EXACT ROBUSTNESS AND POWER STUDIES OF SOME TESTS IN MULTIVARIATE ANALYSIS *

by

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ABSTRACT

Sudjana. Ph.D., Purdue University, August 1973. Exact Robustness and Power Studies of Some Tests in Multivariate Analysis. Major Professor: K.C.S. Pillai

This thesis deals essentially with three areas of study: I) Central and non-central distributions and moments of some test criteria in connection with three basic hypotheses in multivariate analysis, II) Exact robustness, and III) Power studies of the criteria. The test statistics considered are: 1) Hotelling's trace, 2) Pillai's trace, 3) Wilks' criterion, 4) Roy's largest root, and 5) Wilks-Lawley statistic. The hypotheses are: (A) equality of covariance matrices of two p-variate normal populations, (B) equality of p-dimensional mean vectors in ℓ p-variate normal populations having a common unknown covariance matrix, and (C) independence between a p-set and a q-set in a (p+q)-variate normal population.

Pillai's density of the characteristic roots of $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ under violations was used in Chapter I to obtain the distribution of 1), -under a condition-, 3), and 4) in two forms. Also, the moments of 1) and the m.g.f. of 2) are obtained there. Earlier results on these obtained by previous authors are shown to be special cases.

Chapter II is first devoted to deriving the general form of the exact distributions of the first four criteria in the two-roots case in a suitable form for computation. Then, these distributions are used to study the robustness of tests of (A) when normality assumption is violated and of (B) when the assumption of a common covariance matrix is disturbed. It is observed that for (A) there is an indication that the tests are not robust against non-normality and for (B), the powers of tests show modest changes for small deviations of the equality of covariance matrices. Unlike the results obtained earlier by other authors, the study here are of exact nature except for some assumptions made in the model.

Modifying the method of Pillai and Young, the null distribution of 1) for three and four roots are obtained in Chapter III. The expressions hold for all non-negative integral values of an argument m, unlike those of Pillai and Young who considered each integral value of m separately, and are of much simpler forms. The non-null distribution of 1), however, is obtained only for the three-roots case and its explicit expression is given for m=0 in Chapter IV. For this case the powers of 1) are computed for (A), (B) and (C). Comparisons of powers are made with those of Roy's largest root. It is believed that this is the first time exact powers for Hotelling's trace was obtained for the three-roots case.

An expression suggested by Pillai for the distribution of 1) is studied in Chapter V. For the number of roots equals two, his formula reduces to the exact case. For larger number of roots, the exactness has not been verified. For these cases, percentage points are computed and comparisons made with the exact points. Powers of 1) are also computed, for the number of roots greater than two, in view of (B).

The criterion 5) is studied in Chapter VI. The general form of its distribution is derived there in two forms; one, in terms of Meijer's G-function and the other by convolution approach. For the null case, a relation is established between the densities of 3) and 5). A power study of 5) in the two-roots case is carried out for the three tests and comparisons are made with powers of other four criteria.

Summary, conclusion and recommendations are given in Chapter VII.

CHAPTER I

SOME DISTRIBUTION PROBLEMS CONCERNING THE CHARACTERISTIC ROOTS OF $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ UNDER VIOLATIONS

1. Introduction

Let $\underset{\sim}{S}_1(p\times p)$ be a matrix variate distributed $W(p,n_1,\underset{\sim}{\Sigma}_1,\underset{\sim}{\Omega})$, i.e. non-central Wishart on n_1 degrees of freedom with non-centrality parameter $\underset{\sim}{\Omega}$ and covariance matrix $\underset{\sim}{\Sigma}_1$ independently of $\underset{\sim}{S}_2(p\times p)$ distributed central Wishart $W(p,n_2,\underset{\sim}{\Sigma}_2,\underset{\sim}{0})$. Then Pillai [31,32] has obtained the distribution of the latent roots of $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ under certain violations. In this present work Pillai's distribution and method has been used to derive the following:

1) the density function of T, n_2 times Hotelling's T_0^2 [13,14],
2) the moments of T,
3) the moment generating function of Pillai's trace [25],
4) the density function of Wilks' criterion [45], and
5) the density function of Roy's largest root [43] in two forms.

These results are useful in studying the exact robustness of at least two multivariate hypotheses, namely:

(A) equality of covariance matrices of two p-variate normal populations when normality assumption is disturbed, and

(B) equality of p-dimensional mean vectors in ℓ p-variate normal populations under violation of the assumption of a common covariance matrix.

The exact robustness studies are carried out in the next chapter.

2. Preliminaries

For the derivation of some results concerning the latent roots of $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ which will be obtained in the next sections we need some mathematical results mostly on functions of symmetric matrix argument. Herz [11] studied and derived many results on some special functions of symmetric matrix argument such as: gamma, beta, hypergeometric, Laguerre and Bessel functions. For our need however, we will use special functions which have been defined differently by Constantine [3] and James [17], and for our convenience some results are stated in this section.

a. Hypergeometric functions and zonal polynomials. We start with the definition of the hypergeometric function of a symmetric matrix $\underset{\sim}{S}$(pxp)

$$(2.1)\qquad {}_qF_r(a_1,\ldots,a_q;b_1,\ldots,b_r;\underset{\sim}{S}) = \sum_{k=0}^{\infty}\sum_{\kappa}\frac{(a_1)_\kappa\cdots(a_q)_\kappa}{(b_1)_\kappa\cdots(b_r)_\kappa}\frac{C_\kappa(\underset{\sim}{S})}{k!},$$

where $a_1,\ldots,a_q,b_1,\ldots,b_r$ are real or complex constants and the multivariate coefficient $(a)_\kappa$ is defined by

$$(2.2)\qquad (a)_\kappa = \prod_{i=1}^{p}\left(a - \frac{1}{2}(i-1)\right)_{k_i},\quad (a)_k = a(a+1)\ldots(a+k-1),\quad (a)_0=1,$$

k_i are the components of the partition κ of k such that $\kappa = (k_1,\ldots,k_p)$ into not more than p components with $k_1 \geq \ldots \geq k_p \geq 0$ such that $k = k_1+\ldots+k_p$. In (2.1), it is assumed that $q \leq r + 1$, otherwise the series may only converge for $\underset{\sim}{S} = \underset{\sim}{0}$. For $q = r + 1$, the series converge for $||\underset{\sim}{S}|| < 1$, where $||\underset{\sim}{S}||$ denotes the maximum of the absolute values of latent roots of $\underset{\sim}{S}$. For $q \leq r$, the series converge for all $\underset{\sim}{S}$. The $C_\kappa(\underset{\sim}{S})$, called the zonal polynomial of $\underset{\sim}{S}$, is expressible in terms of elementary symmetric functions (e.s.f.) of the latent roots of $\underset{\sim}{S}$ and is a symmetric homogeneous polynomial of degree k in the latent roots of $\underset{\sim}{S}$. James [17] also defines the zonal polynomial $Z_\kappa(\underset{\sim}{S})$, zonal polynomial with a different normalizing constant, which is related to $C_\kappa(\underset{\sim}{S})$ by

(2.4) $$C_\kappa(\underset{\sim}{S}) = [\chi_{[2\kappa]}(1)2^k k!/(2k)!]Z_\kappa(\underset{\sim}{S}),$$

where $\chi_{[\kappa]}(1)$ is defined by

(2.5) $$\chi_{[\kappa]}(1) = k! \prod_{i<j}^{p} (k_i-k_j-i+j)/\prod_{i=1}^{p} (k_i+p-i)!$$

and for $\kappa = (k_1,\ldots,k_p)$ we have $(2\kappa) = (2k_1,\ldots,2k_p)$. If $\underset{\sim}{S} = \underset{\sim}{I}_p$, identity matrix of order p (unless otherwise stated, by $\underset{\sim}{I}$ is to be meant $\underset{\sim}{I}_p$), Constantine [3] proved that

(2.6) $$Z_\kappa(\underset{\sim}{I}_p) = 2^k(\tfrac{1}{2}\,p)_\kappa \;.$$

The values of $Z_\kappa(\underset{\sim}{S})$ in terms of e.s.f. and sums of powers of the latent roots of $\underset{\sim}{S}$ up to order six may be found in [17] and up to order twelve were prepared by A. T. James and A. M. Parkhurst (private communication to Professor Pillai).

Constantine [4] defined the product of two zonal polynomials having the same argument in the form

(2.6a) $$C_\kappa(\underset{\sim}{S})C_\nu(\underset{\sim}{S}) = \sum_\delta q^\delta_{\kappa,\nu}\, C_\delta(\underset{\sim}{S}),$$

where the summation is over all partition δ of d such that d = k + n and $q^\delta_{\kappa,\nu}$ are constants whose values may be found in the work of Khatri and Pillai [20].

The hypergeometric function of two matrix arguments $\underset{\sim}{S}$(pxp) and $\underset{\sim}{T}$(pxp) is defined by

$$(2.7)\qquad {}_qF_r(a_1,\ldots,a_q;b_1,\ldots,b_r;\underset{\sim}{S},\underset{\sim}{T})$$

$$= \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_\kappa\cdots(a_q)_\kappa}{(b_1)_\kappa\cdots(b_r)_\kappa} \cdot \frac{C_\kappa(\underset{\sim}{S})C_\kappa(\underset{\sim}{T})}{C_\kappa(\underset{\sim}{I})k!},$$

where all symbols are now obvious from the previous discussion.

Let $\underset{\sim}{S}$(pxp) $> \underset{\sim}{0}$, ($\underset{\sim}{S} > \underset{\sim}{0}$ is to be meant that $\underset{\sim}{S}$ is positive definite matrix), $\underset{\sim}{Z}$(pxp) a complex matrix whose real part is positive definite symmetric. Then (Herz [11], James [17])

$$(2.8)\qquad \{\Gamma_p(a)\}^{-1} \int_{\underset{\sim}{S}>\underset{\sim}{0}} e^{-\mathrm{tr}\,\underset{\sim}{S}}|\underset{\sim}{S}|^{a-m}\,{}_qF_r(a_1,\ldots,a_q;b_1,\ldots,b_r;\underset{\sim}{Z}\underset{\sim}{S})d\underset{\sim}{S}$$

$$= {}_{q+1}F_r(a_1,\ldots,a_q,a;b_1,\ldots,b_r;\underset{\sim}{Z}),$$

where the multivariate gamma function $\Gamma_p(a)$ is defined by

$$(2.9)\qquad \Gamma_p(a) = \pi^{\frac{1}{2}n} \prod_{i=1}^{p} \Gamma(a-\tfrac{1}{2}(i-1)),$$

and $m = \frac{1}{2}(p+1)$, $n = \frac{1}{2}p(p-1)$, (unless otherwise stated, in the sequel we let $m = \frac{1}{2}(p+1)$ and $n = \frac{1}{2}p(p-1)$), and the integration is over the space of positive definite matrices $\underset{\sim}{S}$. If in addition to the above hypothesis we also have $\underset{\sim}{R}$(pxp) a symmetric matrix, then (James [17])

$$(2.10)\qquad \{\Gamma_p(a)\}^{-1} \int_{\underset{\sim}{S}>\underset{\sim}{0}} e^{-\mathrm{tr}\,\underset{\sim}{S}}|\underset{\sim}{S}|^{a-m}\,{}_qF_r(a_1,\ldots,a_q;b_1,\ldots,b_r;\underset{\sim}{Z}\underset{\sim}{S},\underset{\sim}{R})d\underset{\sim}{S}$$

$$= {}_{q+1}F_r(a_1,\ldots,a_q,a;b_1,\ldots,b_r;\underset{\sim}{Z},\underset{\sim}{R}).$$

From (2.8) and (2.10) we now have the inverse Laplace transforms (James [17]) respectively as

$$(2.11)\quad \frac{2^n\Gamma_p(b)}{(2\pi i)^{pm}} \int_{\mathrm{Re}\,Z>0} e^{\mathrm{tr}\,Z}|Z|^{-b}\,{}_qF_r(a_1,\ldots,a_q;b_1,\ldots,b_r;SZ^{-1})dZ$$

$$= {}_qF_{r+1}(a_1,\ldots,a_q;b_1,\ldots,b_r,b;S),$$

and

$$(2.12)\quad \frac{2^n\Gamma_p(b)}{(2\pi i)^{pm}} \int_{\mathrm{Re}\,Z>0} e^{\mathrm{tr}\,Z}|Z|^{-b}\,{}_qF_r(a_1,\ldots,a_q;b_1,\ldots,b_r;SZ^{-1},T)dZ$$

$$= {}_qF_{r+1}(a_1,\ldots,a_q;b_1,\ldots,b_r,b;S,T),$$

where the integration is taken over $Z = X + iY$ with $X = X_0$ for some positive definite matrix X_0 and Y ranges over all real symmetric matrices.

Some special cases of the hypergeometric function are of interest to be noticed. They are

$$(2.13)\qquad {}_0F_0(S) = \sum_{k=0}^{\infty}\sum_{\kappa}[C_\kappa(S)/k!] = e^{\mathrm{tr}\,S},$$

$$(2.14)\qquad {}_1F_0(a;S) = |I-S|^{-a},$$

and for Re a > m-1, Re b > m-1 and Re(b-a) > m-1 (Herz [11], James [17])

$$(2.15)\qquad {}_1F_1(a;b;S) = \Gamma_p(b)[\Gamma_p(a)\Gamma_p(b-a)]^{-1}$$

$$\cdot\int_0^I e^{\mathrm{tr}\,ST}|T|^{a-m}|I-T|^{b-a-m}\,dT.$$

Some results of integrations involving the zonal polynomials are as follows:

Let $Z(p\times p)$ be a complex matrix as stated above and $T(p\times p)$ be any arbitrary complex symmetric matrix. Constantine [3] proved that

$$(2.16)\qquad \int_{S>0} e^{-\mathrm{tr}\,ZS}|S|^{a-m}C_\kappa(ST)dS = \Gamma_p(a,\kappa)|Z|^{-a}C_\kappa(TZ^{-1}),$$

where $\mathrm{Re}(a) > \frac{1}{2}(p-1)$ and $\Gamma_p(a,\kappa)$ is defined by

(2.17) $$\Gamma_p(a,\kappa) = (a)_\kappa \Gamma_p(a),$$

and Khatri [18] obtained the relation

(2.18) $$\int_{S>0} e^{-\mathrm{tr}\, ZS} |S|^{a-m} C_\kappa(TS^{-1}) dS = \Gamma_p(a,-\kappa) |Z|^{-a} C_\kappa(ZT),$$

where $\mathrm{Re}(a) > \frac{1}{2}(p-1) + k_1$ and $\Gamma_p(a,-\kappa)$ is defined by

(2.19) $$\Gamma_p(a,-\kappa) = \pi^{\frac{1}{2}n} \prod_{i=1}^{p} \Gamma(a-k_i - \tfrac{1}{2}(p-i)).$$

Further, Constantine [3] proved that for $R(p\mathrm{x}p)$ positive definite

(2.20) $$\int_0^I |S|^{a-m} |I-S|^{b-m} C_\kappa(RS) dS = \frac{\Gamma_p(a,\kappa)\Gamma_p(b)}{\Gamma_p(a+b,\kappa)} C_\kappa(R),$$

from which follows (Khatri [18]) the result

(2.21) $$\int |R|^{t-m} |I-R|^{a-m} \prod_{i>j} (r_i - r_j) C_\kappa(R) dR$$
$$= \Gamma_p(t,\kappa)\Gamma_p(a)\Gamma_p(\tfrac{1}{2}p) C_\kappa(I) / [\pi^{\frac{1}{2}p^2} \Gamma_p(t+a,\kappa)],$$

where $R = \mathrm{diag}(r_1,\ldots,r_p)$ such that $0 < r_1 <\ldots< r_p < 1$.

Now expand the hypergeometric functions in (2.11), we directly obtain

(2.22) $$\frac{2^n}{(2\pi i)^{pm}} \int_{\mathrm{Re}Z>0} e^{\mathrm{tr}\, Z} |Z|^{-b} C_\kappa(SZ^{-1}) dZ = \frac{C_\kappa(S)}{\Gamma_p(b,\kappa)}.$$

From this, letting $Z = S^{\frac{1}{2}}\, T\, S^{\frac{1}{2}}$, the result (9) of Constantine [4] is obtained, where in obtaining it he used the inverse Laplace transform of the zonal polynomial.

The method of diagonalizing and dediagonalizing a symmetric matrix by an orthogonal transformation $H \in \mathcal{O}(p)$, $\mathcal{O}(p)$ being an orthogonal group of $\underset{\sim}{H}$(pxp), is quite useful. Some basic results are

$$(2.23)\qquad \int_{\mathcal{O}(p)} C_\kappa(\underset{\sim}{R}\underset{\sim}{H}\underset{\sim}{S}\underset{\sim}{H}')d\underset{\sim}{H} = \{C_\kappa(\underset{\sim}{R})C_\kappa(\underset{\sim}{S})/C_\kappa(\underset{\sim}{I})\},$$

was proved by James [17] and he also proved in [17]

$$(2.24)\qquad \int_{\mathcal{O}(p)} |\underset{\sim}{I}-\underset{\sim}{R}\underset{\sim}{H}\underset{\sim}{S}\underset{\sim}{H}'|^{-a}\, d\underset{\sim}{H} = {}_1F_0(a;\underset{\sim}{R},\underset{\sim}{S}),$$

where $d\underset{\sim}{H}$ stands for the invariant or Haar measure on the orthogonal group $\mathcal{O}(p)$ normalized so that the measure of the whole group is unity and $\underset{\sim}{H}\underset{\sim}{H}'=\underset{\sim}{I}$. Otherwise, we have (Constantine [3])

$$(2.24a)\qquad \int_{\mathcal{O}(p)} d\underset{\sim}{H} = 2^p\pi^{\frac{1}{2}p^2}/\Gamma_p(\tfrac{1}{2}p).$$

The following is due to Khatri [19].

If $g(\underset{\sim}{R})$ is any function of the e.s.f. of $\underset{\sim}{R}$ such that $g(\underset{\sim}{R}) \geq 0$ and $t \geq 0$, then

$$(2.25)\qquad \int_{\mathcal{O}(p)} |\underset{\sim}{I} + \underset{\sim}{\Lambda}^{-1}\underset{\sim}{H}\underset{\sim}{R}\underset{\sim}{H}'|^{-a}\, d\underset{\sim}{H}$$

$$= |\underset{\sim}{I} + \underset{\sim}{\Lambda}^{-1}tg(\underset{\sim}{R})|^{-a}\ {}_1F_0(a;(\underset{\sim}{\Lambda}+tg(\underset{\sim}{R})\underset{\sim}{I})^{-1},\ tg(\underset{\sim}{R})\underset{\sim}{I}-\underset{\sim}{R}).$$

b. <u>Laguerre polynomials and Bessel function.</u> Specializing the definition of Herz [11] for the generalized Laguerre polynomials, Constantine [4] used the zonal polynomials as a basis for symmetric functions and he defined the Laguerre polynomials by

$$(2.26)\qquad e^{-\mathrm{tr}\,\underset{\sim}{S}}L_\kappa^\gamma(\underset{\sim}{S}) = \int_{\underset{\sim}{R}>\underset{\sim}{0}} e^{-\mathrm{tr}\,\underset{\sim}{R}}|\underset{\sim}{R}|^\gamma C_\kappa(\underset{\sim}{R})A_\gamma(\underset{\sim}{R}\underset{\sim}{S})d\underset{\sim}{R},$$

where $\gamma > -1$ and $A_\gamma(\underset{\sim}{R})$ is the generalized Bessel function which has the definition

$$(2.27)\qquad A_\gamma(R) = \frac{2^n}{(2\pi i)^{pm}} \int_{\mathrm{Re}Z>0} e^{\mathrm{tr}\,Z}\, e^{-\mathrm{tr}\,RZ^{-1}} |Z|^{\gamma-m}\, dZ,$$

and has the expansion

$$(2.28)\qquad A_\gamma(R) = [\Gamma_p(\gamma+m)]^{-1} \sum_{k=0}^{\infty} \sum_{\kappa} [C_\kappa(-R)/\{(\gamma+m)_\kappa k!\}].$$

For $S(p\times p)$ and $Z(p\times p)$ complex matrix such that $\mathrm{Re}Z > 0$ and $\gamma > -1$, by substituting (2.27) in (2.26), the relation

$$(2.29)\qquad L_\kappa^\gamma(S) = [2^n/(2\pi i)^{pm}]\Gamma_p(\gamma+m,\kappa) \cdot \int_{\mathrm{Re}Z>0} e^{\mathrm{tr}\,Z} |Z|^{-\gamma-m} C_\kappa(I-SZ^{-1})dZ,$$

was obtained by Constantine [4].

Making use of (2.22) in (2.29)

$$(2.30)\qquad L_\kappa^\gamma(S) = (\gamma+m)_\kappa C_\kappa(I) \sum_{d=0}^{k} \sum_{\delta} \frac{(-1)^d a_{\kappa,\delta} C_\delta(S)}{(\gamma+m)_\delta C_\delta(I)},$$

where in obtaining (2.30), the relation (Constantine [4])

$$(2.31)\qquad \frac{C_\kappa(I+A)}{C_\kappa(I)} = \sum_{d=0}^{k} \sum_{\delta} [a_{\kappa,\delta} C_\delta(A)/C_\delta(I)]$$

has been used. The constants $a_{\kappa,\delta}$ may be found in [4] up to order 4 and for order k = 5(1)8 are tabulated by Pillai and Jouris [39].

For $S = 0$, we have

$$(2.32)\qquad L_\kappa^\gamma(0) = (\gamma+m)_\kappa C_\kappa(I).$$

The moment generating function for the Laguerre polynomials (Constantine [4]) is given by

$$(2.33)\qquad |I-Z|^{-\gamma-m} \int_{\mathcal{O}(p)} \exp[\mathrm{tr}(-SH'Z(I-Z)^{-1}H)]dH = \sum_{k=0}^{\infty} \sum_{\kappa} [L_\kappa^\gamma(S) C_\kappa(Z)/\{k! C_\kappa(I)\}],$$

holds for $||\underset{\sim}{Z}|| < 1$, and for the estimate of the Laguerre polynomials he showed that

$$(2.34) \qquad |L_\kappa^\gamma(\underset{\sim}{S})| \leq (\gamma+m)_\kappa C_\kappa(\underset{\sim}{I})e^{\mathrm{tr}\,\underset{\sim}{S}}.$$

c. <u>Mellin transform and Meijer's G-function</u>. Let s be any complex variate and f(x) be a function of a real variable x, such that

$$(2.35) \qquad F(s) = \int_0^\infty x^{s-1} f(x)dx$$

exists, then under certain conditions (see Consul [5])

$$(2.36) \qquad f(x) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} x^{-s} F(s)ds.$$

F(s) in (2.35) is called the Mellin transform of f(x) and f(x) in (2.36) is the inverse Mellin transform of F(s). Further, if $f_1(x)$ and $f_2(x)$ are the inverse Mellin transform of $F_1(s)$ and $F_2(s)$ respectively, then the inverse Mellin transform of $F_1(s) \cdot F_2(s)$ is given by

$$(2.37) \qquad (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} x^{-s} F_1(s)F_2(s)ds = \int_0^\infty f_1(u)f_2(x/u)(du/u).$$

Using these properties, Consul [6] evaluate some values of Meijer's G-function (Meijer [24]) which is defined by

$$(2.38) \qquad G_{p,q}^{m,n}\left(x\middle|\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right)$$

$$= (2\pi i)^{-1} \int_C \frac{\prod_{j=1}^m \Gamma(b_j-s) \prod_{j=1}^n \Gamma(1-a_j+s)}{\prod_{j=m+1}^q \Gamma(1-b_j+s) \prod_{j=m+1}^p \Gamma(a_j-s)} x^s\, ds,$$

where an empty product is interpreted as unity and C is a curve separating the singularities of $\prod_{j=1}^{m} \Gamma(b_j-s)$ from those of $\prod_{j=1}^{n} \Gamma(1-a_j+s)$, $q \geq 1$, $0 \leq n \leq p \leq q$, $0 \leq m \leq q$; $x \neq 0$ and $|x| < 1$ if $q = p$; $x \neq 0$ if $q > p$. The Meijer's G-function (2.38) can be expressed as a sum of a finite number of hypergeometric functions as follows (Meijer [23]):

$$(2.39) \quad G_{p,q}^{m,n}\left(x\Big|{}^{a_1,\ldots,a_p}_{b_1,\ldots,b_q}\right) = \sum_{h=1}^{m} \Big[\Big\{\prod_{\substack{j=1\\ j\neq h}}^{m} \Gamma(b_j-b_h) \prod_{j=1}^{n} \Gamma(1+b_h-a_j)\Big\}\Big/ \Big\{\prod_{j=m+1}^{q} \Gamma(1+b_h-b_j) \prod_{j=n+1}^{p} \Gamma(a_j-b_h)\Big\}\Big] x^{b_h} \cdot {}_pF_{q-1}(1+b_h-a_1,\ldots,1+b_h-a_p; 1+b_h-b_1,\ldots * \ldots,1+b_h-b_q; (-1)^{p-m-n}\, x),$$

where the asterisk denotes that the number $1+b_h-b_h$ is omitted in the sequence $1+b_h-b_1,\ldots,1+b_h-b_q$. Consul [7] has shown that the value of the G-function for $p=q=m=2$ and $n=0$ is

$$(2.40) \quad G_{2,2}^{2,0}\left(x\Big|{}^{a_1,a_2}_{b_1,b_2}\right) = [x^{b_1}(1-x)^{a-1}/\Gamma(a)] \cdot {}_2F_1(a_2-b_2, a_1-b_2; a; 1-x), \quad 0 < x < 1,$$

where $a = a_1+a_2-b_1-b_2$, and for $p=q=m=1$, $n=0$

$$(2.41) \quad G_{1,1}^{1,0}\left(x\Big|{}^{a}_{b}\right) = x^{b}(1-x)^{a-b-1}/\Gamma(a-b); \quad 0 < x < 1.$$

3. The Density Function of T

In this section the density function of $T = \lambda \operatorname{tr} S_1 S_2^{-1}$, $\lambda > 0$, is derived, where S_1(pxp) is distributed $W(p, n_1, \Sigma_1, \Omega)$, i.e. non-central Wishart distribution on n_1 d.f. with non-centrality parameter Ω and covariance matrix Σ_1, and S_2(pxp) independently distributed central Wishart $W(p, n_2, \Sigma_2, 0)$. We assume that $n_1, n_2 \geq p$ so that we have p non-zero latent roots of $S_1 S_2^{-1}$. We further assume that Ω is of a "random" nature in the sense that it can be diagonalized by an orthogonal transformation $H \in \mathcal{O}(p)$ and integrated with respect to H over $\mathcal{O}(p)$. This implies diagonalization of the parameter matrix by H and putting a Haar prior on H leaving the characteristic roots non-random.

Theorem 3.1. Let S_1(pxp) and S_2(pxp) be independently distributed, S_1 having $W(p, n_1, \Sigma_1, \Omega)$ and S_2 having $W(p, n_2, \Sigma_2, 0)$. If $T = \lambda \operatorname{tr} S_1 S_2^{-1}$, $\lambda > 0$, and Ω "random", then the density function of T is given by

$$(3.1) \qquad f(T) = \{\Gamma_p(\tfrac{1}{2}\nu)/\Gamma_p(\tfrac{1}{2}n_2)\}|\lambda\Lambda|^{-\frac{1}{2}n_1} e^{-\operatorname{tr}\Omega} T^{\frac{1}{2}pn_1-1} \cdot \sum_{k=0}^{\infty}\sum_{\kappa} \frac{(\tfrac{1}{2}\nu)_\kappa (-T)^k C_\kappa(\lambda^{-1}\Lambda^{-1}) L_\kappa^{\gamma}(\Omega)}{k!\,\Gamma(\tfrac{1}{2}pn_1+k)\, C_\kappa(I)},$$

where $\nu = n_1+n_2$, $\Lambda = \operatorname{diag}(\lambda_1,\ldots,\lambda_p)$, $\lambda_1,\ldots,\lambda_p$ being the latent roots of $\Sigma_1\Sigma_2^{-1}$, $0 < \lambda_1 < \ldots < \lambda_p < \infty$, $\gamma = \frac{1}{2}(n_1-p-1)$ and $|T/(\lambda\lambda_1)| < 1$.

Proof: The joint density of S_1 and S_2 is given by

$$\{\Gamma_p(\tfrac{1}{2}n_1)\Gamma_p(\tfrac{1}{2}n_2)|2\Sigma_1|^{\frac{1}{2}n_1}|2\Sigma_2|^{\frac{1}{2}n_2}\}^{-1} e^{-\operatorname{tr}\Omega} e^{-\frac{1}{2}\operatorname{tr}\Sigma_1^{-1}S_1} \cdot |S_1|^{\frac{1}{2}(n_1-p-1)} e^{-\frac{1}{2}\operatorname{tr}\Sigma_2^{-1}S_2} |S_2|^{\frac{1}{2}(n_2-p-1)} {}_0F_1(\tfrac{1}{2}n_1; \tfrac{1}{2}\Sigma_1^{-1}\Omega S_1).$$

Let us now transform $\underset{\sim}{A}_1 = \frac{1}{2}\,\underset{\sim}{\Sigma}_1^{-\frac{1}{2}}\underset{\sim}{S}_1\underset{\sim}{\Sigma}_1^{-\frac{1}{2}}$ and $\underset{\sim}{A}_2 = \frac{1}{2}\,\underset{\sim}{\Sigma}_1^{-\frac{1}{2}}\underset{\sim}{S}_2\underset{\sim}{\Sigma}_1^{-\frac{1}{2}}$, then the joint density of $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$ is given by ($\underset{\sim}{\Sigma}_1^{-\frac{1}{2}}$(pxp) being symmetric positive definite like other matrices of the form $\underset{\sim}{A}^{\frac{1}{2}}$ defined later)

$$C\,|\underset{\sim}{\Lambda}|^{\frac{1}{2}n_2}e^{-\mathrm{tr}\Lambda\underset{\sim}{A}_2}|\underset{\sim}{A}_2|^{\frac{1}{2}(n_2-p-1)}|\underset{\sim}{A}_1|^{\frac{1}{2}(n_1-p-1)}e^{-\mathrm{tr}\underset{\sim}{A}_1}{}_0F_1(\tfrac{1}{2}n_1;\underset{\sim}{\Omega}\underset{\sim}{A}_1),$$

where $C = \{\Gamma_p(\frac{1}{2}n_1)\Gamma_p(\frac{1}{2}n_2)\}^{-1}e^{-\mathrm{tr}\underset{\sim}{\Omega}}$ and $\underset{\sim}{\Lambda} = \underset{\sim}{\Sigma}_1^{\frac{1}{2}}\underset{\sim}{\Sigma}_2^{-1}\underset{\sim}{\Sigma}_1^{\frac{1}{2}}$. Note that the latent roots of $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ are invariant under the above transformations and also under the transformations $\underset{\sim}{B}_1 = \underset{\sim}{\Lambda}^{\frac{1}{2}}\underset{\sim}{A}_1\underset{\sim}{\Lambda}^{\frac{1}{2}}$ and $\underset{\sim}{B}_2 = \underset{\sim}{\Lambda}^{\frac{1}{2}}\underset{\sim}{A}_2\underset{\sim}{\Lambda}^{\frac{1}{2}}$. Using these, we obtain the joint density of $\underset{\sim}{B}_1$ and $\underset{\sim}{B}_2$ in the form

$$(3.2)\quad C\,|\underset{\sim}{\Lambda}|^{-\frac{1}{2}n_1}e^{-\mathrm{tr}\underset{\sim}{B}_2}|\underset{\sim}{B}_2|^{\frac{1}{2}(n_2-p-1)}|\underset{\sim}{B}_1|^{\frac{1}{2}(n_1-p-1)}e^{-\mathrm{tr}\underset{\sim}{\Lambda}^{-1}\underset{\sim}{B}_1}{}_0F_1(\tfrac{1}{2}n_1;\underset{\sim}{\Lambda}^{-\frac{1}{2}}\underset{\sim}{\Omega}\underset{\sim}{\Lambda}^{-\frac{1}{2}}\underset{\sim}{B}_1).$$

Now, the Laplace transform of $T = \lambda\ \mathrm{tr}\ \underset{\sim}{S}_1\underset{\sim}{S}_2^{-1} = \lambda\ \mathrm{tr}\ \underset{\sim}{B}_1\underset{\sim}{B}_2^{-1}$ is given by $E(\exp(-t\,\lambda\ \mathrm{tr}\ \underset{\sim}{B}_1\underset{\sim}{B}_2^{-1}))$. To find this transform we multiply (3.2) by $\exp(-t\,\lambda\ \mathrm{tr}\ \underset{\sim}{B}_1\underset{\sim}{B}_2^{-1})$ and integrate $\underset{\sim}{B}_1$ out using (2.16). We obtain the Laplace transform of T in the form

$$(3.3)\quad C\,\Gamma_p(\tfrac{1}{2}n_1)\int_{\underset{\sim}{B}_2>\underset{\sim}{0}} e^{-\mathrm{tr}\underset{\sim}{B}_2}|\underset{\sim}{B}_2|^{\frac{1}{2}(n_2-p-1)}|\underset{\sim}{I}+t\,\lambda\,\underset{\sim}{\Lambda}\,\underset{\sim}{B}_2|^{-\frac{1}{2}n_1}$$

$$\cdot\,{}_0F_0(\underset{\sim}{\Omega}(\underset{\sim}{I} + t\,\lambda\,\underset{\sim}{\Lambda}^{\frac{1}{2}}\,\underset{\sim}{B}_2^{-1}\underset{\sim}{\Lambda}^{\frac{1}{2}})^{-1})\ d\underset{\sim}{B}_2.$$

Letting $\underset{\sim}{D} = \underset{\sim}{\Lambda}^{\frac{1}{2}}\ \underset{\sim}{B}_2\ \underset{\sim}{\Lambda}^{\frac{1}{2}}$, the above expression becomes

$$(3.4)\quad C\cdot\Gamma_p(\tfrac{1}{2}n_1)|\Lambda|^{\frac{1}{2}n_2}(t\lambda)^{-\frac{1}{2}pn_1}\int_{D>0} e^{-\mathrm{tr}\Lambda D}|D|^{\frac{1}{2}(n_1+n_2-p-1)}$$

$$\cdot|I+(t\lambda)^{-1}D|^{-\frac{1}{2}n_1}\,{}_0F_0(\Omega(t\lambda)^{-1}D(I+(t\lambda)^{-1}D)^{-1})dD.$$

Using the assumption of Ω "random" and then (2.33), from (3.4) we have the Laplace transform of T as

$$(3.5)\qquad g(t) = C_1\cdot(t\lambda)^{-\frac{1}{2}pn_1}\int_{D>0} e^{-\mathrm{tr}\Lambda D}|D|^{\frac{1}{2}(n_1+n_2-p-1)}$$

$$\cdot\sum_{k=0}^{\infty}\sum_{\kappa}[\{L_\kappa^\gamma(\Omega)C_\kappa(-(t\lambda)^{-1}D)\}/\{k!C_\kappa(I)\}]dD,$$

where $\gamma = \frac{1}{2}(n_1-p-1)$ and $C_1 = C\cdot\Gamma_p(\frac{1}{2}n_1)|\Lambda|^{\frac{1}{2}n_2}$.

For fixed D and $\mathrm{Re}(t) = c$ sufficiently large, the series in (3.5) can be integrated term by term with respect to t (see Constantine [4]) and use the fact that if f(T) is the Laplace inversion of g(t) then

$$f(T) = (2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} e^{tT} g(t)dt.$$

Noting that

$$(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} e^{tT}\, t^{-\frac{1}{2}pn_1-k}\, dt = T^{\frac{1}{2}pn_1+k-1}/\Gamma(\tfrac{1}{2}pn_1+k),$$

we obtain the density of T in the form

$$(3.6)\qquad C_1\cdot\lambda^{-\frac{1}{2}pn_1}T^{\frac{1}{2}pn_1-1}\int_{D>0} e^{-\mathrm{tr}\Lambda D}|D|^{\frac{1}{2}(n_1+n_2-p-1)}$$

$$\cdot\sum_{k=0}^{\infty}\sum_{\kappa}\frac{L_\kappa^\gamma(\Omega)C_\kappa(-T\lambda^{-1}D)}{k!\Gamma(\frac{1}{2}pn_1+k)C_\kappa(I)}\,dD.$$

Now apply the estimate (2.34) of $L_\kappa^\gamma(\Omega)$. Then the integral in (3.6) is bounded by

$$\Gamma_p(\tfrac{1}{2}(n_1+n_2))|\Lambda|^{-\frac{1}{2}(n_1+n_2)} e^{\mathrm{tr}\Omega} {}_1F_0(\tfrac{1}{2}(n_1+n_2);-T(\lambda\Lambda)^{-1}),$$

which, by the definition of ${}_1F_0$, is convergent only for $||-T(\lambda\Lambda)^{-1}|| < 1$ or $|T/(\lambda\lambda_1)| < 1$, where λ_1 is the minimum latent root of Λ. Hence for this region of convergence, we can integrate D out in (3.6) using (2.16) to obtain the density function of T as stated in (3.1).

Formula (3.1) will give special cases as follows:

a. For $\Omega = 0$ and applying (2.32) we have the result of Khatri [19] formula (9).

b. For $\Lambda = I$ and $\lambda = 1$, we obtain Theorem 4 of Constantine [4].

Using the definition of Herz [11] for symmetric function of matrix argument $f(A) = f(HAH')$ for all $H \in \mathcal{O}(p)$, we have the following:

Corollary 3.1: The density of T is a symmetric function of Λ.

4. The Moments of T

In order to obtain the kth moment of T, we first note that

$$T^k = (\lambda \mathrm{tr} S_1 S_2^{-1})^k = (\lambda \mathrm{tr} B_1 B_2^{-1})^k = \lambda^k \sum_\kappa C_\kappa(B_1 B_2^{-1}),$$

where B_1 and B_2 are as in section 3 and we will use the joint density of B_1 and B_2 in (3.2). Multiplying (3.2) by $\lambda^k \sum_\kappa C_\kappa(B_1B_2^{-1})$ and integrating out B_2 using (2.18) we have

$$(4.1)\qquad C\cdot|\Lambda|^{-\frac{1}{2}n_1}\lambda^k\, e^{-\mathrm{tr}\Lambda^{-1}B_1}|B_1|^{\frac{1}{2}(n_1-p-1)}\,{}_0F_1(\tfrac{1}{2}n_1;\Lambda^{-\frac{1}{2}}\Omega\Lambda^{-\frac{1}{2}}B_1)$$

$$\cdot\sum_\kappa \Gamma_p(\tfrac{1}{2}n_2,-\kappa)C_\kappa(B_1),$$

where $C = [\Gamma_p(\frac{1}{2}n_1)\Gamma_p(\frac{1}{2}n_2)]^{-1}\, e^{-\mathrm{tr}\Omega}$.

Applying (2.1) and (2.28) we obtain the expression

$$(4.2)\qquad C\cdot\Gamma_p(\tfrac{1}{2}n_1)|\Lambda|^{-\frac{1}{2}n_1}\lambda^k e^{-\mathrm{tr}\Lambda^{-1}B_1}|B_1|^{\frac{1}{2}(n_1-p-1)}$$

$$\cdot\, A_\gamma(-\Lambda^{-\frac{1}{2}}\Omega\Lambda^{-\frac{1}{2}}B_1)\sum_\kappa \Gamma_p(\tfrac{1}{2}n_2,-\kappa)C_\kappa(B_1),$$

where $\gamma = \frac{1}{2}(n_1-p-1)$.

Let us now apply the transformation $D = \Lambda^{-\frac{1}{2}}B_1\Lambda^{-\frac{1}{2}}$.

Then the expression in (4.2) becomes

$$(4.3)\qquad C\cdot\Gamma_p(\tfrac{1}{2}n_1)\lambda^k e^{-\mathrm{tr}D}|D|^{\frac{1}{2}(n_1-p-1)}A_\gamma(-\Omega D)\sum_\kappa\Gamma_p(\tfrac{1}{2}n_2,-\kappa)C_\kappa(\Lambda D).$$

Now assume Λ "random" as explained in the previous section. Hence we can transform Λ by an orthogonal transformation $H \in \mathcal{O}(p)$ and then integrate H over $\mathcal{O}(p)$ using (2.23). The result is

$$(4.4)\qquad C\cdot\Gamma_p(\tfrac{1}{2}n_1)\lambda^k e^{-\mathrm{tr}D}|D|^{\frac{1}{2}(n_1-p-1)}A_\gamma(-\Omega D)$$

$$\cdot\sum_\kappa\{\Gamma_p(\tfrac{1}{2}n_2,-\kappa)C_\kappa(\Lambda)C_\kappa(D)/C_\kappa(I)\}.$$

Upon integrating D out using (2.26) we have the expression for the moments of T as

(4.5) $E(T^k) = \{\Gamma_p(\frac{1}{2} n_2)\}^{-1} \sum_\kappa \{\Gamma_p(\frac{1}{2}n_2,-\kappa) C_\kappa(\lambda \underset{\sim}{\Lambda}) L_\kappa^\gamma(-\underset{\sim}{\Omega})/C_\kappa(\underset{\sim}{I})\}.$

Finally, using the fact that $\Gamma_p(t,-\kappa) = \{(-1)^k \Gamma_p(t)\}/(-t+\frac{1}{2}(p+1))_\kappa$, we obtain the moments of T in the form

(4.6) $$E(T^k) = (-1)^k \sum_\kappa \frac{C_\kappa(\lambda \underset{\sim}{\Lambda}) L_\kappa^\gamma(-\underset{\sim}{\Omega})}{(\frac{1}{2}(p+1-n_2))_\kappa C_\kappa(\underset{\sim}{I})}, \quad \gamma = \frac{1}{2}(n_1-p-1),$$

which exists only for $n_2 > 2k + p - 1$.

Formula (4.6) will give special cases for special values of $\underset{\sim}{\Omega}$ and $\underset{\sim}{\Lambda}$. If we let $\underset{\sim}{\Omega} = \underset{\sim}{0}$, (4.6) gives

(4.7) $$E(T^k) = (-1)^k \sum_\kappa \{(\tfrac{1}{2} n_1)_\kappa C_\kappa(\lambda \underset{\sim}{\Lambda})/(\tfrac{1}{2}(p+1-n_2))_\kappa\},$$

which is the result of Khatri [19]; except that his formula contains an error in the denominator. The correct denominator for his expression should be $(\frac{1}{2}(p-n_2+1))_J$. Substitutions of $\underset{\sim}{\Lambda} = \underset{\sim}{I}$ and $\lambda = 1$ in (4.6) will give the result of Constantine [4] formula (38). Of course, substituting $\underset{\sim}{\Omega} = \underset{\sim}{0}$, $\underset{\sim}{\Lambda} = \underset{\sim}{I}$ and $\lambda = 1$ we have the kth moment of T in the central case as

(4.8) $$E(T^k) = (-1)^k \sum_\kappa [(\tfrac{1}{2} n_1)_\kappa C_\kappa(\underset{\sim}{I})/(\tfrac{1}{2}(p+1-n_2))_\kappa].$$

For illustration, let us compute the mean and the variance of T in the general case from (4.6). Replacing $L_\kappa^\gamma(-\underset{\sim}{\Omega})$ by zonal polynomials as expressed in (2.30) and making use of the tables in [4] and [17] we obtain the mean of T

(4.9) $$E(T) = \lambda a_1(pn_1+2b_1)/[p(n_2-p-1)],$$

and the variance of T

$$(4.10)\qquad \mathrm{Var}(T) = \lambda^2\Big[\frac{n_1(n_1+2)a_{21}}{3(n_2-p-1)(n_2-p-3)}\{1+\frac{4b_1}{pn_1}+\frac{4b_{21}}{pn_1(n_1+2)(p+2)}\}$$

$$+\frac{4n_1(n_1-1)a_2}{3(n_2-p)(n_2-p-1)}\{1+\frac{4b_1}{pn_1}+\frac{8b_2}{pn_1(n_1-1)(p-1)}\}$$

$$-\Big(\frac{n_1a_1}{n_2-p-1}\Big)^2\{1+\frac{4b_1}{pn_1}+\frac{4b_1^2}{p^2n_1^2}\}\Big],$$

where $a_{21} = 3a_1^2 - 4a_2$, $b_{21} = 3b_1^2 - 4b_2$ with a_1, a_2 are first and second e.s.f. of $\Sigma_1\Sigma_2^{-1}$ and b_1, b_2 are those of Ω respectively. The mean and the variance of T in the central case may be obtained from (4.9) and (4.10) respectively by substituting $a_1 = p$, $a_2 = \frac{1}{2}p(p-1)$ and $b_1 = b_2 = 0$.

5. Moment Generating Function of $V^{(p)}$

To obtain the moment generating function of Pillai's criterion $V^{(p)}$ which is defined by $V^{(p)} = \mathrm{tr}[\lambda R(I+\lambda R)^{-1}]$ where $R = \mathrm{diag}(r_1,\ldots,r_p)$, $r_1,\ldots,r_p$ being the latent roots of $S_1S_2^{-1}$, we will start from the result of Pillai [31,32]. Let $L = \lambda R(I+\lambda R)^{-1}$ there and dediagonalize L. After using (2.11), the m.g.f. of $V^{(p)}$ may be written as

$$(5.1)\qquad E(e^{tV^{(p)}}) = C\cdot\int_{\mathrm{Re}Z>0} e^{\mathrm{tr}Z}|Z|^{-\frac{1}{2}n_1}\int_{S>0} e^{-\mathrm{tr}S}|S|^{\frac{1}{2}(n_1+n_2-p-1)}$$

$$\cdot\int_{L>0}^{I}\exp[\mathrm{tr}(tI+S(I-\lambda^{-1}\Lambda^{-\frac{1}{2}}(I-Z^{-1}\Omega)\Lambda^{-\frac{1}{2}}))L]$$

$$\cdot|L|^{\frac{1}{2}(n_1-p-1)}|I-L|^{\frac{1}{2}(n_2-p-1)}\,dL\,dS\,dZ,$$

where $C = [2^{\frac{1}{2}p(p-1)}/\{(2\pi i)^{\frac{1}{2}p(p+1)}\Gamma_p(\frac{1}{2}n_2)\}]|\lambda\Lambda|^{-\frac{1}{2}n_1}e^{-\mathrm{tr}\Omega}$.

