On the L^p norms of stochastic integrals and other martingales

by

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Mimeograph Series #452

May 1976

[1]Supported by an N.S.F. Grant.

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1. Introduction. Let X_t, $0 \leq t < \infty$, be standard Brownian motion. It has recently been proved that there exist absolute positive constants A_p, $0 < p < \infty$, and a_p, $1 < p < \infty$, such that if T is stopping time for X_t then

$$E|X_T|^p \leq A_p ET^{p/2}, \qquad 0 < p < \infty, \tag{1.1}$$

and

$$a_p ET^{p/2} \leq E|X_T|^p, \text{ if } 1 < p < \infty \text{ and } ET^{p/2} < \infty. \tag{1.2}$$

For the exponents $p > 1$ these inequalities are due to D. L. Burkholder in [4] and P. W. Millar in [11]. Inequality (1.2) was extended to the exponents $0 < p \leq 1$ independently by Burkholder and R. F. Gundy in [6] and A. A. Novikov in [13]. The paper [5] is a good general source of information about these and related results.

Here a proof of (1.1) and (1.2) is given which yields the best possible values for the constants a_p and A_p. For $p = 2n$, n an integer, they are respectively z_{2n}^{*2n} and z_{2n}^{2n}, where z_{2n}^* and z_{2n} are the smallest and largest positive zeros of the Hermite polynomial of order 2n. For $p = 4$ this has already been proved by Novikov in [14], and is is well known that the best values for a_2 and A_2 are 1. Bounds for a_p and A_p may be found in [5], [6], [8], and [14]. The constants found here will be shown to be best possible in inequalities related to (1.1) and (1.2) involving stochastic integrals, stopped random walk, and Haar series.

[1]Supported by an N.S.F. Grant.

For example, let $\varphi_1, \varphi_2, \ldots$ be the complete orthonormal system of Haar functions on the Lebesgue unit interval. Let $\lambda_1, \lambda_2, \ldots$ be real numbers, such that $\sum_{i=1}^{\infty} \lambda_i \varphi_i$ converges. Let $f = \sum_{i=1}^{\infty} \lambda_i \varphi_i$ and $S(f) = (\sum_{i=1}^{\infty} (\lambda_i \varphi_i)^2)^{\frac{1}{2}}$. Then there are constants d_p and D_p such that

$$(1.3) \qquad \int_0^1 |f|^p dx \leq D_p \int_0^1 S(f)^p dx, \quad 0 < p < \infty,$$

and

$$(1.4) \qquad d_p \int_0^1 |f|^p dx \leq \int_0^1 S(f)^p dx, \quad \text{if } 1 < p < \infty \text{ and } \int_0^1 S(f)^p dx < \infty.$$

For the exponents $p > 1$ these inequalities are due to R.E.A.C. Paley [12], who proved an equivalent Walsh series form. Marcinkiewicz [9] noted the Haar series version. For the exponents $0 < p \leq 1$, inequality (1.3) was proved by Burkholder and Gundy in [6]. It will be shown that if $p \geq 2$ the best constant for d_p is the same as the one we find for a_p and for $0 < p \leq 2$ the best constant for D_p is the one found for A_p. We have no idea what the constants are for the missing exponents.

Let $D_p(x)$, $-\infty < x < \infty$, be the parabolic cylinder functions of parameter p, and let $M(-p/2, 1/2, z^2/2)$

$$= M_p(z) = \sum_{m=0}^{\infty} (-2z^2)^m (\tfrac{p}{2})(\tfrac{p}{2} - 1)(\tfrac{p}{2} - 2) \cdots (\tfrac{p}{2} - m + 1)/2m!$$

be the confluent hypergeometric function. See [1] as a general reference for these functions. We note that if $n \geq 1$ is a positive integer the zeros of M_{2n} and D_{2n} are exactly the zeros of He_{2n}, the Hermite polynomial. Let z_p^* be the smallest positive 0 of M_p and let z_p be the largest positive 0 of D_p. We prove the following theorem.

