Some Sample Path Properties of Renewal Processes

by

Kenneth Wade Fischer

Purdue University

Department of Statistics
Division of Mathematical Science
Mimeograph Series #332

August 1973

ABSTRACT

Fischer, Kenneth Wade. Ph.D. Purdue University, August 1973. Some Sample Path Properties of Renewal Processes. Major Professor: David H. Root.

Let $\{S_n\}$ be a renewal process, that is, $S_n = \sum_{i=1}^{n} X_i$ where $X_1, X_2, \ldots$ are independent identically distributed non-negative random variables with finite mean μ and distribution function F. Also, let f be a non-negative, bounded, Lebesgue measurable function on $[0,\infty)$. If F is non-singular, then the following dichotomy holds. Either $E(\sum_{n=1}^{\infty} f(S_n)) < \infty$ or $\sum_{n=1}^{\infty} f(S_n) = \infty$ a.s. according as $\int_0^{\infty} f(t)dt$ is finite or infinite. If F is arithmetic with mass concentrated on $0, \lambda, 2\lambda, \ldots$, then the same dichotomy holds. The criterion is then the convergence or divergence of $\sum_{k=1}^{\infty} f(\lambda k)$. Analogues of these results hold if $\{S_n\}$ is a random walk with drift. There also is an analogue for Brownian Motion. In particular, if $\{S(t): t \geq 0\}$ is Brownian Motion with drift $\mu > 0$, then $E(\int_0^{\infty} f(S(t))dt) < \infty$ or $\int_0^{\infty} f(S(t))dt=\infty$ a.s. according as $\int_0^{\infty} f(s)ds$ is finite or infinite. In the special case $f = I_A$, the indicator function of a Lebesgue measurable set A, the above results deal with the number of visits of $\{S_n\}$ to A and the amount of time $\{S(t): t \geq 0\}$ spends in A.

In the case $\sum_{n=1}^{\infty} f(S_n) = \infty$ a.s., Chapter II discusses the existence of a rate of divergence for $\sum_{k=1}^{n} f(S_k)$. If F is non-singular and there exist M and $\delta > 0$ such that $\int_x^{x+M} f(t)dt \geq \delta$ for all $x \geq 0$, then $\frac{\sum_{k=1}^{n} f(S_k)}{\frac{1}{\mu}\int_0^{\mu n} f(t)dt} \overset{a.s.}{\to} 1$ as $n \to \infty$. Analogues of this rate theorem for the arithmetic case and for random walk and Brownian Motion are also presented.

CHAPTER I

This chapter considers the convergence or divergence of an infinite sum of values taken at an infinite sequence of random points. The random sequence of points $S_1, S_2, \ldots, S_n, \ldots$ will be a renewal process, that is, $S_n = \sum_{i=1}^{n} X_i$ where the X_i's are non-negative independent identically distributed random variables. Let F be the common distribution function of the X_i's (called the waiting time distribution of the renewal process), and assume that $\mu = E(X_i) < \infty$. Also assume that the function f is a Lebesgue measurable function defined on $[0,\infty)$ and satisfying $0 \leq f \leq B$.

The infinite sum under consideration is $\sum_{n=1}^{\infty} f(S_n)$. The event $\{\sum_{n=1}^{\infty} f(S_n) = \infty\}$ is a permutable event. So by the Hewitt-Savage zero-one law either $\sum^{\infty} f(S_n) = \infty$ a.s. or $\sum^{\infty} f(S_n) < \infty$ a.s. We will show that under very weak assumptions on F that this dichotomy has a simple equivalent dichotomy. Namely, $\sum^{\infty} f(S_n)$ is finite a.s. or infinite a.s. according as $\int_0^{\infty} f(t)dt$ is finite or infinite. In fact, this dichotomy is even stronger as is indicated by Theorem 1. Later we will show that analogous results hold for random walk and Brownian motion processes.

This problem has been considered before by Chung and Derman ([2]) and later by Stanley Sawyer (unpublished) in the special case $f = I_A$, the indicator function of the Lebesgue measurable set A. In this case $\sum^{\infty} f(S_n)$ is the number of renewals in the set A and $\int_0^{\infty} f(t)dt = mA$, the Lebesgue measure of A. The result then says that the number of renewals in A is either finite or infinite a.s. according as mA is finite or infinite. Chung and Derman obtain this result with the assumption that F has a bounded, continuous density which is in L^r for some $r > 1$. Sawyer assumes only that F has a directly Riemann integrable density.

<u>Theorem 1</u>. Let $\{S_n\}$ be a renewal process with waiting time distribution F and finite expected waiting time μ. Let f be a Lebesgue measurable function on $[0,\infty)$ satisfying $0 \leq f \leq B$.

(a) If F is non-singular, then $\int_0^{\infty} f(t)dt < \infty$ implies that $E(\sum^{\infty} f(S_n)) < \infty$ and $\int_0^{\infty} f(t)dt = \infty$ implies that $\sum^{\infty} f(S_n) = \infty$ a.s.

(b) If F is an arithmetic distribution with its mass concentrated on the points $0,\lambda,2\lambda,\ldots$ then $\sum_{k=0}^{\infty} f(\lambda k) < \infty$ implies that $E(\sum^{\infty} f(S_n)) < \infty$ and $\sum_{k=0}^{\infty} f(\lambda k) = \infty$ implies that $\sum^{\infty} f(S_n) = \infty$ a.s.

The proof of the theorem requires a fundamental result of renewal theory. Let $U(x) = \sum_{n=1}^{\infty} F^{*n}(x)$. This function provides a measure (the renewal measure) on $[0,\infty)$ defined on intervals by $U((a,b])) = U(b) - U(a)$. Throughout the paper, for any increasing function D and any Lebesgue measurable set S, we denote the measure of S induced by D as D(S).

<u>Renewal Theorem</u>. If F is arithmetic with mass p_k at λk, then U is purely atomic with mass $u_k = \sum_{n=1}^{\infty} p_k^{*n}$ at λk, and $u_k \to \frac{\lambda}{\mu}$ as $k \to \infty$.

If F is not arithmetic, then for any $h > 0$ $U(t-h,t] \to \frac{h}{\mu}$ as $t \to \infty$. (See [5], p.347.)

Proof of Theorem 1, $\sum_{k=0}^{\infty} f(\lambda k) < \infty$, $\int_0^{\infty} f(t)dt < \infty$

In the arithmetic case $u_k \to \frac{\lambda}{\mu}$. So u_k is bounded, say by M.

$$E(\sum_{n=1}^{\infty} f(S_n)) = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} f(\lambda k)p_k^{*n}$$

$$= \sum_{k=0}^{\infty} f(\lambda k)u_k$$

$$\leq M \sum_{k=0}^{\infty} f(\lambda k),$$

which is finite if $\sum_{k=0}^{\infty} f(\lambda k) < \infty$.

The same proof works in the case that F has a density which is sufficiently well-behaved to provide a bounded renewal density. However, we need not require this much of F. The following lemma shows that under a much weaker condition on F, the renewal measure is essentially a measure with a bounded density.

Lemma 1. If F' is bounded away from zero on some interval, then the renewal measure can be written as a sum of two measures, one finite and the other with a bounded density.

Proof of the lemma.

Assume that $F'(x) \geq \alpha > 0$ for $x \in [a,a+h]$. Let $g(x) = \alpha I_{[a,a+h]}(x)$, $G(x) = \int_0^x g(t)dt$, and $H(x) = F(x) - G(x)$. H is an increasing function:

$$H(y) - H(x) = F(y) - F(x) - \int_x^y g(t)dt$$

$$\geq F(y) - F(x) - \int_x^y F'(t)dt$$

$$\geq 0$$

$$U(x) = \sum_{n=1}^{\infty} F^{*n}(x) = \sum_{n=1}^{\infty} (G(x) + H(x))^{*n}$$

$$= \sum_{n=1}^{\infty} \sum_{j=0}^{n} \binom{n}{j} G^{*j} * H^{*n-j}(x)$$

$$= G*M(x) + \sum_{n=1}^{\infty} H^{*n}(x),$$

where $M(x) = \sum_{n=1}^{\infty} \sum_{j=1}^{n} \binom{n}{j} G^{*j-1} * H^{*n-j}(x)$.

$G*M(x)$ has density $\int_0^x g(x-y)dM(y)$, which we will now show is bounded.

$$\int_0^x g(x-y)dM(y) = \alpha\, M[x-a-h, x-a].$$

$$U(z, z+\frac{h}{2}) = G*M(z, z+\frac{h}{2}) + \sum H^{*n}(z, z+\frac{h}{2})$$

$$\geq G*M(z, z+\frac{h}{2})$$

$$= \int_z^{z+\frac{h}{2}} \int_0^x g(x-y)dM(y)dx$$

$$= \alpha \int_z^{z+\frac{h}{2}} M(x-a-h, x-a)dx$$

$$\geq \frac{\alpha h}{2} M(z-a-\frac{h}{2}, z-a),$$

since $x \in [z, z+\frac{h}{2}]$ implies that $[x-a-h, x-a] \supset [z-a-\frac{h}{2}, z-a]$.

By applying this inequality, we have

$$\int_0^x g(x-y)\,dM(y) = \alpha[M(x-a-h,x-a-\tfrac{h}{2}) + M(x-a-\tfrac{h}{2},x-a)]$$

$$\leq \frac{2}{h}[U(x-\tfrac{h}{2},x) + U(x,x+\tfrac{h}{2})] = \frac{2}{h}\,U(x-\tfrac{h}{2},x+\tfrac{h}{2}).$$

By the Renewal Theorem, $U(x-\frac{h}{2},x+\frac{h}{2})$ is bounded. Therefore $G*M$ has a bounded density. It remains only to show that $\sum_{n=1}^{\infty} H^{*n}(x)$ is a finite measure.

$$\sum_{n=1}^{\infty} H^{*n}(\infty) \leq \sum_{n=1}^{\infty} (H(\infty))^n < \infty,$$

since $H(\infty) = 1-\alpha h < 1$.

<u>Proof of Theorem 1, $\int_0^\infty f(t)dt < \infty$</u>

We now assume that F is non-singular, that is, $\int_0^\infty F'(t)dt > 0$. Clearly we may truncate F' so that its integral is still positive. In particular, let $V(t) = F'(t)I_{\{x:F'(x) \leq M\}}(t)$ for an M so that $\int_0^\infty V(t)dt > 0$. Now let $D(x) = F(x) - \int_0^x V(t)dt$.

$$F^{*2}(x) = \int_0^x V^{*2}(t)dt + 2\int_0^x F(x-t)dD(t) + D^{*2}(x)$$

and therefore $(F^{*2})'(t) \geq V^{*2}(t)$. Since V is bounded V^{*2} is continuous, and $(F^{*2})'$ therefore is bounded away from zero on some interval. The following equality allows us to exploit this fact.

$$E(\sum_{n=0}^{\infty} f(S_n)) = E(\sum_{n=0}^{\infty} f(S_{2n}) + \sum_{n=0}^{\infty} E(f(S_{2n+1})\,|S_{2n}))$$

$$= E(\sum_{n=0}^{\infty} h(S_{2n}))$$

where $h(s) = f(s) + \int_0^\infty f(s+t)dF(t)$. $\int_0^\infty h(s)ds < \infty$ if $\int_0^\infty f(s)ds < \infty$, and S_{2n} is a renewal process with waiting time distribution F^{*2}. So we may assume without loss of generality that the waiting time distribution F has (almost everywhere) a derivative which is bounded away from zero on an interval. Lemma 1 now applies. Suppose the decomposition provided is $U(x) = \int_0^x W(t)dt + Z(x)$ with $0 \leq W \leq A$ and $Z(\infty) < \infty$.

$$E(\sum_{n=1}^{\infty} f(S_n)) = \sum^{\infty} \int_0^{\infty} f(s)dF^{*n}(s) = \int_0^{\infty} f(s)dU(s)$$

$$= \int_0^{\infty} f(s)W(s)ds + \int_0^{\infty} f(s)dZ(s) \leq A \int_0^{\infty} f(s)ds + BZ(\infty),$$

which is finite if $\int_0^{\infty} f(s)ds < \infty$.

<u>Proof of Theorem 1,</u> $\sum_{k=0}^{\infty} f(\lambda k) = \infty$

Let $E_n = \{\omega \in \Omega : \exists k \text{ such that } S_k = \lambda n\}$, where Ω is the underlying probability space on which $\{S_k\}$ is defined.

(1) $$\sum^{\infty} f(S_n) \geq \sum^{\infty} f(\lambda n) I_{E_n} .$$

By considering the last renewal at λn,

$$P(E_n) = \sum_{k=1}^{\infty} P(S_k = \lambda n,\ X_{k+1} > 0)$$

$$= \sum^{\infty} P(S_k = \lambda n)P(X_1 > 0) = P(X_1 > 0)u_n .$$

Since $u_n \to \frac{\lambda}{\mu}$ as $n \to \infty$, by the Renewal Theorem, there is a $c > 0$ and an N such that

(2) $$P(E_n) \geq c \text{ for } n \geq N.$$

We can now apply a generalization of a lemma due to Spitzer ([9], p. 317).

<u>Lemma 2.</u> Let $Y_1, Y_2, \ldots, Y_n, \ldots$ be a sequence of non-negative random variables. If $E(\sum_{i=1}^{\infty} Y_i) = \infty$ and $\underline{\lim}_{n\to\infty} \dfrac{E[(\sum^n Y_i)^2]}{(E(\sum^n Y_i))^2} \leq \dfrac{1}{c}$, then $P(\sum^{\infty} Y_i = \infty) \geq c$.