After integrating $\underset{\sim}{L}$ out using (2.15) we get

$$(5.2) \qquad E(e^{tV^{(p)}}) = C_1 \cdot \int_{Re\underset{\sim}{Z}>\underset{\sim}{0}} e^{tr\underset{\sim}{Z}} |\underset{\sim}{Z}|^{-\frac{1}{2} n_1} \int_{\underset{\sim}{S}>\underset{\sim}{0}} e^{-tr\underset{\sim}{S}} |\underset{\sim}{S}|^{\frac{1}{2}(n_1+n_2-p-1)}$$

$$\cdot \; {}_1F_1(\tfrac{1}{2} n_1; \tfrac{1}{2}(n_1+n_2); t\underset{\sim}{I}+\underset{\sim}{S}(\underset{\sim}{I}-\lambda^{-1}\underset{\sim}{\Lambda}^{-\frac{1}{2}}(\underset{\sim}{I}-\underset{\sim}{Z}^{-1}\underset{\sim}{\Omega})\underset{\sim}{\Lambda}^{-\frac{1}{2}}))d\underset{\sim}{S}d\underset{\sim}{Z}$$

where $C_1 = C \cdot \Gamma_p(\frac{1}{2} n_1)/\Gamma_p(\frac{1}{2}(n_1+n_2))$.

Now use (2.1) and (2.31) and then integrate $\underset{\sim}{S}$ out using (2.16). The right hand side of (5.2) becomes

$$C_1 \cdot \Gamma_p(\tfrac{1}{2} \nu) \int_{Re\underset{\sim}{Z}>\underset{\sim}{0}} e^{tr\underset{\sim}{Z}} |\underset{\sim}{Z}|^{-\frac{1}{2} n_1} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2} n_1)_\kappa C_\kappa(\underset{\sim}{I})}{k!(\frac{1}{2} \nu)_\kappa}$$

$$\cdot \sum_{d=0}^{k} \sum_{\delta} \{a_{\kappa,\delta} t^{k-d} (\tfrac{1}{2} \nu)_\delta C_\delta(\underset{\sim}{I}-\lambda^{-1}\underset{\sim}{\Lambda}^{-\frac{1}{2}}(\underset{\sim}{I}-\underset{\sim}{Z}^{-1}\underset{\sim}{\Omega})\underset{\sim}{\Lambda}^{-\frac{1}{2}})/C_\delta(\underset{\sim}{I})\}d\underset{\sim}{Z},$$

where $\nu = n_1+n_2$. Further, we make use of (2.31) once more to $C_\delta(\underset{\sim}{I}-\lambda^{-1}\underset{\sim}{\Lambda}^{-\frac{1}{2}}(\underset{\sim}{I}-\underset{\sim}{Z}^{-1}\underset{\sim}{\Omega})\underset{\sim}{\Lambda}^{-\frac{1}{2}})$ and then assume $\underset{\sim}{\Lambda}$ "random". Using these and finally (2.22) we obtain the m.g.f. of $V^{(p)}$ in the form

$$(5.3) \qquad E(e^{tV^{(p)}}) = e^{-tr\underset{\sim}{\Omega}} |\lambda\underset{\sim}{\Lambda}|^{-\frac{1}{2} n_1} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2} n_1)_\kappa C_\kappa(\underset{\sim}{I})}{k!(\frac{1}{2} \nu)_\kappa}$$

$$\cdot \sum_{d=0}^{k} \sum_{\delta} a_{\kappa,\delta} (\tfrac{1}{2} \nu)_\delta t^{k-d} \sum_{n=0}^{d} \sum_{\nu} \frac{a_{\kappa,\nu} C_\nu(-\lambda^{-1}\underset{\sim}{\Lambda}^{-1}) L_\nu^\gamma(\underset{\sim}{\Omega})}{(\frac{1}{2} n_1)_\nu C_\nu(\underset{\sim}{I}) C_\nu(\underset{\sim}{I})},$$

where $\nu = n_1+n_2$ and $\gamma = \frac{1}{2}(n_1-p-1)$.

For $\underset{\sim}{\Omega} = \underset{\sim}{0}$, (5.3) gives the result of Khatri [19] with a correction of his expression given by Pillai [30]. Another special case which can be derived from (5.3) is formula (3.5) of Pillai [30] if in (5.3) we let $\underset{\sim}{\Lambda} = \underset{\sim}{I}$ and $\lambda = 1$.

6. The Density Function of $W^{(p)}$

In terms of the characteristic roots r_i, $i = 1,2,\ldots,p$, of $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$, the Wilks' criterion $W^{(p)}$ is defined by $W^{(p)} = |\underset{\sim}{I}+\lambda\underset{\sim}{R}|^{-1}$. However, in the derivation of the density of $W^{(p)}$ we consider $W^{(p)} = |\underset{\sim}{I}-\underset{\sim}{L}|$, i.e. we let $\underset{\sim}{L} = \lambda\underset{\sim}{R}(\underset{\sim}{I}+\lambda\underset{\sim}{R})^{-1}$, where now $\underset{\sim}{L} = \mathrm{diag}(\ell_1,\ldots,\ell_p)$. From the theorem of Pillai [31,32], the joint density of $\ell_1,\ell_2,\ldots,\ell_p$ is given by

$$(6.1)\qquad C\cdot|\underset{\sim}{L}|^{\frac{1}{2}(n_1-p-1)}\,|\underset{\sim}{I}-\underset{\sim}{L}|^{\frac{1}{2}(n_2-p-1)}\prod_{i>j}(\ell_i-\ell_j)$$

$$\cdot\sum_{k=0}^{\infty}\sum_{\kappa}[(\tfrac{1}{2}(n_1+n_2))_\kappa C_\kappa(\underset{\sim}{L})/k!]\cdot(\Sigma_p),$$

where $C = [\pi^{\frac{1}{2}p^2}\Gamma_p(\frac{1}{2}(n_1+n_2))/\{\Gamma_p(\frac{1}{2}n_1)\Gamma_p(\frac{1}{2}n_2)\Gamma_p(\frac{1}{2}p)\}]e^{-\mathrm{tr}\underset{\sim}{\Omega}}|\lambda\underset{\sim}{\Lambda}|^{-\frac{1}{2}n_1}$,

$$(6.2)\qquad (\Sigma_p) = \sum_{d=0}^{k}\sum_{\delta}\frac{a_{\kappa,\delta}C_\delta(-\lambda^{-1}\underset{\sim}{\Lambda}^{-1})L_\delta^\gamma(\underset{\sim}{\Omega})}{(\frac{1}{2}n_1)_\delta C_\delta(\underset{\sim}{I})C_\delta(\underset{\sim}{I})},\quad \gamma = \tfrac{1}{2}(n_1-p-1).$$

To obtain the density function of $W^{(p)}$, first we find its hth moment and then use the results on inverse Mellin transform. The hth moment of $W^{(p)}$ is given by

$$(6.3)\qquad E((W^{(p)})^h) = C\cdot\int_{L>0}^{I}|\underset{\sim}{L}|^{\frac{1}{2}(n_1-p-1)}\,|\underset{\sim}{I}-\underset{\sim}{L}|^{\frac{1}{2}(n_2+h-p-1)}$$

$$\cdot\sum_{k=0}^{\infty}\sum_{\kappa}[(\tfrac{1}{2}(n_1+n_2))_\kappa C_\kappa(\underset{\sim}{L})/k!]d\underset{\sim}{L}\cdot(\Sigma_p).$$

The expression in (6.3) is obtained from (6.1) after dediagonalizing $\underset{\sim}{L}$. Now integrate $\underset{\sim}{L}$ out using (2.20), we obtain the hth moment of $W^{(p)}$ in the form

$$(6.4)\quad E((W^{(p)})^h) = [\Gamma_p(\tfrac{1}{2}(n_1+n_2))/\Gamma_p(\tfrac{1}{2}n_2)]e^{-\mathrm{tr}\underset{\sim}{\Omega}}|\lambda\underset{\sim}{\Lambda}|^{-\frac{1}{2}n_1}$$

$$\cdot \sum_{k=0}^{\infty}\sum_{\kappa} \frac{(\frac{1}{2}(n_1+n_2))_\kappa(\frac{1}{2}n_1)_\kappa C_\kappa(\underset{\sim}{I})}{k!} \cdot \frac{\prod_{i=1}^{p}\Gamma(r+b_i)}{\prod_{i=1}^{p}\Gamma(r+a_i)} \cdot (\Sigma_p),$$

where $r = \frac{1}{2}n_2 + h - \frac{1}{2}(p-1)$, $b_i = \frac{1}{2}(i-1)$ and $a_i = \frac{1}{2}n_1 + k_{p-i+1} + b_i$. Finally, applying (2.36) we get the density function of $W^{(p)}$ as

$$(6.5)\quad f(W^{(p)}) = [\Gamma_p(\tfrac{1}{2}(n_1+n_2))/\Gamma_p(\tfrac{1}{2}n_2)]e^{-\mathrm{tr}\underset{\sim}{\Omega}}|\lambda\underset{\sim}{\Lambda}|^{-\frac{1}{2}n_1}$$

$$\cdot \sum_{k=0}^{\infty}\sum_{\kappa} \frac{(\frac{1}{2}(n_1+n_2))_\kappa(\frac{1}{2}n_1)_\kappa C_\kappa(\underset{\sim}{I})}{k!} (W^{(p)})^{\frac{1}{2}(n_2-p-1)}$$

$$\cdot (2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} (W^{(p)})^{-r} \frac{\prod_{i=1}^{p}\Gamma(r+b_i)}{\prod_{i=1}^{p}\Gamma(r+a_i)} dr\cdot(\Sigma_p).$$

But the integral in (6.5) is expressible in terms of Meijer's G-function (2.38), so that we obtain the following:

<u>Theorem 6.1</u>: Let $\underset{\sim}{S}_1$ and $\underset{\sim}{S}_2$ be as in Theorem 3.1 and let $\underset{\sim}{R} = \mathrm{diag}(r_1,\ldots,r_p)$ where $r_1,\ldots,r_p$ are the latent roots of $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$. If $\underset{\sim}{\Sigma}_1^{\frac{1}{2}}\underset{\sim}{\Sigma}_2^{-1}\underset{\sim}{\Sigma}_1^{\frac{1}{2}}$ "random", then the density function of $W^{(p)}=|\underset{\sim}{I}+\lambda\underset{\sim}{R}|^{-1}$, $\lambda > 0$, is given by

$$(6.6)\qquad f(W^{(p)}) = [\Gamma_p(\tfrac{1}{2}(n_1+n_2))/\Gamma_p(\tfrac{1}{2}n_2)]e^{-\mathrm{tr}\Omega}|\lambda\Lambda|^{-\frac{1}{2}n_1}$$

$$\cdot\{W^{(p)}\}^{\frac{1}{2}(n_2-p-1)}\sum_{k=0}^{\infty}\sum_{\kappa}[(\tfrac{1}{2}(n_1+n_2))_\kappa(\tfrac{1}{2}n_1)_\kappa C_\kappa(I)/k!]$$

$$\cdot\, G^{p,0}_{p,p}\left(W^{(p)}\,\middle|\,{a_1,\ldots,a_p \atop b_1,\ldots,b_p}\right)\cdot(\Sigma_p),$$

where $\Lambda = \mathrm{diag}(\lambda_1,\ldots,\lambda_p)$, $\lambda_1,\ldots,\lambda_p$ being the latent roots of $\Sigma_1\Sigma_2^{-1}$, $a_i = \frac{1}{2}n_1+k_{p-i+1} + b_i$, $b_i = \frac{1}{2}(i-1)$ and (Σ_p) is as in (6.2).

Special Cases:

a. Substituting $\Omega = 0$, $\Lambda = I$ and $\lambda = 1$ into (6.4) and (6.6) we have the hth moment and the density of $W^{(p)}$ respectively as

$$(6.7)\qquad E((W^{(p)})^h) = \frac{\Gamma_p(\frac{1}{2}(n_1+n_2))}{\Gamma_p(\frac{1}{2}n_2)}\cdot\frac{\prod_{i=1}^{p}\Gamma(\frac{1}{2}(n_2-p+i)+h)}{\prod_{i=1}^{p}\Gamma(\frac{1}{2}(n_1+n_2-p+i)+h)},$$

and

$$(6.8)\qquad f(W^{(p)}) = \frac{\Gamma_p(\frac{1}{2}(n_1+n_2))}{\Gamma_p(\frac{1}{2}n_2)}\{W^{(p)}\}^{\frac{1}{2}(n_2-p-1)}G^{p,0}_{p,p}\left(W^{(p)}\,\middle|\,{a_1,\ldots,a_p \atop b_1,\ldots,b_p}\right)$$

where $a_i = \frac{1}{2}n_1+b_i$ and $b_i = \frac{1}{2}(i-1)$.

These are the results of Consul [7].

b. For $\Omega = 0$ and after using (2.32), formula (6.6) gives the result of Pillai, Al-Ani and Jouris [42] formula (4.7) for testing the hypothesis H_0: $\lambda\Lambda = I$, $\lambda > 0$ being given.

c. If in (6.6) we let $\Lambda = I$, $\lambda = 1$ and use (2.30), then formula (5.2) of Pillai, Al-Ani and Jouris [42] is obtained. Note that, in this case $\ell_1,\ldots,\ell_p$ are the latent roots of the determinantal equation $|\underset{\sim}{S}_1-\ell(\underset{\sim}{S}_1+\underset{\sim}{S}_2)| = 0$, and $W^{(p)}$ is the Wilks' criterion for MANOVA.

d. Letting $p = 2$ in (6.6) and using (2.40) we obtain the density of $W^{(2)}$ as

$$(6.9) \qquad f(W^{(2)}) = [\Gamma_2(\tfrac{1}{2}(n_1+n_2))/\Gamma_2(\tfrac{1}{2}n_2)]e^{-(w_1+w_2)}(\lambda^2\lambda_1\lambda_2)^{-\frac{1}{2}n_1}$$

$$\cdot\{W^{(2)}\}^{\frac{1}{2}(n_2-3)}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\frac{1}{2}(n_1+n_2))_\kappa(\frac{1}{2}n_1)_\kappa}{k!\,\Gamma(n_1+k)}C_\kappa(\underset{\sim}{I}_2)(1-W^{(2)})^{n_1+k-1}$$

$$\cdot\ {}_2F_1(\tfrac{1}{2}n_1+k_1,\ \tfrac{1}{2}(n_1-1)+k_2;n_1+k;1-W^{(2)})\cdot(\Sigma_2),$$

where w_1,w_2 are the latent roots of $\underset{\sim}{\Omega}(2x2)$, λ_1,λ_2 are those of $\underset{\sim}{\Lambda} = \mathrm{diag}(\lambda_1,\lambda_2)$ and Σ_2 is as in (6.2) by taking all matrices involved there by those corresponding matrices of order 2. Let us integrate (6.9) for $0 < W^{(2)} \leq w$ (see Pillai, Al-Ani and Jouris [42]) to obtain the c.d.f. of $W^{(2)}$, and use the relation (Consul [6])

$$(6.10) \qquad (d^n/dz^n)(z^{c-1}{}_2F_1(a,b;c;z)) = (c-n)_n z^{c-n-1}{}_2F_1(a,b;c-n;z)$$

After integrating (6.9) by parts $a_1 = \frac{1}{2}n_1 + k_2$ times when n_1 is even and noting that $\kappa = (k_1,k_2)$ we get the c.d.f. of $W^{(2)}$ as

$$(6.11) \qquad F(W^{(2)}) = e^{-(w_1+w_2)}(\lambda^2\lambda_1\lambda_2)^{-\frac{1}{2}n_1}(W^{(2)})^{\frac{1}{2}(n_1-1)}$$

$$\cdot\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\frac{1}{2}n_1)_\kappa C_\kappa(\underset{\sim}{I}_2)}{k!}\left[\frac{\Gamma_2(\frac{1}{2}(n_1+n_2))(\frac{1}{2}(n_1+n_2))_\kappa}{\Gamma_2(\frac{1}{2}n_2)\Gamma(n_1+k)}\right.$$

$$\cdot \sum_{r=0}^{a} \{(n_1+k-r)_r / (\tfrac{1}{2}(n_2-1))_{r+1}\}(W^{(2)})^r (1-W^{(2)})^{n_1+k-r-1}$$

$$\cdot {}_2F_1(\tfrac{1}{2}n_1+k_1,\ \tfrac{1}{2}(n_1-1)+k_2;\ n_1+k-r; 1-W^{(2)}) + I_{W^{(2)}}(\tfrac{1}{2}n_2, b)] \cdot (\Sigma_2)$$

where $a = a_1-1$, $b = a_2-b_2$, $a_2 = \frac{1}{2}n_1+k_1+\frac{1}{2}$ and $b_2 = \frac{1}{2}$.

In the case n_1 is odd, after integrating (6.9) by parts a_2 times, we obtain the c.d.f. of $W^{(2)}$ as in (6.11) but now $a = a_2-1$ and $b = a_1-b_2$.

In the next chapter, the c.d.f. of $W^{(2)}$ is also obtained using a different method and it seems that the expression is simpler there.

7. The Density Function of the Largest Root

In this section we derive two expressions for the density function of the largest root r_p of $S_1S_2^{-1}$. In obtaining this density we start from the joint density of the roots $r_1,\ldots,r_p$ of $S_1S_2^{-1}$ which is given in [31,32] by the formula

$$(7.1) \qquad C_1 |R|^{\frac{1}{2}(n_1-p-1)} \prod_{i<j}(r_i-r_j) \int_{\mathrm{Re}Z>0} e^{\mathrm{tr}Z}|Z|^{-\frac{1}{2}n_1}$$

$$\cdot \int_{\mathcal{O}(p)} 2^{-p}|I+HRH'\Lambda^{-\frac{1}{2}}(I-W)\Lambda^{-\frac{1}{2}}|^{-\frac{1}{2}(n_1+n_2)} dHdZ,$$

where $R = \mathrm{diag}(r_1,\ldots,r_p)$, $W = \Omega^{\frac{1}{2}}Z^{-1}\Omega^{\frac{1}{2}}$ and

$$C_1 = e^{-\mathrm{tr}\Omega}|\Lambda|^{-\frac{1}{2}n_1} 2^{\frac{1}{2}p(p-1)} \Gamma_p(\tfrac{1}{2}(n_1+n_2))/[(2\pi i)^{\frac{1}{2}p(p+1)} \Gamma_p(\tfrac{1}{2}n_2)].$$

Now use (2.25) by taking $g(F) = r_p$. Applying (2.24a), the second integral in (7.1) becomes

$$[\pi^{\frac{1}{2}p^2}/\Gamma_p(\tfrac{1}{2}p)]\,|\underset{\sim}{I}+\underset{\sim}{\Lambda}^{-\frac{1}{2}}(\underset{\sim}{I}-\underset{\sim}{W})\underset{\sim}{\Lambda}^{-\frac{1}{2}}r_p|^{-\frac{1}{2}(n_1+n_2)}$$

$$\cdot\,{}_1F_0(\tfrac{1}{2}(n_1+n_2);r_p^{-1}(\underset{\sim}{I}+\underset{\sim}{\Lambda}^{-\frac{1}{2}}(\underset{\sim}{I}-\underset{\sim}{W})\underset{\sim}{\Lambda}^{-\frac{1}{2}}r_p^{-1})^{-1},r_p\underset{\sim}{I}-\underset{\sim}{R}).$$

Let $y_i = r_i/r_p$, $i = 1,2,\ldots,p-1$ and $\underset{\sim}{Y} = \mathrm{diag}(y_1,\ldots,y_{p-1})$, then the joint density of r_p, y_1, $y_2,\ldots,y_{p-1}$, where $0 < r_p < \infty$, $0 < y_1 < y_2 < \ldots < y_{p-1} < 1$, is given by

$$(7.2)\qquad C_2\, r_p^{\frac{1}{2}pn_1-1}|\underset{\sim}{Y}|^{\frac{1}{2}(n_1-p-1)}|\underset{\sim}{I}_{p-1}-\underset{\sim}{Y}|\prod_{i>j}^{p-1}(y_i-y_j)$$

$$\cdot\int_{\mathrm{Re}\underset{\sim}{Z}} e^{\mathrm{tr}\underset{\sim}{Z}}|\underset{\sim}{Z}|^{-\frac{1}{2}n_1}|\underset{\sim}{I}+\underset{\sim}{\Lambda}^{-\frac{1}{2}}(\underset{\sim}{I}-\underset{\sim}{W})\underset{\sim}{\Lambda}^{-\frac{1}{2}}r_p|^{-\frac{1}{2}(n_1+n_2)}$$

$$\cdot\,{}_1F_0(\tfrac{1}{2}(n_1+n_2);(\underset{\sim}{I}+\underset{\sim}{\Lambda}^{\frac{1}{2}}(\underset{\sim}{I}-\underset{\sim}{W})^{-1}\underset{\sim}{\Lambda}^{\frac{1}{2}}r_p^{-1})^{-1},\ \underset{\sim}{I}_{p-1}-\underset{\sim}{Y})d\underset{\sim}{Z}$$

where $\underset{\sim}{I}_{p-1}$ is the identity matrix of order p-1, $\underset{\sim}{I}$ is that of order p and

$$C_2 = C_1\cdot\pi^{\frac{1}{2}p^2}/\Gamma_p(\tfrac{1}{2}p).$$

Expand ${}_1F_0$ in terms of zonal polynomials and for the integration with respect to $\underset{\sim}{Y}$ we apply (2.21). Then we obtain the density function of the largest root r_p of the form

$$(7.3)\qquad C_3\cdot r_p^{\frac{1}{2}pn_1-1}\sum_{k=0}^{\infty}\frac{(\frac{1}{2}(n_1+n_2))_\kappa\Gamma_{p-1}(\frac{1}{2}p+1,\kappa)\,C_\kappa(\underset{\sim}{I}_{p-1})}{k!\,\Gamma_{p-1}(\frac{1}{2}(n_1+p+1),\kappa)C_\kappa(\underset{\sim}{I})}$$

$$\cdot\int_{\mathrm{Re}\underset{\sim}{Z}>\underset{\sim}{0}} e^{\mathrm{tr}\underset{\sim}{Z}}|\underset{\sim}{Z}|^{-\frac{1}{2}n_1}|\underset{\sim}{I}+\underset{\sim}{\Lambda}^{-\frac{1}{2}}(\underset{\sim}{I}-\underset{\sim}{W})\underset{\sim}{\Lambda}^{-\frac{1}{2}}r_p|^{-\frac{1}{2}(n_1+n_2)}$$

$$\cdot\ C_\kappa[\{I+\Lambda^{\frac{1}{2}}(I-W)^{-1}\Lambda^{\frac{1}{2}}\ r_p^{-1}\}^{-1}]\ dZ,$$

where $C_3 = C_2\cdot\Gamma_{p-1}(\frac{1}{2}(n_1-1))\Gamma_{p-1}(\frac{1}{2}(p-1))/\pi^{\frac{1}{2}(p-1)^2}$.

The integral in (7.3) can be written as

$$\int_{\mathrm{Re}Z>0} e^{\mathrm{tr}Z}|Z|^{-\frac{1}{2}n_1}|I+\Lambda^{-\frac{1}{2}}(I-W)\Lambda^{-\frac{1}{2}}\ r_p|^{-\frac{1}{2}(n_1+n_2)}$$

$$\cdot\ C_\kappa[\{\Lambda^{-\frac{1}{2}}(I-W)\Lambda^{-\frac{1}{2}}\ r_p\}\{I+\Lambda^{-\frac{1}{2}}(I-W)\Lambda^{-\frac{1}{2}}\ r_p\}^{-1}]\ dZ.$$

Applying (2.16), the above integral becomes

$$[\Gamma_p(\tfrac{1}{2}(n_1+n_2),\kappa)]^{-1}\int_{\mathrm{Re}Z>0} e^{\mathrm{tr}Z}|Z|^{-\frac{1}{2}n_1}|I-W|^{-\frac{1}{2}(n_1+n_2)}$$

$$\cdot\int_{S>0}\exp[-\mathrm{tr}\{S(I+\Lambda^{-\frac{1}{2}}(I-W)\Lambda^{-\frac{1}{2}}\ r_p)\}]|S|^{\frac{1}{2}(n_1+n_2-p-1)}$$

$$\cdot\ C_\kappa[S(\Lambda^{-\frac{1}{2}}(I-W)\Lambda^{-\frac{1}{2}}\ r_p]\ dS\ dZ.$$

Let us take $B = r_p\Lambda^{-\frac{1}{2}}\ S\Lambda^{-\frac{1}{2}}$. After making a substitution and necessary rearrangement, the above expression becomes

$$(7.4)\quad \frac{|r_p\Lambda^{-1}|^{-\frac{1}{2}(n_1+n_2)}}{\Gamma_p(\frac{1}{2}(n_1+n_2),\kappa)}\sum_{n=0}^{\infty}\sum_{\nu}\frac{(-1)^n}{n!}\sum_{\delta} g^{\delta}_{\kappa,\nu}\int_{\mathrm{Re}Z>0} e^{\mathrm{tr}Z}|Z|^{-\frac{1}{2}n_1}$$

$$\cdot\int_{B>0} e^{-\mathrm{tr}\ r_p^{-1}\Lambda B}|B|^{\frac{1}{2}(n_1+n_2-p-1)}C_\delta[(I-W)B]dBdZ,$$

where $g^{\delta}_{\kappa,\nu}$ are constants and $\sum_{i=1}^{p} \delta_i = k+n$,

$\delta = (\delta_1,\ldots,\delta_p)$, $\delta_p \geq \cdots \geq \delta_1 \geq 0$.

Now integrate $\underset{\sim}{B}$ out using (2.16), then (7.4) reduces to

$$(7.5)\quad [\Gamma_p(\tfrac{1}{2}(n_1+n_2),\kappa)]^{-1} \sum_{n=0}^{\infty} \sum_{\nu} \frac{(-1)^n}{n!} \sum_{\delta} g^{\delta}_{\kappa,\nu} \Gamma_p(\tfrac{1}{2}(n_1+n_2),\delta)$$

$$\cdot \int_{\mathrm{Re}\underset{\sim}{Z}>\underset{\sim}{0}} e^{\mathrm{tr}\underset{\sim}{Z}} |\underset{\sim}{Z}|^{-\frac{1}{2}n_1} C_{\delta}[(\underset{\sim}{I}-\underset{\sim}{W})(r_p\underset{\sim}{\Lambda}^{-1})]d\underset{\sim}{Z}.$$

Diagonalizing $(r_p\underset{\sim}{\Lambda}^{-1})$ by an orthogonal transformation $\underset{\sim}{H}$ and integrating over $\mathcal{O}(p)$ and finally applying (2.29), we obtain the density function of r_p as stated in the following theorem.

<u>Theorem 7.1</u>: Let $\underset{\sim}{S}_1$, $\underset{\sim}{S}_2$ and $\underset{\sim}{\Sigma}_1^{\frac{1}{2}}\underset{\sim}{\Sigma}_2^{-1}\underset{\sim}{\Sigma}_1^{\frac{1}{2}}$ be as in theorem 6.1 and let r_p be the largest latent root of $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$. Then the density function of r_p is given by

$$(7.6)\quad C\, e^{\mathrm{tr}\underset{\sim}{\Omega}} |\underset{\sim}{\Lambda}|^{-\frac{1}{2}n_1} r_p^{\frac{1}{2}pn_1-1} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}p+1)_{\kappa}(\frac{1}{2}(p-1))_{\kappa}}{k!(\frac{1}{2}(n_1+p+1))_{\kappa}(\frac{1}{2}p)_{\kappa}}$$

$$\cdot \sum_{n=0}^{\infty} \sum_{\nu} \frac{(-1)^n}{n!} \sum_{\delta} g^{\delta}_{\kappa,\nu} \frac{(\frac{1}{2}(n_1+n_2))_{\delta} C_{\delta}(r_p\underset{\sim}{\Lambda}^{-1}) L^{\gamma}_{\delta}(\underset{\sim}{\Omega})}{(\frac{1}{2}n_1)_{\delta} C_{\delta}(\underset{\sim}{I})},$$

where $\underset{\sim}{\Lambda} = \mathrm{diag}(\lambda_1,\ldots,\lambda_p)$, $\lambda_1,\ldots,\lambda_p$ are the latent roots of $\underset{\sim}{\Sigma}_1\underset{\sim}{\Sigma}_2^{-1}$, $\gamma = \frac{1}{2}(n_1-p-1)$ and

$$(7.7)\quad C = \frac{\Gamma(\frac{1}{2})\Gamma_p(\frac{1}{2}(n_1+n_2))\Gamma_{p-1}(\frac{1}{2}p+1)}{\Gamma(\frac{1}{2}p)\Gamma(\frac{1}{2}n_1)\Gamma_p(\frac{1}{2}n_2)\Gamma_{p-1}(\frac{1}{2}(n_1+p+1))}.$$

Note that in obtaining (7.6) and (7.7), the relations $\Gamma_p(a,\kappa) = (a)_\kappa \Gamma_p(a)$ and

$$(7.8) \qquad C_\kappa(\underset{\sim}{I}_{p-1})/C_\kappa(\underset{\sim}{I}_p) = (\tfrac{1}{2}(p-1))_\kappa/(\tfrac{1}{2}p)_\kappa$$

have been used.

The density in (7.6) may not converge for all r_p, $0 < r_p < \infty$. It can be seen from the following fact. Put $\underset{\sim}{\Omega} = \underset{\sim}{0}$ and set $n = 0$. The density (7.6) reduces to

$$(7.9) \qquad C\cdot|\underset{\sim}{\Lambda}|^{-\frac{1}{2}n_1} r_p^{\frac{1}{2}pn_1-1} \; {}_3F_2(\tfrac{1}{2}(n_1+n_2),\ \tfrac{1}{2}p+1,\ \tfrac{1}{2}(p-1);$$

$$\tfrac{1}{2}(n_1+p+1),\ \tfrac{1}{2}p;\ r_p\underset{\sim}{\Lambda}^{-1}).$$

By the definition of ${}_qF_r$, (7.9) converges only for $||r_p\underset{\sim}{\Lambda}^{-1}|| < 1$ or $|r_p/\lambda_1| < 1$ where λ_1 is the minimum latent root of $\underset{\sim}{\Lambda}$.

To obtain the second expression of the density of r_p, let us return to the formula (7.4). The second integral in (7.4) can be written as

$$(7.10) \qquad \int_{\underset{\sim}{B}>\underset{\sim}{0}} e^{-tr[(\underset{\sim}{I}+r_p^{-1}\underset{\sim}{\Lambda})\underset{\sim}{B}]} \, e^{tr\underset{\sim}{B}} |\underset{\sim}{B}|^{\frac{1}{2}(n_1+n_2-p-1)} \cdot C_\delta[(\underset{\sim}{I}-\underset{\sim}{W})\underset{\sim}{B}]dB.$$

We shall split $(\underset{\sim}{I}-\underset{\sim}{W})$ from $\underset{\sim}{B}$ in the zonal polynomial $C_\delta((\underset{\sim}{I}-\underset{\sim}{W})\underset{\sim}{B})$ as follows. Expand the first exponent in (7.10) and diagonalize $(\underset{\sim}{I}+r_p^{-1}\underset{\sim}{\Lambda})$ then apply (2.23). Further diagonalize $\underset{\sim}{B}$ to separate $(\underset{\sim}{I}-\underset{\sim}{W})$ from $\underset{\sim}{B}$. After dediagonalizing $\underset{\sim}{B}$ and using (2.23), the integral in (7.10) now becomes

$$(7.11) \qquad \int_{\underset{\sim}{B}>\underset{\sim}{0}} e^{-tr[(\underset{\sim}{I}+r_p^{-1}\underset{\sim}{\Lambda})\underset{\sim}{B}]} \; e^{tr\underset{\sim}{B}} |\underset{\sim}{B}|^{\frac{1}{2}(n_1+n_2-p-1)}$$

$$\cdot [C_\delta(\underset{\sim}{B})C_\delta(\underset{\sim}{I}-\underset{\sim}{W})/C_\delta(\underset{\sim}{I})] \; d\underset{\sim}{B},$$

where to note that $(\underset{\sim}{I}+r_p^{-1}\underset{\sim}{\Lambda})$ now is a diagonal matrix $\mathrm{diag}(1+r_p^{-1}\lambda_1,\ldots,1+r_p^{-1}\lambda_p)$, $\lambda_1,\ldots,\lambda_p$ are the latent roots of $\underset{\sim}{\Lambda}$. Expand the second exponent and use (2.6a) to the expression in (7.11). Then (7.4) now becomes

$$\frac{|\underset{\sim}{I}+r_p\underset{\sim}{\Lambda}^{-1}|^{-\frac{1}{2}(n_1+n_2)}}{\Gamma_p(\frac{1}{2}(n_1+n_2),\kappa)} \sum_{n=0}^{\infty} \sum_{\nu} \frac{(-1)^n}{n!} \sum_{\delta} g^{\delta}_{\kappa,\nu}[C_\delta(\underset{\sim}{I})]^{-1}$$

$$\cdot \int_{Re\underset{\sim}{Z}>\underset{\sim}{0}} e^{tr\underset{\sim}{Z}}|\underset{\sim}{Z}|^{-\frac{1}{2}n_1} C_\delta(\underset{\sim}{I}-\underset{\sim}{W})d\underset{\sim}{Z}$$

$$\cdot \sum_{t=0}^{\infty} \sum_{\tau} (t!)^{-1} \sum_{\mu} g^{\mu}_{\delta,\tau}\Gamma_p(\tfrac{1}{2}(n_1+n_2),\mu)C_\mu[(\underset{\sim}{I}+r_p^{-1}\underset{\sim}{\Lambda})^{-1}].$$

Replacing $\underset{\sim}{W}$ by $\underset{\sim}{\Omega}^{\frac{1}{2}}\underset{\sim}{Z}^{-1}\underset{\sim}{\Omega}^{\frac{1}{2}}$ and making use of (2.29) we finally obtain the following :

<u>Theorem 7.2:</u> Let $\underset{\sim}{S}_1,\underset{\sim}{S}_2$ and r_p be as before. If $\underset{\sim}{\Sigma}_1^{\frac{1}{2}}\underset{\sim}{\Sigma}_2^{-1}\underset{\sim}{\Sigma}_1^{\frac{1}{2}}$ "random", then the density of r_p is given by

$$(7.12) \qquad C\cdot e^{-tr\underset{\sim}{\Omega}}|\underset{\sim}{\Lambda}|^{-\frac{1}{2}n_1}|\underset{\sim}{I}+r_p\underset{\sim}{\Lambda}^{-1}|^{-\frac{1}{2}(n_1+n_2)} r_p^{\frac{1}{2}pn_1-1}$$

$$\cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}p+1)_\kappa(\frac{1}{2}(p-1))_\kappa}{k!(\frac{1}{2}(n_1+p+1))_\kappa(\frac{1}{2}p)_\kappa} \sum_{n=0}^{\infty} \sum_{\nu} \frac{(-1)^n}{n!} \sum_{\delta} g^{\delta}_{\kappa,\nu} \frac{L^{\gamma}_{\delta}(\underset{\sim}{\Omega})}{(\frac{1}{2}n_1)_\delta C_\delta(\underset{\sim}{I})}$$

$$\cdot \sum_{t=0}^{\infty} \sum_{\tau} (t!)^{-1} \sum_{\mu} g^{\mu}_{\delta,\tau}(\tfrac{1}{2}(n_1+n_2))_\mu C_\mu[(\underset{\sim}{I}+r_p^{-1}\underset{\sim}{\Lambda})^{-1}],$$

where $(\underset{\sim}{I}+r_p^{-1}\underset{\sim}{\Lambda}) = \text{diag}(1+r_p^{-1}\lambda_1,\ldots,1+r_p^{-1}\lambda_p)$, $\lambda_1,\ldots,\lambda_p$ are the latent roots of $\underset{\sim}{\Sigma}_1\underset{\sim}{\Sigma}_2^{-1}$, $\gamma = \frac{1}{2}(n_1-p-1)$ and C is as in (7.7).

For n=t=0 and $\underset{\sim}{\Omega}=\underset{\sim}{0}$, (7.12) reduces to (16) of Khatri [19].

CHAPTER II

ROBUSTNESS STUDIES OF TESTS IN MULTIVARIATE ANALYSIS

1. Introduction

Consider a $p \times f_1$ matrix variate $\underset{\sim}{U}$ and a $p \times f_2$ matrix variate $\underset{\sim}{V}$, $p \leq f_1$ and $p \leq f_2$, where the columns are all independently normally distributed with covariance matrix $\underset{\sim}{\Sigma}_1$ and $\underset{\sim}{\Sigma}_2$ respectively, while $E(\underset{\sim}{U}) = \underset{\sim}{M}$ and $E(\underset{\sim}{V}) = \underset{\sim}{0}$. It is well known that the density of $\underset{\sim}{S}_1 = \underset{\sim}{U}\underset{\sim}{U}'$ is a non-central Wishart $W(p, f_1, \underset{\sim}{\Sigma}_1, \underset{\sim}{\Omega})$ where $\underset{\sim}{\Omega} = \frac{1}{2} \underset{\sim}{M}\underset{\sim}{M}' \underset{\sim}{\Sigma}_1^{-1}$ and $\underset{\sim}{S}_2 = \underset{\sim}{V}\underset{\sim}{V}'$ is a central Wishart $W(p, f_2, \underset{\sim}{\Sigma}_2, \underset{\sim}{0})$, where $\underset{\sim}{S}_1$ and $\underset{\sim}{S}_2$ are independently distributed. Under the assumption that $\Sigma_1^{\frac{1}{2}} \Sigma_2^{-1} \Sigma_1^{\frac{1}{2}}$ is "random", Pillai [31,32] has obtained the joint density of the latent roots $r_1, r_2, \ldots, r_p$ of $\underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$ in the form

$$(1.1) \qquad C(p,m,n) e^{-tr\underset{\sim}{\Omega}} |\underset{\sim}{\Lambda}|^{-\frac{1}{2} f_1} |\underset{\sim}{R}|^m |\underset{\sim}{I} + \lambda \underset{\sim}{R}|^{-\frac{1}{2} \nu} \prod_{i>j} (r_i - r_j)$$

$$\cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}\nu)_\kappa}{k!} C_\kappa\{\lambda \underset{\sim}{R}(\underset{\sim}{I} + \lambda \underset{\sim}{R})^{-1}\}. \quad F_p,$$

where F_p denotes the function of matrix arguments of order p defined by

$$(1.2) \qquad F_p = \sum_{d=0}^{k} \sum_{\delta} \frac{a_{\kappa,\delta} C_\delta(-\lambda^{-1} \underset{\sim}{\Lambda}^{-1}) L_\delta^m(\underset{\sim}{\Omega})}{(\frac{1}{2} f_1)_\delta C_\delta(\underset{\sim}{I}) C_\delta(\underset{\sim}{I})},$$

$$(1.3)\quad C(p,m,n) = \frac{\pi^{\frac{1}{2}p} \prod_{i=1}^{p} \Gamma(\frac{1}{2}(2m+2n+p+i+2))}{\prod_{i=1}^{p} \{\Gamma(\frac{1}{2}(2m+i+1))\Gamma(\frac{1}{2}(2n+i+1))\Gamma(\frac{1}{2}i)\}},$$

$m = \frac{1}{2}(f_1-p-1)$, $n = \frac{1}{2}(f_2-p-1)$, $\nu = f_1+f_2$, $\lambda > 0$, $\Lambda = \mathrm{diag}(\lambda_1,\ldots,\lambda_p)$, $\lambda_1,\ldots,\lambda_p$ being the latent roots of $\Sigma_1\Sigma_2^{-1}$ and $R = \mathrm{diag}(r_1,\ldots,r_p)$, $0 < r_1 <\ldots< r_p < \infty$.

The meaning of other symbols in (1.1), (1.2) and (1.3) are obvious from Chapter I.

Let us consider the following two hypotheses:

(A) H_0: $\Sigma_1 = \Sigma_2$ or equivalently H_0: $\lambda_i = 1$, $i = 1,\ldots,p$, against

H_1: $\lambda_i \geq 1$, $\sum_{i=1}^{p} \lambda_i > p$

(B) H_0: $\Omega = 0$ given $\Sigma_1 = \Sigma_2 = \Sigma$ (unknown) against

H_1: $\Omega \neq 0$ given $\Sigma_1 = \Sigma_2 = \Sigma$ (unknown).

Some studies on the tests of these hypotheses have been carried out by Pillai and Jayachandran [37] for $p = 2$ based on the powers of four criteria:

1) Roy's largest root r_p,

2) Wilks' criterion $W^{(p)} = \prod_{i=1}^{p} (1+r_i)^{-1}$,

3) The criterion $U^{(p)} = \sum_{i=1}^{p} r_i$, a constant times Hotelling's T_0^2 and

4) Pillai's criterion $V^{(p)} = \sum_{i=1}^{p} \{r_i/(1+r_i)\}$.

For $p = 2$ and 3, Pillai and Dotson [35] have studied the powers of individual characteristic roots for test of (B) and Pillai and Al-Ani [33] for test of (A).

In this chapter an attempt is made to study the robustness of the above four criteria for test of (A) when the assumption of normality is violated and of (B) when that of common covariance matrix is disturbed. This is achieved by the use of density (1.1).

In the next sections, we first derive the exact non-central distributions of these criteria for p = 2 based on (1.1) and then compute lower tail probabilities of $W^{(p)}$ and upper tail probabilities of the other criteria in view of tests of (A) and (B). A few inferences are drawn on the basis of the tabulations.

2. Non-Central Distribution of $U^{(2)}$

For obtaining the exact non-central c.d.f. of $U^{(2)}$, let us use (1.1) putting $\lambda = 1$ and $p = 2$. Then r_1 and r_2 are jointly distributed as

$$(2.1)\qquad C(2,m,n)e^{-(w_1+w_2)}(\lambda_1\lambda_2)^{-\frac{1}{2}f_1}\sum_{k=0}^{\infty}\sum_{\kappa}\{(\tfrac{1}{2}\nu)_\kappa\}/k!$$
$$\cdot(r_1r_2)^m[(1+r_1)(1+r_2)]^{-q}(r_2-r_1)C_\kappa\begin{pmatrix} r_1/(1+r_1) & 0 \\ 0 & r_2/(1+r_2)\end{pmatrix}\cdot F_2,$$

where $q = m+n+3$ and F_2 is obtained from (1.2) with $p = 2$. However, we also know that for any positive definite matrix $\underset{\sim}{A}_2$ of order 2, the following relation holds:

$$(2.2)\qquad C_\kappa(\underset{\sim}{A}_2) = \sum_{r+2s=k} b_\kappa(r,s)a_1^r a_2^s\ ,$$

where a_1 and a_2 are respectively the first and the second elementary symmetric functions in the latent roots of $\underset{\sim}{A}_2$ and $b_\kappa(r,s)$ is easily obtained from formula (2.4) of Chapter I and from the table of James [17].

The values of the coefficients $b_\kappa(r,s)$ up to k = 6 are tabulated in the Appendix A.

Now let us take $x = U^{(2)} = r_1 + r_2$ and $y = r_1 r_2$. We have the Jacobian $(r_2-r_1)^{-1}$ and the range of y is from 0 to $x^2/4$. Upon substitution, (2.1) gives the term in the joint density of x and y corresponding to partition κ of k

$$(2.3) \qquad \sum_{r+2s=k} b_\kappa(r,s) \sum_{i=0}^{r} \binom{r}{i} 2^{r-i} \frac{x^i y^{m+s+r-i}}{(1+x+y)^{q+r+s}} .$$

If in (2.3) we integrate y out, then we have the integral of the form

$$h_{i,j,\ell}(x) = \int_0^{x^2/4} \frac{x^i y^{m+j}}{(1+x+y)^{q+\ell}} \, dy ,$$

where $j = s+r-i$ and $\ell = r+s$.

However, it is easy to check that the following relation holds:

$$h_{i,j,\ell} = h_{i-1,j,\ell-1} - h_{i-1,j,\ell} - h_{i-1,j+1,\ell}$$

so that the density function of $U^{(2)}$ is expressible in terms of the integral of the form

$$h_{a,b}(x) = \int_0^{x^2/4} \frac{y^{m+a}}{(1+x+y)^{q+b}} \, dy, \quad q = m+n+3.$$

The c.d.f. of $U^{(2)}$ then is expressible in terms of incomplete beta functions of the form

$$(2.4) \quad B_{ab}(u) = (n+b-a+1)^{-1}\{2B_w(2(m+a+1),\ 2(n+b-a+1)+1)$$

$$- (1+u)^{a-n-b-1} B_z(m+a+1, n+b-a+2)\},$$

where $w = u/(2+u)$ and $z = w^2$.

Note that this type of integration also appears in the work of Pillai and Jayachandran [37] but there is an omission of a factor 2 in the last beta function there and the notation would be clearer there if j is replaced by j + 3.

The term in the c.d.f. of $U^{(2)}$ corresponding to partition κ of k may be written as

$$\sum_{r+2s=k} b_\kappa(r,s) H_{rs}(U^{(2)}),$$

where

$$H_{rs}(U^{(2)}) = \sum_{i=0}^{r} \binom{r}{i} 2^{r-i} \int_0^{U^{(2)}} \int_0^{x^2/4} \frac{x^i y^{m+j}}{(1+x+y)^{q+\ell}} \, dy\, dx, \tag{2.5}$$

with $q = m+n+3$, $j = r+s-i$, $\ell = r+s$ and the double integrals in (2.5) is expressible in terms of incomplete beta functions of the form (2.4). Using the above we state the following theorem:

<u>Theorem 2.1</u>: Let $\underset{\sim}{S}_1(p\times p)$ and $\underset{\sim}{S}_2(p\times p)$ be independently distributed where $\underset{\sim}{S}_1$ is distributed $W(p,f_1,\underset{\sim}{\Sigma}_1,\underset{\sim}{\Omega})$ and $\underset{\sim}{S}_2$ having $W(p,f_2,\underset{\sim}{\Sigma}_2,\underset{\sim}{0})$. If $U^{(p)} = tr\, \underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$, then, under the assumption $\Sigma_1^{\frac{1}{2}} \Sigma_2^{-1} \Sigma_1^{\frac{1}{2}}$ "random", the exact non-central distribution of $U^{(2)}$ is given by

$$C(2,m,n)\, e^{-(w_1+w_2)} (\lambda_1\lambda_2)^{-\frac{1}{2} f_1} \tag{2.6}$$

$$\cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}\nu)_\kappa}{k!} \sum_{r+2s=k} b_\kappa(r,s) H_{rs}(U^{(2)}) \cdot F_2,$$

where λ_1, λ_2 are the latent roots of $\underset{\sim}{\Sigma}_1 \underset{\sim}{\Sigma}_2^{-1}$, w_1, w_2 are those of $\underset{\sim}{\Omega}$, $b_\kappa(r,s)$ are constants discussed above, $\nu = f_1+f_2$, F_2 is as in (1.2) and $H_{rs}(U^{(2)})$ as in (2.5).

