ASYMPTOTIC FORMULAE FOR THE C.D.F. OF HOTELLING'S TRACE AND ROBUSTNESS STUDIES FOR THREE TESTS OF HYPOTHESES UNDER VIOLATIONS

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1. Introduction.

Let $m\underset{\sim}{S}_1$ and $n\underset{\sim}{S}_2$ have independent central Wishart distributions $W(m,p,\underset{\sim}{\Sigma}_1)$ and $W(n,p,\underset{\sim}{\Sigma}_2)$ respectively. Chattopadhyay and Pillai [1], and Pillai and Saweris [15], [16], (denoted by C-P and P-S respectively hereafter) have studied asymptotic expansions for the c.d.f. and percentiles of $T = m\,\mathrm{tr}\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ up to terms of order n^{-1} and n^{-2} respectively. While C-P consider the deviation parameter in the form $(\underset{\sim}{F}) = \mathrm{tr}\underset{\sim}{F} = \mathrm{tr}(\underset{\sim}{\Sigma}_1\underset{\sim}{\Sigma}_2^{-1} - \underset{\sim}{I})$, P-S expressed it in the form $(\underset{\sim}{F}s) = \mathrm{tr}\underset{\sim}{F}^s$, $s \geq 1$ including terms involving $f_{ij}f_{k\ell}$ in the n^{-1} order term which were neglected by C-P, where f_{ij} is the (i,j)th element of $\underset{\sim}{F}$. Further, P-S [16] extended the work of Chattopadhyay [2] who derived an asymptotic expansion up to terms of order n^{-1} for the c.d.f. and percentile of Hotelling's trace when $m\underset{\sim}{S}_1$ has non-central Wishart distribution $W(m,p,\underset{\sim}{\Sigma}_1,\underset{\sim}{\Omega})$ with non-centrality parameter matrix $\underset{\sim}{\Omega}$, and $n\underset{\sim}{S}_2$ (as before), $W(n,p,\underset{\sim}{\Sigma}_2)$. The extension further was to include the $f_{ij}f_{k\ell}$ terms neglected by Chattopadhyay. In this paper, the asymptotic expansion of Hotelling's trace in the non-central-central case is extended to include terms of order n^{-2}. However, since the number of terms involved in the earlier expansions is extremely large, and painfully so for numerical study, a method is obtained to limit the number of terms in the expansion to a manageable level by neglecting terms of higher powers in some Taylor expansions. However, it should be pointed out that all the above study has been carried out under the assumption $|ch_i \underset{\sim}{F}| < 1$ $i = 1,\ldots,p$ where ch_i denotes the ith characteristic root.

In addition, the asymptotic expansion obtained for the non-central-central case has been extended to the case of canonical correlation by making $\underset{\sim}{\Omega}$ random. When $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$, this reduces to the usual canonical correlation situation, but when $\underset{\sim}{\Sigma}_1 \neq \underset{\sim}{\Sigma}_2$, the distribution reflects a non-normality situation as explained below.

It may be noted that $L(\underset{\sim}{\Sigma}_1,\underset{\sim}{\Sigma}_2,\underset{\sim}{\Omega}) = W(m,p,\underset{\sim}{\Sigma}_1,\underset{\sim}{\Omega})W(n,p,\underset{\sim}{\Sigma}_2)$ can be written in the form

$$(1.1)\quad L(\underset{\sim}{\Sigma}_1,\underset{\sim}{\Sigma}_1,\underset{\sim}{\Omega})|\underset{\sim}{\Sigma}_1\underset{\sim}{\Sigma}_2^{-1}|^{1/2\ n}{}_0F_0(\tfrac{1}{2}(\underset{\sim}{\Sigma}_1^{-1}-\underset{\sim}{\Sigma}_2^{-1})\underset{\sim}{S}_2),$$

where ${}_0F_0(\underset{\sim}{T})$ denotes the hypergeometric function of the matrix variate $\underset{\sim}{T}$, [8].Now $L(\underset{\sim}{\Sigma}_1,\underset{\sim}{\Sigma}_1,\underset{\sim}{\Omega})$ leads to the distribution of the sample canonical correlation coefficients in the normal non-central case when $\underset{\sim}{\Omega}$ is made random, and the expression in (1.1) equals $L(\underset{\sim}{\Sigma}_1,\underset{\sim}{\Sigma}_1,\underset{\sim}{\Omega})$ when $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$. If $\underset{\sim}{\Sigma}_1 \neq \underset{\sim}{\Sigma}_2$, the series ${}_0F_0$ leads to a non-normal situation in the sense of Edgeworth series type expansions in the univariate case.

Numerical results on powers of Hotelling's trace for (i) test of equality of two covariance matrices (ii) MANOVA and (iii) canonical correlation are obtained from the asymptotic expansions for studying the robustness of Hotelling's trace against violations of normality for (i) and (iii) and equality of covariance matrices for (ii). The results are presented in Tables 2 and 4 while Tables 1 and 3 give some comparison of the accuracy of the expansion using terms up to order (1), (n^{-1}) and (n^{-2}).

2. <u>The method of asymptotic expansion</u>. Let $\underset{\sim}{A} = \underset{\sim}{\Sigma}_2^{-1}$, $\underset{\sim}{B} = \underset{\sim}{\Sigma}_1^{-1}$, then tr $\underset{\sim}{B}\underset{\sim}{S}_1$ is distributed as a non-central chi-square with mp degrees of freedom and non-centrality, tr $\underset{\sim}{\Omega} = \omega^2$, i.e.

$$(2.1)\quad \Pr\{\mathrm{tr}\ \underset{\sim}{B}\underset{\sim}{S}_1 \leq \theta\} = e^{-\frac{\omega^2}{2}} \sum_{J=0}^{\infty} \frac{(\frac{\omega^2}{2})^J}{J!2^{\zeta+J}\Gamma(\zeta+J)} \int_0^\theta x^{\zeta+J-1} e^{-\frac{x}{2}} dx$$

$$= G_{mp}(\theta,\omega^2).$$

Let

$$(2.2)\qquad G(\theta) = \Pr\{\mathrm{tr}\ \underset{\sim}{S}_2^{-1}\underset{\sim}{S}_1 \leq \theta\},$$

then $G(\theta)$ can be expressed as

$$G(\theta) = E_{\underset{\sim}{S}_2}\{\Pr[\mathrm{tr}\ \underset{\sim}{S}_2^{-1}\underset{\sim}{S}_1 \leq \theta]|S_2\}.$$

By Taylor's expansion to expand $\Pr[\mathrm{tr}\ \underset{\sim}{S}_2^{-1}\underset{\sim}{S}_1 \leq \theta|S_2]$ about an origin $\underset{\sim}{A}^{-1} = (\sigma_{ij})$, $i,j = 1,\ldots,p$,

we get $\quad \Pr\{\mathrm{tr}\ \underset{\sim}{S}_2^{-1}\underset{\sim}{S}_1 \leq \theta|S_2\} = \exp\{\mathrm{tr}(\underset{\sim}{S}_2-\underset{\sim}{A}^{-1}\underset{\sim}{\partial})\}\cdot\Pr\{\mathrm{tr}\ \underset{\sim}{A}\underset{\sim}{S}_1 \leq \theta\}$,

where $\underset{\sim}{\partial}(p\mathrm{x}p) = (\frac{1}{2}(1+\delta_{ij})\frac{\partial}{\partial\sigma_{ij}})$ and δ_{ij} is the Kronecker delta. Hence

$$(2.3)\qquad G(\theta) = \Theta \cdot \Pr\{\mathrm{tr}\ \underset{\sim}{A}\underset{\sim}{S}_1 \leq \theta\},$$

where $\Theta = \exp(-\mathrm{tr}\ \underset{\sim}{A}^{-1}\underset{\sim}{\partial})|I - \frac{2}{n}\underset{\sim}{A}^{-1}\underset{\sim}{\partial}|^{-\frac{n}{2}}$.

Using [9],[19] we have

$$(2.4)\ \Theta = 1 + \frac{1}{n}\sum\sigma_{rs}\sigma_{tu}\partial_{st}\partial_{ur} + \frac{1}{n^2}\Big[\frac{4}{3}\sum\sigma_{rs}\sigma_{tu}\sigma_{\nu w}\partial_{st}\partial_{u\nu}\partial_{w\nu} +$$

$$\frac{1}{2}\sum\sigma_{rs}\sigma_{tu}\sigma_{\nu w}\sigma_{xy}\partial_{st}\partial_{ur}\partial_{wx}\partial_{y\nu}\Big] + O(n^{-3}),$$

where Σ denotes the summation over all suffixes r, s, t,... each of which ranges from 1 to p. Now, in order to evaluate $\partial_{st}\partial_{u\nu}\Pr\{\mathrm{Tr}\ \underset{\sim}{A}\underset{\sim}{S}_1 \leq \theta\}$ etc. we will use the perturbation technique [6],[7].

$$(2.5)\qquad \text{Let}\quad J = \Pr\{\mathrm{tr}(\underset{\sim}{A}^{-1} + \underset{\sim}{\varepsilon})^{-1}\underset{\sim}{S}_1 \leq \theta\},$$

where $\underset{\sim}{\varepsilon}$ is a pxp symmetric matrix near $\underset{\sim}{O}$(pxp). By [4], [19] (2.6) $J = |I-X\Delta|^{-\frac{m}{2}} \exp[-\omega^2/2] \exp\{\frac{1}{2} E\, tr(\underset{\sim}{I}-\underset{\sim}{X}\Delta)^{-1}\underset{\sim}{\Omega}\} G_{mp}(\theta,0)$,

where $\Delta = E-1$ and $X = \underset{\sim}{B}^{-1}(\underset{\sim}{A}^{-1}+\underset{\sim}{\varepsilon})^{-1} - \underset{\sim}{I}$. In view of the following:

$$|\underset{\sim}{I}-\underset{\sim}{X}\Delta|^{-\frac{m}{2}} = \exp\{tr(-\frac{m}{2}\underset{\sim}{X} + \frac{m}{4}\underset{\sim}{X}^2 - \frac{m}{6}\underset{\sim}{X}^3+\ldots)\},$$

and $e^{\frac{\omega^2}{2}(E-1)} G_{mp}(\theta,0) = G_{mp}(\theta,\omega^2)$, we get

$$J = \exp\{Q_0+Q_1E+Q_2E^2+\ldots\}\; G_{mp}(\theta,\omega^2),$$

where $Q_0 = tr\, m(-\underset{\sim}{X}/2 + \underset{\sim}{X}^2/4 - \underset{\sim}{X}^3/6+\ldots)$

$$Q_1 = \frac{1}{2}\, tr\, \underset{\sim}{\Omega}\, (-\underset{\sim}{X}+\underset{\sim}{X}^2-\underset{\sim}{X}^3+\ldots) + tr\, \frac{m}{2}(\underset{\sim}{X}-\underset{\sim}{X}^2+\underset{\sim}{X}^3-\ldots),$$

$$Q_2 = \frac{1}{2}\, tr\, \underset{\sim}{\Omega}\, (\underset{\sim}{X}-2\underset{\sim}{X}^2+3\underset{\sim}{X}^3-\ldots) + tr\, m(\underset{\sim}{X}^2/4-\underset{\sim}{X}^3/2+3\underset{\sim}{X}^4/4-\ldots),$$

$$Q_3 = \frac{1}{2}\, tr\, \underset{\sim}{\Omega}\, (\underset{\sim}{X}^2-3\underset{\sim}{X}^3+6\underset{\sim}{X}^4-\ldots) + tr(m\underset{\sim}{X}^3/6 - m\underset{\sim}{X}^4/2+\ldots),$$

$$Q_4 = \frac{1}{2}\, tr\, \underset{\sim}{\Omega}\, (\underset{\sim}{X}^3-4\underset{\sim}{X}^4+\ldots) + tr\, m(\underset{\sim}{X}^4/8-\ldots), \text{ etc.}$$

Hence

$$J = \{[1+Q_0+Q_0^2/2! + Q_0^3/3! + Q_0^4/4!+\ldots] + [Q_1+Q_0Q_1+Q_0^2Q_1/2+\ldots]E$$

(2.7) $$+ [Q_2+Q_0Q_2+Q_1^2/2+\ldots]E^2+\ldots\}\; G_{mp}(\theta,\omega^2).$$

On the other hand, by Taylor's expansion we have

$$J = \{1 + \sum\varepsilon_{rs}\partial_{rs} + (1/2!)\sum \varepsilon_{rs}\varepsilon_{tu}\partial_{rs}\partial_{tu}+\ldots\}\;\; Pr\{tr\, \underset{\sim}{A}\underset{\sim}{S}_1 \le \theta\}.$$

Now, $\underset{\sim}{X} = \underset{\sim}{B}^{-1}(\underset{\sim}{A}^{-1}+\underset{\sim}{\varepsilon})^{-1} - \underset{\sim}{I}$

$$= (\underset{\sim}{B}^{-1}\underset{\sim}{A}-\underset{\sim}{I}) - \sum\varepsilon_{rs}(\underset{\sim}{B}^{-1}\underset{\sim}{A})(\underset{\sim}{A}_{rs}^{-1}\underset{\sim}{A}) + \sum\varepsilon_{rs}\varepsilon_{tu}(\underset{\sim}{B}^{-1}\underset{\sim}{A})(\underset{\sim}{A}_{rs}^{-1}\underset{\sim}{A})(\underset{\sim}{A}_{tu}^{-1}\underset{\sim}{A}) - \ldots$$

$$= \underset{\sim}{F} - \sum\varepsilon_{rs}(\underset{\sim}{F}+\underset{\sim}{I})(\underset{\sim}{A}_{rs}^{-1}\underset{\sim}{A}) + \sum\varepsilon_{rs}\varepsilon_{tu}(\underset{\sim}{F}+\underset{\sim}{I})(\underset{\sim}{A}_{rs}^{-1}\underset{\sim}{A})(\underset{\sim}{A}_{tu}^{-1}\underset{\sim}{A}) - \ldots,$$

where $\underset{\sim}{F} = \underset{\sim}{B}^{-1}\underset{\sim}{A} - \underset{\sim}{I}$ is such that $|Ch_i\underset{\sim}{F}| < 1$, $i = 1,\ldots,p$, and $\underset{\sim}{A}_{rs}^{-1}$ is the pxp matrix with (ij)th element $\frac{1}{2}(\delta_{ri}\delta_{sj} + \delta_{rj}\delta_{si})$.