Theorem 1.1. <u>The largest possible value for</u> a_p <u>such that</u> (1.2) <u>holds for all stopping times</u> T <u>satisfying</u> $ET^{p/2} < \infty$ <u>is</u> z_p^{*p} <u>for</u> $p \geq 2$ <u>and</u> z_p^p <u>for</u> $1 < p \leq 2$. <u>The smallest possible value for</u> A_p <u>such that</u> (1.1) <u>holds for all stopping times</u> T <u>is</u> z_p^p <u>for</u> $p \geq 2$ <u>and</u> z_p^{*p} <u>for</u> $0 < p \leq 2$.

The examples which show that the values for $a_p (A_p)$ given in Theorem 1.1 cannot be replaced by larger (smaller) values will be based on some results of A. A. Novikov and Larry Shepp on square root stopping boundaries. For $p = 2$, these are Novikov's examples. Let

$$t_a = \inf\{t > 0: \ |X_t| = a\sqrt{t+1}\}, \ a > 0,$$

and

$$s_a = \inf\{t > 0: \ X_t = a\sqrt{t} - 1\}, \ a > 0.$$

Shepp, in [15], proves that $Et_a^p < \infty$ if $a < z_{2p}^*$ and that $E\, Et_{z_{2p}^*}^p = \infty$, $p > 0$. Novikov proves in [14] that $Es_a^p < \infty$ if $a > z_{2p}$ and $Es_{z_{2p}}^p = \infty$, $p > 1/2$. Noting that $t_a \to t_{z_{2p}^*}$ a.e. as $a \uparrow z_{2p}^*$ we get that $\lim_{a \to z_{2p}^*} Et_a^p = \infty$, so that

$$\lim_{a \uparrow z_{2p}^*} E|X_{t_a}|^{2p}/Et_a^p = \lim_{a \uparrow z_{2p}^*} E(a^2(t_a+1))^p/Et_a^p = z_{2p}^{*2p}, \ p > 0.$$

Similarly,

$$\lim_{a \downarrow z_{2p}} E|X_{s_a}|^{2p}/Es_a^p = z_{2p}^{2p}, \qquad p > 1/2.$$

Together, these supply all the examples needed.

A natural way to find the best possible value for, say, A_p, is to find the time T which maximizes $E|X_T|^p/ET^{p/2}$ and then evaluate this quotent. In fact this is a natural way to try to prove (1.1). Unfortunately, such times do not exist. However, under the constraints $T \geq 1$, $ET^{p/2} = M > 1$, the time which maximizes the above ratio does exist and is of the form

$$T_M = \inf\{t \geq 1: \ |X_t| \geq C(M,p)t^{\frac{1}{2}}\}, \quad \text{if } 0 < p < 2,$$

and

$$T_M = \inf\{t \geq 1: \ |X_t| \leq B(M,p)t^{\frac{1}{2}}\}, \quad \text{if } 2 < p < \infty,$$

where C(M,p) and B(M,p) are constants. As $M \to \infty$, $C(M,p) \to z_p^*$ and $B(M,p) \to z_p$ and the ratios $E|X_{T_M}|^p/ET_M^{p/2}$ approach z_p^{*p}, if $0 < p < 2$, and z_p^p, if $2 < p < \infty$, which can be shown to imply Theorem 1.1. This is the proof used in this paper as initially submitted. The referee, to whom I am indebted, and who wishes to remain anonymous, suggested a proof which is shorter and more direct. His proof will now be given.

2. Proof of Theorem 1.1. We concentrate for the time being on A_p, $p > 2$. It has already been shown that no smaller value than z_p^p will do for A_p. To show that z_p^p is an acceptable value for A_p it will be shown that if $C > z_p^p$ and T is a bounded stopping time then

$$E(|X_T|^p - CT^{p/2}) \leq 0. \tag{2.1}$$

Now define $f(t,x) = |x|^p - Ct^{p/2}$ for $t \geq 0$, $-\infty < x < \infty$. Suppose for a moment that the truth of (2.1) is known. Define $v(t,x) = \sup_{T \in J} E_{t,x} f(T, \bar{X}_T)$, where J is the class of all bounded stopping times and $E_{t,x}$ denotes expectation taken with respect to Brownian motion started at time t and height x ($E_{0,0}$ is shortened to E, as in (2.1)). Clearly $v(t,x) \geq f(t,x)$, and $v(0,0) = 0$. It is not hard to show that $v(t,x) < \infty$ for all t,x, that v is continuous, and that $v(t,\bar{X}_t)$, $t \geq 0$, is a supermartingale, which is equivalent to $v_t + \frac{1}{2} v_{xx} \leq 0$, at least where v is smooth enough.