<u>Proof of Lemma 2</u>

Let $B_{n,\varepsilon} = \{\sum_{i=1}^{k} Y_i \geq \varepsilon \sum_{i=1}^{k} EY_i \text{ for some } k \geq n\}$, $\varepsilon < 1$.

$$P(B_{n,\varepsilon}) \geq \frac{\{E[(\sum^n Y_i) I_{B_{n,\varepsilon}}]\}^2}{E[(\sum^n Y_i)^2]} \qquad \text{(by Holder's inequality)}$$

$$= \frac{\{E(\sum^n Y_i) - E(\sum^n Y_i I_{B^c_{n,\varepsilon}})\}^2}{E[(\sum^n Y_i)^2]} \geq (1-\varepsilon)^2 \frac{(E(\sum^n Y_i))^2}{E[(\sum^n Y_i)^2]}$$

$$P(\sum^{\infty} Y_i = \infty) \geq P(\sum^n Y_i \geq \varepsilon \sum^n EY_i \text{ i.o.}) \qquad (\text{since } \sum^{\infty} EY_i = \infty)$$

$$= P(\bigcap_{n=1}^{\infty} B_{n,\varepsilon}) = \lim_{n\to\infty} P(B_{n,\varepsilon})$$

$$\geq (1-\varepsilon)^2 \overline{\lim_{n\to\infty}} \frac{(\sum^n EY_i)^2}{E[(\sum^n Y_i)^2]} \geq (1-\varepsilon)^2 c$$

Therefore $P(\sum^{\infty} Y_i = \infty) \geq c$.

Now we will verify that the conditions of Lemma 2 are satisfied with $Y_n = f(\lambda n) I_{E_n}$, and the result will follow from (1).

$$E\left(\sum^{\infty} f(\lambda n) I_{E_n}\right) = \sum^{\infty} f(\lambda n) P(E_n) \geq c \sum^{\infty} f(\lambda n) = \infty$$

by hypothesis.

By (2), $P(E_iE_j) \leq P(E_i) \leq \frac{1}{c} P(E_i)P(E_j)$ if $j \geq N$.

$$\frac{E\left[\left(\sum_{i=1}^{n} f(\lambda i) I_{E_i}\right)^2\right]}{\left(\sum^{n} Ef(\lambda i) I_{E_i}\right)^2} = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} f(\lambda i) f(\lambda j) P(E_iE_j)}{\sum_{i=1}^{n} \sum_{j=1}^{n} f(\lambda i) f(\lambda j) P(E_i)P(E_j)}$$

$$\leq \frac{\sum_{i=1}^{N-1} \sum_{j=1}^{N-1} f(\lambda i) f(\lambda j) P(E_iE_j)}{\left(\sum^{n} Ef(\lambda i) I_{E_i}\right)^2} + \frac{\sum^{n}_{i \text{ or } j \geq N} \sum^{n} f(\lambda i) f(\lambda j) P(E_iE_j)}{\sum^{n}_{i \text{ or } j \geq N} \sum^{n} f(\lambda i) f(\lambda j) P(E_i)P(E_j)}.$$

$$< \frac{(N-1)^2 B^2}{\left(\sum^{n} Ef(\lambda i) I_{E_i}\right)^2} + \frac{1}{c}$$

Letting $n \to \infty$, $$\overline{\lim_{n\to\infty}} \frac{E\left(\sum^{n} f(\lambda i) I_{E_i}\right)^2}{\left(\sum^{n} Ef(\lambda i) I_{E_i}\right)} \leq \frac{1}{c}.$$

By Lemma 2 and (1), $P\left(\sum^{\infty} f(S_n) = \infty\right) \geq c > 0$. By the Hewitt-savage zero-one law, $\sum^{\infty} f(S_n) = \infty$ a.s.

<u>Proof of Theorem 1, $\int_0^{\infty} f(t)dt = \infty$</u>

We now assume that F is non-singular. However, noting that $\sum^{\infty} f(S_n) \geq \sum^{\infty} f(S_{2n})$ and that S_{2n} is a renewal process with waiting time distribution F^{*2}, we may, as before, assume without loss of generality that $F' \geq \alpha > 0$ on $[a,a+h]$.

The proof of the infinite part requires a result due to Doob ([3], p. 323). First we will define an object which appears several times in the paper.

Definition. If $Y_1, Y_2, \ldots, Y_n \ldots$ is a sequence of random variables and $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n, \ldots$ is an increasing sequence of sigma-fields such that Y_n is $\mathcal{F}_n$-measurable, then $\{Y_n, \mathcal{F}_n\}_{n=1}^{\infty}$ will be called a stochastic sequence.

Lemma 3 (Doob) If $\{Y_n, \mathcal{F}_n\}_{n=1}^{\infty}$ is a uniformly bounded, non-negative stochastic sequence, then $\sum_{n=1}^{\infty} Y_n(\omega)$ converges for almost all $\omega \in \Omega$ for which $\sum_{n=1}^{\infty} E(Y_{n+1}|\mathcal{F}_n)(\omega)$ converges, and conversely.

By the lemma it is sufficient to show that $\sum^{\infty} E(f(S_{n+1})|S_n) = \infty$ a.s.

$$\sum^{\infty} E(f(S_{n+1})|S_n) = \sum^{\infty} \int_0^{\infty} f(S_n+t)dF(t) \geq \alpha \sum^{\infty} \int_{S_n}^{S_n+h} f(t+a)dt.$$

Let $A_n = (\frac{(n-1)h}{2}, \frac{nh}{2})$ and $E_n = \{\omega \in \Omega : \exists k \text{ such that } S_k \in A_n\}$.

$$(3) \quad \sum^{\infty} E(f(S_{n+1})|S_n) \geq \alpha \sum^{\infty} (\min_{S \in A_n} \int_S^{S+h} f(t+a)dt) I_{E_n}$$

$$\geq \alpha \sum^{\infty} (\int_{A_{n+1}} f(t+a)dt) I_{E_n} .$$

We can now apply Lemma 2 to conclude that the last series is infinite a.s. if $\int_0^{\infty} f(t)dt = \infty$. The same proof as in the arithmetic case works. The crucial facts are that $\sum^{\infty} \int_{A_{n+1}} f(t+a)dt = \infty$, which is obvious, and that $P(E_n) \geq c$ for some $c > 0$ and for n sufficiently large. By considering the last visit of $\{S_k\}$ to A_n, we have

$$P(E_n) = \sum_{r=1}^{\infty} \int_{A_n} P(S_{r+k} \in A_n^c;\ k = 1,2,\ldots | S_r=s)dF^{*r}(s)$$

$$\geq \sum_{r=1}^{\infty} \int_{A_n} P(X_{r+1} > \frac{h}{2})dF^{*r}(s) = P(X_1 > \frac{h}{2})U(A_n).$$

By the Renewal Theorem, there is a $c > 0$ and an N such that $P(E_n) \geq c$ for $n \geq N$. So by Lemma 2 and (3), $P(\sum^{\infty} E(f(S_{n+1})\ S_n) = \infty) \geq c > 0$, and hence, by the Hewitt-Savage zero-one law, $\sum^{\infty} E(f(S_{n+1})\ S_n) = \infty$ a.s.

The Random Walk Case

The assumption of Theorem 1 that $X_i \geq 0$ will now be removed. In this case the process $S_n = \sum_{i=1}^{n} X_i$ is called a random walk, and the X_i's are called the step sizes.

Theorem 2. Let $\{S_n\}$ be a random walk with step size distribution F and expected step size μ, $0 < \mu < \infty$. Let $\{L_k\}$ be the sequence of strictly ascending ladder epochs. Let f be a Lebesgue measurable function defined on $(-\infty,\infty)$ and satisfying $0 \leq f \leq B$.

(a) If F is non-singular, then

(i) $\int_0^{\infty} f(s)ds < \infty$ and $E(\sum_{n=1}^{L_1-1} \int_{S_n}^{0} f(s)ds) < \infty$ imply $E(\sum_{n=1}^{\infty} f(S_n)) < \infty$.

(ii) $E(\sum_{n=1}^{L_1-1} \int_{S_n}^{0} f(s)ds) = \infty$ implies $E(\sum_{n=1}^{\infty} f(S_n)) = \infty$.

(iii) $\int_0^{\infty} f(s)ds < \infty$ implies $\sum_{n=1}^{\infty} f(S_n) < \infty$ a.s.

(iv) $\int_0^{\infty} f(s)ds = \infty$ implies $\sum_{n=1}^{\infty} f(S_n) = \infty$ a.s.

(b) If F is arithmetic with mass concentrated at the points $0, \pm\lambda, \pm 2\lambda, \ldots$, then

(i) $\sum_{k=0}^{\infty} f(\lambda k) < \infty$ and $E(\sum_{n=1}^{L_1-1} \sum_{k=\frac{S_n}{\lambda}}^{-1} f(\lambda k)) < \infty$ imply $E(\sum_{n=1}^{\infty} f(S_n)) < \infty$.

(ii) $E(\sum_{n=1}^{L_1-1} \sum_{k=\frac{S_n}{\lambda}}^{-1} f(\lambda k)) = \infty$ implies $E(\sum_{n=1}^{\infty} f(S_n)) = \infty$.

(iii) $\sum_{k=0}^{\infty} f(\lambda k) < \infty$ implies $\sum_{n=1}^{\infty} f(S_n) < \infty$ a.s.

(iv) $\sum_{k=0}^{\infty} f(\lambda k) = \infty$ implies $\sum_{n=1}^{\infty} f(S_n) = \infty$ a.s.

Proof:

The proofs of the non-singular case and the arithmetic case are the same. We will consider only the non-singular case.

$$
\begin{aligned}
E(\sum_{n=1}^{\infty} f(S_n)) &= E(\sum_{k=0}^{\infty} E(\sum_{n=L_k+1}^{L_{k+1}} f(s_n) | S_{L_k})) \\
&= E(\sum_{k=0}^{\infty} g(S_{L_k})),
\end{aligned} \tag{4}
$$

where

$$g(s) = E(\sum_{n=L_k+1}^{L_{k+1}} f(S_n) | S_{L_k} = s) = E(\sum_{n=1}^{L_1} f(s+S_n)).$$

$g \leq BE(L_1) < \infty$ since $\mu > 0$, and $\{S_{L_k}\}_{k=1}^{\infty}$ is a renewal process with expected waiting time $\mu E(L_1) < \infty$. (See [4], p. 380.) It is also easily seen that $\{S_{L_k}\}$ has a non-singular waiting time distribution, if F is non-singular. Therefore, from (4) and Theorem 1, $E(\sum_{n=1}^{\infty} f(S_n)) < \infty$ if and only if $\int_0^{\infty} g(s)ds < \infty$.

$$\int_0^\infty g(s)ds = \int_0^\infty E(\sum_{n=1}^{L_1} f(s+S_n))ds$$

$$(5) \qquad = E(\sum_{n=1}^{L_1} \int_{S_n}^\infty f(s)ds)$$

$$= E(\sum_{n=1}^{L_1-1} \int_{S_n}^0 f(s)ds)+E(L_1-1)\int_0^\infty f(s)ds+E(\int_{S_{L_1}}^\infty f(s)ds)$$

(i) and (ii) follow from (5) and the fact that $E(L_1) < \infty$.

(iii). By the Strong Law of Large Numbers the number of visits of $\{S_n\}$ to $(-\infty,0)$ is finite a.s. So, for the purpose of proving (iii), we may assume that $f(x) = 0$ for $x < 0$. With this assumption (5) implies $\int_0^\infty g(s)ds \leq E(L_1)\int_0^\infty f(s)ds < \infty$. $\int_0^\infty g(s)ds < \infty$ implies $E(\sum_{n=1}^\infty f(S_n)) < \infty$, and hence $\sum_{n=1}^\infty f(S_n) < \infty$ a.s.

(iv). By Theorem 1, $\sum_{n=1}^\infty f(S_n) \geq \sum_{k=1}^\infty f(S_{L_k}) = \infty$ a.s. if $\int_0^\infty f(s)ds = \infty$.

Applications

As was mentioned at the beginning of the chapter, the special case $f = I_A$ provides some interesting results. Let $N_A = \sum_{n=1}^\infty I_A(S_n)$, the number of visits of $\{S_n\}$ to A.

Corollary 1. Let $\{S_n\}$ be a renewal process with waiting time distribution F and finite expected waiting time μ.

(a) If F is non-singular and A is a Lebesgue measurable set, $A \subset [0,\infty)$, then $m(A) < \infty$ implies $E(N_A) < \infty$ and $m(A) = \infty$ implies $N_A = \infty$ a.s.

(b) Let F be arithmetic with mass concentrated at the points $0,\lambda,2\lambda,\ldots$ and let $A \subset \{0,\lambda,2\lambda,\ldots\}$. If A is an infinite set, then $N_A = \infty$ a.s.

<u>Corollary 2</u>. Let $\{S_n\}$ be a random walk with step size distribution F and expected step size μ, $0 < \mu < \infty$. Let L_1 be the first strictly ascending ladder epoch.

(a) If F is non-singular and A is a Lebesgue measurable set, then

(i) $m(A \cap (0,\infty)) < \infty$ and $E(\sum_{n=1}^{L_1-1} m(A \cap (S_n,0))) < \infty$ imply $E(N_A) < \infty$.

(ii) $E(\sum_{n=1}^{L_1-1} m(A \cap (S_n,0))) = \infty$ implies $E(N_A) = \infty$.

(iii) $m(A \cap (0,\infty)) < \infty$ implies $N_A < \infty$ a.s.

(iv) $m(A \cap (0,\infty)) = \infty$ implies $N_A = \infty$ a.s.

(b) For any set $C \subset \{0,\pm\lambda,\pm2\lambda,\ldots\}$ let M(C) denote the number of members of C. If F is arithmetic with mass concentrated at the points $0,\pm\lambda,\pm2\lambda,\ldots$ and $A \subset \{0,\pm\lambda,\pm2\lambda,\ldots\}$, then

(i) $M\{\lambda k \in A: \ k \geq 0\} < \infty$ $E(\sum_{n=1}^{L_1-1} M\{\lambda k \in A: \ S_n \leq \lambda k < 0\}) < \infty$ imply $E(N_A) < \infty$.

