.7784	0.7585	0.7261	0.6848
3	.4	1.4423	1.4266	1.3877	1.3288	1.2583
		3.5441	3.2426	2.8768	2.4792	2.0908
		0.4808	0.4755	0.4626	0.4429	0.4194
	.7	1.7844	1.7578	1.6920	1.5915	1.4701
		4.3959	4.0606	4.6299	3.1363	2.6292
		0.5948	0.5859	0.5640	0.5305	0.4900
			k = 5			
2	.4	1.3993	1.3893	1.3622	1.3172	1.2587
		4.1254	4.7752	3.3491	2.8800	2.4125
		0.6997	0.6947	0.6811	0.6586	0.6294
	.7	1.6145	1.6015	1.5653	1.5033	1.4198
		4.7653	4.3894	3.9297	3.4130	2.8799
		0.8072	0.8007	0.7827	0.7516	0.7099
3	.4	1.5055	1.4904	1.4512	1.3887	1.3108
		4.4422	4.0725	3.6089	3.0886	2.5649
		.5018	.4968	.4837	.4629	.4369
	.7	1.8882	1.8635	1.7988	1.6946	1.5627
		5.5835	5.1660	4.6218	3.9837	3.3107
		0.6294	0.6212	0.5996	0.5649	0.5209
4	.4	1.5230	1.5067	1.4649	1.3990	1.3177
		4.4949	4.1216	3.6502	3.1198	2.5859
		0.3808	0.3767	0.3662	0.3498	0.3294
	.7	1.9692	1.9392	1.8631	1.7437	1.5964
		5.8267	5.3950	4.8181	4.1356	3.4161
		0.4923	0.4848	0.4658	0.4359	0.3991

$$+(k-1)\sum_{p=k-m}^{k-2}\binom{k-2}{p}\int_{-\infty}^{\infty}\{\Phi(y)\}^{p}\{\Phi(y+d-\frac{\sqrt{n}\delta}{\sigma})-\Phi(y-\frac{\sqrt{n}\delta}{\sigma})\}\times$$

$$\{\Phi(y+d)-\Phi(y)\}^{k-2-p}\,d\Phi(y)$$

The same two tables also list

(1) The expected sum of the ranks of the selected populations

$$=\sum_{i=1}^{k}(k+1-i)p_{\underset{\sim}{\mu}}^{n}(i)$$

(2) The expected proportion of selected populations $= E_{\underset{\sim}{\mu}}[S(n)]/m$ for their respective underlying configurations.

A similar choice of d* can be made corresponding to $L_3(\underset{\sim}{\mu},d)$.

2.2 Selection from Gamma Populations for Scale Parameters

The object in this section is to formulate a restricted subset selection procedure for selecting the normal distribution with smallest variance. The problem for selecting in terms of the largest variance is analagous. The more general problem of selecting the gamma population with the smallest scale parameter will be studied first.

Suppose $\pi_i \sim \Gamma(r,\lambda_i)$ and that π_i is characterized by $\lambda_i (\Rightarrow \Lambda = (0,\infty))$. Furthermore suppose iid random variables $\{X_{ij}\}_{j=1}^{n}$ are observed from π_i and hence X_{ij} has cdf

$$F_i(y) = E_r\left(\frac{y}{\lambda_i}\right) = \int_0^{y/\lambda_i}\frac{x^{r-1}e^{-x}}{\Gamma(r)}\,dx$$

It is assumed r is known. Let

$0 < \lambda_{[1]} \leq \cdots \leq \lambda_{[k]} < \infty$ be the ordered λ_i's ,

(2.2.1) $T_{in} = \frac{1}{n} \sum_{j=1}^{n} X_{ij} = \bar{X}_i$,

$d > 1$ and

(2.2.2) $R(n)$: Select $\pi_i \Leftrightarrow \bar{X}_i \leq \min\{d^{1/n} \bar{X}_{[1]}, \bar{X}_{[m]}\}$.

The object is to select $\pi_{(1)}$, the population with parameter $\lambda_{[1]}$.

<u>Goal G'</u>: Given P*, $\delta\varepsilon(0,1)$, $\{R(n)\}$ (i.e. $d > 1$ and m) find the smallest n such that:

(2.2.3) $$P_{\underset{\sim}{\lambda}}[CS|R(n)] \geq P^* \ \forall \underset{\sim}{\lambda}\varepsilon\Omega(\delta) = \{\underset{\sim}{\lambda}|\lambda_{[1]} \leq \delta\lambda_{[2]}\}.$$

The event $[CS|R(n)]$ stands for the selection of any restricted subset containing $\pi_{(1)}$.

<u>Theorem 2.2.1.</u>

(2.2.4) $$\inf_{\Omega(\delta)} P_{\underset{\sim}{\lambda}}[CS|R(n)] = \int_0^\infty \{1-E_{nr}(yd^{-1/n}\delta)\}^{k-1} I\left(\frac{1-E_{nr}(y\delta)}{1-E_{nr}(yd^{-1/n}\delta)}; k-m,m\right) dE_{nr}(y)$$

<u>Proof</u>. The proof follows from the observation that $G_n(y/\lambda_i) = E_{nr}(\frac{ny}{\lambda_i})$.

Analagous to the normal means procedure

(2.2.5) (i) $$\lim_{\delta\to 0} \inf_{\Omega(\delta)} P[CS|R(n)] = 1$$

(2.2.6) (ii) $$\lim_{d\to\infty} \inf_{\Omega(\delta)} P[CS|R(n)] = (k-m)\binom{k-1}{k-m} \int_0^\infty [E_{nr}(w/\delta)][E_{nr}(w)]^{m-1} [1-E_{nr}(w)]^{k-m-1} dE_{nr}(w)$$

which follows the usual pattern established for formulation I and II in Section 1.6.

As remarked in Section 2.1, in a practical problem where some of d,δ and n are flexible within certain bounds, the goal of interest may really be:

Goal G: Given P^*, m and also possibly n and/or $0 < \delta \leq 1$ define a procedure $R(n)$ based on a sample of common size n from each population which selects at most m populations, which contains $\pi_{(1)}$ and which satisfies (2.2.3).

In this case, the proposed procedure $R(n)$ is just (2.2.2) where $d > 1$ and n are constants to be determined.

Remark 2.2.1. Equation (2.2.6) shows that it may be impossible to satisfy (2.2.3) for certain combinations of P^*,n and δ. In this case the requirements of the experiment on some (or all) of δ,n and P^* must be relaxed.

G and $R(n)$ reduce to the goals and rules of Bechhofer-Sobel [9] and Gupta-Sobel [23] for appropriate choices of the defining parameters.

Since $H_n(x) = d^{-1/n}x$ and $u(\lambda) = \lambda/\delta$ satisfy the regularity conditions and hypotheses of Chapter I, all the usual results hold. In particular:

(i) For any $0 < \delta < 1$, $\inf_{\Omega(\delta)} P[CS|R(n)]=\eta(1,n) \to 1$ as $n\to\infty$ since $1\varepsilon\Lambda$

(ii) For any $\underset{\sim}{\lambda}\varepsilon\Omega$ such that $\lambda_{[1]} < \lambda_{[2]}$

$$S(n) \overset{P}{\to} 1 \text{ as } n\to\infty$$

$$p^n_{\underset{\sim}{\lambda}}(i) \to \begin{cases} 1, & i = 1 \\ 0, & i > 1 \end{cases} \quad \text{as } n\to\infty$$

(iii) For any $\underset{\sim}{\lambda}\varepsilon\Omega$

$$E_{\underset{\sim}{\lambda}}[S(n)] = \sum_{i=1}^{k} \sum_{p=k-m}^{k-1} \sum_{\nu=1}^{\binom{k-1}{p}} \int_0^\infty \prod_{j\varepsilon \mathcal{P}_\nu^p(i)} \{1-E_{nr}(\frac{y\lambda_{[i]}}{\lambda_{[j]}})\} \times \prod_{j\varepsilon \bar{\mathcal{P}}_\nu^p(i)} \{E_{nr}(\frac{y\lambda_{[i]}}{\lambda_{[j]}})-E_{nr}(\frac{y\lambda_{[i]}}{d^{1/n}\lambda_{[j]}})\}dE_{nr}(y)$$

(iv) $$\sup_{\Omega} E_{\underset{\sim}{\lambda}}[S(n)]=k\int_0^\infty \{1-E_{nr}(yd^{-1/n})\}^{k-1} I(\frac{1-E_{nr}(y)}{1-E_{nr}(yd^{-1/n})};k-m,m)dE_{nr}(y)$$

Application to Selection of Variances of Normal Populations.

Let $\pi_i \sim N(\mu_i,\sigma_i^2)$ for $i = 1,\dots,k$ where μ_i is either known or unknown and σ_i^2 is unknown. Also let

(1) $0 < \sigma_{[1]}^2 \leq \dots \leq \sigma_{[k]}^2 < \infty$ be the ordered variances

(2) $$s_i^2 = \begin{cases} \frac{1}{n}\sum_{j=1}^{n} (X_{ij}-\mu_i)^2 & \text{, if } \mu_i \text{ is known} \\ \frac{1}{n-1}\sum_{j=1}^{n} (X_{ij}-\bar{X}_i)^2 & \text{, if } \mu_i \text{ is unknown} \end{cases}$$

(3) $\Omega(\delta) = \{\underset{\sim}{\sigma}^2 = (\sigma_1^2,\dots,\sigma_k^2) \mid \sigma_{[1]}^2 \leq \delta\sigma_{[2]}^2\}$ $(0 < \delta < 1)$

Choose samples from each π_i so that each s_i^2 has the same number of degrees of freedom, say ν

$$\Rightarrow \frac{\nu s_i^2}{2\sigma_i^2} \sim \Gamma(\nu/2,\ 1)$$

Goal G: Given P*,m and also possibly n and/or $0 < \delta \leq 1$ define a procedure R(n) based on the $\{s_i^2\}$ which selects at most m populations and satisfies $P_{\underset{\sim}{\sigma}^2}[CS|R(n)] \geq P^* \ \forall \underset{\sim}{\sigma}^2\varepsilon\Omega(\delta)$.

The proposed procedure is:

R(n): Select $\pi_i \Leftrightarrow s_i^2 \leq \min\{d^{1/n} s_{[1]}^2, s_{[m]}^2\}$

Theorem 2.2.2.

$$\inf_{\Omega(\delta)} P[CS|R(n)] = \int_0^\infty \{1-E_{\nu/2}(y\delta d^{-1/n})\}^{k-1} I\left(\frac{1-E_{\nu/2}(y\delta)}{1-E_{\nu/2}(y\delta d^{-1/n})}; k-m, m\right) dE_{\nu/2}(y)$$

The choice of d can be made by considerations similar to those of Section 2.1.

2.3 Selection in Terms of Generalized Variances

Suppose π_i, $i = 1,\ldots,k$, is distributed as p-variate normal with mean vector $\underset{\sim}{\mu}_i$ and covariance matrix $\underset{\sim}{\Sigma}_i$ ($\pi_i \sim N_p(\underset{\sim}{\mu}_i, \underset{\sim}{\Sigma}_i)$) where it is assumed $\underset{\sim}{\Sigma}_i$ is positive definite. For each π_i the experimenter is interested in the measure of dispersion specified by $\lambda_i = |\underset{\sim}{\Sigma}_i|$, the generalized variance of π_i. The following terminology will be used

(i) $\Omega(\mathcal{F})$ is the set of all possible k vectors of cdf's of p-variate normal random variables with positive definite variance-covariance matrices.

(ii) If $0 < \lambda_{[1]} \leq \cdots \leq \lambda_{[k]} < \infty$ are the ordered λ_i's $\Rightarrow \lambda_i \in \Lambda = (0,\infty)$

(iii) $S_{in} = \frac{1}{n-1} \sum_{j=1}^{n} (X_{ij} - \bar{X}_i)(X_{ij} - \bar{X}_i)'$ is the sample variance-covariance matrix ($n > p$ assumed).

(iv) $T_{in} = |S_{in}|$

It is known T_{in} has the same distribution as $\frac{\lambda_i}{(n-1)^p} \prod_{j=1}^{p} Y_{ij}$ where $\{Y_{ij}\}_{j=1}^p$ are independent random variables with $Y_{ij} \sim \chi^2(n-j)$.

However for any $j = 1,\ldots,p$ it is known $\chi^2(n-j)/(n-1) \xrightarrow{p} 1$ and hence $T_{in} \xrightarrow{p} \lambda_i$. So the elements of the present problem have been brought under the structure specified in Chapter I.

Assume the experimenter's interest lies in selecting $\pi_{(k)}$. For any $d > 1$ define

<u>R(n)</u>: Select $\pi_i \Leftrightarrow T_i \geq \max\{T_{[k-m+1]}, d^{-1/n} T_{[k]}\}$

As usual, the goal is to find n so that $P_{\underset{\sim}{\lambda}}[CS|R(n)] \geq P^*$ wherever $\underset{\sim}{\lambda}\varepsilon\Omega(\delta) = \{\underset{\sim}{\lambda}|\lambda_{[k]} \geq \delta\lambda_{[k-1]}\}(\delta>1)$. If

$\mathcal{L}_n(y) = P[\prod_{j=1}^{p} (\frac{Y_{ij}}{n-1}) \leq y]$ where the $\{Y_{ij}\}_{j=1}^{n}$ are as above

$\Rightarrow G_n(y/\lambda_i) = P[|S_{in}| \leq y] = \mathcal{L}_n(\frac{y}{\lambda_i})$ is a scale parameter family.

Since $h_n(x) = d^{1/n} x$ it follows that

(1) For any $n \geq 1$, $\inf_{\Omega(\delta)} P[CS|R(n)] = \int_0^\infty \{\mathcal{L}_n(yd^{1/n}\delta)\}^{k-1} I(\frac{\mathcal{L}_n(y\delta)}{\mathcal{L}_n(yd^{1/n}\delta)};k-m,m)$

$d\mathcal{L}_n(y)$

$= \psi(1,n)$

(2) $\inf_{\Omega(\delta)} P[CS|R(n)] = \psi(1,n) \to 1$ as $n\to\infty$ since $1\varepsilon\Lambda$

(3) For any $\underset{\sim}{\lambda} \ni \lambda_{[k]} > \lambda_{[k-1]} \Rightarrow S(n) \xrightarrow{p} 1$ as $n\to\infty$.

<u>Remark 2.3.1</u>. The following are special cases of the general results.

(i) $p = 1$

The univariate case is discussed in detail in Section 2.2.