Conversly, suppose that there is a function $u(t,x)$, $-\infty < x < \infty$, $t > 0$, such that $\lim_{t \downarrow 0} u(t,0) = 0$, $u(t,x) \geq f(t,x)$, and $u(t,\bar{X}_t)$ is a supermartingale (under $P_{a,b}$ for all $a > 0$,b). Then if γ is a bounded stopping time, and $\varepsilon > 0$,

$$u(\varepsilon,0) \geq E_{\varepsilon,0}\, u(\gamma,\bar{X}_\gamma) \geq E_{\varepsilon,0} f(\gamma,\bar{X}_\gamma) = E_{\varepsilon,0}(|\bar{X}_\gamma|^p - C\gamma^{p/2}).$$

Since $u(\varepsilon,0) \to 0$ as $\varepsilon \to 0$, this gives the truth of (2.1) for all bounded stopping times T.

Thus (2.1) for bounded stopping times T is equivalent to the existence of a function u with these properties. We will actually exhibit such a function u. The function constructed will satisfy

(2.1) $$u(t,x) \geq f(t,x),$$

and

(2.2) $$u_t + \frac{1}{2}u_{xx} \leq 0,$$

or, more precisely, u_t will be continuous and u_{xx} will be defined and continuous everywhere except the lines $x = \pm k\sqrt{t}$ for one number k and will be bounded on compact sets, and (2.2) will be satisfied everywhere except these lines. This is sufficient to guarantee that $u(t,X_t)$ is a supermartingale, since u grows no faster than a polynomial in x and t. By Brownian scaling we can look for a function of the form $u(t,x) = t^{p/2}V(x/\sqrt{t})$. If we call $g(x) = |x|^p - C$, then (2.1) and (2.2) become

(2.3) $$V(x) \geq g(x),$$

and

(2.4) $$V'' - xV' + pV \leq 0.$$

Let $\phi(z) = e^{z^2/4}D_p(z)$. Then ϕ satisfies $\phi'' - x\phi' + p\phi = 0$, since $D_p(z)$ satisfies $y'' + [p + 1/2 - x^2/4]y = 0$, and

$$\phi(x) = x^p - \frac{p(p-1)}{2}\, x^{p-2} + O(x^{p-3}) \text{ as } x \to \infty.$$

(Equation 19.8.1, p. 689 of [1]). Let $F(\lambda,x) = \lambda\phi(x) - g(x)$. For $\lambda = 1$, it has a root larger than z_p. This is because $F(1,z_p) = C^p - z_p^p > 0$, while $F(1,x)$ is negative for large x. As λ increases from 1, $F(x,\lambda)$ is positive for large x, and thus a new root appears. If λ is very large, $F(x,\lambda)$ is positive for all $x \geq z_p$. Let λ^* be the largest λ such that $F(x,\lambda) = 0$ for some $x \in (z_p,\infty)$, and let k be one of these roots. Then $F(\lambda^*,k) = 0$, $F'(\lambda^*,k) = 0$, and $F''(\lambda^*,k) \geq 0$. Now define $V(x)$ by $g(x)$ for

$|x| \leq k$ and by $\lambda^*\phi(x)$ for $|x| \geq k$. Then V is differentiable and twice differentiable everywhere but k. By construction $V'' - xV' + pV = 0$ if $|x| \geq k$. All that is left to show is $g'' - xg' + pg \leq 0$ for $|x| \leq k$. Since $g(x) = |x|^p - C$, this amounts to verifying $g''(k) - kg'(k) + pg(k) \leq 0$, which holds because

$$\begin{aligned} 0 \leq F''(\lambda^*,k) &= \lambda^*\phi''(k) - g''(k) \\ &= k\lambda^*\phi'(k) - p\lambda^*\phi(k) - g''(k) \\ &= kg'(k) - pg(k) - g''(k). \end{aligned}$$

This completes the proof. It is not difficult to show that the function $u(t,x) = t^{p/2}V(x/\sqrt{t})$ is the least super parabolic majorant of $f(t,x)$.