Let $tr(A_{rs}^{-1}A) = (rs)$; $tr(A_{rs}^{-1}A)(A_{tu}^{-1}A) = (rs|tu)$,

$$tr(F)(A_{rs}^{-1}A)(A_{tu}^{-1}A) = (F|rs|tu), \text{ etc.}$$

Now substituting X in (2.7) and comparing term by term $\varepsilon_{rs}, \varepsilon_{rs}\varepsilon_{tu}, \ldots$ with (2.8), we can find $\partial_{st}\partial_{ur} \Pr\{AS_1 \leq \theta\}$ etc. We first consider X, and X^2 terms and get

$$\begin{aligned}
&\partial_{rs}\partial_{tu} \Pr\{tr\, AS_1 \leq \theta\} = \{(F+I|rs|tu)(E-1)m + \tfrac{1}{2}[2(F|F+I|rs|tu) \\
(2.9)\quad &+ (F+I|rs|F+I|tu)](E-1)^2m + (\Omega|F+I|rs|tu)E(E-1) \\
&+ [2(\Omega F|F+I|rs|tu) + (\Omega|F+I|rs|F+I|tu)]E(E-1)^2 \\
&+ (1/4)[2(F)(F+I|rs|tu) + (F+I|rs)(F+I|tu)](E-1)^2m^2 \\
&+ (1/4)[2(\Omega F)(\Omega|F+I|rs|tu) + (\Omega|F+I|rs)(\Omega|F+I|tu)]E^2(E-1)^2 \\
&+ \tfrac{1}{2}[(F)(\Omega|F+I|rs|tu) + (F+I|rs|tu)(\Omega F) + (F+I|rs)(\Omega|F+I|tu)]E(E-1)^2m \\
&\cdot G_{mp}(\theta,\omega^2)
\end{aligned}$$

In a similar manner expressions have been obtained for $\partial_{rs}\partial_{tu}\partial_{w\nu} \Pr\{tr\, AS_1 \leq \theta\}$ and $\partial_{rs}\partial_{tu}\partial_{w\nu}\partial_{xy} \Pr\{tr\, AS_1 \leq \theta\}$. It may be pointed out that in the above expression each term represents the average of all permutations of rs, tu, wν, xy, for example, $(F|rs|tu)$ represents $\frac{1}{2}[(F|rs|tu) + (F|tu|rs)]$ and so forth. Now, we notice the following:

$$(2.10)\quad \left.\begin{aligned}
&\sum\sigma_{st}\sigma_{ur}(G|rs|tu) = \tfrac{1}{2}(p+1)(G),\ \sum\sigma_{st}\sigma_{ur}(G|rs|H|tu) = \tfrac{1}{2}(GH) + \tfrac{1}{2}(G)(H), \\
&\sum\sigma_{st}\sigma_{ur}(G|rs)(H|tu) = (GH),\ \sum\sigma_{rt}\sigma_{uw}\sigma_{\nu s}(G|rs|tu|w\nu) = (p^2+3p+4)(G)/8, \text{etc.}
\end{aligned}\right\}$$

(see appendix for other results)

where G, H are any non-singular pxp matrices.

Now, use (2.9) and (2.10) in (2.4), we get

$$R_1 = \sum \sigma_{st}\sigma_{ur}\partial_{rs}\partial_{tu} \Pr\{\mathrm{tr}\, \underset{\sim}{A}\underset{\sim}{S}_1 \le \theta_1\}$$

(2.11) $= \{\frac{1}{2} m[\frac{1}{2} q_{23}(\underset{\sim}{F}^2) + \frac{1}{2}(\underset{\sim}{F})^2 + q_{11}(\underset{\sim}{F}) - \frac{1}{2}p\cdot q_{11}] + (m^2/4)[q_{11}(\underset{\sim}{F})^2$
$+ (\underset{\sim}{F}^2) + q_{12}^1(\underset{\sim}{F})+P]\}\ G_{mp}(\theta,\omega^2) + \{\frac{1}{2}[q_{23}(\underset{\sim}{\Omega}\underset{\sim}{F}^2) + q_{23}(\underset{\sim}{\Omega}\underset{\sim}{F}) +$
$+ (\underset{\sim}{\Omega}\underset{\sim}{F})(\underset{\sim}{F}) + (\underset{\sim}{\Omega})(\underset{\sim}{F})] + (m/2)[q_{11}(\underset{\sim}{F})(\underset{\sim}{\Omega}\underset{\sim}{F}) + \frac{1}{2} q_{11}(\underset{\sim}{F})(\underset{\sim}{\Omega}) + (\underset{\sim}{\Omega}\underset{\sim}{F}^2)$
$+ \frac{1}{2} q_{14}^1(\underset{\sim}{\Omega}\underset{\sim}{F}) + (\underset{\sim}{\Omega}) - 3q_{11}(\underset{\sim}{F}) - q_{23}(\underset{\sim}{F}^2) - (\underset{\sim}{F})^2] - \frac{1}{2}m^2[q_{11}(\underset{\sim}{F})^2 + q_{12}^1(\underset{\sim}{F})$
$+ (\underset{\sim}{F}^2) + P]\}\ EG_{mp}(\theta,\omega^2) + (1/4)\{\mathrm{tr}\ \underset{\sim}{\Omega}^2(\underset{\sim}{F}+\underset{\sim}{I})^2 - 4q_{23}(\underset{\sim}{\Omega}\underset{\sim}{F}^2) - 4(\underset{\sim}{\Omega}\underset{\sim}{F})(\underset{\sim}{F})$
$- 2q_{57}(\underset{\sim}{\Omega}\underset{\sim}{F}) + q_{11}(\underset{\sim}{\Omega}\underset{\sim}{F})^2 - 2q_{11}(\underset{\sim}{\Omega}) + q_{11}(\underset{\sim}{\Omega}\underset{\sim}{F})(\underset{\sim}{\Omega}) - 4(\underset{\sim}{\Omega})(\underset{\sim}{F}) + m[q_{23}(\underset{\sim}{F}^2)$
$+ 4q_{11}(\underset{\sim}{F}) + (\underset{\sim}{F})^2 - 4q_{11}(\underset{\sim}{\Omega}\underset{\sim}{F})(\underset{\sim}{F}) - 2q_{11}(\underset{\sim}{F})(\underset{\sim}{\Omega}) - 4(\underset{\sim}{\Omega}\underset{\sim}{F}^2) - 2q_{14}^1(\underset{\sim}{\Omega}\underset{\sim}{F})$
$- 4(\underset{\sim}{\Omega}) + P\cdot q_{11}] + m^2[q_{11}(\underset{\sim}{F})^2 + (\underset{\sim}{F}^2) + q_{12}^1(\underset{\sim}{F}) + P]\}\ E^2G_{mp}(\theta,\omega^2)$
$+ \frac{1}{2}\{q_{23}(\underset{\sim}{\Omega}\underset{\sim}{F}^2) + q_{34}(\underset{\sim}{\Omega}\underset{\sim}{F}) + q_{11}(\underset{\sim}{\Omega}) - q_{11}(\underset{\sim}{\Omega}\underset{\sim}{F})^2 - q_{11}(\underset{\sim}{\Omega}\underset{\sim}{F})(\underset{\sim}{\Omega}) + (\underset{\sim}{\Omega}\underset{\sim}{F})(\underset{\sim}{F})$
$+ (\underset{\sim}{\Omega})(\underset{\sim}{F}) - \mathrm{tr}\underset{\sim}{\Omega}^2(\underset{\sim}{F}+\underset{\sim}{I})^2 + \frac{1}{2}m[2q_{11}(\underset{\sim}{F})(\underset{\sim}{\Omega}\underset{\sim}{F}) + q_{14}^1(\underset{\sim}{\Omega}\underset{\sim}{F}) + q_{11}(\underset{\sim}{F})(\underset{\sim}{\Omega})$
$+ 2(\underset{\sim}{\Omega}\underset{\sim}{F}^2) + 2(\underset{\sim}{\Omega})]\}\ E^3G_{mp}(\theta,\omega^2) + (1/4)\{q_{11}(\underset{\sim}{\Omega}\underset{\sim}{F})^2 + q_{11}(\underset{\sim}{\Omega}\underset{\sim}{F})(\underset{\sim}{\Omega})$
$+ \mathrm{tr}\underset{\sim}{\Omega}^2(\underset{\sim}{F}+\underset{\sim}{I})^2\}\ E^4G_{mp}(\theta,\omega^2),$

$$R_2 = \sum \sigma_{rt}\sigma_{uw}\sigma_{vs}\partial_{rs}\partial_{tu}\partial_{wv} \Pr\{\mathrm{tr}\, \underset{\sim}{A}\underset{\sim}{S}_1 \le \theta\}$$

(2.12) $= -3\{(m/8)[q_{48}^1(\underset{\sim}{F}^2) + q_{12}(\underset{\sim}{F})^2 + 2q_{34}^1(\underset{\sim}{F})] + (m^2/16)[2q_{12}(\underset{\sim}{F}^2) + q_{36}^1(\underset{\sim}{F})^2$
$+ (p^3+q_{12,8}^3)(\underset{\sim}{F}) + 4p\cdot q_{11}]\}G_{mp}(\theta,\omega^2) - (3/4)\{q_{48}^1(\underset{\sim}{\Omega}\underset{\sim}{F}^2)$
$+ \frac{1}{2} q_{11,20}^3(\underset{\sim}{\Omega}\underset{\sim}{F}) + q_{12}(\underset{\sim}{\Omega}\underset{\sim}{F})(\underset{\sim}{F}) + q_{12}(\underset{\sim}{F})(\underset{\sim}{\Omega}) + \frac{1}{2} q_{34}^1(\underset{\sim}{\Omega}) + m[-\frac{1}{2} pq_{34}^1 + 2q_{11}(\underset{\sim}{\Omega})$
$+ (p^3+q_{16,16}^3)(\underset{\sim}{\Omega}\underset{\sim}{F})/4 + q_{36}^1(\underset{\sim}{F})(\underset{\sim}{\Omega}\underset{\sim}{F})/2 + q_{38}^1(\underset{\sim}{F})(\underset{\sim}{\Omega})/4 + q_{12}(\underset{\sim}{\Omega}\underset{\sim}{F}^2)$
$- q_{12}(\underset{\sim}{F})^2 - 5q_{34}^1(\underset{\sim}{F})/2 - q_{48}^1(\underset{\sim}{F}^2)] - (m^2/2)[4pq_{11} + 2q_{12}(\underset{\sim}{F}^2) + q_{36}^1(\underset{\sim}{F})^2$
$+ (p^3+q_{12,8}^3)(\underset{\sim}{F})]\}\ EG_{mp}(\theta,\omega^2) - (3/8)\{\frac{1}{2} q_{36}^1(\underset{\sim}{\Omega}\underset{\sim}{F})^2 + \frac{1}{2} q_{38}^1(\underset{\sim}{\Omega}\underset{\sim}{F})(\underset{\sim}{\Omega})$
$- 4q_{48}^1(\underset{\sim}{\Omega}\underset{\sim}{F}^2) - 4q_{12}(\underset{\sim}{\Omega}\underset{\sim}{F})(\underset{\sim}{F}) - q_{25,44}^7(\underset{\sim}{\Omega}\underset{\sim}{F}) - 3q_{34}^1(\underset{\sim}{\Omega}) + (\underset{\sim}{\Omega})^2 - 4q_{12}(\underset{\sim}{\Omega})(\underset{\sim}{F})$
$+ q_{12}\ \mathrm{tr}\ \underset{\sim}{\Omega}^2(\underset{\sim}{F}+\underset{\sim}{I})^2 + m[pq_{34}^1 + p_{48}^1(\underset{\sim}{F}^2) + q_{12}(\underset{\sim}{F})^2 + 3q_{34}^1(\underset{\sim}{\Omega}) - q_{38}^1(\underset{\sim}{F})(\underset{\sim}{\Omega})$
$- 2q_{36}^1(\underset{\sim}{F})(\underset{\sim}{\Omega}\underset{\sim}{F}) - 4q_{12}(\underset{\sim}{\Omega}\underset{\sim}{F}^2) - (p^3+q_{16,16}^3)(\underset{\sim}{\Omega}\underset{\sim}{F}) - 8q_{11}(\underset{\sim}{\Omega})] + m^2[2pq_{11}$
$+ \frac{1}{2}q_{36}^1(\underset{\sim}{F})^2 + \frac{1}{2}(p^3+q_{12,8}^3)(\underset{\sim}{F}) + q_{12}(\underset{\sim}{F}^2)]\}\ E^2G_{mp}(\theta,\omega^2)$
$- (3/4)\{q_{48}^1(\underset{\sim}{\Omega}\underset{\sim}{F}^2) + q_{7,12}^2(\underset{\sim}{\Omega}\underset{\sim}{F}) - \frac{1}{2} q_{36}^1(\underset{\sim}{\Omega}\underset{\sim}{F})^2 - (\underset{\sim}{\Omega})^2 - \frac{1}{2}q_{38}^1(\underset{\sim}{\Omega}\underset{\sim}{F})(\underset{\sim}{\Omega}) +$
$+ q_{34}^1(\underset{\sim}{\Omega}) + q_{12}(\underset{\sim}{\Omega}\underset{\sim}{F})(\underset{\sim}{F}) + q_{12}(\underset{\sim}{\Omega})(\underset{\sim}{F}) - q_{12}\mathrm{tr}\underset{\sim}{\Omega}^2(\underset{\sim}{F}+\underset{\sim}{I})^2 + (m/4)[2q_{36}^1(\underset{\sim}{F})(\underset{\sim}{\Omega}\underset{\sim}{F})$
$+ q_{38}^1(\underset{\sim}{F})(\underset{\sim}{\Omega}) + (p^3+q_{16,16}^3)(\underset{\sim}{\Omega}\underset{\sim}{F}) + 4q_{12}(\underset{\sim}{\Omega}\underset{\sim}{F}^2) + 8q_{11}(\underset{\sim}{\Omega})]\}\ E^3G_{mp}(\theta,\omega^2)$
$- (3/16)\{q_{36}^1(\underset{\sim}{\Omega}\underset{\sim}{F})^2 + 2q_{12}\mathrm{tr}\underset{\sim}{\Omega}^2(\underset{\sim}{F}+\underset{\sim}{I})^2 + q_{38}^1(\underset{\sim}{\Omega}\underset{\sim}{F})(\underset{\sim}{\Omega}) + 2(\underset{\sim}{\Omega})^2\}\ E^4G_{mp}(\theta,\omega^2),$

