A NOTE ON OPTIMAL SUBSET SELECTION PROCEDURES

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Summary

A Note on Optimal Subset Selection Procedures

Abbreviates Title: Optimal Subset Selection

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A result for constructing an "optimal" selection rule for selecting a subset of $k(\geq 2)$ populations is given. Attention is restricted to the class of rules for which the infimum of the probability of a correct selection, over a subset of the parameter space, is guaranteed to be a specified number. In this class a rule is derived which minimizes the supremum of the expected size of the selected subset.

Key Words and Phrases

Subset selection, restricted minimax AMS 1970 Subject Classifications. Primary 62F07; Secondary 62G30.

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Let $\pi_1, \pi_2, \ldots, \pi_k$ represent $k(\geq 2)$ independent populations (treatments) and let $X_{i1}, \ldots, X_{in_i}$ be n_i independent random observations from π_i. The quality of the ith population π_i is characterized by a real-valued parameter θ_i, usually unknown. Let $\Omega = \{\underline{\theta} | \underline{\theta}' = (\theta_1, \ldots, \theta_k)\}$ denote the parameter space. Let $\tau_{ij} = \tau_{ij}(\underline{\theta})$ be a measure of separation between π_i and π_j. We assume that there exists a monotonically nonincreasing function h such that $\tau_{ji} = h(\tau_{ij})$. Let $\Omega_i = \{\underline{\theta} | \tau_{ij} \geq \tau_0, \forall j \neq i\}$, $1 \leq i \leq k$, and $\Omega_0 = \Omega - \bar{\Omega}$, where $\bar{\Omega} = \bigcup_{i=1}^{k} \Omega_i$. For this problem, we assume τ_0 and τ_{ii} as known with $\tau_0 > \tau_{ii}$ for all i. Let $\tau_i = \min_{j \neq i} \tau_{ij}$, $1 \leq i \leq k$. We define $\tau^* = \max_{1 \leq \ell \leq k} \tau_\ell$. The population associated with τ^* will be called the best population. It should be pointed out that if $\underline{\theta} \in \Omega_i$, then $\tau_i \geq \tau_j$ for all j, since for some j, $j \neq i$, $\tau_{ji} = h(\tau_{ij}) \leq h(\tau_0) \leq h(\tau_{ii}) = \tau_{ii} < \tau_0$. Thus if $\theta \in \Omega_i$, then π_i is the best population. A selection of a subset containing the best population is called a correct selection (CS). In case of tie of the populations corresponding to τ^* any one of them is "tagged" as the best population.

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To illustrate the above notation, we assume that independent observations are drawn from π_i which has a noraml distribution with unknown mean θ_i $(i = 1,\ldots,k)$ and known variance σ^2. We define $\tau_{ij} = \theta_i - \theta_j$; then $\tau_i = \theta_i - \theta_{[k]}$ if $\theta_i < \theta_{[k]}$ and $\tau_i = \theta_i - \theta_{[k-1]}$ if $\theta_i = \theta_{[k]}$, where $\theta_{[1]} \le \cdots \le \theta_{[k]}$. In this case, $\tau_{ii} = 0$ for all i and the population with the largest mean, $\theta_{[k]}$, is the best. If, instead, $\tau_{ij} = \theta_j - \theta_i$ then the population with the smallest mean, $\theta_{[1]}$, would be the best. In the above example, $h(t) = -t$, which is a decreasing function.