It can be seen from (2.6) that the distribution of $U^{(2)}$ depends on the availability of expansion of the zonal polynomials. The expansion up to order six is given below, derived from (2.6) using tables of zonal polynomials found in James [17], the tables of constants $a_{\kappa,\delta}$ prepared by Constantine [4] and Pillai and Jouris [39]. Note here, that in obtaining this expansion, first we have to replace the Laguerre polynomial $L_\delta^\gamma(\underset{\sim}{\Omega})$ by the zonal polynomial using formula (2.30) of Chapter I and then carry out the computations. Stating the result, the c.d.f. of $U^{(2)}$ may be written as

$$(2.7) \qquad F(U^{(2)}) = C\cdot \sum_{j=0}^{6} \sum_{i=0}^{j} (-1)^{i+j}\, D_{ij} H_{ij}(U^{(2)}),$$

where $H_{ij}(U^{(2)})$ is as described in (2.4),

$$(2.8) \qquad C = C(2,m,n)e^{-(w_1+w_2)}(\lambda_1\lambda_2)^{-\frac{1}{2}f_1}$$

and the coefficients D_{ij}'s are given in the appendix B.

3. Non-Central Distribution of $V^{(2)}$

Let $\underset{\sim}{L} = \underset{\sim}{\Lambda}\underset{\sim}{R}(\underset{\sim}{I}+\underset{\sim}{\Lambda}\underset{\sim}{R})^{-1}$ in (1.1). Then the joint density of $\ell_1,\ell_2,\ldots,\ell_p$, $0 < \ell_1 <\ldots< \ell_p < 1$ is given by

$$(3.1) \qquad C(p,m,n)e^{-\mathrm{tr}\underset{\sim}{\Omega}}\,|\lambda\underset{\sim}{\Lambda}|^{-\frac{1}{2}f_1}|\underset{\sim}{L}|^m|\underset{\sim}{I}-\underset{\sim}{L}|^n \prod_{i>j}(\ell_i-\ell_j)$$

$$\cdot \sum_{k=0}^{\infty}\sum_{\kappa} \frac{(\frac{1}{2}\nu)_\kappa}{k!} C_\kappa(\underset{\sim}{L}).F_p,$$

where $\underset{\sim}{L} = \mathrm{diag}(\ell_1,\ldots,\ell_p)$ and the meaning of other symbols are as before. The Pillai's criterion $V^{(p)}$ is now defined by $V^{(p)} = \sum_{i=1}^{p} \ell_i$.

For the case $p = 2$, $V^{(2)} = \ell_1+\ell_2$ and ℓ_1 and ℓ_2 are jointly distributed as (taking $\lambda = 1$)

$$(3.2)\qquad C(2,m,n)\, e^{-(w_1+w_2)} (\lambda_1\lambda_2)^{-\frac{1}{2} f_1}$$

$$\cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}\nu)_\kappa}{k!} (\ell_1\ell_2)^m(1-\ell_1)^n(1-\ell_2)^n(\ell_2-\ell_1)C_\kappa\begin{pmatrix}\ell_1 & 0\\ 0 & \ell_2\end{pmatrix}.F_2.$$

Now, let $x = V^{(2)} = \ell_1+\ell_2$ and $y = \ell_1\ell_2$. Then from (3.2) we have the term in the joint density of x and y corresponding to partition κ of k

$$(3.3)\qquad \sum_{r+2s=k} b_\kappa(r,s)\ x^r y^{m+s}(1-x+y)^n.$$

Integrating y out first and then x, (3.3) gives the term corresponding to partition κ of k in the c.d.f. of $V^{(2)}$ as

$$\sum_{r+2s=k} b_\kappa(r,s)\ F_{rs}(V^{(2)}),$$

where

$$(3.4)\qquad F_{rs}(V^{(2)}) = \int_0^{V^{(2)}} \int_0^{x^2/4} x^r y^{m+s} (1-x+y)^n dy\, dx.$$

The integral in (3.2) is expressible in terms of incomplete beta functions if we carry out the computation by integration by parts. For $0 < V^{(2)} < 1$,

$$(3.5)\qquad F_{rs}(V^{(2)}) = \frac{2^{r+1}}{m+s+1} \sum_{i=0}^{n} (-1)^i R_i B_V(a+4i+2,b-4i-2),$$

and for $1 \le V^{(2)} \le 2$,

$$(3.6) \qquad F_{rs}(V^{(2)}) = \frac{2^{r+1}}{m+s+1} \sum_{i=0}^{n} (-1)^i R_i B_{0.5}(a+4i+2, b-4i-2)$$

$$+ \frac{2^{r+1}}{n+1} \sum_{i=0}^{m+s} (-1)^i P_i [B_v(a,b) - B_{0.5}(a,b)],$$

where $v = \frac{1}{2} V^{(2)}$, $a = 2m+2s-2i+r+1$, $b = 2n+2i+3$,

$$R_i = \prod_{j=1}^{i} \{(n+1-j)/(m+s+1+j)\}, \quad R_0 = 1$$

and $$P_i = \prod_{j=1}^{i} \{(m+s+1-j)/(n+1+j)\}, \quad P_0 = 1.$$

In view of the above, we state the following:

Theorem 3.1: Let $\underset{\sim}{S}_1$, $\underset{\sim}{S}_2$ and $\underset{\sim}{\Sigma}_1^{\frac{1}{2}} \underset{\sim}{\Sigma}_2^{-1} \underset{\sim}{\Sigma}_1^{\frac{1}{2}}$ be as in Theorem 2.1 and $V^{(2)}$ be as defined above. Then the exact non-central distribution of $V^{(2)}$ is expressible in terms of incomplete beta functions and is given by

$$(3.7) \qquad C(2,m,n)\, e^{-(w_1+w_2)} (\lambda_1 \lambda_2)^{-\frac{1}{2} f_1}$$

$$\cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}\nu)_\kappa}{k!} \sum_{r+2s=k} b_\kappa(r,s)\, F_{rs}(V^{(2)}, F_2,$$

where the meaning of all symbols are as before and $F_{rs}(V^{(2)})$ is as in (3.5) and (3.6).

Upon expanding the series in (3.7) up to the sixth order and combining the like terms, we obtain the c.d.f. of $V^{(2)}$ in the form

$$(3.8) \qquad F(V^{(2)}) = C. \sum_{i+2j=k=0}^{6} E_{ij} F_{ij}(V^{(2)}),$$

where in the summation, only integral solutions (i,j) of $i + 2j = k$ are taken, C is as in (2.8), $F_{ij}(V^{(2)})$ as described in (3.5) and (3.6) and the coefficients E_{ij}'s are given below:

$E_{00} = 1$ | $E_{01} = A_{22} - 4A_{21}$

$E_{10} = A_{11}$ | $E_{11} = A_{32} - 12A_{31}$

$E_{20} = 3A_{21}$ | $E_{21} = 3A_{42} - 120A_{41}$

$E_{30} = 5A_{31}$ | $E_{31} = 5A_{52} - 280A_{51}$

$E_{40} = 35A_{41}$ | $E_{41} = 35A_{62} - 1260A_{61}$

$E_{50} = 63A_{51}$ | $E_{02} = A_{43} - 4A_{42} + 48A_{41}$

$E_{60} = 231A_{61}$ | $E_{12} = A_{53} - 12A_{52} + 240A_{51}$

$E_{22} = 3A_{63} - 120A_{62} + 1680A_{61}$ and $E_{03} = A_{64} - 4A_{63} + 48A_{62} - 320A_{61}$,

where the A_{ij}'s are given in the appendix B.

4. Non-Central Distribution of $W^{(2)}$

Using ℓ_1 and ℓ_2 defined in Section 3, the Wilks' criterion $W^{(2)}$ is given by $W^{(2)} = (1-\ell_1)\,(1-\ell_2)$. As in the previous sections, to obtain the c.d.f. of $W^{(2)}$, we shall use the joint density of $x = W^{(2)} = (1-\ell_1)\,(1-\ell_2)$ and $y = \ell_1\ell_2$. The Jacobian of this transformation is $(\ell_2-\ell_1)^{-1}$ and the range of y is from 0 to $(1-\sqrt{x})^2$. After making substitutions in (3.2), we will have the term corresponds to partition κ of k in the form

$$(4.1) \qquad \sum_{r+2s=k} b_\kappa(r,s)\; x^n y^{m+s}\; (1-x+y)^r$$

Integrating y out first and then x, (4.1) reduces to

$$\sum_{r+2s=k} b_\kappa(r,s)\; G_{rs}(W^{(2)}),$$

where

$$(4.2) \qquad G_{rs}(W^{(2)}) = \int_0^{W^{(2)}} \int_0^{(1-\sqrt{x})^2} x^n y^{m+s} (1-x+y)^r \, dy\, dx,$$

which is expressible in terms of incomplete beta function as

$$(4.3) \qquad G_{rs}(W^{(2)}) = \sum_{i=0}^{r} \frac{(-1)^i 2^{r-i+1}}{m+s+1} Q_i B_z(2n+2, 2m+2s+r+i+3),$$

where $Q_i = \prod_{j=1}^{i} \{(r+1-j)/(m+s+1+j)\}$, $Q_0 = 1$ and $z = (W^{(2)})^{1/2}$.

Note that (4.3) is obtained easily from (4.2) by integration by parts. Now we have the following:

Theorem 4.1: Let $\underset{\sim}{S}_1$, $\underset{\sim}{S}_2$ and $\underset{\sim}{\Sigma}_1^{\frac{1}{2}}\underset{\sim}{\Sigma}_2^{-1}\underset{\sim}{\Sigma}_1^{\frac{1}{2}}$ be as in Theorem 2.1, and $W^{(2)}$ be as defined above. Then the exact non-central c.d.f. of $W^{(2)}$ is expressible in terms of incomplete beta function and is given by

$$(4.4) \qquad C(2,m,n)\ e^{-(w_1+w_2)}\ (\lambda_1\lambda_2)^{-\frac{1}{2} f_1}$$

$$\cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}\nu)_\kappa}{k!} \sum_{r+2s=k} b_\kappa(r,s) G_{rs}(W^{(2)}).F_2,$$

where the meaning of all symbols are as before and $G_{rs}(W^{(2)})$ is as in (4.3).

The c.d.f. of $W^{(2)}$ using up to $k = 6$ is given by

$$(4.5) \qquad F(W^{(2)}) = C\cdot \sum_{i+2j=k=0}^{6} E_{ij} G_{ij}(W^{(2)}),$$

where C and E_{ij}'s are as in Section 3 and $G_{ij}(W^{(2)})$ is as defined in (4.2).

Remarks: 1. The c.d.f. of $W^{(2)}$ has also been obtained in Chapter I, Section 6. Comparing with the one obtained there, the c.d.f. in (4.4) is much simpler and more convenient for numerical calculations.

2. If $w_1 = w_2 = 0$, the distributions (2.7), (3.8) and (4.5) reduce to the results of Pillai and Jayachandran [38].

5. Non-Central Distribution of the Largest or Smallest Root

To obtain the distribution of the largest root ℓ_2 we will start from the joint density of ℓ_1 and ℓ_2, $0 < \ell_1 < \ell_2 < 1$, which is described in (3.2). Considering only the expression containing ℓ_1 and ℓ_2, then from (3.2) we have the term corresponding to partition κ of k

$$(5.1) \qquad \sum_{r+2s=k} b_\kappa(r,s) \sum_{t=0}^{r} \binom{r}{t} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \ell_2^h (1-\ell_2)^n \{\ell_2 \ell_1^q - \ell_1^{q+1}\},$$

where $h=m+r+s-t$ and $q=m+i+s+t$. Integrating ℓ_1 out from $0 < \ell_1 < \ell_2$, we easily obtain the density of the largest root ℓ_2 in the form

$$(5.2) \qquad C(2,m,n) e^{-(w_1+w_2)} (\lambda_1\lambda_2)^{-\frac{1}{2} f_1}$$

$$\cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}\nu)_\kappa}{k!} \sum_{r+2s=k} b_\kappa(r,s) p_{rs}(\ell_2) . F_2,$$

for $0 < \ell_2 < 1$ and zero otherwise, where

$$(5.3) \qquad p_{rs}(\ell_2) = \sum_{t=0}^{r} \binom{r}{t} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \{a(a+1)\}^{-1} \ell_2^b (1-\ell_2)^n,$$

where $a=m+i+1+s+t$ and $b=2(m+s+1)+r+i$.

To get the c.d.f. of the largest root, we integrate (5.2) with respect to ℓ_2. From (5.3) we have after integration

$$(5.4) \qquad P_{rs}(\ell_2) = \sum_{t=0}^{r} \binom{r}{t} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \{a(a+1)\}^{-1} B_{\ell_2}(b+1,n+1),$$

where $B_x(p,q)$ is the incomplete beta function. From the above we have

Theorem 5.1: Let $\underset{\sim}{S}_1, \underset{\sim}{S}_2$ and $\underset{\sim}{\Sigma}_1^{\frac{1}{2}}\underset{\sim}{\Sigma}_2^{-1}\underset{\sim}{\Sigma}_1^{\frac{1}{2}}$ be as stated in Theorem 2.1 and ℓ_p be the largest root of $\underset{\sim}{S}_1(\underset{\sim}{S}_1+\underset{\sim}{S}_2)^{-1}$. Then the non-central distribution of ℓ_2 is expressible in terms of incomplete beta function and has the form

$$(5.5) \qquad C(2,m,n)e^{-(w_1+w_2)}(\lambda_1\lambda_2)^{-\frac{1}{2}f_1}$$

$$\cdot \sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\frac{1}{2}\nu)_\kappa}{k!}\sum_{r+2s=k} b_\kappa(r,s).P_{rs}(\ell_2).F_2,$$

where $P_{rs}(\ell_2)$ is as stated in (5.4) and the other symbols are interpreted as before.

Again, expanding the series up to the sixth order we have the c.d.f. of ℓ_2 as follows:

$$(5.6) \qquad F(\ell_2) = C.\sum_{i+2j=k=0}^{6} E_{ij}P_{ij}(\ell_2),$$

where C and E_{ij}'s are as in Section 3 and $P_{ij}(\ell_2)$ is as described in

the method employed by Pillai [27], it is possible to express the density of ℓ_2 in terms of hypergeometric series. However, the expression (5.2) above seems preferable especially for computational purposes.

Using similar approach as above we can obtain the following:

Theorem 5.2: Under the assumptions of theorem 5.1, the non-central c.d.f. of smallest root ℓ_1 is given by

$$(5.7) \qquad C(2,m,n)e^{-(w_1+w_2)}(\lambda_1\lambda_2)^{-\frac{1}{2}f_1}$$

$$\cdot \sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\frac{1}{2}\nu)_\kappa}{k!}\sum_{r+2s=k} b_\kappa(r,s)Q_{rs}(\ell_1).F_2,$$

where the meaning of the symbols are as before, and

$$(5.8) \qquad Q_{rs}(\ell_1) = \sum_{t=0}^{r}\binom{r}{t}\sum_{i=0}^{q}(-1)^i\binom{q}{i}(n+i+1)^{-1}(n+i+2)^{-1}B_{\ell_1}(a,b),$$

where $a=m+r+s+1$, $b=2n+i+3$ and $q=m+r+s-t$.

Its expansion up to the sixth order for the c.d.f. of ℓ_1 is

$$(5.9) \qquad F(\ell_1) = C\cdot\sum_{i+2j=k=0}^{6} E_{ij}Q_{ij}(\ell_1).$$

It may be pointed out here that (5.7) can not be obtained from (5.5) by transformations. However, if $w_1 = w_2 = 0$ and $\lambda_1 = \lambda_2 = 1$, i.e. in the central case, we can obtain (5.7) from (5.5) by performing transformation $\ell_2 \to (1-\ell_1)$ and $m \leftrightarrow n$ as was done in the work of Pillai [27].

6. Non-Central Distribution of $U^{(2)}$, $V^{(2)}$, $W^{(2)}$ and Individual Latent Roots for Test of Independence

Methods similar to those used in the previous sections can be employed to obtain the non-central distribution of $U^{(2)}$, $V^{(2)}$, $W^{(2)}$, ℓ_2 and ℓ_1 for testing the hypothesis:

(C) independence between a p-set and a q-set of variates in a (p+q)-variate normal population. Let the columns of $\binom{X}{Y}$ be ν

independent normal (p+q)-variates, $p \leq q$, $p + q \leq \nu$, with zero means and covariance matrix

$$\underset{\sim}{\Sigma} = \begin{pmatrix} \underset{\sim}{\Sigma}_{11}(p\times p) & \underset{\sim}{\Sigma}_{12}(p\times q) \\ \underset{\sim}{\Sigma}'_{12}(q\times p) & \underset{\sim}{\Sigma}_{22}(q\times q) \end{pmatrix} .$$

If $\underset{\sim}{R} = \mathrm{diag}(r_1,\ldots,r_p)$, $r_1^2, r_2^2, \ldots, r_p^2$, are the latent roots of $\underset{\sim}{X}\underset{\sim}{Y}'(\underset{\sim}{Y}\underset{\sim}{Y}')^{-1}\underset{\sim}{Y}\underset{\sim}{X}'(\underset{\sim}{X}\underset{\sim}{X}')^{-1}$ and $\underset{\sim}{P} = \mathrm{diag}(\rho_1,\ldots,\rho_p)$, $\rho_1^2, \rho_2^2, \ldots, \rho_p^2$, are the latent roots of $\underset{\sim}{\Sigma}_{12}\underset{\sim}{\Sigma}_{22}^{-1}\,\underset{\sim}{\Sigma}'_{12}\underset{\sim}{\Sigma}_{11}^{-1}$, then the joint density function of $r_1^2, r_2^2, \ldots, r_p^2$ is given by Constantine [3], James [17] in the form

$$(6.1) \qquad C(p,m,n)\,|\underset{\sim}{I}-\underset{\sim}{P}^2|^{\frac{1}{2}\nu}\,|\underset{\sim}{R}^2|^m|\underset{\sim}{I}-\underset{\sim}{R}^2|^n \prod_{i>j}(r_i^2-r_j^2)$$

$$\cdot {}_2F_1(\tfrac{1}{2}\nu, \tfrac{1}{2}\nu; \tfrac{1}{2}f_1; \underset{\sim}{P}^2, \underset{\sim}{R}^2),\ 0 < r_1^2 \leq \ldots \leq r_p^2 < 1,$$

where $f_1 = q$, $m = \frac{1}{2}(q-p-1)$, $n = \frac{1}{2}(\nu-q-p-1)$ and $C(p,m,n)$ is as stated in (1.3). Starting from (6.1) we can obtain the c.d.f.'s of $U^{(2)}$, $V^{(2)}$, $W^{(2)}$, r_2^2 and r_1^2 ($U^{(2)}$, $V^{(2)}$, $W^{(2)}$ are as defined in Section 1 but the r_i's must be replaced by r_i^2's). However, since the expressions of the distributions up to the order six have been obtained by Pillai and Jayachandran [37] and Pillai and Al-Ani [33], only general form of the distributions are stated here for the completion of the discussion. The results, for $U^{(2)}$, $V^{(2)}$, $W^{(2)}$, r_2^2 and r_1^2 are respectively:

$$(6.2) \qquad F(U^{(2)}) = Q\cdot \sum_{r+2s=k} b_\kappa(r,s)H_{rs}(U^{(2)}),$$

$$(6.3) \qquad F(V^{(2)}) = Q\cdot \sum_{r+2s=k} b_\kappa(r,s)F_{rs}(V^{(2)}),$$

$$(6.4)\qquad F(W^{(2)}) = Q\cdot \sum_{r+2s=k} b_\kappa(r,s)\ G_{rs}(W^{(2)}),$$

$$(6.5)\qquad F(r_2^2) = Q\cdot \sum_{r+2s=k} b_\kappa(r,s)\ P_{rs}(r_2^2),$$

and finally

$$(6.6)\qquad F(r_1^2) = Q\cdot \sum_{r+2s=k} b_\kappa(r,s)\ Q_{rs}(r_1^2),$$

where

$$(6.7)\qquad Q = C\cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\{(\frac{1}{2}\nu)_\kappa\}^2\ C_\kappa(\underset{\sim}{P}^2)}{k!(\frac{1}{2}f_1)_\kappa C_\kappa(\underset{\sim}{I})},$$

$\underset{\sim}{P}^2 = \mathrm{diag}(\rho_1^2, \rho_2^2)$ and the meaning of the other symbols are as in the previous sections.

7. Numerical Study of Robustness Based on Four Criteria

Let us now use the distributions obtained in the previous sections to study the robustness of the tests concerning the two hypotheses stated earlier, namely:

(A) equality of covariance matrices in two p-variate normal populations, and

(B) equality of p-dimensional mean vector in ℓ p-variate normal populations having a common covariance matrix.

For this purpose, some numerical values of upper tail probabilities of $U^{(2)}$, $V^{(2)}$ and the largest root r_2 and lower tail probabilities of $W^{(2)}$ have been calculated. In these calculations, the upper/lower 5 per cent points of the respective criteria under the null hypotheses of (A) and (B) have been used and were taken from the tables prepared by Pillai and Jayachandran [37] and Pillai and Al-Ani [33]. All computations were carried out on the CDC 6500 Computer at the Purdue University Computing Center.

For various values of w_1, w_2 (i.e. the latent roots of $\underset{\sim}{\Omega}$ for $p = 2$) and λ_1, λ_2 (i.e. the latent roots of $\underset{\sim}{\Sigma}_1\underset{\sim}{\Sigma}_2^{-1}$ for $p = 2$) these upper/lower tail probabilities are tabulated using different values of $m = (n_1-p-1)/2$ and $n = (n_2-p-1)/2$. Table II (1) presents those probabilities for $m = 0$ and Table II (2) for $m = 2$. In both cases the values of $n = 5$, 15 and 40. In both tables the powers of the test of (A) assuming $\underset{\sim}{\Omega} = \underset{\sim}{0}$ and those of the test of (B) assuming $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$ are also presented.

From the tabulations it appears that

1) For hypothesis (A), the powers of tests based on all four criteria show considerable change even for small deviations of (w_1,w_2) from (0,0). The changes of powers become larger for bigger deviations of (w_1,w_2). This probably is indicative that the tests are not robust against non-normality.
2) For hypothesis (B), the powers of tests based on all four criteria show modest changes for small deviations of (λ_1,λ_2) from (1,1) but changes become pronounced as (λ_1,λ_2) deviate more from (1,1).
3) Tabulations do not reveal any advantage of one test statistic over the others in regard to either hypothesis from the point of view of robustness. It is likely that tabulations for larger deviations may bring more light on this problem.

It may be pointed out that Ito [15] and Ito and Schull [16] have also studied similar cases but their results are based on the large sample theory. The results obtained here are of an exact nature except for some of the assumptions made in the model.

Table II (1)

Upper/lower tail probabilities of four criteria

$\alpha = 0.05,\ m = 0$

w_1	w_2	λ_1	λ_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
				m = 0, n = 5			
0	0.001	1	1.001	0.050106	0.050112	0.050109	0.050100
				0.050079	0.050084	0.050082	0.050073
				0.050026	0.050028	0.050027	0.050026
		1	1.01	0.050823	0.050869	0.050852	0.050769
				0.050796	0.050840	0.050824	0.050744
		1.001	1.009	0.050823	0.050869	0.058516	0.050768
				0.050796	0.050841	0.050824	0.050743
		1.005	1.015	0.051625	0.051717	0.051682	0.051515
				0.051598	0.051688	0.051654	0.051490
		1.025	1.025	0.054066	0.054301	0.054213	0.053783
				0.054038	0.054272	0.054184	0.053757
		1	1.05	0.054085	0.054280	0.054217	0.053821
				0.054057	0.054250	0.054188	0.053795
		1	1.1	0.058328	0.058646	0.058571	0.057811
				0.058298	0.058614	0.058538	0.05778
		1.05	1.25	0.0767	0.0776	0.0774	0.0749
				0.0766	0.0775	0.0774	0.0749
		1	1.3	0.0770	0.0771	0.0774	0.0756
				0.0769	0.0771	0.0774	0.0755

Entries in 2nd row denote powers of the test H_0: $\underset{\sim}{\Sigma}_1=\underset{\sim}{\Sigma}_2$ assuming $\underset{\sim}{\Omega}=\underset{\sim}{0}$

Entries in 3rd row denote powers of the test H_0: $\underset{\sim}{\Omega}=\underset{\sim}{0}$ assuming $\underset{\sim}{\Sigma}_1=\underset{\sim}{\Sigma}_2$.

Table II (1), cont.

w_1	w_2	λ_1	λ_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
				m = 0, n = 15			
0	0.001	1	1.001	0.050133	0.050134	0.050134	0.050121
				0.050099	0.050100	0.050100	0.050093
				0.050033	0.050033	0.050033	0.050027
		1	1.01	0.051035	0.051043	0.051041	0.050967
				0.051001	0.051009	0.051007	0.050935
		1.001	1.009	0.051034	0.051043	0.051041	0.050966
				0.051001	0.051009	0.051007	0.050934
		1.005	1.015	0.052045	0.052063	0.052058	0.051912
				0.052011	0.052028	0.052023	0.051880
		1.025	1.025	0.055132	0.055179	0.055165	0.054795
				0.055096	0.055143	0.055130	0.054762
		1	1.05	0.055162	0.055190	0.055187	0.054858
				0.055127	0.055154	0.055151	0.054825
		1	1.1	0.060585	0.060604	0.060619	0.060013
				0.060546	0.060565	0.060580	0.05998
		1.05	1.25	0.0844	0.0844	0.0845	0.0824
				0.0843	0.0844	0.0844	0.0824
		1	1.3	0.0849	0.0845	0.0847	0.0835
				0.0848	0.0845	0.0847	0.0835
				m = 0, n = 40			
0	0.001	1	1.001	0.050145	0.050145	0.050145	0.050132
				0.050108	0.050109	0.050109	0.050102
				0.050036	0.050036	0.050036	0.050029
		1	1.01	0.051131	0.051132	0.051132	0.051063
				0.051094	0.051095	0.051095	0.051028
		1.001	1.009	0.051130	0.051131	0.051131	0.051062
				0.051093	0.051095	0.051094	0.051027
		1.005	1.015	0.052235	0.052238	0.052237	0.052104
				0.052198	0.052201	0.052200	0.052069
		1.025	1.025	0.055617	0.055625	0.055623	0.055285
				0.055577	0.055586	0.055584	0.055248
		1	1.05	0.055652	0.055652	0.055654	0.055361
				0.055613	0.055613	0.055615	0.055324
		1	1.1	0.061617	0.061601	0.061613	0.061087
				0.061574	0.061558	0.061571	0.06105
		1.05	1.25	0.0880	0.0879	0.0879	0.0861
				0.0879	0.0878	0.0878	0.0861
		1	1.3	0.0885	0.0883	0.0883	0.0874
				0.0885	0.0882	0.0883	0.0874

Table II (1), cont.

w_1	w_2	λ_1	λ_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
				m = 0, n = 5			
0	0.01	1	1.001	0.050344	0.050364	0.050356	0.050322
				0.050079	0.050084	0.050082	0.050073
				0.050264	0.050280	0.050274	0.050248
		1	1.01	0.051064	0.051123	0.051101	0.050993
		1.001	1.009	0.051064	0.051124	0.051101	0.050993
		1.005	1.015	0.051868	0.051975	0.051935	0.051742
		1.025	1.025	0.054319	0.054569	0.054475	0.054018
		1	1.05	0.054338	0.054547	0.054479	0.054056
		1	1.1	0.058596	0.058930	0.058849	0.058060
		1.05	1.25	0.0770	0.0774	0.0778	0.0752
		1	1.3	0.0773	0.0775	0.0778	0.0759
				m = 0, n = 15			
0	0.01	1	1.001	0.050432	0.050436	0.050435	0.050401
				0.050099	0.050100	0.050100	0.050093
				0.050332	0.050335	0.050334	0.050307
		1	1.01	0.051338	0.051349	0.051346	0.051251
		1.001	1.009	0.051338	0.051349	0.051346	0.051250
		1.005	1.015	0.052353	0.052373	0.052367	0.052200
		1.025	1.025	0.055453	0.055503	0.055489	0.055095
		1	1.05	0.055483	0.055514	0.055510	0.055158
		1	1.1	0.060928	0.060951	0.060965	0.060333
		1.05	1.25	0.0848	0.0848	0.0849	0.0828
		1	1.3	0.0853	0.0850	0.0852	0.0839
				m = 0, n = 40			
0	0.01	1	1.001	0.050472	0.050473	0.050472	0.050440
				0.050108	0.050109	0.050109	0.050102
				0.050363	0.050363	0.050363	0.050337
		1	1.01	0.051462	0.051464	0.051464	0.051376
		1.001	1.009	0.051462	0.051463	0.051463	0.051374
		1.005	1.015	0.052572	0.052575	0.052574	0.052421
		1.025	1.025	0.055969	0.055978	0.055975	0.055616
		1	1.05	0.056004	0.056005	0.056007	0.055693
		1	1.1	0.061995	0.061980	0.061993	0.061442
		1.05	1.25	0.0884	0.0884	0.0884	0.0866
		1	1.3	0.0890	0.0888	0.0888	0.0879

Table II (1), cont.

w_1	w_2	λ_1	λ_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
				m = 0, n = 5			
0.001	0.009	1	1.001	0.05036	0.05038	0.05033	0.05035
				0.050079	0.05008	0.05008	0.05007
				0.05028	0.05029	0.05025	0.05027
		1	1.01	0.05108	0.05114	0.05108	0.05102
		1.001	1.009	0.05108	0.05114	0.05108	0.05102
		1.005	1.015	0.05189	0.05199	0.05191	0.05177
		1.025	1.025	0.0543	0.0546	0.0545	0.0540
		1	1.05	0.0543	0.0546	0.0545	0.0541
		1	1.1	0.0586	0.0589	0.0588	0.0581
		1.05	1.25	0.077	0.078	0.078	0.075
		1	1.3	0.077	0.077	0.078	0.076
				m = 0, n = 15			
0.001	0.009	1	1.001	0.05045	0.05045	0.05040	0.05042
				0.05010	0.05010	0.05010	0.05009
				0.05035	0.05035	0.05030	0.05033
		1	1.01	0.05135	0.05136	0.05132	0.05127
		1.001	1.009	0.05135	0.05136	0.05132	0.05127
		1.005	1.015	0.05236	0.05238	0.05234	0.05222
		1.025	1.025	0.0555	0.0555	0.0555	0.0551
		1	1.05	0.0555	0.0555	0.0555	0.0552
		1	1.1	0.0609	0.0610	0.0609	0.0603
		1.05	1.25	0.085	0.085	0.085	0.083
		1	1.3	0.085	0.085	0.085	0.084
				m = 0, n = 40			
0.001	0.009	1	1.001	0.05048	0.05048	0.05046	0.05044
				0.05011	0.05011	0.05011	0.05010
				0.05037	0.05037	0.05033	0.05035
		1	1.01	0.05147	0.05147	0.05143	0.05139
		1.001	1.009	0.05147	0.05147	0.05143	0.05139
		1.005	1.015	0.05258	0.05258	0.05254	0.05244
		1.025	1.025	0.0560	0.0560	0.0560	0.0556
		1	1.05	0.0560	0.0560	0.0560	0.0557
		1	1.1	0.0620	0.0620	0.0620	0.0615
		1.05	1.25	0.088	0.088	0.088	0.087
		1	1.3	0.089	0.089	0.089	0.088

Table II (1), cont.

w_1	w_2	λ_1	λ_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
				m = 0, n = 5			
0	0.1	1	1.001	0.052750	0.052897	0.052844	0.052567
				0.050079	0.050084	0.050082	0.050073
				0.052667	0.052809	0.052758	0.052490
		1	1.01	0.053497	0.053686	0.053618	0.053262
		1.001	1.009	0.053497	0.053686	0.053617	0.053262
		1.005	1.005	0.054332	0.054570	0.054483	0.054038
		1.025	1.025	0.056874	0.057262	0.057120	0.056394
		1	1.05	0.056893	0.057239	0.057124	0.056432
		1	1.1	0.06131	0.06178	0.06166	0.06057
		1.05	1.25	0.081	0.081	0.081	0.080
		1	1.3	0.081	0.081	0.081	0.080
				m = 0, n = 15			
0	0.1	1	1.001	0.053462	0.053488	0.053482	0.053246
				0.050099	0.050100	0.050100	0.050093
				0.053357	0.053383	0.053377	0.053149
		1	1.01	0.054408	0.054442	0.054434	0.054132
		1.001	1.009	0.054408	0.054442	0.054433	0.054129
		1.005	1.015	0.055467	0.055512	0.055500	0.055121
		1.025	1.025	0.058703	0.058778	0.058757	0.058135
		1	1.05	0.058733	0.058788	0.058779	0.058199
		1	1.1	0.06440	0.06445	0.06446	0.06358
		1.05	1.25	0.089	0.090	0.090	0.088
		1	1.3	0.090	0.090	0.090	0.088
				m = 0, n = 40			
0	0.1	1	1.001	0.053785	0.053788	0.053788	0.053575
				0.050108	0.050109	0.050109	0.050102
				0.053670	0.053673	0.053673	0.053467
		1	1.01	0.054821	0.054826	0.054826	0.054552
		1.001	1.009	0.054821	0.054826	0.054825	0.054551
		1.005	1.015	0.055983	0.055989	0.055988	0.055645
		1.025	1.025	0.059536	0.059548	0.059546	0.058980
		1	1.05	0.059572	0.059575	0.059577	0.059058
		1	1.1	0.06582	0.06581	0.06582	0.06504
		1.05	1.25	0.093	0.094	0.094	0.092
		1	1.3	0.094	0.094	0.094	0.092

Table II (1), cont.

w_1	w_2	λ_1	λ_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
				m = 0, n = 5			
0	0.5	1	1.001	0.063932	0.064464	0.064338	0.063065
				0.050079	0.050084	0.050082	0.050073
				0.063836	0.064362	0.064237	0.062977
		1	1.01	0.064803	0.065382	0.065241	0.063869
		1.001	1.009	0.064802	0.065383	0.065241	0.063868
		1.005	1.015	0.065775	0.06410	0.066251	0.064764
		1.025	1.025	0.06873	0.06954	0.06932	0.06748
		1	1.05	0.06875	0.06951	0.06932	0.06752
		1	1.1	0.0739	0.0748	0.0746	0.0723
		1.05	1.25	0.096	0.096	0.096	0.093
		1	1.3	0.096	0.096	0.096	0.093
				m = 0, n = 15			
0	0.5	1	1.001	0.067712	0.067744	0.067770	0.066757
				0.050099	0.050100	0.050100	0.050093
				0.067588	0.067619	0.067644	0.066642
		1	1.01	0.068839	0.068880	0.068904	0.067804
		1.001	1.009	0.068839	0.068880	0.068903	0.067803
		1.005	1.015	0.070099	0.070151	0.070172	0.068973
		1.025	1.025	0.07394	0.07402	0.07404	0.07253
		1	1.05	0.07397	0.07403	0.07406	0.07259
		1	1.1	0.0806	0.0807	0.0808	0.0789
		1.05	1.25	0.109	0.110	0.110	0.107
		1	1.3	0.110	0.110	0.110	0.107
				m = 0, n = 40			
0	0.5	1	1.001	0.069442	0.069415	0.069437	0.068559
				0.050108	0.050109	0.050109	0.050102
				0.069304	0.069277	0.069299	0.068430
		1	1.01	0.070688	0.070662	0.070683	0.069726
		1.001	1.009	0.070687	0.070661	0.070683	0.069724
		1.005	1.015	0.07208	0.07206	0.07208	0.07103
		1.025	1.025	0.07633	0.07631	0.07633	0.07500
		1	1.05	0.07637	0.07634	0.07636	0.07508
		1	1.1	0.0838	0.0837	0.0838	0.0821
		1.05	1.25	0.116	0.115	0.115	0.114
		1	1.3	0.116	0.115	0.115	0.114

Table II (1), cont.

w_1	w_2	λ_1	λ_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
				m = 0, n = 5			
0	1	1	1.001	0.07899	0.07958	0.07967	0.07733
				0.050079	0.050084	0.050082	0.050073
				0.078873	0.079463	0.079538	0.077229
		1	1.01	0.08001	0.08066	0.08074	0.07828
		1.001	1.009	0.08001	0.08066	0.08074	0.07828
		1.005	1.02	0.08174	0.08247	0.08253	0.07986
		1.025	1.025	0.0846	0.0855	0.0855	0.0825
		1	1.05	0.0847	0.0855	0.0855	0.0826
		1	1.1	0.091	0.091	0.091	0.088
				m = 0, n = 15			
0	1	1	1.001	0.08723	0.08706	0.08725	0.08554
				0.050099	0.050100	0.050100	0.050093
				0.087084	0.086913	0.087101	0.085403
		1	1.01	0.08859	0.08843	0.08862	0.08679
		1.001	1.009	0.08859	0.08843	0.08861	0.08679
		1.005	1.02	0.09086	0.09072	0.09090	0.08890
		1.025	1.025	0.0947	0.0946	0.0948	0.0924
		1	1.05	0.0947	0.0946	0.0948	0.0925
		1	1.1	0.103	0.103	0.103	0.100
				m = 0, n = 40			
0	1	1	1.001	0.09104	0.09088	0.09098	0.08958
				0.050108	0.050109	0.050109	0.050102
				0.090873	0.090717	0.090823	0.089426
		1	1.01	0.09255	0.09239	0.09249	0.09099
		1.001	1.009	0.09255	0.09239	0.09249	0.09099
		1.005	1.02	0.09508	0.0949	0.09502	0.09335
		1.025	1.025	0.0994	0.0992	0.0993	0.0973
		1	1.05	0.0994	0.0992	0.0993	0.0974
		1	1.1	0.108	0.108	0.108	0.106

Table II (1), cont.

w_1	w_2	λ_1	λ_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
			m = 0, n = 5				
1	1	1	1.001	0.1124	0.1203	0.1164	0.1049
				0.0501	0.0501	0.0501	0.0501
				0.1122	0.1201	0.1164	0.1048
		1	1.01	0.1138	0.1217	0.1178	0.1062
		1.001	1.009	0.1138	0.1217	0.1178	0.1062
		1.005	1.02	0.116	0.124	0.120	0.108
		1.025	1.025	0.120	0.128	0.124	0.112
		1	1.05	0.120	0.128	0.124	0.112
		1	1.1	0.128	0.138	0.132	0.119
			m = 0, n = 15				
1	1	1	1.001	0.1314	0.1341	0.1327	0.1214
				0.0501	0.0501	0.0501	0.0501
				0.1312	0.1339	0.1326	0.1213
		1	1.01	0.1332	0.1359	0.1345	0.1231
		1.001	1.009	0.1332	0.1359	0.1345	0.1231
		1.005	1.02	0.136	0.139	0.138	0.126
		1.025	1.025	0.141	0.144	0.143	0.130
		1	1.05	0.141	0.144	0.143	0.130
		1	1.1	0.152	0.153	0.153	0.140
			m = 0, n = 40				
1	1	1	1.001	0.1402	0.1412	0.1406	0.1295
				0.0501	0.0501	0.0501	0.0501
				0.1400	0.1409	0.1405	0.1293
		1	1.01	0.1422	0.1432	0.1426	0.1314
		1.001	1.009	0.1422	0.1432	0.1426	0.1314
		1.005	1.02	0.146	0.147	0.146	0.134
		1.025	1.025	0.151	0.152	0.152	0.140
		1	1.05	0.151	0.152	0.152	0.140
		1	1.1	0.163	0.163	0.163	0.151

Table II (2)

Upper/lower tail probabilities of four criteria

$\alpha = 0.05$, $m = 2$

w_1	w_2	λ_1	λ_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
				$m = 2$, $n = 5$			
0	0.001	1	1.001	0.050109	0.050122	0.050118	0.050100
				0.050095	0.050107	0.050103	0.050084
				0.050014	0.050015	0.050015	0.050012
		1	1.01	0.050970	0.051089	0.051048	0.050862
				0.050956	0.051074	0.051033	0.050850
		1.001	1.009	0.050970	0.051090	0.051048	0.050861
				0.050956	0.051075	0.051033	0.050849
		1.005	1.015	0.051936	0.052179	0.052094	0.051713
				0.051922	0.052163	0.052079	0.051701
		1.025	1.025	0.054894	0.055526	0.055306	0.054310
				0.054879	0.055510	0.055290	0.054297
		1	1.05	0.054915	0.055454	0.055287	0.054371
				0.054901	0.055437	0.055271	0.054358
		1	1.1	0.06012	0.06106	0.06082	0.05904
				0.060100	0.061036	0.060793	0.05901
		1.05	1.25	0.084	0.087	0.085	0.080
				0.084	0.087	0.085	0.080
		1	1.3	0.085	0.086	0.085	0.081
				0.084	0.086	0.085	0.081

Entries in 2nd row denote powers of the test H_0: $\tilde{\Sigma}_1=\tilde{\Sigma}_2$ assuming $\tilde{\Omega}=\tilde{0}$.

Entries in 3rd row denote powers of the test H_0: $\tilde{\Omega}=\tilde{0}$ assuming $\tilde{\Sigma}_1=\tilde{\Sigma}_2$.