(ii) $p = 2$

It is well known $2(\prod_{j=1}^{2} \chi^2(n-j))^{1/2} \sim \chi^2(2(n-2))$

$$\Rightarrow \mathcal{L}_n(y) = P[\prod_{j=1}^{2} (\frac{\chi^2(n-j)}{n-1}) \leq y] \qquad , y > 0$$

$$= P[\chi^2(2(n-2)) \leq 2(n-1)\sqrt{y}] \qquad , y > 0$$

$$= \int_0^{2(n-1)\sqrt{y}} \frac{x^{n-3}e^{-x/2}dx}{\Gamma(n-2)2^{n-2}} \qquad , y > 0$$

$$= \int_0^{\sqrt{y}} \frac{[(n-1)w]^{n-3}e^{-w(n-1)}}{\Gamma(n-2)} dw$$

$= E_n^1(\sqrt{y})$ where $E_n^1(\cdot)$ is the cdf of a gamma random variable with parameters $(n-2)$ and $1/(n-1)$.

$$\Rightarrow \inf_{\Omega(\delta)} P_{\underset{\sim}{\lambda}}[CS|R(n)] = \int_0^\infty \{E_n^1(wd^{1/2n}\sqrt{\delta})\}^{k-1} I(\frac{E_n^1(w\sqrt{\delta})}{E_n^1(wd^{1/2n}\sqrt{\delta})};k-m,m) \, dE_n^1(w)$$

(iii) $p > 2$

Since the distribution of $|S_{in}|$ can not be expressed in a reasonably simple form when $p > 2$ the following approximation suggested by Hoel [34] will be used. The distribution of $(\prod_{j=1}^{p} Y_{ij})^{1/p}$ can be approximated by a gamma random variable with density

$$e_n^2(y) = \begin{cases} \dfrac{\lambda^{\frac{p(n-p)}{2}} y^{[\frac{p}{2}(n-p)-1]} e^{-\lambda y}}{\Gamma(\frac{p}{2}(n-p))} & , y > 0 \\ 0 & , y \leq 0 \end{cases}$$

where $\lambda = (\frac{p}{2})[1 - \frac{(p-1)(p-2)}{2n}]^{1/p}$

Gnanadesikan and Gupta [19] have studied this approximation and found it to be quite accurate when n is large relative to p. For p=1 and 2 it is exact.

$$\Rightarrow \mathcal{L}_n(y) = P\left[\prod_{j=1}^{p} \left(\frac{Y_{ij}}{n-1}\right) \leq y\right] \doteq E_n^2((n-1)y^{1/p})$$

$$\text{where } E_n^2(y) = \int_0^y e_n^2(w)\,dw$$

To find the n (approximate) guaranteeing a given P* level the following equation must be solved.

$$P^* = \int_0^\infty \{E_n^2(yd^{1/np}\delta^{1/p})\}^{k-1} I\left(\frac{E_n^2(y\delta^{1/p})}{E_n^2(yd^{1/pn}\delta^{1/p})};k-m,m\right) dE_n^2(y).$$

2.4 Selection from Noncentral Chi Square Populations

Suppose π_i is a non central chi-square population with p degrees of freedom and non centrality parameter λ_i (abbreviated $\chi^2(p,\lambda_i)$). Each π_i is characterized by the scalar λ_i and it is assumed $\pi_{(1)}$ is of interest. If the random variables $\{X_{ij}\}_{j=1}^n$ are observed from π_i then each X_{ij} has cdf

(2.4.1) $F_p(x/\lambda_i) = e^{\frac{-\lambda_i}{2}} \sum_{j=0}^{\infty} \frac{\lambda_i^j}{2^j j!} E_{p+2j}(x)$, x>0 and $\lambda_i \geq 0$ where

$$E_q(x) = \int_0^x \frac{y^{\frac{q}{2}-1} e^{-y/2}}{\Gamma(q/2)2^{q/2}}\,dy.$$

$F_p(x/\lambda_i)$ is a convex mixture of central χ^2 cdf's. The following properties of this distribution are well known.

(2.4.2) $\frac{\partial F_p(x/\lambda)}{\partial \lambda} = \frac{1}{2}[F_{p+2}(x/\lambda) - F_p(x/\lambda)] = -f_{p+2}(x/\lambda)$ where

$$f_p(x/\lambda) = \frac{dF_p(x/\lambda)}{dx} \quad \text{is the density of } F_p(x/\lambda).$$

(2.4.3) $\frac{f_{p+2}(x/\lambda)}{f_p(x/\lambda)}$ is non increasing in λ.

The following lemma due to Lehmann [37] will be needed.

Lemma 2.4.1. Let $h(z) = \sum_{j=0}^{\infty} b_j z^j / \sum_{j=0}^{\infty} a_j z^j$ where the constants a_j, $b_j \geq 0$ and where $\Sigma a_j z^j$, $\Sigma b_j z^j$ both converge $\forall\, z \geq 0$. If $\{b_j/a_j\}$ is monotone in j then h(z) is a monotone function of z in the same direction.

An application of Lemma 2.4.1 shows that the following monotonicity properties hold:

(2.4.4) $\frac{\lambda f_{p+2}(x/\lambda)}{f_p(x/\lambda)}$ is non decreasing in λ for fixed x,p.

(2.4.5) $\frac{f_{p+2}(x/\lambda)}{x f_p(x/\lambda)}$ is $\begin{cases} \text{non increasing in } x \text{ for fixed } \lambda \text{ and } p. \\ \text{non increasing in } (x\lambda) \text{ for fixed } p. \end{cases}$

Following Alam and Rizvi [1] the preference zone is taken to be

$$\Omega(u) = \Omega_1 \cap \Omega_2 \text{ where}$$

$$\begin{cases} \Omega_1 = \{\underset{\sim}{\lambda} \mid \lambda_{[2]} - \lambda_{[1]} \geq \delta_1\} & (\delta_1 > 0) \\ \Omega_2 = \{\underset{\sim}{\lambda} \mid \lambda_{[2]} \geq \delta_2\, \lambda_{[1]}\} & (\delta_2 > 1) \end{cases}$$

(2.4.6) $\Rightarrow \Omega(u) = \{\underset{\sim}{\lambda} \mid \lambda_{[2]} \geq \max\{\lambda_{[1]} + \delta_1, \delta_2\, \lambda_{[1]}\}$

The remainder of the problem is formalized in the notation of Chapter I by:

(i) $\Lambda = [0,\infty)$ since $0 \le \lambda_{[1]} \le \cdots \le \lambda_{[k]} < \infty$

(ii) $$u(\lambda) = \begin{cases} \lambda+\delta_1 & , 0 \le \lambda \le \delta_1/(\delta_2-1) \\ \delta_2\lambda & , \lambda \ge \delta_1/(\delta_2-1) \end{cases}$$

gives (2.4.6)

(iii) $H_n(x) = b^{1/n}x$ where $0 < b < 1$

(iv) $T_{in} = \frac{1}{n}\sum_{j=1}^{n} X_{ij}$

Now $T_{in} \xrightarrow{p} p + \lambda_i$ as $n\to\infty$ by WLLN and hence selection in terms of T_{in} is equivalent to selection in terms of a consistent sequence of estimators for λ_i. As noted the goal of interest is to find $\pi_{(1)}$ although selection for $\pi_{(k)}$ is a completely analagous problem. The proposed rule is:

<u>R'(n)</u>: Select $\pi_i \Leftrightarrow T_i \le \min\{T_{[m]}, T_{[1]}\, b^{-1/n}\}$

<u>Goal G'</u>: Given P^*,m, $\delta_1 > 0$, $\delta_2 > 1$ and the sequence of rules $\{R'(n)\}$ find the smallest common sample size n needed to achieve

$$P_{\underset{\sim}{\lambda}}[CS|R'(n)] \ge P^* \; \forall \; \underset{\sim}{\lambda}\varepsilon\Omega(u).$$

Since $n\, T_{in} \sim \chi^2(np, n\lambda_i)$

$\Rightarrow G_n(y/\lambda_i) = P[T_{in} \le y] = F_{np}(ny/n\lambda_i)$ and $E_n^\lambda = [0,\infty)\ \forall\ \lambda\varepsilon\Lambda$.

Furthermore since the non central χ^2 family of densities has MLR in x it follows that $\forall n$ $\{G_n(y/\lambda)|\lambda\varepsilon\Lambda\}$ forms a stochastically increasing family. Hence by Theorem (1.5.2):

$$\inf_{\Omega(u)} P[CS|R'(n)] = \inf_{\lambda\ge 0} \eta(\lambda,n) \text{ where}$$

$$\eta(\lambda,n)=\begin{cases}\displaystyle\int_0^\infty \{1-F_{np}(yb^{1/n}/n(\lambda+\delta_1))\}^{k-1} I\left(\frac{1-F_{np}(y/n(\lambda+\delta_1))}{1-F_{np}(yb^{1/n}/n(\lambda+\delta_1))};k-m,m\right) dF_{np}(y/n\lambda), & 0 \le \lambda \le \frac{\delta_1}{\delta_2-1} \\ \displaystyle\int_0^\infty \{1-F_{np}(yb^{1/n}/n\lambda\delta_2)\}^{k-1} I\left(\frac{1-F_{np}(y/n\lambda\delta_2)}{1-F_{np}(yb^{1/n}/n\lambda\delta_2)};k-m,m\right) dF_{np}(y/n\lambda), & \lambda > \frac{\delta_1}{\delta_2-1}\end{cases}$$

Theorem 2.4.1. For any $b\varepsilon(\delta_2^{-1},1)$ and $n \ge 1$

$$\eta(\lambda,n) \text{ is } \begin{cases}\text{non increasing in } \lambda \text{ on } [0,\ \delta_1/(\delta_2-1)) \\ \text{non decreasing in } \lambda \text{ on } (\delta_1/(\delta_2-1),\infty)\end{cases}$$

Hence $\inf_{\lambda\ge 0} \eta(\lambda,n) = \eta(\delta_1/(\delta_2-1),n)$

Proof. The proof is an application of Theorem 1.5.3. From the proof of that theorem note that if (1.5.10) and (1.5.11) are both non positive it follows that $\eta(\lambda,n)$ is non increasing in λ.

Case A: $\lambda\varepsilon[0,\delta_1/(\delta_2-1))$

Using (2.4.2) the equation (1.5.10) can be seen to reduce to

(2.4.7) $$f_{np}(y/n\lambda)f_{np+2}(b^{1/n}y/n(\lambda+\delta_1))-b^{1/n}f_{np}(b^{1/n}y/n(\lambda+\delta_1))f_{np+2}(y/n\lambda) \le 0.$$

But $b\varepsilon(\delta_2^{-1},\ 1)$

$$\Rightarrow b(\delta_2-1) > 1-b \Rightarrow \frac{\delta_1}{(\delta_2-1)} < \frac{b\delta_1}{(1-b)}$$

So finally we see that for $\lambda\varepsilon[0,\delta_1/(\delta_2-1))$, $n \ge 1$ and $y > 0$

$$\lambda < \frac{b\delta_1}{1-b} < \frac{b^{1/n}\delta_1}{(1-b^{1/n})}$$

$$\Rightarrow \lambda(1-b^{1/n}) < b^{1/n}\delta_1$$

$$\Rightarrow b^{1/n}(\lambda+\delta_1) > \lambda$$

$$\Rightarrow b^{1/n}yn(\lambda+\delta_1) > yn\lambda$$

But from (2.4.5) $\frac{1}{y}\ \frac{f_{p+2}(y/\lambda)}{f_p(y/\lambda)}$ is non increasing in $(y\lambda)$

for fixed $p \Rightarrow$ (2.4.7) holds.

Similarily it can be seen (1.5.11) reduces to

$$(2.4.8)\quad f_{np}(y/n\lambda)f_{np+2}(y/n(\lambda+\delta_1)-f_{np}(y/n(\lambda+\delta_1))f_{np+2}(y/n\lambda) \leq 0.$$

Now $\delta_1 > 0$

$\Rightarrow n(\lambda+\delta_1) > n\lambda$ and applying result (2.4.3)

$$\frac{f_{np+2}(y/n(\lambda+\delta_1))}{f_{np}(y/n(\lambda+\delta_1))} \leq \frac{f_{np+2}(y/n\lambda)}{f_{np}(y/n\lambda)} \Rightarrow \text{(2.4.8) holds.}$$

So finally it can be seen that $\eta(\lambda,n)$ is non increasing on $[0, \delta_1/(\delta_2-1))$.

Case B: $\lambda > \delta_1/(\delta_2-1)$

Using arguments similar to the above together with (2.4.4) and (2.4.5) it can easily be seen that (1.5.10) and (1.5.11) both hold for $\lambda\varepsilon(\delta_1/(\delta_2-1),\infty)$ and hence $\eta(\lambda,n)$ is non decreasing in this range.

The last part of the theorem is now obvious and this completes the proof.

Remark 2.4.1. The final result for $1 > b > \delta_2^{-1}$ is that

$$(2.4.9)\quad \inf_{\Omega(u)} P_{\underset{\sim}{\lambda}}[CS|R'(n)]=\int_0^\infty \{1-F_{np}(b^{1/n}y/\frac{n\delta_1\delta_2}{(\delta_2-1)})\}^{k-1} I(\frac{1-F_{np}(y/\frac{\delta_1\delta_2 n}{\delta_2-1})}{1-F_{np}(b^{1/n}y/\frac{\delta_1\delta_2 n}{(\delta_2-1)})};k-m,m)dF_{np}(y/\frac{\delta_1 n}{(\delta_2-1)})$$

Hereafter it will be assumed that $\delta_2^{-1} < b < 1$ so that (2.4.9) holds. Now since $\delta_1\delta_2/(\delta_2-1)\varepsilon\Lambda$ the hypotheses of Theorem 1.5.4 holds and

$$\inf_{\Omega(u)} P_{\underset{\sim}{\lambda}}[CS|R'(n)] \to 1 \text{ as } n\to\infty.$$

Thus all P* levels are attainable in goal G'. All other usual properties hold for R'(n). In particular (1.5.18) holds (as verified by Gupta-Panchapakesan [33]) and hence

$$\sup_{\Omega} E_{\underset{\sim}{\lambda}}[S'(n)] = \sup_{\lambda\geq 0} \gamma'(\lambda,n) \text{ where}$$

$$\gamma'(\lambda,n)=k\int_0^\infty \{1-F_{np}(b^{1/n}y/n\lambda)\}^{k-1} I(\frac{1-F_{np}(y/n\lambda)}{1-F_{np}(b^{1/n}/n\lambda)};k-m,m)dF_{np}(y/n\lambda)$$

<u>Corollary 2.4.1.</u>

$$\sup_{\Omega} E_{\underset{\sim}{\lambda}}[S'(n)] = m$$

<u>Proof.</u> Since (1.5.18) holds for this problem $\gamma'(\lambda,n)$ is non decreasing in λ and hence $\sup_{\Omega} E[S'(n)] = \lim_{\lambda\to\infty} \gamma'(\lambda,n)$.

First note the following limiting behavior of $\chi^2(p,\lambda)$ random variables. Suppose Y_1,Y_2 are iid $\chi^2(p,\lambda)$ and $b\varepsilon(0,1)$.

$$P[bY_1 > Y_2]=P[\frac{b(Y_1-\mu)}{\sigma} - \frac{(Y_2-\mu)}{\sigma} \geq \frac{(1-b)\mu}{\sigma}]$$

where $\mu = E[Y_1] = p+\lambda$

$\sigma^2 = \mathrm{Var}(Y_1) = 2p+4\lambda$

$$\leq P\left[\left|\frac{b(Y_1-\mu)}{\sigma} - \frac{(Y_2-\mu)}{\sigma}\right| \geq \frac{(1-b)\mu}{\sigma}\right]$$

$$\leq \frac{(1+b^2)\ (2p+4\lambda)}{(1-b)^2\ (p+\lambda)^2} \text{ by Chebyshev's Inequality}$$

$\to 0$ as $\lambda\to\infty$

Now $\gamma'(\lambda,n)$ can be rewritten as

$$\gamma'(\lambda,n) = k \sum_{p=k-m}^{k-1} \binom{k-1}{p} \mathcal{J}_p(\lambda) \text{ where}$$

$$\mathcal{J}_p(\lambda) = \int_0^\infty \{1-F_{np}(y/n\lambda)\}^p\{F_{np}(y/n\lambda)-F_{np}(yb^{1/n}/n\lambda)\}^{k-1-p}dF_{np}(y/n\lambda)$$

$$= P[X_1 \leq X_j \text{ for } j=2,\ldots,p+1;\ b^{1/n}X_1 \leq X_j \leq X_1 \text{ for } j=p+2,\ldots,k]$$

$$= \begin{cases} P[\bigcap_{j=2}^{p+1} A_j \bigcap_{j=p+2}^{k} A_j^c B_j] & , \quad p=k-m,\ldots,k-2 \\ P[\bigcap_{j=2}^{k} A_j] & , \quad p=k-1 \end{cases}$$

where: $X_1,\ldots,X_k$ are iid $\chi^2(p,\lambda)$

$A_j=[X_1 \leq X_j]$

$B_j=[b^{1/n}\ X_1 \leq X_j]$

By the first part of the proof $P[\bigcap_{j=p+2}^{k} B_j] \to 1$ as $\lambda\to\infty$.