and

$$R_3 = \sum\sigma_{st}\sigma_{ur}\sigma_{wx}\sigma_{\nu y}\partial_{rs}\partial_{tu}\partial_{w\nu}\partial_{xy}\ \Pr\{\mathrm{tr}\ \underset{\sim}{A}\underset{\sim}{S}_1 \le \theta\}$$

$$\begin{aligned}
(2.13) = &\{(m/2)[q_{58}(\underset{\sim}{F})^2+q^5_{17,25}(\underset{\sim}{F}^2) + 6q^2_{55}(\underset{\sim}{F})+pq^2_{55}] + (m^2/4)[q^5_{12,17}(\underset{\sim}{F})^2 \\
&+ 2q_{58}(\underset{\sim}{F}^2) + 2(3p^3+q^7_{22,16})(\underset{\sim}{F})+p(p+1)q^1_{1,16}]\}\ G_{mp}(\theta,\omega^2) \\
&+ \{q^5_{17,25}(\underset{\sim}{\Omega}\underset{\sim}{F}^2) + q_{58}(\underset{\sim}{\Omega}\underset{\sim}{F})(\underset{\sim}{F}) + q_{58}(\underset{\sim}{\Omega})(\underset{\sim}{F}) + q^9_{27,35}(\underset{\sim}{\Omega}\underset{\sim}{F}) + 2q^2_{55}(\underset{\sim}{\Omega}) \\
&+ m[q_{58}(\underset{\sim}{\Omega}\underset{\sim}{F}^2) + \tfrac{1}{2}q^5_{12,17}(\underset{\sim}{\Omega}\underset{\sim}{F})(\underset{\sim}{F}) + \tfrac{1}{2}q^3_{7,12}(\underset{\sim}{\Omega})(\underset{\sim}{F}) + \tfrac{1}{2}(3p^3+q^7_{32,32})(\underset{\sim}{\Omega}\underset{\sim}{F}) \\
&+ \tfrac{1}{2}q_{11}q^1_{1,16}(\underset{\sim}{\Omega}) - q^5_{17.25}(\underset{\sim}{F}^2)-q_{58}(\underset{\sim}{F})^2-7q^2_{55}(\underset{\sim}{F})-2pq^2_{55}]-m^2[\tfrac{1}{2}\ q^5_{12,17}(\underset{\sim}{F})^2 \\
&+ q_{58}(\underset{\sim}{F}^2) + (3p^3+q^7_{22,16})(\underset{\sim}{F})+pq_{11}q^1_{1,16}/2]\}\ EG_{mp}(\theta,\omega^2) \\
&+ \{\tfrac{1}{2}[\tfrac{1}{2}q^5_{12,17}(\underset{\sim}{\Omega}\underset{\sim}{F})^2+q^3_{7,12}(\underset{\sim}{\Omega}\underset{\sim}{F})(\underset{\sim}{\Omega}) + \tfrac{1}{2}q^1_{27}(\underset{\sim}{\Omega})^2+q_{58}\mathrm{tr}\ \underset{\sim}{\Omega}^2(\underset{\sim}{F}+\underset{\sim}{I})^2-10q^2_{55}(\underset{\sim}{\Omega}) \\
&- 2q^{20}_{59,75}(\underset{\sim}{\Omega}\underset{\sim}{F})-4q_{58}(\underset{\sim}{\Omega})(\underset{\sim}{F})-4q_{58}(\underset{\sim}{\Omega}\underset{\sim}{F})(\underset{\sim}{F})-4q^5_{17,25}(\underset{\sim}{\Omega}\underset{\sim}{F}^2)] \\
&+ m[(3/2)pq^2_{55}+4q^2_{55}(\underset{\sim}{F})+\tfrac{1}{2}q_{58}(\underset{\sim}{F})^2+\tfrac{1}{2}q^5_{17,25}(\underset{\sim}{F}^2)-q^5_{12,17}(\underset{\sim}{\Omega}\underset{\sim}{F})(\underset{\sim}{F}) \\
&- 2q_{58}(\underset{\sim}{\Omega}\underset{\sim}{F}^2) - q^3_{7,12}(\underset{\sim}{\Omega})(\underset{\sim}{F}) - (3p^3+q^7_{32,32})(\underset{\sim}{\Omega}\underset{\sim}{F}) - q_{11}q^1_{1,16}(\underset{\sim}{\Omega})] + \tfrac{1}{2}m^2 \\
&[\tfrac{1}{2}q^5_{12,17}(\underset{\sim}{F})^2 + q_{58}(\underset{\sim}{F}^2) + (3p^3+q^7_{22,16})(\underset{\sim}{F}) + \tfrac{1}{2}pq_{11}q^1_{1,16}]\}\ E^2G_{mp}(\theta,\omega^2) \\
&+ \{q^5_{17,25}(\underset{\sim}{\Omega}\underset{\sim}{F}^2) + q_{58}(\underset{\sim}{\Omega}\underset{\sim}{F})(\underset{\sim}{F}) + q^{11}_{32,40}(\underset{\sim}{\Omega}\underset{\sim}{F}) + q_{58}(\underset{\sim}{\Omega})(\underset{\sim}{F}) + 3q^2_{55}(\underset{\sim}{\Omega}) \\
&- \tfrac{1}{2}q^5_{12,17}(\underset{\sim}{\Omega}\underset{\sim}{F})^2 - q^3_{7,12}(\underset{\sim}{\Omega}\underset{\sim}{F})(\underset{\sim}{\Omega}) - \tfrac{1}{2}\ q^1_{27}(\underset{\sim}{\Omega})^2 - q_{58}\mathrm{tr}\underset{\sim}{\Omega}^2(\underset{\sim}{F}+\underset{\sim}{I})^2 + (m/2) \\
&[q^5_{12,17}(\underset{\sim}{\Omega}\underset{\sim}{F})(\underset{\sim}{F}) + 2q_{58}(\underset{\sim}{\Omega}\underset{\sim}{F}^2) + q^3_{7,12}(\underset{\sim}{\Omega})(\underset{\sim}{F}) + (3p^3+q^7_{32,32})(\underset{\sim}{\Omega}\underset{\sim}{F}) \\
&+ q_{11}q^1_{1,16}(\underset{\sim}{\Omega})]\}\ E^3G_{mp}(\theta,\omega^2) + \tfrac{1}{2}\ \{\tfrac{1}{2}q^5_{12,17}(\underset{\sim}{\Omega}\underset{\sim}{F})^2 + q^3_{7,12}(\underset{\sim}{\Omega}\underset{\sim}{F})(\underset{\sim}{\Omega}) \\
&+ \tfrac{1}{2}\ q^1_{27}(\underset{\sim}{\Omega})^2 + q_{58}\mathrm{tr}\underset{\sim}{\Omega}^2(\underset{\sim}{F}+\underset{\sim}{I})^2\}\ E^4G_{mp}(\theta,\omega^2),
\end{aligned}$$

where $q_{ij} = ip+j$ and $q^i_{jk} = ip^2+jp+k$. Now substituting (2.11), (2.12) and (2.13) in (2.4), we get the following result:

<u>Theorem 1</u>. Let $m\underset{\sim}{S}_1$ and $n\underset{\sim}{S}_2$ be independently distributed $w(m,p,\underset{\sim}{B}^{-1},\underset{\sim}{\Omega})$, $w(n,p,\underset{\sim}{A}^{-1})$ respectively, and let $|\mathrm{Ch}_i F| < 1$ $i = 1,\ldots,p$, where $\underset{\sim}{B}^{-1}\underset{\sim}{A} = \underset{\sim}{I}+\underset{\sim}{F}$, then an asymptotic expansion for the c.d.f. of tr $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ is given by

$$(2.14)\ G(\theta) = \Pr\{\mathrm{tr}\ \underset{\sim}{A}\underset{\sim}{S}_1 \le \theta\} + R_1/n + [(4/3)R_2 + \tfrac{1}{2}R_3]/n^2 + O(n^{-3}),$$

where R_1, R_2 and R_3 are defined in (2.11), (2.12) and (2.13) respectively. Now $\mathrm{tr}\, \underset{\sim}{A}\underset{\sim}{S}_1 = \sum_{i=1}^{p} \lambda_i\, Z_i^2(\omega_i^2)$, where $Z_i^2(i=1,\ldots,p)$ are independently distributed as non-central chi-squares with m degrees of freedom each, $\sum_{\lambda=1}^{p} \omega_i^2 = \omega^2$ and λ_i, $i = 1,\ldots,p$, are the characteristic roots of $\underset{\sim}{\Sigma}_1 \underset{\sim}{\Sigma}_2^{-1} = \underset{\sim}{I}+\underset{\sim}{F}$.

Special case. If $\lambda_1 = \lambda_2 =\ldots=\lambda_p = \lambda$, then $\mathrm{tr}\, \underset{\sim}{A}\underset{\sim}{S}_1 = \lambda\, \mathrm{tr}\, \underset{\sim}{X}'\underset{\sim}{\Sigma}_1^{-1} \underset{\sim}{X}$, which is distributed as a non-central chi-square with mp degrees of freedom and non-centrality ω^2.

3. Further improvement. Noting that $E^r G_{mp}(\theta,\omega^2) = G_{mp+2r}(\theta,\omega^2)$, we can calculate the approximate value of $G(\theta)$ by theorem 1 of previous section. But, the approximation although reasonably adequate numerically, could be further improved without increasing the number of terms considerably. In order to do this, notice $\underset{\sim}{F}$ is small, so the terms which really contribute in (2.11), (2.12) and (2.13) are the terms independent of $\underset{\sim}{F}$ or involving the first power of $\underset{\sim}{F}$. So, we use all terms in $\underset{\sim}{X}^3$ and $\underset{\sim}{X}^4$ which are independent of $\underset{\sim}{F}$ and of the first power of $\underset{\sim}{F}$ in $\underset{\sim}{X}^3$. After tedious calculation, the extra terms in $\sum \sigma_{rs}\sigma_{ur}\partial_{rs}\partial_{tu}\, \Pr\{\mathrm{tr}\, \underset{\sim}{A}\underset{\sim}{S}_1 \le \theta\}$ included are

$$R_1 = \{-A_{11}+(3A_{11}-A_{12})E+(-3A_{11}+3A_{12}-A_{13})E^2+(A_{11}-3A_{12}+3A_{13}-A_{14})E^3$$

$$(3.1) \quad + (A_{12}-3A_{13}+3A_{14})E^4 + (A_{13}-3A_{14})E^5 + A_{14}E^6\}G_{mp}(\theta,\omega^2),$$

where $A_{11} = (1/8)[4q_{11}m + q_{14}^1 m^2 + pm^3](\underset{\sim}{F})$

$A_{12} = \frac{1}{2}[q_{23}(\underset{\sim}{\Omega}\underset{\sim}{F})+(\underset{\sim}{\Omega})(\underset{\sim}{F})] + (m/8)[q_{1,12}^1(\underset{\sim}{\Omega}\underset{\sim}{F})+2q_{11}(\underset{\sim}{\Omega})(\underset{\sim}{F})] + (m^2/8)[p(\underset{\sim}{\Omega}\underset{\sim}{F})+2(\underset{\sim}{\Omega})(\underset{\sim}{F})]$,

$A_{13} = (1/4)[q_{11}(\underset{\sim}{\Omega})(\underset{\sim}{\Omega}\underset{\sim}{F}) + 4(\underset{\sim}{\Omega}^2\underset{\sim}{F})] + (m/8)[(\underset{\sim}{\Omega}^2)(\underset{\sim}{F})+2(\underset{\sim}{\Omega})(\underset{\sim}{\Omega}\underset{\sim}{F})]$,

and $\quad A_{14} = (\underset{\sim}{\Omega}^2)(\underset{\sim}{\Omega}\underset{\sim}{F})/8$.