(ii) $E(\sum_{n=1}^{L_1-1} M\{\lambda k \in A: \ S_n \leq \lambda k < 0\}) = \infty$ implies $E(N_A) = \infty$.

(iii) $M\{\lambda k \in A: \ k \geq 0\} = \infty$ implies $N_A = \infty$ a.s.

If we specialize further to the case $A = (-\infty,0)$, then N_A is the number of visits below zero by a random walk with positive drift. (i) and (ii) of Corollary 2 now combine to give:

Corollary 3. Let $\{S_n\}$ be a random walk with expected step size μ, $0 < \mu < \infty$, and either non-singular or arithmetic step size distribution. Let L_1 be the first strictly ascending ladder epoch. $E(N_{(-\infty,0)}) < \infty$ if and only if $E(\sum_{n=1}^{L_1-1} |S_n|) < \infty$.

We now use Corollary 3 to provide an example of the fact that, although the number of visits below zero by a random walk with positive drift is finite a.s., the expected number of such visits may be infinite.

Example. Let $\mu = E(X_i) > 0$, $E((X_i^-)^2) = \infty$, and $X_i \leq 1$. $X_1 < -\sqrt{t}$ implies $\sum_{n=1}^{L_1-1} |S_n| > \frac{t}{2}$. Therefore

$$E(2\sum_{n=1}^{L_1-1} |S_n|) = \int_0^\infty P(\sum_{n=1}^{L_1-1} |S_n| > \tfrac{t}{2})dt$$

$$\geq \int_0^\infty P((X_1^-)^2 > t)dt$$

$$= E((X_1^-)^2) = \infty.$$

So by Corollary 3, $E(N_{(-\infty,0)}) = \infty$.

The Brownian Motion Case

Theorem 3. Let $\{S(t): t \geq 0\}$ be Brownian Motion with drift $\mu > 0$ and f a Lebesgue measurable function defined on $(-\infty,\infty)$ and satisfying $0 \leq f \leq B$. $\int_0^\infty f(s)ds < \infty$ implies that $E(\int_0^\infty f(S(t))dt) < \infty$, and $\int_0^\infty f(s)ds = \infty$ implies that $\int_0^\infty f(S(t))dt = \infty$ a.s.

Proof:

$$\int_0^\infty f(S(t))dt = \sum_{n=0}^\infty \int_n^{n+1} f(S(t))dt$$

$\int_n^{n+1} f(S(t)) \leq B$. So by Lemma 3, $\int_0^\infty f(S(t))dt = \infty$ a.s. if $\sum_{n=0}^\infty E(\int_n^{n+1} f(S(t))dt|S(n)) = \infty$ a.s.

(6) $$\sum_{n=0}^\infty E(\int_n^{n+1} f(S(t))dt|S(n)) = \sum_{n=0}^\infty g(S(n)),$$

where

$$g(s) = E(\int_n^{n+1} f(S(t))dt|S(n)=s) = E(\int_0^1 f(s+S(t))dt).$$

By definition of Brownian Motion, $S(n)$ is a random walk with step sizes normally distributed with mean μ. If $\int_0^\infty f(s)ds = \infty$, then

(7) $$\int_0^\infty g(s)ds = \int_0^\infty E(\int_0^1 f(s+S(t))dt)ds = E(\int_0^1 \int_{S(t)}^\infty f(s)dsdt)$$

$$\geq \int_0^\infty f(s)ds - BE(\sup_{0\leq t\leq 1} |S(t)|) = \infty,$$

since $E(\sup_{0\leq t\leq 1} |S(t)|) < \infty$. Therefore, by Theorem 2 (a) (iv) and (6), $\int_0^\infty f(S(t))dt = \infty$ a.s. if $\int_0^\infty f(s)ds = \infty$.

Taking expectations in (6), we have

(8) $$E(\int_0^\infty f(S(t))dt) = E(\sum_{n=0}^\infty g(S_n)).$$

As in (7), if $\int_0^\infty f(s)ds < \infty$, then

$$\int_0^\infty g(s)ds \leq \int_0^\infty f(s)ds + BE(\sup_{0\leq t\leq 1} |S(t)|) < \infty.$$

Letting L_1 be the first strictly ascending ladder epoch of $S(n)$,

$$E\left(\sum_{n=1}^{L_1-1} \int_{S_n}^{0} g(s)ds\right) \leq BE\left(\sum_{n=1}^{L_1-1} |S(n)|\right),$$

which is finite, by Corollary 3, if $N_{(-\infty,0)} = \sum_{n=1}^{\infty} I_{(-\infty,0)}(S(n))$ has finite expected value. Since $S(n)$ is normally distributed it is easily shown that $E(N_{(-\infty,0)}) = \sum_{n=1}^{\infty} P(S(n) < 0) < \infty$. So, by Theorem 2 (a) (i) and (8), $\int_0^\infty f(s)ds < \infty$ implies $E(\int_0^\infty f(S(t))dt) < \infty$.

In the special case $f = I_A$, A a Lebesgue measurable set,

$$\int_0^\infty f(S(t))dt = m\{t \geq 0: \ S(t) \in A\} ,$$

or the amount of time spent by the process in A.

<u>Corollary 4</u>. Let $\{S(t): t > 0\}$ be Brownian Motion with drift $\mu > 0$ and A a Lebesgue measurable set. $m(A \cap (0,\infty)) < \infty$ implies $E(m\{t \geq 0: \ S(t) \in A\}) < \infty$, and $m(A \cap (0,\infty)) = \infty$ implies $m\{t \geq 0: \ S(t) \in A\} = \infty$ a.s.

<u>Corollary 5</u>. If $\{S(t): \ t \geq 0\}$ is Brownian Motion with drift μ, $-\infty < \mu < \infty$, then for any $\varepsilon > 0$

$$E(m\{t: \ \frac{S(t)}{t} \in (\mu-\varepsilon,\mu+\varepsilon)^c\}) < \infty .$$

<u>Proof</u>:

$E(m\{t: \ \frac{S(t)}{t} < \ \mu-\varepsilon\}) = E(m\{t: \ S(t) - (\mu-\varepsilon)t < 0\}) < \infty$, by Corollary 4, since $S(t)-(\mu-\varepsilon)t$ is Brownian Motion with drift $\varepsilon>0$. By symmetry, we also have $E(m\{t: \ \frac{S(t)}{t} > \mu+\varepsilon)\} < \infty$.

CHAPTER II

In Chapter I it was shown that $\sum^{\infty} f(S_n) = \infty$ a.s. if and only if $\int_0^{\infty} f(t)dt = \infty$. This fact suggests the following question: In the case $\int_0^{\infty} f(t)dt = \infty$, does there exist a rate of divergence for $\sum_{j=1}^{n} f(S_j)$ in terms of the partial integrals of f. The answer, in general, is no, and a counterexample is provided below. However, if in addition to the assumptions of Theorem 1, f is not allowed to be small on progressively longer intervals, then there is such a rate of divergence.

Theorem 4. Let $\{S_n\}$ be a renewal process with waiting time distribution F and finite mean μ. Let f be a non-negative, bounded, Lebesgue measurable function.

(a) If F is a non-singular distribution, and there exist M and δ such that $\int_x^{x+M} f(t)dt \geq \delta > 0$ for all $x \geq 0$, then

$$\frac{\sum^{n} f(S_j)}{\frac{1}{\mu}\int_0^{\mu n} f(t)dt} \overset{a.s.}{\rightarrow} 1 \quad \text{as } n \to \infty.$$

(b) If F is an arithmetic distribution with mass p_k at λk, k=0,1,2,..., and there exist m and δ such that

$\sum_{k=0}^{\infty} f(s+\lambda k)p_k^{*m} \geq \delta > 0$ for $s=0,1,2,\ldots$, then $\dfrac{\sum^{n} f(S_j)}{\frac{1}{\mu}\sum_{j=1}^{[\mu n]^*} f(\lambda j)} \xrightarrow{a.s.} 1.$

Before beginning the proof of Theorem 4 we will show that the additional assumption on f is needed. Let some waiting times X_i have an absolutely continuous distribution with $\mu = E(X_i) = \sigma^2(X_i) = 1$. By alternatively taking f to be 0 and then 1 on progressively longer intervals, it is possible to construct a function f such that $\dfrac{n}{\int_0^{n^2} f(t)dt} \geq 1$, $f(x) = 0$ for $x \in [n^2, n^2+n]$, and $f(x) = 1$ for $x \in [n^2+n, n^2+3n]$ for an infinite number of n. For such an n

$$P\left(\left|\frac{\sum_{j=1}^{n^2+n} f(S_j)}{\int_0^{n^2+n} f(t)dt} - \frac{\sum_{j=1}^{n^2} f(S_j)}{\int_0^{n^2} f(t)dt}\right| \geq 1\right)$$

$$\geq P(S_{n^2} \in [n^2+n, n^2+2n],\ S_{n^2+n} \leq n^2+3n)$$

$$\geq P(S_{n^2} \in [n^2+n, n^2+2n])\ P(S_n \leq n)$$

$$= P\left(1 \leq \frac{S_{n^2} - n^2}{n} \leq 2\right)\ P\left(\frac{S_n - n}{\sqrt{n}} \leq 0\right)$$

$\not\to 0$ as $n \to \infty$ by the Central Limit Theorem. So in this case $\dfrac{\sum^{n} f(S_j)}{\int_0^n f(t)dt}$ does not even converge in probability.

---

* [k] denotes the greatest integer less than k.

The following theorem is the basic tool for the proof of Theorem 4.

Theorem 5. Let $\{Y_k, \mathscr{F}_k\}$ be a non-negative stochastic sequence. Assume that there exists a distribution function G satisfying

$$P(Y_k \le c | \mathscr{F}_{k-1}) \ge G(c) \text{ for all } c \ge 0 \text{ and } k \ge 1$$

and

$$\int_0^\infty (1-G(c))dc < \infty.$$

(a) Then $\frac{1}{n} \sum_{k=1}^{n} (Y_k - E(Y_k | \mathscr{F}_{k-m})) \overset{a.s.}{\to} 0$ as $n \to \infty$, $m = 1,2,\ldots$.

(For $i \le 0$, $\mathscr{F}_i$ is defined to be $\{\phi,\Omega\}$ so that $E(Y_k|\mathscr{F}_i) = E(Y_k)$.)

(b) If, in addition, for some $\gamma > 0$ $E(Y_k|\mathscr{F}_{k-m}) \ge \gamma$ (or $E(Y_{k+m}|\mathscr{F}_k) \ge \gamma$) for $k = 1,2,\ldots$, then

$$\frac{\sum_{k=1}^{n} Y_k}{\sum_{k=1}^{n} E(Y_k|\mathscr{F}_{k-m})} \overset{a.s.}{\to} 1 \quad \text{as } n \to \infty,$$

or in a more convenient form

$$\frac{\sum_{k=1}^{n} Y_k}{\sum_{k=1}^{n} E(Y_{k+m}|\mathscr{F}_k)} \overset{a.s.}{\to} 1 \quad \text{as } n \to \infty.$$

Proof:

First consider the case m=1. Let $\bar{Y}_k = Y_k I_{\{Y_k \le k\}}$.

$$\sum_{k=1}^{\infty} P(\bar{Y}_k \ne Y_k) = \sum_{k=1}^{\infty} P(Y_k > k) \le \sum_{k=1}^{\infty} (1-G(k)) \le \int_0^\infty (1-G(c))dc < \infty.$$

Therefore, by the Borel-Cantelli Lemma, $\bar{Y}_k = Y_k$ except finitely often. Hence

$$\frac{1}{n}\sum_{k=1}^{n}(Y_k - \bar{Y}_k) \overset{a.s.}{\to} 0 \quad \text{as } n \to \infty.$$

Let Z be a random variable with distribution G.

$$E(Z) = \int_0^\infty (1-G(c))dc < \infty.$$

$$\frac{1}{n}\sum_{k=1}^{n}(E(Y_k|\mathcal{F}_{k-1}) - E(\bar{Y}_k|\mathcal{F}_{k-1})) = \frac{1}{n}\sum_{k=1}^{n}E(Y_k I_{\{Y_k>k\}}|\mathcal{F}_{k-1})$$

$$\leq \frac{1}{n}\sum_{k=1}^{n}E(ZI_{\{Z>k\}}) \to 0 \quad \text{as } n \to \infty,$$

since $E(ZI_{\{Z>k\}}) \to 0$ as $k \to \infty$ by the Dominated Convergence Theorem.

So it suffices to show that

$$\frac{1}{n}\sum_{k=1}^{n}(\bar{Y}_k - E(\bar{Y}_k|\mathcal{F}_{k-1})) \overset{a.s.}{\to} 0 \quad \text{as } n \to \infty \quad \text{or,}$$

by Kronecker's Lemma, that $\sum_{k=1}^{\infty}\left(\frac{\bar{Y}_k - E(\bar{Y}_k|\mathcal{F}_{k-1})}{k}\right)$ converges a.s.