Also note $P[\bigcap_{j=2}^{p+1} A_j \bigcap_{j=p+2}^{k} A_j^c]=P[X_1 \leq X_j$ for $j=2,\ldots,p+1$; $X_1 > X_j$ for $j=p+2,\ldots,k]$

$$= \frac{(k-1-p)!p!}{k!} = \frac{1}{k\binom{k-1}{p}}$$

and $P[\bigcap_{j=2}^{k} A_j] = \frac{1}{k} = \frac{1}{k\binom{k-1}{k-1}}$

So we obtain

$$\mathcal{I}_p(\lambda) = \begin{cases} P[\bigcap_{j=2}^{p+1} A_j \bigcap_{j=p+2}^{k} A_j^c B_j] \to \frac{1}{k\binom{k-1}{p}} & \text{as } \lambda\to\infty \\ P[\bigcap_{j=2}^{k} A_j] = \frac{1}{k\binom{k-1}{k-1}} & \forall\lambda \end{cases}$$

$$\Rightarrow \gamma'(\lambda,n)=k\sum_{p=k-m}^{k-1}\binom{k-1}{p}\mathcal{I}_p(\lambda) \to k\sum_{p=k-m}^{k-1}\binom{k-1}{p}\frac{1}{k\binom{k-1}{p}} = m \text{ as } \lambda\to\infty$$

and this completes the proof of the corollary.

Application I: Selection of the population with shortest Mahalanobis distance to the origin.

Let $\pi_i \sim N_p(\underset{\sim}{\mu}_i,\underset{\sim}{\Sigma}_i)$ for i=1,...,k where $\underset{\sim}{\Sigma}_i$ is known. Let $\lambda_i = \underset{\sim}{\mu}_i'\underset{\sim}{\Sigma}_i^{-1}\underset{\sim}{\mu}_i$ be the Mahalanobis distance of π_i to the origin and $0 \le \lambda_{[1]} \le \cdots \le \lambda_{[k]} < \infty$ be the ordered λ_i's. The object is to find $\pi_{(1)}$, the population with shortest distance to the origin. Let $\{W_{ij}\}_{j=1}^{n}$ be a sample from $\pi_{(i)}$ and

$$X_{ij} = W_{ij}' \underset{\sim}{\Sigma}_i^{-1} W_{ij} \sim \chi^2(p,\lambda_i)$$

and

$$T_{in} = \frac{1}{n}\sum_{j=1}^{n} X_{ij} \overset{P}{\to} p+\lambda_i$$

Given an indifference zone specified by $\delta_1 > 0$ and $\delta_2 \ge 1$ and also P* and m take any $b\varepsilon(\delta_2^{-1},1)$ and let

R'(n): Select $\pi_i \Leftrightarrow T_i \le \min\{T_{[m]}, b^{-1/n} T_{[1]}\}$. The sample size needed to attain $P_{\tilde{\lambda}}[CS|R'(n)] \ge P^* \forall \tilde{\lambda} \epsilon \Omega(u)$ is the smallest n satisfying

$$P^* = \int_0^\infty \{1-F_{np}(b^{1/n} y/\frac{n\delta_1\delta_2}{(\delta_2-1)})\}^{k-1} I(\frac{1-F_{np}(y/\frac{n\delta_1\delta_2}{(\delta_2-1)})}{1-F_{np}(b^{1/n} y/\frac{n\delta_1\delta_2}{(\delta_2-1)})};k-m,m)\, dF_{np}(y/\frac{n\delta_1}{(\delta_2-1)})$$

Application II: Comparison of Multivariate Normal Populations with a Control.

Let $\pi_i \sim N_p(\tilde{\mu}_i, \tilde{\Sigma})$ for i=0,1,...,k and suppose $\tilde{\mu}_0$ and $\tilde{\Sigma}$ are known but $\tilde{\mu}_1,\ldots,\tilde{\mu}_k$ are unknown. π_0 may be thought of as a standard or control population and it is desired to select that populations which is closest to π_0 in the sense of Mahalanobis distance. If $\lambda_i = (\tilde{\mu}_i - \tilde{\mu}_0)'\tilde{\Sigma}^{-1}(\tilde{\mu}_i - \tilde{\mu}_0)$ for i=1,...,k then the population with parameter $\lambda_{[1]}$ is being sought. If, as before, $\{W_{ij}\}_{j=1}^n$ are iid from π_i let

$$X_{ij} = (W_{ij} - \tilde{\mu}_0)'\tilde{\Sigma}^{-1}(W_{ij} - \tilde{\mu}_0)$$

and then the problem falls into the framework of this section.

2.5 Selection in Terms of Non Central $\mathcal{F}$ Distributions

Suppose π_i is characterized by a scalar $\lambda_i \varepsilon [0,\infty)$ and the sequence of statistics $\{T_{in}\}$ defined in terms of $\{X_{ij}\}$ from π_i satisfies

(1) T_{in} has a non central F cdf with p and q degrees of freedom and non centrality parameter λ_i (abbreviated $F_{p,q}(y/\lambda_i)$). Both p and q may depend on n but as usual, the dependence is suppressed for ease of notation.

(2) $T_{in} \overset{P}{\to} \lambda_i$ as $n \to \infty$

Remark 2.5.1. If the noncentrality parameter of T_{in} is not λ_i but $r(\lambda_i)$ where $r(\lambda)$ is a strictly increasing differentiable function of λ, the conclusions below still hold.

Again let

$$(2.5.1) \qquad \Omega(p) = \Omega_1' \cap \Omega_2' \text{ where}$$

$$\begin{cases} \Omega_1' = \{\underset{\sim}{\lambda} | \lambda_{[k]} \geq \delta_2 \lambda_{[k-1]}\} & (\delta_2 > 1) \\ \Omega_2' = \{\underset{\sim}{\lambda} | \lambda_{[k]} - \lambda_{[k-1]} \geq \delta_1\} & (\delta_1 > 0) \end{cases}$$

so that $\Omega(p)$ is analagous to $\Omega(u)$ of Section 2.4.

$$\Rightarrow \Omega(p) = \{\underset{\sim}{\lambda} | \lambda_{[k]} \geq \max\{\delta_2 \lambda_{[k-1]}, \lambda_{[k-1]} - \delta_1\}\}$$

$$\Rightarrow p(\underset{\sim}{\lambda}) = \begin{cases} \lambda - \delta_1 & , \quad 0 \leq \lambda \leq \delta_1 \delta_2 / (\delta_2 - 1) \\ \delta_2^{-1} \lambda & , \quad \delta_1 \delta_2 / (\delta_2 - 1) < \lambda \end{cases}$$

$$\Rightarrow \Lambda' = [\delta_1, \infty).$$

The remaining elements of the problem can be stated in the notation of Chapter I as

(i) $\Lambda = [0,\infty)$

(ii) $G_n(y/\lambda) = F_{p,q}(y/\lambda)$

and

$$g_n(y/\lambda)=f_{p,q}(y/\lambda)= \frac{e^{-\lambda/2}}{\Gamma(q/2)} \sum_{r=0}^{\infty} \frac{y^{p/2+r-1}\Gamma(p/2+q/2+r)\lambda^r}{(1+y)^{p/2+q/2+r}\Gamma(p/2+r)2^r r!}, y>0$$

(iii) $h_n(x) = d^{1/n}x \quad (d > 1)$

The following are known properties of $F_{p,q}(y/\lambda)$ ($q > 2$ will be assumed throughout this section).

(2.5.2) $$\frac{\partial F_{p,q}(y/\lambda)}{\partial\lambda} = \frac{-1}{q-2} f_{p+2,q-2}(y/\lambda)$$

(2.5.3) $f_{p,q}(y/\lambda)$ has MLR in y

(2.5.4) $\dfrac{\lambda f_{p+2,q-2}(y/\lambda)}{f_{p,q}(y/\lambda)}$ is non decreasing in λ for fixed $y>0$, p and q.

(2.5.5) $\dfrac{f_{p+2,q-2}(y/\lambda)}{f_{p,q}(y/\lambda)}$ is non increasing in λ for fixed $y>0$, p and q.

In addition, an application of Lemma 2.4.1 shows that

$\dfrac{f_{p+2,q-2}(y/\lambda)}{y f_{p,q}(y/\lambda)}$ is non increasing in both

(a) y for fixed λ, p and q.
(b) $(\frac{y\lambda}{1+y})$ for fixed p and q.

As usual the rule is defined by

(2.5.8) R(n): Select $\pi_i \Leftrightarrow T_i \geq \max\{T_{[k-m+1]}, d^{-1/n} T_{[k]}\}$.

The goal is to find n satisfying

(2.5.9) $$P_{\underset{\sim}{\lambda}}[CS|R(n)] \geq P^* \quad \underset{\sim}{\lambda}\varepsilon\Omega(p)$$

Using (2.5.3) it follows that

$$\inf_{\Omega(p)} P_{\underset{\sim}{\lambda}}[CS|R(n)] = \inf_{\lambda \geq \delta_1} \psi(\lambda,n) \text{ where}$$

$$\psi(\lambda,n) = \begin{cases} \int_0^\infty \{F_{p,q}(yd^{1/n}/\lambda-\delta_1)\}^{k-1} I\left(\frac{F_{p,q}(y/\lambda-\delta_1)}{F_{p,q}(yd^{1/n}/\lambda-\delta_1)};k-m,m\right) dF_{p,q}(y/\lambda), & \lambda\varepsilon I_1 \\ \int_0^\infty \{F_{p,q}(yd^{1/n}/\delta_2^{-1}\lambda)\}^{k-1} I\left(\frac{F_{p,q}(y/\delta_2^{-1}\lambda)}{F_{p,q}(yd^{1/n}/\delta_2^{-1}\lambda)};k-m,m\right) dF_{p,q}(y/\lambda), & \lambda\varepsilon I_2 \end{cases}$$

where $I_1 = [\delta_1, \delta_1\delta_2/(\delta_2-1)]$

$I_2 = [\delta_1\delta_2/(\delta_2-1), \infty)$

Theorem 2.5.1. For any $1 < d < \delta_2$ and $n \geq 1$

$$\psi(\lambda,n) \text{ is } \begin{cases} \text{non increasing in } \lambda \text{ on } I_1 \\ \text{non decreasing in } \lambda \text{ on } I_2. \end{cases}$$

Hence $\inf_{\lambda\geq\delta_1} \psi(\lambda,n) = \psi(\delta_1\delta_2/(\delta_2-1),n)$

Proof. The proof is an application of Theorem 1.2.3 and generally follows along the lines of the proof of Theorem 2.4.1. However the details of proving that (1.2.8) is non positive on I_1 are sufficiently different to warrant being displayed.

Using (2.5.2) it can be seen that (1.2.8) being non positive reduces to

$$\frac{d^{1/n}}{q-2}\, f_{p+2,q-2}(y/\lambda) f_{p,q}(yd^{1/n}/\lambda-\delta_1) \le \frac{1}{q-2}\, f_{p+2,q-2}(y/\lambda-\delta_1)$$

$$f_{p,q}(y/\lambda)\,\forall\, y>0 \text{ and } \lambda\varepsilon I_1$$

$$(2.5.10) \Leftrightarrow \frac{1}{y}\,\frac{f_{p+2,q-2}(y/\lambda)}{f_{p,q}(y/\lambda)} \le \frac{1}{d^{1/n}y}\,\frac{f_{p+2,q-2}(yd^{1/n}/\lambda-\delta_1)}{f_{p,q}(yd^{1/n}/\lambda-\delta_1)}, \forall\, y>0 \text{ and }$$

$$\lambda\varepsilon I_1$$

But $1 < d < \delta_2$

$$\Rightarrow \frac{\delta_1\delta_2}{(\delta_2-1)} < \frac{\delta_1 d}{(d-1)} \quad \text{as } \delta_1 > 0$$

$$\Rightarrow \lambda < (\delta_1 d)/(d-1) \qquad \forall \lambda\varepsilon I_1$$

$$\Rightarrow \lambda < \frac{\delta_1 d^{1/n}}{d^{1/n}-1} \qquad \forall\lambda\varepsilon I_1 \text{ and } n \ge 1 \text{ since } \frac{d}{d-1} \le \frac{d^{1/n}}{d^{1/n}-1} \quad \forall\, n \ge 1$$

$$\Rightarrow \lambda < \frac{\delta_1 d^{1/n}(1+y)}{(d^{1/n}-1)} \quad \forall\, y>0,\ \lambda\varepsilon I_1 \text{ and } n \ge 1$$

$$\Rightarrow \lambda > d^{1/n}(\lambda-\delta_1)-\delta_1 d^{1/n}y$$

$$\Rightarrow \lambda(1+d^{1/n}y) > (1+y)d^{1/n}(\lambda-\delta_1)$$

$$\Rightarrow \frac{\lambda y}{(1+y)} > \frac{d^{1/n}y(\lambda-\delta_1)}{(1+yd^{1/n})} \quad \forall\, y>0,\ \lambda\varepsilon I_1 \text{ and } n \ge 1$$

But since $\frac{1}{y}\,\frac{f_{p+2,q-2}(y/\lambda)}{f_{p,q}(y/\lambda)}$ is $\downarrow$ in $\frac{\lambda y}{(1+y)} \Rightarrow$ (2.5.10) holds.

The remaining parts of the proof are straightforward and thus the proof is completed.

Remark 2.5.2. For the remainder of the section it will be assumed that $1 < d < \delta_2$ so that

$$\inf_{\Omega(p)} P_{\underset{\sim}{\lambda}}[CS|R(n)] = \int_0^\infty \{F_{p,q}(yd^{1/n}/\frac{\delta_1}{\delta_2-1})\}^{k-1} I(\frac{F_{p,q}(y/\frac{\delta_1}{\delta_2-1})}{F_{p,q}(yd^{1/n}/\frac{\delta_1}{\delta_2-1})};k-m,m) \, dF_{p,q}(y/\frac{\delta_1\delta_2}{\delta_2-1})$$

Furthermore the non central F distribution satisfies all other regularity conditions and hypotheses of Chapter I so that all previous results hold. In particular

$$\inf_{\Omega(p)} P_{\underset{\sim}{\lambda}}[CS|R(n)] = \psi(\frac{\delta_1\delta_2}{(\delta_2-1)},n) \to 1 \text{ as } n\to\infty \text{ since } \frac{\delta_1\delta_2}{(\delta_2-1)} \in \Lambda$$

$$\sup_{\Omega} E[S(n)] = \sup_{\lambda\geq 0} \gamma(\lambda,n) \text{ where}$$

$$\gamma(\lambda,n) = k\int_0^\infty \{F_{p,q}(d^{1/n}y/\lambda)\}^{k-1} I(\frac{F_{p,q}(y/\lambda)}{F_{p,q}(d^{1/n}y/\lambda)};k-m,m)\,dF_{p,q}(y/\lambda)$$

Lemma 2.5.1. For fixed p and q, $F_{p,q}(x/\lambda) \to H_q(x)$ uniformly on $(0,\infty)$

(2.5.11) as $\lambda\to\infty$ where $H_q(x) = P[q/\chi^2(q) \leq x]$.