Further, the extra terms in $\sum \sigma_{rt}\sigma_{uw}\sigma_{\nu s}\partial_{rs}\partial_{tu}\partial_{w\nu}$ Pr{tr $\underset{\sim}{A}\underset{\sim}{S}_1 \leq \theta$} are given by

$$R_2' = -\{-A_{21}+(3A_{21}-A_{22})E+(-3A_{21}+3A_{22}-A_{23})E^2+(A_{21}-3A_{22}+3A_{23}-A_{24})E^3$$

$$(3.2)\quad + (A_{22}-3A_{23}+3A_{24})E^4+(A_{23}-3A_{24})E^5+A_{24}E^6\}G_{mp}(\theta,\omega^2),$$

where $A_{21} = (m/8)[9q^1_{34}(\underset{\sim}{F})+pq^1_{34}]+(m^2/16)[3(p^3+p^3_{18,14})(\underset{\sim}{F})+6pq_{11}]$
$+ (m^3/8)[3q^1_{11}(\underset{\sim}{F})+p]$,

$A_{22} = (3/8)[q^3_{15,36}(\underset{\sim}{\Omega}\underset{\sim}{F})+6q_{12}(\underset{\sim}{\Omega})(\underset{\sim}{F})+q^1_{34}(\underset{\sim}{\Omega})]+(3m/8)[\frac{1}{2}(p^3+q^3_{38,42})(\underset{\sim}{\Omega}\underset{\sim}{F})$
$+ q^1_{38}(\underset{\sim}{\Omega})(\underset{\sim}{F})+3q_{11}(\underset{\sim}{\Omega})]+(3m^2/8)[q^1_{13}(\underset{\sim}{\Omega}\underset{\sim}{F})+2q_{11}(\underset{\sim}{\Omega})(\underset{\sim}{F})+(\underset{\sim}{\Omega})]$,

$A_{23} = (3/8)[7q_{12}(\underset{\sim}{\Omega}^2\underset{\sim}{F})+q^1_{3,11}(\underset{\sim}{\Omega})(\underset{\sim}{\Omega}\underset{\sim}{F})+q_{12}(\underset{\sim}{\Omega}^2)+(\underset{\sim}{\Omega})^2]+(3m/8)$
$[2q_{11}(\underset{\sim}{\Omega})(\underset{\sim}{\Omega}\underset{\sim}{F})+\frac{1}{2}q_{12}(\underset{\sim}{\Omega}^2)(\underset{\sim}{F})+\frac{1}{2}(\underset{\sim}{\Omega})^2(\underset{\sim}{F})+3(\underset{\sim}{\Omega}^2\underset{\sim}{F})+(\underset{\sim}{\Omega}^2)]$,

and

$A_{24} = (1/16)[3q_{12}(\underset{\sim}{\Omega}^2)(\underset{\sim}{\Omega}\underset{\sim}{F})+3(\underset{\sim}{\Omega})^2(\underset{\sim}{\Omega}\underset{\sim}{F})+6(\underset{\sim}{\Omega}^3\underset{\sim}{F})+2(\underset{\sim}{\Omega}^3)]$.

Those terms in $\sum \sigma_{st}\sigma_{ur}\sigma_{wx}\sigma_{\nu y}\partial_{rs}\partial_{tu}\partial_{w\nu}\partial_{xy}$ Pr{tr $\underset{\sim}{A}\underset{\sim}{S}_1 \leq \theta$} are given by

$$R_3' = \{-A_{31}+A_{35}+(3A_{31}-A_{32}-4A_{35}+A_{36})E+(-3A_{31}+3A_{32}-A_{33}+6A_{35}-4A_{36}$$

$$(3.3)\quad + A_{37})E^2+(A_{31}-3A_{32}+3A_{33}-A_{34}-4A_{35}+6A_{36}-4A_{37}+A_{38})E^3+(A_{32}-3A_{33}$$

$$+ 3A_{34}+A_{35}-4A_{36}+6A_{37}-4A_{38}+A_{39})E^4+(A_{33}-3A_{34}+A_{36}-4A_{37}+$$

$$6A_{38}-4A_{39})E^5+(A_{34}+A_{37}-4A_{38}+6A_{39})E^6+(A_{38}-4A_{39})E^7+A_{39}E^8\}G_{mp}(\theta,\omega^2),$$

where $A_{31} = m[6q^2_{55}(\underset{\sim}{F})+pq^2_{55}] + (m^2/4)[(11p^3+5q^5_{20,16})(\underset{\sim}{F})+pq_{11}q^1_{1,16}] +$
$(m^3/8)[(p^4+2p^3+q^{23}_{22,24})(\underset{\sim}{F})+2pq^1_{14}]$,

$A_{32} = q^{18}_{61,90}(\underset{\sim}{\Omega}\underset{\sim}{F})+q_{18,29}(\underset{\sim}{\Omega})(\underset{\sim}{F})+3q^2_{55}(\underset{\sim}{\Omega})+(m/4)[2q^9_{29,40}(\underset{\sim}{\Omega})(\underset{\sim}{F}) +$
$(15p^3+q^{33}_{212,232})(\underset{\sim}{\Omega}\underset{\sim}{F})+3q_{11}q^1_{1,16}(\underset{\sim}{\Omega})] + (m^2/8)[(p^4+2p^3+q^{27}_{26,72})(\underset{\sim}{\Omega}\underset{\sim}{F})$
$+ 2(p^3+q^2_{21,20})(\underset{\sim}{\Omega})(\underset{\sim}{F}) + 6q^1_{14}(\underset{\sim}{\Omega})]$,

$A_{33} = \frac{1}{2}[q^{11}_{25,50}(\underset{\sim}{\Omega})(\underset{\sim}{\Omega}\underset{\sim}{F})+10q_{58}(\underset{\sim}{\Omega}^2\underset{\sim}{F})+q^1_{27}(\underset{\sim}{\Omega})^2+2q_{58}(\underset{\sim}{\Omega}^2)] + (m/8)[6q_{23}(\underset{\sim}{\Omega}^2)(\underset{\sim}{F})$
$+ q^1_{27}(\underset{\sim}{\Omega})^2(\underset{\sim}{F})+2(p^3+q^2_{23,22})(\underset{\sim}{\Omega})(\underset{\sim}{\Omega}\underset{\sim}{F}) + 4q^1_{1,18}(\underset{\sim}{\Omega}^2\underset{\sim}{F})+2q^1_{1,12}(\underset{\sim}{\Omega}^2)+4q_{11}(\underset{\sim}{\Omega})^2]$.

$A_{34} = (1/4)[3q_{23}(\Omega^2)(\Omega F) + \frac{1}{2} q^1_{27}(\Omega)^2(\Omega F)+2q_{11}(\Omega)(\Omega^2 F)+12(\Omega^3 F)$
$\quad + q_{11}(\Omega^2)(\Omega)+4(\Omega^3)]$

$A_{35} = (1/4)mpq^2_{55} + (m^2/16)pq_{11}q^1_{1,20} + (m^3/8)pq^1_{14} + m^4p^2/16$

$A_{36} = q^2_{55}(\Omega)+(m/4)q_{11}q^1_{1,20}(\Omega)+(m^2/2)q^1_{14}(\Omega)+(pm^3/4)(\Omega),$

$A_{37} = (1/4)[2q_{58}(\Omega^2) + q^1_{27}(\Omega)^2]+(m/8)[4q_{11}(\Omega)^2+q^1_{1,20}(\Omega^2)]+(m^2/8)[p(\Omega^2)+2(\Omega)^2],$

$A_{38} = (1/4)[(p+1)(\Omega^2)(\Omega)+4(\Omega^3)+m(\Omega^2)(\Omega)]$, and $A_{39} = (\Omega^2)^2/16$

Then we have the following theorem:

Theorem 2. Under the same assumptions as in theorem 1, an asymptotic expansion for the c.d.f. of tr $S_1S_2^{-1}$ is given by

(3.4) $G(\theta) = \Pr\{\mathrm{tr}\, AS_1 \leq \theta\} + (R_1+R_1')/n + [(3/4)(R_2+R_2') + \frac{1}{2}(R_3+R_3')]/n^2+O(n^{-3})$

where R_1', R_2' and R_3' are defined in (3.1), (3.2) and (3.3) respectively.

4. An asymptotic expansion for c.d.f. of Hotelling's trace in the canonical correlation case. Now let us consider Ω in the formula (3.4) for the c.d.f. of Hotelling's trace to be random, i.e. let $\Omega = \Sigma_1^{-1/2} MYY'M'\Sigma_1^{-1/2}$ where YY' has a central Wishart distribution $W(n+m, m, \Sigma_3)$ given by

(4.1) $\{\Gamma_m[\frac{1}{2}(m+n)]|2\Sigma_3|^{\frac{1}{2}(m+n)}\}^{-1}|YY'|^{\frac{1}{2}(n-1)} e^{-\frac{1}{2}\mathrm{tr}\Sigma_3^{-1}YY'}$.

Now multiply (3.4) by (4.1) and integrate with respect to YY' over $YY' > 0$. We can obtain an asymptotic expansion for the c.d.f. of Hotelling's trace in the canonical correlation case. We have to consider the integration term by term. But the term $\Pr\{\mathrm{tr}\, AS_1 \leq \theta\}$ cannot be easily expressed in terms of Ω. However, in some special cases it is easy to do so, for example, when $\lambda_1 = \lambda_2=\ldots=\lambda_p$. In this case, we write

(4.2) $\Pr\{\mathrm{tr}\, AS_1 \leq \theta\} = \exp(-\mathrm{tr}\Omega/2) \sum_{i=0}^{\infty} k_i(\mathrm{tr}\Omega)^i$

where k_i can be easily worked out from (2.2). Now multiply (4.2) by (4.1) and integrate over $\underset{\sim}{S} = \underset{\sim}{Y}\underset{\sim}{Y}' > 0$, we get

$$(4.3) \qquad \sum_{i=0}^{\infty} \frac{k_i}{\Gamma_m(\frac{1}{2}(m+n))\ 2^{m(m+n)/2}} \int_{T>0} e^{-\frac{1}{2}\,tr(\underset{\sim}{I}+\underset{\sim}{\Omega})\underset{\sim}{T}} (tr\underset{\sim}{\Omega}\underset{\sim}{T})^i |\underset{\sim}{T}|^{\frac{1}{2}(n-1)} d\underset{\sim}{T},$$

where $\underset{\sim}{\Omega}_1 = \underset{\sim}{\Sigma}_3^{1/2} \underset{\sim}{M}'\underset{\sim}{\Sigma}_1^{-1} \underset{\sim}{M}\underset{\sim}{\Sigma}_3^{1/2}$. The characteristic roots of $(\underset{\sim}{I}+\underset{\sim}{\Omega}_1)^{-1}\underset{\sim}{\Omega}_1$ are the canonical correlations ρ_i^2's. See [3], [13].

Now let us consider the second and third terms in (3.4) i.e. $(R_1+R_1')/n + [(4/3)(R_2+R_2') + \frac{1}{2}(R_3+R_3')]/n^2$
After multiplying these by (4.1) and integrating over $\underset{\sim}{Y}\underset{\sim}{Y}' > 0$, we get the following result:

$$(CR_1+CR_1')/n + [(4/3)(CR_2+CR_2') + \frac{1}{2}(CR_3+CR_3')]/n^2,$$

where CR_1, CR_1', CR_2, CR_2', CR_3, and CR_3' are given in the appendix.

However, for integrating out $\underset{\sim}{Y}\underset{\sim}{Y}'$, the following lemmas were useful:

Lemma 4.1

Let $\underset{\sim}{F}$ be an mxm arbitrary symmetric matrix and k a positive integer, then

$$\int_{\underset{\sim}{S}>0} e^{-tr\,\underset{\sim}{S}} [tr\ \underset{\sim}{S}\underset{\sim}{F}]^k |\underset{\sim}{S}|^{t-\frac{1}{2}(m+1)}\, d\underset{\sim}{S}$$

$$= k!\Gamma_m(t)\{\frac{t}{k}(\underset{\sim}{F}^k) + \frac{t^2}{2!}\Big[\sum_{\substack{j_1+j_2=k\\ j_1,j_2\neq 0}} \frac{(\underset{\sim}{F}^{j_1})(\underset{\sim}{F}^{j_2})}{j_1 j_2}\Big]+\dots+$$

$$\frac{t^i}{i!}\Big[\sum_{\substack{j_1+\dots+j_i=k\\ j_1,\dots,j_i\neq 0}} \frac{(\underset{\sim}{F}^{j_1})\dots(\underset{\sim}{F}^{j_i})}{j_1 j_2,\dots,j_i}\Big]+\dots+\frac{t^k}{k!}(\underset{\sim}{F})^k$$

where $(\underset{\sim}{F}^{j_i}) = tr\ \underset{\sim}{F}^{j_i}$.