For any stochastic sequence $\{W_k, \mathcal{F}_k\}$, $\sum_{k=1}^{\infty} W_k$ converges a.s. if $\sum_{k=1}^{\infty} E(W_k|\mathcal{F}_{k-1})$ converges a.s. and $\sum_{k=1}^{\infty}\sigma^2(W_k) < \infty$. (See [5], p. 387).

$$\sum_{k=1}^{\infty} E\left(\frac{\bar{Y}_k - E(\bar{Y}_k|\mathcal{F}_{k-1})}{k}\bigg|\mathcal{F}_{k-1}\right) = 0.$$

$$\sum_{k=1}^{\infty}\sigma^2\left(\frac{\bar{Y}_k - E(\bar{Y}_k|\mathcal{F}_{k-1})}{k}\right) \leq \sum_{k=1}^{\infty}\frac{1}{k^2}[E(\bar{Y}_k^2) + (E(Z))^2]. \qquad \sum_{k=1}^{\infty}\frac{1}{k^2}(E(Z))^2 < \infty.$$

$$E(\bar{Y}_k^2) = E(Y_k^2 I_{\{Y_k \leq k\}}) = \int_0^\infty P(Y_k^2 I_{\{Y_k \leq k\}} > c)dc$$

$$\leq \int_0^{k^2} P(Y_k > \sqrt{c})dc \leq \int_0^{k^2}(1-G(\sqrt{c}))dc = 2\int_0^k c(1-G(c))dc$$

$$= \int_0^k c^2 dG(c) + k^2(1-G(k)) = E(Z^2 I_{\{Z \leq k\}}) + k^2(1-G(k))$$

$E(Z) < \infty$ implies $\sum_{k=1}^{\infty} \frac{1}{k^2} E(Z^2 I_{\{Z \leq k\}}) < \infty$. (See [5], p. 239.)

$$\sum_{k=1}^{\infty} \frac{1}{k^2} k^2 (1-G(k)) \leq \int_0^{\infty} (1-G(c))dc < \infty.$$

Therefore $\sum_{k=1}^{\infty} \sigma^2 \left(\frac{\bar{Y}_k - E(\bar{Y}_k | \mathscr{F}_{k-1})}{k}\right) < \infty$, and part (a) is proved for the case m=1.

The proof now proceeds by induction on m.

$$\frac{1}{n} \sum_{k=1}^{n} (Y_k - E(Y_k | \mathscr{F}_{k-(m+1)})) = \frac{1}{n} \sum_{k=1}^{n} (Y_k - E(Y_k | \mathscr{F}_{k-m}))$$

(2.1)
$$+ \frac{1}{n} \sum_{k=1}^{n} (E(Y_k | \mathscr{F}_{k-m}) - E(E(Y_k | \mathscr{F}_{k-m}) | \mathscr{F}_{k-(m+1)})).$$

$$E(Y_k | \mathscr{F}_{k-m}) = E(E(Y_k | \mathscr{F}_{k-1}) | \mathscr{F}_{k-m})$$
$$\leq E(E(Z) | \mathscr{F}_{k-m}) \leq E(Z) < \infty.$$

So $\{E(Y_k | \mathscr{F}_{k-m}), \mathscr{F}_{k-m}\}$ is a stochastic sequence, which is bounded in an even stronger sense than that required by Theorem 5. Therefore the second term in (2.1) converges to zero by applying Theorem 5 for the special case (m=1) already proven true. The first term in (2.1) converges to zero by the induction hypothesis. This concludes the proof of part (a).

Part (b) follows from part (a) by elementary properties of sequences of numbers.

Proof of Theorem 4

We may assume that $f \leq 1$.

By applying Theorem 5b we will show that it suffices to prove Theorem 4 with f replaced by a function g_m, which has some useful properties. Let $g_m(s) = E(f(S_{k+m}) | S_k = s)$. In the non-singular case, if m is sufficiently large, there exists $\alpha > 0$ such that $(F^{*m})' \geq \alpha$ on some interval of length M, say [a,a+M]. Therefore,

$$g_m(s) = \int_0^\infty f(s+t)\,dF^{*m}(t) \geq \alpha \int_a^{a+M} f(s+t)\,dt \geq \alpha\,\delta > 0$$

by hypothesis. So by Theorem 5b, $\dfrac{\sum^n f(S_k)}{\sum^n g_m(S_k)} \overset{a.s.}{\to} 1$ as $n \to \infty$. We also have $\dfrac{\int_0^{\mu n} f(s)\,ds}{\int_0^{\mu n} g_m(s)\,ds} \to 1$ as $n \to \infty$, since $\int_0^{\mu n} g_m(s)\,ds \geq \alpha\,\delta\,n$ and

$$\left|\int_0^{\mu n} g_m(s)\,ds - \int_0^{\mu n} f(s)\,ds\right| = \left|\int_0^{\mu n} E(f(s+S_m))\,ds - \int_0^{\mu n} f(s)\,ds\right|$$

$$= \left|E\left(\int_{S_m}^{S_m+\mu n} f(s)\,ds - \int_0^{\mu n} f(s)\,ds\right)\right|$$

$$\leq \left|E\left(\int_{\mu n}^{\mu n+S_m} f(s)\,ds - \int_0^{S_m} f(s)\,ds\right) I_{\{S_m \leq \mu n\}}\right| + \mu n P(S_m > \mu n)$$

$$\leq E(S_m) + \mu n P(S_m > \mu n)\ .$$

Therefore, $\dfrac{\sum^n g_m(S_k)}{\frac{1}{\mu}\int_0^{\mu n} g_m(s)\,ds} \overset{a.s.}{\to} 1$ implies that $\dfrac{\sum^n f(S_k)}{\frac{1}{\mu}\int_0^{\mu n} f(s)\,ds} \overset{a.s.}{\to} 1$.

In the arithmetic case the same argument shows that we need only consider $\dfrac{\sum^n g_m(S_k)}{\frac{1}{\mu}\sum^{[\mu n]} g_m(\lambda k)}$, where $g_m(s) = \sum_{k=1}^\infty f(s+\lambda k)p_k^{*m}$. In this case $g_m \geq \delta > 0$ by hypothesis.

So in either case we may assume without loss of generality that f is bounded away from zero, say $f \geq \gamma > 0$.

Now assuming that F is non-singular and $f \geq \gamma > 0$, we continue to apply the technique used above to replace f by a uniformly continuous function h_m. As in Lemma 1 of Chapter I, F can be written

$$F(x) = \int_0^x g(t)dt + H(x), \text{ where } g(t) = F'(t)I_{\{F'(t) \leq N\}}(t) \text{ and}$$

$H(\infty) < 1$. $F^{*m}(x) = \int_0^x q_m(t)dt + \beta_m(x)$,

where
$$q_m(t) = \sum_{j=2}^{m} \binom{m}{j} g^{*j} * H^{*m-j}(t)$$

$$= g^{*2} * \Big(\sum_{j=2}^{m} \binom{m}{j} g^{*j-2} * H^{*m-j}\Big)(t)$$

and

$$\beta_m(x) = mG * H^{*m-1}(x) + H^{*m}(x).$$

q_m is continuous, since g^{*2} is continuous. $\beta_m(\infty) \to 0$ as $m \to \infty$, since $H(\infty) < 1$.

By applying Theorem 5b we have seen that f may be replaced by

$$g_m(s) = \int_0^\infty f(s+t)dF^{*m}(t) = h_m(s) + \ell_m(s),$$

where

$$h_m(s) = \int_0^\infty f(s+t)q_m(t)dt \text{ and } \ell_m(s) = \int_0^\infty f(s+t)d\beta_m(t).$$

$$(2.2) \quad \frac{\sum^n f(S_k)}{\frac{1}{\mu}\int_0^{\mu n} f(s)ds} = \frac{\sum^n f(S_k)}{\sum^n g_m(S_k)} \cdot \frac{\sum^n g_m(S_k)}{\sum^n h_m(S_k)} \cdot \frac{\sum^n h_m(S_k)}{\frac{1}{\mu}\int_0^{\mu n} h_m(s)ds} \cdot \frac{\frac{1}{\mu}\int_0^{\mu n} h_m(s)ds}{\frac{1}{\mu}\int_0^{\mu n} g_m(s)ds}.$$

$$\frac{\frac{1}{\mu}\int_0^{\mu n} g_m(s)\,ds}{\frac{1}{\mu}\int_0^{\mu n} f(s)\,ds} \; .$$

The first and last factors converge to unity for any m , by the previous discussion.

$$\left|\frac{\sum^n h_m(S_k)}{\sum^n g_m(S_k)} - 1\right| = \frac{\sum^n \ell_m(S_k)}{\sum^n g_m(S_k)} \leq \frac{n\beta_m(\infty)}{n\gamma} = \frac{\beta_m(\infty)}{\gamma} ,$$

which goes to zero as $m \to \infty$. Similarly

$$\left|\frac{\int_0^{\mu n} h_m(s)\,ds}{\int_0^{\mu n} g_m(s)\,ds} - 1\right| \leq \frac{\beta_m(\infty)}{\gamma} \to 0 \text{ as } m \to \infty.$$

Therefore if for the third factor in (2.2)

$$\frac{\sum^n h_m(S_k)}{\frac{1}{\mu}\int_0^{\mu n} h(s)\,ds} \overset{a.s.}{\to} 1 \quad \text{as } n \to \infty ,$$

then $$\frac{\sum^n f(S_k)}{\frac{1}{\mu}\int_0^{\mu n} f(s)\,ds} \overset{a.s.}{\to} 1 \quad \text{as } n \to \infty.$$

It is easily shown that $h_m(s) = \int_0^\infty f(s+t)q_m(t)\,dt$ is uniformly continuous on $[0,\infty)$, since f is bounded and q_m is continous. So we may assume without loss of generality that f is uniformly continuous on $[0,\infty)$.

Existence of a Rate of Divergence

If there does exist a sequence of constants $\{c_n\}$ such that $\frac{1}{c_n}\sum^n f(S_j) \overset{a.s.}{\to} 1$, then $\sum^n f(S_j(\omega_1))$ and $\sum^n f(S_j(\omega_2))$ grow at the same rate for almost all pairs $(\omega_1,\omega_2) \in \Omega x \Omega$, where Ω is the underlying probability space. We will use the converse of this statement to show that there is a rate of divergence. For any $\omega=(\omega_1,\omega_2) \in \Omega x\Omega$, let $S_j(\omega) = S_j(\omega_1)$ and $S_j'(\omega) = S_j(\omega_2)$, so that $\{S_j\}$ and $\{S_j'\}$ are independent renewal processes with the same distribution. If

$$\frac{\sum^n f(S_j)}{\sum^n f(S_j')} \to 1$$

for almost every $\omega \in \Omega x\Omega$, then for some particular $\omega_2^* \in \Omega$

$$\frac{\sum^n f(S_j(\omega_1,\omega_2^*))}{\sum^n f(S_j'(\omega_1,\omega_2^*))} = \frac{\sum^n f(S_j(\omega_1))}{\sum^n f(S_j(\omega_2^*))} \to 1$$

for almost every $\omega_1 \in \Omega$. That is, the constants $c_n = \sum^n f(S_j(\omega_2^*))$ serve as a rate of divergence. Therefore it will be our goal to show that

$$\frac{\sum^n f(S_j)}{\sum^n f(S_j')} \overset{a.s.}{\to} 1.$$

The proof of this fact is greatly simplified if the waiting times are assumed to have an arithmetic distribution, and the idea behind the proof for the non-singular case stands out clearly in the proof of the arithmetic case.

The Arithmetic Case

Let $X_j = S_j - S_{j-1}$ and $X_j' = S_j' - S_{j-1}'$, where S_j and S_j' are the renewal processes defined above. Now we define a Markov process

$\{M_j\}$ and some associated stopping variables N_k, T_k, and T_k'.

Let $M_o=0$, $U_o=0$, $D_o=0$ and let $\{A_j\}_{j=1}^{\infty}$ be a sequence of independent events such that $P(A_j) = P(A_j^c) = \frac{1}{2}$ and A_j is independent of $\{M_1, M_2, \ldots, M_{j-1}\}$. Let

$$U_j = \sum_{i=1}^{j} I_{\{M_{i-1}<0\}\cup\{M_{i-1}=0, A_i\}}, \quad D_j = \sum_{i=1}^{j} I_{\{M_{i-1}>0\}\cup\{M_{i-1}=0, A_i^c\}}$$

$$M_{j+1} = \begin{cases} M_j + X_{U_j+1} & \text{if} \quad M_j < 0 \\ M_j - X'_{D_j+1} & \text{if} \quad M_j > 0 \\ M_j + X_{U_j+1} I_{A_{j+1}} - X'_{D_j+1} I_{A^c_{j+1}} & \text{if} \quad M_j = 0, \end{cases}$$

N_k = kth j such that $M_j = 0$, $U_{N_k} = T_k$, $D_{N_k} = T_k'$.

The Markov process $\{M_j\}$ and its return times to zero $\{N_k\}$ imitate the procedure of searching out the increasing sequence of values which are attained by both renewal processes $\{S_j\}$ and $\{S_j'\}$. Note that $M_j = \sum_{i=1}^{U_j} X_i - \sum_{i=1}^{D_j} X_i' = S_{U_j} - S'_{D_j}$. At each j, M_{j+1} is formed as follows: (a) If $M_j < 0$ or equivalently $S_{U_j} < S'_{D_j}$, then add X_{U_j+1}. (b) If $M_j > 0$ or equivalently $S_{U_j} > S'_{D_j}$, then subtract X'_{D_j+1}. (c) If $M_j = 0$ or equivalently $S_{U_j} = S_{D_j}$, then we choose, with probability $\frac{1}{2}$ each, to either add X_{U_j} or subtract X'_{D_j+1}. In any case we seek the next j such that $M_j = 0$. For at this sequence of return times to zero $\{N_k\}$, $M_{N_k} = S_{T_k} - S'_{T_k'}$ and hence $S_{T_k} = S'_{T_k'}$. So we record another value attained by both renewal processes.

With these definitions it is obvious that each of $\{N_k - N_{k-1}\}$, $\{T_k - T_{k-1}\}$, and $\{T'_k - T'_{k-1}\}$ is a sequence of independent identically distributed random variables. It is also clear from symmetry that $T_k - T_{k-1}$ and $T'_k - T'_{k-1}$ have the same distribution.

We can now begin the proof. We will show first that

$$\frac{\sum_{k=1}^{n} \sum_{j=T_{k-1}+1}^{T_k} f(S_j)^{*}}{\sum_{k=1}^{n} \sum_{j=T'_{k-1}+1}^{T'_k} f(S'_j)} \overset{a.s.}{\to} 1$$ as $n \to \infty$, and then that this convergence implies $\frac{\sum^n f(S_j)}{\sum^n f(S'_j)} \to 1$.