Let the observed sample vector be denoted by $\underset{\sim}{X}' = (\underset{\sim}{X}_1',\ldots,\underset{\sim}{X}_k')$ where $\underset{\sim}{X}_i$ has components $X_{i1},\ldots,X_{in_i}$, $i = 1,\ldots,k$. Let $\delta = (\delta_1,\ldots,\delta_k)$ be a selection procedure where $\delta_i(\underset{\sim}{x})$ is the probability of selecting π_i $(1 \le i \le k)$ based on the observed vector $\underset{\sim}{X} = \underset{\sim}{x}$. As measures of goodness of a selection rule, consider two quantities (cf. Lehmann [5]) $R(\underset{\sim}{\theta},\delta)$ and $S(\underset{\sim}{\theta},\delta)$. We define

$S(\underset{\sim}{\theta},\delta) = P_{\underset{\sim}{\theta}}(CS|\delta)$ and $R(\underset{\sim}{\theta},\delta) = \sum_{i=1}^{k} R^{(i)}(\underset{\sim}{\theta},\delta_i)$, where $R^{(i)}(\underset{\sim}{\theta},\delta_i) =$ P{Selecting $\pi_i|\delta$}. Thus $R(\underset{\sim}{\theta},\delta)$ is the expected size of the selected subset. For a specified γ, $(0 < \gamma < 1)$, we restrict attention to the class $\mathscr{C}$ of all δ such that

$$S(\underset{\sim}{\theta},\delta) \ge \gamma \text{ for } \underset{\sim}{\theta} \in \bar{\Omega}. \tag{1}$$

We are interested in constructing an optimal procedure δ^0 in $\mathscr{C}$ which minimizes the supremum of $R(\underset{\sim}{\theta},\delta)$ over Ω for all $\delta \in \mathscr{C}$, i.e.,

$$\sup_{\underset{\sim}{\theta}\in\Omega} R(\underset{\sim}{\theta},\delta^0) = \min_{\delta\in\mathscr{C}} \sup_{\underset{\sim}{\theta}\in\Omega} R(\underset{\sim}{\theta},\delta). \tag{2}$$

Remark: For some basic results and the motivation of the subset selection approach, reference can be made to Gupta [4]. Some (different) optimality results assuming a slippage configuration are given by Studden [7] for the exponential family. Recently Bjørnstad [2] has obtained some results on the minimaxity aspects of the procedures of Gupta [4], Seal [6] and Studden [7].

We restrict attention to those selection procedures which depend on the observations only through a sufficient statistic for $\underline{\theta}$.

Let the statistic Z_{ij} be based on the n_i and n_j independent observations from π_i and π_j $(i,j = 1,2,\ldots,k)$, respectively, and suppose that for any i, the statistic $\underline{Z}_i' = (Z_{i1},\ldots,Z_{ik})$ is invariant sufficient under a transformation group G and let $\underline{\tau}_i' = (\tau_{i1},\ldots,\tau_{ik})$ be a maximal invariant under the induced group $\bar{G}$. It is well known (see Ferguson [3]) that the distribution of $\underline{Z}_i$ depends only on $\underline{\tau}_i$. For any i, let the joint density of $Z_{ij}, \forall\ j \neq i$, be $p_{\underline{\theta}}(\underline{z}_i)$. Let $p_{\underline{\theta}}(\underline{z}_i)$ be denoted by $p_0(\underline{z}_i)$ when $\tau_{i1} = \ldots = \tau_{ik} = \tau_{ii}$ = constant and by $p_i(\underline{z}_i)$ when $\tau_{i1} = \ldots = \tau_{ik} = \tau_0$, $1 \leq i \leq k$. In the normal means example, a choice of Z_{ij} might be $\bar{X}_i - \bar{X}_j$, where $\bar{X}_i = \frac{1}{n_i}\sum_{\ell=1}^{n_i} X_{i\ell}$ and $\bar{X}_j = \frac{1}{n_j}\sum_{\ell=1}^{n_j} X_{j\ell}$. Let ν be a σ-finite measure on $\mathbb{R}^{k-1}$.

Now we state and prove a theorem which provides a solution to the restricted minimax problem as stated in (1) and (2) (cf. Lehmann [5]).