To show that the value for A_p, $0 < p < 2$, is z_p^{*p}, we again need to show that if $C > z_p^{*p}$ then there is a function V satisfying (2.3) and (2.4). Again let $\phi(z) = M(-\frac{1}{2}p, \frac{1}{2}, \frac{1}{2}z^2)$. As before, $\phi'' - x\phi' + p\phi = 0$. Consider $F(x,\lambda) = \lambda\phi(x) - g(x)$. Then $F(z_p^*,\lambda) = C^p - z_p^{*p} > 0$ for all λ, while $F(0,-C) = 0$. Also, $F(x,0) > 0$ on $[0, z_p^*]$. Thus there is a largest λ, say λ^*, such that $F(x,\lambda) = 0$ for some $x \in [0,z_p^*]$, and a corresponding value $k \in [0,z_p^*]$ satisfying

$$F(\lambda^*,k) = 0,$$

$$F'(\lambda^*,k) = 0, \text{ and}$$

$$F''(\lambda^*,k) \geq 0.$$

We now define $V(x) = \lambda^*\phi(x)$ for $|x| \leq k$ and $V(x) = g(x)$ for $|x| \geq k$. To verify $V''(x) - xV'(x) + pV(x) \leq 0$ for $|x| \geq k$ the same trick as before works for $p \in (1,2)$, while $g''(x) - xg'(x) + pg(x) \leq 0$ for $0 < p \leq 1$, $x \neq 0$.

The other cases will be sketched briefly, since most of the details are similar. We note that the truth of inequality (1.2) for any value of a_p implies that this inequality holds for all stopping times T satisfying $ET^{p/2} < \infty$. For, using inequality (1.1), $ET^{p/2} < \infty$ implies

$$\lim_{t\to\infty} E|X_{T\wedge t}|^p \leq A_p ET^{p/2} < \infty,$$

where ^ denotes minimum,
so by an inequality of Doob (see [7], Chapter VII, Theorem 3.4, and page 354) $E\sup_{t\geq 0}|X_{T\wedge t}|^p < \infty$ and thus $\lim_{t\to\infty} E|X_{T\wedge t}|^p = E|X_T|^p$.

To show that z_p^{*p} is an acceptable value for a_p, $2 < p < \infty$, it can be shown that if $c < z_p^{*p}$ there is a function V(x) satisfying $V(x) \geq c - |x|^p$ and $V'' - xV' + pV \leq 0$ at all except one point x. The form of this function is $V(x) = c - |x|^p$ if $|x| \geq k$ $(k < z_p^*)$ and $V(x) = \lambda^* M(-p/2, 1/2, x^2/2)$ if $|x| \leq k$.

For a_p, $1 < p < 2$, the form of V is $V(x) = c - |x|^p$ if $|x| \leq k$ and $V(x) = \lambda^* e^{x^2/4} D_p(x)$ if $|x| \geq k$.

3. Other martingales. Let X_t, $t \geq 0$, be standard Brownian motion and let $f(t,\omega)$ be a non-anticipating function satisfying $\int_0^\infty f(t,\omega)^2 dt < \infty$ a.s.. Then there is a standard Brownian motion Z_t, $0 \leq t < \infty$, and a stopping time $T = T_f$ for Z_t such that T and $\int_0^\infty f(t,\omega)^2 dt$ have the same distribution and Z_t and $\int_0^\infty f(t,\omega) dX_t$ have the same distribution (See McKean, [10], p. 29). Thus (1.1) implies

$$(3.1) \qquad E\left|\int_0^\infty f(t,\omega) dX_t\right|^p \leq A_p E\left(\int_0^\infty f(t,\omega)^2 dt\right)^{p/2}, \qquad 0 < p < \infty,$$

while (1.2) gives

$$(3.2) \qquad a_p E\left(\int_0^\infty f(t,\omega)^2 dt\right)^{p/2} \leq E\left|\int_0^\infty f(t,\omega) dX_t\right|^p, \text{ if } 1 < p < \infty \text{ and}$$

$$E\left(\int_0^\infty f(t,\omega)^2 dt\right)^p < \infty,$$

where any values of A_p and a_p such that (1.1) and (1.2) hold suffice here. Since for any stopping time T we can write $T = \int_0^\infty I[0,T]^2(s)ds$, where I is the indicator function, and $X_T = \int_0^\infty I(0,T)dX_s$, Theorem 1.1 implies the following theorem.