Let $\mathscr{F}_k$ be the σ-field generated by $S_1, S_2, \ldots, S_{T_k}$ and $S'_1, S'_2, \ldots, S'_{T'_k}$. Now apply Theorem 5b with $Y_k = \sum_{j=T_{k-1}+1}^{T_k} f(S_j)$. Since $f \geq \gamma > 0$, $E(Y_k | \mathscr{F}_{k-1}) \geq \gamma E(T_k - T_{k-1} | \mathscr{F}_{k-1}) = \gamma E(T_1) > 0$. Since $f \leq 1$, $Y_k \leq T_k - T_{k-1} \leq N_k - N_{k-1}$, and hence $P(Y_k \leq c | \mathscr{F}_{k-1}) \geq P(N_k - N_{k-1} \leq c | \mathscr{F}_{k-1}) = P(N_1 \leq c)$. In order to verify all the hypothesis of Theorem 5b, it remains only to show that $\int_0^\infty P(N_1 > c)dc = E(N_1) < \infty$.

It is easily verified that zero is a recurrent state of the Markov process, and that the Markov process $\{|M_j|\}$ has stationary initial probabilities π_j, where

$$\pi_0 = \frac{(1-p_0)}{2\mu} \; ; \; \pi_j = \frac{1}{\mu} \left(\sum_{k=j+1}^{\infty} p_k + \frac{p_j}{2} \right), \; j = 1,2,\ldots \; .$$

Therefore $E(N_1) = \dfrac{2\mu}{(1-p_0)} < \infty$.

---

* If $T_k = T_{k-1}$, then $\sum_{j=T_{k-1}+1}^{T_k} f(S_j)$ is defined to be 0.

So, by Theorem 5b, we have

$$\frac{\sum_{k=1}^{m} \sum_{j=T_{k-1}+1}^{T_k} f(S_j)}{\sum_{k=1}^{m} E(\sum_{j=T_{k-1}+1}^{T_k} f(S_j) | \mathscr{F}_{k-1})} \overset{a.s.}{\to} 1 \text{ as } m \to \infty.$$

Similarly

$$\frac{\sum_{k=1}^{m} \sum_{j=T'_{k-1}+1}^{T'_k} f(S'_j)}{\sum_{k=1}^{m} E(\sum_{j=T'_{k-1}+1}^{T'_k} f(S'_j) | \mathscr{F}_{k-1})} \overset{a.s.}{\to} 1 \text{ as } m \to \infty.$$

$$E(\sum_{j=T_{k-1}+1}^{T_k} f(S_j) | \mathscr{F}_{k-1}) = h(S_{T_{k-1}}), \text{ where}$$

$$h(s) = E(\sum_{j=T_{k-1}+1}^{T_k} f(S_j) | S_{T_{k-1}} = s) = E(\sum_{j=1}^{T_1} f(s+S_j)).$$

We get the same result for the terms of the other denominator.

$$E(\sum_{j=T'_{k-1}+1}^{T'_k} f(S'_j) | S'_{T'_{k-1}} = s) = E(\sum_{j=1}^{T'_1} f(s+S'_j))$$

$$= E(\sum_{j=1}^{T_1} f(s+S_j)) = h(s),$$

since $(S_1, S_2, \ldots, S_{T_1})$ and $(S'_1, S'_2, \ldots, S'_{T'_1})$ have the same distribution.

$$E[\sum_{j=T'_{k-1}+1}^{T'_k} f(S'_j) | \mathscr{F}_{k-1}] = h(S'_{T'_{k-1}}) = h(S_{T_{k-1}})$$

$$= E[\sum_{j=T_{k-1}+1}^{T_k} f(S_j) | \mathscr{F}_{k-1}],$$

since $S'_{T'_{k-1}} = S_{T_{k-1}}$.

Combining these results we have

$$\frac{\sum_{j=1}^{T_m} f(S_j)}{\sum_{j=1}^{T'_m} f(S_j)} = \frac{\sum_{k=1}^{m} \sum_{j=T_{k-1}+1}^{T_k} f(S_j)}{\sum_{k=1}^{m} \sum_{j=T'_{k-1}+1}^{T'_k} f(S'_j)} \xrightarrow{a.s.} 1 \quad \text{as } m \to \infty.$$

For any n there exists an m so that $T'_m \leq n < T'_{m+1}$.

$$(2.3) \quad \frac{\sum^{n} f(S_j)}{\sum^{n} f(S'_j)} \leq \frac{\sum_{j=1}^{T_m} f(S_j) + \sum_{j=T_m}^{T'_m} f(S_j) + \sum_{j=T'_m}^{n} f(S_j)}{\sum_{j=1}^{T'_m} f(S'_j)}$$

$$\leq \frac{\sum^{T_m} f(S_j)}{\sum^{T'_m} f(S'_j)} + \frac{|T_m - T'_m|}{\gamma T'_m} + \frac{T'_{m+1} - T'_m}{\gamma T'_m}$$

$$\frac{|T_m - T'_m|}{T'_m} = \left| \frac{\frac{T_m}{S_{T_m}}}{\frac{T'_m}{S'_{T'_m}}} - 1 \right| \xrightarrow{a.s.} 0 \quad \text{by the Strong Law of Large Numbers.}$$

$$\frac{T'_{m+1} - T'_m}{T'_m} = \frac{\frac{T'_{m+1}}{m} - \frac{T'_m}{m}}{\frac{T'_m}{m}} \xrightarrow{a.s.} 0 \quad \text{by the Strong Law of Large Numbers.}$$

Letting n and hence $m \to \infty$ in (2.3), $\overline{\lim_{n\to\infty}} \frac{\sum^{n} f(S_j)}{\sum^{n} f(S'_j)} \leq 1.$

$$(2.4) \quad \frac{\sum^{n} f(S_j)}{\sum^{n} f(S_j')} \geq \frac{\sum_{j=1}^{T_{m+1}} f(S_j) + \sum_{j=T_{m+1}}^{T_m'} f(S_j) - \sum_{j=n}^{T_{m+1}'} f(S_j)}{\sum_{j=1}^{T_{m+1}'} f(S_j')}$$

$$\geq \frac{\sum^{T_{m+1}} f(S_j)}{\sum^{T_{m+1}'} f(S_j')} - \frac{|T_{m+1}'-T_{m+1}|}{T_{m+1}'} - \frac{T_{m+1}-T_m'}{T_{m+1}'}$$

Letting n and $m \to \infty$ in (2.4), $\lim\limits_{n\to\infty} \frac{\sum^{n} f(S_j)}{\sum^{n} f(S_j')} \geq 1$. Therefore

$\frac{\sum^{n} f(S_j)}{\sum^{n} f(S_j)} \overset{a.s.}{\to} 1$. This completes the proof of the existence of a rate of divergence in the arithmetic case.

The Non-Singular Case

Now, following the outline of the proof for the arithmetic case, we will show that there is a rate of divergence in the non-sinular case. The Markov process $\{M_j\}$ and U_j and D_j are defined exactly as before. However, since the event $\{M_j=0\}$ may have probability zero for all j, new stopping variables are defined. For any $\varepsilon>0$, let $N_k(\varepsilon)$ = the kth j such that $M_j \in [-\varepsilon,\varepsilon]$, and $T_k(\varepsilon) = U_{N_k(\varepsilon)}$ and $T_k'(\varepsilon) = D_{N_k(\varepsilon)}$. $\{M_j\}$ and $\{N_k(\varepsilon)\}$ play the same roles as in the arithmetic case. However, it is no longer true that each of $\{N_k(\varepsilon)-N_{k-1}(\varepsilon)\}$, $\{T_k(\varepsilon)-T_{k-1}(\varepsilon)\}$, and $\{T_k'(\varepsilon)-T_{k-1}'(\varepsilon)\}$ is a sequence of independent identically distributed random variables. Also, we now have $S_{T_k}-S'_{T_k'} = M_{N_k} \in [-\varepsilon,\varepsilon]$, rather than $S_{T_k} = S'_{T'_k}$. Even with these differences, the situation is similar enough to the arithmetic case that the same idea provides a proof.

Again let $\mathscr{F}_k$ be the sigma-field generated by $S_1, S_2, \ldots, S_{T_k}$ and $S'_1, S'_2, \ldots, S'_{T'_k}$. In this case we will apply Theorem 5b with

$Y_k = \sum_{j=T_{k-1}(\varepsilon)+1}^{T_k(\varepsilon)} f(S_j)$. Recalling that we may assume $0 < \gamma \le f \le 1$,

$Y_k \ge (T_k(\varepsilon) - T_{k-1}(\varepsilon))\gamma$, and hence

$$E(Y_k|\mathscr{F}_{k-r}) = E(E(Y_k|\mathscr{F}_{k-1})|\mathscr{F}_{k-r})$$

$$\ge \gamma E(E(T_k(\varepsilon) - T_{k-1}(\varepsilon)\ M_{N_{k-1}}(\varepsilon))|\mathscr{F}_{k-r}).$$

Note that $T_k(\varepsilon) - T_{k-1}(\varepsilon)$ is the number of upward jumps of the process $\{M_j\}$ between the times of the k-1st and the kth entries of the process into $[-\varepsilon, \varepsilon]$. It is clear from this definition, that for ε sufficiently small there exists $\psi(\varepsilon) > 0$ such that $E(T_k(\varepsilon) - T_{k-1}(\varepsilon)|\ M_{N_{k-1}}(\varepsilon)) \ge \psi(\varepsilon)$ for all k. Therefore

(2.5) $\quad E(Y_k|\mathscr{F}_{k-r}) \ge \gamma\psi(\varepsilon) > 0$ for all k and r.

$Y_k \le T_k(\varepsilon) - T_{k-1}(\varepsilon) \le N_k(\varepsilon) - N_{k-1}(\varepsilon)$ implies

$P(Y_k \le c|\mathscr{F}_{k-1}) \ge P(N_k(\varepsilon) - N_{k-1}(\varepsilon) \le c|M_{N_{k-1}})$. Let $Q(\varepsilon)$ = the first $j \ge 1$ such that $M_j \in [0, \varepsilon]$.

$$P(N_k(\varepsilon) - N_{k-1}(\varepsilon) \le c|M_{N_{k-1}} = x) = P(N_1(\varepsilon) \le c|M_0 = x)$$

$$\ge P(Q(\varepsilon) \le c|M_0 = \varepsilon) \text{ for } x \in [0, \varepsilon].$$

The last inequality follows from the reasoning below.

Consider two sample paths of the Markov process $\{M_j\}$ which are identical except that one starts at ε and the other starts at some $x \in [0,\varepsilon]$. Refer to these paths as the ε-path and the x-path, respectively. The two paths will remain in phase, that is, the ε-path will remain ε-x units above the x-path, except in the event that the ε-path enters $[0,\varepsilon]$ and the x-path enters $[-\varepsilon,0)$. For in this case the upper path would be headed down and the lower path would be headed upward. As long as the two paths do remain in phase, if the ε-path enters $[0,\varepsilon]$ then the x-path must enter $[-\varepsilon,\varepsilon]$ at the same time. So in any case the x-path enters $[-\varepsilon,\varepsilon]$ no later than the ε-path enters $[0,\varepsilon]$, or equivalently, $N_1(\varepsilon)$ evaluated on any x-path is less than or equal to $Q(\varepsilon)$ evaluated on the corresponding ε-path. Therefore $P(N_1(\varepsilon) \leq c|M_0=x) \geq P(Q(\varepsilon) \leq c|M_0=\varepsilon)$ for $x \in [0,\varepsilon]$.

Similarly for $x \in [-\varepsilon,0]$,

$P(N_1(\varepsilon) \leq c|M_0=x) \geq P(R(\varepsilon) \leq c|M_0 = -\varepsilon)$,

where $R(\varepsilon)$=the first $j \geq 1$ such that $M_j \in [-\varepsilon,0]$. By the symmetry about zero of the Markov process, $P(R(\varepsilon) \leq c|M_0=-\varepsilon)=P(Q(\varepsilon) \leq c|M_0=\varepsilon)$.

Therefore

$$(2.6) \qquad P(Y_k \leq c|\mathscr{F}_{k-1}) \geq P(Q(\varepsilon) \leq c|M_0=\varepsilon).$$

In order to complete the verification of the hypotheses of Theorem 5b, we need only show that $\int_0^\infty (1-G(c))dc < \infty$, where $G(c) = P(Q(\varepsilon) \leq c|M_0=\varepsilon)$, or equivalently, that $E(Q(\varepsilon)|M_0=\varepsilon) < \infty$.