Theorem: Suppose that for any i, $p_i(\underline{z}_i)/p_0(\underline{z}_i)$ is nondecreasing in $\underline{z}_i$. If $R(\underline{\theta},\delta^0)$ is maximized at $\tau_{ij} = \tau_{ii}$ = constant, for all i,j, where δ^0 is given by

$$\delta_i^0(\underline{z}_i) = \begin{cases} 1 & \text{if} \quad p_i(\underline{z}_i) > c\, p_0(\underline{z}_i), \\ \lambda_i & \qquad\qquad = \qquad\qquad , \\ 0 & \qquad\qquad < \qquad\qquad , \end{cases}$$

such that $c(>0)$ and λ_i are determined by $\int \delta_i^0 p_i = \gamma$, $1 \leq i \leq k$. Then $\delta^0 = (\delta_1^0, \ldots, \delta_k^0)$ minimizes $\sup_{\underline{\theta} \in \Omega} R(\underline{\theta}, \delta)$ subject to $\inf_{\theta \in \bar{\Omega}} S(\underline{\theta}, \delta) \geq \gamma$.

Proof. For any $\delta \in \mathscr{C}$,

$\underline{\theta} \in \bar{\Omega}$ implies $\underline{\theta} \in \Omega_i$ for some i, thus

$$S(\underline{\theta}, \delta) = \int \delta_i(\underline{z}_i) p_{\underline{\theta}}(\underline{z}_i) d\nu(\underline{z}_i) \geq \min_{1 \leq i \leq k} \inf_{\underline{\theta} \in \Omega_i} \int \delta_i(\underline{z}_i) p_{\underline{\theta}}(\underline{z}_i) d\nu(\underline{z}_i).$$

We have

$$\inf_{\underline{\theta} \in \bar{\Omega}} S(\underline{\theta}, \delta) = \min_{1 \leq i \leq k} \inf_{\underline{\theta} \in \Omega_i} \int \delta_i(\underline{z}_i) p_{\underline{\theta}}(\underline{z}_i) d\nu(\underline{z}_i).$$

Hence for any $\delta \in \mathscr{C}$, $\inf_{\underline{\theta} \in \Omega_i} \int \delta_i(\underline{z}_i) p_{\underline{\theta}}(\underline{z}_i) d\nu(\underline{z}_i) \geq \gamma$, $1 \leq i \leq k$, and by the assumption that $\int \delta_i^0 p_i = \gamma$, it follows that

$$\int (\delta_i - \delta_i^0)(p_i - cp_0) \leq 0$$

which implies

$$\int \delta_i^0 \, p_0 \leq \int \delta_i p_0.$$

By our assumption, $\delta_i^0(\underline{z}_i)$ is nondecreasing in $\underline{z}_i$, hence

$$\inf_{\underline{\theta} \in \bar{\Omega}} S(\underline{\theta}, \delta^0) = \min_{1 < i < k} \int \delta_i^0 p_i = \gamma.$$

If $R(\underline{\theta},\delta^0)$ is maximized at $\tau_{ij} = \tau_{ii}$ = constant, for all i,j, then

$$\sup_{\underline{\theta}\in\Omega} R(\underline{\theta},\delta) \geq \sum_{i=1}^{k} \int \delta_i p_0 \geq \sum_{i=1}^{k} \int \delta_i^0 p_0 = \sup_{\underline{\theta}\in\Omega} R(\underline{\theta},\delta^0),$$

which completes the proof.

As an application of the preceding result, consider the following example:

<u>Example</u>: Let $\pi_1,\pi_2,\ldots,\pi_k$ be k independent normal populations with means $\theta_1,\ldots,\theta_k$ and common known variance $\sigma^2 = 1$. The ordered θ_i's are denoted by $\theta_{[1]} \leq \ldots \leq \theta_{[k]}$. It is assumed that there is no prior knowledge of the correct pairing of the ordered and the unordered θ_i's. Our goal is to select a nonempty subset of the k populations so as to include the population associated with $\theta_{[k]}$.