Proof. Now $F_{p,q}(x/\lambda) = \sum_{j=0}^{\infty} (\frac{e^{-\lambda/2}\lambda^j}{\lambda^j 2^j})\, E_{p+2j,q}(x)$ where

$$E_{p+2r,q}(x) = P[X(r)/Y \leq x] \text{ where } X(r) \sim \frac{\chi^2(p+2r)}{p+2r},\ Y\sim\frac{\chi^2(q)}{q}$$

and $X(r)$, Y are independent.

Since $X(r) \overset{P}{\to} 1$ as $r\to\infty$, $\frac{X(r)}{Y} \overset{\mathcal{L}}{\to} q/\chi^2(q)$ as $r\to\infty$. Furthermore the convergence is uniform on $(0,\infty)$ since the limiting distribution is continuous.

So given $\epsilon > 0$

(a) $\exists R \ni \forall r \geq R \Rightarrow |E_{p+2r}(x)-H_q(x)| < \epsilon/2 \quad \forall x \varepsilon (0,\infty)$

(b) $\exists \lambda_0 \ni \forall \lambda \geq \lambda_0 \Rightarrow \sum_{j=0}^{R} \frac{e^{-\lambda/2}\lambda^j}{2^j j!} < \epsilon/2.$

Hence $\forall \lambda \geq \lambda_0$ and $x \varepsilon (0,\infty)$

$$|F_{p,q}(x/\lambda) - H_q(x)|$$

$$\leq \sum_{j=0}^{R} \frac{e^{-\lambda/2}\lambda^j}{j!2^j}|E_{p+2j,q}(x)-H_q(x)| + \sum_{j=R+1}^{\infty} \frac{e^{-\lambda/2}\lambda^j}{2^j j!}|E_{p+2j,q}(x)-H_q(x)|$$

$$\leq \sum_{j=0}^{R} \frac{e^{-\lambda/2}\lambda^j}{2^j j!} + \epsilon/2 \sum_{j=R+1}^{\infty} \frac{e^{-\lambda/2}\lambda^j}{j!2^j} \leq \epsilon$$

Theorem 2.5.2.

$$\sup_{\Omega} E_{\underset{\sim}{\lambda}}[S(n)] = \int_0^{\infty} \{H_q(d^{1/n}x)\}^{k-1} I\left(\frac{H_q(x)}{H_q(d^{1/n}y)};k-m,m\right) dH_q(x)$$

where $H_q(x)$ is defined by (2.5.11).

Proof. Since the conditions of Corollary 1.4.2 hold, it follows that $\gamma(\lambda,n)$ is non decreasing in

$$\Rightarrow \sup_{\Omega} E_{\underset{\sim}{\lambda}}[S(n)] = \lim_{\lambda\to\infty} \gamma(\lambda,n).$$

$\gamma(\lambda,n) = \sum_{p=k-m}^{k-1} \binom{k-1}{p} \int_0^{\infty} A_p(y,\lambda)dF_{p,q}(y/\lambda)$ where

$$A_p(y,\lambda)=\{F_{p,q}(y/\lambda)\}^p\{F_{p,q}(d^{1/n}y/\lambda)-F_{p,q}(y/\lambda)\}^{k-1-p}$$

$\Rightarrow$ it suffices to prove $\forall p = k-m,\ldots,k-1$ that

$$\int_0^\infty A_p(y,\lambda)dF_{p,q}(y/\lambda) \to \int B_p(y)dH_q(y) \text{ as } \lambda\to\infty$$

where $B_p(y) = \{H_q(y)\}^p\{H_q(d^{1/n}y) - H_q(y)\}^{k-1-p}$

Since $F_{p,q}(y/\lambda) \to H_q(y)$ uniformly on $(0,\infty)$ as $\lambda\to\infty \Rightarrow$ given $\epsilon>0 \ \exists \ \lambda_1 \ni$

$$|A_p(y,\lambda)-B_q(y)| < \epsilon/2 \ \forall\lambda \geq \lambda_1 \text{ and } y\varepsilon(0,\infty).$$

Also since $B_q(y)$ is bounded and continuous on $(0,\infty) \Rightarrow \exists\lambda_2 > \lambda_1 \ni$

$$\left|\int_0^\infty B_p(y)dF_{p,q}(y/\lambda)-\int_0^\infty B_p(y)dH_q(y)\right| < \epsilon/2 \ \forall\lambda \geq \lambda_2 \text{ by the}$$

Helly-Bray Theorem. So, finally, it can be seen that $\forall\lambda \geq \lambda_2$

$$\left|\int_0^\infty A_p(y,\lambda)dF_{p,q}(y/\lambda)-\int_0^\infty B_p(y)dH_q(y)\right|$$

$$\leq \left|\int_0^\infty A_p(y,\lambda)dF_{p,q}(y/\lambda)-\int_0^\infty B_p(y)dF_{p,q}(y/\lambda)\right|+\left|\int_0^\infty B_p(y)dF_{p,q}(y/\lambda)-\int_0^\infty B_p(y)dH_q(y)\right|$$

$$\leq \int_0^\infty |A_p(y,\lambda)-B_p(y)|dF_{p,q}(y/\lambda)+\epsilon/2 \leq \epsilon/2 \int_0^\infty dF_{p,q}(y/\lambda)+\epsilon/2 = \epsilon$$

and the proof is completed.

Remark 2.5.3. Alam and Rizvi [1] have stated that for the special case m=k, $\sup_\Omega E_{\underset{\sim}{\lambda}}[S(n)]=k$. Theorem 2.5.2 shows their calculations to be in error and that $\sup_\Omega E_{\underset{\sim}{\lambda}}[S(n)]=\int_0^\infty \{H_q(d^{1/n}x)\}^{k-1}dH_q(x)$. Their result for the non central χ^2 case, $\sup_\Omega E_{\underset{\sim}{\lambda}}[S(n)]=k$, is obtained as a special case of the general result in Corollary 2.4.1.

Application: Selection of the Population with Largest Mahalanobis Distance.

For i=1,...,k suppose $\pi_i \sim N_p(\underset{\sim}{\mu}_i, \underset{\sim}{\Sigma}_i)$ where both $\underset{\sim}{\mu}_i$ and $\underset{\sim}{\Sigma}_i$ are unknown. Let $\lambda_i = \underset{\sim}{\mu}_i' \underset{\sim}{\Sigma}_i^{-1} \underset{\sim}{\mu}_i$ and $0 \leq \lambda_{[1]} \leq \cdots \leq \lambda_{[k]}$ be the ordered λ_i's. It is desired to find $\pi_{(k)}$ the population having farthest Mahalanobis distance from the origin. Take

$T_{in} = \bar{X}_i' S_i^{-1} \bar{X}_i$ where

$$\begin{cases} \bar{X}_i = \frac{1}{n} \sum_{j=1}^{n} X_{ij} \text{ is the sample mean from } \pi_i \text{ and} \\ S_i = \frac{1}{n} \sum_{j=1}^{n} (X_{ij}-\bar{X}_i)(X_{ij}-\bar{X}_i)' \text{ is the sample covariance matrix.} \end{cases}$$

It is known $\frac{(n-p)}{p}(\bar{X}_i' S_i^{-1} \bar{X}_i)$ has cdf $F_{p,n-p}(y/n\lambda_i)$ and hence

$$G_n(y/\lambda_i) = F_{p,n-p}(\tfrac{(n-p)y}{p}/n\lambda_i).$$

Furthermore since $\bar{X}_i \overset{p}{\to} \underset{\sim}{\mu}_i$ and $S_i \overset{p}{\to} \Sigma_i$ it follows that $T_{in} = \bar{X}_i' S_i^{-1} \bar{X}_i \overset{p}{\to} \mu_i' \Sigma_i^{-1} \mu_i$. Choosing $\Omega(p)$ as in (2.5.1) and $1 < d < \delta_2$ define the selection procedure.

R(n): Select $\pi_i \Leftrightarrow T_i \geq \max\{T_{[k-m+1]}, d^{-1/n} T_{[k]}\}$.

Since n is merely a scale factor in the function $r(\lambda) = n\lambda$, the proof of Theorem 2.5.1 shows

$$\text{(2.5.12)} \quad \inf_{\Omega(p)} P_{\underset{\sim}{\lambda}}[CS|R(n)] = \int_0^\infty \{F_{p,n-p}(yd^{1/n}/\tfrac{n\delta_1}{(\delta_2-1)})\}^{k-1} I\left(\frac{F_{p,n-p}(y/\frac{n\delta_1}{(\delta_2-1)})}{F_{p,n-p}(yd^{1/n}/\frac{n\delta_1}{(\delta_2-1)})}; k-m, m\right) dF_{p,n-p}(y/\tfrac{n\delta_1\delta_2}{(\delta_2-1)})$$

and n should be chosen to make the right hand side of (2.5.12) equal to P*. Selection for $\pi_{(1)}$ is completely analagous.

2.6 Selection from Uniform Populations

The examples discussed thus far have dealt only with the so called 'regular' case in which the support of $G_n(y/\lambda)$ did not depend on λ. The next example illustrates the non regular case and hence Theorems 1.2.3 and 1.4.3 are not applicable in this problem.

Let π_i have a uniform distribution on $[0,\lambda_i]$ ($\mathcal{U}[0,\lambda_i]$) for $i=1,\ldots,k$ where $\lambda_i \varepsilon (0,\infty)$ and suppose $\{X_{ij}\}_{j=1}^{n}$ are iid from π_i. Take

(2.6.1) $$T_{in} = \max_{1 \leq j \leq n} X_{ij}.$$

It is known T_{in} is a complete sufficient statistic for λ_i and the sequence $\{T_{in}\}$ is a consistent sequence of estimators for λ_i.

Formally the elements of the problem can be stated in the language of Chapter I as

$$\Lambda = (0,\infty)$$

$$G_n(y/\lambda_i) = \begin{cases} 0 & , \ y \leq 0 \\ (y/\lambda_i)^n & , \ 0 < y < \lambda_i \\ 1 & , \ y \geq \lambda_i \end{cases}$$

$$p(\lambda) = \delta\lambda \quad , \quad 0 < \delta < 1$$

$$\Omega = \{\underset{\sim}{\lambda} | \lambda_i \varepsilon \Lambda \forall_i\}$$

$$\Omega(p) = \{\underset{\sim}{\lambda} | \lambda_{[k-1]} \leq \delta\lambda_{[k]}\}$$

It can easily be verified that for each $n, \{G_n(y/\lambda) \mid \lambda\varepsilon(0,\infty)\}$ forms a stochastically increasing family. Now pick any $d > 1$ and define the rule:

(2.6.2) <u>R(n)</u>: Select $\pi_i \Leftrightarrow T_i \geq \max\{d^{-1/n}T_{[k]}, T_{[k-m+1]}\}$

<u>Theorem 2.6.1.</u>

$$\inf_{\Omega(p)} P[CS|R(n)] = \inf_{\Omega^o(p)} P[CS|R(n)]$$

(2.6.3) $$= 1+\delta^n\{[\frac{1+kd-k}{kd}]I(1/d;k-m,m) + \frac{m}{k} I(1-1/d;m+1,k-m)-1\}$$

<u>Proof</u>. Since $\{G_n(y/\lambda) \mid \lambda\varepsilon\Lambda\}$ is a stochastically increasing family, Theorem 1.2.2 applies and hence the problem is reduced to evaluating the one dimensional infimum

$$\inf_{\Omega^o(p)} P[CS|R(n)] = \inf_{\lambda\varepsilon(0,\infty)} \psi(\lambda,n) \text{ where}$$

$$\psi(\lambda,n) = \sum_{i=k-m}^{k-1} \binom{k-1}{i} \int_{-\infty}^{\infty} \{G_n(y/\delta\lambda)\}^i \{G_n(d_n y/\delta\lambda) - G_n(y/\delta\lambda)\}^{k-1-i} dG_n(y/\lambda)$$

and $d_n = d^{1/n}$

But

$\psi(\lambda,n) = T_1(\lambda) + T_2(\lambda) + T_3(\lambda)$ where

$$T_1(\lambda) = \sum_{i=k-m}^{k-1} \binom{k-i}{i} \int_0^{\delta\lambda/d_n} \{(\frac{y}{\delta\lambda})^n\}^i \{(\frac{d_n y}{\delta\lambda})^n - (\frac{y}{\delta\lambda})^n\}^{k-1-i} \, d(y/\lambda)^n$$

$$T_2(\lambda) = \sum_{i=k-m}^{k-1} \binom{k-1}{i} \int_{\delta\lambda/d_n}^{\delta\lambda} \{(\frac{y}{\delta\lambda})^n\}^i \{1-(\frac{y}{\delta\lambda})^n\}^{k-1-i} \, d(y/\lambda)^n$$

$$T_3(\lambda) = \int_{\delta\lambda}^{\lambda} d(y/\lambda)^n$$

Making the change of variables $y = \lambda w$ shows

$$\psi(\lambda,n) = \psi(1,n) \ \forall \lambda.$$

Now

$$T_1(1) = \delta^n \sum_{i=k-m}^{k-1} \binom{k-1}{i} \int_0^{1/d} z^i (dz-z)^{k-1-i}\, dz$$

$$= \delta^n \sum_{i=k-m}^{k-1} \binom{k-1}{i} (d-1)^{k-1-i} \frac{z^k}{k} \Big|_0^{1/d}$$

$$= \frac{\delta^n}{kd} \sum_{i=k-m}^{k-1} \binom{k-1}{i} (1-1/d)^{k-1-i} (1/d)^i$$

$$= \frac{\delta^n}{kd} I(1/d;k-m,m)$$

$$T_2(1) = \delta^n \int_{1/d}^{1} \sum_{i=k-m}^{k-1} \binom{k-1}{i} z^i (1-z)^{k-1-i}\, dz$$

$$= C\delta^n \int_{1/d}^{1} \int_0^z x^{k-m-1} (1-x)^{m-1}\, dx\, dz$$

where $C = (k-m) \binom{k-1}{k-m}$

$$= C\delta^n (1-1/d) \int_0^{1/d} x^{k-m-1} (1-x)^{m-1}\, dx + C\delta^n \int_{1/d}^{1} x^{k-m-1} (1-x)^m dx$$

$$= \delta^n \{(1-1/d) I(1/d;k-m,m) + m/k\, I(1-1/d;m+1,k-m)\}$$

$T_3(1) = 1-\delta^n$ and adding $T_1(1)$, $T_2(1)$ and $T_3(1)$ gives the result.

Remark 2.6.1. Since $1 \varepsilon \Lambda$ the sequence of rules $\{R(n)\}$ is consistent wrt $\Omega(p)$. Also the usual results regarding the alternate formulations I and II again hold:

$$\lim_{\delta \to 0} \inf_{\Omega(p)} P[CS|R(n)] = 1$$

$$\lim_{d \to \infty} \inf_{\Omega(p)} P[CS|R(n)] = 1-\delta^n+\frac{\delta^n m}{k} < 1 \text{ for } 1 \le m < k.$$

For each rule R(n) the properties of strong monotonicity and unbiasedness hold. Now let S(n) be the number of populations selected by the rule R(n).