If F is non-singular, $\{M_j\}$ has a stationary distribution π with density

$$\pi'(x) = \begin{cases} \frac{1-F(x)}{2\mu}, & y \geq 0 \\ \frac{1-F(-x)}{2\mu}, & y < 0 \end{cases} \quad \text{(See [6], p. 11.)}$$

If π is used as an initial distribution for the Markov process (i.e. $P(M_0 \leq c) = \pi(c)$), then

$$\frac{1}{\pi[0,\varepsilon]} = E(Q(\varepsilon)|M_0 \in [0,\varepsilon]) \quad \text{(See [1], p.123.)}$$

$$\geq E(Q(\varepsilon)I_{\{Q(\varepsilon)>2\}}|M_0 \in [0,\varepsilon])$$

$$= E(E(Q(\varepsilon)I_{\{Q(\varepsilon)>2\}}|M_1,M_2)|M_0 \in [0,\varepsilon]).$$

Let $\quad H(x,y) = P(M_1 \leq x,\ M_2 \leq y|M_0 \in [0,\varepsilon]).$

$$\frac{1}{\pi[0,\varepsilon]} \geq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(Q(\varepsilon)I_{\{Q(\varepsilon)>2\}}|M_1=x,M_2=y)\,dH(x,y)$$

$$= \int\int_{x,y \in [0,\varepsilon]^c} E(Q(\varepsilon)|M_1=x,M_2=y)\,dH(x,y)$$

$$= \int\int_{x,y \in [0,\varepsilon]^c} (2+E(Q(\varepsilon)|M_0=y))\,dH(x,y).$$

For ε sufficiently small $H(x,y)$ has a density which is positive if x is in a certain sub-interval of $(-\infty,0]$ (in particular, those values which can be reached by the process in one downward jump from any point in $[0,\varepsilon]$) and y is in a certain interval of the form $[\varepsilon,\varepsilon+a]$ (in particular, those values which can be reached from $[0,\varepsilon]$ by one downward jump followed by an upward jump). Therefore from the integrand of the last integral, we have $E(Q(\varepsilon)|M_0=y) < \infty$ for almost every $y \in [\varepsilon,\varepsilon+a]$. $y \geq \varepsilon$ implies $Q(y) \leq Q(\varepsilon)$. So $E(Q(y)|M_0=y) \leq E(Q(\varepsilon)|M_0=y) < \infty$, and hence $E(Q(y)|M_0=y) < \infty$ for almost every $y \geq 0$ in a neighborhood of zero. This is sufficient for our purposes; that is, the hypotheses of Theorem 5b are now verified for

$$Y_k = \sum_{j=T_{k-1}(\varepsilon)+1}^{T_k(\varepsilon)} f(S_j) \text{ and almost every } \varepsilon>0 \text{ in a neighborhood of zero.}$$

So, by Theorem 5b, we have for any r (the ε's are omitted to simplify the notation)

$$(2.7) \qquad \frac{\sum_{k=1}^{m} \sum_{j=T_{k-1}+1}^{T_k} f(S_j)}{\sum_{k=1}^{m} E\left(\sum_{j=T_{k-1}+1}^{T_k} f(S_j) \mid \mathscr{F}_{k-r}\right)} \overset{a.s.}{\to} 1 \quad \text{as } m \to \infty .$$

By symmetry, for any r

$$(2.8) \qquad \frac{\sum_{k=1}^{m} \sum_{j=T'_{k-1}+1}^{T'_k} f(S'_j)}{\sum_{k=1}^{m} E\left(\sum_{j=T'_{k-1}+1}^{T'_k} f(S'_j) \mid \mathscr{F}_{k-r}\right)} \overset{a.s.}{\to} 1 \quad \text{as } m \to \infty .$$

We will now show that the ratio of the denominators of (2.7) and (2.8) can be made arbitrarily close to one by taking r large. First set

$$G_r^{s-s'}(c) = P(M_{N_{r-1}} \le c \mid M_0 = s-s') \qquad \text{and}$$

$$(2.9) \quad h(r,s,c) = E\left(\sum_{j=T_{r-1}+1}^{T_r} f\left(s+S_{T_{r-1}} + \sum_{i=T_{r-1}+1}^{j} X_i\right) \Big| M_{N_{r-1}} = c\right).$$

$$E\left(\sum_{j=T_{k-1}+1}^{T_k} f(S_j) \mid \mathscr{F}_{k-r}\right) = g_r\left(S_{T_{k-r}}, S'_{T'_{k-r}}\right),$$

where

$$g_r(s,s') = E(\sum_{j=T_{k-1}+1}^{T_k} f(S_j) \mid S_{T_{k-r}}=s,\ S'_{T'_{k-r}}=s')$$

$$= E(\sum_{j=T_{r-1}+1}^{T_r} f(s+S_j) \mid M_0=s-s')$$

$$= E(E(\sum_{j=T_{r-1}+1}^{T_r} f(s+S_j) \mid M_{N_{r-1}}) \mid M_0=s-s')$$

$$= \int_{-\varepsilon}^{\varepsilon} h(r,s,c)\,dG_r^{s-s'}(c) \ .$$

Now consider the denominator of (2.8).

$$E(\sum_{j=T'_{k-1}+1}^{T'_k} f(S'_j) \mid S_{T_{k-r}}=s,\ S'_{T'_{k-r}}=s')$$

$$= E(\sum_{j=T'_{r-1}+1}^{T'_r} f(s'+S'_j) \mid M_0=s-s')$$

$$= \int_{-\varepsilon}^{\varepsilon} E(\sum_{j=T'_{r-1}+1}^{T'_r} f(s'+S'_{T'_{r-1}} + \sum_{i=T'_{r-1}+1}^{j} X'_i) \mid M_{N_{r-1}}=c)\,dG_r^{s-s'}(c)$$

$$= \int_{-\varepsilon}^{\varepsilon} E(\sum_{j=T_{r-1}+1}^{T_r} f(s'+S'_{T'_{r-1}} + \sum_{i=T_{r-1}+1}^{j} X_i) \mid M_{N_{r-1}} = -c)\,dG_r^{s-s'}(c).$$

The last equality follows from the symmetry about zero of the Markov process and the symmetry in the roles of $\{X_i\}$ and $\{X'_i\}$.

The last expression will be shown to be approximately $\int_{-\varepsilon}^{\varepsilon} h(r,s,-c)\,dG^{s-s'}(c)$. Since f can be assumed to be bounded away from zero and uniformly continuous on $[0,\infty)$, and

$|(s'+S'_{T'_{r-1}})-(s+S_{T_{r-1}})| \leq 2\varepsilon$, we have for any $\Delta > 0$ and ε sufficiently small

$$1 - \Delta \leq \frac{f(s+S_{T_{r-1}} + \sum_{i=T_{r-1}+1}^{j} X_i)}{f(s'+S'_{T'_{r-1}} + \sum_{i=T_{r-1}+1}^{j} X_i)} \leq 1 + \Delta$$

$$1 - \Delta \leq \frac{\int_{-\varepsilon}^{\varepsilon} E(\sum_{j=T_{r-1}+1}^{T_r} f(s+S_{T_{r-1}} + \sum_{i=T_{r-1}+1}^{j} X_i) | M_{N_{r-1}} = -c) dG_r^{s-s'}(c)}{\int_{-\varepsilon}^{\varepsilon} E(\sum_{j=T_{r-1}+1}^{T_r} f(s'+S'_{T'_{r-1}} + \sum_{i=T_{r-1}+1}^{j} X_i) | M_{N_{r-1}} = -c) dG_r^{s-s'}(c)} \leq 1+\Delta$$

$$1 - \Delta \leq \frac{q_r(s,s')}{e_r(s,s')} \leq 1 + \Delta ,$$

where

$$q_r(s,s') = \int_{-\varepsilon}^{\varepsilon} E(\sum_{j=T_{r-1}+1}^{T_r} f(s+S_{T_{r-1}} + \sum_{i=T_{r-1}+1}^{j} X_i) | M_{N_{r-1}} = -c) dG_r^{s-s'}(c)$$

$$= \int_{-\varepsilon}^{\varepsilon} h(r,s,-c) dG_r^{s-s'}(c) \quad ,$$

and

$$e_r(s,s') = \int_{-\varepsilon}^{\varepsilon} E(\sum_{j=T_{r-1}+1}^{T_r} f(s'+S'_{T'_{r-1}} + \sum_{i=T_{r-1}+1}^{j} X_i) | M_{N_{r-1}} = -c) dG_r^{s-s'}(c).$$

$$1 - \Delta \leq \frac{\sum_{k=1}^{m} q_r(S_{T_{k-r}}, S'_{T'_{k-r}})}{\sum_{k=1}^{m} e_r(S_{T_{k-r}}, S'_{T'_{k-r}})} \leq 1 + \Delta \quad \text{for } \varepsilon$$

sufficiently small.

For large r, $G_r^{s-s'}$ is approximately a symmetric distribution function, so that $q_r(s,s') = \int_{-\varepsilon}^{\varepsilon} h(r,s,-c)dG_r^{s-s'}(c)$ is approximately equal to $g_r(s,s') = \int_{-\varepsilon}^{\varepsilon} h(r,s,c)dG_r^{s-s'}(c)$ for large r. This fact and (2.10) combine to give the desired estimate of the ratio of the denominators of (2.7) and (2.8). The following lemma makes these approximations rigorous.

Lemma 4. There exists a stationary distribution Π for the Markov process $\{M_{N_r}\}_{r=0}^{\infty}$ such that there exists a $\beta > 0$ and an $\eta < 1$ such that

$$(2.11) \qquad |P(M_{N_{r-1}} \in A | M_0 = s-s') - \Pi(A)| \leq \beta\eta^r$$

for any Borel set A. Furthermore Π is a symmetric distribution.

Proof: By a Theorem of Doob ([3], Case (b), p. 197) if a Markov process has stationary transition probabilities $P(A|x)$, whose absolutely continuous component has a density $p(y|x)$ such that $p(y|x) \geq \alpha > 0$ for all x in the state space and y in some interval, then there exists a Π and an $\eta < 1$ such that $|P^{(r)}(A|x) - \Pi(A)| \leq \beta\eta^r$ for all A and all x in the state space. If we apply this theorem to the Markov process $\{M_{N_r}\}_{r=0}^{\infty}$ we obtain (2.11). The theorem applies to this process if $\frac{dG_1^x(c)}{dc}$ is bounded away from zero for $x, c \in [-\varepsilon, \varepsilon]$.

Recall that $F'(x) \geq \alpha > 0$ for $x \in [a, a+h]$. Without loss of generality we may assume that $a > 0$. For $x < 0$, $c \in [-\varepsilon, \varepsilon]$, and $\varepsilon < \min\{\frac{a}{2}, \frac{h}{9}\}$

$$\frac{1}{\Delta} P(M_{N_1} \in [c,c+\Delta] | M_0=x) \geq \frac{1}{\Delta} P(N_1=2, M_2 \in [c,c+\Delta] | M_1=x)$$

$$= \frac{1}{\Delta} P(x+X_1 > \varepsilon, x+X_1-X_1' \in [c,c+\Delta])$$

$$= \frac{1}{\Delta} \int_{\varepsilon-x}^{\infty} [F(y+x-c)-F(y+x-c-\Delta)]dF(y)$$

$$\geq \frac{1}{\Delta} \int_{\varepsilon-x}^{\infty} \int_{y+x-c-\Delta}^{y+x-c} F'(w)dwdF(y).$$

$[y+x-c-\Delta, y+x-c] \subset [a,a+h]$ if and only if $a-x+c+\Delta \leq y \leq a+h-x+c$. $a > 2\varepsilon$ and $c > -\varepsilon$ implies $\varepsilon-x \leq a-x+c+\Delta$. Therefore

$$\frac{1}{\Delta} \int_{\varepsilon-x}^{\infty} \int_{y+x-c-\Delta}^{y+x-c} F'(w)dwdF(y) \geq \frac{1}{\Delta} \int_{a-x+c+\Delta}^{a+h-x+c} \alpha\Delta dF(y)$$

$$\geq \alpha\int_{a-x+c+\Delta}^{a+h-x+c} F'(y)dy .$$

$a+h-x+c \geq a+h-\varepsilon \geq a + \frac{8h}{9}$ and

$a-x+c+\Delta \leq a+3\varepsilon \leq a + \frac{h}{3}$.

$$\alpha\int_{a-x+c+\Delta}^{a+h-x+c} F'(y)dy \geq \alpha\int_{a+\frac{h}{3}}^{a+\frac{8h}{9}} F'(y)dy \geq \alpha^2\left(\frac{5h}{9}\right)$$

$$\frac{1}{\Delta} P(M_{N_1} \in [c,c+\Delta] | M_0=x) \geq \alpha^2\left(\frac{5h}{9}\right) \quad \text{for } x < 0,\ c \in [-\varepsilon,\varepsilon].$$

Letting $\Delta \to 0$

$$\frac{dG_1^x(c)}{dc} \geq \alpha^2\left(\frac{5h}{9}\right) > 0 \quad \text{for } x < 0,\ c \in [-\varepsilon,\varepsilon].$$

By a similar argument it follows that $\frac{dG_1^x(c)}{dc}$ is bounded away from zero for $x > 0$ and $c \in [-\varepsilon,\varepsilon]$. This proves the first part of the lemma.

$$P(M_{N_r} \leq c \mid M_0 = s-s') = P(-M_{N_r} \leq c \mid M_0 = s'-s)$$

$$= P(M_{N_r} \geq -c \mid M_0 - s'-s).$$

Letting $r \to \infty$, we have $\Pi[-\varepsilon,c] = \Pi[-c,\varepsilon]$. So Π is symmetric.

Now we can show that

$$|g_r(s,s') - q_r(s,s')|$$

$$= \left|\int_{-\varepsilon}^{\varepsilon} h(r,s,c)\,dG_r^{s-s'}(c) - \int_{-\varepsilon}^{\varepsilon} h(r,s,-c)\,dG_r^{s-s'}(c)\right|$$

is small if r is large.

$$h(r,s,c) \leq E(T_r - T_{r-1} \mid M_{N_{r-1}} = c) \leq E[Q(\varepsilon) \mid M_0 = \varepsilon] < \infty.$$

Let B be an upper bound for h and let

$$Q_n(r,s,c) = \sum_{k=0}^{n-1} \frac{Bk}{n} I_{\{c:\ \frac{Bk}{n} \leq h(r,s,c) < \frac{B(k+1)}{n}\}}$$

$$= \left|\int_{-\varepsilon}^{\varepsilon} h(r,s,c)\,dG_r^{s-s'}(c) - \int_{-\varepsilon}^{\varepsilon} Q_n(r,s,c)\,dG_r^{s-s'}(c)\right| \leq \int_{-\varepsilon}^{\varepsilon} \frac{B}{n}\,dG_r^{s-s'}(c) = \frac{B}{n}.$$

$$\left|\int_{-\varepsilon}^{\varepsilon} Q_n(r,s,c)\,dG_r^{s-s'}(c) - \int_{-\varepsilon}^{\varepsilon} Q_n(r,s,c)\,d\Pi(c)\right|$$

$$= \sum_{k=0}^{n-1} \frac{Bk}{n} |G_r^{s-s'} - \Pi|(\{c:\ \frac{Bk}{n} \leq h(r,s,c) \leq \frac{B(k+1)}{n}\})$$

$$\leq \sum_{k=0}^{n-1} \frac{Bk}{n} \beta\eta^r = \frac{B}{n} \frac{(n-1)n}{2} \beta\eta^r = \frac{B\gamma}{2} (n-1)\eta^r .$$

$$\left|\int_{-\varepsilon}^{\varepsilon} Q_n(r,s,c)\,d\Pi(c) - \int_{-\varepsilon}^{\varepsilon} h(r,s,c)\,d\Pi(c)\right| \leq \int_{-\varepsilon}^{\varepsilon} \frac{B}{n}\,d\Pi(c) = \frac{B}{n}.$$

Therefore

$$\left|\int_{-\varepsilon}^{\varepsilon} h(r,s,c)\,dG_r^{s-s'}(c) - \int_{-\varepsilon}^{\varepsilon} h(r,s,c)\,d\Pi(c)\right| \leq \frac{2B}{n} + \frac{B\beta}{2} (n-1)\eta^r .$$

First take n large and then r large. We have

$$\left|\int_{-\varepsilon}^{\varepsilon} h(r,s,c)\,dG_r^{s-s'}(c) - \int_{-\varepsilon}^{\varepsilon} h(r,s,c)\,d\Pi(c)\right| \leq \frac{\varepsilon'}{2}$$

for r sufficiently large.