Proof. We can take q small enough, such that $\underset{\sim}{I}-q\underset{\sim}{F} > 0$ by [3],

$$\int_{\underset{\sim}{S}>\underset{\sim}{0}} e^{tr-(\underset{\sim}{I}-q\underset{\sim}{F})\underset{\sim}{S}} |\underset{\sim}{S}|^{t-\frac{1}{2}(m+1)} d\underset{\sim}{S} = \Gamma_m(t)|\underset{\sim}{I}-q\underset{\sim}{F}|^{-t}.$$ Now,

$$\text{Left hand side} = \int_{\underset{\sim}{S}>\underset{\sim}{0}} e^{-tr\underset{\sim}{S}}[1 + (\underset{\sim}{S}\underset{\sim}{F})q + \frac{(\underset{\sim}{S}\underset{\sim}{F})^2}{2!} q^2+\dots] \quad |\underset{\sim}{S}|^{t-\frac{1}{2}(m+1)} d\underset{\sim}{S}$$

$$\text{Right hand side} = \Gamma_m(t)\, e^{-t \log|\underset{\sim}{I}-q\underset{\sim}{F}|} = \Gamma_m(t)\, e^{t(\underset{\sim}{F})q + \frac{t(\underset{\sim}{F}^2)}{2} q^2+\dots}$$

$$= \Gamma_m(t)\{1 + t(\underset{\sim}{F})q + [\tfrac{t}{2}(\underset{\sim}{F}^2) + \tfrac{t^2}{2}(\underset{\sim}{F})^2]q^2+\dots\}.$$

Further, compare the coefficient of q^k. We get the lemma.

Lemma 4.2

Let $\underset{\sim}{L}$ be an mxm symmetric matrix and $\underset{\sim}{H}$ be an mxm p.d. symmetric matrix, then

$$\int_{\underset{\sim}{S}>\underset{\sim}{0}} e^{-tr\, \underset{\sim}{H}\underset{\sim}{S}} [tr\, \underset{\sim}{S}]^k (tr\, \underset{\sim}{S}\underset{\sim}{L}) |\underset{\sim}{S}|^{t-\frac{1}{2}(m+1)} d\underset{\sim}{S}$$

$$= k!\Gamma_m(t)|\underset{\sim}{H}|^{-t}\{t(\underset{\sim}{H}^{-k-1}\underset{\sim}{L}) + \frac{t^2}{2!}\Big[\sum_{\substack{j_1+j_2=k+1\\ j_1,j_2\neq 0}} \frac{j_1(\underset{\sim}{H}^{-j_1}\underset{\sim}{L})(\underset{\sim}{H}^{-j_2}) + j_2(\underset{\sim}{H}^{-j_2}\underset{\sim}{L})(\underset{\sim}{H}^{-j_1})}{j_1 j_2}\Big]$$

$$+\dots+ \frac{t^i}{i!}\Big[\sum_{\substack{j_1+\dots+j_i=k+1\\ j_1,\dots,j_i\neq 0}} \frac{\sum^1 j_1(\underset{\sim}{H}^{-j_1}\underset{\sim}{L})(\underset{\sim}{H}^{-j_2})(\underset{\sim}{H}^{-j_3})\dots(\underset{\sim}{H}^{-j_i})}{j_1\cdots j_i}\Big]$$

$$+\dots+ \frac{t^{k+1}}{(k)!}(\underset{\sim}{H}^{-1}\underset{\sim}{L})(\underset{\sim}{H}^{-1})^k\},$$

where $\sum^1 j_1(\underset{\sim}{H}^{-j_1}\underset{\sim}{L})(\underset{\sim}{H}^{-j_2})\dots(\underset{\sim}{H}^{-j_i}) = j_1(\underset{\sim}{H}^{-j_1}\underset{\sim}{L})(\underset{\sim}{H}^{-j_2})\dots$

$$\dots(\underset{\sim}{H}^{-j_i}) + j_2(\underset{\sim}{H}^{-j_2}\underset{\sim}{L})(\underset{\sim}{H}^{-j_1})(\underset{\sim}{H}^{-j_3})\dots(\underset{\sim}{H}^{-j_i})$$

$$+\dots+ j_s(\underset{\sim}{H}^{-j_s}\underset{\sim}{L})(\underset{\sim}{H}^{-j_1})\dots(\underset{\sim}{H}^{-j_{s-1}})(\underset{\sim}{H}^{-j_{s+1}})\dots(\underset{\sim}{H}^{-j_i}) + \dots + j_i(\underset{\sim}{H}^{-j_i}\underset{\sim}{L})(\underset{\sim}{H}^{-j_1})\dots(\underset{\sim}{H}^{-j_{i-1}}).$$

Proof. Since H is p.d. symmetric, so $H^{-\frac{1}{2}} L H^{-\frac{1}{2}}$ is symmetric. Now

$$\int_{S>0} e^{-\mathrm{tr}\, HS}[\mathrm{tr}\, S\{qI+L\}]^{k+1}|S|^{t-\frac{1}{2}(m+1)}\, dS$$

$$= |H|^{-t}\int_{S>0} e^{-\mathrm{tr}\, S}[\mathrm{tr}\, H^{-\frac{1}{2}}SH^{-\frac{1}{2}}\{qI+L\}]^{k+1}|S|^{t-\frac{1}{2}(m+1)}\, dS$$

$$= (k+1)!\Gamma_m(t)|H|^{-t}\{\frac{t}{k+1}(U^{k+1}) + \frac{t^2}{2!}[\sum_{\substack{j_1+j_2=k+1\\ j_1,j_2\neq 0}} \frac{(U^{j_1})(U^{j_2})}{j_1 j_2}]$$

$$+\ldots+ \frac{t^{k+1}}{(k+1)!}(U)^{k+1}\},\quad \text{by Lemma 4.1}$$

where $U = H^{-1}\{qI+L\}$. Now compare the coefficient of q^k. We get lemma 4.2.

Lemma 4.3. Under the assumptions as in Lemma 4.2, we have

$$\frac{1}{k!}\int_{S>0} e^{-\mathrm{tr}\, HS}[\mathrm{tr}\, S]^k(\mathrm{tr}\, SL)^2|S|^{t-\frac{1}{2}(m+1)}\, dS$$

$$= 2\Gamma_m(t)|H|^{-t}\{\frac{t(k+1)}{2}(H^{-k-2}L^2)+ \frac{t^2}{2!}\sum_{\substack{j_1+j_2=k+2\\ j_1,j_2\neq 0}} \frac{1}{j_1 j_2}\cdot [\frac{j_1(j_1-1)}{2}(H^{-j_1}L^2)(H^{-j_2})$$

$$+ \frac{j_2(j_2-1)}{2}(H^{-j_2}L^2)(H^{-j_1}) + j_1 j_2(H^{-j_1}L)(H^{-j_2}L)]+\ldots+$$

$$\frac{t^i}{i!}\sum_{\substack{j_1+\ldots+j_i=k+2\\ j_1,j_2\ldots j_i\neq 0}} \frac{1}{j_1\cdots j_i}[\sum{}' \frac{j_1(j_1-1)}{2}(H^{-j_1}L^2)(H^{-j_2})\ldots(H^{-j_i})$$

$$+ \sum{}'' j_1 j_2(H^{-j_1}L)(H^{-j_2}L)(H^{-j_3})\ldots(H^{-j_i})]$$

$$+\ldots+ \frac{t^{k+2}}{(k+2)!}[\frac{(k+2)(k+1)}{2}(H^{-1}L)^2(H^{-1})^i]\}$$

where $\sum'$, $\sum''$ have similar definitions as in lemma 4.2.

Proof. Follows as in Lemma 4.2.

Using the above 3 lemmas, we can integrate most of the terms in (2.11), (2.12), (2.13), but we still need to evaluate

$$\text{(4.4)} \qquad \int_{\underset{\sim}{S}>\underset{\sim}{0}} e^{-\mathrm{tr}\,\underset{\sim}{H}\underset{\sim}{S}}[\mathrm{tr}\,\underset{\sim}{S}]^k(\mathrm{tr}\,\underset{\sim}{S}^i)|\underset{\sim}{S}|^{t-\frac{1}{2}(m+1)}d\underset{\sim}{S}$$

for $i = 2, 3$. This seems to be very difficult. But we can use zonal polynomials [8], [12] to evaluate (4.4) for small values of k. For example, since $\mathrm{tr}\,\underset{\sim}{S}^2 = C_{(2)}(\underset{\sim}{S}) - \frac{1}{2}C_{(1^2)}(\underset{\sim}{S}) = [\mathrm{tr}\,\underset{\sim}{S}]^2 - (3/2)C_{(1^2)}(\underset{\sim}{S})$, and $A_J = [\mathrm{tr}\,\underset{\sim}{S}]^J = \sum_\kappa C_\kappa(\underset{\sim}{S})$,
then $A_J[\mathrm{tr}\,\underset{\sim}{S}^2] = A_{J+2} - (3/2)\,C_{(1^2)}(\underset{\sim}{S})[\sum_\kappa C_\kappa(\underset{\sim}{S})]$.

Now $C_\kappa(S)C_\tau(S)$ can be expressed as $\sum_\delta g^\delta_{\kappa,\tau}C_\delta(S)$, $\sum_{i=1}^m \delta_i = k+t$, and tabulations are available for the $g^\delta_{\kappa,\tau}$ coefficients for small values of k and t (See [11]).

Then (4.4) becomes a linear compound of integrals of the form

$$\int_{\underset{\sim}{S}>\underset{\sim}{0}} e^{-\mathrm{tr}\,\underset{\sim}{H}\underset{\sim}{S}}|\underset{\sim}{S}|^{t-\frac{1}{2}(m+1)}C_\kappa(\underset{\sim}{S})d\underset{\sim}{S}.$$

This integral is evaluated in [3]. Similarly, we can evaluate the following integral for small values of k:

$$\int_{\underset{\sim}{S}>\underset{\sim}{0}} e^{-\mathrm{tr}\,\underset{\sim}{H}\underset{\sim}{S}}(\mathrm{tr}\,\underset{\sim}{S})^k(\mathrm{tr}\,\underset{\sim}{S}^3)|\underset{\sim}{S}|^{t-\frac{1}{2}(m+1)}C_\kappa(\underset{\sim}{S})d\underset{\sim}{S}.$$

5. Numerical results. By using theorem 2 in Section 3, we have calculated the powers of the T-test under violations of hypothesis (i) for a) p=2, m=3, 7, n=83, $\alpha = .05$ and b) p=3, m=4, n=84, $\alpha = .05$, and hypothesis (ii) for p=2, m=3, n=63, 83 and $\alpha = .05$, while deviation matrix, F, has equal deviation parameters, i.e. $f_1 = f_2$, where f_i's are the characteristic roots of $\underset{\sim}{F}$. The results are presented in Tables 1 to 4. The exact powers given there are taken from Pillai and Jaychandran [14] for p=2 and Pillai and Sudjana [18] for p=3.

From the power tabulations the following appear to emerge:

1. For hypothesis (i) i.e. $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$, the power tabulations in Table 2 show considerable change even for small deviations of $\underset{\sim}{\Omega}$ from $\underset{\sim}{0}$ and the difference in the powers remains approximately of the same magnitude irrespective of the values of $(\underset{\sim}{F})$ for a given (m,n). The changes of powers become larger for larger deviations of $\underset{\sim}{\Omega}$. This is probably indicative that the test is not robust against non-normality.

2. For hypothesis (ii) i.e. MANOVA, Table 2 shows modest changes of powers for small values of the deviation parameter $(\underset{\sim}{F})$ but changes become pronounced as $(\underset{\sim}{F})$ deviates more from 0. However the tabulations are carried out under the assumption that $|Ch_i\underset{\sim}{F}| < 1$, $i = 1,\ldots,p$.

3. For hypothesis (iii) again the changes in Table 4 show only modest changes for small values of $(\underset{\sim}{F})$ and small values of p^2 indicating probably that slight non-normality does not affect the test seriously in the neighborhood of the hypothesis of independence. However for larger values of $(\underset{\sim}{F})$ the changes in powers become pronounced.

The findings for hypotheses (i) and (ii) are in agreement with those of Pillai and Sudjana [17] computed based on Pillai [13] where the latter introduced his concept of the parameter matrices being partially random (denoted "random"). The inference regarding hypothesis (iii) reflects the findings of Gayen [5] in the bivariate case.