Theorem 2.6.2. For fixed $0 < \lambda_{[2]} \le \lambda_{[3]} \le \cdots \le \lambda_{[k]} < \infty$, $E_{\underset{\sim}{\lambda}}[S(n)]$ is non decreasing in $\lambda_{[1]}$ on $(0, \lambda_{[2]}]$.

Proof. The method of proof will be to directly show that $\frac{dE_{\lambda}[S(n)]}{d\lambda_{[1]}} \ge 0$ on $(0, \lambda_{[2]}]$ and the argument will use the fact that the support of $G_n(y/\lambda)$ depends on λ. Throughout this proof $\lambda_{[i]}$ will be written as λ_i. Using the first part of the proof of Theorem 1.4.3 it can be seen, after some cancellation, that

$$E_{\underset{\sim}{\lambda}}[S(n)]=1-\sum_{p=k-m}^{k-2} \sum_{r=2}^{k} \sum_{\nu=1}^{\binom{k-2}{p}} \int_{-\infty}^{\infty} \prod_{j\in\mathcal{P}_\nu^p(1,r)} \mathcal{G}_j(y) \prod_{j\in\bar{\mathcal{P}}_\nu^p(1,r)} \mathcal{D}_j(y)\mathcal{G}_1(y)\, d\mathcal{D}_r(y)$$

$$+\sum_{p=k-m}^{k-2} \sum_{r=2}^{k} \sum_{\nu=1}^{\binom{k-2}{p}} \int_{-\infty}^{\infty} \prod_{j\in\mathcal{P}_\nu^p(1,r)} \mathcal{G}_j(y) \prod_{j\in\bar{\mathcal{P}}_\nu^p(1,r)} \mathcal{D}_j(y)\mathcal{D}_1(y)\, d\mathcal{G}_r(y)$$

where

$$\begin{cases} \mathcal{G}_j(y) = G_n(y/\lambda_j) \\ \mathcal{D}_j(y) = G_n(d^{1/n}y/\lambda_j) - G_n(y/\lambda_j) \end{cases}$$

Using the fact that

$$\mathcal{D}_r(y) = \begin{cases} 0 & , \quad y \le 0 \text{ or } y > \lambda_r \\ (y/\lambda_r)^n(d-1) & , \quad 0 < y \le \lambda_r/d_n \\ 1-(y/\lambda_r)^n & , \quad \lambda_r/d_n < y \le \lambda_r \end{cases}$$

and $$\mathcal{E}_j(y) = \begin{cases} 0 & \quad y \le 0 \\ (y/\lambda_i)^n & , \quad 0 < y < \lambda_j \\ 1 & , \quad y > \lambda_j \end{cases}$$

it can be shown after further cancellation that:

$$\begin{aligned} E_{\tilde{\lambda}}[S(n)] &= 1 \\ &+ \sum_{r=2}^{R} \sum_{p=k-m}^{k-2} \sum_{\nu=1}^{\binom{k-2}{p}} \Bigg\{ \int_{\frac{\lambda_1}{d_n}}^{\frac{\lambda_r}{d_n}} \underset{\mathcal{P}^p_\nu(1,r)}{\pi} \mathcal{E}_j(y) \underset{\bar{\mathcal{P}}^p_\nu(1,r)}{\pi} \mathcal{D}_j(y)[1-d(y/\lambda_1)^n] \frac{ny^{n-1}}{\lambda_r^n}\,dy \\ &\qquad + \int_{\frac{\lambda_r}{d_n}}^{\lambda_r} \underset{\mathcal{P}^p_\nu(1,r)}{\pi} \mathcal{E}_j(y) \underset{\bar{\mathcal{P}}^p_\nu(1,r)}{\pi} \mathcal{D}_j(y) \frac{ny^{n-1}}{(\lambda_r)^n}\,dy \Bigg\} \\ &+ \sum_{r=R+1}^{k} \sum_{p=k-m}^{k-2} \sum_{\nu=1}^{\binom{k-2}{p}} \Bigg\{ \int_{\lambda_1/d_n}^{\lambda_1} \underset{\mathcal{P}^p_\nu(1,r)}{\pi} \mathcal{E}_j(y) \underset{\bar{\mathcal{P}}^p_\nu(1,r)}{\pi} \mathcal{D}_j(y) \frac{ny^{n-1}}{(\lambda_r)^n}[1-d(y/\lambda_1)^n]\,dy \\ &\qquad - \int_{\lambda_1}^{\lambda_r/d_n} \underset{\mathcal{P}^p_\nu(1,r)}{\pi} \mathcal{E}_j(y) \underset{\bar{\mathcal{P}}^p_\nu(1,r)}{\pi} \mathcal{D}_j(y) \frac{ny^{n-1}}{(\lambda_r)^n}[d-1]\,dy \\ &\qquad + \int_{\lambda_r/d_n}^{\lambda_r} \underset{\mathcal{P}^p_\nu(1,r)}{\pi} \mathcal{E}_j(y) \underset{\bar{\mathcal{P}}^p_\nu(1,r)}{\pi} \mathcal{D}_j(y) \frac{ny^{n-1}}{(\lambda_r)^n}\,dy \Bigg\} \end{aligned}$$

where R satisfies $\lambda_R \leq d_n\lambda_1 < \lambda_{R+1}$ and the second sum is zero if R=k. So finally taking derivatives wrt λ_1(i.e. $\lambda_{[1]}$) it is seen that:

$$\frac{dE_\lambda[S(n)]}{d\lambda_1} = \sum_{r=2}^{k} \sum_{p=k-m}^{k-2} \sum_{\nu=1}^{\binom{k-2}{p}} \int_{\lambda_1/d}^{C_r} \prod_{j\in\mathscr{P}_\nu^p(1,r)} \mathscr{S}_j(y) \prod_{j\in\bar{\mathscr{P}}_\nu^p(1,r)} \mathscr{D}_j(y) \frac{n^2 dy^{2n-1}}{(\lambda_1\lambda_r)^n\lambda_1}\, dy \geq 0$$

where $C_r = \min\{\lambda_1, \lambda_r/d^{1/n}\}$ and hence the required monotonicity holds.

Now by virtue of the fact that $\dfrac{dE_\lambda[S(n)]}{d\lambda_{[1]}} \geq 0$ on $(0, \lambda_{[2]}]$, Corollary 1.4.2 applies to give the result.

(2.6.4) $$\sup_{\Omega} E_{\underset{\sim}{\lambda}}[S(n)] = k\{[\frac{1+kd-k}{kd}]I(1/d;k-m,m) + \frac{m}{k}I(1-1/d;m+1,k-m)\}$$

The supremum takes place at a point where all components are equal and by arguments similar to those in the proof of Theorem 2.6.1 this one dimensional supremum is independent of the common λ. The corresponding results for T(n), the member of non best populations selected, are similar to the above.

CHAPTER III

A GENERALIZED GOAL FOR RESTRICTED SUBSET SELECTION PROCEDURES

3.1 Formulation of the Problem

Another goal which some experimenters may desire to achieve is that of selecting at least one of the t best populations using a restricted subset selection procedure. The structure of the populations $\{\pi_i\}$ and the terminology for this chapter will be the same as that introduced in Section 1.1 except where explicitly noted. If $\pi_{(k-t+1)},\ldots,\pi_{(k)}$ are the t best populations corresponding to $\lambda_{[k-t+1]},\ldots,\lambda_{[k]}$, the goal in this chapter is to select a subset containing at least one of these t best populations.

The following terminology will be needed in the next sections:

$$\Omega(\mathcal{F}) = \{\underset{\sim}{F}=(F_1,\ldots,F_k)|F_i(\cdot)=F(\cdot|\lambda_i) \text{ and } \lambda_i \varepsilon \Lambda\}$$

$$\Omega^t(p) = \{\underset{\sim}{\lambda}|\lambda_{[k-t]} \leq p(\lambda_{[k-t+1]})\}$$

$$\Omega_0^t(p) = \{\underset{\sim}{\lambda}|\lambda_{[1]}=\lambda_{[k-t]}=p(\lambda_{[k-t+1]})=p(\lambda_{[k]})\}$$

$$\Omega_0 = \{\underset{\sim}{\lambda}|\lambda_{[1]}=\lambda_{[k]}\}$$

Intuitively it is clear that the subset selection approach is applicable in certain cases while for others the indifference zone approach is needed. When $t > (k-m)$ the subset selection approach is usable since by choosing the maximum possible number of populations, m, a correct selection can be guaranteed. When

When $t \leq (k-m)$ and even if all m populations are chosen it is still possible for all t best populations to remain unselected. Hence an indifference zone must be imposed on the parameter space and the probability of a correct selection is only maintained over the preference zone.

A. Subset Selection Approach $(t > k-m)$

Let H be the class of functions defined in (1.6.2) and $\forall h \varepsilon H$ let

(3.1.1) R(h): Select $\pi_i \Leftrightarrow T_i \geq \max\{T_{[k-m+1]}, h^{-1}(T_{[k]})\}$.

Goal G_A: Given P*,k,m and t find $h \varepsilon H$ such that

(3.1.2) $$P_{\underset{\sim}{\lambda}}[CS|R(h)] \geq P^* \ \forall \underset{\sim}{\lambda} \varepsilon \Omega$$

In this case $[CS|R(h)]$ stands for the selection of any restricted subset containing at least one of $\pi_{(k-t+1)},\ldots,\pi_{(k)}$. It will be shown that G_A is always attainable for some $h\varepsilon H$ under the assumptions (1.6.2).

B. Indifference Zone Approach $(t \leq k-m)$

Let $\{h_n(\cdot)\}$ be a sequence of functions satisfying (1.1.3) and let

(3.1.3) R(n): Select $\pi_i \Leftrightarrow T_i \geq \max\{T_{[k-m+1]}, h_n^{-1}(T_{[k]})\}$.

which is the same rule as that proposed for the original problem of Chapter I(t = 1).

Goal G_B: Given P^*, m, t, $\{h_n(\cdot)\}$ and $p(\lambda)$ find the smallest common sample size n needed to satisfy the probability requirement:

$$(3.1.4) \qquad P_{\underset{\sim}{\lambda}}[CS|R(n)] \geq P^* \quad \forall \underset{\sim}{\lambda} \varepsilon \Omega^t(p).$$

The event $[CS|R(n)]$ has the same meaning as in G_A. The theorems in the next section will show that the indifference zone is needed in order to attain all P^* levels whenever $k-m \geq t$. Furthermore it will be shown that under the hypotheses of Theorem 3.3.1 any P^* level can be attained by choosing n sufficiently large.

In a practical situation the goal may be somewhat more general, to select both the sequence $\{h_n(\cdot)\}$ and then the sample size n.

3.2 Infimum of the Probability of a Correct Selection

Since the form of the rules (3.1.1) for $h \varepsilon H$ and (3.1.3) for $h_j \varepsilon \{h_n\}$ are the same and since a correct selection occurs for either iff at least one of the t best populations is selected, the following lemma is applicable in both cases. The symbol R denotes any rule (3.1.1) or (3.1.3).

Lemma 3.2.1.

$$(3.2.1) \quad P_{\underset{\sim}{\lambda}}[CS|R] \text{ is } \begin{cases} \uparrow \text{ in } \lambda_{[i]} \text{ for any } i \varepsilon \mathcal{S} \text{ when all other } \lambda_{[j]}\text{'s are fixed.} \\ \downarrow \text{ in } \lambda_{[i]} \text{ for any } i \varepsilon \mathcal{S}^c \text{ when all other } \lambda_{[j]}\text{'s are fixed.} \end{cases}$$

where

$$\begin{cases} \mathcal{S} = \{k-t+1, \dots, k\} \\ \mathcal{S}^c = \{1, \dots, k-t\} \end{cases}$$

Proof. Since $P_{\lambda}[CS|R] = E_{\lambda}[\eta(T)]$ where

$$\eta(T) = \begin{cases} 1 \ , & T_{(j)} \geq \max\{T_{[k-m+1]}, h^{-1}(T_{[k]})\} \text{ for some } j \varepsilon \mathcal{D} \\ 0 \ , & \text{otherwise} \end{cases}$$

it suffices to show

$$\eta(T) \text{ is } \begin{cases} \text{A.} \ \downarrow \text{ in } T_{(i)} \text{ for any } i \varepsilon \mathcal{D}^c \text{ when all other } T_{(j)}\text{'s are fixed} \\ \qquad \text{and} \\ \text{B.} \ \uparrow \text{ in } T_{(\ell)} \text{ for any } \ell \varepsilon \mathcal{D} \text{ when all other } T_{(j)}\text{'s are fixed.} \end{cases}$$

Case A: If $i \varepsilon \mathcal{D}^c$, $\eta(T) = 0$ and T, T' satisfy $T'_{(i)} > T_{(i)}$ and $T'_{(j)} = T_{(j)} \forall j \neq i$, it must be shown that $\eta(T') = 0$.

$$\eta(T) = 0 \Leftrightarrow T_{(j)} < \max\{T_{[k-m+1]}, h^{-1}(T_{[k]})\} \ \forall j \varepsilon \mathcal{D}$$

(1) If $t < k-m+1$, there are two possible cases

(i) If $h(T_{[k-m+1]}) > T_{[k]} \Rightarrow \forall j \varepsilon \mathcal{D}, T_{(j)} < T_{[k-m+1]}$

Now $T'_{[k-m+1]} \geq T_{[k-m+1]}$

$$\Rightarrow T'_{(j)} = T_{(j)} < T'_{[k-m+1]} \ \forall j \varepsilon \mathcal{D}$$

$$\Rightarrow h(T'_{(j)}) < h(T'_{[k-m+1]}) \leq \max\{T'_{[k]}, h(T'_{[k-m+1]})\}$$

$$\Rightarrow \eta(T') = 0$$

(ii) If $h(T_{[k-m+1]}) \leq T_{[k]} \Rightarrow h(T_{(j)}) < T_{[k]} \ \forall j \varepsilon \mathcal{D}$

Again $T_{(i)} < T'_{(i)} \Rightarrow T_{[k]} \leq T'_{[k]}$

$$\Rightarrow h(T'_{(j)}) = h(T_{(j)}) < T'_{[k]} \leq \max\{T'_{[k]}, h(T'_{[k-m+1]})\} \ \forall j \varepsilon \mathcal{D}$$

$$\Rightarrow \eta(T') = 0.$$

(2) If $t \geq k-m+1 \Rightarrow T_{[k]} > h(T_{[k-m+1]}) \Rightarrow$ (ii) applies

Case B: If $i \varepsilon \mathcal{B}$, $\eta(\underset{\sim}{T}) = 1$, $T'_{(i)} > T_{(i)}$ and $T'_{(j)} = T_{(j)}$ $\forall j \neq i$, it must be shown that $\eta(\underset{\sim}{T'}) = 1$.

$$\eta(\underset{\sim}{T}) = 1 \Leftrightarrow \exists j_0 \varepsilon \mathcal{B} \ni h(T_{(j_0)}) \geq \max\{h(T_{[k-m+1]}), T_{[k]}\}$$

$$\Leftrightarrow h(T_{(j_0)}) \geq T_{[k]} \text{ and } T_{(j_0)} \geq T_{[k-m+1]}.$$

Two subcases arise.