By the same argument we have

$$\left|\int_{-\varepsilon}^{\varepsilon} h(r,s,-c)\,dG_r^{s-s'}(c) - \int_{-\varepsilon}^{\varepsilon} h(r,s,-c)\,d\Pi(c)\right| \leq \frac{\varepsilon'}{2}$$

for r sufficiently large.

Therefore for r large

$$|g_r(s,s')-q_r(s,s')| = \left|\int_{-\varepsilon}^{\varepsilon} h(r,s,c)\,dG_r^{s-s'}(c) - \int_{-\varepsilon}^{\varepsilon} h(r,s,-c)\,dG_r^{s-s'}(c)\right|$$

$$\leq \varepsilon' + \left|\int_{-\varepsilon}^{\varepsilon} h(r,s,c)\,d\Pi(c) - \int_{-\varepsilon}^{\varepsilon} h(r,s,-c)\,d\Pi(c)\right| = \varepsilon',$$

since Π is symmetric about zero.

$h(r,s,c) \geq E(\gamma(T_r-T_{r-1})/M_{N_{r-1}}=c) \geq \gamma\psi(\varepsilon) > 0$, by (2.5).

$$\left|\frac{g_r(s,s')}{q_r(s,s')} - 1\right| = \left|\frac{\int_{-\varepsilon}^{\varepsilon} h(r,s,c)\,dG_r^{s-s'}(c)}{\int_{-\varepsilon}^{\varepsilon} h(r,s,-c)\,dG_r^{s-s'}(c)} - 1\right| \leq \frac{\varepsilon'}{\gamma\psi(\varepsilon)} = \Delta$$

for the appropriate ε'. Therefore

$$(2.11) \quad 1-\Delta \leq \frac{\sum_{r-1}^{m} g_r(S_{T_{r-k}}, S'_{T'_{r-k}})}{\sum_{k=1}^{m} q_r(S_{T_{r-k}}, S'_{T'_{r-k}})} \leq 1+\Delta\ .$$

From (2.10) and (2.11), for any $\Delta > 0$ there exists ε small and r large so that for all m

$$(2.12)\quad \frac{\sum_{k=1}^{m} E\left(\sum_{j=T_{k-1}+1}^{T_k} f(S_j)\,|\,\mathscr{F}_{k-r}\right)}{\sum_{k=1}^{m} E\left(\sum_{j=T'_{k-1}+1}^{T'_k} f(S'_j)\,|\,\mathscr{F}_{k-r}\right)} = \frac{\sum_{k=1}^{m} g_r(S_{T_{k-r}}, S'_{T'_{k-r}})}{\sum_{k=1}^{m} e_r(S_{T_{k-r}}, S'_{T'_{k-r}})}$$

$$= \frac{\sum_{k=1}^{m} g_r(S_{T_{k-r}}, S'_{T'_{k-r}})}{\sum_{k=1}^{m} q_r(S_{T_{k-r}}, S'_{T'_{k-r}})} \cdot \frac{\sum_{k=1}^{m} q_r(S_{T_{k-r}}, S'_{T'_{k-r}})}{\sum_{k=1}^{m} e_r(S_{T_{k-r}}, S'_{T'_{k-r}})}$$

$$\in [(1-\Delta)^2,\ (1+\Delta)^2].$$

$$\frac{\sum_{j=1}^{T_m} f(S_j)}{\sum_{j=1}^{T'_m} f(S'_j)} = \frac{\sum_{k=1}^{m} \sum_{j=T_{k-1}+1}^{T_k} f(S_j)}{\sum_{k=1}^{m} E\left(\sum_{j=T_{k-1}+1}^{T_k} f(S_j)\,|\,\mathscr{F}_{k-r}\right)} \cdot \frac{\sum_{k=1}^{m} E\left(\sum_{j=T'_{k-1}+1}^{T'_k} f(S'_j)\,|\,\mathscr{F}_{k-r}\right)}{\sum_{k=1}^{m} \sum_{j=T'_{k-1}+1}^{T'_k} f(S'_j)} \cdot$$

$$\cdot \frac{\sum_{k=1}^{m} E\left(\sum_{j=T_{k-1}+1}^{T_k} f(S_j)\,|\,\mathscr{F}_{k-r}\right)}{\sum_{k=1}^{m} E\left(\sum_{j=T'_{k-1}+1}^{T'_k} f(S'_j)\,|\,\mathscr{F}_{k-r}\right)} .$$

By (2.7), (2.8), and (2.12)

$$1 - \Delta \le \varliminf_{m\to\infty} \frac{\sum_{j=1}^{T_m} f(S_j)}{\sum_{j=1}^{T'_m} f(S'_j)} \le \varlimsup_{n\to\infty} \frac{\sum_{j=1}^{T_m} f(S_j)}{\sum_{j=1}^{T'_m} f(S'_j)} \le 1 + \Delta.$$

From inequalities (2.3) and (2.4), in order to prove

$\frac{\sum^{n} f(S_j)}{\sum^{n} f(S'_j)} \to 1$, we need only show that $\frac{T_m}{T'_m} \overset{a.s.}{\to} 1$ and $\frac{T'_{m+1}}{T'_m} \overset{a.s.}{\to} 1$.

Without the arithmetic case assumption we no longer have $S_{T_k} = S'_{T'_k}$, and $\{T_k - T_{k-1}\}$ and $\{T'_k - T'_{k-1}\}$ are no longer independent sequences.

$$\frac{T_m}{T'_m} = \frac{\dfrac{S_{T_m}}{S'_{T'_m}} \cdot \dfrac{S'_{T'_m}}{T'_m}}{\dfrac{S_{T_m}}{T_m}} \quad \text{and} \quad \left|\frac{S_{T_m} - S'_{T'_m}}{S'_{T'_m}}\right| \le \frac{\varepsilon}{S'_{T'_m}} \to 0$$

Therefore by the strong law of large numbers $\frac{T_m}{T'_m} \to 1$.

$$\frac{T'_{m+1}}{T'_m} = \frac{\sum^{m+1} (T_k - T_{k-1})}{\sum^{m} (T_k - T_{k-1})} .$$

We have already shown indirectly (see page 31) that the hypotheses of Theorem 5b are satisfied for $Y_k = T_k - T_{k-1}$. Therefore

$$\frac{\sum_{k=1}^{m} (T_k - T_{k-1})}{\sum_{k=1}^{m} E(T_k - T_{k-1} | M_{N_{k-1}})} \overset{a.s.}{\to} 1 \text{ as } m \to \infty .$$

$$\gamma\psi(\varepsilon) \le E(T_k - T_{k-1} | M_{N_{k-1}}) \le E(Q(\varepsilon) | M_0 = \varepsilon) \quad \text{implies}$$

$$\frac{\sum_{k=1}^{m+1} E(T_k - T_{k-1} | M_{N_{k-1}})}{\sum_{k=1}^{m} E(T_k - T_{k-1} | M_{N_{k-1}})} \overset{a.s.}{\to} 1 .$$

Therefore $\frac{T'_{m+1}}{T'_m} \to 1$.

So we have shown that $\frac{\sum^n f(S_j)}{\sum^n f(S'_j)} \overset{a.s.}{\to} 1$ and hence that there does exist a rate of divergence for $\sum_{j=1}^{n} f(S_j)$.

The Rate of Divergence

We have now demonstrated the existence of a sequence $\{c_n\}$ such that $\frac{1}{c_n} \sum^{n} f(S_j) \overset{a.s.}{\to} 1$. Furthermore it is obvious from their construction that $\gamma n \leq c_n \leq n$ under the assumption $\gamma \leq f \leq 1$.

$$E\left(\frac{\sum^{n} f(S_k)}{c_n} I_{\{\frac{1}{c_n}\sum^{n} f(S_k) > a\}}\right) = 0 \qquad \text{for } a > \frac{1}{\gamma},$$

since in this case

$$\{\frac{1}{c_n} \sum^{n} f(S_k) > a\} \subset \{\sum^{n} f(S_k) > n\} = \phi.$$

Therefore $\frac{1}{c_n} \sum_{k=1}^{n} f(S_k)$ is a uniformly integrable sequence, and hence

$$\frac{1}{c_n} \sum^{n} E(f(S_k)) \to 1 \qquad \text{as } n \to \infty.$$

So the sequence $\sum^{n} E(f(S_k))$ serves as a rate of divergence. We will now show that $\dfrac{\sum^{n} E(f(S_k))}{\frac{1}{\mu}\int_0^{\mu n} f(s)ds} \to 1$ in the non-singular case, so that the sequence $\frac{1}{\mu}\int_0^{\mu n} f(s)ds$ also serves as a rate of divergence. The same argument will show that $\frac{1}{\mu}\sum_{k=1}^{[\mu n]} f(\lambda k)$ serves as a rate of divergence in the arithmetic case.

First, let $U(t) = \sum_{k=1}^{\infty} F^{*k}(t)$ and $U_n(t) = \sum_{k=1}^{n} F^{*k}(t)$.

$$\frac{1}{\mu}\int_0^{\mu n} f(t)dt - \sum_{k=1}^{n} E(f(S_k)) = \frac{1}{\mu}\int_0^{\mu n} f(t)dt - \int_0^{\infty} f(t)dU_n(t)$$

$$(2.13)\qquad = (\frac{1}{\mu}\int_0^{\mu n} f(t)dt - \int_0^{\mu n} f(t)dU(t))$$

$$+ (\int_0^{\mu n} f(t)dU(t) - \int_0^{\mu n} f(t)dU_n(t)) - \int_{\mu n}^{\infty} f(t)dU_n(t).$$

We will now show that each of the three terms of the last expression is $o(n)$.

By the Renewal Theorem, for any h and $\varepsilon > 0$ there exists a $J(\varepsilon)$ such that for $t > J(\varepsilon)$, $|U(t+h,t) - \frac{h}{\mu}| < \varepsilon$. Since we also have $0 \leq f \leq 1$,

$$(2.14)\qquad |\frac{1}{\mu}\int_0^{\mu n} f(t)dt - \int_0^{\mu n} f(t)dU(t)| \leq J(\varepsilon)+\mu n\varepsilon \quad \text{for } n > J(\varepsilon).$$

$$\int_0^{\mu n} f(t)dU(t) - \int_0^{\mu n} f(t)dU_n(t) = \sum_{k=n+1}^{\infty} \int_0^{\mu n} f(t)dF^{*k}(t)$$

$$\leq \sum_{k=n+1}^{\infty} P(S_k \leq \mu n) = E(\sum_{k=n+1}^{\infty} P(S_k \leq \mu n | S_n))$$

$$= E(\sum_{j=1}^{\infty} F^{*j}(\mu n - S_n)) = E(U(\mu n - S_n)) .$$

By the Renewal Theorem, there exist constants c_1 and c_2 such that $U(t) \leq c_1+c_2t$ for $t \geq 0$. Therefore

$$(2.15)\qquad |\int_0^{\mu n} f(t)dU(t) - \int_0^{\mu n} f(t)dU_n(t)| \leq c_1+c_2E(|S_n-\mu n|).$$

$$\int_{\mu n}^{\infty} f(t)dU_n(t) \leq \sum_{k=1}^{n} P(S_k \geq \mu n)$$

$$\leq 1 + E(\sum_{k=1}^{n-1} P(S_k \geq \mu n | S_n)).$$

For $k=1,2,\ldots,n-1$

$$P(S_k \geq \mu n | S_n=s) = P(X_{k+1}+\ldots+X_n \leq s-\mu n) = F^{*n-k}(s-\mu n) .$$

So

$$\int_{\mu n}^{\infty} f(t)\,dU_n(t) \le 1 + E\Big(\sum_{k=1}^{n-1} F^{*n-k}(S_n-\mu n)\Big) \tag{2.16}$$

$$\le 1 + E(U(S_n-\mu n)) \le 1+c_1+c_2\,E(|S_n-\mu n|).$$

Combining (2.13) through (2.16)

$$\frac{\left|\frac{1}{\mu}\int_0^{\mu n} f(t)\,dt - \sum_{k=1}^{n} E(f(S_k))\right|}{\sum_{k=1}^{n} E(f(S_k))} \le \frac{1}{\gamma n}(J(\varepsilon)+\mu n\varepsilon+1+2c_1+2c_2E(|S_n-\mu n|) \to \mu\varepsilon \text{ as } n\to\infty.$$

Since ε is arbitrary $\dfrac{\sum_{k=1}^{n} E(f(S_k))}{\frac{1}{\mu}\int_0^{\mu n} f(t)\,dt} \to 1$.

This concludes the proof of Theorem 4.

The Random Walk Case

With an additional assumption, Theorem 4 can be extended to the random walk case.

Theorem 6. Let $\{S_n\}$ be a random walk with step size distribution F and mean μ, $0 < \mu < \infty$. Let $\{L_k\}$ be the sequence of strictly ascending ladder epochs, and assume $E\big(\sum_{j=1}^{L_1-1} |S_j|\big) < \infty$. Let f be a non-negative, bounded, Lebesgue measurable function on $(-\infty,\infty)$.