Table 1

Powers of T test for comparison of accuracy of the expansion (3.4) for α=.05 when $f_1=f_2$

p=2, m=3, n=83					
ω_2	($\underset{\sim}{F}$)	0(1)	0(1/n)	0($1/n^2$)	Exact
.00001	.05	.04279	.05489	.0556	.055577 (ω_2=0)
.001	.1	.04799	.06091	.0617	.050036 ($\underset{\sim}{F}=\underset{\sim}{0}$)
.005	.005	.03856	.04999	.05073	
.01	.3	.07198	.08842	.0894	.050363 ($\underset{\sim}{F}=\underset{\sim}{0}$)
.05	.01	.04048	.05223	.05296	
.1	.5	.10517	.12604	.127	.053670 ($\underset{\sim}{F}=\underset{\sim}{0}$)
.5	.01	.05614	.07007	.07069	.069304 ($\underset{\sim}{F}=\underset{\sim}{0}$)
1	.00001	.07434	.09039	.09088	.090873 ($\underset{\sim}{F}=\underset{\sim}{0}$)
2	.001	.12018	.13998	.1401	

p=3, m=4, n=84							
ω_1	ω_2	ω_3	($\underset{\sim}{F}$)	0(1)	0(1/n)	0($1/n^2$)	Exact
0	0	.0001	.5	.07942	.1090	.113	.1103 (ω_3=0)
0	0	.001	.00001	.031072	.047492	.050025	.071409 ($\underset{\sim}{F}=\underset{\sim}{0}$)
0	0	.825	.00001	.04760	.06895	.07147	
.125	.250	.5	.005	.04922	.07099	.07352	
.275	.275	.275	.01	.04859	.07018	.07273	
0	0	1	.5	.11817	.1552	.159	
0	1	1	.1	.09247	.1233	.117	
0	0	2	.05	.08491	.1145	.126	
1	1	1	.0001	.10912	.1427	.145	.143 ($\underset{\sim}{F}=\underset{\sim}{0}$)

Table 2

Powers of T test under violations for α=.05 when $f_1=f_2$
p=2, m=3, n=83, ω_1=0

(F) \ ω_2	.001	.005	.01	.05
.001	.05014 (.050108) [.050036]	.05029	.05047 [.0503626]	.05194
.005	.05058	.05073	.05091	.05239
.01	.05114 (.051094)	.05129	.05147	.05296
.05	.0557 (.055577)	.0559	.0561	.0576
.1	.0617 (.061438)	.0619	.0621	.0638
.3	.0889 (.0885)	.0891	.0894	.0915
.5	.121 (.1170)	.121	.121	.123

(F) \ ω_2	.1	.3	.5	1	2
.001	.05379 [.0536702]	.06143	.06945 [.0693042]	.09104 [.0908734]	.1401 [.13994]
.005	.05425	.06194	.07000	.0917	.141
.01	.05483	.06258	.07069	.0926	.142
.05	.0596	.0678	.0764	.0994	.151
.1	.0659	.0747	.0839	.108	.163
.3	.0943	.105	.117	.147	.211
.5	.127	.140	.154	.189	.262

Entries ( ), [ ] are for ω_2=0 and (F)=0 respectively.

Table 2
(continued)
p=2, m=7, n=83

$(F) \backslash \omega_2$	.001	.005	.01	.05
.001	.05018 (.050149)	.05026	.05037 [.0502135]	.05123
.005	.05080	.05089	.05099	.05187
.01	.05159 (.051508)	.05167	.05178	.05266
.05	.0581	.0582	.0584	.0593
.1	.0670 (.066400)	.0671	.0673	.0683
.3	.1099 (.108)	.1101	.1103	.1118
.5	.164 (.154)	.164	.164	.166

$(F) \backslash \omega_2$	.1	.3	.5	1	2
.001	.05232 [.0521565]	.05679	.06146 [.0612521]	.07391 [.0736673]	.1022 [.10186]
.005	.05291	.05748	.06218	.0747	.1032
.01	.05378	.0583	.0631	.0758	.1045
.05	.0605	.0655	.0707	.0844	.1152
.1	.0697	.0752	.0809	.0959	.129
.3	.1137	.1215	.129	.150	.194
.5	.169	.179	.189	.215	.269

Table 2
(continued)
p=3, m=4, n=84

($\underset{\sim}{F}$)	$\omega_1 = \omega_2 = 0$				
	ω_3=.0001	ω_3=.001	ω_3=.825	ω_3=1	ω_3=2
.0001	.05001	.05003	.07148 [.071409]	.07651 [.076422]	.1082 [.1081]
.001	.05010	.05012	.07159	.07663	.1084
.005	.05049	.05051	.07210	.07716	.109
.01	.05099	.05101	.07274	.07783	.110
.05	.0550	.0551	.0779	.0833	.117
.1	.0604	.0604	.0848	.0904	.126
.3	.0845	.0846	.115	.122	.165
.5	.113	.113	.151	.159	.208

($\underset{\sim}{F}$)	ω_1=.125 ω_2=.25, ω_3=.5	ω_1=.275 ω_2=.275, ω_3=.275	ω_1=0 ω_2=1, ω_3=1	ω_1=1 ω_2=1, ω_3=1
.0001	.07289 [.072810]	.07147 [.071393]	.1082 [.1080]	.145 [.143]
.001	.07300	.07158	.1083	.145
.005	.07352	.07209	.1090	.146
.01	.07416	.07273	.1099	.147
.05	.0795	.0779	.117	.155
.1	.0864	.0848	.126	.166
.3	.117	.115	.165	.211
.5	.153	.151	.208	.261

Entries [ ] are for ($\underset{\sim}{F}$) = 0

Table 3

Powers of T test for comparison of accuracy of the expansion (4.3) for $p=2$, $m=3$, $\alpha=.05$ when $f_1=f_2$ and $\rho_1^2=\rho_2^2$

$2\rho_1^2$	(F)	0(1)	0(1/n)	$0(1/n^2)$	Exact
			n = 63		
.0025	.000001	.03949	.05477	.05602	.0560038 ($\underset{\sim}{F}=\underset{\sim}{0}$)
.003	.01	.04152	.05720	.05845	
.01	.5	.1331	.1630	.1640	
.05	.005	.1863	.217	.216	
.1	.001	.404	.434	.432	
			n = 83		
.0025	.5	.1124	.1334	.1343	
.003	.000001	.04638	.05900	.05969	.0596840 ($\underset{\sim}{F}=\underset{\sim}{0}$)
.01	.01	.07033	.08591	.08646	
.05	.001	.258	.282	.281	
.1	.005	.547	.566	.564	

Table 4

Powers of T test under violations for $p=2$, $m=3$, $\alpha=.05$ when $f_1=f_2$ and $\rho_1^2=\rho_2^2$

($\underset{\sim}{F}$) $2\rho_1^2$	.0025	.003	.01	.05	.1
		n = 63			
.000001	.05602 [.0560038]	.05726 [.0572373]	.07569 [.0756270]	.2146	.431 [.432]
.001	.05614	.05738	.07583	.215	.432
.005	.05661	.05785	.07641	.216	.433
.01	.05720	.05845	.07714	.217	.435
.05	.06203	.06336	.08308	.229	.450
.1	.06839	.06981	.09085	.243	.467
.5	.1305	.1327	.1640	.358	.588
		n = 83			
.000001	.05803 [.0580230]	.05969 [.0596840]	.08487 [.0848456]	.281	.562
.001	.05815	.05982	.08502	.281	.563
.005	.05864	.06031	.08566	.282	.564
.01	.05925	.06094	.08646	.284	.566
.05	.06426	.06605	.09297	.297	.581
.1	.07083	.07275	.1014	.313	.598
.5	.1343	.1372	.1793	.437	.708

The entries [ ] are for (F) = 0

References

[1] Chattopadhyay, A.K. and Pillai, K.C.S. (1971). Asymptotic formulae for the distribution of some criteria for test of equality of covariance matrices. Journal of Multivariate Analysis, Vol. 1, 215-231.

[2] Chattopadhyay, A.K. (1972). Non-null distribution of Hotelling's generalized T_0^2 statistic. ARL 72-0033, Wright Patterson Air Force Base, Ohio.

[3] Constantine, A.G. (1963). Some non-central distribution problems in multivariate analysis. Ann. Math. Statist. 34, 1270-1285.

[4] Constantine, A.G. (1966). The distribution of Hotelling's generalized T_0^2. Ann. Math. Statist. 37, 215-225.

[5] Gayen, A. K. (1951). The frequency distribution of the product-moment correlation coefficient in random samples of any size drawn from non-normal universes. Biometrika, 38, 219-247.

[6] Ito, K. (1956). Asymptotic formulae for the distribution of Hotelling's generalized T_0^2. Ann. Math. Statist. 27, 1091-1105.

[7] Ito, K. (1960). Asymptotic formulae for the distribution of Hotelling's generalized T_0^2 statistic II. Ann. Math. Statist. 31, 1148-1153.

[8] James, A.T. (1964). Distribution of matrix variates and latent roots derived from normal samples. Ann. Math. Statist. 35, 475-501.

[9] James, G.S. (1951). The comparison of several groups of observations when the ratios of the population variances are unknown. Biometrika, 38, 324-329.

[10] James, G.S. (1954). Tests of linear hypothesis in the univariate and multivariate analysis when the ratios of the population variance are unknown. Biometrika, 41, 19-43.

[11] Khatri, C.G. and Pillai, K.C.S. (1968). On the non-central distribution of two test criteria in multivariate analysis of variance. Ann. Math. Statist. 39, 215-226.

[12] Parkhurst, A.M. and James, A.T. Selected tables in Mathematical Statistics Vol. 2. Edited by the Institute of Math. Stat.

[13] Pillai, K.C.S. (1975). The distribution of the characteristic roots of $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ under violations. Ann. Statistics 3, (to appear).

[14] Pillai, K.C.S. and Jayachandran, K. (1967). Power comparison of tests of two multivariate hypothesis based on four criteria. Biometrika, 54, 195-210.

[15] Pillai, K.C.S. and Nashat B. Saweris (1973). Asymptotic formulae for the distribution of Hotelling's trace for tests of equality of two covariance matrices. Mimeograph Series No. 343, Department of Statistics, Purdue University.

[16] Pillai, K.C.S. and Nashat B. Saweris (1974). Asymptotic formulae for the percentiles and c.d.f. of Hotelling's trace under violations. Mimeograph Series No. 382. Department of Statistics, Purdue University.

[17] Pillai, K.C.S. and Sudjana (1975). Exact Robustness Studies of tests of two multivariate hypotheses based on four criteria and their distribution problems under violations. Ann. Statistics, 3 (to appear).

[18] Pillai, K.C.S. and Sudjana (1974). On the distribution of Hotelling's trace and power comparison. Communications in Statistics, 3, 433-454.

[19] Siotani, M. (1971). An asymptotic expansion of the non-null distribution of Hotelling's generalized T_0^2-statistic. Ann. Math. Statist. 42, 560-571.

Appendix 1

All the following expressions are the average of all permutations of rs, tu, $w\nu$ and xy.

$$\sum \sigma_{rt}\sigma_{uw}\sigma_{\nu s}(\mathbf{G}|rs)(\mathbf{H}|tu|w\nu) = (p+2)(\mathbf{HG})/4 + (\mathbf{G})(\mathbf{H})/4$$

$$\sum \sigma_{rt}\sigma_{uw}\sigma_{\nu s}(\mathbf{G}|rs|\mathbf{H}|tu|w\nu) = (p+4)(\mathbf{GH})/8 + (p+2)(\mathbf{G})(\mathbf{H})/8$$

$$\sum \sigma_{rt}\sigma_{uw}\sigma_{\nu s}(\mathbf{G}|rs)(\mathbf{H}|tu)(\mathbf{L}|w\ \ ) = (\mathbf{GHL})$$

$$\sum \sigma_{st}\sigma_{ur}\sigma_{wx}\sigma_{\nu y}(\mathbf{G}|rs|tu|w\nu|xy) = (2p^2+5p+5)(\mathbf{G})/12$$

$$\sum \sigma_{st}\sigma_{ur}\sigma_{wx}\sigma_{\nu y}(\mathbf{G}|rs|\mathbf{H}|tu|w\nu|xy) = (2p+3)(\mathbf{G})(\mathbf{H})/12 + (2p+5)(\mathbf{GH})/12$$

$$\sum \sigma_{st}\sigma_{ur}\sigma_{wx}\sigma_{\nu y}(\mathbf{G}|rs|tu|\mathbf{H}|w\nu|xy) = (p^2+3p+5)(\mathbf{GH})/12 + (p+2)(\mathbf{G})(\mathbf{H})/12$$

$$\sum \sigma_{st}\sigma_{ur}\sigma_{wx}\sigma_{\nu y}(\mathbf{G}|rs)(\mathbf{H}|tu|w\nu|xy) = (2p+3)(\mathbf{GH})/6 + (\mathbf{G})(\mathbf{H})/6$$

$$\sum \sigma_{st}\sigma_{ur}\sigma_{wx}\sigma_{\nu y}(\mathbf{G}|rs|tu)(\mathbf{H}|w\nu|xy) = (p^2+2p+3)(\mathbf{G})(\mathbf{H})/12 + (p+2)(\mathbf{GH})/6$$

$$\sum \sigma_{st}\sigma_{ur}\sigma_{wx}\sigma_{\nu y}(\mathbf{G}|rs)(\mathbf{H}|tu)(\mathbf{L}|w\nu|xy) = (p+1)(\mathbf{GH})(\mathbf{L})/6 + 2(\mathbf{GHL})/3$$

$$\sum \sigma_{st}\sigma_{ur}\sigma_{wx}\sigma_{\nu y}(\mathbf{G}|rs)(\mathbf{H}|tu)(\mathbf{L}|w\nu)(\mathbf{M}|xy) = (1/3)[(\mathbf{GH})(\mathbf{LM})+2(\mathbf{GL})(\mathbf{HM})]$$

where $\mathbf{G}$, $\mathbf{H}$, $\mathbf{L}$ and $\mathbf{M}$ are pxp nonsingular matrices.