(1) If $i = j_0 \Rightarrow T'_{(j_0)} > T_{(j_0)}$ and by arguments similar to the above it can be shown $T'_{(j_0)} \geq T'_{[k-m+1]}$ and $h(T'_{(j_0)}) \geq T'_{[k]}$

$$\Rightarrow \eta(\underset{\sim}{T'}) = 1 \text{ since } j_0 \varepsilon \mathcal{B}$$

(2) If $i \neq j_0 \Rightarrow T'_{(j_0)} = T_{(j_0)}$. Again by considering different subcases it can be shown that either

(a) $T'_{(j_0)} \geq \max\{T'_{[k-m+1]}, h^{-1}(T'_{[k]})\}$

or

(b) $T'_{(i)} \geq \max\{T'_{[k-m+1]}, h^{-1}(T'_{[k]})\}$

$$\Rightarrow \eta(\underset{\sim}{T'}) = 1 \text{ since } \{j_0, i\} \subset \mathcal{B}.$$

This completes the proof.

Hence the infimum of the probability of a correct selection occurs when for all $i \varepsilon \mathcal{B}^c$, $\lambda_{[i]}$ is as large as possible and when for all $j \varepsilon \mathcal{B}$, $\lambda_{[j]}$ is as small as possible.

Theorem 3.2.1.

For $t \leq k-m$, $\inf\limits_{\Omega^t(p)} P_{\underset{\sim}{\lambda}}[CS|R] = \inf\limits_{\Omega_0^t(p)} P_{\underset{\sim}{\lambda}}[CS|R]$

For $t > k-m$, $\inf\limits_{\Omega} P_{\underset{\sim}{\lambda}}[CS|R] = \inf\limits_{\Omega_0} P_{\underset{\sim}{\lambda}}[CS|R]$

The next object is to obtain an explicit expression for the infimum so that

A. When $k-m < t$, questions concerning the choice of $h \varepsilon H$ to attain G_A can be answered and

B. When $k-m \geq t$, questions concerning the consistency of $\{R(n)\}$ can be answered and computations of required sample sizes for G_B can be performed.

The main interest here is in Case B although Case A will also be studied.

Theorem 3.2.2. For any $t=1,\ldots,k$ such that $t \leq k-m$ and any $R(n)$ of form (3.1.3)

$$\inf_{\Omega^t(p)} P_{\underset{\sim}{\lambda}}[CS|R(n)] = \inf_{\lambda \varepsilon \Lambda'} \beta(\lambda,t,n) \text{ where}$$

$$(3.2.2) \quad \beta(\lambda,t,n) = \int_{E_n^\lambda} \{G_n(h_n(y)/p(\lambda))\}^{k-t} I\left(\frac{G_n(y/p(\lambda))}{G_n(h_n(y)/p(\lambda))}; k-m-t+1, m\right) d\{G_n(y/\lambda)\}^t$$

For any $t=1,\ldots,k$ such that $t > k-m$ and any $R(h)$ as in (3.1.1)

$$\inf_{\Omega} P_{\underset{\sim}{\lambda}}[CS|R(h)] = \inf_{\lambda \varepsilon \Lambda} \phi(\lambda,t,h) \text{ where}$$

$$(3.2.3) \quad \phi(\lambda,t,h) = t\int_{-\infty}^{\infty} \{G_n(y/\lambda)\}^{t-1}\{G_n(h(y)/\lambda)\}^{k-t} dG_n(y/\lambda)$$

Proof. From Theorem 3.2.1 it follows that

$$\inf_{\Omega^t(p)} P[CS|R(n)] = \inf_{\lambda \varepsilon \Lambda'} P[\text{at least one of } \{T_{(k-t+1)},\ldots,T_{(k)}\} \geq \max\{T_{[k-m+1]}, e_n(T_{[k]})\}$$

where
$$\begin{cases} T_{[1]} \leq \cdots \leq T_{[k]} \text{ are the ordered } T_i\text{'s} \\ e_n(y) = h_n^{-1}(y) \\ T_{(k-t+1)},\ldots,T_{(k)} \text{ are iid w/cdf } G_n(\cdot/\lambda) \\ T_{(1)},\ldots,T_{(k)} \text{ are iid w/cdf } G_n(\cdot/p(\lambda)) \end{cases}$$

Let $Z = \max\{T_{(k-t+1)},\ldots,T_{(k)}\}$

$$\Rightarrow P[\text{at least one of } \{T_{(k-t+1)},\ldots,T_{(k)}\} \geq \max\{T_{[k-m+1]}, e_n(T_{[k]})\}]$$

$$= P[Z \geq \max\{T_{[k-m+1]}, e_n(T_{[k]})\}]$$

$$= P[Z \geq T_{[k-m+1]}, h_n(Z) \geq T_{[k]}]$$

$$= \sum_{i=k-m+1}^{k} P[Z=T_{[i]}, h_n(T_{[i]}) \geq T_{[k]}] \text{ since } [Z \geq T_{[k-m+1]}] = \bigcup_{i=k-m+1}^{k} [Z=T_{[i]}]$$

$$(3.2.4) \quad = \sum_{i=k-m+1}^{k} \sum_{p=k-t+1}^{k} P[Z=T_{(p)}=T_{[i]}, h_n(T_{(p)}) \geq T_{[k]}]$$

But $[Z = T_{(p)} = T_{[i]}]$

$$= \left[\begin{matrix} T_{(p)} > T_{(\ell)} \text{ for } (i-t)\ \ell\text{'s w/}\ell\varepsilon\mathcal{S}^c; T_{(p)} > T_{(\ell)}\ \forall\ \ell\varepsilon\mathcal{S}-\{p\}; \\ T_{(p)} < T_{(\ell)} \text{ for } (k-i)\ \ell\text{'s w/}\ell\varepsilon\mathcal{S}^c \end{matrix}\right]$$

$$= \sum_{\nu=1}^{\binom{k-T}{i-T}} [[T_{(p)} > T_{(j)}\ \forall\ j\varepsilon\mathcal{S}-\{p\} \text{ and } \forall j\varepsilon\mathcal{A}_\nu^{i-t}(\mathcal{S}); T_{(p)} < T_{(j)}\ \forall\ j\varepsilon\bar{\mathcal{A}}_\nu^{i-t}(\mathcal{S})].$$

So (3.2.4) can be rewritten as

$$\sum_{i=k-m+1}^{k} \sum_{p=k-t+1}^{k} \sum_{\nu=1}^{\binom{k-t}{i-t}} P \left[\begin{array}{l} T_{(j)} < T_{(p)} \ \forall\, j \varepsilon \mathcal{S}-\{p\} \text{ and } \forall\, j \varepsilon \mathcal{S}_{\nu}^{i-t}(\mathcal{S}); \\ \qquad\qquad h_n(T_{(p)}) > T_{(j)} \ \forall\, j \neq p; \\ T_{(j)} > T_{(p)} \ \forall\, j \varepsilon \bar{\mathcal{S}}_{\nu}^{i-t}(\mathcal{S}) \end{array} \right]$$

$$= \sum_{i=k-m+1}^{k} \sum_{p=k-T+1}^{k} \sum_{\nu=1}^{\binom{k-T}{i-T}} P \left[\begin{array}{l} T_{(j)} < T_{(p)} \ \forall\, j \varepsilon \mathcal{S}-\{p\} \text{ and } \forall\, j \varepsilon \mathcal{S}_{\nu}^{i-t}(\mathcal{S}); \\ T_{(p)} < T_{(j)} < h_n(T_{(p)}) \forall\, j \varepsilon \bar{\mathcal{S}}_{\nu}^{i-t}(\mathcal{S}) \end{array} \right]$$

$$(3.2.5) \quad = \sum_{i=k-m+1}^{k} t\binom{k-t}{i-t} \int_{E_n^{\lambda}} \{G_n(y/\lambda)\}^{t-1} \{G_n(y/p(\lambda))\}^{i-t} \{G_n(h_n(y)/p(\lambda)) - G_n(y/p(\lambda))\}^{k-i} \, dG_n(y/\lambda)$$

$$= \beta(\lambda,t,n).$$

The proof of the second assertion follows along the lines of the first after noting that $[Z \geq \max\{T_{[k-m+1]}, e_n(T_{[k]})\}] = [Z \geq e_n(T_{[k]})]$ since $t > k-m$. This completes the proof of the theorem.

<u>Remark 3.2.1</u>. When t=1

$$\beta(\lambda,1,n) = \int_{E_n^{\lambda}} \{G_n(h_n(y)/p(\lambda))\}^{k-1} I\left(\frac{G_n(y/p(\lambda))}{G_n(h_n(y)/p(\lambda))}; k-m-1+1,\ m\right) d\{G_n(y/\lambda)\}^{1}$$

$$= \psi(\lambda,n).$$

The evaluation of the k dimensional infimum has been reduced to the evaluation of a one dimensional infimum. In the location and scale parameter cases $\beta(\lambda,t,n)$ is independent of λ and hence the infimum of the probability of correct selection is completely evaluated. In the general case the following theorems allow evaluation.

<u>Theorem 3.2.3</u>. When $G_n(y)$, $h_n(y)$ and $p(\lambda)$ are sufficiently smooth so as to satisfy the hypotheses of Theorem 1.2.3 then $\beta(\lambda,t,n)$ is nondecreasing in λ.

<u>Proof</u>. $\beta(\lambda,t,n)$ can be rewritten in the following form

$$\beta(\lambda,t,n) = \int_{E_n} \{G_n(h_n(y)/p(\lambda))\}^{k-t} I(K_n(y,\lambda);k-t-m+1,m) \quad dG_n^*(y/\lambda)$$

where

$$\begin{cases} G_n^*(y/\lambda) = \{G_n(y/\lambda)\}^t \\[2ex] K_n(y,\lambda) = \dfrac{G_n(y/p(\lambda))}{G_n(h_n(y)/p(\lambda))} \end{cases}$$

$$= E_\lambda[\{G_n(h_n(y)/p(\lambda))\}^{k-t} I(K_n(y,\lambda);k-m-t+1,m)]$$

Now using Lemma 1.2.2 and the same method of proof as in Theorem 1.2.3 where

$$\frac{\partial G_n^*(y/\lambda)}{\partial\lambda} = t\{G_n(y/\lambda)\}^{t-1} \frac{\partial G_n(y/\lambda)}{\partial\lambda}$$

$$g_n^*(y/\lambda) = t\{G_n(y/\lambda)\}^{t-1} g_n(y/\lambda)$$

the result follows.

Remark 3.2.2. Since the hypotheses of Theorem 1.2.3 are satisfied for the first five examples of Chapter II, the present theorem allows complete evaluation of the infimum of the probability of correct selection for these cases.

Remark 3.2.3. In the case $k-m < t$ since $\phi(\lambda,t,h)=E_\lambda[\{G_n(h(y)/\lambda)\}^{k=t}]$ where the expectation is taken wrt $G_n^*(y/\lambda)$ above, some additional computation shows that if $E_n^\lambda = E_n$ $\forall\lambda\epsilon\Lambda$ and (1.2.7) holds when $p(\lambda)=\lambda$ and

$$(3.2.6)\quad g_n(y/\lambda)\ \frac{\partial G_n(h_n(y)/\lambda)}{\partial\lambda} - \frac{\partial G_n(y/\lambda)}{\partial\lambda}\ g_n(h_n(y)/\lambda)h_n'(y)\geq 0 \text{ ae in } E_n$$

then $\phi(\lambda,t,h)$ is non decreasing in λ. This result is a generalization of Theorem 2.2 of Gupta-Panchapakesan [33].

3.3 Properties of {R(n)}

Since the sequence of procedures {R(n)} proposed for the present problem is the same as that studied in Chapter I some of the questions concerning its properties can immediately be answered from the work of Section 1.3. However, the most important question is not answered by the earlier results-namely given a P* level can n be found to achieve (3.1.2). Note that the formal definition of a consistent sequence of rules is unchanged but now the event $[CS|R(n)]$ has a different meaning than before.

Theorem 3.3.1. If there exists $\lambda_0\epsilon\Lambda'$ and $N \geq 1$ such that

$$(3.3.1)\quad \inf_{\lambda\epsilon\Lambda'} \beta(\lambda,t,n) = \beta(\lambda_0,t,n)\ \forall n \geq N$$

then {R(n)} is consistent wrt $\Omega^t(p)$.

Proof. Under the assumption of the hypothesis (3.3.1) and by Theorem (3.2.2) it follows that

$$\inf_{\Omega^t(p)} P[CS|R(n)] = \beta(\lambda_0,t,n) \text{ for } n \geq N$$

Hence it suffices to show $\beta(\lambda_0,t,n) \to 1$ as $n\to\infty$. Since $p(\lambda_0)<\lambda_0$ there exists $\alpha \ni p(\lambda_0) < \alpha < \lambda_0$

Claim A: For any $k-m+1 \leq i < k$

$$\int \prod_{j=1}^{3} f_j^n(y)\,dG_n(y/\lambda_0) \qquad 0 \text{ as } n\to\infty \tag{3.3.2}$$

where

$$\begin{cases} f_1^n(y) = \{G_n(y/\lambda_0)\}^t \\ f_2^n(y) = \{G_n(y/p(\lambda_0))\}^{i-t} \\ f_3^n(y) = \{G_n(h_n(y)/p(\lambda_0))-G_n(y/p(\lambda_0))\}^{k-i} \end{cases}$$

Given $\epsilon > 0$ pick $M > N$ so that $\forall n \geq M \Rightarrow \begin{cases} G_n(\alpha/\lambda_0) < \epsilon/2 \\ G_n(\alpha/p(\lambda_0)) > 1 - \epsilon/2 \end{cases}$

$$\begin{aligned} \Rightarrow \forall y \geq \alpha, \ \prod_{j=1}^{3} f_j^n(y) &\leq f_3^n(y) \\ &\leq \{G_n(h_n(y)/p(\lambda_0))-G_n(y/p(\lambda_0))\} \\ &\leq 1-G_n(y/p(\lambda_0)) \\ &\leq 1-G_n(\alpha/p(\lambda_0)) \\ &\leq \epsilon/2 \qquad \forall_n \geq M \end{aligned}$$

So

$$\begin{aligned} 0 \leq \int_{-\infty}^{\infty} \prod_{j=1}^{3} f_j^n(y)\,dG_n(y/\lambda_0) &\leq \int_{-\infty}^{\alpha} 1\,dG_n(y/\lambda_0)+\int_{\alpha}^{\infty} \epsilon/2\ dG_n(y/\lambda_0) \\ &\leq G_n(\alpha/\lambda_0) + \epsilon/2\cdot 1 \leq \epsilon \ \forall_n \geq M \end{aligned}$$

Claim B:

$$(3.3.3) \quad t\int_{-\infty}^{\infty} \prod_{j=1}^{2} \ell_j^n(y)\,dG_n(y/\lambda_0) \to 1 \text{ as } n\to\infty$$

$$\text{where } \begin{cases} \ell_1^n(y) = \{G_n(y/\lambda_0)\}^{t-1} \\ \ell_2^n(y) = \{G_n(y/p(\lambda_0))\}^{k-t} \end{cases}$$

Note that for every $n \geq 1$

$$\Rightarrow t\int \prod_{j=1}^{2} \ell_j^n(y)\,dG_n(y/\lambda_0) \leq t\int_{-\infty}^{\infty} \{G_n(y/\lambda_0)\}^{t-1}\, dG_n(y/\lambda_0) = 1$$

Now given $1 > \epsilon' > 0$ choose ϵ so that $(1-\epsilon)^{k-t}(1-\epsilon^t) = 1-\epsilon'$ and then choose $M > N$ so that for every $n \geq M$

$$\Rightarrow \begin{cases} G_n(\alpha/\lambda_0) < \epsilon \\ G_n(\alpha/p(\lambda_0)) > 1-\epsilon \end{cases}$$

$$\Rightarrow \quad \ell_2^n(y) \geq \{G_n(\alpha/p(\lambda_0))\}^{k-t} \geq (1-\epsilon)^{k-t} \;\forall y \geq \alpha,\ n \geq M$$

So finally it follows that for every $n \geq M$

$$1 \geq t\int_{-\infty}^{\infty} \prod_{j=1}^{2} \ell_j^n(y)\,dG_n(y/\lambda_0) \geq t\int_{\alpha}^{\infty} \prod_{j=1}^{2} \ell_j^n(y)\,dG_n(y/\lambda_0)$$

$$\geq t\int_{\alpha}^{\infty} \{G_n(y/\lambda_0)\}^{t-1}(1-\epsilon)^{k-t}\,dG_n(y/\lambda_0)$$

$$\geq (1-\epsilon)^{k-t}[1-(G_n(\alpha/\lambda_0))^t]$$

$$\geq (1-\epsilon)^{k-t}[1-\epsilon^t] = 1-\epsilon'$$

Since (3.2.3) shows that $\beta(\lambda_0,t,n)$ is just the sum of (m-1) terms having form (3.3.2) and one term having form (3.3.3) the above argument completes the proof.

