SOME RESULTS ON SUBSET SELECTION PROBLEMS*

by

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INTRODUCTION

The shortcomings of the classical tests of homogeneity, i.e., testing the hypothesis of equality of parameters, have long been known. The only question answered by such a test is whether there is any difference at all among the available populations. Bahadur [2], Mosteller [58] and Paulson [61] were among the earliest research workers to recognize this and to formulate the problem as a multiple decision problem concerned with the selection and ordering of k populations.

In the two decades since these early papers, ranking and selection problems have become an active area of statistical research. There have been two approaches to these problems, the 'indifference zone' approach and the 'subset selection' approach. In the first approach, a single population (or a fixed number of populations) is chosen and is guaranteed to be the one of interest with a fixed probability P^* whenever the unknown parameters lie outside some subset, or zone of indifference, of the entire parameter space. This formulation is due to Bechhofer [11]. Other contributions to this problem are Bechhofer and Sobel [16], Bechhofer, Dunnett and Sobel [14], Sobel and Huyett [75], Chambers and Jarratt [20], Barr and Rizvi [7], Eaton [26], and Mahamunulu [56]. A quite adequate bibliography may be found in Santner [69] and Bechhofer, Kiefer and Sobel [15].

The second approach assumes no a priori information about the parameter space. A single population is not necessarily chosen; rather a subset of the given k populations is selected depending on the outcome of the experiment. It is guaranteed to contain the population of interest with probability P^*, the basic probability requirement in these procedures. This 'subset selection' formulation is due to Gupta [38], [41]. Some recent contributions to this aspect of the problem are Gnanadesikan [30], Gnanadesikan and Gupta [31], Gupta and McDonald [44], McDonald [57], Panchapakesan [60], Gupta and Panchapakasan [45], Gupta and Studden [46], Santner [69], Huang [50], Huang [49], Gupta and Huang [42].

The sequential and multistage aspects of the ranking and selection problems have been explored by Bechhofer, Dunnett and Sobel [14], Bechhofer [12], Bechhofer and Blumenthal [13], and Paulson [62], [63], [64], [65]. Nearly all of this work in sequential and multistage procedures is based on the indifference zone approach. Barron and Gupta [9] and Huang [50] consider sequential procedures using the subset selection approach.

An optimum theory was developed for the first approach by Bahadur [2], Bahadur and Goodman [3], Lehmann [55] and Eaton [26]. Contributions toward optimum properties of subset selection approach have also been made by Goel and Rubin [33], Govindarajulu and Harvey [36], Gupta [39], Deely and Gupta [25], Lehmann [54], Robbins [68], Seal [70], [71], [72] and Studden [77].

The main purpose of this thesis is to study some problems using the subset selection approach and make some contributions.

Chapter I deals with some selection and ranking procedures for the smallest unknown parameter of k Poisson populations. In Section 1.2, a procedure is derived to select a subset containing the best of several Poisson populations. In Section 1.3, a procedure conditioned on the total sum of the observations is proposed. A different selection procedure of the type suggested by Seal is considered in Section 1.4. In Section 1.5, selection of populations better than a standard is discussed. An application to a test of homogeneity is described in Section 1.6. Tables related to the selection procedures are given at the end of this chapter. These tables give the necessary constants to carry out the procedure and also evaluate the efficiency of the procedure in terms of the probability of a correct selection and the expected proportion in the selected subset under specified configurations of parameters.

Chapter II discusses some results on subset selection procedures for double exponential (Laplace) distributions. Section 2.1 deals with some characteristics and use of this distribution as a model. In Section 2.2, a selection procedure for the location parameters is proposed and studied using the subset selection approach. Also selection with respect to largest location parameter using the indifference zone approach is considered in Section 2.3. Section 2.4 gives a discussion of selecting the t-best populations. In Section 2.5 a procedure is proposed for subset selection with respect to the scale parameter. In Section 2.6, a test of homogeneity is given which is based on the sample median range. The distribution of a statistic associated with the procedure in Section 2.2 is considered in Section 2.7. Tables of the upper percentage points of $Y = \max_{1 \leq i \leq p} (X_i - X_o)$ where $X_o, X_1, \ldots, X_p$ are independent and identically distributed Laplace random variables with scale parameter unity are given at the end of this chapter.

In Chapter III, the subset selection approach is used to the problem of classification of k univariate normal populations. In Section 3.2, two classification rules with respect to the mean are proposed according as the k populations have (i) common known variance σ^2 and (ii) common unknown variance σ^2. In Section 3.3, a classification rule with respect to the variance is given. These rules might not classify π_o as any one of the k populations. Hence different classification procedures, with respect to the mean and the reciprocal of the coefficient of variation, which classify π_o as at least one of the k populations are proposed and studied in Section 3.4 and Section 3.5, respectively.

Chapter IV deals with some selection procedures for the negative binomial populations. A statistic of type $c \max_{1 \leq j \leq k} X_j - X_i$ is used in Section 4.2 where X_i denotes the number of failures before the r_ith success from the ith negative binomial population. In Section 4.3, a rule based on the same statistic as in Section 4.2 but conditioned on ΣX_i, the total number of observations, is investigated. The problem of selecting all populations better than a standard is considered in Section 4.4. An application is given in Section 4.5. For k = 2 and various values of r, t and P^*, the tables of the constants $c_{15}(t)$ required for the procedure in Section 4.3 are given at the end of the chapter.