Let $\bar{X}_i$, $1 \leq i \leq k$, denote the sample means of independent samples of size n from these populations. The likelihood function of $\underline{\theta}$ is then

$$p_{\underline{\theta}}(\underline{x}) = \prod_{i=1}^{k} p_{\theta_i}(\bar{x}_i),$$

where $$p_{\theta_i}(\bar{x}_i) = \frac{\sqrt{n}}{\sqrt{2\pi}} e^{-\frac{n}{2}(\bar{x}_i-\theta_i)^2}, \quad 1 \leq i \leq k.$$ Let

$\tau_{ij} = \tau_{ij}(\underline{\theta}) = \theta_i-\theta_j$, $1 \leq i,j \leq k$, $\tau_0 = \Delta > 0$, $\bar{\Omega} = \{\underline{\theta} | \theta_{[k]}-\theta_{[k-1]} \geq \Delta\}$

and $Z_{ij} = \bar{X}_i-\bar{X}_j$ $1 \leq i, j \leq k$. Let $\underline{z}_i' = (z_{i1},\ldots,z_{ik})$, $\underline{\tau}_i' = (\tau_{i1},\ldots,\tau_{ik})$, then since $Z_{ii} = 0$ and $\tau_{ii} = 0$, $\forall i$, the joint density of Z_{ij}, $j \neq i$, is given by

$$p_{\underline{\theta}}(\underline{z}_i) = (2\pi)^{\frac{k-1}{2}} |\Sigma|^{-1/2} \exp\{-(\underline{z}_i-\underline{\tau}_i)'\Sigma^{-1}(\underline{z}_i-\underline{\tau}_i)\},$$

where $\sum_{(k-1)x(k-1)} = \frac{1}{n}\begin{pmatrix} 2 & & 1 \\ & \cdot & \\ 1 & & \cdot 2 \end{pmatrix}$ is the covariance matrix of Z_{ij}'s.

Since

$$\frac{p_i(\underline{z}_i)}{p_0(\underline{z}_i)} = \exp\{\underline{z}_i'\Sigma^{-1}\underline{\Delta} + \underline{\Delta}'\Sigma^{-1}\underline{z}_i - \underline{\Delta}'\Sigma^{-1}\underline{\Delta}\} = \exp\{\frac{n\Delta}{k}(z_{i1} + \ldots + z_{ik})\}$$

is nondecreasing in z_{ij}, $j \neq i$, where $\underline{\Delta}' = (\Delta,\ldots,\Delta)$. And

$$\frac{p_i(\underline{z}_i)}{p_0(\underline{z}_i)} > c$$

is equivalent to

$$\bar{x}_i > \frac{1}{k-1} \sum_{j \neq i} \bar{x}_j + d.$$

Since $R(\underline{\theta},\delta^0) = \sum_{i=1}^{k} P\{\bar{X}_i > \frac{1}{k-1} \sum_{j \neq i} \bar{X}_j + d\}$ is the expected size of the selected subset for Seal's average-type procedure δ^0 [6], the following result of Berger [1] and Bjørnstad [2] applies

$$\sup_{\underline{\theta}\in\Omega} R(\underline{\theta},\delta^0) = R(\underline{\theta},\delta^0) \text{ iff } \inf_{\underline{\theta}\in\Omega} S(\underline{\theta},\delta^0) \geq \frac{k-1}{k} .$$

Since the right hand side is equivalent to $\Phi(\sqrt{\frac{k-1}{k}}\sqrt{n}\, d) \leq \frac{1}{k}$, the left hand side for every fixed $\Delta > 0$ holds if and only if

$$\gamma = 1 - \Phi(\sqrt{\frac{k-1}{k}}\sqrt{n}\,(d - \Delta)) \geq 1 - \Phi(\Phi^{-1}(\frac{1}{k}) - \sqrt{\frac{k-1}{k}}\sqrt{n}\,\Delta),$$

where $\Phi(\cdot)$ is the cdf of the standard normal. Therefore, if for $\Delta > 0$, γ is the chosen in such a way that the preceding inequality holds, then the result of the theorem can be applied.

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