Inequalities and Asymptotic Bounds for Ramsey Numbers II

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0. Introduction

Professor Erdös has stimulated a great deal of interest in extremal problems in graph theory, [2], [3]. In particular, his work on Ramsey's Theorem which dates back to 1935 in his joint paper with Szekeres [4] has influenced a large amount of research. To this date no one has improved on Erdös lower bounds for Ramsey numbers [1] which was established by the ingenius probabilistic methods which he invented.

This paper is devoted to obtaining upper bounds for Ramsey numbers. Our methods are an extension of the methods of Graver and Yackel [5]. The main result which we obtain is that the Ramsey number $R(n_1, n_2)$ satisfies the inequality

$$R(n_1, n_2) \le C \left(\frac{\log \log n_2}{\log n_2}\right)^{n_1-2} n_2^{n_1-1}$$

where C is bounded for all n_1. This inequality is only of value when n_1 is fixed and n_2 is large. In particular, it does not cover the case of $R(n, n)$ which was treated in Yackel, [6].

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1. Definitions

We consider Ramsey's Theorem as it pertains to partitions of pairs of elements of a finite set S into two disjoint classes. Our presentation will use the graph theoretic representation in which the pairs of elements of S in one class determine the edges of the graph.

Definition 1. $I(G)$, the independence number of the graph G, is the maximum number of points of G that can be chosen so that no two are joined by an edge.

Definition 2. $C(G)$, the clique number of the graph G, is the maximum number of points in any complete subgraph of G.

Definition 3. G is a Ramsey (n_1, n_2)-graph if $n_1 > C(G)$ and $n_2 > I(G)$.

Definition 4. $R(n_1, n_2)$ is the largest integer such that there is a Ramsey (n_1, n_2)-graph on $R(n_1, n_2)$ points.

Our primary concern in this paper is to study the local connectedness of a Ramsey (n_1, n_2)-graph and to use this connectedness in obtaining bounds for $R(n_1, n_2)$. To facilitate this study we define several symbols for notation to be used throughout this paper.

Definition 5. With respect to a given independent set H of G the <u>support</u> of a point is that subset of H adjacent (joined by an edge) to that point. A <u>i-point</u> is a point of G for which the support contains exactly i members.

2. Basic Inequalities

Our main inequality results from the application of Proposition 6, p. 140 of Graver and Yackel [5]. We will first obtain extensions of Lemma 9, p. 155 of [5], for which it will be necessary to study the connectedness of a Ramsey (n_1, n_2)-graph. The objective in studying connectedness is to estimate the intersection of the support of i-points.

We now prove several lemmas for the purpose of estimating the intersection of support of i-points. To that end we let $p(i; n_1, n_2)$ be the maximum number of i-points with respect to an independent $(n_2 - 1)$ set for a Ramsey (n_1, n_2)-graph. We also denote by $e(i, j; n_1, n_2)$ the maximum number of edges which join two i-points, for which the intersection of the support of the two points is j, all with respect to an independent $(n_2 - 1)$ set for a Ramsey (n_1, n_2)-graph.

Throughout we will assume that n_1 is much smaller than n_2.

Lemma 1. $e(i, 0; n_1, n_2) \le p(i; n_1, n_2)R(n_1 - 1, n_2)/2$.

Proof: This is an upper bound for the number of edges joining two i-points.

Lemma 2. $e(i, 1; n_1, n_2) \le ip(i; n_1, n_2)R(n_1 - 2, n_2)/2$.

Proof: Each i-point is adjacent to i elements of the independent set. Among the i-points adjacent to any one element of the independent set the maximum valence is $R(n_1 - 2, n_2)$ since the set of all points adjacent to any point in a Ramsey (n_1, n_2)-graph is a Ramsey $(n_1 - 1, n_2)$-graph and the valence of any point of a Ramsey $(n_1 - 1, n_2)$-graph is at most $R(n_1 - 2, n_2)$. Thus $ip(i; n_1, n_2)R(n_1 - 2, n_2)/2$ gives an upper bound for the number of edges between two i-points both of which have common support of at least one point. This completes the proof of the lemma.

Lemma 3.

$$e(i, 2; n_1, n_2) \leq \frac{p(i; n_1, n_2)}{2} \binom{i}{2} R(n_1 - 2, n_2) \frac{a(n_1 - 1, n_2)}{n_2 - 1}$$

where $a(n_1 - 1, n_2)$ is the average support of the points in a Ramsey $(n_1 - 1, n_2)$-graph with respect to an independent $(n_2 - 1)$ set.