Appendix 2

$$
\begin{aligned}
CR_1 = {} & \{\tfrac{1}{2}m[\tfrac{1}{2}q_{23}(\underset{\sim}{F}^2) + \tfrac{1}{2}(\underset{\sim}{F})^2 + q_{11}(\underset{\sim}{F}) - \tfrac{1}{2}pq_{11}] + (m^2/4)[q_{11}(\underset{\sim}{F})^2 + (\underset{\sim}{F}^2) \\
& + q_{12}^1(\underset{\sim}{F})+p]\}D_{0000}^0(\underset{\sim}{I}) \\
& + \{\tfrac{1}{2}[q_{23}D_{1100}^1(\underset{\sim}{F}^2)+q_{23}D_{1100}^1(\underset{\sim}{F})+(\underset{\sim}{F})D_{1100}^1(\underset{\sim}{F})+(\underset{\sim}{F})D_{1100}^1(\underset{\sim}{I})] \\
& + (m/2)[q_{11}(\underset{\sim}{F})D_{1100}^1(\underset{\sim}{F}) + \tfrac{1}{2}q_{11}(\underset{\sim}{F})D_{1100}^1(\underset{\sim}{I}) + D_{1100}^1(\underset{\sim}{F}^2) + \tfrac{1}{2}q_{14}^1D_{1100}^1(\underset{\sim}{F}) \\
& + D_{1100}^1(\underset{\sim}{I}) - 3q_{11}(\underset{\sim}{F})D_{0000}^1(\underset{\sim}{I}) - q_{23}(\underset{\sim}{F}^2)D_{0000}^1(\underset{\sim}{I})-(\underset{\sim}{F})^2D_{0000}^1(\underset{\sim}{I})] \\
& - \tfrac{1}{2}m^2[q_{11}(\underset{\sim}{F})^2+q_{12}^1(\underset{\sim}{F}) + (\underset{\sim}{F}^2)+p]D_{0000}^1(\underset{\sim}{I})\} + (1/4)\{D_{2100}^2((\underset{\sim}{F}+\underset{\sim}{I})^2) \\
& - 4q_{23}D_{1100}^2(\underset{\sim}{F}^2)-4(\underset{\sim}{F})D_{1100}^2(\underset{\sim}{F}) - 2q_{57}D_{1100}^2(\underset{\sim}{F})+q_{11}D_{1200}^2(\underset{\sim}{F})-2q_{11}D_{1100}^2(\underset{\sim}{I}) \\
& + q_{11}D_{1111}^2(\underset{\sim}{F})-4(\underset{\sim}{F})D_{1100}^2(\underset{\sim}{I})+m[q_{23}(\underset{\sim}{F}^2)+4q_{11}(\underset{\sim}{F})+(\underset{\sim}{F})^2+pq_{11}]D_{0000}^2(\underset{\sim}{I}) \\
& - 2m[2q_{11}(\underset{\sim}{F})D_{1100}^2(\underset{\sim}{F})+q_{11}(\underset{\sim}{F})D_{1100}^2(\underset{\sim}{I})+2D_{1100}^2(\underset{\sim}{F}^2)+q_{14}^1D_{1100}^2(\underset{\sim}{F})+2D_{1100}^2(\underset{\sim}{F})] \\
& + m^2[q_{11}(\underset{\sim}{F})^2+(\underset{\sim}{F}^2)+q_{12}^1(\underset{\sim}{F})+p]D_{0000}^2(\underset{\sim}{I})\} + \tfrac{1}{2}\{q_{23}D_{1100}^3(\underset{\sim}{F}^2)+q_{34}D_{1100}^3(\underset{\sim}{F}) \\
& + q_{11}D_{1100}^3(\underset{\sim}{I})-q_{11}D_{1200}^3(\underset{\sim}{F})-q_{11}D_{1111}^3(\underset{\sim}{F})+(\underset{\sim}{F})D_{1100}^3(\underset{\sim}{F})+(\underset{\sim}{F})D_{1100}^3(\underset{\sim}{I}) \\
& - D_{2100}^3((\underset{\sim}{F}+\underset{\sim}{I})^2) + \tfrac{1}{2}m[2q_{11}(\underset{\sim}{F})D_{1100}^3(\underset{\sim}{F})+q_{14}^1D_{1100}^3(\underset{\sim}{F})+q_{11}(\underset{\sim}{F})D_{1100}^3(\underset{\sim}{I}) \\
& + 2D_{1100}^3(\underset{\sim}{F}^2) + 2D_{1100}^3(\underset{\sim}{I})] + (1/4)\{q_{11}D_{1200}^4(\underset{\sim}{F})+q_{11}D_{1111}^4(\underset{\sim}{F}) \\
& + D_{2100}^4((\underset{\sim}{F}+\underset{\sim}{I})^2)\},
\end{aligned}
$$

$$
\begin{aligned}
CR_2 = {} & -3\{(m/8)[q_{48}^1(\underset{\sim}{F}^2)+q_{12}(\underset{\sim}{F})^2+2q_{34}^1(\underset{\sim}{F})]+(m^2/16)[2q_{12}(\underset{\sim}{F}^2)+q_{36}^1(\underset{\sim}{F})^2 \\
& + (p^3+q_{12,8}^3)(\underset{\sim}{F})+4pq_{11}]\}D_{0000}^0(\underset{\sim}{I})-(3/4)\{q_{48}^1D_{1100}^1(\underset{\sim}{F}^2)+\tfrac{1}{2}q_{11,20}^3D_{1100}^1(\underset{\sim}{F}) \\
& + q_{12}(\underset{\sim}{F})D_{1100}^1(\underset{\sim}{F})+q_{12}(\underset{\sim}{F})D_{1100}^1(\underset{\sim}{I})+\tfrac{1}{2}q_{34}^1D_{1100}^1(\underset{\sim}{I})+m[2q_{11}D_{1100}^1(\underset{\sim}{I}) \\
& + (p^3+q_{16,16}^3)D_{1100}^1(\underset{\sim}{F})/4+q_{36}^1(\underset{\sim}{F})D_{1100}^1(\underset{\sim}{F})/2+q_{12}D_{1100}^1(\underset{\sim}{F}^2)+q_{38}^1(\underset{\sim}{F})D_{1100}^1(\underset{\sim}{I})/4] \\
& - m[q_{12}(\underset{\sim}{F})^2+5q_{34}^1(\underset{\sim}{F})/2-q_{48}^1(\underset{\sim}{F}^2)]D_{0000}^1(\underset{\sim}{I})-(m^2/2)[4pq_{11}+2q_{12}(\underset{\sim}{F}^2) \\
& + (p^3+q_{12,8}^3)(\underset{\sim}{F})+q_{36}^1(\underset{\sim}{F})^2]D_{0000}^1(\underset{\sim}{I}) - \tfrac{1}{2}\,pq_{34}^1D_{0000}^1(\underset{\sim}{I}) \\
& - (3/8)\{\tfrac{1}{2}q_{36}^1D_{1200}^2(\underset{\sim}{F})+\tfrac{1}{2}q_{38}^1D_{1111}^2(\underset{\sim}{F})-4q_{48}^1D_{1100}^2(\underset{\sim}{F}^2)-4q_{12}(\underset{\sim}{F})D_{1100}^2(\underset{\sim}{F}) \\
& - q_{25,44}^7D_{1100}^2(\underset{\sim}{F})-3q_{34}^1D_{1100}^2(\underset{\sim}{I})+D_{1200}^2(\underset{\sim}{I})-4q_{12}(\underset{\sim}{F})D_{1100}^2(\underset{\sim}{I})+q_{12}D_{2100}^2((\underset{\sim}{F}+\underset{\sim}{I})^2) \\
& + m[pq_{34}^1+q_{48}^1(\underset{\sim}{F}^2)+q_{12}(\underset{\sim}{F}^2)+3q_{34}^1(\underset{\sim}{F})]D_{0000}^2(\underset{\sim}{I})-m[8q_{11}D_{1100}^2(\underset{\sim}{I}) \\
& + (p^3+q_{16,16}^3)D_{1100}^2(\underset{\sim}{F})+4q_{12}D_{1100}^2(\underset{\sim}{F}^2)+2q_{36}^1(\underset{\sim}{F})D_{1100}^2(\underset{\sim}{F})+q_{38}^1(\underset{\sim}{F})D_{1100}^2(\underset{\sim}{I})]
\end{aligned}
$$

$$+ m^2[2pq_{11} + \tfrac{1}{2}q^1_{36}(\underset{\sim}{F})^2+q_{12}(\underset{\sim}{F}^2) + \tfrac{1}{2}(p^3+q^3_{12,8})(\underset{\sim}{F})]D^2_{0000}(\underset{\sim}{I})\}$$
$$- (3/4)\{q^1_{48}D^3_{1100}(\underset{\sim}{F}^2)+q^2_{7,12}D^3_{1100}(\underset{\sim}{F})-\tfrac{1}{2}q^1_{36}D^3_{1200}(\underset{\sim}{F})-D^3_{1200}(\underset{\sim}{I})$$
$$- \tfrac{1}{2}q^1_{38}D^3_{1111}(\underset{\sim}{F})+q^1_{34}D^3_{1100}(\underset{\sim}{I})+q_{12}(\underset{\sim}{F})D^3_{1100}(\underset{\sim}{F})+q_{12}(\underset{\sim}{F})D^3_{1100}(\underset{\sim}{I})-q_{12}D^3_{2100}((\underset{\sim}{F}+\underset{\sim}{I}))$$
$$+ (m/4)[2q^1_{36}(\underset{\sim}{F})D^3_{1100}(\underset{\sim}{F})+q^1_{38}(\underset{\sim}{F})D^3_{1100}(\underset{\sim}{I})+(p^3+q^3_{16,16})D^3_{1100}(\underset{\sim}{F})+4q_{12}D^3_{1100}(\underset{\sim}{F}^2)$$
$$+ 8q_{11}D^3_{1100}(\underset{\sim}{I})]\} - (3/16)\{q^1_{36}D^4_{1200}(\underset{\sim}{F})+2q_{12}D^4_{2100}((\underset{\sim}{F}+\underset{\sim}{I})^2)+q^1_{38}D^4_{1111}(\underset{\sim}{F})$$
$$+ D^4_{1200}(\underset{\sim}{I})\},$$

$$CR_3 = \{(m/2)[q_{58}(\underset{\sim}{F})^2+q^5_{17,25}(\underset{\sim}{F}^2)+6q^2_{55}(\underset{\sim}{F})+pq^2_{55}]+(m^2/4)[q^5_{12,17}(\underset{\sim}{F})^2$$
$$+ 2q_{58}(\underset{\sim}{F}^2)+2(3p^3+q^7_{22,16})(\underset{\sim}{F})+pq_{11}q^1_{1,16}]\}D^0_{0000}(\underset{\sim}{I})+\{q^5_{17,25}D^1_{1100}(\underset{\sim}{F}^2)$$
$$+ q_{58}(\underset{\sim}{F})D^1_{1100}(\underset{\sim}{F})+q_{58}(\underset{\sim}{F})D^1_{1100}(\underset{\sim}{I})+q^9_{27,35}D^1_{1100}(\underset{\sim}{F})+2q^2_{55}D^1_{1100}(\underset{\sim}{I})$$
$$+ m[q^{58}D^1_{1100}(\underset{\sim}{F}^2)+\tfrac{1}{2}q^5_{12,17}(\underset{\sim}{F})D^1_{1100}(\underset{\sim}{F})+\tfrac{1}{2}q^3_{7,12}(\underset{\sim}{F})D^1_{1100}(\underset{\sim}{I})+\tfrac{1}{2}(3p^3+q^7_{32,32})D^1_{1100}(\underset{\sim}{F})$$
$$+ \tfrac{1}{2}q_{11}q^1_{1,16}D^1_{1100}(\underset{\sim}{I})]-m[\, q^5_{17,25}(\underset{\sim}{F}^2)+q_{58}(\underset{\sim}{F})^2+7q^2_{55}(\underset{\sim}{F})-2pq^2_{55}]D^1_{0000}(\underset{\sim}{I})$$
$$- m^2[\tfrac{1}{2}q^5_{12,17}(\underset{\sim}{F})^2+q_{58}(\underset{\sim}{F}^2)+(3p^3+q^7_{22,16})(\underset{\sim}{F})+pq_{11}q^1_{1,16}/2]D^1_{0000}(\underset{\sim}{I})\}$$
$$+ \{\tfrac{1}{2}[\tfrac{1}{2}q^5_{12,17}D^2_{1200}(\underset{\sim}{F})+q^3_{7,12}D^2_{1111}(\underset{\sim}{F})+\tfrac{1}{2}q^1_{2,7}D^2_{1200}(\underset{\sim}{I})+q_{58}D^2_{2100}((\underset{\sim}{F}+\underset{\sim}{I})^2)$$
$$- 10q^2_{55}D^2_{1100}(\underset{\sim}{I})-2q^{20}_{59,75}D^2_{1100}(\underset{\sim}{F})-4q_{58}(\underset{\sim}{F})D^2_{1100}(\underset{\sim}{I})-4q_{58}(\underset{\sim}{F})D^2_{1100}(\underset{\sim}{F})$$
$$- 4q^5_{17,25}D^2_{1100}(\underset{\sim}{F}^2)]+m[(3/2)pq^2_{55}+4q^2_{55}(\underset{\sim}{F})+\tfrac{1}{2}q_{58}(\underset{\sim}{F})^2+\tfrac{1}{2}q^5_{17,25}(\underset{\sim}{F}^2)]D^2_{0000}(\underset{\sim}{I})$$
$$- m[q^5_{12,17}(\underset{\sim}{F})D^2_{1100}(\underset{\sim}{F})+2q_{58}D^2_{1100}(\underset{\sim}{F}^2)+q^3_{7,12}(\underset{\sim}{F})D^2_{1100}(\underset{\sim}{I})+q_{11}q^1_{1,16}D^2_{1100}(\underset{\sim}{I})$$
$$+ (3p^3+q^7_{32,32})D^2_{1100}(\underset{\sim}{F})]+\tfrac{1}{2}m^2[\tfrac{1}{2}q^5_{12,17}(\underset{\sim}{F})^2+q_{58}(\underset{\sim}{F}^2)+(3p^3+q^7_{22,16})(\underset{\sim}{F})$$
$$+ \tfrac{1}{2}pq_{11}q^1_{1,16}]D^2_{0000}(\underset{\sim}{I})\} +\{q^5_{17,25}D^3_{1100}(\underset{\sim}{F}^2)+q_{58}(\underset{\sim}{F})D^3_{1100}(\underset{\sim}{F})+q^{11}_{32,40}D^3_{1100}(\underset{\sim}{F})$$
$$+ q_{58}(F)D^3_{1100}(\underset{\sim}{I})+3q^2_{55}D^3_{1100}(\underset{\sim}{I}) - \tfrac{1}{2}q^5_{12,17}D^3_{1200}(\underset{\sim}{F})-q^3_{7,12}D^3_{1111}(\underset{\sim}{F})$$
$$- \tfrac{1}{2}q^1_{27}D^3_{1200}(\underset{\sim}{I})-q_{58}D^3_{2100}((\underset{\sim}{F}+\underset{\sim}{I})^2)+(m/2)[q^5_{12,17}(\underset{\sim}{F})D^3_{1100}(\underset{\sim}{F})+2q_{58}D^3_{1100}(\underset{\sim}{F}^2)$$
$$+ q^3_{7,12}(\underset{\sim}{F})D^3_{1100}(\underset{\sim}{I})+(3p^3+q^7_{32,32})D^3_{1100}(\underset{\sim}{F})+q_{11}q^1_{1,16}D^3_{1100}(\underset{\sim}{I})]\}$$
$$+ \tfrac{1}{2}\{\tfrac{1}{2}q^5_{12,17}D^4_{1200}(\underset{\sim}{F})+q^3_{7,12}D^4_{1111}(\underset{\sim}{F})+\tfrac{1}{2}q^1_{27}D^4_{1200}(\underset{\sim}{I})+q_{58}D^4_{2100}((\underset{\sim}{F}+\underset{\sim}{I})^2)\},$$