Table II (2), cont.

w_1	w_2	λ_1	λ_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
				m = 2, n = 15			
0	0.001	1	1.001	0.050150	0.050153	0.050152	0.050137
				0.050130	0.050134	0.050133	0.050115
				0.050019	0.050019	0.050019	0.050017
		1	1.01	0.051339	0.051368	0.051360	0.051192
				0.051320	0.051349	0.051341	0.051175
		1.001	1.009	0.051338	0.051369	0.051360	0.051191
				0.051319	0.051349	0.051341	0.051174
		1.005	1.015	0.052677	0.052740	0.052722	0.052374
				0.052658	0.052720	0.052702	0.052357
		1.025	1.025	0.056812	0.056980	0.056931	0.056005
				0.056791	0.056958	0.056910	0.055987
		1	1.05	0.056847	0.056949	0.056936	0.056129
				0.056827	0.056928	0.056915	0.056111
		1	1.1	0.06424	0.06433	0.06437	0.06287
				0.064218	0.064302	0.064347	0.06284
		1.05	1.25	0.099	0.099	0.098	0.094
				0.099	0.099	0.098	0.094
		1	1.3	0.100	0.099	0.098	0.096
				0.100	0.098	0.098	0.096
				m = 2, n = 40			
0	0.001	1	1.001	0.050171	0.050172	0.050171	0.050160
				0.050149	0.050150	0.050150	0.050134
				0.050021	0.050021	0.050021	0.050019
		1	1.01	0.051530	0.051535	0.051534	0.051383
				0.051508	0.051513	0.051512	0.051363
		1.001	1.009	0.051529	0.051535	0.051533	0.051381
				0.051507	0.051513	0.051511	0.051361
		1.005	1.015	0.053063	0.053074	0.053071	0.052754
				0.053040	0.053052	0.053049	0.052734
		1.025	1.025	0.057816	0.057850	0.057841	0.056983
				0.057792	0.057826	0.057817	0.056962
		1	1.05	0.057859	0.057859	0.057868	0.057152
				0.057835	0.057835	0.057844	0.057130
		1	1.1	0.06643	0.06636	0.06641	0.06513
				0.066400	0.066336	0.066388	0.06509
		1.05	1.25	0.107	0.107	0.106	0.102
				0.107	0.107	0.106	0.102
		1	1.3	0.108	0.107	0.106	0.105
				0.108	0.107	0.106	0.105

Table II (2), cont.

w_1	w_2	λ_1	λ_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
				<u>m = 2</u>, <u>n = 5</u>			
0	0.01	1	1.001	0.050231	0.050260	0.050250	0.050208
				0.050095	0.050107	0.050103	0.050084
				0.050136	0.050153	0.050147	0.050124
		1	1.01	0.051094	0.051229	0.051183	0.050971
		1.001	1.009	0.051094	0.051230	0.051183	0.050970
		1.005	1.015	0.052062	0.052321	0.052231	0.051824
		1.025	1.025	0.055026	0.05567	0.055449	0.054426
		1	1.05	0.055048	0.055603	0.055430	0.054487
		1	1.1	0.06026	0.06122	0.06097	0.05916
		1.05	1.25	0.084	0.087	0.086	0.080
		1	1.3	0.085	0.086	0.086	0.082
				<u>m = 2</u>, <u>n = 15</u>			
0	0.01	1	1.001	0.050318	0.050326	0.050324	0.050287
				0.050130	0.050134	0.050133	0.050115
				0.050187	0.050192	0.050190	0.050170
		1	1.01	0.051510	0.051544	0.051535	0.051344
		1.001	1.009	0.051510	0.051545	0.051535	0.051342
		1.005	1.015	0.052853	0.052919	0.052900	0.052528
		1.025	1.025	0.056998	0.057170	0.057120	0.056168
		1	1.05	0.057033	0.057139	0.057125	0.056292
		1	1.1	0.06445	0.06454	0.06458	0.06305
		1.05	1.25	0.099	0.099	0.099	0.094
		1	1.3	0.099	0.099	0.099	0.097
				<u>m = 2</u>, <u>n = 40</u>			
0	0.01	1	1.001	0.050363	0.050365	0.050365	0.050333
				0.050149	0.050150	0.050150	0.050134
				0.050214	0.050214	0.050214	0.050199
		1	1.01	0.051726	0.051732	0.051731	0.051559
		1.001	1.009	0.051726	0.051732	0.051731	0.051557
		1.005	1.015	0.053263	0.053276	0.053273	0.052933
		1.025	1.025	0.058030	0.058065	0.058056	0.057173
		1	1.05	0.058073	0.058074	0.058083	0.057342
		1	1.1	0.06664	0.06660	0.06665	0.06534
		1.05	1.25	0.107	0.107	0.106	0.102
		1	1.3	0.108	0.107	0.106	0.106

Table II (2), cont.

w_1	w_2	λ_1	λ_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
				m = 2, n = 5			
0.001	0.009	1	1.001	0.05036	0.05037	0.05035	0.05019
				0.05010	0.05010	0.05010	0.05008
				0.05026	0.05027	0.05027	0.05008
		1	1.01	0.0512	0.0513	0.0511	0.0511
		1.001	1.009	0.0512	0.0513	0.0511	0.0511
		1.005	1.015	0.0522	0.0524	0.0522	0.0520
		1.025	1.025	0.0551	0.0557	0.0554	0.0545
		1	1.05	0.0551	0.0557	0.0554	0.0546
		1	1.1	0.060	0.061	0.061	0.059
		1.05	1.25	0.084	0.087	0.086	0.080
		1	1.3	0.085	0.086	0.086	0.082
				m = 2, n = 15			
0.001	0.009	1	1.001	0.05042	0.05042	0.05041	0.05024
				0.05013	0.05013	0.05013	0.05012
				0.05029	0.05029	0.05029	0.05010
		1	1.01	0.0516	0.0516	0.0515	0.0515
		1.001	1.009	0.0516	0.0516	0.0515	0.0515
		1.005	1.015	0.0529	0.0530	0.0528	0.0526
		1.025	1.025	0.0571	0.0572	0.0571	0.0563
		1	1.05	0.0571	0.0572	0.0571	0.0564
		1	1.1	0.0645	0.0646	0.0645	0.0631
		1.05	1.25	0.099	0.099	0.098	0.094
		1	1.3	0.099	0.099	0.098	0.097
				m = 2, n = 40			
0.001	0.009	1	1.001	0.05045	0.05045	0.05045	0.05027
				0.05015	0.05015	0.05015	0.05013
				0.05030	0.05030	0.05031	0.05012
		1	1.01	0.0518	0.0518	0.0517	0.0516
		1.001	1.009	0.0518	0.0518	0.0517	0.0516
		1.005	1.015	0.0533	0.0534	0.0532	0.0530
		1.025	1.025	0.0581	0.0581	0.0580	0.0573
		1	1.05	0.0581	0.0581	0.0580	0.0574
		1	1.1	0.0667	0.0667	0.0666	0.0654
		1.05	1.25	0.107	0.107	0.106	0.102
		1	1.3	0.108	0.107	0.106	0.106

Table II (2), cont.

w_1	w_2	λ_1	λ_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
				<u>m = 2</u>, <u>n = 5</u>			
0	0.1	1	1.001	0.051464	0.051644	0.051582	0.051299
				0.050095	0.050107	0.050103	0.050084
				0.051367	0.051534	0.051477	0.051213
		1	1.01	0.052345	0.052634	0.052535	0.052076
		1.001	1.009	0.052344	0.052635	0.052535	0.052075
		1.005	1.015	0.053332	0.053749	0.053606	0.052944
		1.025	1.025	0.05636	0.05718	0.05689	0.05559
		1	1.05	0.05638	0.05710	0.05687	0.05565
		1	1.1	0.0616	0.0628	0.0625	0.0604
		1.05	1.25	0.086	0.091	0.088	0.082
		1	1.3	0.086	0.088	0.087	0.083
				<u>m = 2</u>, <u>n = 15</u>			
0	0.1	1	1.001	0.052022	0.052066	0.052054	0.051798
				0.050130	0.050134	0.050133	0.050115
				0.051887	0.051928	0.051917	0.051679
		1	1.01	0.053245	0.053317	0.053298	0.052881
		1.001	1.009	0.053244	0.053317	0.053298	0.052879
		1.005	1.015	0.054622	0.054728	0.054699	0.054092
		1.025	1.025	0.05887	0.05909	0.05903	0.05782
		1	1.05	0.05891	0.05906	0.05903	0.05794
		1	1.1	0.0665	0.0663	0.0667	0.0648
		1.05	1.25	0.102	0.102	0.101	0.097
		1	1.3	0.103	0.103	0.103	0.099
				<u>m = 2</u>, <u>n = 40</u>			
0	0.1	1	1.001	0.052311	0.052318	0.052317	0.052085
				0.050149	0.050150	0.050150	0.050134
				0.052157	0.052164	0.052162	0.051947
		1	1.01	0.053713	0.053725	0.053723	0.053343
		1.001	1.009	0.053772	0.053725	0.053722	0.053341
		1.005	1.015	0.055294	0.055313	0.053090	0.054754
		1.025	1.025	0.06019	0.06024	0.06023	0.05910
		1	1.05	0.06024	0.06024	0.06065	0.05927
		1	1.1	0.0691	0.0690	0.0690	0.0675
		1.05	1.25	0.111	0.110	0.110	0.105
		1	1.3	0.112	0.111	0.110	0.109

Table II (2), cont.

w_1	w_2	λ_1	λ_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
			m = 2, n = 5				
0	0.5	1	1.001	0.057131	0.057902	0.057666	0.056342
				0.050095	0.050107	0.050103	0.050084
				0.057025	0.057782	0.057550	0.056249
		1	1.01	0.058091	0.058985	0.058709	0.057182
		1.001	1.009	0.058091	0.058986	0.058709	0.057181
		1.005	1.015	0.059168	0.060205	0.059880	0.058120
		1.025	1.025	0.06246	0.06385	0.06345	0.06098
		1	1.05	0.06248	0.06387	0.06347	0.06105
		1	1.1	0.0683	0.0701	0.0696	0.0662
		1.05	1.25	0.095	0.090	0.096	0.090
		1	1.3	0.096	0.098	0.097	0.092
			m = 2, n = 15				
0	0.5	1	1.001	0.059949	0.060087	0.060074	0.058910
				0.050130	0.050134	0.050133	0.050115
				0.059800	0.059934	0.059921	0.058778
		1	1.01	0.061313	0.061482	0.061461	0.060105
		1.001	1.009	0.061312	0.061482	0.061461	0.060103
		1.005	1.015	0.062847	0.063054	0.063023	0.061443
		1.025	1.025	0.06757	0.06790	0.06784	0.06555
		1	1.05	0.06761	0.06787	0.06784	0.06568
		1	1.1	0.0760	0.0763	0.0763	0.0733
		1.05	1.25	0.115	0.116	0.114	0.108
		1	1.3	0.116	0.115	0.115	0.111
			m = 2, n = 40				
0	0.5	1	1.001	0.061426	0.061422	0.061437	0.060405
				0.050149	0.050150	0.050150	0.050134
				0.061252	0.061248	0.061262	0.060251
		1	1.01	0.063006	0.063007	0.063021	0.061811
		1.001	1.009	0.063006	0.063007	0.063021	0.061808
		1.005	1.015	0.064786	0.064794	0.064807	0.063385
		1.025	1.025	0.07032	0.07032	0.07033	0.06823
		1	1.05	0.07033	0.07033	0.07035	0.06841
		1	1.1	0.082	0.0801	0.0802	0.0775
		1.05	1.25	0.126	0.126	0.124	0.119
		1	1.3	0.127	0.126	0.126	0.123

Table II (2), cont.

w_1	w_2	λ_1	λ_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
				m = 2, n = 5			
0	1	1	1.001	0.06464	0.06596	0.06564	0.06309
				0.05010	0.05011	0.05010	0.05008
				0.064523	0.065823	0.065504	0.06299
		1	1.01	0.06570	0.06716	0.06680	0.06401
		1.001	1.009	0.06570	0.06716	0.06680	0.06401
		1.005	1.02	0.0675	0.0691	0.0688	0.0656
		1.025	1.025	0.0705	0.0727	0.0721	0.0682
		1	1.05	0.0706	0.0726	0.0721	0.0683
		1	1.1	0.077	0.080	0.079	0.074
				m = 2, n = 15			
0	1	1	1.001	0.07067	0.07075	0.07084	0.06870
				0.05013	0.05013	0.05013	0.05012
				0.070496	0.070579	0.070666	0.06855
		1	1.01	0.07221	0.07233	0.07241	0.07004
		1.001	1.009	0.07221	0.07233	0.07241	0.07004
		1.005	1.02	0.0748	0.0750	0.0751	0.0723
		1.025	1.025	0.0793	0.0796	0.0796	0.0762
		1	1.05	0.0793	0.0796	0.0796	0.0763
		1	1.1	0.089	0.089	0.089	0.085
				m = 2, n = 40			
0	1	1	1.001	0.07387	0.07375	0.07384	0.07203
				0.05012	0.05015	0.05015	0.05013
				0.073667	0.073556	0.073641	0.07185
		1	1.01	0.07568	0.07556	0.07565	0.07362
		1.001	1.009	0.07567	0.07556	0.07565	0.07362
		1.005	1.02	0.0787	0.0786	0.0787	0.0763
		1.025	1.025	0.0840	0.0839	0.0840	0.0809
		1	1.05	0.0840	0.0839	0.0840	0.0809
		1	1.1	0.095	0.095	0.095	0.091

Table II (2), cont.

w_1	w_2	λ_1	λ_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
				m = 2, n = 5			
1	1	1	1.001	0.0809	0.0861	0.0840	0.0764
				0.0501	0.0501	0.0501	0.0501
				0.0808	0.0858	0.0839	0.0763
		1	1.01	0.0822	0.0875	0.0854	0.0775
		1.001	1.009	0.0822	0.0875	0.0854	0.0775
		1.005	1.02	0.084	0.090	0.088	0.079
		1.025	1.025	0.088	0.094	0.092	0.083
		1	1.05	0.088	0.094	0.092	0.083
		1	1.1	0.096	0.099	0.099	0.090
				m = 2, n = 15			
1	1	1	1.001	0.0947	0.0964	0.0957	0.0878
				0.0501	0.0501	0.0501	0.0501
				0.0944	0.0961	0.0961	0.0876
		1	1.01	0.0966	0.0984	0.0976	0.0895
		1.001	1.009	0.0966	0.0984	0.0976	0.0895
		1.005	1.02	0.100	0.102	0.101	0.092
		1.025	1.025	0.105	0.107	0.106	0.097
		1	1.05	0.106	0.108	0.106	0.097
		1	1.1	0.117	0.117	0.117	0.108
				m = 2, n = 40			
1	1	1	1.001	0.1021	0.1026	0.1023	0.0946
				0.0501	0.0502	0.0502	0.0501
				0.1018	0.1023	0.1021	0.0944
		1	1.01	0.1044	0.1049	0.1046	0.0966
		1.001	1.009	0.1044	0.1049	0.1046	0.0966
		1.005	1.02	0.108	0.109	0.108	0.100
		1.025	1.025	0.115	0.115	0.115	0.106
		1	1.05	0.115	0.115	0.115	0.106
		1	1.1	0.129	0.129	0.129	0.118

CHAPTER III

THE EXACT NULL DISTRIBUTION OF HOTELLING'S TRACE

1. Introduction

Let the matrices $\underset{\sim}{S}_1(p\times p)$ and $\underset{\sim}{S}_2(p\times p)$ be independently distributed central Wishart $W(p,n_i,\underset{\sim}{\Sigma},\underset{\sim}{0})$, $i=1,2$, and let $U^{(p)} = \text{tr}\ \underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ (i.e. n_2 times Hotelling's T_0^2). The exact null distribution of $U^{(p)}$ has been studied by several authors. Hotelling [14] obtained the distribution of T_0^2 for p=2 and Pillai and Chang [34] developed the distribution of $U^{(3)}$ as a slowly convergent infinite series. Further, Pillai and Young [41], through inverse Laplace transform, obtained the null distribution of $U^{(3)}$ for m=0(1)5 and of $U^{(4)}$ for m=0(1)2, where $m=(n_1-p-1)/2$. Constantine [4] obtained the density of $U^{(p)}$ in a series involving zonal polynomials which converges only for $|U^{(p)}| < 1$. Davis [8] has shown that the density function of $U^{(p)}$ satisfies an ordinary differential equation of order p with regular singularities at $U^{(p)} = 0,-1,-2,\ldots,-p$ and infinity and later he [9] extended the series of Constantine [4] by analytic continuation using a system of linear differential equations. Krishnaiah and Chang [21] derived the Laplace transform of $U^{(p)}$ in terms of double integrals and illustrated the derivation of the density for p=2 and m=1.

In this chapter, the Pillai and Young approach of the inverse Laplace transform has been modified to yield the density and distribution functions of $U^{(p)}$ for p=3 and p=4 in a much simpler form than obtained before, and hold for all non-negative integers m unlike in the Pillai and Young approach where individual values of m were considered.

2. Preliminaries

Let us assume that we have k variables $x_1,\ldots,x_k$ and form the product $\prod_{i>j}(x_i-x_j)$. It can be shown that this product equals to the Vandermonde's determinant

$$\begin{vmatrix} x_k^{k-1} & x_k^{k-2} & \cdots & x_k & 1 \\ \vdots & & & & \vdots \\ x_1^{k-1} & x_1^{k-2} & \cdots & x_1 & 1 \end{vmatrix} . \tag{2.1}$$

Next, let us consider the integral of the type

$$\int\cdots\int_{\mathscr{B}} \exp[-t\sum_{i=1}^{k} x_i^{-1}] \prod_{i=1}^{k}\{x_i^n(1-x_i)^m\} \prod_{i>j}(x_i-x_j)\prod_{i=1}^{k} dx_i , \tag{2.2}$$

where n, m are >-1, t>0 independent of the x's and $\mathscr{B}=\{(x_1,\ldots,x_k)\}\,|\,0\leq x_1<\cdots<x_k\leq x\}$. Expressing the product $\prod_{i>j}(x_i-x_j)$ by (2.1) and applying the properties of the determinant, (2.2) can now be written as:

(2.3) $U(x:r_k,m;\cdots;r_1,m;t)$

$$= \begin{vmatrix} \int_0^x e^{-t/x_k} x_k^{r_k}(1-x_k)^m \, dx_k & \cdots & \int_0^x e^{-t/x_k} x_k^{r_1}(1-x_k)^m \, dx_k \\ \vdots & & \vdots \\ \int_0^{x_2} e^{-t/x_1} x_1^{r_k}(1-x_1)^m \, dx_1 & \cdots & \int_0^{x_2} e^{-t/x_1} x_1^{r_1}(1-x_1)^m \, dx_1 \end{vmatrix}$$

where $r_i = n+i-1$, $i=1,2,\ldots,k$. The determinant in (2.3) will involve integrals of the type

$$(2.4) \qquad A(x':r,m;F;t) = \int_0^{x'} e^{-t/y} F(y) y^r (1-y)^m \, dy,$$

where F(y) is a function of y such that the integral exists. In our case F(y) could be of the form

$$(2.5) \qquad \int_0^y e^{-t/x_{k-1}} x_{k-1}^{r_{k-1}}(1-x_{k-1})^m \, dx_{k-1} \cdots \int_0^{x_2} e^{-t/x_1} x_1^{r_1}(1-x_1)^m \, dx_1,$$

and in this case we will denote (2.4) by $A(x':r,m;r_{k-1},m;\cdots;r_1,m;t)$. Analogous results to those obtained by Pillai [26] will be stated below with brief proofs only for the completeness of the arguments. Writing

$$x^r(1-x)^m e^{-t/x} \text{ into the form } x^{r+m}\{(1-x)/x\}^m e^{-t/x}$$

and performing the integration by parts by integrating x^{r+m} and differentiating $\{(1-x)/x\}^m F(x)e^{-t/x}$ we have

<u>Lemma 2.1</u>: $A(x':r,m;F';t) = (r+m+1)^{-1}[A_0(x':r+1,m;F;t)$

$- A(x':r+1,m;F';t) - tA(x':r-1,m;F;t) + mA(x':r,m-1;F;t)]$,

where

$$A_0(x':r+1,m;F;t) = y^{r+1}(1-y)^m F(y) e^{-t/y} \Big|_{y=0}^{y=x'}$$

and $$F'(y) = \frac{d}{dy} F(y).$$

Now, using (2.4) with F(y) in (2.5), it can be shown (by interchanging the order of integration in the first term on the left hand side) that

$$A(x':r_2,m;r_1,m;t)+A(x':r_1,m;r_2,m;t) = A(x':r_1,m;t)A(x':r_2,m;t)$$

and in general we have

Lemma 2.2: If σ is any permutation of $(1,2,\ldots,k)$ then

$$\sum_{\sigma} A(x':r_{\sigma(k)},m;\ldots;r_{\sigma(1)},m;t) = \prod_{j=1}^{k} A(x':r_j,m;t),$$

where the summation is over all possible permutations.

Furthermore, if we let $U(x:r_k',m';t';\ldots;r_1',m',t')^{(i)}$ to denote the determinant in (2.3) when the indices of the ith row alone are different from those of the other rows, where the indices of the ith row are $r_k',m',t';\ldots;r_1',m',t'$, we have the following:

Lemma 2.3:

$$\sum_{i=1}^{k} (-1)^{i-1} U(x:r_k',m',t';\ldots;r_1',m';t')^{(i)} =$$

$$= \sum_{j=k}^{1} (-1)^{k-j} A(x:r_j',m';t')U(x:r_k,m;\ldots;r_{j+1},m;r_{j-1},m;\ldots;r_1,m;t).$$

Having these lemmas, we are now ready to evaluate the determinant in (2.3) and we have the following theorem:

Theorem 2.1: The reduction formula to evaluate the determinant (2.3) is given by

$$(2.6)\quad U(x{:}r_k,m;\ldots;r_1,m;t) = (r_k+m+1)^{-1}[A^{(k)}+B^{(k)}-tC^{(k)}+mD^{(k)}],$$

where

$$A^{(k)} = y^{r_k+1}(1-y)^m e^{-t/y}\Big|_{y=0}^{y=x}\cdot U(x{:}r_{k-1},m;\ldots;r_1,m;t)$$

$$B^{(k)} = 2\sum_{j=1}^{k-1}(-1)^{k+j}\, A(x{:}r_k+r_j+1,2m;2t)$$

$$\cdot U(x{:}r_{k-1},m;\ldots;r_{j+1},m;r_{j-1},m;\ldots;r_1,m;t)$$

$$C^{(k)} = U(x{:}r_k-1,m;r_{k-1},m;\ldots;r_1,m;t)$$

$$D^{(k)} = U(x{:}r_k,m-1;r_{k-1},m;\ldots;r_1,m;t)$$

Proof: Expand the determinant by the first column. Integrate by parts the term involving the element from the ith row and first column with respect to x_{k-i+1} using Lemma (2.1). Add the results obtained and then apply the above lemmas.

Now, let us consider the determinant of the form

$$(2.7)\quad D(n{:}q_p,m;\ldots;q_1,m;t)$$

$$= \begin{vmatrix} \int_0^1 e^{-t/x_p} x_p^{n+q_p}(1-x_p)^m\,dx_p & \ldots & \int_0^1 e^{-t/x_p} x_p^{n+q_1}(1-x_p)^m\,dx_p \\ \vdots & & \vdots \\ \int_0^{x_2} e^{-t/x_1} x_1^{n+q_p}(1-x_1)^m\,dx_1 & \ldots & \int_0^{x_2} e^{-t/x_1} x_1^{n+q_1}(1-x_1)^m\,dx_1 \end{vmatrix}$$

We see that (2.7) is a special case of (2.3) if in (2.3) we let x=1 and k=p, so that q_i=i-1, i=1,2,...,p. Therefore, Theorem 2.1 will give the following:

Corollary: The reduction formula to evaluate the determinant (2.7) is given by

$$(2.8)\quad D(n:q_p,m;\ldots;q_1,m;t) = (n+m+q_p+1)^{-1}[E^{(p)}+F^{(p)}-tG^{(p)}+mH^{(p)}],$$

where

$$E^{(p)} = x^{n+q_p+1}(1-x)^m e^{-t/x} \Big|_{x=0}^{x=1} \cdot D(n:q_{p-1},m;\ldots;q_1,m;t)$$

$$F^{(p)} = e^{-2t} \sum_{j=1}^{p-1} (-1)^{p+j}\, g(m,n:q_p+q_j+3,2;t)$$

$$\cdot D(n:q_{p-1},m;\ldots;q_{j+1},m;q_{j-1},m;\ldots;q_1,m;t)$$

where

$$(2.9)\quad g(m,n;a,b;t) = \int_0^\infty \frac{e^{-tx}(x/b)^{bm}dx}{(1+x/b)^{b(m+n)+a}},$$

$$G^{(p)} = D(n:q_p-1,m;q_{p-1},m;\ldots;q_1,m;t)$$

$$H^{(p)} = D(n:q_p,m-1;q_{p-1},m;\ldots;q_1,m;t).$$

Proof: Let x=1 in (2.6) and replace r_i by $n+q_i$, $i=1,2,\ldots,p$, so that $D(n:q_p,m;\ldots;q_1,m;t) = U(1:n+q_p,m;\ldots;n+q_1,m;t)$. The result follows if we note that

$$A(1:2n+q_p+q_j+1,2m;2t) = \int_0^1 e^{-2t/x}\, x^{2n+q_p+q_j+1}(1-x)^{2m}\,dx$$

$$= \frac{1}{2}\, e^{-2t}\, g(m,n:q_p+q_j+3,2;t).$$

For our work in the sequel, we will use the properties of the determinant D and the integrals A and g as follows:

1. If any two of q_j's are equal, then the value of the determinant D is zero.

2. If $q_j = j-1$, j positive integer, then

(2.10) $D(n:q_p,m-1;q_{p-1},m;\ldots;q_1,m;t)=D(n:q_p,m-1;q_{p-1},m-1;\ldots;q_1,m-1;t)$.

3. For non-negative integer k, we have

(2.11) $D(n:k,m;t)=A(1:n+k,m;t) = e^{-t}g(m,n:k+2,1;t)$,

and for $0 \leq q_1 < q_2 <\ldots< q_p$, q_i integer, we have

(2.12) $D(n:q_p,m;\ldots;q_1,m;t) = D(n+q_1:q_p-q_1,m;\ldots;0,m;t)$.

As for the g function, we only need the following properties:

4. For non-negative integer c, $g(m,n+c:a,b;t) = g(m,n:a+bc,b;t)$.

5. By integration by parts we obtain

(2.13) $g(0,n:a,b;t) = t^{-1}[1-\{(bn+a)/b\}g(0,n:a+1,b;t)]$

and for $m \neq 0$

(2.14) $g(m,n:a,2;t) = t^{-1}[mg(m-1,n:a+1,2;t)$

$-mg(m-1,n:a+2,2;t) - \frac{1}{2}\{2(m+n)+a\}g(m,n:a+1,2;t)]$.

Finally, let us consider the integral of the type

(2.15) $h(u:m,m';n,n';a,a';b,b')=\int_0^u \frac{(t/b)^{bm}\{(u-t)/b'\}^{b'm'}dt}{(1+t/b)^g\{1+(u-t)/b'\}^{g'}}$,

where $g=b(m+n)+a$ and $g'=b'(m'+n')+a'$.

The above integral can be thrown into the integral of the form

(2.16) $I(u:a,b,m;c,d,n) = \int_0^u \frac{dx}{(a+bx)^m(c+dx)^n}$.

Considering only non-negative integers m,n and real numbers $a>0$, $c>0$, $b\neq 0$ and $d\neq 0$, then we have

(2.17) $$I(u:a,b,m;c,d,n) = \frac{A_1}{b} \ln(\frac{a+bu}{a}) + \frac{B_1}{d} \ln(\frac{c+du}{c})$$

$$- \sum_{i=2}^{m} \frac{A_i}{b(i-1)} \{(a+bu)^{1-i} - a^{1-i}\} + \sum_{j=2}^{n} \frac{B_j}{d(j-1)} \{(c+du)^{1-j} - c^{1-j}\},$$

where

(2.18) $$A_i = \{(-b)^n (bd)^{m-i} \prod_{s=1}^{m-i} (n+s-1)\}/\{(m-i)!(ad-bc)^{n+m-i}\}$$

(2.19) $$B_j = \{d^m (bd)^{n-j} \prod_{s=1}^{n-j} (m+s-1)\}/\{(n-j)!(ad-bc)^{n+m-j}\},$$

where the empty product is to be interpreted as unity. Using the expression in (2.17), we now have

(2.20) $$h(u:m,m';n,n';a,a';b,b') = b^p b'^{p'}$$

$$\cdot \sum_{i=0}^{bm} \sum_{j=0}^{b'm'} (-b)^i (-b')^j \binom{bm}{i} \binom{b'm'}{j} I(u:b,1,p+i;b'+u,-1,p'+j),$$

where p=bn+a, p'=b'n'+a' and bm,b'm' are non-negative integers. Integrating h from 0 to u, $0 < u < \infty$, we obtain

(2.21) $$J(u:m,m';n,n';a,a';b,b') = \int_0^u h(x:m,m';n,n';a,a';b,b')dx$$

$$= \sum_{i=0}^{bm} \sum_{j=0}^{b'm'} (-1)^{i+j} \binom{bm}{i} \binom{b'm'}{j} \{b'/(p'+j-1)\} \cdot [\{b/(p+i-1)\}\{1-(1+u/b)^{-(p+i-1)}\}$$

$$- b^{p+i} b'^{p'+j-1} I(u:b,1,p+i;b'+u,-1,p'+j-1)],$$

where p=bn+a, p'=b'n'+a', and the I function is as in (2.17).

3. Some Values of the Determinant D

Now, let us first illustrate the use of the reduction formula (2.8) for p=2. Noting that $q_j=j-1$, j=1,2, then for m=0 we easily can see that (2.8) gives

$$(3.1)\quad D(n:1,0;0,0;t) = e^{-2t}(n+2)^{-1}\{g(0,n:2,1;t)-g(0,n:4,2;t)\}.$$

In obtaining (3.1), property 1 and (2.11) have been used. For m≠0, use (2.10) and then (2.8) repeatedly (m-1) times. Applying also (3.1) we obtain the general formula for p=2 as

$$(3.2)\quad D(n:1,m;0,m;t) = \frac{e^{-2t}}{m+n+2}\,[Q(m)g(0,n:2,1;t)-\sum_{i=0}^{m} Q(i)g(m-i,n:4,2;t)],$$

where g(m,n:a,b;t) is as in (2.9) and

$$(3.3)\quad Q(i) = Q(m,q,i) = \prod_{j=1}^{i} \{(m+1-j)/(q-1-j)\},\ Q(0) = 1$$

with q=m+n+3.

For p=3 and m=0, after using (3.1) and (2.11), we have

$$(3.4)\quad D(n:2,0;1,0;0,0;t) = e^{-3t}(n+3)^{-1}[(n+2)^{-1}\{g(0,n:2,1;t)-g(0,n:4,2;t)\}$$
$$+g(0,n:5,2;t)g(0,n:3,1;t)-g(0,n:6,2;t)g(0,n:2,1;t)].$$

Repeating the use of (2.8), then (2.11) and (3.4) we finally obtain the general expression for the determinant D(n:2,m;1,m;0,m;t) as

$$(3.5)\quad D(n:2,m;1,m;0,m;t) = \frac{e^{-3t}}{m+n+3}\,[\frac{Q(m)}{n+2}\,\{g(0,m:2,1;t)-g(0,n:4,2;t)\}$$
$$+\sum_{i=0}^{m} Q(i)\{g(m-i,n:5,2;t)g(m-i,n:3,1;t)$$
$$-g(m-i,n:6,2;t)g(m-i,n:2,1;t)\}],$$

where Q(i), as before, is as defined in (3.3) but now we have to take q=m+n+4.

Finally, let us obtain the expression of the determinant D for p=4. First we will need the following values of the determinants:

$$(3.6)\quad D(n:2,m;1,m;t) = D(n+1:1,m;0,m;t)$$

$$= \frac{e^{-2t}}{m+n+3}\left[Q(m)g(0,n:3,1;t) - \sum_{i=0}^{m} Q(i)g(m-i,n:6,2;t)\right],$$

where Q(i) as in (3.3) with q=m+n+4. Further, we also need to compute D(n:2,m;0,m;t). If m=0, we have

$$D(n:2,0;0,0;t) = e^{-2t}(n+3)^{-1}[g(0,n:2,1;t)-g(0,n:5,2;t)$$

$$- (n+2)^{-1}t\{g(0,n:2,1;t)-g(0,n:4,2;t)\}].$$

Now apply (2.13) to eliminate t. The above expression becomes

$$(3.7)\quad D(n:2,0;0,0;t) = \frac{e^{-2t}}{n+3}[g(0,n:2,1;t)-2g(0,n:5,2;t)+g(0,n:3,1;t)].$$

For m≠0, the determinant D(n:2,m;0,m;t) is given by the expression

$$(m+n+3)^{-1}[-e^{-2t}g(m,n:5,2;t)-tD(n:1,m;0,m;t)+mD(n:2,m-1;0,m;t)].$$

We have to eliminate t in the second term. For this, we can see from (3.2) that the relations (2.13) and (2.14) are applicable. The second term in the above expression now becomes

$$\frac{e^{-2t}}{m+n+2}\left[-(n+2)Q(m)g(0,n:3,1;t)+\sum_{i=0}^{m}(n+m-i+2)Q(i)g(m-i,n:5,2;t)\right.$$

$$\left.-\sum_{i=0}^{m}(m-i)Q(i)\{g(m-1-i,n:5,2;t)-g(m-1-i,n:6,2;t)\}\right].$$

As for the last term, it is easily seen that

$$mD(n:2,m-1;0,m;t)=m\{D(n:1,m-1;0,m-1;t)-D(n:1,m;0,m;t)\}$$

$$=e^{-2t}[g(m,n:4,2;t)+\frac{n+2}{m+n+2}\{Q(m)g(0,n:2,1;t)-\sum_{i=0}^{m}Q(i)g(m-i,n:4,2;t)\}].$$

Hence, the value of the determinant D(n:2,m;0,m;t) is given by

$$(3.8)\quad D(n:2,m;0,m;t)=\frac{e^{-2t}}{q(q-1)}[(q-1)\{g(m,n:4,2;t)-g(m,n:5,2;t)\}$$
$$+(n+2)Q(m)\{g(0,n:2,1;t)+g(0,n:3,1;t)$$
$$-\sum_{i=0}^{m}\{(n+2)Q(i)g(m-i,n:4,2;t)+(n+m-i+2)Q(i)g(m-i,n:5,2;t)\}$$
$$+\sum_{i=0}^{m}(m-i)Q(i)\{g(m-1-i,n:5,2;t)-g(m-1-i,n:6,2;t)\}],$$

where q=m+n+3 and Q(i) as in (3.3).

Repeating the use of (2.8) we finally obtain the expression of the determinant D for p=4 in the form

$$(3.9)\quad D(n:3,m;2,m;1,m;0,m;t)$$
$$=e^{-4t}(n+m+4)^{-1}[Q(m,q,m)e^{3t}D(n:2,0;1,0;0,0;t)$$
$$-\sum_{i=0}^{m}Q(m,q,i)\{g(m-i,n:6,2;t)e^{2t}D(n:2,m-i;1,m-i;t)$$
$$-g(m-i,n:7,2;t)e^{2t}D(n:2,m-i;0,m-i;t)$$
$$+g(m-i,n:8,2;t)e^{2t}D(n:1,m-i;0,m-i,t)\}],$$

where q=m+n+5, the determinants on the right hand side are, respectively, as defined in (3.4), (3.6), (3.8) and (3.2).

4. The Exact Null Density Function of $U^{(p)}$

Let $\underset{\sim}{S}_1(p\times p)$ be distributed $W(p,n_1,\underset{\sim}{\Sigma},\underset{\sim}{\Omega})$, i.e. non-central Wishart distribution on n_1 d.f. with non-centrality $\underset{\sim}{\Omega}$ and covariance matrix $\underset{\sim}{\Sigma}$, independently of $\underset{\sim}{S}_2(p\times p)$ central Wishart $W(p,n_2,\underset{\sim}{\Sigma},\underset{\sim}{0})$. If $r_1,r_2,\ldots,r_p$ are the characteristic roots of $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$, then the joint density function of $r_1,r_2,\ldots,r_p$, $0 < r_1 <\ldots< r_p < \infty$, when $\underset{\sim}{\Omega}=\underset{\sim}{0}$ is given by

$$(4.1)\qquad f(r_1,\ldots,r_p) = C_p \prod_{i=1}^{p} \{r_i^m/(1+r_i)^q\} \prod_{i>j} (r_i-r_j),$$

where $q=m+n+p+1$, $m = \frac{1}{2}(n_1-p-1)$, $n = \frac{1}{2}(n_2-p-1)$ and

$$(4.2)\qquad C_p = C(p,m,n) = \pi^{\frac{1}{2}p} \prod_{i=1}^{p} \left[\frac{\Gamma(\frac{1}{2}(2m+2n+p+i+2))}{\Gamma(\frac{1}{2}(2m+i+1))\Gamma(\frac{1}{2}(2n+i+1))\Gamma(\frac{1}{2}i)}\right].$$

To find the density function of $U^{(p)} = \sum_{i=1}^{p} r_i = \mathrm{tr}\, \underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$, we will use the Laplace transform of $U^{(p)}$ with respect to $f(r_1,\ldots,r_p)$. It is given by

$$L(t:p,m,n) = E[\exp(-tU^{(p)})] = E[\exp(-t \sum_{i=1}^{p} r_i)]$$

$$= C_p \int_{\mathscr{A}}\cdots\int \exp(-t \sum_{i=1}^{p} r_i) \prod_{i=1}^{p} \{r_i^m/(1+r_i)^q\} \prod_{i>j} (r_i-r_j) \prod_{i=1}^{p} dr_i ,$$

where $\mathscr{A} = \{(r_1,\ldots,r_p)|0 < r_1 <\ldots< r_p < \infty\}$. Let now

$x_i = (1+r_{p-i+1})^{-1}$, $i=1,2,\ldots,p$, then

$$L(t:p,m,n)=C_p e^{pt} \int_{\mathscr{B}}\cdots\int \exp(-t \sum_{i=1}^{p} x_i^{-1}) \prod_{i=1}^{p} \{x_i^n(1-x_i)^m\} \prod_{i>j} (x_i-x_j) \prod_{i=1}^{p} dx_i ,$$

where $\mathscr{B} = \{(x_1,\ldots,x_p)|0 < x_1 <\ldots< x_p < 1\}$. Applying the results in Section 2, the above Laplace transform can be written as:

(4.3) $L(t:p,m,n) = C_p e^{pt} D(n:q_p,m;\ldots;q_1,m;t)$,

where $D(n:q_p,m;\ldots;q_1,m;t)$ is exactly as in (2.7) with $q_j=j-1$, $j=1,2,\ldots,p$. By the uniqueness of the Laplace transform, (4.3) will give the density of $U^{(p)}$ if we take its inverse. Therefore let us denote by $D^*(n:q_p,m;\ldots;q_1,m;t)$ the inverse Laplace transform of $e^{pt}D(n:q_p,m;\ldots;q_1,m;t)$. Then the density function of $U^{(p)}$ can be written in the form

(4.4) $f(U^{(p)}) = C_p D^*(n:q_p,m;\ldots;q_1,m;U^{(p)})$.

To obtain the density $f(U^{(p)})$ explicitly, we can see from the expression of the determinant D that we have to obtain the inverse Laplace transform of g function described in (2.9) and that of the product of g's. If $g^*(u:m,n;a,b)$ denotes the inverse Laplace transform of $g(m,n:a,b;t)$, then

(4.5) $g^*(u:m,n;a,b) = (u/b)^{bm}/(1+u/b)^{b(m+n)+a}$.

Furthermore, by the convolution property for the Laplace transform we also have that the inverse Laplace transform of the product $g(m,n:a,b;t)g(m',n':a',b';t)$ is given by the function $h(u:m,m';n,n';a,a';b,b')$ described in (2.15), whose value is given in (2.20).

Now, let us first obtain the density function of $U^{(p)}$ for p=2. From (3.2) we can see easily that

(4.6) $$D^*(n:1,m;0,m;u) = (m+n+2)^{-1}[Q(m)g^*(u:0,n;2,1) - \sum_{i=0}^{m} Q(i)g^*(u:m-i,n;4,2)],$$

so that the density of $U^{(2)}$, after using (4.5), is given by

$$(4.7)\quad f(U^{(2)}) = (m+n+2)^{-1}C(2,m,n)[Q(m,q,m)(1+U^{(2)})^{-(n+2)} - \sum_{i=0}^{m}\{Q(m,q,i)(\tfrac{1}{2}U^{(2)})^{2(m-i)}/(1+\tfrac{1}{2}U^{(2)})^{2(m-i+n)+4}\}],$$

for $0 < U^{(2)} < \infty$ and zero otherwise, where Q(m,q,i) is as defined in (3.3) with q=m+n+3 and $C(p,m,n) = C_p$ in (4.2) for p=2.

The second density that can be obtained here is that of $U^{(3)}$. From (3.5) we can find D*(n:2,m;1,m;0,m;u) so that the density of $U^{(3)}$ is given by

$$(4.8)\quad f(U^{(3)}) = (n+m+3)^{-1}C(3,m,n) \cdot [(n+2)^{-1}Q(m,q,m)\{g^*(U^{(3)}:0,n;2,1)-g^*(U^{(3)}:0,n;4,2)\} + \sum_{i=0}^{m} Q(m,q,i)\{h(U^{(3)}:m-i,m-i;n,n;5,3;2,1) - h(U^{(3)}:m-i,m-i;n,n;6,2;2,1)\}],$$

for $0 < U^{(3)} < \infty$ and zero otherwise, where q=m+n+4, $g^*(U^{(3)}:m,n;a,b)$ is as in (4.5) and $h(U^{(3)}:m,m';n,n';a,a';b,b')$ is as described in (2.20).

Finally, from (3.9) we also can obtain the density function of $U^{(4)}$. To shorten the expression we let $D_1^*(u;i)$ be the inverse Laplace transform of $g(m-i,n:6,2;t)e^{2t}D(n:2,m-i;1,m-i;t)$, so that

$$(4.9)\quad D_1^*(u;i)=(n+m-i+3)^{-1}\cdot[Q(m-i,q+1,m-i)h(u:m-i,0;n,n;6,3;2,1) - \sum_{j=0}^{m-i} Q(m-i,q+1;j)h(u:m-i,m-i-j;n,n;6,6;2,2)],$$

and $D_2^*(u;i)$ be that of $g(m-i,n:7,2;t)e^{2t}D(n:2,m-i;0,m-i;t)$, so that

(4.10) $$D_2^*(u;i)=(n+m-i+2)^{-1}(n+m-i+3)^{-1}$$

$$\cdot[(n+m-i+2)\{h(u:m-i,m-i;n,n;7,4;2,2)-h(u:m-i,m-i;n,n;7,5;2,2)\}$$

$$+(n+2)Q(m-i,q,m-i)\{h(u:m-i,0;n,n;7,2;2,1)+h(u:m-i,0;n,n;7,3;2,1)\}$$

$$-(n+2)\sum_{j=0}^{m-i}Q(m-i,q,j)h(u:m-i,m-i-j;n,n;7,4;2,2)$$

$$-\sum_{j=0}^{m-i}(n+m-i-j+2)Q(m-i,q,j)h(u:m-i,m-i-j;n,n;7,5;2,2)$$

$$+\sum_{j=0}^{m-i}(m-i-j)Q(m-i,q,j)\{h(u:m-i,m-1-i-j;n,n;7,5;2,2)$$

$$-h(u:m-i,m-1-i-j;n,n;7,6;2,2)\}],$$

where $q=m+n+3$.

Also we let the inverse Laplace transform of

$g(m-i,n:8,2;t)e^{2t}D(n:1,m-i;0,m-i;t)$ be $D_3^*(u;i)$, then

(4.11) $$D_3^*(u;i)=(m-i+n+2)^{-1}\cdot[Q(m-i,q,m-i)h(u:m-i,0;n,n;8,2;2,1)$$

$$-\sum_{j=0}^{m-i}Q(m-i,q,j)h(u:m-i,m-i-j;n,n;8,4;2,2)],$$

and that of $e^{3t}D(n:2,0;1,0;0,0;t)$ be $D_4^*(u)$. Upon simplification we have

(4.12) $$D_4^*(u)=(n+3)^{-1}(2n+5)^{-1}[(n+2)^{-1}g^*(u:0,n;2,1)$$

$$-(n+2)^{-1}(2n+5)g^*(u:0,n;4,2)+2g^*(u:0,n;5,2)+h(u:0,0;n,n;5,3;2,1)].$$

The density of $U^{(4)}$ therefore can be written as

(4.13) $$f(U^{(4)})=(n+m+4)^{-1}C(4,m,n)[Q(m,q,m)D_4^*(U^{(4)})$$

$$-\sum_{i=0}^{m}Q(m,q,i)\{D_1^*(U^{(4)};i)-D_2^*(U^{(4)};i)+D_3^*(U^{(4)};i)\}],$$

holds for $0 < U^{(4)} < \infty$ and is defined to be zero otherwise.

5. The Distribution Function of $U^{(p)}$

If $F(U^{(p)})$ denotes the c.d.f. of $U^{(p)}$ then upon integrating (4.4) we obtain the general form of the c.d.f. of $U^{(p)}$ as:

$$(5.1)\quad F(U^{(p)}) = C(p,m,n) \int_0^{U^{(p)}} D^*(n:q_p,m;\ldots;q_1,m;x)dx.$$

Thus, we see that the c.d.f. of $U^{(p)}$ for p=2,3 and 4 can be obtained by performing the integration of $g^*(u:m,n;a,b)$ and $h(u:m,m';n,n';a,a';b,b')$ with respect to u from 0 to $U^{(p)}$. Starting from (4.7), i.e., the density function of $U^{(2)}$, we readily obtain the c.d.f. of $U^{(2)}$ which can be written in the form:

$$(5.2)\quad F(U^{(2)}) = \frac{C(2,m,n)}{m+n+2} [Q(m)B_v(1,n+1)-2\sum_{i=0}^{m} Q(i)B_w(a,b)],$$

where $a=2m-2i+1$, $b=2n+3$, $v=U^{(2)}/(1+U^{(2)})$, $w=U^{(2)}/(2+U^{(2)})$ and $B_z(\alpha,\beta)$ denotes the incomplete beta function.

Cumulative distribution function (5.2) is an alternate form for the one obtained by Hotelling [14] which is given by

$$(5.3)\quad P = I_w(2m+2n,2n+3)$$

$$- \frac{(2m+2n+4)!((1-w)/(1+w))^{n+1}}{2(2m+1)!(2n+2)!} B_v(m+1,n+2),$$

where $w=U^{(2)}/(2+U^{(2)})$, $v=w^2$ and $I_z(\alpha,\beta) = B_z(\alpha,\beta)/B(\alpha,\beta)$ with $B(\alpha,\beta)$ is the complete beta function.

To obtain the c.d.f. of $U^{(3)}$ we will need the following results:

$$(5.5)\quad \int_0^u g^*(x:0,n;a,b)dx = \frac{b}{bn+a-1} (1-g^*(u:0,n;a-1,b))$$

and $\int_0^u h(x:m,m';n,n';a,a';b,b')dx$ which has been stated in (2.21).

Integrating $f(U^{(3)})$ in (4.8) and applying the results (5.5) and (2.21) we obtain the c.d.f. of $U^{(3)}$ as stated in the following theorem:

Theorem 5.1: The exact null distribution function of $U^{(3)}$ is given by

$$(5.6)\quad F(U^{(3)}) = \frac{C(3,m,n)}{m+n+3}\Big[\frac{Q(m,q+1;m)}{n+2}\{(n+1)^{-1}(2n+3)^{-1} - (n+1)^{-1}(1+U^{(3)})^{-(n+1)} + 2(2n+3)^{-1}(1+\tfrac{1}{2}U^{(3)})^{-(2n+3)}\}$$
$$+ \sum_{i=0}^{m} Q(m,q+1;i)\{J(U^{(3)}:m-i,m-i;n,n;5,3;2,1) - J(U^{(3)}:m-i,m-i,n,n;6,2;2,1)\}\Big],$$

for $0 < U^{(3)} < \infty$ and zero otherwise, where $Q(m,q;i)$ is as in (3.3), $q=m+n+3$ and $J(U^{(3)}:m,m';n,n';a,a';b,b')$ is as in (2.21).

Pillai and Young [41] have obtained the c.d.f. of $U^{(3)}$ for small values of m, i.e., for m=0(1,5). Their expression is so complex that it is not possible to write it down explicitly since for their c.d.f. they need tables of constant coefficients and of values of the integrals in the determinants separately for each values of m, a and b they used. Note also that there is an error in formula (5.5) of Pillai and Young [41] for the expression of the distribution of $U^{(3)}$ in the first summation. The right one should be $\ell+1$. The c.d.f. of $U^{(3)}$ obtained here, as we can see from (5.6) above, holds for all non-negative integers m and its expression is simpler.

Now, let us integrate $D_1^*(u;i), D_2^*(u;i), D_3^*(u;i)$ and $D_4^*(u)$ with respect to u. Using the results and the expressions in (2.21) and (5.5) we have

$$(5.7)\quad J_1(u;i) = (n+m-i+3)^{-1}\cdot[Q(m-i,q+1,m-i)J(u:m-i,0;n,n;6,3;2,1) - \sum_{j=0}^{m-i} Q(m-i,q+1,j)J(u:m-i,m-i-j;n,n;6,6;2,2)],$$

$$(5.8)\quad J_2(u;i) = (n+m-i+2)^{-1}(n+m-i+3)^{-1}$$

$$\cdot[(n+m-i+2)\{J(u:m-i,m-i;n,n;7,4;2,2)-J(u:m-i,m-i;n,n;7,5;2,2)\}$$

$$+(n+2)Q(m-i,q,m-i)\{J(u:m-i,0;n,n;7,2;2,1)+J(u:m-i,0;n,n;7,3;2,1)\}$$

$$-(n+2)\sum_{j=0}^{m-i} Q(m-i,q,j)J(u:m-i,m-i-j;n,n;7,4;2,2)$$

$$-\sum_{j=0}^{m-i}(n+m-i-j+2)Q(m-i,q,j)J(u:m-i,m-i-j;n,n;7,5;2,2)$$

$$+\sum_{j=0}^{m-i}(m-i-j)Q(m-i,q,j)\{J(u:m-i,m-1-i-j;n,n;7,5;2,2)$$

$$-J(u:m-i,m-1-i-j;n,n;7,6;2,2)\}],$$

$$(5.9)\quad J_3(u;i) = (n+m-i+2)^{-1}[Q(m-i,q,m-i)J(u:m-i,0;n,n;8,2;2,1) - \sum_{j=0}^{m-i} Q(m-i,q,j)J(u:m-i,m-i-j;n,n;8,4;2,2)]$$

and finally, after using (5.5) we also have

$$(5.10)\quad J_4(u) = (n+2)^{-1}(n+3)^{-1}(2n+5)^{-1}[-(2n+1)(n+1)^{-1}(2n+3)^{-1}$$

$$-(n+1)^{-1}g^*(u:0,n;1,1)+2(2n+5)(2n+3)^{-1}g^*(u:0,n;3,2)$$

$$-2g^*(u:0,n;4,2)+(n+2)J(u:0,0;n,n;5,3;2,1)].$$

Having the above expressions, we can perform the integration on $f(U^{(4)})$ in (4.13) to obtain the following result:

Theorem 5.2: The exact null distribution function of $U^{(4)}$ is given by

(5.11) $F(U^{(4)}) = (m+n+4)^{-1}C(4,m,n)[Q(m,q,m)J_4(U^{(4)})$

$$- \sum_{i=0}^{m} Q(m,q,i)\{J_1(U^{(4)};i) - J_2(U^{(4)};i) + J_3(U^{(4)};i)\}],$$

for $0 < U^{(4)} < \infty$ and zero otherwise, where $J_1(U^{(4)};i)$, $J_2(U^{(4)};i)$, $J_3(U^{(4)};i)$ and $J_4(U^{(4)})$ are as in (5.7), (5.8), (5.9) and (5.10) respectively, $Q(m,q,i)$ is as in (3.3) with $q=m+n+5$.