Theorem 3.1. <u>The best possible values for</u> a_p <u>and</u> A_p <u>in</u> (3.1) <u>and</u> (3.2) <u>are those given in the statement of Theorem</u> 1.1.

Next discrete martingales will be considered. If Z_t, $0 \le t < \infty$, is a standard Brownian motion and if $\tau_a = \inf\{t > 0: \ |Z_t| = a\}$ then $E\tau_a = a$. Symmetry also gives $E(\tau_a | Z_{\tau_a} = a) = E(\tau_a | Z_{\tau_a} = -a) = a$. Now let $d_1, d_2, \ldots, d_n$, be any martingale difference sequence such that each d_i takes on only a finite number of values and also such that $P(d_i = a | d_1, \ldots, d_{i-1})$ $= P(d_i = -a | d_1, \ldots, d_{i-1})$ for each of these values a and all i. Let W_t, $0 \le t < \infty$, be a standard Brownian motion which is independent of $(d_1, \ldots, d_n)$. Define T_i, $0 \le i \le n$ by putting $T_0 = 0$ and, for $i > 0$, saying that $T_i = \inf\{t > T_{i-1}: \ |W_t - W_{T_{i-1}}| = a\}$ on $\{|d_i| = a\}$. Then $E(T_i - T_{i-1} | d_1, \ldots, d_n) = a^2$ on $\{|d_i| = a\}$, so that

$$E(T_n | d_1, \ldots, d_n) = \sum_{i=1}^n d_i^2. \tag{3.3}$$

Also, $(W_{T_1}, W_{T_2} - W_{T_1}, \ldots, W_{T_n} - W_{T_{n-1}}) \stackrel{d}{=} (f_1, f_2, \ldots, f_n)$ where $f_k = d_1 + \ldots + d_k$. Now (3.3) gives $ET_n^q \le E(\sum_{i=1}^n d_i^2)^q$ if $q \le 1$, and $ET_n^q \ge (E \sum_{i=1}^n d_i^2)^q$ if $q \ge 1$.

Thus, using Theorem 1.1,

$$E|f_n|^p = E|W_{T_n}|^p \le z_p^{*p} \, ET_n^{p/2} \le z_p^{*p} \, E(\sum_{i=1}^n d_i^2)^{p/2}, \qquad 0 < p \le 2, \tag{3.4}$$

and similarly

$$z_p^{*p} \, E(\sum_{i=1}^n d_i^2)^{p/2} \le E|f_n|^p, \qquad 2 \le p < \infty. \tag{3.5}$$

Equations (3.4) and (3.5) and an easy approximation argument (omitted) give the following theorem.

Theorem 3.2. *Let* $d_1, d_2, \ldots$ *be a martingale difference sequence satisfying* $P(d_n > a | d_1, \ldots, d_{n-1}) = P(d_n < -a | d_1, \ldots, d_{n-1})$ a.e. *for each integer* n *and each positive real number* a. *Suppose* Σd_n *converges, and put* $f = \sum_{i=1}^{\infty} d_i$, $S(f) = (\Sigma d_i^2)^{\frac{1}{2}}$. Then

$$E|f|^p \leq z_p^{*p} \, ES(f)^p, \qquad 0 < p \leq 2, \tag{3.6}$$

and

$$z_p^{*p} \, ES(f)^p \leq E|f|^p, \quad \text{if } 2 \leq p < \infty \text{ and } ES(f)^p < \infty. \tag{3.7}$$

