A new "definition" of the integral

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Abstract: We give a one-sentence definition of the Lebesgue-Stieltjes integral which is easy to use, conveys the intuitive concept, and has direct generalizations to include many Riemann- and Lebesgue-type integrals.

Among the criteria of a good definition of a mathematical concept are conciseness and the conveyance of an understanding of the concept. The usual treatments of Lebesgue type integration frequently fail to meet either of these criteria. We claim that the definition given here satisfies these criteria, and we also illustrate the ease of direct generalization of the definition.

We define:

$a = \int f d\mu$ if for every $\varepsilon > 0$ there exist:

a disjoint covering $\{S_i: i \in I\}$ of measurable sets,

a sequence $\{v_i: i \in I\}$ of values, and

a sequence $\{e_i: i \in I\}$ of errors, such that

(1) $|f(x) - v_i| \leq e_i$ for all $x \in S_i$,

(2) $|a - \Sigma v_i \mu(S_i)| < \varepsilon$,

(3) $\Sigma e_i \mu(S_i) < \varepsilon$.

The sums are in the sense of unconditional convergence and $0 \cdot \infty = 0$ wherever it occurs.

If we take I to be the integers, this gives the usual Lebesgue integral. If we require I to be finite, strengthen (1) to hold for all $x \in \overline{S}_i$, and require the S_i to be intervals we get the Riemann integral.

If μ is a signed measure, we can obtain the usual integral by replacing μ by $|\mu|$ in (3) and requiring, as we can, that only finitely many $v_i \neq 0$.

Furthermore, we do not need f or μ to be real-valued. If f takes values in the linear space F, μ takes values in the linear space M, and there is a bilinear mapping * from $F \times M$ to a linear space G and suitable magnitude functions, the modifications of (1), (2), and (3) are immediate. We can obtain the various integrals of Banach-space valued functions in this way, and the spectral theorem in Hilbert space is easily obtained in this form, even becoming a Riemann type integral for bounded Hermitian operators.

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ABSTRACT

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Among the criteria of a good definition of a mathematical concept are conciseness and the conveyance of an understanding of the concept. The usual treatments of Lebesgue type integration frequently fail to meet either of these criteria. We claim that the definition given here satisfies these criteria, and we also illustrate the ease of direct generalization of the definition.

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