As in the case of $U^{(3)}$, Pillai and Young [41] have provided the c.d.f. of $U^{(4)}$ with necessary tables of constants and integrals for m=0(1)2. The c.d.f. of $U^{(4)}$ obtained here has simpler expression and valid for all non-negative integers m.

Remark: The density and c.d.f. of $U^{(p)}$ become more complicated as p becomes larger. Even for p=5, convolutions of three independent beta type variables are involved and the expressions become very cumbersome.

CHAPTER IV

THE EXACT NON-CENTRAL DISTRIBUTION OF HOTELLING'S TRACE

1. Introduction

Consider the three basic hypotheses (A), (B) and (C) as stated in Chapter II (see Section 1, 6 and 7). Using the matrix variates $\underset{\sim}{U}$ and $\underset{\sim}{V}$ stated in Section 1, $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ in Section 6, Chapter II, under certain assumption on the population parameter, the distributions of the four criteria (see Section 1 and 6 for the definitions) have been obtained. A few inferences have been drawn there on the basis of the tabulations of lower/upper tail probabilities of the criteria. In particular, the distribution and inferences concerning the criterion $U^{(p)}$ for p=2 were considered.

In this chapter, starting from Pillai's distribution [31 ,32] as stated in (1.1) Chapter II the general form of the distribution of $U^{(p)}$ for p=3 is obtained using the inverse Laplace transform and the determinant of Pillai [28]. Numerical calculations of upper tail probabilities of $U^{(3)}$ are tabulated only in view of test of (A) assuming the mean vectors zero, of (B) in the case of a common covariance matrix and of (C). A few inferences are drawn on the basis of the tabulations.

2. Preliminaries

In obtaining the distribution of $U^{(3)}$ we will use some mathematical results stated in the previous chapter. In this connection we will introduce the following notation: We will write, for example, I(2.9), to denote formula (2.9) of Chapter I. In addition, we also will need some results on Vandermonde determinant due to Pillai [28] which are stated below.

Lemma 2.1: Let $V(g_p,g_{p-1},\dots,g_1);g_j \geq 0$, $j=1,2,\dots,p$, denote the determinant

$$(2.1)\quad V(g_p,\dots,g_1) = \begin{vmatrix} x_p^{g_p} & x_p^{g_{p-1}} & \dots & x_p^{g_1} \\ \vdots & & & \vdots \\ x_1^{g_p} & x_1^{g_{p-1}} & \dots & x_1^{g_1} \end{vmatrix} .$$

If $d_r, r \leq p$, denotes the rth elementary symmetric function in p x's, then

Result (i).

$$(2.2)\qquad d_r\, V(g_p,\dots,g_1) = \Sigma' V(g_p',\dots,g_1'),$$

where $g_j' = g_j + \delta$, $j=1,2,\dots,p$; $\delta = 0,1$ and Σ' denotes the sum over the $\binom{p}{r}$ combinations of p g's taken r at a time for which r indices $g_j' = g_j+1$ such that $\delta=1$ while for other indices $g_j'=g_j$ such that $\delta=0$.

Result (ii).

$$(2.3)\qquad d_r d_h V(g_p,\dots,g_1) = \Sigma'' V(g_p'',\dots,g_1'').$$

where $h \leq p$, $g_j''=g_j'+\delta$, $j=1,2,\dots,p$; $\delta=0,1$ and Σ'' denotes summation over $\binom{p}{r}\binom{p}{h}$ terms obtained by taking h at a time of the p g's in each V in Σ' in (2.2) for which h indices $g_j''=g_j'+1$ while for other indices $g_j''=g_j'$.

Result (iii).

$(d_r)^k(d_h)^\ell\ V(g_p,\ldots,g_1)$, k, $\ell \geq 0$, can be expressed as a sum of $\binom{p}{r}^k\binom{p}{h}^\ell$ determinants obtained by performing on $V(g_p,\ldots,g_1)$ in any order (i) k times and (ii) ℓ times with $r=h$.

However, if at least two of the indices in any determinant are equal, the corresponding term in the summation vanishes.

3. Non-Central Distribution of $U^{(3)}$ for test (A) and (B)

In II(1.1), let us put $p=3$. Then we have the joint density of r_1, r_2, r_3, $0 < r_1 < r_2 < r_3 < \infty$, as

$$(3.1)\qquad C(3,m,n)e^{-\mathrm{tr}\,\underset{\sim}{\Omega}_3}|\underset{\sim}{\Lambda}_3|^{-\frac{1}{2}f_1}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\frac{1}{2}\nu)_\kappa}{k!}$$

$$\cdot|\underset{\sim}{R}_3|^m|\underset{\sim}{I}_3+\underset{\sim}{R}_3|^{-\frac{1}{2}\nu}\prod_{i>j}(r_i-r_j)C_\kappa[\underset{\sim}{R}_3(\underset{\sim}{I}_3+\underset{\sim}{R}_3)^{-1}].F_3,$$

where $\underset{\sim}{R}_3$ denotes a matrix of order 3 and the meaning of the other symbols are as explained in Section 1, Chapter II.

Now let $\ell_i = r_i/(1+r_i)$, $i=1,2,3$, $0 < \ell_1 < \ell_2 < \ell_3 < 1$. Then from (3.1) we easily can see that the term corresponding to the partition κ of k containing the variables ℓ_i, $i=1,2,3$, is

$$(3.2)\qquad \prod_{i=1}^{3}\{\ell_i^m(1-\ell_i)^n\}\prod_{i>j}(\ell_i-\ell_j)C_\kappa\begin{pmatrix}\ell_1 & 0 & 0\\ 0 & \ell_2 & 0\\ 0 & 0 & \ell_3\end{pmatrix}.$$

Proceeding as in Section 2 of Chapter II, (3.2) becomes

$$(3.3)\qquad \sum_{r+2s+3t=k} b_\kappa(r,s,t)\prod_{i=1}^{3}\{\ell_i^m(1-\ell_i)^n\}\prod_{i>j}(\ell_i-\ell_j)a_1^r a_2^s a_3^t,$$

where a_i (i=1,2,3) is the ith elementary symmetric function (e.s.f.) in the ℓ_i's and $b_\kappa(r,s,t)$ are constants whose values up to k=6 are given in appendix C. For ease of writing, we will consider only (3.3) and take the Laplace transform of $U^{(3)} = \sum_{i=1}^{3} (\ell_i/(1-\ell_i))$. After making a change of variable $x_i = 1-\ell_i$, i=1,2,3, then for $0 < x_1 < x_2 < x_3 < 1$ we have

$$(3.4) \qquad \sum_{r+2s+3t=k} b_\kappa(r,s,t)e^{3z} \iiint_{\mathscr{R}} e^{-z\Sigma x_i^{-1}} \prod_{i=1}^{3} \{(1-x_i)^m x_i{}^n\} \cdot \prod_{i>j} (x_i-x_j) C_1^r C_2^s C_3^t \prod_{i=1}^{3} dx_i ,$$

where $\mathscr{R} = \{(x_1,x_2,x_3) | 0 < x_1 < x_2 < x_3 < 1\}$, $C_1 = 3-d_1$, $C_2 = 3-2d_1 + d_2$ and $C_3 = 1-d_1+d_2-d_3$ and where d_i (i=1,2,3) is the ith e.s.f. in the x_i's. Now expand the factor $C_1^r C_2^s C_3^t$ in (3.4) to get the expression

$$(3.5) \qquad \sum_{r+2s+3t=k} b_\kappa(r,s,t)e^{3z} \iiint_{\mathscr{R}} e^{-z\Sigma x_i^{-1}} \prod_{i=1}^{3} \{(1-x_i)^m x_i^n\} \prod_{i>j} (x_i-x_j) \cdot c(r,s,t) d_1^\alpha d_2^\beta d_3^\gamma \prod_{i=1}^{3} dx_i ,$$

where in the above expression

$$(3.6) \qquad c(r,s,t) = \sum_{i_1=0}^{r} \sum_{i_2=0}^{s} \sum_{i_3=0}^{s-i_2} \sum_{i_4=0}^{t} \sum_{i_5=0}^{t-i_4} \sum_{i_6=0}^{b} (-1)^{\alpha+\gamma} 2^{i_3} 3^a \cdot \binom{r}{i} \binom{s}{i_2 i_3} \binom{t}{i_4 i_5 i_6},$$

where $\alpha = i_1+i_3+i_6$, $\beta = i_2+i_5$, $\gamma = i_4$, $a = r+s-i_1-i_2-i_3$ and $b = t-i_4-i_5$. But now we can apply Lemma 2.1 to the expression

$d_1^\alpha d_2^\beta d_3^\gamma \prod_{i>j}(x_i-x_j)$, so that we, therefore, are able to express the triple integral in (3.5) in terms of the determinant of the form III(2.7), i.e. of the form $D(n:q_3,m;q_2,m;q_1,m;z)$. For example, for k=0, i.e. the first term of the series in (3.5), the triple integral in (3.5) becomes $D(n:2,m;1,m;0,m;z)$. For k=1, we have that $c(r,s,t)d_1^\alpha d_2^\beta d_3^\gamma = 3-d_1$ so that after using Lemma 2.1, the triple integral in (3.5) becomes $3D(n:2,m;1,m;0,m;z)-D(n:3,m;1,m;0,m;z)$. The use of Lemma 2.1 can be carried out for each k to find the distribution of $U^{(3)}$. To obtain the general formulation, let us set

$$(3.7)\qquad \psi_{ab}(z) = \iiint_{\mathscr{R}} e^{-z\Sigma x_i^{-1}} \prod_{i=1}^{3}\{(1-x_i)^m x_i^n\} \prod_{i>j}(x_i-x_j) \cdot c(r,s,t)d_1^\alpha d_2^\beta d_3^\gamma \prod_{i=1}^{3} dx_i,$$

where the indices a and b are to be chosen appropriately according to the value of k. Then the term corresponding to the partition κ of k can be written as

$$(3.8)\qquad \sum_{r+2s+3t=k} b_\kappa(r,s,t)e^{3z}\,\psi_{ab}(z).$$

Following the method in Chapter III, let us take $\psi^*_{ab}(z)$ to be the inverse Laplace transform of $e^{3z}\psi_{ab}(z)$. Then it is clear that $\psi^*_{ab}(z)$ is a linear combination of $D^*(n:q_3,m;q_2,m;q_1,m;z)$, where $D^*(n:q_3,m;q_2,m;q_1,m;z)$ is the inverse Laplace transform of $e^{3z}D(n:q_3,m;q_2,m;q_1,m;z)$. The density of $U^{(3)}$, therefore can be written in the form

$$(3.9) \quad f(U^{(3)}) = C(3,m,n)e^{-tr\underset{\sim}{\Omega}_3}|\underset{\sim}{\Lambda}_3|^{-\frac{1}{2}f_1}$$

$$\cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}\nu)}{k!} \sum_{r+2s+3t=k} b_{\kappa}(r,s,t)\psi^*_{ab}(U^{(3)}).F_3,$$

where the meaning of all symbols are now obvious from the previous discussion. The c.d.f. of $U^{(3)}$ can be obtained from (3.9) by performing the integration on each $D^*(n:q_3,m;q_2,m;q_1,m;x)$ with respect to x with the limits from 0 to $U^{(3)}$. This c.d.f. is expressible in terms of g* and J functions whose values are stated in Section 2, Chapter III.

Remark: For $\underset{\sim}{\Omega}_3=\underset{\sim}{0}$ and $\underset{\sim}{\Lambda}_3=\underset{\sim}{I}_3$, (3.9) reduces to III(4.8).

Using the theory explained above, let us find the non-central c.d.f. of $U^{(3)}$ up to sixth degree. Expand the series in (3.9) and carry out the necessary calculations. The result is, for the density of $U^{(3)}$

$$(3.10) \quad f(U^{(3)}) = C(3,m,n)e^{-b_1} a_3^{-\frac{1}{2}f}$$

$$\cdot [\psi^*(U^{(3)}) + \sum_{i=1}^{6} \sum_{j=1}^{7} A'_{ij}\psi^*_{ij}(U^{(3)})],$$

where b_1 is the first esf in the latent roots of $\underset{\sim}{\Omega}$, a_3 is third esf of those of $\underset{\sim}{\Lambda}$, f=2m+4, the A'_{ij}'s, ψ^* and ψ^*_{ij}'s are given in the appendices D and E respectively. Integrating each ψ^*_{ij} from 0 to $U^{(3)}$ we then have the non-central c.d.f. of $U^{(3)}$ as

$$(3.11) \quad F(U^{(3)}) = C(3,m,n)e^{-b_1} a_3^{-\frac{1}{2}f}$$

$$\cdot [\psi^{**}(U^{(3)}) + \sum_{i=1}^{6} \sum_{j=1}^{7} A'_{ij}\psi^{**}_{ij}(U^{(3)})],$$

where the ψ^{**}_{ij}'s are given in the appendix F.

4. Non Central Distribution of $U^{(3)}$ for Test (C)

Let p=3 in II(6.1). Then the joint density of r_1^2, r_2^2 and r_3^2, $0 < r_1^2 < r_2^2 < r_3^2 < 1$ is

$$(4.1)\quad C(3,m,n)\,|\underset{\sim}{I}_3-\underset{\sim}{P}_3^2|^{\frac{1}{2}\nu} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{(\frac{1}{2}\nu)_\kappa(\frac{1}{2}\nu)_\kappa C_\kappa(\underset{\sim}{P}_3^2)}{(\frac{1}{2}f_1)_\kappa k!\,C_\kappa(\underset{\sim}{I}_3)}$$

$$\cdot \prod_{i=1}^{3}\{(r_i^2)^m(1-r_i^2)^n\} \prod_{i>j}(r_i^2-r_j^2)\,C_\kappa\begin{pmatrix} r_1^2 & 0 & 0 \\ 0 & r_2^2 & 0 \\ 0 & 0 & r_3^2 \end{pmatrix}.$$

where $\underset{\sim}{A}_3$ denotes the matrix of order 3.

Using the method in Section 3, we obtain the density function of $U^{(3)}$ in the non-central case as:

$$(4.2)\quad f(U^{(3)}) = C(3,m,n)\,|\underset{\sim}{I}_3-\underset{\sim}{P}_3^2|^{\frac{1}{2}\nu}$$

$$\cdot \sum_{k=0}^{\infty}\sum_{\kappa} \frac{(\frac{1}{2}\nu)_\kappa(\frac{1}{2}\nu)_\kappa C_\kappa(\underset{\sim}{P}_3^2)}{(\frac{1}{2}f_1)_\kappa k!\,C_\kappa(\underset{\sim}{I}_3)} \sum_{r+2s+3t=k} b_\kappa(r,s,t)\,\psi_{ab}^{*}(U^{(3)}),$$

where now the meaning of all symbols are obvious from Section 3.

As in the previous section, the c.d.f. of $U^{(3)}$ is readily obtained by performing the integration on $D^*(n{:}q_3,m;q_2,m;q_1,m;x)$ and this c.d.f. is expressible in terms of g* and J functions as discussed in Chapter III. Using (4.2) we can obtain the c.d.f. of $U^{(3)}$ up to the sixth degree in a similar way as in Section 3. However this also can be obtained easier from (3.11) by making necessary changes. The result is

$$(4.3)\quad F(U^{(3)}) = C(3,m,n)\{(1-\rho_1^2)(1-\rho_2^2)(1-\rho_3^2)\}^{\frac{1}{2}\nu}$$

$$\cdot\ [\psi^{**}(U^{(3)}) + \sum_{i=1}^{6}\sum_{j=1}^{7} A''_{ij}\,\psi_{ij}^{**}(U^{(3)})],$$

holds for $0 < U^{(3)} < \infty$ and the coefficients A''_{ij}'s are obtained from A'_{ij}'s in appendix D by multiplying A'_{ij} by 2^{-i}, substituting $a_1=a_2=3$, $a_3=1$ there, squaring each factor containing ν and defining $b_1 = \rho_1^2+\rho_2^2+\rho_3^2$, $b_2 = \rho_1^2\rho_2^2+\rho_1^2\rho_3^2+\rho_2^2\rho_3^2$ and $b_3 = (\rho_1\rho_2\rho_3)^2$. The ψ^{**} and ψ^{**}_{ij}'s are the same as in the appendix F.

5. Power Function for Test (B)

Let us now first consider the hypothesis (B) for two 3-variate normal populations by assuming a common covariance matrix. In this case the non-central c.d.f. of $U^{(3)}$ is obtained from (3.11) by making substitution $\lambda_1=\lambda_2=\lambda_3=1$. For illustration, this c.d.f. is used to compute the powers of test (B) based on $U^{(3)}$ for various values of w_1, w_2, w_3 and m=0, n=5, 15, 40 and $\alpha = 0.05$. The upper percentage points for these calculations were obtained from III(5.6). The tabulations are presented in Table IV(1) for larger deviations of parameters and in Table IV(2) for small deviations.

From the tabulations we observe that

1. With respect to individual population characteristic roots, $U^{(3)}$ possesses monotonicity of power.
2. For $w_1+w_2+w_3$ = constant, it seems that the power of $U^{(3)}$ decreases as the three roots tend to be equal.

Comparison with the individual roots criteria may be made. It has been shown by Pillai and Dotson [35] that in general the largest root appears to be a more powerful test than the other individual characteristic roots. Therefore only comparison with the largest root criterion is considered here. Powers of test (B) based on the largest root were taken from Pillai and Dotson [35]. For small deviations of parameters, it appears that the criterion $U^{(3)}$ seems to have more power than the largest root. For larger deviations however, Table IV(1) reveals a few exceptions. In the two-roots case, Pillai and Jayachandran [37] have also shown this property.

Table IV (1)

Powers of $U^{(3)}$ for testing $(w_1,w_2,w_3)=(0,0,0)$ against different simple alternatives and for larger deviations, $\alpha = 0.05$

w_1	w_2	w_3	$U^{(3)}$	Largest Root	$U^{(3)}$	Largest Root
			m=0,n=5		m=0,n=40	
0	0	5	0.157	0.149	0.229	0.229
0	0	6	0.185	0.175	0.275	0.278
2	2	2	0.180	0.175	0.271	0.238
0	0	8	0.243	0.233	0.372	0.381
2	2	4	0.243	0.256	0.372	0.339
0	0	10	0.306	0.297	0.470	0.485

Table IV (2)

Powers of $U^{(3)}$ test for testing $(w_1,w_2,w_3) = (0,0,0)$ against different simple alternatives, $\alpha = 0.05$, $m = 0$

w_1	w_2	w_3	$U^{(3)}$	Largest Root
		n = 5		
0	0	0.003	0.0500474	0.0500411
0	0.0015	0.0015	0.0500474	0.0500411
0	0	0.111	0.0517683	0.0515351
0.001	0.01	0.1	0.0517681	0.0515327
0.037	0.037	0.037	0.0517673	0.0515261
0	0	0.825	0.063878	0.062179
0.125	0.250	0.50	0.064726	0.062493
0.275	0.275	0.275	0.063828	0.061680
0	0	1	0.067037	0.064989
0	0	2	0.0865	0.0825
0	1	1	0.0864	0.0803
0	0	3	0.108	0.103
1	1	1	0.106	0.097
		n = 15		
0	0	0.003	0.0500634	0.0500553
0	0.0015	0.0015	0.0500634	0.0500552
0	0	0.111	0.0523743	0.0520740
0.001	0.01	0.1	0.0523742	0.0520693
0.037	0.037	0.037	0.0523736	0.0520567
0	0	0.825	0.068949	0.066896
0.125	0.250	0.50	0.070163	0.067096
0.275	0.275	0.275	0.068917	0.065929
0	0	1	0.073349	0.070916
0	0	2	0.1010	0.0967
0	1	1	0.1009	0.0924
0	0	3	0.132	0.127
1	1	1	0.130	0.115
		n = 40		
0	0	0.003	0.0500711	0.0500629
0	0.0015	0.0015	0.0500711	0.0500628
0	0	0.111	0.0526629	0.0523641
0.001	0.01	0.1	0.0526628	0.0523579
0.037	0.037	0.037	0.0526626	0.0523414
0	0	0.825	0.071409	0.069505
0.125	0.250	0.50	0.072810	0.069607
0.275	0.275	0.275	0.071393	0.068239
0	0	1	0.076422	0.074213
0	0	2	0.1081	0.1048
0	1	1	0.1080	0.0992
0	0	3	0.144	0.141
1	1	1	0.143	0.126

6. Power Function for Test (A)

Now, we consider the hypothesis (A). The non-central distribution of $U^{(3)}$ in this case, again can be obtained from (3.11) by making substitution $w_1=w_2=w_3=0$. This c.d.f. is used to compute powers of test (A) based on $U^{(3)}$ for various values of $\lambda_1,\lambda_2,\lambda_3$ and m=0, n=5, 15, 40 and $\alpha=0.05$. The powers are tabulated in Table IV(3) together with those based on the largest root (taken from [35]) for comparison. Similar conclusions may be obtained as in the Section 5.

Table IV (3)

Powers of $U^{(3)}$ test for testing $(\lambda_1,\lambda_2,\lambda_3)=(1,1,1)$ against different simple alternatives, $\alpha=0.05$, m=0

λ_1	λ_2	λ_3	$U^{(3)}$	Largest Root
		n = 5		
1	1	1.001	0.050063	0.050054
1	1	1.15	0.0506015	0.05891
1	1.05	1.1	0.06004	0.05868
1.05	1.05	1.05	0.05998	0.05856
1	1	1.5	0.0886	0.0848
1	1.25	1.25	0.0875	0.0821
1	1	2	0.147	0.132
		n = 15		
1	1	1.001	0.050085	0.050073
1	1	1.15	0.063873	0.06239
1	1.05	1.1	0.06369	0.06194
1.05	1.05	1.05	0.06360	0.06171
1	1	1.5	0.1045	0.1010
1	1.25	1.25	0.1027	0.0958
1	1	2	0.176	0.172
		n = 40		
1	1	1.001	0.050095	0.050083
1	1	1.15	0.065683	0.06432
1	1.05	1.1	0.065467	0.06373
1.05	1.05	1.05	0.065358	0.06344
1	1	1.5	0.1123	0.1101
1	1.25	1.25	0.1103	0.1035
1	1	2	0.194	0.194

7. Power Function for Test (C)

The non-central c.d.f. of $U^{(3)}$ for testing hypothesis (C), i.e. for testing the independence between two sets of multivariate normal populations, is discussed in Section 4 and has the form as described in (4.3). Powers of $U^{(3)}$ for this case are tabulated in Table IV(4) for larger deviations and in Table IV(5) for small deviations of parameters. The powers for the largest root are also tabulated in the last column of Table IV(5). Similar conclusions may be made.

Table IV (4)

Powers of $U^{(3)}$ for testing $(\rho_1^*, \rho_2, \rho_3) = (0,0,0)$ against different simple alternatives and for larger deviations, $\alpha=0.05$, $m=0$

ρ_1^2	ρ_2^2	ρ_3^2	n=5	n=15
0	0	0.1	0.087	0.177
0.01	0.01	0.08	0.085	0.173
0.005	0.015	0.08	0.085	0.173
0	0	0.15	0.113	0.275
0.05	0.05	0.05	0.104	0.242
0	0	0.2	0.146	0.393
0	0	0.25	0.187	0.521

Table IV (5)

Powers of $U^{(3)}$ test for testing $(\rho_1,\rho_2,\rho_3)=(0,0,0)$ against different simple alternative hypotheses, $\alpha=0.05$, m=0.

ρ_1^2	ρ_2^2	ρ_3^2	$U^{(3)}$	Largest Root
			n = 5	
0	0	0.0001	0.0500284	0.0500246
0	0.00005	0.00005	0.0500284	0.0500246
0	0	0.00375	0.0510763	0.0509339
0	0.00125	0.00250	0.0510740	0.0509299
0.00125	0.00125	0.00125	0.0510728	0.0509279
0	0	0.012	0.0535192	0.0530614
0.002	0.002	0.008	0.0534916	0.0530146
0.002	0.004	0.006	0.0534855	0.0530043
0.004	0.004	0.004	0.0534824	0.0529991
0.000001	0.0001	0.01	0.0529469	0.0525613
0.01	0.01	0.08	0.0854	0.08029
			n = 15	
0	0	0.0001	0.0500804	0.0500700
0	0.00005	0.00005	0.0500804	0.0500700
0	0	0.00375	0.0530723	0.0526869
0	0.00125	0.00250	0.0530652	0.0526625
0.00125	0.00125	0.00125	0.0530617	0.0526503
0	0	0.012	0.0602569	0.0590592
0.002	0.002	0.008	0.0601721	0.0587702
0.002	0.004	0.006	0.0601533	0.0587062
0.004	0.004	0.004	0.060144	0.0586745
0.000001	0.0001	0.01	0.058548	0.0575272
0.01	0.01	0.08	0.173	0.1550
			n = 40	
0	0	0.0001	0.0502086	0.0501845
0	0.00005	0.00005	0.0502086	0.0501844
0	0	0.00375	0.0581553	0.0573030
0	0.000125	0.00250	0.0581362	0.0571531
0.00125	0.00125	0.00125	0.0581267	0.0570788
0	0	0.012	0.0785062	0.076201
0.002	0.002	0.008	0.0782652	0.074423
0.002	0.004	0.006	0.0782098	0.074031
0.004	0.004	0.004	0.07818	0.073838
0.000001	0.0001	0.01	0.073521	0.071453
0.01	0.01	0.08	0.478	0.04119

CHAPTER V

THE DISTRIBUTION OF TRACE $S_1S_2^{-1}$ UNDER EQUALITY OF COVARIANCE MATRICES

1. Introduction

Let us return to the two independent matrix variates $S_1(p\times p)$ and $S_2(p\times p)$ but now S_1 having non-central Wishart $W(p,n_1,\Sigma,\Omega)$ and S_2, central Wishart $W(p,n_2,\Sigma,0)$. Consider also the statistic $T = U^{(p)} = \mathrm{tr}\ S_1S_2^{-1}$ as defined in the previous chapters. For $n_1=1$, Hotelling [12] derived the distribution of T (here T is the generalization of Student's t) and for p=2, in [14] he obtained the null, i.e. if $\Omega=0$, distribution of T as stated by Constantine [4] in the following form

$$(1.1)\qquad C_2\ \frac{(T/2)^{n_1-1}}{(1+T/2)^n}\ {}_2F_1(1,\tfrac{1}{2}\ n;\tfrac{1}{2}(n_1+1);\ \frac{(T/2)^2}{(1+T/2)^2}),$$

where

$$(1.2)\qquad C_2 = \Gamma(n-1)/\{2\Gamma(n_1)\Gamma(n_2-1)\},$$

$n = n_1+n_2$ and ${}_2F_1$ is the Gaussian hypergeometric function. In Chapter II the exact non-central distribution of T for p=2 was obtained and used for robustness studies of various tests. The exact null distribution of T for p=3 and p=4 were obtained in Chapter III and its non-central distribution for p=3 was derived in Chapter IV including the necessary tables of constants and integrals for m=0 to illustrate its usefulness. In Section 3 of Chapter I, see also Constantine [4], the density of T has been obtained and has the form (put $\underset{\sim}{\Lambda} = \underset{\sim}{I}$ and $\lambda = 1$ there)

$$(1.3) \qquad C_p T^{\frac{1}{2} pn_1 - 1} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(-T)^k (\frac{1}{2} n)_\kappa}{(\frac{1}{2} pn_1)_\kappa k!} L_\kappa^\gamma(\underset{\sim}{\Omega}), \quad |T| < 1,$$

where

$$(1.4) \qquad C_p = e^{-\mathrm{tr}\underset{\sim}{\Omega}} \Gamma_p(\tfrac{1}{2} n) / \{\Gamma(\tfrac{1}{2} pn_1) \Gamma_p(\tfrac{1}{2} n_2)\},$$

$n = n_1 + n_2$ and $\gamma = (n_1 - p - 1)/2$. (For the meaning of the symbols in (1.3) and (1.4), see Section 2 Chapter I).

Even though (1.3) is stated to be convergent for $|T| < 1$, numerical calculations using terms up to k=10 did not exhibit adequate convergence in that range. Therefore, Pillai [32] suggested a form of the density of T which will be given in the next section which reduces to the exact case (1.1) for p=2. Further investigation on this density is carried out in this chapter for larger values of p to see its usefulness and power tabulations of the statistic T are obtained. Based on these tabulations a few inferences are drawn.

2. The Distribution of $T = \mathrm{tr}\, \underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$

The series in (1.3) can be written as

$$(2.1) \qquad C_p \, T^{\frac{1}{2} pn_1 - 1} \sum_{k=0}^{\infty} (-1)^k D_k (T/p)^k,$$

where

$$(2.2) \qquad D_k = [p^k / \{(\tfrac{1}{2} pn_1)_\kappa k!\}] \sum_{\kappa} (\tfrac{1}{2} n)_\kappa L_\kappa^\gamma(\underset{\sim}{\Omega}).$$

Writing only first few terms of (2.1), we have

$$(2.3)\qquad C_p T^{\frac{1}{2}pn_1-1}\{1-\frac{1}{2}np(1-(2a_1/pn_1))(T/p)+\ldots\},$$

where a_1 is the first esf in the latent roots of $\underset{\sim}{\Omega}$. In view of the forms (2.1) and (2.3) and some considerations of the Laplace transform of T (Constantine [4]), Pillai suggested the form [32]

$$(2.4)\qquad C_p \frac{T^{\frac{1}{2}pn_1-1}}{(1+T/p)^{\frac{1}{2}np}} \sum_{k=0}^{\infty}(-1)^k E_k \frac{(T/p)^k}{(1+T/p)^k}$$

for the density of T. Now expand (2.4) in terms of (T/p) and compare with (2.1). We obtain the coefficient E_k expressed in terms of D_k. After simplifications, the E_k has the form

$$(2.5)\qquad E_k = D_k - \sum_{r=0}^{k-1}[\{\prod_{j=r}^{k-1}(\frac{1}{2}np+j)/(k-r)!\}E_r],$$

where D_k is as in (2.2) and $E_0 = 1$. Upon integrating T form 0 to u, the c.d.f. of T can be expressed in terms of an incomplete beta series and we have the following:

Let $\underset{\sim}{S}_1, \underset{\sim}{S}_2$ and T be as stated in the Introduction. As an improved form of the distribution of T, is as below:

$$(2.6)\qquad P(T \leq u) = C_p \cdot p^{\frac{1}{2}pn_1} \sum_{k=0}^{\infty}(-1)^k E_k B_w(\frac{1}{2}pn_1+k, \frac{1}{2}pn_2),$$

where $w = u/(p+u)$, C_p and E_k are as in (1.4) and (2.5) respectively and $B_y(a,b)$ denotes the incomplete beta function. The null distribution of T still has the form (2.6) but now D_k has the expression

$$(2.7)\qquad D_k = [p^k/\{(\frac{1}{2}pn_1)_\kappa k!\}]\sum_\kappa (\frac{1}{2}n_1)_\kappa (\frac{1}{2}n)_\kappa C_\kappa(\underset{\sim}{I}),$$

and the coefficient $E_1 = 0$.

For p=2 and $\underset{\sim}{\Omega}=\underset{\sim}{0}$, formula (2.4) with E_k in (2.5) reduces to the form given in (1.1). For higher values of p the zonal polynomials involved in D_k have to be computed in order to make use of (2.6). With the help of the tables of zonal polynomials James [17] and the tables prepared by James and Parkhurst (private communication to Professor Pillai), the coefficients D_k for the central case were calculated using terms up to k=10 and those for the non-central case up to k=6. These coefficients are given in appendices G and H respectively. Numerical investigations are carried out in the next sections.

3. Tabulations of Percentage Points and Comparisons

Some calculations were carried out to illustrate the usefulness of the c.d.f. of T in (2.6). The total probabilities (sum of the terms used in (2.6)) in the central case and upper percentage points were calculated using terms upto either k=8 or 10. Table V(1) gives these upper percentage points for p=3 and 4 and α=0.05 and 0.01, and the exact percentage points taken from Pillai and Young [41] for comparisons. It may be seen from this table that (2.6) is very close to the exact. Table V(2) contains the values of upper percentage points of T for p=5 and p=6 with α=0.05 and 0.01. It may be mentioned that in these calculations, first the total probabilities were computed and then the pairs of n_1 and n_2 were determined that give the total probabilities reasonably good. For instance, for p=5, the pair n_1=6 and n_2=50 gives total probability 0.99834, and n_1=8 and n_2=60 give 0.99509. Total probabilities converge to unity as n_2 becomes larger. For $p=6$ and the pair $n_1=6$, $n_2=60$ the total probability is 0.99588 and that for $n_1=8$, $n_2=80$ is 0.99507. Then for those pairs of n_1 and n_2 the upper percentage points were calculated.

Table V (1)

Comparison of percentage points using terms up to k=8 or k=10 in (2.6) and exact points

			Upper 5% Points		Upper 1% Points	
p=3		n_2	Eq.(2.6)	Exact	Eq.(2.6)	Exact
n_1=4	(k=8)	14	2.4672	2.4959	3.6256	3.6581
		24	1.1504	1.1540	1.5238	1.5581
	(k=10)	34	0.74668	0.74702	0.97765	0.98145
		44	0.55169	0.55174	0.71450	0.71518
		64	0.36208	0.36208	0.46310	0.46316
		84	0.26939	0.26939	0.34234	0.34235
		104	0.21447	0.21447	0.27149	0.27150
		124	0.17814	0.17814	0.22493	0.22493
		164	0.13306	0.13306	0.16747	0.16747
		204	0.10618	0.10618	0.13340	0.13340
n_1=10	(k=8)	24	2.4317	2.4640	3.1839	3.1187
	(k=10)	34	1.5563	1.5827	1.9558	1.9452
		44	1.1637	1.1642	0.14104	1.4103
		64	0.76063	0.76070	0.90841	0.90860
		84	0.56470	0.56471	0.66981	0.66986
		104	0.44897	0.44897	0.53038	0.53039
		124	0.37259	0.37259	0.43896	0.43896
		164	0.27798	0.27798	0.32640	0.32640
		204	0.22168	0.22168	0.25977	0.25977
p=4						
n_1=5	(k=8)	25	1.6994	1.7377	2.1515	2.2474
		35	1.1136	1.1217	1.3806	1.4114
	(k=10)	45	0.82538	0.82732	1.0173	1.0272
		65	0.54205	0.54221	0.66315	0.66455
		85	0.40312	0.40315	0.49077	0.49100
		105	0.32083	0.32083	0.38924	0.38929
		125	0.26643	0.26643	0.32248	0.32248
		165	0.19894	0.19894	0.24006	0.24006
		205	0.15873	0.15873	0.19120	0.19120
n_1=9	(k=10)	35	1.8099	1.8363	2.1271	2.2222
		45	1.3443	1.3517	1.5818	1.6136
		65	0.88314	0.88405	1.0369	1.0416
		85	0.65646	0.65663	0.76774	0.76869
		105	0.52220	0.52224	0.60887	0.60905
		125	0.43348	0.43349	0.50422	0.50430
		165	0.32352	0.32352	0.37521	0.37521
		205	0.25805	0.25805	0.29874	0.29874

Table V (2)

Some percentage points calculated from (2.6)

for p=5 and p=6

n_2	$p=5, n_1=6$ $\alpha=0.05$	$\alpha=0.01$	n_2	$p=6, n_1=6$ $\alpha=0.05$	$\alpha=0.01$
50	1.04	1.26	60	1.00	1.21
60	0.842	1.005	80	0.72	0.85
80	0.6089	0.7209	100	0.561	0.655
100	0.4766	0.5617	120	0.4601	0.5347
120	0.39142	0.4604	150	0.3622	0.4201
160	0.288327	0.338188	180	0.2986	0.3459
200	0.228208	0.267183	200	0.26729	0.30934
240	0.188833	0.220816	240	0.22097	0.25542
260	0.173836	0.203192	260	0.20335	0.23495
300	0.150008	0.175205	300	0.17538	0.20249
	$p=5, n_1=8$			$p=6, n_1=8$	
60	1.08	1.24	80	0.93	1.05
80	0.78	0.90	100	0.72	0.82
100	0.608	0.703	120	0.588	0.672
120	0.499	0.576	150	0.463	0.528
160	0.3675	0.4234	180	0.3817	0.4347
200	0.2908	0.3345	200	0.3417	0.3891
240	0.24062	0.27637	240	0.2824	0.3210
260	0.22150	0.25429	260	0.2599	0.2955
300	0.191136	0.219317	300	0.22418	0.25467

4. Tabulations of Powers

Using the coefficients D_k up to k=6 the powers of T were calculated for various values of p,n_1,n_2 and $\underset{\sim}{\Omega}$ in connection with testing the hypothesis (B), i.e. of the equality of p-dimensional mean vectors of ℓ p-variate normal populations having an unknown common covariance matrix. In these calculations, first the total probabilities were obtained accurate at least up to three decimals and then the powers of T with the same accuracy. For the calculations of powers the percentage points tabulated in section 3 have been used.

Table V(3) contains powers of T for p=3 using $\alpha=0.05$ and $\alpha=0.01$ for testing H_0: $(w_1,w_2,w_3) = (0,0,0)$ against different simple alternative hypotheses. Tables V(4), V(5) and V(6) contain those of T for p=4,5 and 6 respectively taking $\alpha=0.05$.

Observing the tables V(3) up to V(6), it seems that the power of T possesses monotonicity property with respect to the individual population root.

Table V (3)

Powers of T for p=3 for testing $(w_1,w_2,w_3)=(0,0,0)$

against different simple alternative hypotheses

$\alpha=0.05$ and $\alpha=0.01$

w_1	w_2	w_3	Power		Power	
			5%	1%	5%	1%
			$n_1=4,n_2=84$		$n_1=8,n_2=84$	
0	0	0.001	0.050	0.010	0.050	0.010
0	0	0.003	0.050	0.010	0.050	0.010
0	0.0015	0.0015	0.050	0.010	0.050	0.010
0.001	0.001	0.001	0.050	0.010	0.050	0.010
0.01	0.02	0.03	0.053	0.011	-	-
0	0	0.111	0.056	0.012	-	-
0.001	0.01	0.1	0.056	0.012	-	-
			$n_1=4,n_2=124$		$n_1=8,n_2=124$	
0	0	0.001	0.0500	0.0100	0.050	0.010
0	0	0.003	0.0501	0.0100	0.050	0.010
0	0.0015	0.0015	0.0501	0.0100	0.050	0.010
0.001	0.001	0.001	0.0501	0.0100	0.050	0.010
0.01	0.02	0.03	0.053	0.011	0.052	0.010
0	0	0.111	0.056	0.012	0.054	0.011
0.001	0.01	0.1	0.056	0.012	0.054	0.011
			$n_1=4,n_2=204$		$n_1=8,n_2=204$	
0	0	0.001	0.05005	0.01001	0.0500	0.0100
0	0	0.003	0.05015	0.01004	0.0501	0.0100

Table V (3), cont.

w_1	w_2	w_3	Power		Power	
			5%	1%	5%	1%
0	0.0015	0.0015	0.05015	0.01004	0.0501	0.0100
0.001	0.001	0.001	0.05013	0.01003	0.0501	0.0100
0.01	0.02	0.03	0.0530	0.0109	0.052	0.011
0	0	0.111	0.0557	0.0116	0.054	0.011
0.001	0.01	0.1	0.0557	0.0116	0.054	0.011
0.1	0.1	0.2	0.072	0.017	0.064	0.014
0	0	0.5	0.078	0.019	0.068	0.015
0	0.25	0.25	0.078	0.019	0.068	0.015
0.1	0.2	0.2	0.078	0.019	0.068	0.015
0	0	0.6	0.084	0.021	-	-
0.2	0.2	0.2	0.084	0.021	-	-
			$n_1=4, n_2=300$		$n_1=8, n_2=300$	
0	0	0.001	0.05005	0.01002	0.0500	0.0100
0	0	0.003	0.05015	0.01004	0.0501	0.0100
0	0.0015	0.0015	0.05015	0.01004	0.0501	0.0100
0.001	0.001	0.001	0.05013	0.01003	0.0501	0.0100
0.01	0.02	0.03	0.0530	0.0109	0.0520	0.0106
0	0	0.111	0.0557	0.0116	0.0538	0.0111
0.001	0.01	0.1	0.0557	0.0116	0.0538	0.0111
0.1	0.1	0.2	0.072	0.017	0.064	0.014
0	0	0.5	0.078	0.019	0.068	0.015
0	0.25	0.25	0.078	0.019	0.068	0.015
0.1	0.2	0.2	0.078	0.019	0.068	0.015
0	0	0.6	0.084	0.021	0.073	0.017
0.2	0.2	0.2	0.084	0.021	0.073	0.017

Table V (4)

Powers of T for p=4 for testing $(w_1,w_2,w_3,w_4)=(0,0,0,0)$

against different simple alternative hypotheses

$\alpha=0.05$

w_1	w_2	w_3	w_4	Power	Power
				$n_1=4,n_2=125$	$n_1=6,n_2=125$
0	0	0	0.0001	0.0500	0.050
0	0	0	0.001	0.0500	0.050
0	0	0.0005	0.0005	0.0500	0.050
0	0.001	0.001	0.001	0.0501	0.050
0.001	0.001	0.001	0.001	0.0502	0.050
0.001	0.002	0.003	0.004	0.0504	0.050
0.001	0.001	0.004	0.004	0.0504	0.050
0.01	0.02	0.03	0.04	0.054	-
0.02	0.02	0.03	0.03	0.054	-
0	0	0	0.1	0.054	-
				$n_1=4,n_2=205$	$n_1=6,n_2=205$
0	0	0	0.0001	0.05000	0.0500
0	0	0	0.001	0.05004	0.0500
0	0	0.0005	0.0005	0.05004	0.0500
0	0.001	0.001	0.001	0.05013	0.0501
0.001	0.001	0.001	0.001	0.0502	0.0501
0.001	0.002	0.003	0.004	0.0504	0.050
0.001	0.001	0.004	0.004	0.0504	0.050
0.01	0.02	0.03	0.04	0.054	0.053
0.02	0.02	0.03	0.03	0.054	0.053
0	0	0	0.1	0.054	0.053
0.1	0.1	0.1	0.2	0.073	0.068

Table V (4), Cont.

w_1	w_2	w_3	w_4	Power $n_1=4, n_2=300$	Power $n_1=6, n_2=300$
0	0	0	0.0001	0.05000	0.05000
0	0	0	0.001	0.05004	0.0500
0	0	0.0005	0.0005	0.05004	0.0500
0	0.001	0.001	0.001	0.05013	0.0501
0.001	0.001	0.001	0.001	0.05017	0.0501
0.001	0.002	0.003	0.004	0.05042	0.0503
0.001	0.001	0.004	0.004	0.05042	0.0503
0.01	0.02	0.03	0.04	0.0543	0.053
0.02	0.02	0.03	0.03	0.0543	0.053
0	0	0	0.1	0.0543	0.053
0.1	0.1	0.1	0.2	0.0734	0.068

Table V (5)

Powers of T for p=5 for testing $(w_1,...,w_5)=(0,...,0)$

against different simple alternative hypotheses

$\alpha=0.05$

w_1	w_2	w_3	w_4	w_5	Power	Power
					$n_1=6,n_2=200$	$n_1=8,n_2=200$
0	0	0	0	0.0001	0.050	0.050
0	0	0	0	0.001	0.050	0.050
0	0	0	0.0005	0.0005	0.050	0.050
0	0	0.001	0.001	0.001	0.050	-
0.001	0.001	0.001	0.001	0.001	0.050	-
0.001	0.002	0.003	0.004	0.005	0.050	-
					$n_1=6,n_2=260$	$n_1=8,n_2=260$
0	0	0	0	0.0001	0.0500	0.050
0	0	0	0	0.001	0.0500	0.050
0	0	0	0.0005	0.0005	0.0500	0.050
0	0	0.001	0.001	0.001	0.0501	0.050
0.001	0.001	0.001	0.001	0.001	0.0502	0.050
0.001	0.002	0.003	0.004	0.005	0.0505	0.050
0	0	0	0	0.015	0.0505	0.050
0.03	0.03	0.03	0.03	0.03	0.055	-
0.015	0.015	0.025	0.025	0.05	0.054	-
					$n_1=6,n_2=300$	$n_1=8,n_2=300$
0	0	0	0	0.0001	0.0500	0.0500
0	0	0	0	0.001	0.0500	0.0500
0	0	0	0.0005	0.0005	0.0500	0.050
0	0	0.001	0.001	0.001	0.0501	0.050
0.001	0.001	0.001	0.001	0.001	0.0502	0.050
0.001	0.002	0.003	0.004	0.005	0.0505	0.050
0	0	0	0	0.015	0.0505	0.050
0.03	0.03	0.03	0.03	0.03	0.055	0.054
0.015	0.015	0.025	0.025	0.05	0.054	0.054

Table V (6)

Powers of T for p=6 for testing $(w_1,...,w_6)=(0,...,0)$

against different simple alternative hypotheses

$\alpha=0.05$

w_1	w_2	w_3	w_4	w_5	w_6	Power	
						$n_1=6, n_2=300$	$n_1=8, n_2=300$
0	0	0	0	0	0.0001	0.050	0.050
0	0	0	0	0	0.001	0.050	0.050
0	0	0	0.0005	0.0005	0.0005	0.050	0.050
0	0	0.001	0.001	0.001	0.002	0.050	0.050
0.001	0.001	0.001	0.001	0.001	0.001	0.050	0.050
0.001	0.002	0.003	0.004	0.005	0.006	0.051	0.051
0.002	0.002	0.004	0.004	0.008	0.008	0.051	0.051

CHAPTER VI

THE DISTRIBUTION OF WILKS-LAWLEY STATISTIC AND ITS POWER STUDIES

1. Introduction

Pillai and Nagarsenker [40] have derived the distribution of a class of statistics in connnection with three basic hypotheses (A), (B) and (C). One of their results is the non-central distribution of the Wilks-Lawley statistic which falls under their class and which is denoted here by $Z^{(p)}$ (denoted by U by Gnanadesikan [10]).