$$CR'_1 = \{-A^0_{11}+(3A^1_{11}-A^1_{12})+(-3A^2_{11}+3A^2_{12}-A^2_{13})+(A^3_{11}-3A^3_{12}+3A^3_{13}-A^3_{14})+(A^4_{12}-3A^4_{13}+3A^4_{14})$$
$$+ (A^5_{13}-3A^5_{14})+A^6_{14}\},$$
$$CR'_2 = \{-A^0_{21}+(3A^1_{21}-A^1_{22})+(-3A^2_{21}+3A^2_{22}-A^2_{23})+(A^3_{21}-3A^3_{22}+3A^3_{23}-A^3_{24})+(A^4_{22}-3A^4_{23}+3A^4_{24})$$
$$+ (A^5_{23}-3A^5_{24})+A^6_{24}\},$$

$$CR_3' = \{(-A^0_{31}+A^0_{35})+(3A^1_{31}-A^1_{32}-4A^1_{35}+A^1_{36})+(-3A^2_{31}+3A^2_{32}-A^2_{33}+6A^2_{35}-4A^2_{36}+A^2_{37})$$
$$+ (A^3_{31}-3A^3_{32}+3A^3_{33}-A^3_{34}-4A^3_{35}+6A^3_{36}-4A^3_{37}+A^3_{38})+(A^4_{32}-3A^4_{33}+3A^4_{34}+A^4_{35}-4A^4_{36}$$
$$+ 6A^4_{37}-4A^4_{38}+A^4_{39})+(A^5_{33}-3A^5_{34}+A^5_{36}-4A^5_{37}+6A^5_{38}-4A^5_{39})+(A^6_{34}+A^6_{37}-4A^6_{38}+6A^6_{39})$$
$$+ (A^7_{38}-4A^7_{39})+A^8_{39}\},$$

where $A^i_{11} = (1/8)[4q_{11}m+q^1_{14}m^2+pm^3](\underset{\sim}{F})D^i_{0000}(\underset{\sim}{I})$,

$$A^i_{12} = (1/2)[q_{23}D^i_{1100}(\underset{\sim}{F})+(\underset{\sim}{F})D^i_{1100}(\underset{\sim}{I})]+(m/8)[q^1_{1,12}D^i_{1100}(\underset{\sim}{F})+2q_{11}(\underset{\sim}{F})D^i_{1100}(\underset{\sim}{I})]$$
$$+ (m^2/8)[pD^i_{1100}(\underset{\sim}{F})+2(\underset{\sim}{F})D^i_{1100}(\underset{\sim}{I})],$$

$$A^i_{13} = (1/4)[q_{11}D^i_{1111}(\underset{\sim}{F})+4D^i_{2100}(\underset{\sim}{F})]+(m/8)[(\underset{\sim}{F})D^i_{2100}(\underset{\sim}{I})+2D^i_{1111}(\underset{\sim}{F})]$$

$$A^i_{14} = (1/8)D^i_{1121}(\underset{\sim}{F}),\ A^i_{21} = \{(m/8)[9q^1_{34}(\underset{\sim}{F})+14p]+(m^2/16)[3(p^3+q^3_{18,14})(\underset{\sim}{F})$$
$$+ 6pq_{11}]+(m^3/8)[3q^1_{11}(\underset{\sim}{F})+p]\}D^i_{0000}(\underset{\sim}{I})$$

$$A^i_{22} = (3/8)[3q^1_{5,12}D^i_{1100}(\underset{\sim}{F})+6q_{12}(\underset{\sim}{F})D^i_{1100}(\underset{\sim}{I})+q^1_{34}D^i_{1100}(\underset{\sim}{I})]+(3m/8)[\tfrac{1}{2}(p^3+q^3_{38,42})$$
$$+ (3m/8)[\tfrac{1}{2}(p^3+q^3_{38,42})D^i_{1100}(\underset{\sim}{F})+q^1_{38}(\underset{\sim}{F})D^i_{1100}(\underset{\sim}{I})+3q_{11}]_{11}D^i_{1100}(\underset{\sim}{I})]$$
$$+ (3m^2/8)[q^1_{13}D^i_{1100}(\underset{\sim}{F})+2q_{11}(\underset{\sim}{F})D^i_{1100}(\underset{\sim}{I})+D^i_{1100}(\underset{\sim}{I})],$$

$$A^i_{23} = (3/8)[7q_{12}D^i_{2100}(\underset{\sim}{F})+q^1_{3,11}D^i_{1111}(\underset{\sim}{F})+q_{12}D^i_{2100}(\underset{\sim}{I})+D^i_{0002}(\underset{\sim}{I})]$$
$$+ (3m/8)[2q_{11}D^i_{1111}(\underset{\sim}{F})+\tfrac{1}{2}q_{12}D^i_{0121}(\underset{\sim}{F})+\tfrac{1}{2}D^i_{0112}(\underset{\sim}{F})+3D^i_{2100}(\underset{\sim}{F})+D^i_{2100}(\underset{\sim}{I})],$$

$$A^i_{24} = (1/16)[3q_{12}D^i_{1121}(\underset{\sim}{F})+3D^i_{1112}(\underset{\sim}{F})+6D^i_{3100}(\underset{\sim}{F})+2D^i_{0031}(\underset{\sim}{I})],$$

$$A^i_{31} = A_{31}D^i_{0000}(\underset{\sim}{I}),\ A^i_{32} = q^{18}_{61,90}D^i_{1100}(\underset{\sim}{F})+q_{18,29}D^i_{0111}(\underset{\sim}{F})+3q^1_{55}D^i_{0011}(\underset{\sim}{I})$$
$$+ (m/4)[2q^9_{29,40}D^i_{0111}(\underset{\sim}{F})+(15p^3+q^{33}_{212,232})D^i_{1100}(\underset{\sim}{F})+3q_{11}q^1_{1,16}D^i_{0011}(\underset{\sim}{I})]$$
$$+ (m^2/8)[6q^1_{14}D^i_{0011}(\underset{\sim}{I})+2q_{11}q^1_{1,20}D^i_{0111}(\underset{\sim}{F})+(p^4+2p^3+q^{27}_{26,72})D^i_{1100}(\underset{\sim}{F})],$$

$$A^i_{33} = \tfrac{1}{2}[q^{11}_{25,50}D^i_{1111}(\underset{\sim}{F})+10q_{58}D^i_{2100}(\underset{\sim}{F})+q^1_{27}D^i_{0012}(\underset{\sim}{I})+2q_{58}D^i_{0021}(\underset{\sim}{I})]$$
$$+ (m/8)[6q_{23}D^i_{0121}(\underset{\sim}{F})+q^1_{27}D^i_{0112}(\underset{\sim}{F})+2(p^3+q^2_{23,22})D^i_{1111}(\underset{\sim}{F})+4q^1_{1,18}D^i_{2100}(\underset{\sim}{F})$$
$$+ 2q^1_{1,12}D^i_{0021}(\underset{\sim}{I})+4q_{11}D^i_{0012}(\underset{\sim}{I})],$$

$$A^i_{34} = (1/4)[3q_{23}D^i_{1121}(\underset{\sim}{F})+\tfrac{1}{2}q^1_{27}D^i_{1112}(\underset{\sim}{F})+2q_{11}D^i_{2111}(\underset{\sim}{F})+12D^i_{3100}(\underset{\sim}{F})+q_{11}D^i_{2111}(\underset{\sim}{I})$$
$$+ 2D^i_{0031}(\underset{\sim}{I})],$$

$$A^i_{35} = A_{35}D^i_{0000}(\underset{\sim}{I}),\ A^i_{36} = [q^2_{55}+(m/4)q_{11}q^1_{1,20}+q^1_{14}m^2/4+pm^3/4]D^i_{1100}(\underset{\sim}{I}),$$

$A_{37}^{i} = [q_{58}/2+mq_{1,20}^{1}/8+pm^2/8]D_{2100}^{i}(\underset{\sim}{I})+[q_{27}^{1}/4+mq_{11}/2+m^2/4]D_{1200}^{i}(\underset{\sim}{I})$,

$A_{38}^{i} = (1/4)[q_{11}D_{2111}^{i}(\underset{\sim}{I})+4D_{0031}^{i}(\underset{\sim}{I})+mD_{2111}^{i}(\underset{\sim}{I})]$, and $A_{39}^{i} = D_{2121}^{i}(\underset{\sim}{I})/16$

where

$$D_{ijk\ell}^{t}(G) = \sum_{J=0}^{\infty} GA(J+t)\cdot(1/2^J J!)[\Gamma_m(\tfrac{1}{2}(m+n))|2\underset{\sim}{\Sigma}_3|^{\frac{1}{2}(m+n)}]^{-1}.$$

$$\int_{\underset{\sim}{S}>\underset{\sim}{0}} e^{-\frac{1}{2}tr(\underset{\sim}{M}\underset{\sim}{\Sigma}_1^{-1}\underset{\sim}{M}'+\underset{\sim}{\Sigma}_3^{-1})\underset{\sim}{S}}[tr(\underset{\sim}{M}\underset{\sim}{\Sigma}_1^{-1}\underset{\sim}{M}'\underset{\sim}{S})^{i}\underset{\sim}{G}]^{j}[tr(\underset{\sim}{M}\underset{\sim}{\Sigma}_1^{-1}\underset{\sim}{M}'\underset{\sim}{S})^{k}]^{\ell}.|\underset{\sim}{S}|^{\frac{1}{2}(n-1)}d\underset{\sim}{S}$$

$$= \sum_{J=0}^{\infty} GA(J+t)[2^J J!]^{-1}[\Gamma_m(\tfrac{1}{2}(m+n))|2\underset{\sim}{\Sigma}_3|^{\frac{1}{2}(m+n)}]^{-1}$$

$$\int_{\underset{\sim}{S}>\underset{\sim}{0}} e^{-\frac{1}{2}tr(\underset{\sim}{I}+\underset{\sim}{\Omega}_1)\underset{\sim}{\Sigma}_3^{-1}\underset{\sim}{S}}[tr(\underset{\sim}{\Omega}_1\underset{\sim}{\Sigma}_3^{-1}\underset{\sim}{S})^{i}\underset{\sim}{G}]^{j}[tr(\underset{\sim}{\Omega}_1\underset{\sim}{\Sigma}_3^{-1}\underset{\sim}{S})^{k}]^{\ell}|\underset{\sim}{S}|^{\frac{1}{2}(n-1)}d\underset{\sim}{S}$$

where $GA(J) = [2^{mp/2+J}\Gamma(mp/2+J)]^{-1}\int_0^{\theta} x^{mp/2+J-1}e^{-x/2}dx$.