Remark 3.3.1. Under the hypothesis that there is a $\lambda_0 \varepsilon \Lambda$ such that the one dimensional infimum can always be evaluated at λ_0 i.e.

$$\inf_{\lambda\varepsilon\Lambda} \phi(\lambda,t,h) = \phi(\lambda_0,t,h)\ \forall h\varepsilon H\ (\lambda_0 \text{ independent of } h)$$

then the following holds

$$(3.3.4) \qquad \lim_{\substack{c\to\infty \\ \text{or} \\ d\to\infty}} \inf_{\Omega} P_{\underset{\sim}{\lambda}}[CS|R(h)] = 1.$$

For a given $h\varepsilon H$ the evaluation of the infimum over Ω may be accomplished using (3.2.3) or by some other method. The point is that c and/or d may always be chosen sufficiently large to attain any P* level. An example will be given in Section 3.4.

The remaining properties to be mentioned are those of the individual rules R(n) rather than the sequence {R(n)} and are stated in terms of the monotonicity properties for the probabilities of selecting individual populations. They are the standard properties defined in Chapter I and are listed here in catalog form for the sake of completeness. Let

$$\begin{cases} p^n_{\underset{\sim}{\lambda}}(i) = P_{\underset{\sim}{\lambda}}[R(n) \text{ selects } \pi_{(i)}] \\ S(n) = \text{Number of populations } R(n) \text{ selects} \\ T(n) = \text{Number of non best populations } R(n) \text{ selects} \end{cases}$$

(i) For any $n \geq 1$ and any $i=1,\ldots,k$ R(n) is strongly monotone in $\pi_{(i)}$.

(ii) For any $n \geq 1$, R(n) is unbiased and monotone.

(iii) If $\lambda_{[k]} > \lambda_{[k-1]} \Rightarrow \lim_{n\to\infty} p^n_{\underset{\sim}{\lambda}}(i) = \begin{cases} 1 & , \; i = k \\ 0 & , \; i \leq i < k \end{cases}$

$$S(n) \overset{p}{\to} 1 \text{ as } n\to\infty$$

$$T(n) \overset{p}{\to} 0 \text{ as } n\to\infty$$

(iv) For an arbitrary $\underset{\sim}{\lambda}\varepsilon\Omega$

$$E_{\underset{\sim}{\lambda}}[S(n)] = \sum_{i=1}^{k} \sum_{p=k-m}^{k-1} \sum_{\nu=1}^{\binom{k-1}{p}} \int_{-\infty}^{\infty} \prod_{j\varepsilon\mathcal{P}^p_\nu(i)} G_n^{(j)}(y) \prod_{j\varepsilon\bar{\mathcal{P}}^p_\nu(i)} \{G_n^{(j)}(h_n(y)) - G_n^{(j)}(y)\} dG_n^{(i)}(y)$$

(v) Under the assumptions of Corollary 1.4.2

$$\sup_{\Omega} E_{\underset{\sim}{\lambda}}[S(n)] = \sup_{\lambda\varepsilon\Lambda} k\int_{-\infty}^{\infty} \{G_n(h_n(y)/\lambda)\}^{k-1} I\left(\frac{G_n(y/\lambda)}{G_n(h_n(y)/\lambda)}; k-m, m\right) dG_n(y/\lambda)$$

Furthermore this last integral is non decreasing in λ. Hence if there is a greatest $\lambda_0\varepsilon\Lambda$

$$\Rightarrow \sup_{\Omega} E[S(n)] = k\int_{-\infty}^{\infty} \{G_n(h_n(y)/\lambda_0)\}^{k-1} I\left(\frac{G_n(y/\lambda_0)}{G_n(h_n(y)/\lambda_0)}; k-m, m\right) dG_n(y/\lambda_0)$$

In the location and scale parameter cases this one dimensional supremum is independent of λ_0.

3.4 Application to the Normal Means Problem

The normal theory example will be discussed in some detail since it is a commonly used model in practical work. The goal in this section will be enlarged from that stated in Section 3.1 to a more practical goal which includes partial specification of the $\{h_n(x)\}$ sequence.

Let $\pi_i \sim N(\mu_i, \sigma^2)$ for $i=1,\dots,k$ and suppose the common σ^2 is known. Also let $\mu_{[1]} \le \dots \le \mu_{[k]}$ be the ordered means, $\pi_{(i)}$ the (unknown) population with mean $\mu_{[i]}$ and let $\pi_{(k-t+1)},\dots,\pi_{(k)}$ be the t best populations.

Goal G: Given P*, m and also possibly n and $\delta \geq 0$ define a selection procedure, R(n), based on n observations from each π_i which selects a subset of populations not exceeding m in size, which contains at least one of $\pi_{(k-t+1)},\dots,\pi_{(k)}$ and satisfies

$$(3.4.1) \quad P_{\underset{\sim}{\mu}}[CS|R(n)] \geq P^* \ \forall \underset{\sim}{\mu} \in \Omega^t(\delta) = \{\underset{\sim}{\mu} | \mu_{[k-t+1]} - \mu_{[k-t]} \geq \delta\}$$

The event [CS|R(n)] occurs iff at least one of $\pi_{(k-t+1)},\dots,\pi_{(k)}$ is included in the selected subset.

The proposed procedure is of the form

$$(3.4.2) \quad \underline{R(n)}: \text{ Select } \pi_i \Leftrightarrow \bar{X}_i \geq \max\{\bar{X}_{[k-m+1]}, \bar{X}_{[k]} - d\sigma/\sqrt{n}\}$$

where the dependence of the sample means on n has been suppressed as usual. Both n and d must be determined before the rule R(n) is completely specified.

Theorem 3.4.1. If $t \leq k-m$ so that the indifference zone approach is used,

$$(3.4.3) \quad \inf_{\Omega^t(\delta)} P[CS|R(n)] = \int_{-\infty}^{\infty} \{\Phi(y+d+\frac{\sqrt{n}\delta}{\sigma})\}^{k-t} I\left(\frac{\Phi(y+\frac{\sqrt{n}\delta}{\sigma})}{\Phi(y+d+\frac{\sqrt{n}\delta}{\sigma})}; k-t-m+1, m\right) d\{\Phi(y)\}^t$$

Proof. Note that $G_n(y/\mu) = P_\mu[\bar{X} \leq y] = \Phi(\frac{\sqrt{n}}{\sigma}(y-\mu))$ and after substituting in (3.2.2) and making the appropriate change of variables the result follows.

Using (3.4.3) and dominated convergence the following can easily be computed:

$$\text{(3.4.4)} \quad \lim_{\delta\to\infty} \inf_{\Omega^t(\delta)} P[CS|R(n)] = 1$$

$$\text{(3.4.5)} \quad \lim_{d\to\infty} \inf_{\Omega^t(p)} P[CS|R(n)] = \sup_{d\geq 0} \inf_{\Omega^t(\delta)} P[CS|R(n)]$$

$$= \int_{-\infty}^{\infty} I(\Phi(y+\frac{\sqrt{n}\delta}{\sigma});k-t-m+1,\ m)\ d\{\Phi(y)\}^t$$

Remark 3.4.1. The fixed subset size rule ($d=+\infty$) which selects the m populations corresponding to the largest m sample means has infimum of probability of correct selection specified by (3.4.5).

If a subset selection type requirement ($\delta=0$) is desired for a rule of form $R(n)$ when $1 \leq t \leq k-m$ the highest P^* level that can be attained is

$$\text{(3.4.6)} \quad \sup_{d\geq 0} \inf_{\Omega} P[CS|R(n)] = 1-\frac{(k-m)_t}{(k)_t}, \qquad 1 \leq m \leq k-t$$

where $n_\ell = n(n-1)\ldots.(n-\ell+1)$

For $t=1$ this reduces to the result m/k which was obtained previously. Equation (3.4.6) shows that if both n and δ are fixed by the experimenter it may be impossible to attain a given P^* level by merely increasing d. It may be necessary to increase n or δ.

Remark 3.4.2. If n is fixed and the subset selection approach is to be used (k-m < t) then take $H = \{h_d(x) | h_d(x) = x+d/\sqrt{n}$ for $d\varepsilon[0,\infty)\}$. The rule corresponding to $h_d \varepsilon H$ is

(3.4.7) R(d): Select $\pi_i \Leftrightarrow \bar{X}_i \geq \max\{\bar{X}_{[k]}-d\sigma/\sqrt{n}, \bar{X}_{[k-m+1]}\}$

From the general result and a change of variables the following can easily be derived

$$(3.4.8) \quad \inf_{\Omega} P[CS|R(d)] = \int_{-\infty}^{\infty} \{\Phi(y+d)\}^{k-t} d\{\Phi(y)\}^t$$

Setting the right hand side of (3.4.8) equal to P* gives the d value satisfying the requirement (3.1.2).

Since R(n) is of the form (3.1.1) with $h_n(x) = x+d\sigma/\sqrt{n}$ satisfying (1.1.3) the general theory applies to give all the usual properties of R(n). A few of these properties are stated for this special case.

(i) For every $d > 0$ and $\delta > 0$, $\lim_{n\to\infty} \inf_{\Omega(\delta)} P[CS|R(n)] = 1$

(ii) For any $i=1,\ldots,k$, $\{R(n)\}$ is strongly monotone in $\pi_{(i)}$.

(iii) For any $\underset{\sim}{\mu}\varepsilon\Omega$ such that $\mu_{[k]} > \mu_{[k-1]} \Rightarrow S(n) \overset{P}{\to} 1$ as $n\to\infty$.

(iv) $\sup_{\Omega} E[S(n)] = k\int_{-\infty}^{\infty} \{\Phi(y+d)\}^{k-1} I(\frac{\Phi(y)}{\Phi(y+d)};k-m,m)d\Phi(y)$

For the purpose of implementing {R(n)}, Table V has been prepared which lists the values of $\sqrt{n}\delta/\sigma$ satisfying

$$(3.4.9) \quad P^* = \int_{-\infty}^{\infty} \{\Phi(y+d+\frac{\sqrt{n}\delta}{\sigma})\}^{k-t} I\left(\frac{\Phi(y+\frac{\sqrt{n}\delta}{\sigma})}{\Phi(y+d+\frac{\sqrt{n}\delta}{\sigma})};k-t-m+1, m\right) \quad d\{\Phi(y)\}^t$$

for various P*,m,k,t and d. In general, for given P* and δ there will be many combinations of d and n satisfying (3.4.9). The choice of d should be made by considerations similar to those in Section 2.1. In particular tables of the following quantities have been constructed for various parameters:

(i) $e(P^*,k,m,d) = n(d)/n(\infty)$ is a measure of the additional sample size needed over and above that for the fixed subset size rule to attain the same probability requirement. (Table VI)

(ii) $E_{\underset{\sim}{\mu}}[S(n)]$ measures the savings realized for the rule R(n) over the fixed size subset rule when $\underset{\sim}{\mu}$ is the true underlying vector of means. (Table III and IV)

Having chosen d the required sample size can now be determined and the experiment performed.

Table V

Lists the value of $\frac{\sqrt{n}}{\sigma}\delta$ needed to attain P* levels .75, .90 and .975 for the rules given by d = .4, .8, 1.2, 1.6 for various k, m and t.

		d	.4			.8		
				P*			P*	
k	m	t	.75	.90	.975	.75	.90	.975
4	2	2	.419	1.121	1.914	.136	.818	1.585
6	2	2	.853	1.498	2.234	.638	1.261	1.960
	3	2	.797	1.457	2.203	.447	1.089	1.828
	4	2	.791	1.453	2.203	.398	1.054	1.804
8	2	2	1.080	1.701	2.406	.894	1.492	2.164
		3	.656	1.250	1.921	.458	1.025	1.664
	3	2	1.015	1.650	2.375	.698	1.308	2.007
		3	.596	1.203	1.898	.263	.848	1.523
	4	2	1.005	1.644	2.375	.627	1.257	1.976
10	2	2	1.230	1.837	2.531	1.061	1.644	2.304
		3	.835	1.410	2.062	.660	1.207	1.828
		4	.546	1.105	1.750	.361	.892	1.492
	3	2	1.162	1.781	2.492	.865	1.460	2.140
		3	.771	1.359	2.031	.462	1.023	1.671
		4	.484	1.058	1.718	.164	.712	1.343
	4	2	1.150	1.773	2.484	.785	1.396	2.093
		3	.759	1.351	2.031	.387	.966	1.632
15	2	2	1.468	2.054	2.726	1.323	1.886	2.523
		3	1.103	1.652	2.281	.954	1.478	2.070
		4	.853	1.380	1.984	.701	1.199	1.765
	3	2	1.395	1.994	2.679	1.130	1.701	2.351
		3	1.031	1.593	2.234	.759	1.293	1.898
		4	.783	1.324	1.945	.505	1.011	1.601
	4	2	1.377	1.982	2.671	1.041	1.625	2.296
		3	1.015	1.582	2.234	.669	1.218	1.843
		4	.767	1.314	1.945	.481	.943	1.554
20	2	2	1.619	2.195	2.851	1.486	2.039	2.664
		3	1.264	1.880	2.406	1.128	1.642	2.214
		4	1.029	1.539	2.125	.891	1.375	1.925
	3	2	1.543	2.128	2.796	1.298	1.853	2.492
		3	1.189	1.736	2.367	.937	1.457	2.039
		4	.953	1.476	2.085	.701	1.187	1.757
	4	2	1.521	2.115	2.789	1.204	1.771	2.421
		3	1.169	1.722	2.359	.842	1.375	1.984
		4	.934	1.466	2.078	.607	1.109	1.695

Table V (Cont.)

k	m	t	d = 1.2, P* = .75	1.2, .90	1.2, .975	d = 1.6, P* = .75	1.6, .90	1.6, .975
4	2	2	*	.605	1.351	*	.476	1.218
6	2	2	.521	1.130	1.812	.464	1.070	1.750
	3	2	.181	.800	1.500	.009	.613	1.289
	4	2	.029	.675	1.414	*	.343	1.062
8	2	2	.802	1.390	2.046	.763	1.347	2.000
		3	.356	.910	1.531	.310	.859	1.476
	3	2	.490	1.076	1.742	.377	.949	1.593
		3	.031	.587	1.218	*	.435	1.039
	4	2	.316	.921	1.617	.104	.687	1.343
10	2	2	.984	1.558	2.203	.954	1.525	2.164
		3	.576	1.113	1.710	.543	1.074	1.664
		4	.270	.789	1.367	.231	.746	1.320
	3	2	.668	1.257	1.898	.601	1.160	1.781
		3	.271	.804	1.406	.173	.693	1.273
		4	*	.472	1.062	*	.343	.906
	4	2	.513	1.095	1.765	.351	.912	1.546
		3	.095	.648	1.281	*	.437	1.031
15	2	2	1.263	1.820	2.445	1.242	1.796	2.414
		3	.892	1.406	1.796	2.870	1.380	1.945
		4	.636	1.125	1.761	.612	1.097	1.640
	3	2	.994	1.541	2.156	.935	1.474	2.078
		3	.617	1.125	1.687	.556	1.052	1.601
		4	.357	.835	1.382	.292	.757	1.281
	4	2	.824	1.377	2.007	.716	1.252	1.859
		3	.444	.960	1.539	.332	.826	1.375
		4	.183	.671	1.242	.062	.525	1.054
20	2	2	1.435	1.980	2.593	1.418	1.962	2.570
		3	1.076	1.582	2.140	1.059	1.560	2.117
		4	.838	1.312	1.843	.820	1.291	1.820
	3	2	1.180	1.716	2.320	1.134	1.664	2.257
		3	.818	1.314	1.859	.771	1.257	1.789
		4	.579	1.039	1.566	.530	.980	1.492
	4	2	1.017	1.554	2.171	.933	1.457	2.054
		3	.650	1.152	1.710	.566	1.050	1.578
		4	.413	.875	1.418	.324	.765	1.277

This choice of d insures that the probability level P can even be attained over all Ω no matter what n is used.