Inequalities (3.6) and (3.7) are true in much greater generality if the constant z_p^{*p} is allowed to be replaced by absolute constants, the best values of which are not known. See [4] and [6]. In particular (3.6) and (3.7) show that an acceptable value for D_p in (1.3) is z_p^{*p}, $0 < p \leq 2$, and for d_p in (1.4) is z_p^{*p}, $2 \leq p < \infty$. As a special case of these inequalities we get that if $X_1, X_2, \ldots$ are independent identically distributed random variables with $P(X_i = +1) = P(X_i = -1) = 1/2$, if $S_n = X_1 + \ldots + X_n$, and if N is a stopping time for S_n then

$$E|S_N|^p \leq z_p^{*p} \, EN^{p/2}, \qquad 0 < p \leq 2, \tag{3.8}$$

and

$$z_p^{*p} \, EN^{p/2} \leq E|S_N|^p, \quad \text{if } 2 \leq p < \infty \text{ and } EN^{p/2} < \infty. \tag{3.9}$$

To show that the constants z_p^{*p} are the best possible in (3.8) and (3.9) (and thus also in (1.3) and (1.4)), examples are needed. All cases are essentially the same, so an example is given to show that z_3^{*3} in (3.9), p = 3, may not be replaced by a larger value. Let W_t, $0 \leq t < \infty$, be standard Brownian motion, and let T be a stopping time in L^∞ such that $E|W_T|^3/ET^{3/2} < z_3^{*3} + \varepsilon$. We can take T to be a truncation of $t_{z_3^*}$ (defined in the introduction). Let $v_1(\delta) = \inf\{t > 0\colon |W_t| = \delta\}$, and, if $i > 1$, $v_i(\delta) = \inf\{t > v_{i-1}(\delta)\colon |W_t - W_{v_{i-1}(\delta)}| = \delta\}$. Let $N(\delta) = N = \inf\{i\colon v_i(\delta) > T\}$. Then since $|W_{v_N(\delta)} - W_T| \le \delta$, we have

$$(3.10) \qquad E|W_{v_N(\delta)}|^3 \longrightarrow E|W_T|^3 \text{ as } \delta \to 0,$$

and also, since $T \in L^\infty$,

$$(3.11) \qquad \overline{\lim}_{\delta\downarrow 0} E \sup_{0\le t\le v_N(\delta)} |W_t|^p < \infty \text{ for each } p > 0.$$

Now let $g_0, g_1, \ldots$ stand for the martingale $0, W_{\min(v_1(\delta), v_N(\delta))}, W_{\min(v_2(\delta), v_N(\delta))}, \ldots$ Then $\lim_{i\to\infty} g_i = g_N$ and $\sum_{i=1}^{N} (g_i - g_{i-1})^2 = \delta^2 N$. Thus, since $\sup_{0\le t\le v_N(\delta)} |W_t| > \sup_i |g_i|$, (3.10) and inequality (3.5) imply $\overline{\lim}_{\delta\downarrow 0} E(\delta^2 N)^p < \infty$ for each $p > 0$. Since $\delta^2 N \to T$ in probability (see [3], ch. 13), this last fact gives

$$(3.12) \qquad E\delta^2 N(\delta)^{3/2} \to ET^{3/2} \text{ as } \delta \to 0.$$

Now $0, g_1/\delta, g_2/\delta, \ldots = h_1, h_2, \ldots$ is fair random walk up to the stopping time $N = N(\delta)$. We have $h_N = W_{v_N(\delta)}/\delta$. In view of (3.10) and (3.12), we have

$$\lim_{\delta\to 0} E|h_N|^3/EN^{3/2} = E|W_T|^3/ET^{3/2} < z_3^{*3} + \varepsilon,$$

the examples desired. The following theorem summarizes these results.

Theorem 3.2. _The inequalities_ (3.8) _and_ (3.6) _do not hold in general if_ z_p^{*p} _is replaced by a smaller value. The smallest possible value for_ D_p, $2 \le p < \infty$, _in_ (1.3) _is_ z_p^{*p}. _The inequalities_ (3.9) _and_ (3.7) _do not hold in general if_ z_p^{*p} _is replaced by a larger value. The largest possible value for_ d_p _in_ (1.4), $0 < p \le 2$, _is_ z_p^{*p}.

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