In this chapter the general form of the distribution of the Wilks-Lawley statistic $Z^{(p)}$ is obtained using Pillai's density of the characteristic roots of $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ under violations. The density is derived in two different forms, one in terms of Meijer's G-function and the other, for small values of p by convolution approach. For the null case, a relation exists between the density of Wilks' statistic $W^{(p)}$ and that of Wilks-Lawley statistic $Z^{(p)}$. For the two-roots case, a power study of the criterion $Z^{(p)}$ is carried out for the three tests and comparisons are made with powers of $U^{(p)}$, Wilks' criterion $W^{(p)}$, Pillai's trace $V^{(p)}$ and the Roy's largest root.

2. Preliminaries

Meijer's G-function stated in section 2 of chapter I will be used in the next sections to derive the distribution of $Z^{(p)}$. Some further results on this function which will be needed later are stated below.

1. Using the notations in section 2, chapter I, we have

$$(2.1)\quad G_{3,3}^{3,0}\left(x\Big|{a_1 a_2 a_3 \atop b_1 b_2 b_3}\right) = \frac{x^{b_1}(1-n)^{a-b-1}}{\Gamma(a-b)} \sum_{r=0}^{\infty} \frac{(a_3-b_3)_r (a_2-b_3)_r}{r!(a-b)_r}$$

$$.(1-x)^r {}_2F_1(a_1-b_2, a_2+a_3-b_2-b_3+r; a-b+r; 1-x),$$

where $a=a_1+a_2+a_3$ and $b = b_1+b_2+b_3$.

Formula (2.1) is proved by using formula (6.1) of Consul [6] and the definitions of hypergeometric and Meijer's G-functions.

2. For 4 pairs of parameters (a_i,b_i), we have

$$(2.2)\quad G_{4,4}^{4,0}\left(x\Big|{a_1 \ldots a_4 \atop b_1 \ldots b_4}\right) = \frac{x^{b_4}(1-x)^{c-d-1}}{\Gamma(a-b)} \sum_{t=0}^{\infty} \frac{(a_4-b_1)_t}{t!}$$

$$.\sum_{r=0}^{\infty} \frac{(a_3-b_3)_r (a_2-b_3)_r}{r!(a-b)_r} (1-x)^{t+r} \frac{\Gamma(a-b+t+r)}{\Gamma(c-d+t+r)}$$

$$.{}_3F_2(a_1-b_2, a_2+a_3-b_2-b_3+r, a-b+t+r; a-b+r, c-d+t+r; 1-x),$$

where $a=a_1+a_2+a_3$, $b=b_1+b_2+b_3$, $c=a+a_4$ and $d=b+b_4$.

The proof of (2.2) is immediate if we apply the formula (4.4), Chapter V of Al-Ani [1].

3. The Density of the Wilks-Lawley Statistic

To obtain the density function of the Wilks-Lawley statistic, let us start from (3.1) of Section 3, Chapter II. Using the roots ℓ_i, $i=1,2,\ldots,p$, defined there, the Wilks-Lawley statistic $Z^{(p)}$ ([22,45]) is given by

$$(3.1)\quad Z^{(p)} = \prod_{i=1}^{p} \ell_i,\ 0 < \ell_1 < \ldots < \ell_p < 1.$$

In view of the cases of (A) and (B), let us derive the general form of the density of $Z^{(p)}$ using its h^{th} moment. We have

$$(3.2)\quad E\{(Z^{(p)})^h\} = C(p,m,n)\ e^{-tr\underset{\sim}{\Omega}}|\lambda\underset{\sim}{\Lambda}|^{-\frac{1}{2}f_1}$$

$$\cdot\int_{\underset{\sim}{0}}^{\underset{\sim}{I}} |\underset{\sim}{L}|^{m+h}|\underset{\sim}{I}-\underset{\sim}{L}|^n \prod_{i>j}(\ell_i-\ell_j) \sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\frac{1}{2}\nu)_\kappa}{k!}C_\kappa(\underset{\sim}{L})d\underset{\sim}{L}.F_p.$$

After using (2.20), chapter I and further simplification we obtain the h^{th} moment of $Z^{(p)}$ in the form

$$(3.3)\quad E\ (Z^{(p)})^h = \frac{\Gamma_p(m+n+p+1)}{\Gamma_p(\frac{1}{2}(2m+p+1))}\ e^{-tr\underset{\sim}{\Omega}}|\lambda\underset{\sim}{\Lambda}|^{-\frac{1}{2}f_1}$$

$$\cdot\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\frac{1}{2}\nu)_\kappa C_\kappa(\underset{\sim}{I})}{k!}\ \frac{\prod_{i=1}^{p}\Gamma(r+b_i)}{\prod_{i=1}^{p}\Gamma(r+a_i)}\ F_p,$$

where $r=m+1+h$, $b_i=\frac{1}{2}(i-1)+k_{p-i+1}$, $a_i=\frac{1}{2}(2n+p+1)+b_i$. By the inverse Mellin transform (2.36), chapter I, we obtain the density of $Z^{(p)}$ in the form

$$(3.4)\quad f(Z^{(p)}) = \frac{\Gamma_p(m+n+p+1)}{\Gamma_p(\frac{1}{2}(2m+p+1))}|\lambda\underset{\sim}{\Lambda}|^{-\frac{1}{2}f_1}\ e^{-tr\underset{\sim}{\Omega}}\ (Z^{(p)})^m$$

$$\cdot\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\frac{1}{2}\nu)_\kappa C_\kappa(\underset{\sim}{I})}{k!}\ (2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty}(Z^{(p)})^{-r}\ \frac{\prod_{i=1}^{p}\Gamma(r+b_i)}{\prod_{i=1}^{p}\Gamma(r+a_i)}\ dr.F_p.$$

Expressing the integral in terms of Meijer's G-function (see (2.38), chapter I), we have the following:

Theorem 3.1: Let $\underset{\sim}{S}_1(p\times p)$ and $\underset{\sim}{S}_2(p\times p)$ be independently distributed, $\underset{\sim}{S}_1$ having $W(p,f_1,\underset{\sim}{\Sigma}_1,\underset{\sim}{\Omega})$ and $\underset{\sim}{S}_2$ having $W(p,f_2,\underset{\sim}{\Sigma}_2,\underset{\sim}{0})$. Let $\ell_1,\ell_2,\ldots,\ell_p$, $0 < \ell_1 <\ldots< \ell_p < 1$, be the roots of $|\underset{\sim}{S}_1-\ell(\underset{\sim}{S}_1+\underset{\sim}{S}_2)| = 0$. If $\underset{\sim}{\Sigma}_1^{\frac{1}{2}}\underset{\sim}{\Sigma}_2^{-1}\underset{\sim}{\Sigma}_1^{\frac{1}{2}}$ is "random" then the density function of $Z^{(p)} = \prod_{i=1}^{p} \ell_i$ is given by

$$(3.5)\quad f(Z^{(p)}) = \frac{\Gamma_p(m+n+p+1)}{\Gamma_p(\frac{1}{2}(2m+p+1))} e^{-\mathrm{tr}\underset{\sim}{\Omega}}|\lambda\underset{\sim}{\Lambda}|^{-\frac{1}{2}f_1}(Z^{(p)})^m$$

$$\cdot\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\frac{1}{2}\nu)_\kappa C_\kappa(\underset{\sim}{I})}{k!}\; G^{p,0}_{p,p}\left(Z^{(p)}\left|\begin{matrix}a_1\ldots a_p\\ b_1\ldots b_p\end{matrix}\right.\right).F_p,$$

for $0 < Z^{(p)} < 1$ and zero otherwise, where $m = \frac{1}{2}(f_1 - p - 1)$, $n = \frac{1}{2}(f_2 - p - 1)$, $\nu = f_1 + f_2$, $a_i = \frac{1}{2}f_2 + b_i$, $b_i = \frac{1}{2}(i - 1) + k_{p-i+1}$ and $\underset{\sim}{\Lambda} = \mathrm{diag}(\lambda_1,\ldots,\lambda_p)$, $\lambda_1, \lambda_2,\ldots,\lambda_p$ are the latent roots of $\underset{\sim}{\Sigma}_1\,\underset{\sim}{\Sigma}_2^{-1}$.

4. Special Cases

Now we derive some special cases of (3.5).

(i) The null density of $Z^{(p)}$ is given by

$$(4.1)\qquad f(Z^{(p)}) = \frac{\Gamma_p(m+n+p+1)}{\Gamma_p(\frac{1}{2}(2m+p+1))}(Z^{(p)})^m\; G^{p,0}_{p,p}\left(Z^{(p)}\left|\begin{matrix}a_1\ldots a_p\\ b_1\ldots b_p\end{matrix}\right.\right)$$

where $b_i = \frac{1}{2}(i - 1)$ and $a_i = \frac{1}{2}(2n + p + 1) + b_i$.

(ii) Putting $p = 1$ in (4.1) and using (2.41) of Chapter I we obtain the beta density function.

(iii) In (3.5), let $p = 2$ and apply (2.40), Chapter I. We have the non-central density of $Z^{(2)}$ in the form

$$(4.2) \qquad F(Z^{(2)}) = \frac{\Gamma_2(m+n+3)}{\Gamma_2(\frac{1}{2}(2m+3))\Gamma(2n+3)} \; e^{-(w_1+w_2)}(\lambda^2\lambda_1\lambda_2)^{-\frac{1}{2}(2m+3)}$$

$$\cdot \sum_{k=0}^{\infty} \sum_{\kappa} [\{(\tfrac{1}{2}\nu)_\kappa\}/k!] C_\kappa(\underset{\sim}{I})(Z^{(2)})^{m+k_2} (1-Z^{(2)})^{2(n+1)}$$

$$\cdot {}_2F_1(\tfrac{1}{2}(2n+3),\; n+1+k_2-k_1;\; 2n+3;\; 1-Z^{(2)}).F_2,$$

where w_1, w_2 are the latent roots of $\underset{\sim}{\Omega}$ for $p = 2$, λ_1, λ_2 are those of $\underset{\sim}{\Lambda}$ and $\underset{\sim}{I}$ is the identity matrix of order 2. Now, let $w_1 = w_2 = 0$, $\lambda = \lambda_1 = \lambda_2 = 1$ in (4.2) and use the relation (Snow[40])

$$(4.3) \qquad {}_2F_1(\tfrac{1}{2}t,\; \tfrac{1}{2}(t-1);\; t;\; 1-y) = 2^{t-1}(1+\sqrt{y})^{1-t},$$

then we obtain the null density of $Z^{(2)}$ as follows:

$$(4.4) \qquad f(Z^{(2)}) = (n+1)^{-1}\, C(2,m,n)(Z^{(2)})^m (1-\sqrt{Z^{(2)}})^{2(n+1)},$$

for $0 < Z^{(2)} < 1$ and zero otherwise. The c.d.f. of $Z^{(2)}$ is given by

$$(4.5) \qquad F(Z^{(2)}) = C_1(m,n)\; B_x(2m+2,\; 2n+3),\; 0 < Z^{(2)} < 1,$$

where $x = (Z^{(2)})^{\frac{1}{2}}$, $B_x(a,b)$ denotes the incomplete beta function and

$$(4.6) \qquad C_1(m,n) = \frac{2T(\frac{1}{2})\Gamma(m+n+2\frac{1}{2})\Gamma(m+n+3)}{\Gamma(m+1)\Gamma(m+1\frac{1}{2})\Gamma(n+\frac{1}{2})\Gamma(n+2)}.$$

From (4.5) we can obtain the upper α percentage points for various values of m and n. For α=0.01 and 0.05 the tables of upper percentage points are tabulated for m= (1)5(5)20, n=5(5)30(10)60(25)100 and are presented in the tables VI(1) and VI(2).

(iv) For p=3, after applying (2.1), we have the non-central density of $Z^{(3)}$ in the form

$$(4.7)\quad f(Z^{(3)}) = \frac{\Gamma_3(m+n+4)}{\Gamma_3(m+2)\Gamma(3n+6)} e^{-tr\underset{\sim}{\Omega}}|\lambda\underset{\sim}{\Lambda}|^{-(m+2)}(Z^{(3)})^m(1-Z^{(3)})^{3n+5}$$

$$\cdot\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\frac{1}{2}\nu)_\kappa C_\kappa(\underset{\sim}{I})}{k!}(Z^{(3)})^{k_3}\sum_{r=0}^{\infty}\frac{(n+2)_r(n+1\frac{1}{2}+k_2-k_1)_r}{r!(3n+6)_r}(1-Z^{(3)})^r$$

$$\cdot{}_2F_1(n+1\tfrac{1}{2}+k_3-k_2, 2n+4+r; 3n+6+r; 1-Z^{(3)}).F_3,$$

where $k_3 \geq k_2 \geq k_1 \geq 0$, $k_1 + k_2 + k_3 = k$ and the matrices involved in (4.7) are of order 3.

(v) Using (2.2) and putting p=4 in (3.5), we obtain the non-central density of $Z^{(4)}$ as given below

$$(4.8)\quad f(Z^{(4)}) = P(4,m,n)\ e^{-tr\underset{\sim}{\Omega}}|\lambda\underset{\sim}{\Lambda}|^{-\frac{1}{2}(2m+5)}(Z^{(4)})^{m+1\frac{1}{2}}(1-Z^{(4)})^{4n+9}$$

$$\cdot\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\frac{1}{2}\nu)_\kappa}{k!}C_\kappa(\underset{\sim}{I})\ (Z^{(4)})^{k_1}\sum_{t=0}^{\infty}\frac{(n+4+k_1-k_4)_t}{t!}$$

$$\cdot\sum_{r=0}^{\infty}\frac{(\frac{1}{2}(2n+5))_r(n+2+k_3-k_2)_r}{r!(\frac{1}{2}(6n+15))_r}(1-Z^{(4)})^{t+r}\frac{\Gamma(\frac{1}{2}(6n+15)+t+r)}{\Gamma(4n+10+t+r)}$$

$$\cdot{}_3F_2(n+2+k_4-k_3, 2n+5+r, \tfrac{1}{2}(6n+15)+t+r; \tfrac{1}{2}(6n+15)+r,$$
$$4n+10+t+r; 1-Z^{(4)}).F_4,$$

where $P(4,m,n) = \dfrac{\Gamma_4(m+n+5)}{\Gamma_4(\frac{1}{2}(2m+5))\Gamma(\frac{1}{2}(6n+15))}$,

$k_4 \geq \ldots \geq k_1 \geq 0$, $k_4 + \ldots + k_1 = k$ and all matrices in (4.8) are of order 4.

5. Alternative Form of the Null Density of $Z^{(p)}$

Let us compare (4.1) and the null density of $W^{(p)}$ obtained from (6.6) chapter I by substituting $\tilde{\Omega} = \tilde{0}$, $\tilde{\Lambda} = \tilde{I}$ and $\lambda = 1$. We see that the following property holds: The null density of $Z^{(p)}$ can be obtained from that of $W^{(p)}$ by making the transformation

(5.1) $(W^{(p)}, f_1, f_2) \to (Z^{(p)}, f_2, f_1)$,

and vice versa. This property obviously also holds for their moments in the central case (put $\tilde{\Omega} = \tilde{0}$, $\tilde{\Lambda} = \tilde{I}$ and $\lambda = 1$ in (3.3) and (6.4) chapter I, then compare them).

Now, assuming the results of Pillai and Gupta [36] on the null density of $W^{(p)}$ and applying the above property, we can obtain the null density of $Z^{(p)}$ for various values of p. Following Pillai and Gupta [36], denote the Wilks' criterion by U_{p,f_2,f_1} (see also Anderson [2], chapter 8) and accordingly we shall denote the Wilks-Lawley statistic by Z_{p,f_1,f_2}. Then, for example, from formula (4.2) of [36] we have the null density of Z_{3,f_1,f_2} at the point z in the form

(5.2) $$f(z) = \{2B(f_2-1, f_1)B(\tfrac{1}{2}(f_2-2), \tfrac{1}{2}f_1)\}^{-1}$$
$$\cdot \sum_{i=1}^{\frac{1}{2}f_1-1} \Bigg[(-1)^{i-1} \binom{f_1-1}{2i-1} \binom{\frac{1}{2}f_1-1}{i} (-\log z) z^{\frac{1}{2}(f_2+2i-4)}$$
$$+ 2 \sum_{\substack{j=0 \\ j \neq 2i-1}}^{f_1-1} \sum_{i=0}^{\frac{1}{2}f_1-1} \frac{(-1)^{i+j}}{(2i-j-1)} \binom{f_1-1}{j} \binom{\frac{1}{2}f_1-1}{i} \left(z^{\frac{1}{2}(f_2+j-3)} - z^{\frac{1}{2}(f_2+2i-4)}\right) \Bigg]$$

for $0 < z < 1$ and zero otherwise. (5.2) is a closed form for f_1 even and an infinite series for f_1 odd. The distribution of Z_{3,f_1,f_2} has the form

(5.3) $$P\{Z_{3,f_1,f_2} \le z\} \quad \{B(f_2 - 1, f_1)\, B(\tfrac{1}{2}(f_2 - 2), \tfrac{1}{2}f_1)\}^{-1} \cdot \left[\sum_{i=0}^{\frac{1}{2}f_1 - 1} \frac{(-1)^{i-1} z^a}{a^2} \binom{\frac{1}{2}f_1 - 1}{i} \binom{f_1 - 1}{2i - 1} (2 - a \log z) \right.$$
$$\left. + 2 \sum_{\substack{j=0 \\ j \neq 2i-1}}^{f_1 - 1} \sum_{i=0}^{\frac{1}{2}f_1 - 1} \frac{(-1)^{i+j}}{2i - j - 1} \binom{f_1 - 1}{j} \binom{\frac{1}{2}f_1 - 1}{i} \left(\frac{z^{\frac{1}{2}b}}{b} - \frac{z^{\frac{1}{2}a}}{a} \right) \right],$$

where $a = f_2 + 2i - 2$ and $b = f_2 + j - 1$. The distributions for $p = 4$, 5 and 6 may be obtained from [36] using the above property.

6. Non-Central Distribution of $Z^{(2)}$

Using the method in section 4, Chapter II we can derive the alternative form for (4.2) of the non-central distribution of $Z^{(2)}$. In (3.1) of section 3 chapter II, let $p = 2$ and expand the zonal polynomial $C_\kappa(\underset{\sim}{L}_2)$. After a change of variables and necessary integrations we obtain the term of the c.d.f. of $Z^{(2)}$ corresponding to the partition κ of k in the form

(6.1) $$\sum_{r + 2s = k} b_\kappa(r,s)\, M_{rs}(Z^{(2)}), \qquad 0 < Z^{(2)} < 1,$$

where

(6.2) $$M_{rs}(Z^{(2)}) = \sum_{i=0}^{r} \frac{2^{r-i+1}}{n+1} Q_i\, B_x\,(2m+2s+r+2-i,\ 2n+3+2i),$$

and where $x = (Z^{(2)})^{1/2}$, $b_\kappa(r,s)$ are constants tabulated in appendix A, and

$$(6.3) \qquad Q_i = \prod_{j=1}^{i} \{(r+1-j)/(n+1+j)\}, \quad Q_0 = 1.$$

Summarizing, we now have the following:

Theorem 6.1: Let $\underset{\sim}{S}_1$ and $\underset{\sim}{S}_2$ be as in theorem 3.1 and $Z^{(p)}$ be as defined earlier. Then the exact non-central distribution of $Z^{(2)}$ has the form

$$(6.4) \qquad F(Z^{(2)}) = C(2,m,n)\ e^{-(w_1+w_2)}\ (\lambda^2\lambda_1\lambda_2)^{-\frac{1}{2}f_1}$$

$$\cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}\nu)_\kappa}{k!} \sum_{r+2s=k} b_\kappa(r,s)\ M_{rs}(Z^{(2)}).\ F_2,$$

where $M_{rs}(Z^{(2)})$ is as in (6.2) and the meaning of the other symbols are as explained in the previous sections.

We can see from (4.2) and (6.4) that the distribution of $Z^{(2)}$ is expressed in terms of infinite series. Therefore, in order to study some properties of the statistic $Z^{(2)}$ we will expand the series and take the first few terms. Taking the terms up to the sixth degree, formula (6.4) will give

$$(6.5) \qquad F(Z^{(2)}) = K. \sum_{i+2j=k=0}^{6} E_{ij}M_{ij}(Z^{(2)}), \qquad 0 < Z^{(2)} < 1,$$

where the coefficients E_{ij}'s are exactly as those in section 3 of chapter II, and

(6.6) $$K = C(2,m,n)e^{-(w_1+w_2)}\,(\lambda^2\lambda_1\lambda_2)^{-\frac{1}{2}f_1},$$

while $M_{ij}(Z^{(2)})$ is as in (6.2).

7. The Distribution of $Z^{(p)}$ for Test of Independence.

As it was mentioned in section 1, Pillai and Nagarsenker [40] have also obtained the distribution of $Z^{(p)}$ in case (C), i.e., for the test of independence, where the distribution is expressed in terms of H - function. Here, the general form of the distribution will be expressed in terms of G - function and since the method is similar to the one used earlier, we will only state the result.

Theorem 7.1: Let $\begin{pmatrix} \underset{\sim}{X} \\ \underset{\sim}{Y} \end{pmatrix}$ be ν independent normal (p+q) - variates ($p \leq q$, $p+q \leq \nu$, $\nu + 1 = n'$ = sample size) with mean zero and covariance matrix

$\underset{\sim}{\Sigma} = \begin{pmatrix} \underset{\sim}{\Sigma}_{11} & \underset{\sim}{\Sigma}_{12} \\ \underset{\sim}{\Sigma}_{21} & \underset{\sim}{\Sigma}_{22} \end{pmatrix}$, $\underset{\sim}{\Sigma}_{21} = \underset{\sim}{\Sigma}_{12}'$. Further, let $r_1^2,\dots,r_p^2$ be the roots of $(\underset{\sim}{X}\underset{\sim}{Y}'\,(\underset{\sim}{Y}\underset{\sim}{Y}')^{-1}\underset{\sim}{Y}\underset{\sim}{X}')(\underset{\sim}{X}\underset{\sim}{X}')^{-1}$ and $\rho_1^2,\dots,\rho_p^2$ be those of $(\underset{\sim}{\Sigma}_{12}\,\underset{\sim}{\Sigma}_{22}^{-1}\,\underset{\sim}{\Sigma}_{21})(\underset{\sim}{\Sigma}_{11}^{-1})$. If $Z^{(p)} = \prod_{i=1}^{p} r_i^2$, then the non-central density function of $Z^{(p)}$ is given by

(7.1) $$f(Z^{(p)}) = \frac{\Gamma_p(m+n+p+1)}{\Gamma_p(\frac{1}{2}(2m+p+1))}\,|\underset{\sim}{I} - \underset{\sim}{P}^2|^{\frac{1}{2}\nu}\,(Z^{(p)})^m$$

$$\cdot\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\{(\frac{1}{2}\nu)_\kappa\}^2\,C_\kappa(\underset{\sim}{P}^2)}{(\frac{1}{2}f_1)_\kappa\,k!}\,G^{p,0}_{p,p}\left(Z^{(p)}\,\middle|\,\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_p\end{matrix}\right),$$

for $0 < Z^{(p)} < 1$ and zero otherwise, where $f_1 = q$, $m = \frac{1}{2}(q-p-1)$, $n = \frac{1}{2}(\nu-q-p-1)$, (if n is defined as $\frac{1}{2}(f_1-p-1)$, $\nu = f_1+f_2$) and $\underset{\sim}{P} = \mathrm{diag}(\rho_1,\ldots,\rho_p)$.

Special cases of (7.1) can be obtained easily. For example, letting $\underset{\sim}{P} = \underset{\sim}{0}$ in (7.1), we get the null density of $Z^{(p)}$ as stated in (4.1). Also, as in the section 4, we can obtain the densities of $Z^{(p)}$ for $p = 1(1)4$. In particular, for $p = 2$, we have the following:

Theorem 7.2: Let all the hypotheses be as stated in theorem 7.1. Then the exact non-central distribution of $Z^{(2)}$, $0 < Z^{(2)} < 1$, has the form

$$(7.2) \qquad F(Z^{(2)}) = C(2,m,n)\{(1-\rho_1^2)(1-\rho_2^2)\}^{\frac{1}{2}\nu} \cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\{(\frac{1}{2}\nu)_\kappa\}^2 C_\kappa(\underset{\sim}{P}^2)}{k!(\frac{1}{2}f_1)_\kappa C_\kappa(\underset{\sim}{I}_2)} \sum_{r+2s=k} b_\kappa(r,s) M_{rs}(Z^{(2)}).$$

Expanding (7.2) up to sixth order, we have the c.d.f. of $Z^{(2)}$ in the non-central case as

$$(7.3) \qquad F(Z^{(2)}) = K_1 \cdot \sum_{i+2j=k=0}^{6} E'_{ij} M_{ij}(Z^{(2)}), \quad 0 < Z^{(2)} < 1,$$

where $K_1 = C(2,m,n)\{(1-\rho_1^2)(1-\rho_2^2)\}^{\frac{1}{2}\nu}$, the coefficients E'_{ij}'s are obtained from the corresponding E_{ij}'s in section 3 of chapter II by multiplying each A_{ij} by 2^{-i}, making substitutions $a_1 = 2$, $a_2 = 1$ there, squaring each factor containing ν and redefining $b_1 = \rho_1^2 + \rho_2^2$ and $b_2 = (\rho_1\rho_2)^2$.

8. Power Studies and Comparisons

For calculation of powers of $Z^{(2)}$ in connection with tests of (A), (B) and (C), upper percentage points computed earlier have been used. Powers of $Z^{(2)}$ for test of (A) assuming $\underset{\sim}{\Omega}=\underset{\sim}{0}$, of (B) assuming $\underset{\sim}{\Sigma}_1=\underset{\sim}{\Sigma}_2$ and of (C) were computed for m=0,2,5; n=5,15,40; α=0.05 and various values of parameters. These powers, together with those obtained by Pillai [29] based on the largest root and by Pillai and Jayachandran [37,38] based on $U^{(2)}$, $V^{(2)}$ and $W^{(2)}$ are tabulated in the tables VI (3), VI (4) and VI (5) for comparisons. General observation for powers of $Z^{(2)}$ for the three hypotheses, give the following:

a. The power of the tests based on $Z^{(2)}$ has monotonicity property with respect to the individual population root.

b. For the sum of the roots constant, the power of $Z^{(2)}$ seems to increase as the two roots tend to be equal.

For small deviations from the hypothesis, the tabulations give the following general comparisons for the five criteria:

1. The Pillai's criterion $V^{(2)}$ seems to have more power than the other four.
2. For m=2,5 and n=5, $Z^{(2)}$ has more power than the criteria $W^{(2)}$, $U^{(2)}$ and the largest root.
3. For m=5, n=15, it seems that $Z^{(2)}$ has more power than $U^{(2)}$ and the largest root but still behind the criteria $V^{(2)}$ and $W^{(2)}$.
4. For other values of m and n observed, the $Z^{(2)}$ criterion stays behind $V^{(2)}$, $W^{(2)}$ and $U^{(2)}$ but still ahead of the largest root except for m=0 and n=15 and 40.

For larger deviation parameters, we may observe from tables VI (6) and VI (7) that properties a and b for $Z^{(2)}$ still hold. It seems that, in general, the power of $Z^{(2)}$ criterion stays behind those of $U^{(2)}$, $V^{(2)}$ and $W^{(2)}$ with some exceptions for m=5 and n=30.

It may be pointed out that individual observation for powers of $V^{(2)}$, $U^{(2)}$, $W^{(2)}$ and the largest root have been carried out, including their comparisons, by Pillai [29] and Pillai and Jayachandran [37].

Table VI (1)

Upper 5% Points of $Z^{(2)}$

n \ m	0	1	2	3	4	5	10	15	20
5	.29673424	.41657228	.49782825	.55803451	.60484422	.64243604	.75689546	.81554456	.85132575
10	.18289247	.27190196	.33940197	.39394742	.43944669	.47819039	.61057184	.68863256	.74042182
15	.13207383	.20151993	.25695830	.30370567	.34415040	.37970553	.50985600	.59376792	.65281621
20	.10333500	.16002228	.20662878	.24693520	.28259484	.31457685	.43713005	.52117542	.58294601
25	.08486196	.13267832	.17275042	.20799094	.23964182	.26842062	.38235462	.46410589	.52623049
30	.07198975	.11330820	.14840229	.17963466	.20799148	.23403402	.33968339	.41815906	.47939217
40	.05523152	.08769393	.11575481	.14112573	.16449878	.18625765	.27759298	.34887509	.40671896
50	.04480123	.07152192	.09487507	.11620209	.13603396	.15465927	.23463706	.29919104	.35304494
60	.03768426	.06038473	.08037437	.09875636	.11596102	.13221877	.20317081	.26185310	.31182629
80	.02859792	.04604375	.06155529	.07594719	.08953150	.10247199	.16017938	.20950767	.25273186
100	.02304194	.03720680	.04987597	.06169564	.07291071	.08364823	.13219343	.17458222	.21243341

Table VI (2)

Upper 1% Points of $Z^{(2)}$

n \ m	0	1	2	3	4	5	10	15	20
5	.38909548	.50293639	.57719575	.63093916	.67204765	.70466251	.80196657	.85075218	.88018512
10	.24624551	.33720843	.40393281	.45672731	.50010268	.53660928	.65874836	.72912012	.77521070
15	.17981937	.25295050	.30960168	.35646244	.39644138	.43120594	.55589358	.63447697	.68904943
20	.14155974	.20224012	.25075053	.29195786	.32793957	.35988110	.47990632	.56043266	.61884576
25	.11670740	.16842258	.21062262	.24710864	.27947297	.30861503	.42183568	.50137777	.56105139
30	.09927149	.14427713	.18153547	.21415734	.24342468	.27005348	.37613581	.45334431	.51283711
40	.07642734	.11211092	.14221649	.16902138	.19344087	.21597591	.30896190	.38015712	.43724402
50	.06212732	09166505	.11688321	.13957694	.16045550	.17990101	.26204468	.32715255	.38085132
60	.05233409	.07752358	.09920526	.11886074	.13706844	.15413627	.22745659	.28704967	.33724396
80	.03978893	.05924148	.07616099	.09164605	.10611905	.11980133	.17990998	.23045741	.27429577
100	.03209488	.04793559	.06180229	.07456827	.08656635	.09796920	.14878428	.19246753	.23109372

Table VI (3)

Powers of five criteria for testing $(w_1, w_2)=(0,0)$ against different simple alternative hypotheses, $\alpha=0.05$

w_1	w_2	$Z^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
			m=0,n=5			
0	0.01	0.0502611	0.0502644	0.0502795	0.0502737	0.0502465
0	0.1	0.0526112	0.0526669	0.0528094	0.0527580	0.0524889
0.05	0.05	0.0526529	0.0526663	0.0528318	0.0527664	0.0524770
0	0.5	0.0630457	0.0638359	0.0643622	0.0642368	0.0629750
0.1	0.4	0.0637098	0.0638298	0.0647178	0.0643725	0.0627864
0.25	0.25	0.0640834	0.0638264	0.0649178	0.0644488	0.0626802
0	1	0.076020	0.0788732	0.0794634	0.0795383	0.077227
0	2	0.1016	0.11219	0.11157	0.11298	0.1092
1	1	0.1175	0.11223	0.12011	0.11636	0.1048
			m=0,n=15			
0	0.01	0.0502967	0.0503320	0.0503350	0.0503341	0.0503112
0	0.1	0.0529737	0.0533574	0.0533829	0.0533766	0.0531527
0.05	0.05	0.0530170	0.0533571	0.0533933	0.0533812	0.0531328
0	0.5	0.0649535	0.0675878	0.0676187	0.0676444	0.0666565
0.1	0.4	0.0656748	0.0675856	0.067851	0.0677187	0.0663320
0.25	0.25	0.0660805	0.0675844	0.0678787	0.0677606	0.0661550
0	1	0.080076	0.0870840	0.0869126	0.0871010	0.085408
0	2	0.1105	0.13112	0.12992	0.13080	0.1286
1	1	0.1278	0.13120	0.13386	0.13264	0.1212

Table VI (3), cont.

w_1	w_2	$Z^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
			m=0,n=40			
0	0.01	0.0503139	0.0503626	0.0503631	0.0503630	0.0503423
0	0.1	0.0531472	0.0536702	0.0536733	0.0536731	0.0534726
0.05	0.05	0.0531936	0.0536700	0.0536778	0.0536752	0.0534484
0	0.5	0.0658977	0.0693042	0.0692771	0.0692987	0.0684364
0.1	0.4	0.0666360	0.0693035	0.0693479	0.0693325	0.0680538
0.25	0.25	0.0670511	0.0693032	0.0693878	0.0693515	0.0678624
0	1	0.082119	0.0908734	0.0907170	0.0908232	0.089434
0	2	0.1151	0.13994	0.13926	0.13965	0.1381
1	1	0.1328	0.13999	0.14092	0.14048	0.1293
			m=2,n=5			
0	0.01	0.0501508	0.0501358	0.0501529	0.0501469	0.0501201
0	0.1	0.0515118	0.0513668	0.0515344	0.0514768	0.0512096
0.05	0.05	0.0515212	0.0513663	0.0515421	0.0514800	0.0512055
0	0.5	0.0576309	0.0570250	0.0577820	0.0575496	0.0562452
0.1	0.4	0.0577835	0.0570175	0.0579065	0.0576017	0.0561786
0.25	0.25	0.0578694	0.0570133	0.0579765	0.0576310	0.0561411
0	1	0.065423	0.0645229	0.0658232	0.0655039	0.062981
0	2	0.0814	0.08089	0.08258	0.08256	0.0779
1	1	0.0854	0.08075	0.08584	0.08394	0.0762

Table VI (3), cont.

w_1	w_2	$Z^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
			m=2, n=15			
0	0.01	0.0501846	0.0501871	0.0501915	0.0501902	0.0501656
0	0.1	0.0518537	0.0518870	0.0519284	0.0519173	0.0516744
0.05	0.05	0.0518648	0.0518866	0.0519337	0.0519195	0.0516660
0	0.5	0.0594319	0.0597982	0.0599335	0.0599209	0.0587728
0.1	0.4	0.0596129	0.0597926	0.0600188	0.0599574	0.0586366
0.25	0.25	0.0597147	0.0597894	0.0600668	0.0599780	0.0585601
0	1	0.069242	0.0704959	0.0705791	0.0706656	0.068546
0	2	0.0899	0.09450	0.09389	0.09453	0.0911
1	1	0.0947	0.09441	0.09614	0.09552	0.0876
			m=2, n=40			
0	0.01	0.0502034	0.0502135	0.0502144	0.0502141	0.0501916
0	0.1	0.0520449	0.0521565	0.0521635	0.0521621	0.0519401
0.05	0.05	0.0520565	0.0521563	0.0521661	0.0521633	0.0519287
0	0.5	0.0604556	0.0612521	0.0612476	0.0612623	0.0602428
0.1	0.4	0.0606453	0.0612492	0.0612899	0.0612812	0.0600583
0.25	0.25	0.0607521	0.0612475	0.0613137	0.0612918	0.0599545
0	1	0.071454	0.0736673	0.0735561	0.0736409	0.071840
0	2	0.0949	0.10186	0.10123	0.10162	0.0990
1	1	0.1000	0.10182	0.10234	0.10210	0.0943

Table VI (3), cont.

w_1	w_2	$Z^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
			m=5, n=5			
0	0.01	0.0500939	0.0500807	0.0500943	0.0500901	0.0500691
0	0.1	0.0509411	0.0508100	0.0509455	0.0509038	0.0506950
0.05	0.05	0.0509441	0.0508097	0.0509483	0.0509051	0.0506933
0	0.5	0.0547506	0.0541284	0.0547795	0.0545883	0.0535538
0.1	0.4	0.0548001	0.0541246	0.0548243	0.0546095	0.0535276
0.25	0.25	0.0548280	0.0541224	0.0548495	0.0546215	0.0535128
0	1	0.059609	0.0584539	0.0596835	0.0593478	0.057306
0	2	0.0697	0.06768	0.06983	0.06937	0.0654
1	1	0.0710	0.06760	0.07103	0.06993	0.0647
			m=5,n=15			
0	0.01	0.0501229	0.0501203	0.0501248	0.0501235	0.0501024
0	0.1	0.0512340	0.0512105	0.0512538	0.0512423	0.0510331
0.05	0.05	0.0512378	0.0512102	0.0512563	0.0512434	0.0510292
0	0.5	0.0562755	0.0562276	0.0564067	0.0563713	0.0553540
0.1	0.4	0.0563396	0.0562239	0.0564479	0.0563897	0.0552908
0.25	0.25	0.0563757	0.0562218	0.0564711	0.0564001	0.0552553
0	1	0.062806	0.0628946	0.0631535	0.0631409	0.061185
0	2	0.0766	0.07754	0.07763	0.07786	0.0743
1	1	0.0784	0.07747	0.07876	0.07836	0.0727

Table VI (3), cont.

w_1	w_2	$Z^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
			m=5, n=40			
0	0.01	0.0501416	0.0501442	0.0501445	0.0501449	0.0501248
0	0.1	0.0514232	0.0514528	0.0514615	0.0514597	0.0512607
0.05	0.05	0.0514274	0.0514526	0.0514631	0.0514604	0.0512548
0	0.5	0.0572756	0.0575132	0.0575293	0.0575362	0.0565885
0.1	0.4	0.0573451	0.0575110	0.0575546	0.0575472	0.0564927
0.25	0.25	0.0573841	0.0575098	0.0575688	0.0575533	0.0564389
0	1	0.064940	0.0656522	0.0655988	0.0656668	0.063901
0	2	0.0814	0.08380	0.08306	0.08370	0.0807
1	1	0.0833	0.08376	0.08407	0.08401	0.0782

Table VI (4)

Powers of five criteria for testing $(\lambda_1, \lambda_2) = (1,1)$

against different simple alternative hypotheses, $\alpha = 0.05$

λ_1	λ_2	$Z^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
			m=0,n=5			
1	1.001	0.050078	0.050079	0.050084	0.050082	0.050073
1.01	1.09	0.057916				
1	1.1	0.057825	0.058298	0.058614	0.058538	0.05778
1.025	1.075	0.058014				
1.05	1.05	0.058078	0.058237	0.058703	0.058525	0.05763
1	1.5	0.0810	0.0976	0.0976	0.0981	0.0955
1.25	1.25	0.0946	0.0964	0.0991	0.0983	0.0926
1	2	0.122	0.155	0.148	0.154	0.152
			m=0,n=15			
1	1.001	0.50089	0.050099	0.050100	0.050100	0.05093
1.01	1.09	0.059061				
1	1.1	0.058968	0.060546	0.060565	0.060580	0.05998
1.025	1.075	0.059163				
1.05	1.05	0.059228	0.060431	0.060527	0.060499	0.05973
1	1.5	0.0950	0.1122	0.1110	0.1119	0.1105
1.25	1.25	0.1017	0.1105	0.1110	0.1109	0.1058
1	2	0.137	0.187	0.184	0.186	0.186

Table VI (4), cont.

λ_1	λ_2	$Z^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
			m=0,n=40			
1	1.001	0.050094	0.050108	0.050109	0.050109	0.050102
1.01	1.09	0.059626				
1	1.1	0.059533	0.061574	0.061558	0.061571	0.06105
1.025	1.075	0.059726				
1.05	1.05	0.059791	0.061438	0.061455	0.061451	0.06075
1	1.5	0.0986	0.1189	0.1184	0.1187	0.1179
1.25	1.25	0.1052	0.1170	0.1171	0.1170	0.1124
1	2	0.146	0.202	0.200	0.201	0.203
			m=2,n=5			
1	1.001	0.050106	0.050095	0.050107	0.050103	0.050084
1.01	1.09	0.060910				
1	1.1	0.060771	0.060100	0.061036	0.060793	0.05901
1.025	1.075	0.061060				
1.05	1.05	0.061157	0.060015	0.061334	0.060878	0.05877
1	1.5	0.106	0.118	0.110	0.112	0.107
1.25	1.25	0.117	0.109	0.118	0.115	0.101
1	2	0.158	0.195	0.180	0.185	0.189

Table VI (4), cont.

λ_1	λ_2	$Z^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
			m=2,n=15			
1	1.001	0.050129	0.050131	0.050134	0.050133	0.050115
1.01	1.09	0.063568				
1	1.1	0.063412	0.064218	0.064302	0.064347	0.06284
1.025	1.075	0.063738				
1.05	1.05	0.063847	0.064081	0.064433	0.064333	0.06234
1	1.5	0.123	0.140	0.136	0.139	0.137
1.25	1.25	0.136	0.138	0.141	0.139	0.126
1	2	0.197	0.263	0.248	0.251	0.262
			m=2,n=40			
1	1.001	0.050142	0.050149	0.050150	0.050150	0.050134
1.01	1.09	0.065093				
1	1.1	0.064935	0.066400	0.066336	0.066388	0.06509
1.025	1.075	0.065264				
1.05	1.05	0.065374	0.066234	0.066304	0.066286	0.06442
1	1.5	0.134	0.157	0.154	0.155	0.155
1.25	1.25	0.147	0.154	0.155	0.155	0.141
1	2	0.225	0.299	0.290	0.291	0.304

Table VI (4), cont.

λ_1	λ_2	$Z^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
			m=5,n=5			
1	1.001	0.050122	0.050105	0.050123	0.050117	0.050089
1	1.09	0.062813				
1	1.1	0.062646	0.061244	0.062739	0.062368	0.05975
1.025	1.075	0.062994				
1.05	1.05	0.063111	0.061157	0.063167	0.062543	0.05944
1	1.5	0.117	0.122	0.121	0.122	0.115
1.25	1.25	0.132	0.119	0.133	0.128	0.106
1	2	0.173	0.248	0.232	0.227	0.239
			m=5,n=15			
1	1.001	0.050160	0.050156	0.050162	0.050161	0.050133
1.01	1.09	0.067195				
1	1.1	0.066983	0.067333	0.067529	0.067594	0.06516
1.025	1.075	0.067426				
1.05	1.05	0.067575	0.067187	0.067873	0.067685	0.06440
1	1.5	0.148	0.168	0.160	0.164	0.163
1.25	1.25	0.165	0.166	0.171	0.169	0.143
1	2	0.226	0.349	0.319	0.298	0.348

Table VI (4), cont.

λ_1	λ_2	$Z^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
			m=5,n=40			
1	1.001	0.050184	0.050187	0.050188	0.050188	0.050162
1.01	1.09	0.070146				
1	1.1	0.069923	0.071161	0.071181	0.071135	0.06902
1.025	1.075	0.070399				
1.05	1.05	0.070545	0.070981	0.071107	0.071098	0.06786
1	1.5	0.170	0.200	0.201	0.196	0.199
1.25	1.25	0.192	0.197	0.199	0.198	0.171
1	2	0.304	0.413	0.372	0.371	0.425

Table VI (5)

Powers of five criteria for testing $(\rho_1, \rho_2) = (0, 0)$

against different simple alternative hypotheses, $\alpha = 0.05$

ρ_1^2	ρ_2^2	$Z^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
			m=0, n=5			
0	0.0001	0.0500418	0.0500423	0.0500447	0.0500437	0.0500394
0	0.0025	0.0510471	0.0510638	0.0511230	0.0511009	0.0509921
0.00125	0.00125	0.0510521	0.0510621	0.0511248	0.0511006	0.0509888
0	0.01	0.0542198	0.0543437	0.0545617	0.0544873	0.0540580
0.005	0.005	0.0542998	0.0543162	0.0545901	0.0544813	0.0540040
0.005	0.01	0.0565214	0.0565494	0.0569695	0.0568029	0.0560686
0.001	0.05	0.0725359	0.0746873	0.0753383	0.0753126	0.0732126
			m=0, n=15			
0	0.0001	0.0501068	0.0501195	0.0501205	0.0501202	0.0501120
0	0.0025	0.0526810	0.0530274	0.0530506	0.0530449	0.0528426
0.00125	0.00125	0.0527133	0.0530225	0.0530542	0.0530438	0.0528223
0	0.01	0.0608567	0.0626238	0.0626665	0.0626733	0.0619218
0.005	0.005	0.0613750	0.0625413	0.0627207	0.0626526	0.0615920
0.005	0.01	0.067450	0.0692811	0.0695676	0.0694583	0.0677713
0.001	0.05	0.109524	0.128431	0.127511	0.128229	0.12552

Table VI (5), cont.

ρ_1^2	ρ_2^2	$Z^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
			m=0, n=40			
0	0.0001	0.0502700	0.0503118	0.0503123	0.0503122	0.050294
0	0.0025	0.0568058	0.0580362	0.0580374	0.0580399	0.057626
0.00125	0.00125	0.0570089	0.0580230	0.0580451	0.0580367	0.057503
0	0.01	0.077842	0.0850749	0.0849585	0.0850333	0.083768
0.005	0.005	0.08108	0.0848456	0.0850549	0.0849607	0.081798
0.005	0.01	0.0987	0.104947	0.1052942	0.1051379	0.099884
0.001	0.05	0.202	0.293	0.292	0.2919	0.2915
			m=2, n=5			
0	0.0001	0.0500302	0.0500271	0.0500306	0.0500294	0.0500240
0	0.0025	0.0507569	0.0506829	0.0597679	0.0507384	0.0506041
0.00125	0.00125	0.0507582	0.0506818	0.0507686	0.0507381	0.0506022
0	0.01	0.0530633	0.0527847	0.0531136	0.0530040	0.0524680
0.005	0.005	0.0530832	0.0527668	0.0531263	0.0529994	0.0524375
0.005	0.01	0.0546709	0.0541946	0.0547384	0.0545478	0.0536927
0.001	0.05	0.0666751	0.0657494	0.0671257	0.0668045	0.0640729

Table VI (5), cont.

ρ_1^2	ρ_2^2	$Z^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
			m=2, n=15			
0	0.0001	0.0500738	0.0500748	0.0500766	0.0500761	0.0500662
0	0.0025	0.0518588	0.0518926	0.0519340	0.0519230	0.0516797
0.00125	0.00125	0.0518671	0.0518894	0.0519364	0.0519223	0.0516688
0	0.01	0.0575972	0.0578622	0.0579827	0.0579658	0.0570277
0.005	0.005	0.0577327	0.0578082	0.0580192	0.0579535	0.0568488
0.005	0.01	0.0618318	0.0619790	0.0622961	0.0622003	0.0604935
0.001	0.05	0.09352	0.098642	0.09794	0.09865	0.09477
			m=2,n=40			
0	0.0001	0.0501831	0.0501922	0.0501930	0.0501927	0.050172
0	0.0025	0.0546473	0.0549336	0.0549436	0.0549437	0.054456
0.00125	0.00125	0.0546993	0.0549250	0.0549492	0.0549419	0.054391
0	0.01	0.069434	0.0713659	0.0712785	0.0713484	0.069681
0.005	0.005	0.070308	0.0712151	0.0713619	0.073104	0.068605
0.005	0.01	0.081767	0.083379	0.083587	0.083517	0.079215
0.001	0.05	0.178	0.203	0.200	0.2018	0.200

Table VI (5), cont.