Proof: Let an arbitrary i-point, z, be given. Choose two of the support points, say x and y. There are at most $R(n_1 - 2, n_2)$ edges with z as one end point and for which the other endpoint has support containing x. Next we find an independent $n_2 - 1$ set among the points adjacent to z and including x, y.

The points adjacent to z form a Ramsey $(n_1 - 1, n_2)$-graph and the average support of those points with respect to the independent set choosen is $a(n_1 - 1, n_2)$. As we consider all points adjacent to z we thus find that

$$\frac{\binom{n_2 - 3}{a(n_1-1, n_2) - 2}}{\binom{n_2 - 2}{a(n_1-1, n_2) - 1}} R(n_1 - 2, n_2)$$

of thosepoints will also have support containing y.

When we take account of all i-points, all pairs x, y and the fact that each i-point is counted twice we find that

$$e(i, 2; n_1, n_2) \le \frac{p(i; n_1, n_2)}{2} \binom{i}{2} R(n_1 - 2, n_2) \frac{a(n_1 - 1, n_2)}{n_2 - 1}$$

as stated.

Lemma 4.

$$e(i, j; n_1, n_2) \le \frac{p(i; n_1, n_2)}{2} \binom{i}{j} R(n_1 - 2, n_2) \left(\frac{a(n_1 - 1, n_2)}{n_2 - 1}\right)^{j-1}.$$

Proof: The argument is the same as in Lemma 3. We must consider all adjacencies and so we obtain an average but with less freedom when j points of support must be common.

For a fixed value of n_1 and as n_2 is taken to be large it is convenient to write

$$R(n_1, n_2) \leq f(n_1)n_2^{n_1-1}$$

see Graver and Yackel [5], or Yackel [6].

Theorem 1: For n_1 a fixed integer and n_2 sufficiently large we have

$$p(i; n_1, n_2) \leq (f(n_1 - 1))^{\frac{i-1}{i}} C\, n_2^{(n_1-1)-\frac{n_1-2}{i}}$$

where C is bounded for all n_1, n_2 and $i \leq \log n_2$.

Proof: As a direct application of propostion 6 in [5] we determine that

$$(1) \quad k\binom{n_2 - 1}{k} \geq p(i; n_1, n_2)\binom{n_2 - 1 - i}{k - i} - \sum_{j=0}^{i} e(i, j; n_1, n_2)\binom{n_2 - 1 - 2i + j}{k - 2i + j} .$$

We leave it as an exercise for the reader to verify that

$$(2) \quad \frac{\sum_{j=0}^{i} e(i, j; n_1, n_2)\binom{n_2 - 1 - 2i + j}{k - 2i + j}}{\binom{n_2 - 1 - i}{k - i}} = e(i, 0; n_1, n_2)\frac{\binom{n_2 - 1 - 2i}{k - 2i}}{\binom{n_2 - 1 - i}{k - i}}(1 + o(1))$$

as n_2 approaches ∞ for n_1 fixed and $i \leq \log n_2$. Lemma 1, Lemma 2, and Lemma 3, together with the fact that $a(n_1, n_2) = o(\log n_2)$ for fixed n_1 as n_2 approaches ∞, suffice for that assertion. If $a(n_1, n_2)$ were not $o(\log n_2)$ then our principal result, Theorem 2, would follow with no more additional work.

To complete the proof for the theorem we need only estimate $e(i, 0; n_1, n_2)$ by $\frac{p(i; n_1, n_2)}{2} f(n_1 - 1)n_2^{n_1-2}$ using Lemma 1 and the remark preceeding the statement of this theorem. Then we make standard estimates of the quantities in (1) using (2) as well to complete the upper bound. In obtaining the final result we must choose the value of k. Thus we choose

$$k = \text{int}\left[\frac{n_2^{1 - \frac{n_1-2}{i}}}{f(n_1 - 1)^{\frac{1}{i}}}\right]$$

in making our final estimates. This completes the Theorem.

3. Asymptotic Bounds

In this section the bounds obtained for $p(i; n_1, n_2)$ are used to determine bounds on $R(n_1, n_2)$. Since the results in section 2 are obtained piecemeal for each $p(i; n_1, n_2)$ there is some work yet to be done in order to find the best bounds available from the results stated in Theorem 1.

Theorem 2. $R(n_1, n_2) \le C \left(\frac{\log \log n_2}{\log n_2}\right)^{n_1-2} \cdot n_2^{n_1-1}$ for large values of n_2, where C is an absolute constant for all n_1.

Proof: For any Ramsey (n_1, n_2)-graph, with respect to an independent $(n_2 - 1)$ set we find

$$(3) \qquad \sum_{i=1}^{n_2-1} ip(i; n_1, n_2) \le n_2 R(n_1 - 1, n_2)$$

by counting edges.

