# A Probabilistic Proof of the Basic Limit Theorem for Null Recurrent Markov Chains

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Hoel, Port, and Stone [3] have given an elegant, elementary proof of the following familiar theorem for discrete time Markov chains with stationary transition probabilities:

THEOREM 1. _If the Markov chain is irreducible, aperiodic, and positive recurrent, then the n-step transition probabilities, $P_{i,j}(n)$, all converge to the stationary probabilities, $\Pi(j)$._

Their proof says, in effect: Let one "particle" start in state i, another start with the stationary distribution, and each, independently of the other, move according to the given transition matrix. The three components of the hypothesis guarantee that the "direct-product" process is irreducible and recurrent so with probability one the two particles will eventually be in the same state at the same time. Letting T be the time until this occurs, we conclude that

(1) $$|P_{i,j}(n) - \Pi(j)| < P(T > n).$$

This proof has great appeal not only because it is essentially probabilistic and much simpler than the usual proof, see e.g. Chung [1], or Feller [2] or Karlin [4], but also because for finite chains we can then get an exponential rate of convergence. That is, the right side of

(1) is easily seen to be less than $(1-a)^n$ for some $a > 0$ which does not depend on the initial state i, and for all n greater than the number of states.

On the other hand, the usual proof covers the null recurrent case as well, while Hoel, Port, and Stone's does not. To remedy this situation we offer a proof in the same spirit as theirs for the null recurrent case:

THEOREM 2. <u>If the Markov chain is irreducible, aperiodic and null recurrent, then the n-step transition probabilities converge to zero.</u>

<u>Proof</u>. The three components of the hypothesis guarantee that the direct-product process--namely the one with transition probabilities

$$Q_{(i,j),(k,\ell)} = P_{i,k} P_{j,\ell}$$

--is irreducible just as before but not necessarily recurrent.

If it is transient, the proof is easy since transience is well-known to be equivalent to

(2) $$\sum_{n=1}^{\infty} Q_{(i,j)(k,\ell)}(n) < \infty \qquad \text{for all } i,j,k,\ell.$$

so, in particular,

$$\sum_{n=1}^{\infty} [P_{i,j}(n)]^2 < \infty \qquad \text{for all } i,j.$$

(Incidentally (2) is easily proved probabilistically; see e.g. Hoel, Port, and Stone [3].)

If it is recurrent, we exploit the existence of a (unique up to multiplicative constant) generalized stationary distribution (i.e. invariant measure) for the original chain, namely the non-negative solution to the system of equations:

$$m(j) = \sum_{\text{all } i} m(i)P_{i,j}$$

Let R(i) "particles" start in state i, for each i, and let each "particle", independently of the others, move according to the given transition matrix. If $ER(i) = m(i)$ for each i, then, as is well-known, the expected number of particles in state i at time n is again m(i) for each i and n. Moreover, since $\sum_i m(i) = \infty$ for any null recurrent chain, it is easy to choose the R(i)'s so that $\sum_i R(i) = \infty$ with probability one.

Now start a "particle" in state $i_0$ and let it move independently of the others. By recurrence of the direct product chain each one of the other "particles" will eventually be in the same state at the same time as this one. Then, roughly speaking all the ones that do so by time n have the same chance as this one of being in state $j_0$ at time n. But the expected number is $m(j_0)$ for all n. Hence $P_{i_0,j_0}(n)$ must go to zero.

Here is a rigorous proof:

Let $\{Y(n): n \geq 0\}$ and $\{X_r(n): r = 1,2,\ldots; n \geq 0\}$ satisfy:

$$Y(0) = i_0$$

$$X_r(0) = i \qquad \text{if } \sum_{j<i} R(j) < r \leq \sum_{j\leq i} R(j)$$

$$P(Y(n+1) = j \mid \{R(i)\}, \{Y(k): k\leq n\}, \{X_r(n)\}) = P_{Y(n),j}$$

$$P(X_r(n+1) = j \mid \{R(i)\}, \{Y(n)\}, \{X_r(k): k\leq n\}, \{X_s(k): s \neq r\}) = P_{X_r(n),j}$$

Let

$N_n$ = number of r for which $Y(k) = X_r(k)$ for some $k \leq n$

$$\{W_t: t \leq N_0\} = \{r: Y(0) = X_r(0)\}$$

$$\{W_t: N_n < t \leq N_{n+1}\} = \{r: Y(n+1) = X_r(n+1), Y(k) \neq X_r(k), k \leq n\}$$

i.e. $W_t$ is the "t-th particle met by" $\{Y(n)\}$.

Then

$$m(j_0) = E(\text{number of } r \text{ for which } X_r(n) = j_0)$$

$$\geq E[\sum_t I_{\{X_{W(t)}(n) = j_0\}} I_{\{N_n \geq t\}}]$$

$$= \sum_t P(X_{W(t)}(n) = j_0 | N_n \geq t) P(N_n \geq t).$$

It is easy to show that

$$P(X_{W(t)}(n) = j_0 | N_n \geq t) = P(Y(n) = j_0 | N_n \geq t).$$

Now

$$P_{i_0,j_0}(n) = P(Y(n) = j_0 | N_n \geq t) P(N_n \geq t) + P(Y(n)=j_0 | N_n < t) P(N_n < t)$$

$$\leq P(Y(n) = j_0 | N_n \geq t) + P(N_n < t).$$

Thus

$$m(j_0) \geq \sum_t [P_{i_0,j_0}(n) - P(N_n < t)] P(N_n \geq t).$$

Since $N_n \uparrow \infty$ as $n \to \infty$, we conclude that $P_{i_0,j_0}(n) \to 0$ as $n \to \infty$. ∎

Here are two related open questions: Does $N_n \uparrow \infty$ even when the direct-product chain is transient? ($EN_n$ does.) For any given null recurrent chain is there a k such that the k-fold direct product chain is transient? (i.e. is $\sum_n [P_{i,j}(n)]^k < \infty$ for some i,j and k?)

## REFERENCES


[1] Chung, K. L. (1967) Markov Chains with Stationary Transition Probabilities. 2nd ed. Springer Verlag, New York.

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