ρ_1^2	ρ_2^2	$Z^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
			m=5,n=5			
0	0.0001	0.0500244	0.0500210	0.0500245	0.0500234	0.0500180
0	0.0025	0.0506128	0.0505270	0.0506156	0.0505883	0.0504521
0.00125	0.00125	0.0506132	0.0505262	0.0506159	0.0505880	0.0504508
0	0.01	0.0524825	0.0521457	0.0524958	0.0523903	0.0518441
0.005	0.005	0.0524895	0.0521324	0.0525007	0.0523859	0.0518229
0.005	0.01	0.0537695	0.0532302	0.0537871	0.0536142	0.0527597
0.001	0.05	0.063598	0.0620581	0.0637059	0.0632872	0.06044
			m=5,n=15			
0	0.0001	0.0500565	0.0500553	0.0500574	0.0500568	0.0500471
0	0.0025	0.0514239	0.0513976	0.0514470	0.0514341	0.0511932
0.00125	0.00125	0.0514270	0.0513951	0.0514483	0.0514335	0.0511864
0	0.01	0.0558280	0.0557810	0.0559488	0.0559152	0.0549688
0.005	0.005	0.0558796	0.0557421	0.0559688	0.0559049	0.0548558
0.005	0.01	0.0589811	0.0587867	0.0591265	0.0590335	0.0574242
0.001	0.05	0.08393	0.08513	0.08511	0.08551	0.08118

Table VI (5), cont.

ρ_1^2	ρ_2^2	$Z^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
			m=5,n=40			
0	0.0001	0.0501359	0.0501384	0.0501391	0.0501391	0.050120
0	0.0025	0.0534524	0.0535390	0.0535552	0.0535538	0.053083
0.00125	0.00125	0.0534718	0.0535329	0.0535587	0.0535523	0.053044
0	0.01	0.064479	0.0651641	0.0651140	0.0651804	0.063465
0.005	0.005	0.064816	0.0650586	0.0651834	0.0651526	0.062814
0.005	0.01	0.073113	0.0735602	0.0727100	0.073695	0.070034
0.001	0.05	0.138	0.156	0.152	0.1550	0.152

Table VI (6)

Powers of $Z^{(2)}$, $U^{(2)}$. $V^{(2)}$ and $W^{(2)}$ test for testing $(w_1,w_2)=(0,0)$ against different simple alternatives for larger deviation parameters,

$\alpha = 0.05$

w_1	w_2	$Z^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
		m = 0, n = 30			
0	5	0.209	0.311	0.307	0.308
2.5	2.5	0.304	0.311	0.317	0.313
0	8	0.292	0.493	0.485	0.487
4	4	0.488	0.495	0.505	0.498
0	10	0.340	0.603	0.594	0.589
5	5	0.600	0.605	0.616	0.603
		m = 2, n = 30			
0	5	0.168	0.204	0.199	0.202
2.5	2.5	0.202	0.204	0.207	0.206
0	8	0.247	0.330	0.317	0.324
4	4	0.328	0.330	0.337	0.336
0	10	0.297	0.417	0.398	0.408
5	5	0.418	0.417	0.440	0.420
		m = 5, n = 30			
0	5	0.134	0.148	0.143	0.146
2.5	2.5	0.147	0.148	0.150	0.149
0	8	0.194	0.230	0.215	0.225
4	4	0.231	0.230	0.234	0.231
0	10	0.236	0.292	0.265	0.284
5	5	0.294	0.292	0.297	0.294

Table VI (7)

Powers of $Z^{(2)}$, $U^{(2)}$, $V^{(2)}$ and $W^{(2)}$ for testing $(\rho_1^2, \rho_2^2) = (0,0)$ against different simple alternatives for larger deviation parameters, $\alpha = 0.05$

ρ_1^2	ρ_2^2	$Z^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
			m = 0, n = 30		
0	0.1	0.274	0.452	0.445	0.447
0.05	0.05	0.420	0.432	0.440	0.434
0	0.15	0.472	0.673	0.664	0.644
			m = 2, n = 30		
0	0.1	0.241	0.321	0.308	0.314
0.05	0.05	0.301	0.303	0.308	0.305
0	0.15	0.353	0.516	0.513	0.489
			m = 5, n = 30		
0	0.1	0.204	0.245	0.228	0.239
0.05	0.05	0.229	0.231	0.235	0.233
0	0.15	0.309	0.401	0.377	0.382

CHAPTER VII

SUMMARY AND CONCLUSION

This dissertation has been concerned with some studies on the criteria: 1) Hotelling's trace, 2) Pillai's trace, 3) Wilks' criterion, 4) Roy's largest root, and 5) Wilks-Lawley statistic, including their central and non-central distributions, in connection with testing three basic hypotheses in multivariate analysis. The hypotheses considered are: (A) equality of covariance matrices of two p-variate normal populations, (B) equality of mean vectors of ℓ p-variate normal populations having a common covariance matrix and (C) independence of two sets of variates distributed multivariate normal.

Chapter I deals with the derivations of the distributions and moments of the first four criteria under violations using Pillai's distribution. The results obtained are the densities of Hotelling's trace under a condition, Wilks' criterion and Roy's largest root, and also the moments of Hotelling's trace and the m.g.f. of Pillai's trace. It was shown also that corresponding results obtained earlier by previous authors are special cases of ours.

Exact robustness studies were carried out in Chapter II for the tests of the hypotheses (A) when normality assumption is violated and of (B) when the assumption of a common covariance matrix is disturbed. For these studies, first general forms of the distributions of the criteria were derived and then some calculations of powers of the criteria were carried out for various values of parameters. The studies were performed in the two-roots case and the hypotheses considered were simple.

The next three chapters were devoted to obtain the distribution of Hotelling's trace for number of roots greater than two. Chapter III deals with the derivation of its distribution in the central case for three and four roots. Here, the Pillai and Young approach of the inverse Laplace transform has been modified to extend their results and to obtain much simpler forms. Its non-central distribution for the three-roots case was obtained for a special value of one of the degrees of freedoms in Chapter IV using a lemma on the Vandermonde determinant due to Pillai and methods similar to those employed in Chapter III. The distribution is expressed as a series which converges reasonably fast. The powers of the criterion in connection with testing the hypotheses of (A) assuming zero mean vectors, of (B) assuming a common covariance matrix and of (C) were computed and comparisons made with those of the largest root. This is the first time exact powers for Hotelling's trace was obtained for the three-roots case. In Chapter V, an expression suggested by Pillai for the density of Hotelling's trace as an incomplete beta series which is an improvement of the one obtained by Constantine has been studied. Pillai's suggested form is exact for p=2 and for higher values of p some upper percentage points were computed and comparisons made with the exact results tabulated by Pillai and Young. Also some powers of the criterion were computed in connection with testing the hypothesis (B) when the covariance matrices are equal.

Finally, the distribution of the fifth criterion, i.e. the Wilks-Lawley statistic, was derived in the last chapter. Its expression is given in two forms, and in the central case, the relation between the distribution of this statistic and that of Wilks' criterion was obtained. The powers of this statistic for the three hypotheses were computed and studied in the two-roots case and comparisons made with other criteria.

In conclusion, this dissertation is an attempt at an exact study of robustness and power of some tests in multivariate analysis. The following are some suggestions for future work:

1. Although there is some advantage in the "randomness" concept introduced by Pillai, it may still be desirable to obtain the distribution of the characteristic roots $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ under violations and carry out the work of the first two chapters without the "randomness" assumption.
2. Exact robustness studies may be extended to the canonical correlation problem.
3. Further investigation of the new distributional form of Hotelling's trace suggested by Pillai may be carried out to establish the exactness or otherwise of the form for number of roots greater than two.

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2. Exact robustness studies may be extended to the canonical correlation problem.
3. Further investigation of the new distributional form of Hotelling's trace suggested by Pillai may be carried out to establish the exactness or otherwise of the form for number of roots greater than two.

LIST OF REFERENCES

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[1] Al-Ani, S. (1968). Some distribution problems concerning characteristic roots and vectors in multivariate analysis. Mimeograph Series No. 162, Department of Statistics, Purdue University.

[2] Anderson, T. W. (1958). Introduction to Multivariate Analysis. John Wiley, New York.

[3] Constantine, A. G. (1963). Some non-central distribution problems in multivariate analysis. Ann. Math. Statist. 34, 1270-1285.

[4] Constantine, A. G. (1966). The distribution of Hotelling's generalized T_o^2. Ann. Math. Statist. 37, 215-225.

[5] Consul, P. C. (1966). On some inverse Mellin integral transforms. Academic Royale Des Science de Belgique 52, 547-561.

[6] Consul, P. C. (1967a). On the exact distributions of likelihood ratio criteria for testing independence of sets of variates under null hypothesis. Ann. Math. Statist. 38, 1160-1169.

[7] Consul, P. C. (1967b). On the exact distribution of the W criterion for testing sphericity in a p-variate normal distribution. Ann. Math. Statist. 38, 1170-1174.

[8] Davis, A. W. (1968). A system of linear differential equations for the distribution of Hotelling's generalized T_o^2. Ann. Math. Statist. 39, 815-832.

[9] Davis, A. W. (1970). Exact distribution of Hotelling's generalized T_o^2. Biometrika, 57, 187-191.

[10] Gnanadesikan, R. et al. (1971). Analysis and design of certain quantitative multiresponse experiments. Pergamon Press, New York.

[11] Herz, C. S. (1955). Bessel functions of matrix argument. Ann. Math. 61, 474-523.

[12] Hotelling, H. (1931). The generalization of Student's ratio. Ann. Math. Statist. 2, 360-378.

[13] Hotelling, H. (1947). Multivariate quality control, illustrated by the air testing of sample bomb sights. Techniques of Statistical Analysis, 11-184. McGraw-Hill, New York.

[14] Hotelling, H. (1951). A generalized T-test and measure of multivariate dispersion. Proc. Second Berkeley Symp. 23-42.

[15] Ito, K. (1969). On the effect of heterroscedasticity and non-normality upon some multivariate test procedures. Multivariate Analysis II. Ed. P. R. Krishnaiah, Academic Press, New York.

[16] Ito, K. and Schull, W. J. (1964). On the robustness of the T_o^2 test in multivariate analysis of variance when variance-covariance matrices are not equal. Biometrika 51, 71-82.

[17] James, A. T. (1964). Distribution of matrix variates and latent roots derived from normal samples. Ann. Math. Statist. 38, 944-948.

[18] Khatri, C. G. (1966). On certain distribution problems based on positive definite quadratic functions in normal vectors. Ann. Math. Statist. 37, 468-479.

[19] Khatri, C. G. (1967). Some distribution problems connected with the characteristic roots of $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$. Ann. Math. Statist. 38, 944-948.

[20] Khatri, C. G. and Pillai, K. C. S. (1968). On the non-central distributions of two test criteria in multivariate analysis of variance. Ann. Math. Statist. 39, 215-226.

[21] Krishnaiah, P. R. and Chang, T. C. (1962). On the exact distributions of the traces of $\underset{\sim}{S}_1(\underset{\sim}{S}_1+\underset{\sim}{S}_2)^{-1}$ and $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$. Sankhya: series A, 1-8.

[22] Lawley, D. N. (1938). A generalization of Fisher's z test. Biometrika, 30, 180-187.

[23] Meijer, C. S. (1946a). On the G-function. II, III, IV. Nederl. Akad. Wetensch. Proc. 49, 344-356, 457-469, 632-641.

[24] Meijer, C. S. (1946b). On the G-function I. Indag. Math. 8, 124-134.

[25] Pillai, K. C. S. (1955). Some new test criteria in multivariate analysis. Ann. Math. Statist. 26, 117-211.

[26] Pillai, K. C. S. (1956). Some results useful in multivariate analysis. Ann. Math. Statist. 27, 1106-1114.

[27] Pillai, K. C. S. (1956). On the distribution of the largest or smallest root of a matrix in multivariate analysis. Biometrika. 43, 122-127.

[28] Pillai, K. C. S. (1964). On the moments of elementary symmetric functions of the roots of two matrices. Ann. Math. Statist. 35, 1704-1712.

[29] Pillai, K. C. S. (1965). On the non-central distributions of the largest roots of two matrices in multivariate analysis. Mimeograph Series No. 51, Department of Statistics, Purdue University.

[30] Pillai, K. C. S. (1968). On the moment generating function of Pillai's $V^{(s)}$ criterion. Ann. Math. Statist. 39, 877-880.

[31] Pillai, K. C. S. (1972). The distribution of the characteristic roots of $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ under violations. Mimeograph Series No. 278, Department of Statistics, Purdue University.

[32] Pillai, K. C. S. (1973). A unified distribution theory for robustness studies of tests in multivariate analysis. Mimeograph Series No. 320, Department of Statistics, Purdue University. (Invited paper presented at the meeting of the Institute of Mathematical Statistics, Ithaca, May 30-June 1, 1973).

[33] Pillai, K. C. S. and Al-Ani, S. (1970). Power comparisons of tests of equality of two covariance matrices based on individual characteristic roots. Jour. Amer. Statist. Ass. 65, 1438-1446.

[34] Pillai, K. C. S. and Chang, T. C. (1968). On the distributions of Hotelling's T_o^2 for three latent roots and the smallest root of a covariance matrix. Mimeograph Series No. 147, Department of Statistics, Purdue University.

[35] Pillai, K. C. S. and Dotson, C. O. (1969). Power comparisons of tests of two multivariate hypotheses based on individual roots. Amer. Inst. Statist. Math. 21, 49-66.

[36] Pillai, K. C. S. and Gupta, A. K. (1969). On the exact distribution of Wilks' criterion. Biometrika 56, 109-118.

[37] Pillai, K. C. S. and Jayachandran, K. (1967). Power comparisons of tests of two multivariate hypotheses based on four criteria. Biometrika, 54, 195-210.

[38] Pillai, K. C. S. and Jayachandran, K. (1968). Power comparisons of tests of equality of two covariance matrices based on four criteria. Biometrika, 55, 335-342.

[39] Pillai, K. C. S. and Jouris, G. M. (1969). On the moments of elementary symmetric functions of the roots of two matrices. Ann. Inst. Statist. Math. 21, 309-320.

[40] Pillai, K. C. S. and Nagarsenker, B. N. (1972). On the distributions of a class of statistics in multivariate analysis. Jour. Multi. Analysis. 2, 96-111.

[41] Pillai, K. C. S. and Young, D. L. (1971). On the exact distribution of Hotelling's generalized T_o^2. Jour. of Mult. Analysis. 1, 90-107.

[42] Pillai, K. C. S., Al-Ani, S. and Jouris, G. M. (1969). On the distributions of the ratios of the roots of a covariance matrix and Wilks' criterion for tests of three hypotheses. Ann. Math. Statist. 40, 2033-2040.

[43] Roy, S. N. (1945). The individual sampling distribution of the maximum, the minimum and any intermediate of the p-statistics on the null hypotheses. Sankhya. 7, 133-158.

[44] Snow, C. (1952). Hypergeometric and Legendre functions with applications to integral equations of potential theory. Nat. Bureau of Standards Appl. Math. series 19.

[45] Wilks, S. S. (1932). Certain generalizations in the analysis of variance. Biometrika, 24, 471-494.

[46] Young, D. L. (1970). Some tests of equality of several covariance matrices. Mimeograph Series No. 241, Department of Statistics, Purdue University.

APPENDICES

Appendix A

The constants $b_{\kappa}(r,s)$ up to $k = 6$ for the c.d.f. of the criteria in the two-roots case.

κ	r \ s	0	1	2
(1)	1			
(2)	2	1		
	0		$-\frac{4}{3}$	
(1^2)	0		$\frac{4}{3}$	
(3)	3	1		
	1		$-\frac{12}{5}$	
(21)	1		$\frac{12}{5}$	
(4)	4	1		
	2		$-\frac{24}{7}$	
	0			$\frac{48}{35}$
(31)	2		$\frac{24}{7}$	
	0			$-\frac{32}{7}$
(2^2)	0			$\frac{16}{5}$

κ	r \ s	0	1	2	3
(5)	5	1			
	3		$-\frac{40}{9}$		
	1			$\frac{80}{21}$	
(41)	3		$\frac{40}{9}$		
	1			$-\frac{32}{3}$	
(32)	1			$\frac{48}{7}$	
(6)	6	1			
	4		$-\frac{60}{11}$		
	2			$\frac{80}{11}$	
	0				$-\frac{320}{231}$
(51)	4		$\frac{60}{11}$		
	2			$-\frac{1440}{77}$	
	0				$\frac{576}{77}$
(42)	2			$\frac{80}{7}$	
	0				$-\frac{320}{21}$
(3^2)	0				$\frac{64}{7}$

Appendix B

The coefficients D_{ij} for the c.d.f. of the criteria in the two-roots case.

$D_{00} = 1+A_{11}+3A_{21}+5A_{31}+35A_{41}+63A_{51}+231A_{61}$;

$D_{01} = A_{11}+6A_{21}+15A_{31}+140A_{41}+315A_{51}+1386A_{61}$;

$D_{11} = A_{11}+2A_{21}+A_{22}+3A_{31}+A_{32}+20A_{41}+3A_{42}+35A_{51}+5A_{52}+126A_{61}+35A_{62}$;

$D_{02} = 3A_{21}+15A_{31}+210A_{41}+630A_{51}+3465A_{61}$;

$D_{12} = 6A_{21}+18A_{31}+A_{32}+180A_{41}+6A_{42}+420A_{51}+15A_{52}+1890A_{61}+140A_{62}$;

$D_{22} = 3A_{21}+3A_{31}+A_{32}+18A_{41}+2A_{42}+A_{43}+30A_{51}+3A_{52}+A_{53}+105A_{61}+20A_{62}+3A_{63}$;

$D_{03} = 5A_{31}+140A_{41}+630A_{51}+4620A_{61}$;

$D_{13} = 15A_{31}+300A_{41}+3A_{42}+1050A_{51}+15A_{52}+6300A_{61}+210A_{62}$;

$D_{23} = 15A_{31}+180A_{41}+6A_{42}+450A_{51}+18A_{52}+A_{53}+2100A_{61}+180A_{62}+6A_{63}$;

$D_{33} = 5A_{31}+20A_{41}+3A_{42}+30A_{51}+3A_{52}+A_{53}+100A_{61}+18A_{62}+2A_{63}+A_{64}$;

$D_{04} = 35A_{41}+315A_{51}+3465A_{61}$;

$D_{14} = 140A_{41}+980A_{51}+5A_{52}+8820A_{61}+140A_{62}$;

$D_{24} = 210A_{41}+1050A_{51}+15A_{52}+7350A_{61}+300A_{62}+3A_{63}$;

$D_{34} = 140A_{41}+420A_{51}+15A_{52}+2100A_{61}+180A_{62}+6A_{63}$;

$D_{44} = 35A_{41}+35A_{51}+5A_{52}+105A_{61}+20A_{62}+3A_{63}$;

$D_{05} = 63A_{51}+1386A_{61}$; $D_{15} = 315A_{51}+5670A_{61}+35A_{62}$;

$D_{25} = 630A_{51}+8820A_{61}+140A_{62}$; $D_{06} = D_{66} = 231A_{61}$;

$D_{35} = 630A_{51}+6300A_{61}+210A_{62}$; $D_{16} = D_{56} = 1386A_{61}$;

$D_{45} = 315A_{51}+1890A_{61}+140A_{62}$; $D_{26} = D_{46} = 3465A_{61}$;

$D_{55} = 63A_{51}+126A_{61}+35A_{62}$; $D_{36} = 4620A_{61}$;

where the coefficients A_{ij} are given by (after letting $m_i = \nu + 2i$ and $n_i = \nu - i$)

$$A_{11} = \frac{1}{2} m_o(1+B_{11}) \; ; \; A_{21} = \frac{m_o m_1}{24} (1+2B_{11}+B_{22})$$

$$A_{22} = \frac{m_o n_1}{6} (1+2B_{11}+B_{21}) \; ; \; A_{31} = \frac{m_o m_1 m_2}{240} (1+3B_{11}+3B_{22}+B_{33})$$

$$A_{32} = \frac{m_o m_1 n_1}{20} (1+3B_{11}+ \frac{4}{3} B_{22} + \frac{5}{3} B_{21}+B_{32}) \; ;$$

$$A_{41} = \frac{m_o m_1 m_2 m_3}{13440} (1+4B_{11}+6B_{22}+4B_{33}+B_{44}) \; ;$$

$$A_{42} = \frac{m_o m_1 m_2 n_1}{336} (1+4B_{11}+ \frac{11}{3} B_{22}+ \frac{7}{3} B_{21}+ \frac{6}{5} B_{33}+ \frac{14}{5} B_{32}+B_{43}) \; ;$$

$$A_{43} = \frac{m_o m_1 n_1 (n_1+2)}{120} (1+4B_{11}+ \frac{8}{3} B_{22}+ \frac{10}{3} B_{21}+4B_{32}+B_{42}) \; ;$$

$$A_{51} = \frac{m_o m_1 m_2 m_3 m_4}{241920} (1+5B_{11}+10B_{22}+10B_{33}+5B_{44}+B_{55}) \; ;$$

$$A_{52} = \frac{m_o m_1 m_2 m_3 n_1}{4320} (1+5B_{11}+7B_{22}+3B_{21}+ \frac{23}{5} B_{33}+ \frac{27}{5} B_{32}+ \frac{8}{7} B_{44}+ \frac{27}{7} B_{43}+B_{54}) \; ;$$

$$A_{53} = \frac{m_o m_1 m_2 n_1 (n_1+2)}{560} (1+5B_{11}+ \frac{16}{3} B_{22}+ \frac{14}{3} B_{21}+ \frac{8}{5} B_{33}+ \frac{42}{5} B_{32}+ \frac{8}{3} B_{43} + \frac{7}{3} B_{42}+B_{53}) \; ;$$

$$A_{61} = \frac{m_o m_1 \cdots m_5}{(14784)(720)} (1+6B_{11}+15B_{22}+20B_{33}+15B_{44}+6B_{55}+B_{66}) \; ;$$

$$A_{62} = \frac{m_o m_1 \cdots m_4 n_1}{295680} (1+6B_{11}+ \frac{34}{3} B_{22}+ \frac{11}{3} B_{21}+ \frac{56}{5} B_{33}+ \frac{44}{5} B_{32}+ \frac{39}{7} B_{44} + \frac{66}{7} B_{43}+ \frac{10}{9} B_{55}+ \frac{44}{9} B_{54}+B_{65})$$

$$A_{63} = \frac{m_o m_1 m_2 m_3 n_1 (n_1+2)}{12096} (1+6B_{11}+9B_{22}+6B_{21}+ \frac{28}{5} B_{33}+ \frac{72}{5} B_{32}+ \frac{48}{35} B_{44}+ \frac{66}{7} B_{43} + \frac{21}{5} B_{42}+ \frac{12}{5} B_{54}+ \frac{18}{5} B_{53}+B_{64})$$

$$A_{64} = \frac{m_o m_1 m_2 n_1 (n_1+2)(n_1+4)}{5040} (1+6B_{11}+8B_{22}+7B_{21}+ \frac{16}{5} B_{33}+ \frac{84}{5} B_{32} +8B_{43} +7B_{42}+6B_{53}+B_{63}) \; ;$$

while the values of B_{ij} are given by (after letting $r_i = f_i + 2i$ and $s_i = f_1 - i$)

$$B_{11} = -\frac{1}{2} a_1 \left(1 - \frac{b_1}{r_o}\right) ; \quad B_{22} = \frac{a_{21}}{8} \left(1 - \frac{2b_1}{r_o} + \frac{b_{21}}{2r_o r_1}\right) ;$$

$$B_{21} = a_{22} \left(1 - \frac{2b_1}{r_o} + \frac{4b_{22}}{r_o s_1}\right) ; \quad B_{33} = \frac{-a_{31}}{16} \left(1 - \frac{3b_1}{r_o} + \frac{3b_{21}}{2r_o r_1} - \frac{b_{31}}{2r_o r_1 r_2}\right) ;$$

$$B_{32} = \frac{-a_{32}}{2} \left(1 - \frac{3b_1}{r_o} + \frac{2b_{21}}{3r_o r_1} + \frac{20b_{22}}{3r_o s_1} - \frac{4b_{32}}{r_o r_1 s_1}\right) ;$$

$$B_{44} = \frac{a_{41}}{128} \left(1 - \frac{4b_1}{r_o} + \frac{3b_{21}}{r_o r_1} - \frac{2b_{31}}{r_o r_1 r_2} + \frac{b_{41}}{8r_o r_1 r_2 r_3}\right) ;$$

$$B_{43} = \frac{a_{42}}{8} \left(1 - \frac{4b_1}{r_o} + \frac{11b_{21}}{6r_o r_1} + \frac{28b_{22}}{3r_o s_1} - \frac{3b_{31}}{5r_o r_1 r_2} - \frac{56b_{32}}{5r_o r_1 s_2} + \frac{2b_{42}}{r_o r_1 r_2 s_1}\right)$$

$$B_{42} = a_{43} \left(1 - \frac{4b_1}{r_o} + \frac{4b_{21}}{3r_o s_1} + \frac{40b_{22}}{3r_o s_1} - \frac{16b_{32}}{r_o r_1 s_1} + \frac{16b_{43}}{r_o r_1 s_1 (s_1+2)}\right) ;$$

$$B_{55} = \frac{-a_{51}}{256} \left(1 - \frac{5b_1}{r_o} + \frac{5b_{21}}{r_o r_1} - \frac{5b_{31}}{r_o r_1 r_2} + \frac{5b_{41}}{8r_o r_1 r_2 r_3} - \frac{b_{51}}{8r_o r_1 \ldots r_4}\right) ;$$

$$B_{54} = \frac{-a_{52}}{16} \left(1 - \frac{5b_1}{r_o} + \frac{7b_{21}}{2r_o r_1} + \frac{12b_{22}}{r_o s_1} - \frac{23b_{31}}{10r_o r_1 r_2} - \frac{108b_{32}}{5r_o r_1 s_1} + \frac{b_{41}}{7r_o r_1 r_2 r_3}\right.$$
$$\left. + \frac{54b_{42}}{7r_o r_1 r_2 s_1} - \frac{2b_{52}}{r_o r_1 r_2 r_3 s_1}\right) ;$$

$$B_{53} = \frac{-a_{53}}{2} \left(1 - \frac{5b_1}{r_o} + \frac{8b_{21}}{3r_o r_1} + \frac{56b_{22}}{3r_o s_1} - \frac{4b_{31}}{5r_o r_1 r_2} - \frac{168b_{32}}{5r_o r_1 s_1} + \frac{16b_{42}}{3r_o r_1 r_2 s_1}\right.$$
$$\left. + \frac{112b_{43}}{3r_o r_1 s_1 (s_1+2)} - \frac{16b_{53}}{r_o r_1 r_2 s_1 (s_1+2)}\right) ;$$

$$B_{66} = \frac{a_{61}}{1024} \left(1 - \frac{6b_1}{r_o} + \frac{15b_{21}}{2r_o r_1} - \frac{10b_{31}}{r_o r_1 r_2} + \frac{15b_{41}}{8r_o r_1 r_2 r_3} - \frac{3b_{51}}{4r_o r_1 \ldots r_4}\right.$$
$$\left. + \frac{b_{61}}{16r_o \ldots r_5}\right) ;$$

$$B_{65} = \frac{a_{62}}{128} \left(1 - \frac{6b_1}{r_o} + \frac{17b_{21}}{3r_o r_1} + \frac{44b_{22}}{3r_o s_1} - \frac{28b_{31}}{5r_o r_1 r_2} - \frac{176b_{32}}{5r_o r_1 s_1} + \frac{39b_{41}}{56r_o r_1 r_2 r_3}\right.$$
$$\left. + \frac{132b_{42}}{7r_o r_1 r_2 s_1} - \frac{5b_{51}}{36r_o r_1 \ldots r_4} - \frac{88b_{52}}{9r_o r_1 r_2 r_3 s_1} + \frac{b_{62}}{2r_o r_1 r_2 r_3 r_4 s_1}\right) ;$$

$$B_{64} = \frac{a_{63}}{8}\Big(1 - \frac{6b_1}{r_o} + \frac{9b_{21}}{2r_or_1} + \frac{24b_{22}}{r_os_1} - \frac{14b_{31}}{5r_or_1r_2} - \frac{288b_{32}}{5r_or_1s_1} + \frac{6b_{41}}{35r_or_1r_2r_3}$$

$$+ \frac{132b_{42}}{7r_or_1r_2s_1} + \frac{336b_{43}}{5r_or_1s_1(s_1+2)} - \frac{24b_{52}}{5r_or_1r_2r_3s_1} - \frac{288b_{53}}{5r_or_1r_2s_1(s_1+2)}$$

$$+ \frac{8b_{63}}{r_or_1r_2r_3s_1(s_1+2)}\Big) ;$$

$$B_{63} = a_{64}\Big(1 - \frac{6b_1}{r_o} + \frac{4b_{21}}{r_or_1} + \frac{28b_{22}}{r_os_1} - \frac{8b_{31}}{5r_or_1r_2} - \frac{336b_{32}}{5r_or_1s_1} + \frac{16b_{42}}{r_or_1r_2s_1}$$

$$+ \frac{112b_{43}}{r_or_1s_1 \; s_1+2)} - \frac{96b_{53}}{r_or_1r_2s_1(s_1+2)} + \frac{64b_{64}}{r_or_1r_2s_1(s_1+2)(s_1+4)}\Big) ;$$

with $a_1 = \lambda_1^{-1}+\lambda_2^{-1}$, $a_2 = (\lambda_1\lambda_2)^{-1}$, and

$a_{21} = 3a_1^2-4a_2$; $a_{22} = a_2$; $a_{31} = 5a_1^3-12a_1a_2$; $a_{32} = a_1a_2$

$a_{41} = 35a_1^4-120a_1^2a_2+48a_2^2$; $a_{42} = 3a_1^2a_2-4a_2^2$; $a_{43} = a_2^2$

$a_{51} = 63a_1^5-280a_1^3a_2+240a_1a_2^2$; $a_{52} = 5a_1^3a_2-12a_1a_2^2$; $a_{53} = a_1a_2^2$;

$a_{61} = 231a_1^6-1260a_1^4a_2+1680a_1^2a_2^2-320a_2^3$; $a_{62} = 35a_1^4a_2-120a_1^2a_2^2+48a_2^3$;

$a_{63} = 3a_1^2a_2^2-4a_2^3$; $a_{64} = a_2^3$;

and $b_1 = w_1+w_2$, $b_2 = w_1w_2$ and b_{ij}'s are obtained from corresponding

a_{ij}'s by replacing a_1 and a_2 by b_1 and b_2 respectively.

Appendix C

The constants $b_\kappa(r,s,t)$ up to $k = 6$ for the c.d.f. of $U^{(3)}$.

Remark: Since $b_\kappa(r,s) = b_\kappa(r,s,o)$, the table below gives only $b_\kappa(r,s,t)$ for $t \neq 0$. Note also $b_{(41^2)}(0,3,0) = 0$.

κ	t	1	1
	r \ s	0	1
(3)	0	$\frac{8}{5}$	
(21)	0	$-\frac{18}{5}$	
(1^3)	0	2	
(4)	1	$\frac{96}{35}$	
(31)	1	$-\frac{80}{21}$	
(2^2)	1	$-\frac{64}{15}$	
(21^2)	1	$\frac{16}{3}$	
(5)	2	$\frac{80}{21}$	
	0		$-\frac{64}{21}$
(41)	2	$-\frac{14}{3}$	
	0		$\frac{40}{3}$
(32)	2	$-\frac{64}{7}$	
	0		$-\frac{160}{21}$
(31^2)	2	10	
	0		$-\frac{40}{3}$
$(2^2 1)$	0		$\frac{32}{3}$

κ	t	1	2	1
	r \ s	0		1
(6)	3	$\frac{160}{33}$		
	1			$-\frac{640}{77}$
	0		$\frac{128}{77}$	
(51)	3	$-\frac{432}{77}$		
	1			$\frac{10944}{385}$
	0		$-\frac{3456}{385}$	
(42)	3	$-\frac{320}{21}$		
	1			$\frac{80}{7}$
	0		$-\frac{40}{7}$	
(41^2)	3	16		
	1			$-\frac{192}{5}$
	0		$\frac{128}{5}$	$-\frac{678}{35}$
(3^2)	1			$-\frac{768}{35}$
	0		$\frac{512}{35}$	
(321)	1			$\frac{144}{5}$
	0		$-\frac{216}{5}$	
(31^3)	0		0	
(2^3)	0		16	

Appendix D

The coefficients A'_{ij} for the distribution of $U^{(3)}$

$A'_{12} = \ldots = A'_{17} = A'_{23} = \ldots = A'_{27} = A'_{34} = \ldots = A'_{37} = A'_{45} = \ldots = A'_{47} = A'_{56} = A'_{57} = 0$; $A'_{11}, A'_{21}, A'_{22}, A'_{31}, A'_{41}, A'_{51}, A'_{61}$ are the corresponding A_{ij}'s as given in Appendix B; $A'_{32} = \frac{1}{2} A_{32}$, $A'_{42} = \frac{1}{3} A_{42}$, $A'_{43} = \frac{1}{3} A_{43}$, $A'_{52} = \frac{1}{4} A_{52}$; $A'_{53} = \frac{1}{9} A_{53}$, $A'_{62} = \frac{1}{5} A_{62}$, $A'_{63} = \frac{1}{2} A_{63}$, where the A_{ij}'s are as in Appendix B and the other A'_{ij}'s are given below (using $m_j = \nu + 2j$ and $n_j = \nu - j$):

$$A'_{33} = \frac{m_o n_1 n_2}{24} (1+3B_{11}+3B_{21}+B_{31}) \;;$$

$$A'_{44} = \frac{m_o m_1 n_1 n_2}{72} (1+4B_{11}+ \frac{5}{3} B_{22}+ \frac{13}{3} B_{21}+ \frac{5}{2} B_{32}+ \frac{3}{2} B_{31}+B_{41}) \;;$$

$$A'_{54} = \frac{m_o m_1 m_2 n_1 n_2}{1152} (1+5B_{11}+ \frac{13}{3} B_{22}+ \frac{17}{3} B_{21}+ \frac{7}{5} B_{33}+ \frac{33}{5} B_{32}+2B_{31}+ \frac{7}{3} B_{43} + \frac{8}{3} B_{41}+B_{52}) \;;$$

$$A'_{55} = \frac{m_o m_1 n_1 n_2 (n_1+2)}{360} (1+5B_{11}+ \frac{10}{3} B_{22}+ \frac{20}{3} B_{21}+ \frac{15}{2} B_{32}+ \frac{5}{2} B_{31}+ \frac{5}{3} B_{42} + \frac{10}{3} B_{41}+B_{51})$$

$$A'_{64} = \frac{m_o m_1 m_2 m_3 n_1 n_2}{14400} (1+6B_{11}+8B_{22}+7B_{21}+ \frac{26}{5} B_{33}+ \frac{123}{10} B_{32}+ \frac{5}{2} B_{31}+ \frac{9}{7} B_{44} + \frac{61}{7} B_{43}+5B_{41}+ \frac{9}{4} B_{54}+ \frac{15}{4} B_{52}+B_{63})$$

$$A'_{65} = \frac{m_o m_1 m_2 n_1 (n_1+2)(n_1+4)}{25200} (1+6B_{11}+8B_{22}+7B_{21}+ \frac{16}{5} B_{33}+ \frac{84}{5} B_{32}+8B_{43}+7B_{42} +6B_{53}+B_{62}) \;;$$

$$A'_{66} = \frac{m_o m_1 m_2 n_1 n_2 (n_1+2)}{3200} (1+6B_{11}+ \frac{19}{3} B_{22}+ \frac{26}{3} B_{21}+ \frac{28}{15} B_{33}+ \frac{74}{5} B_{32}+ \frac{10}{3} B_{31} + \frac{14}{3} B_{43}+ \frac{11}{3} B_{42}+ \frac{20}{3} B_{41}+ \frac{14}{9} B_{53}+ \frac{20}{9} B_{52} + \frac{20}{9} B_{51}+B_{61})$$

$$A'_{67} = \frac{m_o^2 m_1 n_1 (n_1+2) n_2}{2880} (1+6B_{11}+5B_{22}+10B_{21}+15B_{32}+5B_{31}+5B_{42}+10B_{41}+6B_{51}+B_{60}),$$

but the values of B_{ij}'s are now different and must be replaced by (where $r_i = f_1 + 2i$ and $s_i = f_1 - i$):

$$B_{11} = -\frac{a_1}{3}(1 - \frac{2b_1}{3r_o}) \; ; \; B_{22} = \frac{a_{21}}{15}(1 - \frac{4b_1}{3r_o} + \frac{4b_{21}}{15r_or_1}) \; ;$$

$$B_{21} = \frac{a_{22}}{3}(1 - \frac{4b_1}{3r_o} + \frac{4b_{22}}{3r_os_1}) \; ; \; B_{33} = -\frac{a_{31}}{35}(1 - \frac{2b_1}{r_o} + \frac{4b_{21}}{5r_or_1} - \frac{8b_{31}}{35r_or_1r_2}) \; ;$$

$$B_{32} = -\frac{a_{32}}{15}(1 - \frac{2b_1}{r_o} + \frac{16b_{21}}{45r_or_1} + \frac{20b_{22}}{9r_or_1} - \frac{8b_{32}}{15r_or_1s_1}) \; ;$$

$$B_{31} = -a_{33}(1 - \frac{2b_1}{r_o} + \frac{4b_{22}}{r_os_1} - \frac{8b_{33}}{r_os_1s_2}) \; ;$$

$$B_{44} = \frac{a_{41}}{315}(1 - \frac{8b_1}{3r_o} + \frac{8b_{21}}{5r_or_1} - \frac{32b_{31}}{35r_or_1r_2} + \frac{16b_{41}}{315r_or_1r_2r_3}) \; ;$$

$$B_{43} = \frac{a_{42}}{105}(1 - \frac{8b_1}{3r_o} + \frac{44b_{21}}{44r_or_1} + \frac{28b_{22}}{9r_or_1} - \frac{48b_{31}}{175r_or_1r_2} - \frac{122b_{32}}{75r_or_1s_1} + \frac{16b_{42}}{105r_or_1r_2s_1});$$

$$B_{42} = \frac{a_{43}}{15}(1 - \frac{8b_1}{3r_o} + \frac{32b_{21}}{45r_or_1} + \frac{40b_{22}}{9r_os_1} - \frac{32b_{32}}{15r_or_1s_1} + \frac{16b_{43}}{15r_or_1s_1(s_1+2)}) \; ;$$

$$B_{41} = \frac{a_{44}}{3}(1 - \frac{8b_1}{3r_o} + \frac{4b_{21}}{9r_or_1} + \frac{52b_{22}}{9r_os_1} - \frac{4b_{32}}{3r_or_1s_1} - \frac{12b_{33}}{r_os_1s_2} + \frac{16b_{44}}{3r_or_1s_1s_2}) \; ;$$

$$B_{55} = \frac{-a_{51}}{963}(1 - \frac{10b_1}{3r_o} + \frac{8b_{21}}{3r_or_1} - \frac{16b_{31}}{7r_or_1r_2} + \frac{16b_{41}}{63r_or_1r_2r_3} - \frac{32b_{51}}{693r_or_1\cdots r_4}) \; ;$$

$$B_{54} = \frac{-a_{52}}{315}(1 - \frac{10b_1}{3r_o} + \frac{28b_{21}}{15r_or_1} + \frac{4b_{22}}{r_os_1} - \frac{184b_{31}}{175r_or_1r_2} - \frac{216b_{32}}{75r_or_1s_1}$$

$$+ \frac{128b_{41}}{2205r_or_1r_2r_3} + \frac{144b_{42}}{245r_or_1r_2s_1} - \frac{32b_{52}}{315r_or_1r_2r_3s_1}$$

$$B_{53} = \frac{-a_{53}}{105}(1 - \frac{10b_1}{3r_o} + \frac{64b_{21}}{45r_or_1} + \frac{56b_{22}}{9r_os_1} - \frac{64b_{31}}{175r_or_1r_2} - \frac{122b_{32}}{25r_or_1s_1}$$

$$+ \frac{128b_{42}}{315r_o r_1 r_2 s_1} + \frac{122b_{43}}{45r_o r_1 s_1(s_1+2)} - \frac{32b_{53}}{105r_o r_1 r_2 s_1(s_1+2)} \Big) ;$$

$$B_{52} = \frac{-a_{54}}{15} \Big(1 - \frac{10b_1}{3r_o} + \frac{52b_{21}}{45r_o r_1} + \frac{68b_{22}}{9r_o s_1} - \frac{8b_{31}}{r_o r_1 r_2} - \frac{88b_{32}}{25r_o r_1 s_1} - \frac{16b_{33}}{r_o s_1 s_2}$$

$$+ \frac{16b_{42}}{45r_o r_1 r_2 s_1} + \frac{128b_{44}}{9r_o r_1 s_1 s_2} - \frac{32b_{54}}{15r_o r_1 r_2 s_1 s_2} \Big) ;$$

$$B_{51} = \frac{-a_{55}}{3} \Big(1 - \frac{10b_1}{3r_o} + \frac{4b_{21}}{9r_o r_1} + \frac{80b_{22}}{9r_o s_1} - \frac{4b_{32}}{r_o r_1 s_1} - \frac{20b_{33}}{r_o s_1 s_2} + \frac{16b_{43}}{9r_o r_1 s_1(s_1+2)}$$

$$+ \frac{160b_{44}}{9r_o r_1 s_1 s_2} - \frac{32b_{55}}{3r_o r_1 s_1 s_2(s_1+2)}\Big) ;$$

$$B_{66} = \frac{a_{61}}{3003} \Big(1 - \frac{4b_1}{r_o} + \frac{4b_{21}}{r_o r_1} - \frac{32b_{31}}{7r_o r_1 r_2} + \frac{16b_{41}}{21r_o \ldots r_3} - \frac{64b_{51}}{231r_o \ldots r_4}$$

$$+ \frac{64b_{61}}{3003r_o \ldots r_5}\Big) ;$$

$$B_{65} = \frac{a_{62}}{3465} \Big(1 - \frac{4b_1}{r_o} + \frac{136b_{21}}{45r_o r_1} + \frac{44b_{22}}{9r_o s_1} - \frac{192b_{31}}{75r_o r_1 r_2} - \frac{352b_{32}}{75r_o r_1 s_1} + \frac{208b_{41}}{735r_o \ldots r_3}$$

$$+ \frac{352b_{42}}{245r_o r_1 r_2 s_1} - \frac{320b_{51}}{6237r_o \ldots r_4} - \frac{1408b_{52}}{2835r_o \ldots r_3 s_1}$$

$$+ \frac{32b_{62}}{3465r_o \ldots r_4 s_1}\Big) ;$$

$$B_{64} = \frac{a_{63}}{105} \Big(1 - \frac{4b_1}{r_o} + \frac{12b_{21}}{5r_o r_1} + \frac{8b_{22}}{r_o s_1} - \frac{32b_{31}}{25r_o r_1 r_2} - \frac{192b_{32}}{25r_o r_1 s_1} + \frac{256b_{41}}{525r_o \ldots r_3}$$

$$+ \frac{352b_{42}}{245r_o r_1 r_2 s_1} + \frac{112b_{43}}{25r_o r_1 s_1(s_1+2)} - \frac{128b_{52}}{525r_o \ldots r_3 s_1}$$

$$- \frac{192b_{53}}{175r_o r_1 r_2 s_1(s_1+2)} + \frac{64b_{63}}{105r_o \ldots r_3 s_1(s_1+2)}\Big) ;$$

$$B_{63} = \frac{a_{64}}{35} \Big(1 - \frac{4b_1}{r_o} + \frac{32b_{21}}{15r_o r_1} + \frac{28b_{22}}{3r_o s_1} - \frac{208b_{31}}{175r_o r_1 r_2} - \frac{164b_{32}}{25r_o r_1 s_1} - \frac{40b_{33}}{r_o s_1 s_2}$$

$$+ \frac{16b_{41}}{245r_o \ldots r_3} + \frac{976b_{42}}{735r_o r_1 r_2 s_1} + \frac{80b_{44}}{3r_o r_1 s_1 s_2} - \frac{8b_{52}}{35r_o \ldots r_3 s_1}$$

$$- \frac{8b_{54}}{r_o r_1 r_2 s_1 s_2} + \frac{64b_{64}}{35r_o \ldots r_3 s_1 s_2}) \;;$$

$$B_{62} = \frac{a_{65}}{35} (1 - \frac{4b_1}{r_o} + \frac{32b_{21}}{15r_o r_1} - \frac{128b_{31}}{175r_o r_1 r_2} - \frac{672b_{32}}{75r_o r_1 s_1} + \frac{128b_{42}}{105r_o r_1 r_2 s_1} + \frac{28b_{22}}{3r_o s_1}$$

$$+ \frac{112b_{43}}{15r_o r_1 s_1 (s_1+2)} - \frac{64b_{53}}{35r_o r_1 r_2 s_1 (s_1+2)} + \frac{32b_{65}}{35r_o r_1 r_2 s_1 (s_1+2)(s_1+4)}) ;$$

$$B_{61} = \frac{a_{66}}{15} (1 - \frac{4b_1}{r_o} + \frac{76b_{21}}{45r_o r_1} + \frac{104b_{22}}{9r_o s_1} - \frac{32b_{31}}{75r_o r_1 r_2} - \frac{592b_{32}}{75r_o r_1 s_1} - \frac{160b_{33}}{3r_o s_1 s_2}$$

$$+ \frac{32b_{42}}{45r_o r_1 r_2 s_1} + \frac{176b_{43}}{45r_o r_1 s_1 (s_1+2)} + \frac{320b_{44}}{9r_o r_1 s_1 s_2}$$

$$- \frac{64b_{53}}{135r_o r_1 r_2 s_1 (s_1+2)} - \frac{128b_{54}}{27r_o r_1 r_2 s_1 s_2} - \frac{640b_{55}}{27r_o r_1 s_1 s_2 (s_1+2)}$$

$$+ \frac{72b_{66}}{5r_o r_1 r_2 s_1 s_2 (s_1+2)}) \;;$$

$$B_{60} = a_{67} (1 - \frac{4b_1}{r_o} + \frac{4b_{21}}{3r_o r_1} + \frac{40b_{22}}{3r_o s_1} - \frac{8b_{32}}{r_o r_1 s_1} - \frac{80b_{33}}{r_o s_1 s_2} + \frac{16b_{43}}{3r_o r_1 s_1 (s_1+2)}$$

$$+ \frac{160b_{44}}{3r_o r_1 s_1 s_2} - \frac{64b_{55}}{r_o r_1 s_1 s_2 (s_1+2)} + \frac{64b_{67}}{r_o^2 r_1 s_1 s_2 (s_1+2)})$$

with the a_{ij}'s are expressed in terms of the roots $\lambda_1, \lambda_2, \lambda_3$ of $\underset{\sim}{\Sigma}_1 \underset{\sim}{\Sigma}_2^{-1}$ as follows:

$a_1 = \lambda_1^{-1}+\lambda_2^{-1}+\lambda_3^{-1}$; $a_2 = (\lambda_1\lambda_2)^{-1}+(\lambda_1\lambda_3)^{-1}+(\lambda_2\lambda_3)^{-1}$; $a_3 = (\lambda_1\lambda_2\lambda_3)^{-1}$

$a_{21} = 3a_1^2-4a_2$; $a_{22} = a_2$; $a_{31} = 5a_1^3-12a_1a_2+8a_3$; $a_{32} = 2a_1a_2-3a_3$;

$a_{33} = a_3$; $a_{41} = 35a_1^4-120a_1^2a_2+48a_2^2+96a_1a_3$; $a_{42} = 9a_1^2a_2-12a_2^2-10a_1a_3$;

$a_{43} = 3a_2^2-4a_1a_3$; $a_{44} = a_1a_3$; $a_{51} = 63a_1^5-280a_1^3a_2+240a_1a_2^2+240a_1^2a_3-192a_2a_3$;

$a_{52} = 20a_1^3a_2-48a_1a_2^2-21a_1^2a_3+60a_2a_3$; $a_{53} = 9a_1a_2^2-12a_1^2a_3-10a_2a_3$;

$a_{54} = 3a_1^2a_3-4a_2a_3$; $a_{55} = a_2a_3$; $a_{61} = 231a_1^6-1260a_1^4a_2+1680a_1^2a_2^2+1120a_1^3a_3$ $- 320a_2^3-1920a_1a_2a_3+384a_3^2$; $a_{62} = 175a_1^4a_2-600a_1^2a_2^2-180a_1^3a_3+240a_2^3+912a_1a_2a_3$ $- 288a_3^2$; $a_{63} = 6a_1^2a_2^2-8a_1^3a_3-8a_2+6a_1a_2^3a_3-3a_3^2$; $a_{64} = 5a_1^3a_3-12a_1a_2a_3+8a_3^2$;

$a_{65} = 5a_2^3-12a_1a_2a_3+8a_3^2$; $a_{66} = 2a_1a_2a_3-3a_3^2$; $a_{67} = a_3^2$, and the b_{ij}'s are obtained from corresponding a_{ij}'s by replacing a_1, a_2, a_3 by b_1, b_2, b_3 respectively, where $b_1 = w_1+w_2+w_3$, $b_2 = w_1w_2+w_1w_3+w_2w_3$, $b_3 = w_1w_2w_3$ with w_1, w_2, w_3 are the characteristic roots of $\underset{\sim}{\Omega}$.