Since $R(n_1, n_2) = n_2 + \sum_{i=1}^{n_2-1} p(i; n_1, n_2)$ we are interested in finding an upperbound for $\sum_{i=1}^{n_2-1} p(i; n_1, n_2)$. The upperbound can most easily be established by stating the linear programming problem:

Find the maximum of $\sum_{i=1}^{n_2-1} p(i; n_1, n_2)$ where

$$p(i; n_1, n_2) \le (f(n_1 - 1))^{\frac{i-1}{i}} C\, n_2^{(n_1-1)-\frac{n_1-2}{i}}$$

for $i = 1, 2, \ldots, \log n_2$

$$p(i; n_1, n_2) \le n_2 R(n_1 - 1, n_2)/i \quad \text{for } n_2 > i > \log n_2$$

and

$$\sum_{i=1}^{n_2-1} ip(i; n_1, n_2) \le n_2 R(n_1 - 1, n_2).$$

Now, we do not intend to solve this problem but will instead use the dual problem to find an easy and useful bound on the maximum.

We accomplish this by finding a feasible point for the dual problem and estimating the value of the objective function for the dual problem at that point. This procedure gives a reasonable bound on the maximum of the original problem when it is carefully done.

The dual problem states:

Find the minimum of

$$\sum_{i=1}^{\log n_2} z_i(f(n_1 - 1))^{\frac{i-1}{i}} C\, n_2^{(n_1-1)-\frac{n_1-2}{i}}$$

$$+ \sum_{i=\log n_2+1}^{n_2-1} z_i\, n_2 R(n_1 - 1, n_2)/i + z_{n_2}\, n_2 R(n_1 - 1, n_2)$$

where $z_i + iz_{n_2} \geq 1$ for $i = 1, 2, \ldots, n_2 - 1$.

To establish the theorem we propose the choice

$$z_i = \begin{cases} 1 - \frac{1}{M} & \text{if } i \leq M \\ 0 & \text{if } i > M \end{cases} \qquad \text{for } i = 1,2,\ldots,n_2 - 1$$

and $z_{n_2} = \frac{1}{M}$. Next we propose $M = \frac{n_1 - 2}{3} \frac{\log n_2}{\log \log n_2}$ to complete the description of the feasible solution for the minimum problem. Finally, with this choice we find

$$\sum_{i=1}^{M} z_i(f(n_1 - 1))^{\frac{i-1}{i}} C\, n_2^{(n_1-1)-\frac{n_1-2}{i}} + z_{n_2} n_2 R(n_1 - 1, n_2)$$

$$\leq \frac{n_2}{M} R(n_1 - 1, n_2)(1 + o(1)) \quad \text{as} \quad n_2 \to \infty$$

and the theorem is established.

4. Concluding Remarks

It is surprising that the local connectedness properties we studied here in section 2 should give global results. This seems to suggest that a deeper study of the structure of Ramsey graphs would significantly improve these results.

There is clearly no point in attempting to evaluate constants nor in improving the statement of Theorem 2 by more careful optimization of the linear programming problem.

A study of the constructions of Erdös [1] would be of interest to compare the connectedness of his graphs with the results of this paper. This study has not yet been done to my knowledge.

Bibliography

1. P. Erdös (1961). Graph Theory and Probability II. Canad. J. Math. 13, 346-352.

2. P. Erdös (1968). Some recent results on extremal problems in graph theory. Theory of Graphs, Proc. Coll., Tihany, Hungary 1966, Academic Press, 83-98.

3. P. Erdös (1971). Unsolved Problems in Graph Theory and Combinatorial Analysis. Combinatorial Mathematics and its applications, Academic Press, 97-109.

4. P. Erdös and G. Szekeres (1935). A combinatorial problem in geometry. Comp. Math. 2, 463-470.

5. J. Graver and J. Yackel (1968). Some graph theoretic results associated with Ramsey's theorem. J. Combinatorial Theory 4, 125-175.

6. J. Yackel (1972). Inequalities and Asymptotic Bounds for Ramsey Numbers. J. Combinatorial Theory 13, 56-68.

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13. ABSTRACT

When all pairs of elements of a set are partitioned into two disjoint classes there are determined two graphs. This paper studies the maximal sizes of sets which admit partitions of the pairs of elements into two classes satisfying the conditions of Ramsey's Theorem. The connectedness of the related graph is used to obtain new inequalities from which we are able to deduce the upper bound

$$R(n_1,n_2) \le C \left(\frac{\log \log n_2}{\log n_2}\right)^{n_1-2} n_2^{n_1-1}$$

where C is independent of n_1 and n_2.

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