(a) If F is a non-singular distribution, and there exist M and δ such that $\int_x^{x+M} f(t)\,dt \ge \delta > 0$ for all $x \ge 0$, then

$$\frac{\sum_{j=1}^{n} f(S_j)}{\frac{1}{\mu}\int_0^{\mu n} f(s)\,ds} \overset{a.s.}{\rightarrow} 1 \quad \text{as } n \rightarrow \infty.$$

(b) If F is an arithmetic distribution with mass p_k at λk, $k=0, \pm 1, \pm 2, \ldots$, and there exist m and δ such that

$\sum_{k=0}^{\infty} f(s+\lambda k)p_k^{*m} \geq \delta > 0$ for $s=0, \pm 1, \pm 2, \ldots$, then

$$\frac{\sum_{j=1}^{n} f(S_j)}{\frac{1}{\mu}\sum_{j=1}^{[\mu n]} f(\lambda j)} \overset{a.s.}{\rightarrow} 1 \quad \text{as } n \rightarrow \infty,$$

<u>Proof</u>: The proofs for the arithmetic and non-singular cases are the same. We will consider only the non-singular case.

As in the beginning of the proof of Theorem 4, we will show that f may be assumed to be bounded away from zero on $(0,\infty)$. By the same proof $\dfrac{\sum^{n} f(S_k)}{\sum^{n} g_m(S_k)} \overset{a.s.}{\rightarrow} 1$ as $n \rightarrow \infty$, where g_m is now defined by

$$g_m(s) = E(f(S_k)|S_{k-m}=s) = \int_{-\infty}^{\infty} f(s+t)dF^{*m}(s).$$

As before, there exists m and $\gamma > 0$ such that $g_m \geq \gamma$. We also have

$\dfrac{\int_0^{\mu n} f(s)ds}{\int_0^{\mu n} g_m(s)ds} \rightarrow 1$ as $n \rightarrow \infty$, since $\int_0^{\mu n} g(s)ds \geq \gamma n$ and

$$\left|\int_0^{\mu n} g_m(s)ds - \int_0^{\mu n} f(s)ds\right| = \left|E\left(\int_{S_m}^{S_m+\mu n} f(s)ds - \int_0^{\mu n} f(s)ds\right)\right|$$

$$(2.17)\quad \begin{aligned} &\leq \left|E\left(\int_{S_m}^{0} f(s)ds - \int_{\mu n+S_m}^{\mu n} f(s)ds\right)I_{\{S_m<0\}}\right| \\ &+ \left|E\left(\int_{\mu n}^{\mu n+S_m} f(s)ds - \int_0^{S_m} f(s)ds\right)I_{\{0\leq S_m\leq \mu n\}}\right| + \mu nP(S_m>\mu n) \end{aligned}$$

$$\leq E(S_m^-) + E(S_m^+) + \mu nP(S_m>\mu n).$$

Therefore $\dfrac{\sum^n g(S_k)}{\frac{1}{\mu}\int_0^{\mu n} g(s)ds} \to 1$ implies $\dfrac{\sum^n f(S_k)}{\frac{1}{\mu}\int_0^{\mu n} f(s)ds} \to 1$, and we may assume without loss of generality that $0 < \gamma \leq f \leq 1$.

Let $\{L_k\}$ be the sequence of strictly ascending ladder epochs, and let

$$h(s) = E\left(\sum_{j=L_k+1}^{L_{k+1}} f(S_j)\,\middle|\,S_{L_k}=s\right) = E\left(\sum_{j=1}^{L_1} f(s+S_j)\right).$$

$$0 < \gamma\, E(L_1) \leq h(S_{L_k}) \leq E(L_1) < \infty.$$

Therefore, by Theorem 5b,

$$(2.18)\quad \frac{\sum_{j=1}^{L_m} f(S_j)}{\sum_{k=0}^{m} h(S_{L_k})} = \frac{\sum_{k=0}^{m}\sum_{j=L_k+1}^{L_{k+1}} f(S_j)}{\sum_{k=0}^{m} E\left(\sum_{j=L_k+1}^{L_{k+1}} f(S_j)\,\middle|\,S_{L_k}\right)} \overset{a.s.}{\to} 1 \text{ as } m\to\infty.$$

Let $\theta = E(L_1)$. $\{S_{L_k}\}_{k=0}^{\infty}$ is a renewal process with expected waiting time $\theta\mu$. Applying Theorem 4 to this renewal process and the function h,

$$(2.19)\qquad \frac{\sum_{k=0}^{m} h(S_{L_k})}{\frac{1}{\theta\mu}\int_0^{\mu\theta m} h(s)ds} \overset{a.s.}{\to} 1 \quad \text{as } m \to \infty .$$

The number of visits of $\{S_n\}$ to $(-\infty,0)$ is finite a.s., since $\mu > 0$. So we may assume that $f(x) = 0$ for $x < 0$, since a finite number of terms $f(S_k)$ does not change the rate of divergence of $\sum^n f(S_k)$. With this extra assumption we will show

$$(2.20)\qquad \frac{\int_0^{\mu\theta m} h(s)ds}{\theta\int_0^{\mu\theta m} f(s)ds} \to 1 \quad \text{as } m \to \infty.$$

$$\left|\int_0^{\mu\theta m} h(s)ds-\theta\int_0^{\mu\theta m} f(s)ds\right| = \left|E\left(\sum_{j=1}^{L_1} \int_0^{\mu\theta m} f(s+S_j)-\theta\int_0^{\mu\theta m} f(s)ds\right)\right|$$

$$\leq \left|E\left(\sum_{j=1}^{L_1} \left(\int_{S_j}^{S_j+\mu\theta m} f(s)ds-\int_0^{\mu\theta m} f(s)ds\right)\right)\right| .$$

If $j < L_1$, $S_j \leq 0$. If $S_j < -\mu\theta m$,

$$\left|\int_{S_j}^{S_j+\mu\theta m} f(s)ds-\int_0^{\mu\theta m} f(s)ds\right| = \int_0^{\mu\theta m} f(s)ds \leq |S_j| .$$

If $-\mu\theta m \leq S_j \leq 0$,

$$\left|\int_{S_j}^{S_j+\mu\theta m} f(s)ds-\int_0^{\mu\theta m} f(s)ds\right| = \int_{S_j+\mu\theta m}^{\mu\theta m} f(s)ds \leq |S_j| .$$

For $j = L_1$,

$$eE\left(\left|\int_{S_{L_1}}^{S_{L_1}+\mu\theta m} f(s)ds-\int_0^{\mu\theta m} f(s)ds\right|\right) \leq E(S_{L_1} I_{\{S_{L_1}\leq\mu\theta m\}})+\mu\theta m P(S_{L_1} >\mu\theta m).$$

Combining

$$\frac{\left|\int_0^{\mu\theta m} h(s)-\theta\int_0^{\mu\theta m} f(s)ds\right|}{\theta\int_0^{\mu\theta m} f(s)ds} \le \frac{E\left(\sum_{j=1}^{L_1-1}|S_j|\right)+E(S_{L_1})+\mu\theta m P(S_{L_1}>\mu\theta m)}{\mu\theta^2\gamma m}$$

$$\to 0 \quad \text{as } m\to\infty.$$

This proves (2.20).

(2.18) through (2.20) imply

$$\frac{\sum_{j=1}^{L_m} f(S_j)}{\frac{1}{\mu}\int_0^{\mu\theta m} f(s)ds} \overset{a.s.}{\to} 1 \quad \text{as } m\to\infty.$$

For any n there exists an m such that $L_m \le n \le L_{m+1}$, and for such an m

$$\frac{\frac{1}{\mu}\int_0^{\mu n} f(s)ds}{\sum^n f(S_j)} \le \frac{\frac{1}{\mu}\left(\int_0^{\mu\theta m} f(s)ds+\int_{\mu\theta m}^{\mu L_{m+1}} f(s)ds\right)}{\sum^{L_m} f(S_j)}$$

$$\le \frac{\frac{1}{\mu}\int_0^{\mu\theta m} f(s)ds}{\sum^{L_m} f(S_j)} + \frac{|L_{m+1}-\theta m|}{\gamma L_m} \to 1 \quad \text{as } m\to\infty.$$

Therefore $\underline{\lim}_{n\to\infty} \dfrac{\sum^n f(S_j)}{\frac{1}{\mu}\int_0^{\mu n} f(s)ds} \ge 1.$

If $\theta m \le n \le \theta(m+1)$

$$\frac{\sum^{n} f(S_j)}{\frac{1}{\mu}\int_0^{\mu n} f(s)ds} \leq \frac{\sum^{L_m} f(S_j)}{\frac{1}{\mu}\int_0^{\mu\theta m} f(s)ds} + \frac{|\theta(m+1)-L_m|}{\gamma\theta m} \to 1 \text{ as } m \to \infty.$$

Therefore $\overline{\lim}_{n\to\infty} \frac{\sum^{n} f(S_j)}{\frac{1}{\mu}\int_0^{\mu n} f(s)ds} \leq 1.$

The Brownian Motion Case

Theorem 4 also has an analogue for Brownian Motion.

Theorem 7. Let $\{S(t): t \geq 0\}$ be Brownian Motion with drift $\mu > 0$. Let f be a non-negative, bounded, Lebesgue measurable function on $(-\infty,\infty)$ such that there exist M and δ such that $\int_x^{x+M} f(t)dt \geq \delta > 0$ for all $x \geq 0$. Then

$$\frac{\int_0^n f(S(t))dt}{\frac{1}{\mu}\int_0^{\mu n} f(s)ds} \overset{a.s.}{\to} 1 \quad \text{as } n \to \infty.$$

Proof: Since $m\{t: S(t) < 0\} < \infty$ a.s., we may assume without loss of generality that $\int_x^{x+M} f(t)dt \geq \delta$ for all x.

$$\int_0^n f(S(t))dt = \sum_{k=0}^{n} \int_k^{k+1} f(S(t))dt \ ,$$

Let

$$g(s)=E(\int_k^{k+1} f(S(t))dt/S(k)=s)=E(\int_0^1 f(s+S(t))dt)=\int_0^1 E(f(s+S(t)))dt.$$

Let $Q_t(x)$ be the density for $S(t)$. Since $S(t)$ is normally distributed with mean t, there exists $\alpha > 0$ such that $Q_t(x) > \alpha$ for $t \in [0,1]$ and $x \in [0,M]$. Therefore, for all $t \in [0,1]$

$$E(f(s+S(t)))=\int_{-\infty}^{\infty} f(s+x)Q_t(x)dx \geq \alpha\int_{s}^{s+M} f(x)dx \geq \alpha\ \delta > 0.$$

So, by Theorem 5b,

$$(2.21)\quad \frac{\int_0^n f(S(t))dt}{\sum_{k=0}^{n} g(S(k))} = \frac{\sum_{k=0}^{n-1}\int_k^{k+1} f(S(t))dt}{\sum_{k=0}^{n} E(\int_k^{k+1} f(S(t))dt\,|\,S(k))} \overset{a.s.}{\to} 1 \text{ as } n\to\infty.$$

In the proof of Theorem 3 we verified that all the hypotheses of Theorem 6 are satisfied for the random walk $\{S(k)\}$. Therefore

$$(2.22)\quad \frac{\sum_{k=0}^{n} g(S(k))}{\frac{1}{\mu}\int_0^{\mu n} g(s)ds} \overset{a.s.}{\to} 1 \text{ as } n\to\infty .$$

$$(2.23)\quad \frac{\int_0^{\mu n} f(s)ds}{\int_0^{\mu n} g(s)ds} \to 1 \quad \text{as} \quad n\to\infty, \text{ since } \int_0^{\mu n} g(s)ds \geq \alpha\gamma\mu n \text{ and}$$

$$\left|\int_0^{\mu n} g(s)ds-\int_0^{\mu n} f(s)ds\right| = \left|E(\int_0^1\int_0^{\mu n} f(s+S(t))dsdt-\int_0^{\mu n} f(s)ds)\right|$$

$$= \left|E(\int_0^1 (\int_{S(t)}^{S(t)+\mu n} f(s)ds-\int_0^{\mu n} f(s)ds)dt)\right|$$

$$\leq E(\int_0^1 (|S(t)|+\mu n\ I_{\{|S(t)|>\mu n\}})dt)$$

$$\leq E(\sup_{0\leq t\leq 1} |S(t)|) + \mu n\ E(m\{t \in [0,1]: |S(t)| > \mu n\}).$$

(2.21) through (2.23) imply

$$\frac{\int_0^n f(S(t))dt}{\frac{1}{\mu}\int_0^{\mu n} f(s)ds} \overset{a.s.}{\to} 1 \quad \text{as} \quad n\to\infty.$$

BIBLIOGRAPHY

[1] Brieman, Leo (1968). Probability. Addison-Wesley, Reading, Massachusetts.

[2] Chung, K. L. and Derman, C. (1956). Non-recurrent Random Walks. Pacific Journal of Mathematics. (6), 441-447.

[3] Doob, J. L. (1953). Stochastic Processes. John Wiley, New York.

[4] Feller, William (1966). An Introduction to Probability Theory and Its Applications, Volume II. John Wiley, New York.

[5] Loeve, Michel (1955). Probability Theory. D. Van Nostrand, Princeton, New Jersey.

[6] Puri, Prem S. and Rubin, Herman (1970). A Characterization Based on the Absolute Difference of Two I.I.D. Random Variables. Annals of Mathematical Statistics. (41), 2113-2122.

[7] Robbins, Herbert (1953). On the Equidistribution of Sums of Independent Random Variables. Proceedings of the American Mathematical Society. (4), 786-799.

[8] Root, David (1971). A Counterexample in Renewal Theory. Annals of Mathematical Statistics. (42), 1763-1766.

[9] Spitzer, Frank (1964). Principles of Random Walk. D. Van Nostrand, New York.