Appendix E

Expressed in terms of $D^*(n;q_3,m;q_2,m;q_1,m;u)$, to be shortened by $D^*(q_3,q_2,q_1)$, the Ψ^*_{ab} functions are given below:

$\Psi^*(u) = D^*(2,1,0)$; $\Psi^*_{11}(u) = 3D^*(2,1,0)-D^*(3,1,0)$;

$\Psi^*_{21}(u) = 15D^*(2,1,0)-10D^*(3,1,0)+3D^*(4,1,0)-D^*(3,2,0)$;

$\Psi^*_{22}(u) = 3D^*(2,1,0)-2D^*(3,1,0)+D^*(3,2,0)$;

$\Psi^*_{31}(u) = 35D^*(2,1,0)-35D^*(3,1,0)+21D^*(4,1,0)-7D^*(3,2,0)-5D^*(5,1,0)$
$+2D^*(4,2,0)-D^*(3,2,1)$;

$\Psi^*_{32}(u) = 15D^*(2,1,0)-15D^*(3,1,0)+4D^*(4,1,0)+7D^*(3,2,0)-2D^*(4,2,0)$
$+D^*(3,2,1)$;

$\Psi^*_{33}(u) = D^*(2,1,0)-D^*(3,1,0)+D^*(3,2,0)-D^*(3,2,1)$;

$\Psi^*_{41}(u) = 315D^*(2,1,0)-420D^*(3,1,0)+378D^*(4,1,0)-126D^*(3,2,0)-2D^*(4,3,0)$
$+9D^*(4,2,1)+72D^*(4,2,0)-36D^*(3,2,1)-180D^*(5,1,0)-15D^*(5,2,0)$
$+35D^*(6,1,0)$;

$\Psi^*_{42}(u) = 105D^*(2,1,0)-140D^*(3,1,0)+56D^*(3,2,0)+16D^*(3,2,1)+77D^*(4,1,0)$
$-32D^*(4,2,0)-4D^*(4,2,1)-3D^*(4,3,0)-18D^*(5,1,0)+9D^*(5,2,0)$;

$\Psi^*_{43}(u) = 15D^*(2,1,0)-20D^*(3,1,0)+14D^*(3,2,0)+4D^*(3,2,1)+8D^*(4,1,0)$
$-8D^*(4,2,0)-D^*(4,2,1)+3D^*(4,3,0)$;

$\Psi^*_{44}(u) = 3D^*(2,1,0)-4D^*(3,1,0)+4D^*(3,2,0)-4D^*(3,2,1)+D^*(4,1,0)-D^*(4,2,0)$
$+D^*(4,2,1)$;

$\Psi^*_{51}(u) = 693D^*(2,1,0)-1155D^*(3,1,0)-462D^*(3,2,0)-198D^*(3,2,1)$
$+1386D^*(4,1,0)+396D^*(4,2,0)+99D^*(4,2,1)-22D^*(4,3,0)-3D^*(4,3,1)$
$-990D^*(5,1,0)-165D^*(5,2,0)-18D^*(5,2,1)+5D^*(5,3,0)+385D^*(6,1,0)$
$+28D^*(6,2,0)-63D^*(7,1,0)$;

$\Psi^*_{52}(u) = 315D^*(2,1,0)-525D^*(3,1,0)+168D^*(3,2,0)+72D^*(3,2,1)+441D^*(4,1,0)$
$-144D^*(4,2,0)-36D^*(4,2,1)-37D^*(4,3,0)-3D^*(4,3,1)-207D^*(5,1,0)$
$+87D^*(5,2,0)+9D^*(5,2,1)+8D^*(5,3,0)-20D^*(6,2,0)+40D^*(6,1,0)$;

$$\Psi^*_{53}(u) = 105D^*(2,1,0)-175D^*(3,1,0)+126D^*(3,2,0)+54D^*(3,2,1)+112D^*(4,1,0) -108D^*(4,2,0)-27D^*(4,2,1)+41D^*(4,3,0)+4D^*(4,3,1)-24D^*(5,1,0) +24D^*(5,2,0)+3D^*(5,2,1)-9D^*(5,3,0) \;;$$

$$\Psi^*_{54}(u) = 15D^*(2,1,0)-25D^*(3,1,0)+24D^*(3,2,0)-24D^*(3,2,1)+13D^*(4,1,0) -12D^*(4,2,0)+12D^*(4,2,1)-D^*(4,3,0)+D^*(4,3,1)-3D^*(5,1,0) +3D^*(5,2,0)-3D^*(5,2,1) \;;$$

$$\Psi^*_{55}(u) = 3D^*(2,1,0)-5D^*(3,1,0)+6D^*(3,2,0)-6D^*(3,2,1)+2D^*(4,1,0) -3D^*(4,2,0)+3D^*(4,2,1)+D^*(4,3,0)-D^*(4,3,1) \;;$$

$$\Psi^*_{61}(u) = 3003D^*(2,1,0)-6006D^*(3,1,0)-3003D^*(3,2,0)-1716D^*(3,2,1) +9009D^*(4,1,0)+3432D^*(4,2,0)+1287D^*(4,2,1)-286D^*(4,3,0) -78D^*(4,3,1)-D^*(4,3,2)-8580D^*(5,1,0)-2145D^*(5,2,0)-468D^*(5,2,1) +130D^*(5,3,0)+16D^*(5,3,1)-5D^*(5,4,0)+5005D^*(6,1,0)+728D^*(6,2,0) +70D^*(6,2,1)-21D^*(6,3,0)-1638D^*(7,1,0)-105D^*(7,2,0)+231D^*(8,1,0);$$

$$\Psi^*_{62}(u) = 3465D^*(2,1,0)-6930D^*(3,1,0)+1617D^*(3,2,0)+924D^*(3,2,1) +7854D^*(4,1,0)-1848D^*(4,2,0)-693D^*(4,2,1)-1056D^*(4,3,0) -178D^*(4,3,1)+9D^*(4,3,2)-5544D^*(5,1,0)+1881D^*(5,2,0) +384D^*(5,2,1)+458D^*(5,3,0)+32D^*(5,3,1)-10D^*(5,4,0)+2145D^*(6,1,0) -920D^*(6,2,0)-80D^*(6,2,1)-75D^*(6,3,0)-350D^*(7,1,0)+175D^*(7,2,0);$$

$$\Psi^*_{63}(u) = 105D^*(2,1,0)-210D^*(3,1,0)+147D^*(3,2,0)+84D^*(3,2,1)+189D^*(4,1,0) -168D^*(4,2,0)-63D^*(4,2,1)+59D^*(4,3,0)+12D^*(4,3,1)-D^*(4,3,2) -84D^*(5,1,0)+78D^*(5,2,0)+18D^*(5,2,1)-26D^*(5,3,0)-2D^*(5,3,1) -2D^*(5,4,0)+16D^*(6,1,0)-16D^*(6,2,0)-2D^*(6,2,1)+6D^*(6,3,0) \;;$$

$$\Psi^*_{64}(u) = 35D^*(2,1,0)-70D^*(3,1,0)+63D^*(3,2,0)-64D^*(3,2,1)+56D^*(4,1,0) -47D^*(4,2,0)+48D^*(4,2,1)-9D^*(4,3,0)+8D^*(4,3,1)+D^*(4,3,2) -26D^*(5,1,0)+24D^*(5,2,0)-24D^*(5,2,1)+2D^*(5,3,0)-2D^*(5,3,1) +5D^*(6,1,0)-5D^*(6,2,0)+5D^*(6,2,1) \;;$$

$$\Psi^*_{65}(u) = 35D^*(2,1,0)-70D^*(3,1,0)+63D^*(3,2,0)+36D^*(3,2,1)+56D^*(4,1,0)$$
$$-72D^*(4,2,0)-27D^*(4,2,1)+41D^*(4,3,0)+8D^*(4,3,1)+D^*(4,3,2)$$
$$-16D^*(5,1,0)+24D^*(5,2,0)-18D^*(5,3,0)+6D^*(5,2,1)-2D^*(5,3,1)$$
$$+5D^*(5,4,0);$$

$$\Psi^*_{66}(u) = 15D^*(2,1,0)-30D^*(3,1,0)+37D^*(3,2,0)-36D^*(3,2,1)+19D^*(4,1,0)$$
$$-28D^*(4,2,0)+27D^*(4,2,1)+9D^*(4,3,0)-8D^*(4,3,1)-D^*(4,3,2)$$
$$-4D^*(5,1,0)+6D^*(5,2,0)-6D^*(5,2,1)-2D^*(5,3,0)+2D^*(5,3,1);$$

$$\Psi^*_{67}(u) = D^*(2,1,0)-2D^*(3,1,0)+3D^*(3,2,0)-4D^*(3,2,1)+D^*(4,1,0)$$
$$-2D^*(4,2,0)+3D^*(4,2,1)+D^*(4,3,0)-2D^*(4,3,1)+D^*(4,3,2);$$

Appendix F

Below, the values of Ψ^{**}_{ij} are given for m = o, However, if D** denotes the integral of D* from o to u, then Ψ^{**}_{ij} are Ψ^{*}_{ij} in Appendix E replacing single star by double stars. Therefore, only the expressions of D** in terms of g(a,b) = g*(u:o,n;a,b) and h(a,a';b,b') = $b^{bn+a}b'^{b'n'+a'}\, I(u:b,1,bn+a;\ b'+u,-1,b'n'+a')$, see Chapter III, are given. The values of D**(2,1,0) up to D*(7,3,0) are available in the work of Young [46], Appendix A. The additional values are (letting $a_i = n + i$ and $b_i = 2n + i$):

$$\begin{aligned}
D^{**}(8,1,0) = \{&a_1a_2a_4a_5a_6a_7a_8a_9b_3b_{11}\}^{-1}[28a_4a_6a_7a_8-7a_4a_5a_6a_7a_8b_3g(1,1)\\
&+\tfrac{15}{8}(4n^2+38n+91)a_1a_4b_3b_{11}g(2,1)+\tfrac{5}{4}(4n+17)a_1a_2a_4b_3b_{11}g(3,1)\\
&+2a_1a_4a_5a_6a_7a_8b_{11}g(3,2)+\tfrac{3}{4}(4n+15)a_1a_2a_3b_3b_{11}g(4,1)\\
&+a_1a_4a_5a_6a_7b_3b_{11}g(4,2)+\tfrac{3}{2}a_1a_2a_3a_4b_3b_{11}g(5,1)\\
&+a_1a_2a_4a_5a_6b_3b_{11}g(5,2)+\tfrac{1}{2}a_1a_2a_4a_5b_3b_{11}g(6,1)\\
&+\tfrac{1}{2}a_1a_2a_4a_5b_3b_5b_{11}g(6,2)+\tfrac{1}{2}a_1a_2a_3a_4b_3b_5b_{11}g(7,2)\\
&+\tfrac{1}{4}a_1a_2a_3b_3b_5b_7b_{11}g(8,2)+\tfrac{1}{4}a_1a_2a_4b_3b_5b_7b_{11}g(9,2)\\
&-\tfrac{1}{8}(56n^3+812n^2+4130n+7365)a_1a_4b_3b_{11}g(10,2)\\
&-\tfrac{1}{8}(224n^3+3864n^2+22624n+44919)a_1a_4a_5b_3h(11,2;2,1)]
\end{aligned}$$

$$\begin{aligned}
D^{**}(8,2,0) = \{&a_1a_2a_3a_5a_6a_7a_8a_9\}^{-1}[12a_7a_8-3a_2a_5a_7a_8g(1,1)\\
&+\tfrac{15}{4}(4n^2+38n+91)a_1a_5g(2,1)+\tfrac{5}{4}(4n+17)a_1a_2b_9g(3,1)\\
&+\tfrac{3}{2}(4n+15)a_1a_2a_3g(4,1)+2a_1a_5a_6a_7a_8g(4,2)+\tfrac{3}{2}a_1a_2a_3b_7g(5,1)\\
&+2a_1a_2a_5a_6a_7g(5,2)+a_1a_2a_3a_5g(6,1)+a_1a_2a_5a_6b_5g(6,2)\\
&+a_1a_2a_3a_5b_5g(7,2)+\tfrac{1}{2}a_1a_2a_3b_5b_7g(8,2)+\tfrac{1}{2}a_1a_2a_3b_5b_7g(9,2)\\
&-\tfrac{1}{4}(24n^3+364n^2+1922n+3525)a_1a_2g(10,2)\\
&-\tfrac{1}{4}(24n^3+364n^2+1922n+3525)a_1a_5g(11,2)\\
&-\tfrac{1}{8}(192n^3+3096n^2+16512n+28791)a_1a_5h(12,2;2,1)]
\end{aligned}$$

Appendix G

The values of D_k for central case up to k = 10. To shorten the writting we use the following abbreviations: $v = n_1+n_2$, $a_j = pn_1+2j$, $p_j = p+j$, $q_j = p-j$, $r_j = n_1+j$, $s_j = n_1-j$, $t_j = v+j$, $u_j = v-j$.

$D_0 = 1$, $D_1 = vp/2$, $D_2 = \{vp^2/(8a_1)\}(C_{21}+C_{22})$

$D_3 = \{vp^3/(2^3.3!a_1a_2)\}(C_{31}+C_{32}+C_{33})$

$D_4 = \{vp^4/(2^4.4!a_1a_2a_3)\}(C_{41}+\ldots+C_{45})$

$D_5 = \{vp^5/(2^5.5!a_1\ldots a_4)\}(C_{51}+\ldots+C_{57})$

$D_6 = \{vp^6/(2^6.6!a_1\ldots a_5)\}(C_{61}+\ldots+C_{611})$

$D_7 = \{vp^7/(2^7.7!a_1\ldots a_6)\}(C_{71}+\ldots+C_{715})$

$D_8 = \{vp^8/(2^8.8!a_1\ldots a_7)\}(C_{81}+\ldots+C_{822})$

$D_9 = \{vp^9/(2^9.9!a_1\ldots a_8)\}(C_{91}+\ldots+C_{930})$

$D_{10} = \{vp^{10}/(2^{10}.10!a_1\ldots a_9)\}(C_{101}+\ldots+C_{142})$

where the C's are given below:

$C_{21} = r_2t_2p_2/3$, $C_{22} = 2s_1u_1q_1/3$, $C_{31} = r_4t_4p_4C_{21}/15$,

$C_{32} = 9s_1u_1q_1C_{21}/5$, $C_{33} = s_2u_2q_2C_{22}/2$, $C_{41} = r_6t_6p_6C_{31}/7$,

$C_{42} = 20r_4t_4p_4C_{32}/63$, $C_{43} = 2r_1t_1p_1C_{32}/9$, $C_{44} = 8s_2u_2q_2C_{32}/9$,

$C_{45} = 2s_3u_3q_3C_{33}/5$, $C_{51} = r_8t_8p_8C_{41}/9$, $C_{52} = 7r_6t_6p_6C_{42}/36$,

$C_{53} = 5r_4t_4p_4C_{43}/7$, $C_{54} = 25r_4t_4p_4C_{44}/56$, $C_{55} = r_1t_1p_1C_{44}/2$,

$C_{56} = 25s_3u_3q_3C_{44}/42$, $C_{57} = s_4u_4q_4C_{45}/3$, $C_{61} = r_{10}t_{10}p_{10}C_{51}/11$,

$C_{62} = 54r_8t_8p_8C_{52}/385$, $C_{63} = 5r_6t_6p_6C_{53}/18$, $C_{64} = 56r_6t_6p_6C_{54}/225$,

$C_{65} = 2r_3t_3p_3C_{53}/15$, $C_{66} = 27r_1t_1p_1C_{54}/25$, $C_{67} = 7s_3u_3q_3C_{54}/9$,

$C_{68} = n_1vpC_{55}/6$, $C_{69} = 20s_3u_3q_3C_{55}/21$, $C_{610} = 9s_4u_4q_4C_{56}/20$,

$C_{611} = 2s_5u_5q_5C_{57}/7$, $C_{71} = r_{12}t_{12}p_{12}C_{61}/13$, $C_{72} = 77r_{10}t_{10}p_{10}C_{62}/702$,

$C_{73} = 49r_8t_8p_8C_{63}/275$, $C_{74} = 15r_8t_8p_8C_{64}/88$, $C_{75} = 7r_6t_6p_6C_{65}/12$,

$C_{76} = 28r_6t_6p_6C_{66}/81$, $C_{77} = 84r_6t_6p_6C_{67}/275$, $C_{78} = 5r_3t_3p_3C_{66}/27$,

$C_{79} = 7n_1 vpC_{66}/27$, $C_{710} = 98s_3u_3q_3C_{66}/81$, $C_{711} = 14s_4u_4q_4C_{67}/25$,

$C_{712} = 2s_3u_3q_3C_{68}$, $C_{713} = 5s_4u_4q_4C_{69}/8$, $C_{714} = 49s_5u_5q_5C_{610}/135$,

$C_{714} = s_6u_6q_6C_{611}/4$, $C_{81} = r_{14}t_{14}p_{14}C_{71}/15$, $C_{82} = 104r_{12}t_{12}p_{12}C_{72}/1155$,

$C_{83} = 12r_{10}t_{10}p_{10}C_{73}/91$, $C_{84} = 176r_{10}t_{10}p_{10}C_{74}/1365$,

$C_{85} = 8r_8t_8p_8C_{75}/33$, $C_{86} = 7r_8t_8p_8C_{76}/33$, $C_{87} = 55r_8t_8p_8C_{77}/273$,

$C_{88} = 2r_5t_5p_5C_{75}/21$, $C_{89} = 8r_3t_3p_3C_{76}/21$, $C_{810} = n_1 vpC_{76}/3$,

$C_{811} = 16s_3u_3q_3C_{76}/11$, $C_{812} = 4r_6t_6p_6C_{711}/11$, $C_{813} = 8r_3t_3p_3C_{79}/21$,

$C_{814} = 5r_3t_3p_3C_{710}/21$, $C_{815} = 28r_4t_4p_4C_{712}/15$,

$C_{816} = 112r_4r_4p_4C_{713}/75$, $C_{817} = 24s_5u_5q_5C_{711}/55$, $C_{818} = 2s_1u_1q_1C_{712}/15$,

$C_{819} = 20s_4u_4q_4C_{712}/21$, $C_{820} = 7s_5u_5q_5C_{713}/15$, $C_{821} = 32s_6u_6q_6C_{714}/105$,

$C_{822} = 2s_7u_7q_7C_{715}/9$, $C_{91} = r_{16}t_{16}p_{16}C_{81}/17$, $C_{92} = 135r_{14}t_{14}p_{14}C_{82}/1768$,

$C_{93} = 11r_{12}t_{12}p_{12}C_{83}/105$, $C_{94} = 91r_{12}t_{12}p_{12}C_{84}/880$,

$C_{95} = 21r_{10}t_{10}p_{10}C_{85}/130$, $C_{96} = 486r_{10}t_{10}p_{10}C_{86}/3185$,

$C_{97} = 78r_{10}t_{10}p_{10}C_{87}/525$, $C_{98} = 27r_8t_8p_8C_{88}/55$, $C_{99} = 25r_8t_8p_8C_{89}/88$,

$C_{910} = 14r_8t_8p_8C_{810}/55$, $C_{911} = 77r_8t_8p_8C_{811}/312$,

$C_{912} = 297r_8t_8p_8C_{812}/1274$, $C_{913} = 28s_2u_2q_2C_{88}/5$,

$C_{914} = 27r_3t_3p_3C_{810}/35$, $C_{915} = 27r_3t_3p_3C_{811}/56$, $C_{916} = 27n_1vpC_{811}/40$,

$C_{917} = 130r_1t_1p_1C_{813}/49$, $C_{918} = 11r_8t_8p_8C_{817}/28$,

$C_{919} = r_2t_2p_2C_{813}/10$, $C_{920} = 63s_3u_3q_3C_{813}/20$, $C_{921} = 14s_4u_4q_4C_{814}/15$,

$C_{922} = 5s_1u_1q_1C_{815}/28$, $C_{923} = 8s_4u_4q_4C_{815}/7$, $C_{924} = 81r_1t_1p_1C_{817}/28$,

$C_{925} = 5s_6u_6q_6C_{817}/14$, $C_{926} = 27s_4u_4q_4C_{818}/14$, $C_{927} = 5s_5u_5q_5C_{819}/8$,

$C_{928} = 7r_1t_1p_1C_{821}/4$, $C_{929} = 81s_7u_7q_7C_{821}/308$, $C_{930} = s_8u_8q_8C_{822}/5$,

$C_{101} = r_{18}t_{18}p_{18}C_{91}/19$, $C_{102} = 34r_{16}t_{16}p_{16}C_{92}/513$,

$C_{103} = 65r_{14}t_{14}p_{14}C_{93}/748$, $C_{104} = 400r_{14}t_{14}p_{14}C_{94}/4641$,

$C_{105} = 6r_{12}t_{12}p_{12}C_{95}/49$, $C_{106} = 77r_{12}t_{12}p_{12}C_{96}/648$,

$C_{107} = 175r_{12}t_{12}p_{12}C_{97}/1496$, $C_{108} = 25r_{10}t_{10}p_{10}C_{98}/117$,

$C_{109} = 12r_{10}t_{10}p_{10}C_{99}/65$, $C_{110} = 225r_{10}t_{10}p_{10}C_{910}/1274$,

$C_{111} = 468r_{10}t_{10}p_{10}C_{911}/2695$, $C_{112} = 91r_{10}t_{10}p_{10}C_{912}/540$,

$C_{113} = 2r_7t_7p_7C_{98}/27$, $C_{114} = 175s_2u_2q_2C_{98}/27$, $C_{115} = 4n_1vpC_{99}/5$,

$C_{116} = 55s_3u_3q_3C_{99}/26$, $C_{117} = 44s_3u_3q_3C_{910}/13$, $C_{118} = 50s_4u_4q_4C_{911}/49$,

$C_{119} = 26s_5u_5q_5C_{912}/45$, $C_{120} = 25r_5t_5p_5C_{914}/162$,

$C_{121} = 24s_3u_3q_3C_{913}/11$, $C_{122} = 4r_2t_2p_2C_{914}/27$, $C_{123} = 81s_3u_3q_3C_{914}/22$,

$C_{124} = 175s_4u_4q_4C_{915}/162$, $C_{125} = 35s_1u_1q_1C_{916}/162$,

$C_{126} = 250s_4u_4q_4C_{916}/189$, $C_{127} = 33s_5u_5q_5C_{917}/52$,

$C_{128} = 20s_6u_6q_6C_{918}/49$, $C_{129} = 25s_3u_3q_3C_{919}/6$, $C_{130} = 50s_1u_1q_1C_{920}/189$,

$C_{131} = 112s_4u_4q_4C_{920}/81$, $C_{132} = 36s_5u_5q_5C_{921}/55$, $C_{133} = 16s_4u_4q_4C_{922}/7$,

$C_{134} = 225s_5u_5q_5C_{923}/308$, $C_{135} = 35s_6u_6q_6C_{924}/81$,

$C_{136} = 55s_7u_7q_7C_{925}/182$, $C_{137} = s_2u_2q_2C_{926}/9$, $C_{138} = 2s_1u_1q_1C_{927}/5$,

$C_{139} = 7s_6u_6q_6C_{927}/15$, $C_{140} = 24s_7u_7q_7C_{928}/77$,

$C_{141} = 25s_8u_8q_8C_{929}/108$, $C_{142} = 2s_9u_9q_9C_{930}/11$.

Appendix H

The values of D_k for non-central case up to $k = 6$. Here we also use the same abbreviations as in the Appendix G.

$$D_0 = 1 \ , \ D_1 = vpB_{11}/2 \ , \ D_2 = \{vp^2/(8a_1)\}(C_{21}B_{21}+C_{22}B_{22}) \ ,$$

$$D_3 = \{vp^3/(2^3.3!a_1a_2)\}(C_{31}B_{31}+C_{32}B_{32}+C_{33}B_{33}) \ ,$$

$$D_4 = \{vp^4/(2^4.4!a_1a_2a_3)\}(C_{41}B_{41}+\ldots+C_{45}B_{45}) \ ,$$

$$D_5 = \{vp^5/(2^5.5!a_1a_2a_3a_4)\}(C_{51}B_{51}+\ldots+C_{57}B_{57}) \ ,$$

$$D_6 = \{vp^6/(2^6.6!a_1\ldots a_5)\}(C_{61}B_{61}+\ldots+C_{611}B_{611}) \ ,$$

where the C's are as in the Appendix G and the B's are given as follows:

$$B_{11} = 1-2b_{11}/a_o \ , \ B_{21} = 1-4b_{11}/a_o+4b_{21}/(a_or_2p_2) \ ,$$

$$B_{22} = 1-4b_{11}/a_o+8b_{22}/(a_os_1q_1) \ ,$$

$$B_{31} = 1-6b_{11}/a_o+12b_{21}/(a_or_2p_2)-24b_{31}/(a_or_2r_4p_2p_4) \ ,$$

$$B_{32} = 1-6b_{11}/a_o+16b_{21}/(3a_or_2p_2)+40b_{22}/(3a_os_1q_1)-16b_{32}/(a_or_2s_1p_2q_1) \ ,$$

$$B_{33} = 1-6b_{11}/a_o+24b_{22}/(a_os_1q_1)-48b_{33}/(a_os_1s_2q_1q_2) \ ,$$

$$B_{41} = 1-8b_{11}/a_o+24b_{21}/(a_or_2p_2)-96b_{31}/(a_or_2r_4p_2p_4)+48b_{41}/(a_or_2r_4r_6p_2p_4p_6),$$

$$B_{42} = 1-8b_{11}/a_o+44b_{21}/(3a_or_2p_2)+56b_{22}/(3a_os_1q_1)-144b_{31}/(5a_or_2r_4p_2p_4)$$
$$-224b_{32}/(5a_or_2s_1p_2q_1)+32b_{42}/(a_or_2r_4s_1p_2p_4q_1) \ ,$$

$$B_{43} = 1-8b_{11}/a_o+32b_{21}/(3a_or_2p_2)+80b_{22}/(3a_os_1q_1)-64b_{32}/(a_or_2s_1p_2q_1)$$
$$+128b_{43}/(a_or_2s_1r_1p_2q_1p_1) \ ,$$

$$B_{44} = 1-8b_{11}/a_o+20b_{21}/(3a_or_2p_2)+140b_{22}/(3a_os_1q_1)-40b_{32}/(a_or_2s_1p_2q_1)$$
$$-72b_{33}/(a_os_1s_2q_1q_2)+32b_{44}/(a_or_2s_2s_1p_2q_2q_1) \ ,$$

$$B_{45} = 1-8b_{11}/a_o+48b_{22}/(a_os_1q_1)-192b_{33}/(a_os_1s_2q_1q_2)$$
$$+128b_{45}/(a_os_1s_2s_3q_1q_2q_3) \ ,$$

$$B_{51} = 1-10b_{11}/a_o+40b_{21}/(a_or_2p_2)-240b_{31}/(a_or_2r_4p_2p_4)$$
$$+240b_{41}/(a_or_2r_4r_6p_2p_4p_6)-480b_{51}/(a_or_2r_4r_6r_8p_2p_4p_6p_8) \ ,$$

$$B_{52} = 1-10b_{11}/a_o+28b_{21}/(a_or_2p_2)+24b_{22}/(a_os_1q_1)-552b_{31}/(5a_or_2r_4p_2p_4)$$
$$-432b_{32}/(5a_or_2s_1p_2q_1)+384b_{41}/(7a_or_2r_4r_6p_2p_4p_6)$$
$$+864b_{42}/(7a_or_2r_4s_1p_2p_4q_1)-192b_{52}/(a_or_2r_4r_6s_1p_2p_4p_6q_1) ,$$

$$B_{53} = 1-10b_{11}/a_o+64b_{21}/(3a_or_2p_2)+112b_{22}/(3a_os_1q_1)-192b_{31}/(5a_or_2r_4p_2p_4)$$
$$-672b_{32}/(5a_or_2s_1p_2q_1)+256b_{42}/(3a_or_2r_4s_1p_2p_4q_1)$$
$$+896b_{43}/(3a_or_2s_1r_1p_2q_1p_1)-256b_{53}/(a_or_2r_4s_1r_1p_2p_4q_1p_1) ,$$

$$B_{54} = 1-10b_{11}/a_o+52b_{21}/(3a_or_2p_2)+136b_{22}/(3a_os_1q_1)-168b_{31}/(5a_or_2r_4p_2p_4)$$
$$-528b_{32}/(5a_or_2s_1p_2q_1)-96b_{33}/(a_os_1s_2q_1q_2)+224b_{42}/(3a_or_2r_4s_1p_2p_4q_1)$$
$$+256b_{44}/(3a_or_2s_2s_1p_2q_2q_1)-64b_{54}/(a_or_2r_4s_1s_2p_2p_4q_1q_2) ,$$

$$B_{55} = 1-10b_{11}/a_o+40b_{21}/(3a_or_2p_2)+160b_{22}/(3a_os_1q_1)-120b_{32}/(a_or_2s_1p_2q_1)$$
$$-120b_{33}/(a_os_1s_2q_1q_2)+640b_{43}/(3a_or_2s_1r_1p_2q_1q_1)$$
$$+320b_{44}/(3a_or_2s_2s_1p_2q_2q_1)-640b_{55}/(a_or_2s_1r_1s_2p_2q_1p_1q_2) ,$$

$$B_{56} = 1-10b_{11}/a_o+8b_{21}/(a_or_2p_2)+64b_{22}/(a_os_1q_1)-72b_{32}/(a_or_2s_1p_2q_1)$$
$$-264b_{33}/(a_os_1s_2q_1q_2)+576b_{44}/(5a_or_2s_2s_1p_2q_2q_1)$$
$$+896b_{45}/(5a_os_1s_2s_3p_1p_2p_3)-384b_{56}/(a_or_2s_1s_2s_3p_2q_1q_2q_3) ,$$

$$B_{57} = 1-10b_{11}/a_o+80b_{22}/(a_os_1q_1)-480b_{33}/(a_os_1s_2q_1q_2)$$
$$+640b_{45}/(a_os_1s_2s_3q_1q_2q_3)-3840b_{57}/(a_os_1s_2s_3s_4q_1q_2q_3q_4) ,$$

$$B_{61} = 1-12b_{11}/a_o+60b_{21}/(a_or_2p_2)-480b_{31}/(a_or_2r_4p_2p_4)$$
$$+720b_{41}/(a_or_2r_4r_6p_2p_4p_6)-2880b_{51}/(a_or_2r_4r_6r_8p_2p_4p_6p_8)$$
$$+2880b_{61}/(a_or_2r_4r_6r_8r_{10}p_2p_4p_6p_8p_{10}) ,$$

$$B_{62} = 1-12b_{11}/a_o+136b_{21}/(3a_or_2p_2)+88b_{22}/(3a_os_1q_1)-1344b_{31}/(5a_or_2r_4p_2p_4)$$
$$-704b_{32}/(5a_or_2s_1p_2q_1)+1872b_{41}/(7a_or_2r_4r_6p_2p_4p_6)$$
$$+2112b_{42}/(7a_or_2r_4s_1p_2p_4q_1)-1600b_{51}/(3a_or_2r_4r_6r_8p_2p_4p_6p_8)$$
$$-2816b_{52}/(3a_or_2r_4r_6s_1p_2p_4p_6q_1)+384b_{62}/(a_or_2r_4r_6r_8s_1p_2p_4p_6p_8q_1) ,$$

$$B_{63} = 1-12b_{11}/a_o+36b_{21}/(a_o r_2 p_2)+48b_{22}/(a_o s_1 q_1)-672b_{31}/5a_o r_2 r_4 p_2 p_4)$$
$$-1152b_{32}/(5a_o r_2 s_1 p_2 q_1)+2304b_{41}/(35a_o r_2 r_4 r_6 p_2 p_4 p_6)$$
$$+2112b_{42}/(7a_o r_2 r_4 s_1 p_2 p_4 q_1)+(147)(128)b_{43}/(35a_o r_2 s_1 r_1 p_2 q_1 p_1)$$
$$-2304b_{52}/(5a_o r_2 r_4 r_6 s_1 p_2 p_4 p_6 q_1)-4608b_{53}/(5a_o r_2 r_4 s_1 r_1 p_2 p_4 q_1 p_1)$$
$$+4608b_{63}/(a_o r_2 r_4 r_6 s_1 r_1 p_2 p_4 p_6 q_1 p_1) ,$$

$$B_{64} = 1-12b_{11}/a_o+32b_{21}/(a_o r_2 p_2)+56b_{22}/(a_o s_1 q_1)-624b_{31}/(5a_o r_2 r_4 p_2 p_4)$$
$$-984b_{32}/(5a_o r_2 s_1 p_2 q_1)-120b_{33}/(a_o s_1 s_2 q_1 q_2)+432b_{41}/(7a_o r_2 r_4 r_6 p_2 p_4 p_6)$$
$$+1952b_{42}/(7a_o r_2 r_4 s_1 p_2 p_4 q_1)+160b_{44}/(a_o r_2 s_2 s_1 p_2 q_2 q_1)$$
$$-432b_{52}/(a_o r_2 r_4 r_6 s_1 p_2 p_4 p_6 q_1)-240b_{54}/(a_o r_2 r_4 s_1 s_2 p_2 p_4 q_1 q_2)$$
$$+384b_{64}/(a_o r_2 r_4 r_6 s_1 s_2 p_2 p_4 p_6 q_1 q_2) ,$$

$$B_{65} = 1-12b_{21}/a_o+32b_{21}/(a_o r_2 p_2)+56b_{22}/(a_o s_1 q_1)-384b_{31}/(5a_o r_2 r_4 p_2 p_4)$$
$$-1344b_{32}/(5a_o r_2 s_1 p_2 q_1)+256b_{42}/(a_o r_2 r_4 s_1 p_2 p_4 q_1)$$
$$+896b_{43}/(a_o r_2 s_1 r_1 p_2 q_1 p_1)-1536b_{53}/(a_o r_2 r_4 s_1 r_1 p_2 p_4 q_1 p_1)$$
$$+3072b_{65}/(a_o r_2 r_4 s_1 r_1 r_3 p_2 p_4 q_1 p_1 p_3) ,$$

$$B_{66} = 1-12b_{11}/a_o+76b_{21}/(3a_o r_2 p_2)+208b_{22}/(3a_o s_1 q_1)-224b_{31}/(5a_o r_2 r_4 p_2 p_4)$$
$$-1184b_{32}/(5a_o r_2 s_1 p_2 q_1)-160b_{33}/(a_o s_1 s_2 q_1 q_2)+448b_{42}/(3a_o r_2 r_4 s_1 p_2 p_4 q_1)$$
$$+1408b_{43}/(3a_o r_2 s_1 r_1 p_2 q_1 p_1)+640b_{44}/(3a_o r_2 s_2 s_1 p_2 q_2 q_1)$$
$$-2584b_{53}/(9a_o r_2 r_4 s_1 r_1 p_2 p_4 q_1 p_1)-1280b_{54}/(9a_o r_2 r_4 s_1 s_2 p_2 p_4 q_1 q_2)$$
$$-12800b_{55}/(9a_o r_2 s_1 r_1 s_2 p_2 q_1 p_1 q_2)+512b_{66}/(a_o r_2 r_4 s_1 r_1 s_2 p_2 p_4 q_1 p_1 q_2) ,$$

$$B_{67} = 1-12b_{11}/a_o+20b_{21}/(a_o r_2 p_2)+80b_{22}/(a_o s_1 q_1)-192b_{31}/(5a_o r_2 r_4 p_2 p_4)$$
$$-912b_{32}/(5a_o r_2 s_2 p_2 q_1)-336b_{33}/(a_o s_1 s_2 q_1 q_2)+128b_{42}/(a_o r_2 r_4 s_1 p_2 p_4 q_1)$$
$$+1472b_{44}/(5a_o r_2 s_2 s_1 p_2 q_2 q_1)+1152b_{45}/(5a_o s_1 s_2 s_3 q_1 q_2 q_3)$$
$$-1536b_{54}/(7a_o r_2 r_4 s_1 s_2 p_2 p_4 q_1 q_2)-6912b_{56}/(7a_o r_2 s_1 s_2 s_3 p_2 q_1 q_2 q_3)$$
$$+1536b_{67}/(a_o r_2 r_4 s_1 s_2 s_3 p_2 p_4 q_1 q_2 q_3) ,$$

$$B_{68} = 1-12b_{11}/a_o+20b_{21}/(a_o r_2 p_2)+80b_{22}/(a_o s_1 q_1)-240b_{32}/(a_o r_2 s_1 p_2 q_1)$$
$$-240b_{33}/(a_o s_1 s_2 q_1 q_2)+648b_{43}/(a_o r_2 s_1 r_1 p_2 q_1 p_1)+320b_{44}/(a_o r_2 s_2 s_1 p_2 q_2 q_1)$$
$$-3840b_{55}/(a_o r_2 s_1 r_1 s_2 p_2 q_1 p_1 q_2)+7680b_{68}/(a_o^2 r_2 s_1 r_1 s_2 p_2 q_1 p_1 q_2) ,$$

$$B_{69} = 1-12b_{11}/a_o+16b_{21}/(a_or_2p_2)+88b_{22}/(a_os_1q_1)-192b_{32}/(a_or_2s_1p_2q_1)$$
$$-384b_{33}/(a_os_1s_2p_1p_2)+320b_{43}/(a_or_2s_1r_1p_2q_1p_1)$$
$$+1664b_{44}/(5a_or_2s_2s_1p_2q_2q_1)+1344b_{45}/(5a_os_1s_2s_3q_1q_2q_3)$$
$$-1920b_{55}/(a_or_2s_1r_1s_2p_2q_1p_1q_2)-1152b_{56}/(a_or_2s_1s_2s_3p_2q_1q_2q_3)$$
$$+1536b_{69}/(a_or_2s_1r_1s_2s_3p_2q_1p_1q_2q_3) \ ,$$

$$B_{610}= 1-12b_{11}/a_o+28b_{21}/(3a_or_2p_2)+304b_{22}/(3a_os_1q_1)-112b_{32}/(a_or_2s_1p_2q_1)$$
$$-624b_{33}/(a_os_1s_2q_1q_2)+1344b_{44}/(5a_or_2s_2s_1p_2q_2q_1)$$
$$+4224b_{45}/(5a_os_1s_2s_3q_1q_2q_3)-1792b_{56}/(a_or_2s_1s_2s_3p_2q_1q_2q_3)$$
$$-5120b_{57}/(a_os_1s_2s_3s_4q_1q_2q_3q_4)+1536b_{610}/(a_or_2s_1s_2s_3s_4p_2q_1q_2q_3q_4) \ ,$$

$$B_{611}= 1-12b_{11}/a_o+120b_{22}/(a_os_1q_1)-960b_{33}/(a_os_1s_2q_1q_2)$$
$$+1920b_{45}/(a_os_1s_2s_3q_1q_2q_3)-23040b_{57}/(a_os_1s_2s_3s_4q_1q_2q_3q_4)$$
$$+(45)(1024)b_{611}/(a_os_1s_2s_3s_4s_5q_1q_2q_3q_4q_5) \ ,$$

where the quantities b_{ij}'s are:

$$b_{11} = b_1 \ ; \ b_{21} = 3b_1^2 - 4b_2 \ ; \ b_{22} = b_2; \ b_{31} = 5b_1^3-12b_1b_2+8b_3 \ ;$$
$$b_{32} = 2b_1b_2-3b_3 \ ; \ b_{33} = b_3 \ ; \ b_{41} = 35b_1^4-120b_1^2b_2+48b_2^2+96b_1b_3-64b_4$$
$$b_{42} = 9b_1^2b_2-12b_2^2-10b_1b_3+16b_4 \ ; \ b_{43} = 3b_2^2-4b_1b_3+b_4 \ ; \ b_{44} = 5b_1b_3-8b_4 \ ;$$
$$b_{45} = b_4 \ ; \ b_{51} = 63b_1^5-280b_1^3b_2+240b_1b_2^2+240b_1^2b_3-192b_2b_3-192b_1b_4+128b_5$$
$$b_{52} = 20b_1^3b_2-48b_1b_2^2-21b_1^2b_3-60b_2b_3+24b_1b_4-40b_5$$
$$b_{53} = 9b_1b_2^2-12b_1^2b_3-10b_2b_3+18b_1b_4-5b_5 \ ; \ b_{54} = 21b_1^2b_3-28b_2b_3-24b_1b_4+40b_5$$
$$b_{55} = 2b_2b_3-3b_1b_4+b_5 \ ; \ b_{56} = 3b_1b_4-5b_5 \ ; \ b_{57} = b_5$$
$$b_{61} = 231b_1^6-1260b_1^4b_2+1680b_1^2b_2^2+1120b_1^3b_3-320b_2^3-1920b_1b_2b_3-960b_1^2b_4$$
$$+384b_3^2+768b_2b_4+768b_1b_5-512b_6$$
$$b_{62} = 175b_1^4b_2-600b_1^2b_2^2-180b_1^3b_3+240b_2^3+912b_1b_2b_3+192b_1^2b_4-288b_3^2-576b_2b_4$$
$$-224b_1b_5+384b_6$$
$$b_{63} = 6b_1^2b_2^2-8b_1^3b_3-8b_2^3+6b_1b_2b_3+9b_1^2b_4-3b_3^2+8b_2b_4-14b_1b_5+4b_6$$
$$b_{64} = 45b_1^3b_3-108b_1b_2b_3-48b_1^2b_4+72b_3^2+64b_4b_2+56b_1b_5-96b_6$$

$b_{65} = 15b_2^3-36b_1b_2b_3+24b_1^2b_4+24b_3^2-22b_2b_4-8b_1b_5+3b_6$

$b_{66} = 14b_1b_2b_3-21b_1^2b_4-21b_3^2+8b_2b_4+32b_1b_5-12b_6$

$b_{67} = 6b_1^2b_4-8b_2b_4-7b_1b_5+12b_6$

$b_{68} = 3b_3^2-4b_2b_4+b_1b_5$

$b_{69} = 5b_2b_4-8b_1b_5+3b_6$

$b_{610} = 7b_1b_5-12b_6$; $b_{611} = b_6$.

In the above relations, the b_j's are the j^{th} elementary symmetric function in the latent roots of $\underset{\sim}{\Omega}$

VITA

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13. ABSTRACT The following criteria are studied in connection with tests of (A) equality of covariance matrices of two p-variate normal populations, (B) MANOVA and (C) canonical correlations: 1) Hotelling's trace, 2) Pillai's trace, 3) Wilks' criterion, 4) Roy's largest root and 5) Wilks-Lawley statistic. The distribution of 1) -in restricted case-, 3) and 4) in two forms, the moments of 1) and the mgf of 2) are obtained in Chapter I using Pillai's density of the latent roots of $S_1 S_2^{-1}$ under violations. The exact distribution of the first four criteria in the two-roots case are derived in Chapter II in a suitable form for computation. These results are used to study the robustness of tests of (A) against non-normality and (B) against unequal covariance matrices. The study here is of an exact nature and shows that for (A) there is an indication that the tests are not robust against non-normality and for (B), the powers of tests show modest changes for small deviations from the hypothesis. In Chapter III the null distribution of 1) for three and four roots are obtained which hold for all non-negative integral values of an argument m, unlike those of Pillai and Young. Its non-null distribution for m=0 in the three-roots case is derived in Chapter IV and power studies and comparisons are carried out there. In addition, an expression suggested by Pillai for the distribution of 1) is studied in Chapter V, which is exact in the two-roots case. The distribution of 5) is obtained in Chapter VI in two forms. Its power studies in the two-roots case is carried out for the three tests and comparisons are made with powers of the other four criteria. Summary, conclusion and recommendations are given in Chapter VII.

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