Table IV

This table lists $n(d)/n(\infty)$ where $n(a)$ is the sample size necessary for the rule $R(n)$: Select $\pi_i \Leftrightarrow \bar{X}_i \geq \max\{\bar{X}_{[k-m+1]}, \bar{X}_{[k]} - a\sigma/\sqrt{n}\}$ to satisfy $P_{\underset{\sim}{\mu}}[CS/R(n)] = P^* \forall \underset{\sim}{\mu} \epsilon\ \Omega^{t}(\delta)$

		d	.4		.8		1.2		1.6	
k	m	t	.90	.975	.90	.975	.90	.975	.90	.975
4	2	2	9.97	2.98	5.31	2.04	2.90	1.48	1.80	1.21
6	2	2	2.10	1.71	1.49	1.31	1.20	1.12	1.07	1.05
	3	2	11.17	4.00	6.21	2.76	3.35	1.86	1.97	1.37
8	2	2	1.65	1.48	1.27	1.20	1.10	1.07	1.03	1.02
	3	2	3.66	2.53	2.30	1.81	1.55	1.36	1.21	1.14
	4	2	12.94	4.79	7.57	3.32	4.06	2.22	2.26	1.53
10	2	2	1.48	1.39	1.19	1.15	1.07	1.05	1.02	1.01
	3	2	2.62	2.10	1.76	1.55	1.30	1.22	1.11	1.07
	4	2	5.26	3.23	3.26	2.29	2.01	1.63	1.39	1.25
15	2	2	1.32	1.28	1.11	1.10	1.04	1.03	1.01	1.01
		3	1.45	1.39	1.16	1.14	1.05	1.04	1.01	1.01
	3	2	1.91	1.71	1.39	1.32	1.14	1.11	1.04	1.03
	4	2	2.81	2.27	1.89	1.68	1.35	1.28	1.12	1.10
20	2	2	1.26	1.24	1.09	1.08	1.02	1.02	1.01	1.01
		3	1.34	1.30	1.12	1.10	1.04	1.03	1.01	1.01
	3	2	1.68	1.57	1.27	1.24	1.09	1.08	1.03	1.02
		3	1.98	1.80	1.39	1.33	1.13	1.11	1.04	1.03
	4	2	2.25	1.96	1.58	1.48	1.21	1.19	1.07	1.06

BIBLIOGRAPHY

BIBLIOGRAPHY

1. Alam, K. and Rizvi, M. H. (1966). Selection from multivariate normal populations. Ann. Inst. Statist. Math. (18), 307-318.

2. Bahadur, R. R. (1950). On a problem in the theory of k populations. Ann. Math. Statist. (21), 362-375.

3. Barlow, R. E. and Gupta, S. S. (1969). Selection procedures for restricted families of distributions. Ann. Math. Statist. (40), 905-917.

4. Barlow, R. E., Gupta, S. S. and Panchapakesan, S. (1969). On the distribution of the maximum and minimum of ratios of order statistics. Ann. Math. Statist. (40), 918-934.

5. Barr, D. R. and Rizvi, M. H. (1969). Ranking and selection problems of uniform distributions. Trabajos Estadist. (17), 15-31.

6. Barron, A. M. (1969). A class of sequential multiple decision procedures. Department of Statistics, Purdue University, mimeograph series No. 169.

7. Barron, A. M. and Gupta, S. S. (1972). A class of non-eliminating sequential multiple decision procedures. In Operations Research Verfahren (ed. Henn, Künzi and Schubert), pp. 11-37, Verlag Anton Hain Meisenheimams Glan, Germany.

8. Bechhofer, R. E. (1954). A single-sample multiple decision procedure for ranking means of normal populations with known variances. Ann. Math. Statist. (25), 16-39.

9. Bechhofer, R. E. and Sobel, M. (1954). A single sample multiple decision procedure for ranking variances of normal populations. Ann. Math. Statist. (25), 273-289.

10. Bechhofer, R. E., Dunnett, C. W. and Sobel, M. (1954). A two sample multiple decision procedure for ranking means of normal populations with a common unknown variance. Biometrika (41), 170-176.

11. Bechhofer, R. E. (1958). A sequential multiple decision procedure for selecting the best one of several normal populations with a common unknown variance, and its use with various experimental designs. Biometrics (14), 408-429.

12. Bechhofer, R. E., Kiefer, J. and Sobel, M. (1968). Sequential Identification and Ranking Procedures. Statistical Research Monographs, Vol. III, The University of Chicago Press, Chicago.

13. Chambers, M. L. and Jaratt, D. (1964). Use of double sampling for selecting the best population. Biometrika (51), 49-64.

14. Deeley, J. J. (1965). Multiple decision procedures from an empirical Bayes approach. Department of Statistics, Purdue University, mimeograph series No. 45.

15. Desu, M. and Sobel, M. (1968). A fixed subset-size approach to a selection problem. Biometrika (55), 401-410.

16. Desu, M. and Sobel, M. (1971). Non parametric procedures for selecting fixed size subsets. In Statistical Decision Theory and Related Topics (ed. S. S. Gupta and J. Yackel), pp. 255-274, Academic Press, N. Y.

17. Dunnett, C. W. (1955). A multiple comparison procedure for comparing several treatments with a control. J. American Statist. Assoc. (50), 1096-1121.

18. Gnanadesikan, M. (1966). Some selection and ranking procedures for multivariate normal populations. Ph.D. Thesis, Department of Statistics, Purdue University, Lafayette, Indiana.

19. Gnanadesikan, M. and Gupta, S. S. (1970). A selection procedure for multivariate normal distributions in terms of generalized variances. Technometrics (12), 103-117.

20. Gupta, S. S. (1956). On a decision rule for a problem in ranking means. Institute of Statistics, University of North Carolina, Chapel Hill, N. C., mimeograph series No. 150.

21. Gupta, S. S. and Sobel, M. (1958). On selecting a subset which contains all populations better than a standard. Ann. Math. Statist. (29), 235-244.

22. Gupta, S. S. and Sobel, M. (1960). Selecting a subset containing the best of several binomial populations. In Contributions to Probability and Statistics (ed. I. Olkin), pp. 224-248, Stanford Univ. Press, Stanford, Cal.

23. Gupta, S. S. and Sobel, M. (1962a). On selecting a subset containing the population with smallest variance. Biometrika (49), 495-507.

24. Gupta, S. S. and Sobel, M. (1962b). On the smallest of several correlated F statistics. Biometrika (49), 509-523.

25. Gupta, S. S. (1963). On a selection and ranking procedure for gamma populations. Ann. Inst. Statist. Math. (14), 199-216.

26. Gupta, S. S. (1965). On some multiple decision (selection and ranking) rules. Technometrics (7), 225-245.

27. Gupta, S. S. (1966). On selection and ranking procedures for multivariate normal populations using distance functions. In Multivariate Analysis (ed. P. R. Krishnaiah), pp. 457-475, Academic Press, N. Y.

28. Gupta, S. S. and Deverman, J. (1969). On a selection procedure concerning the t-best populations. Technical Report, Sandia Corporation, Livermore, California.

29. Gupta, S. S. and Panchapakesan, S. (1969). Some selection and ranking procedures for multivariate normal populations. In Multivariate Analysis II (ed. P. R. Krishnaiah), pp. 475-505, Academic Press, N. Y.

30. Gupta, S. S. and McDonald, G. C. (1970). On some classes of selection procedures based on ranks. In Nonparametric Techniques in Statistical Inference (ed. M. L. Puri), pp. 491-514, Cambridge University Press, London.

31. Gupta, S. S. and Studden, W. J. (1970). On some selection and ranking procedures with applications to multivariate populations. In Essays in Probability and Statistics (ed. R. C. Bose et. al.), pp. 327-338, University of North Carolina Press, Chapel Hill, North Carolina.

32. Gupta, S. S. and Nagel, K. (1971). On some contributions to multiple decision theory. In Statistical Decision Theory and Related Topics, (ed. S. S. Gupta and J. Yackel), pp. 79-102, Academic Press, N. Y.

33. Gupta, S. S., and Panchapakesan, S. (1972). On a class of subset selection procedures. Ann. Math. Statist. (43), 814-822.

34. Hoel, P. G. (1937). A significance test for component analysis. Ann. Math. Statist. (8), 149-158.

35. Huang, W. T. (1972). Some contributions to sequential and ranking procedures. Department of Statistics, Purdue University, mimeograph series No. 295.

36. Krishnaiah, P. R. and Rizvi, M. H. (1966). Some procedures for selection of multivariate normal populations better than a control. In Multivariate Analysis (ed. P. R. Krishnaiah), pp. 477-490, Academic Press, N. Y.

37. Lehmann, E. L. (1959). Testing Statistical Hypotheses, John Wiley, N. Y.

38. Lehmann, E. L. (1961). Some Model I problems of selection. Ann. Math. Statist. (32), 990-1012.

39. Mahamunulu, D. M. (1967). Some fixed-sample ranking and selection problems. Ann. Math. Statist. (38), 1079-1091.

40. McDonald, G. C. (1969). On some distribution-free ranking and selection procedures. Department of Statistics, Purdue University, mimeograph series No. 174.

41. McDonald, G. C. (1972). Some multiple comparison selection procedures based on ranks. Sankhyā Series A (34), 53-64.

42. Mosteller, F. (1948). A k-sample slippage test for an extreme population. Ann. Math. Statist. (19), 58-65.

43. Nagel, K. (1970). On subset selection rules with certain optimality properties. Department of Statistics, Purdue University, mimeograph series No. 222.

44. Panchapakesan, S. (1969). A class of selection procedures including procedures for restricted families of distributions. Department of Statistics, Purdue University, mimeograph series No. 189.

45. Panchapakesan, S. (1969). Some contributions to multiple decision (selection and ranking) procedures. Department of Statistics, Purdue University, mimeograph series No. 191.

46. Patterson, D. W. (1966). Contributions to the theory of ranking and selection problems. Ph.D. Thesis, State Univ. of Rutgers, New Brunswick, N. J.

47. Paulson, E. (1949). A multiple decision procedure for certain problems in the analysis of variance. Ann. Math. Statist. (20), 95-98.

48. Paulson, E. (1952). On the comparison of several experimental categories with a control. Ann. Math. Statist. (23), 239-246.

49. Paulson, E. (1962). A sequential procedure for comparing several experimental categories with a standard or control. Ann. Math. Stat. (33), 438-443.

50. Paulson, E. (1964). A sequential procedure for selecting the population with largest mean from k normal populations. Ann. Math. Statist. (35), 174-180.

51. Paulson, E. (1967). Sequential procedures for selection of the best one of several binomial populations. Ann. Math. Statist. (38), 117-123.

52. Rizvi, M. H. and Sobel, M. (1967). Nonparametric procedures for selecting a subset containing the population with largest α-quantile. Ann. Math. Statist. (38), 1788-1803.

53. Rizvi, M. H. (1970). Some selection problems involving folded normal distributions. Technometrics (12), 355-369.

54. Seal, K. C. (1955). On a class of decision procedures for ranking means of normal populations. Ann. Math. Statist. (26), 387-398.

55. Seal, K. C. (1958). On ranking parameters of scale in Type III populations. J. Amer. Statist. Assoc. (53), 164-175.

56. Sobel, M. and Huyett, M. M. (1957). Selecting the best one of several binomial populations. Bell Systems Tech. J. (36), 537-576.

57. Sobel, M. (1967). Nonparametric procedures for selecting the t populations with the largest α-quantiles. Ann. Math. Statist. (38), 1804-1816.

58. Sobel, M. (1969). Selecting a subset containing at least one of the t best populations. In Multivariate Analysis II (ed. P. R. Krishnaiah), pp. 515-540, Academic Press, N. Y.

Unclassified

Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)	2a. REPORT SECURITY CLASSIFICATION
Purdue University	Unclassified
	2b. GROUP

3. REPORT TITLE

A Restricted Subset Selection Approach to Ranking and Selection Problems

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Technical Report, May 1973

5. AUTHOR(S) (Last name, first name, initial)

Santner, Thomas J.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
May 1973	125	58

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
N00014-67-A-0226-00014	Mimeo Series #318
b. PROJECT NO.	
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	

10. AVAILABILITY/LIMITATION NOTICES

Distribution of this document is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Office of Naval Research Washington, D.C.

13. ABSTRACT

Let $(\mathcal{X},\mathcal{B},P_i)$ be k populations where $\mathcal{X}$ is a finite dimensional Euclidean space and $P_i \varepsilon \mathcal{P}$ a known family of probability measures. If $\lambda_i=\lambda_i(P_i)$ is a scalar quantity measuring the worth of π_i then it is desired to select the population corresponding to $\lambda_{[k]}$, the largest λ_i.

It is assumed there is a consistent sequence of estimators for λ_i, $\{T_{in}\}$, an upper bound specifying the maximum number of populations to be selected ,m, and a function $p(\lambda)<\lambda$ so that $\Omega(p)=\{\underset{\sim}{\lambda}|\lambda_{[k-1]} \leq p(\lambda_{[k]})\}$ is that set of vectors of λ_i's over which it is desired to detect the best the best population.

A class of rules is proposed which select a random number of populations between one and m and guarantee at least a P* probability of correct selection whenever the true underlying vector of λ_i's is in $\Omega(p)$.

Monotone behavior, asymptotic properties and other features of the rules are studied. Applications to specific populations are made and tables provided. Several generalizations and other formulations are made.

DD FORM 1 JAN 64 1473

Unclassified

Security Classification