CHAPTER I

ON SUBSET SELECTION PROCEDURES FOR POISSON POPULATIONS

1.1 Introduction

Poisson distribution has been used as a model in several statistical problems. As early as 1898, Bortkiewicz [18] used it to fit the data pertaining to the deaths by kicks from horses in a regiment. Poisson process is used as a model in many applied probability problems, for example, for the waiting time, for arrivals of calls at a telephone exchange, for arrivals of radioactive particles at a Geiger counter, etc.

In this chapter our object is to study the problem of comparing k Poisson distributions. Not much work has been done on this problem. More specifically, we consider the problem of selecting a subset of k Poisson populations including the best which is associated with the smallest value of the parameter. Gupta and Huang [43] have considered the selection problem according to the largest value of the parameter. However, a procedure of the type proposed by them does not work for the problem of selection with respect to the smallest parameter. Goel [32] has shown that the usual type of selection procedures do not exist for some values of the probability P^* of a correct selection. In this chapter, we propose a procedure different from that of Gupta and Huang [43] for subset selection which exists for all P^*. The rule is based on a result of Chapman [21] who showed that there is no unbiased estimator of the ratio $\frac{\lambda_1}{\lambda_2}$ with finite variance, where λ_1, λ_2 are expected values of two independent random variables X_1, X_2 with Poisson distributions, but that the estimator $\frac{X_1}{X_2+1}$ is "almost unbiased".

Let $\pi_1, \pi_2, \ldots, \pi_k$ be k independent Poisson populations, i.e. π_i has a Poisson distribution with unknown parameter λ_i, $i = 1,2,\ldots,k$. Suppose that we take n independent observations $X_{i1},\ldots,X_{in}$ from each population π_i, $i=1,\ldots,k$. A sufficient statistic for λ_i is ΣX_{ij}, hence without loss of generality we will assume the sample size to be one. Let $\lambda_{[1]} \leq \lambda_{[2]} \leq \cdots \leq \lambda_{[k]}$ be the ordered values of the parameters; it is assumed that there is no a priori information available about the correct pairing of the ordered $\lambda_{[i]}$ and the k given populations from which observations are taken.

Given any $P^*(\frac{1}{k} < P^* < 1)$, we wish to select a non-empty (small) subset of these k populations such that the subset contains the population corresponding to the parameter $\lambda_{[1]}$ with probability at least P^*, no matter what the configuration of $\lambda_1, \lambda_2, \ldots, \lambda_k$ is. We use the notation CS for correct selection where CS means that the selected subset includes the best population. Therefore we are interested in defining a selection procedures R such that

$$\inf_{\underline{\lambda}\in\Omega} P_{\underline{\lambda}}(CS|R) \geq P^* \qquad (1.1.1)$$

where Ω is the set of all k-tuples $\underline{\lambda} = (\lambda_1,\lambda_2,\ldots,\lambda_k)$, $\lambda_i > 0$, $i = 1,2,\ldots,k$.

Let $X_1, X_2, \ldots, X_k$ denote the independent observations from populations $\pi_1, \pi_2, \ldots, \pi_k$, respectively. Let $X_{(i)}$ be that value of $X_1,\ldots,X_k$ which is associated with $\lambda_{[i]}$; of course $X_{(i)}$ is unknown.

In Section 1.2, we discuss a subset selection rule so as to satisfy the basic probability requirement (1.1.1), and to find an upper bound for the expected subset size. In Section 1.3, we consider a conditional selection procedure conditioned on the total sum of the observations; a method for constructing the constants (conservative) and an upper bound for the expected subset size are derived for this conditional rule. Section 1.4 deals with a different selection procedure of the type suggested by Seal for the normal means problem. We also discuss the Seal type procedure conditioning on the total sum of the observation, in which case the selection constant can be determined precisely so as to satisfy the basic probability requirement. An exact expression for the expected subset size of the conditional Seal type procedure is stated in Theorem 1.4.5. Selection procedures for selecting a subset which contains all populations better than a standard are considered in Section 1.5. An application to a test of homogeneity is mentioned in Section 1.6. Tables related to the selection procedures are given at the end of this chapter.

1.2 The Unconditional Selection Procedure R_1

(A) The Rule R_1 and the Probability of Correct Selection

R_1: Select the population π_i in the subset if and only if

$$X_i \leq c_1 \min_{1\leq j\leq k} x_j + c_1$$

where $c_1 \geq 1$ is to be chosen so as to satisfy the basic probability requirement (1.1.1).

For $i = 1,2,\ldots,k$, let $p_{\underline{\lambda}}(i) = P_{\underline{\lambda}}$ (select population $\pi_{(i)}|R_1$).

<u>Theorem 1.2.1.</u> $P_{\underline{\lambda}}(i)$ is a decreasing function in $\lambda_{[i]}$ when all other λ's are fixed and $p_{\underline{\lambda}}(i)$ is an increasing function in $\lambda_{[j]}$, $j \neq i$, when all other λ's are fixed.

<u>Proof</u>. $p_{\underline{\lambda}}(i) = P_{\underline{\lambda}}(\text{select population } \pi_{(i)}|R_1)$

$$= P_{\underline{\lambda}}(X_{(i)} \le c_1 \min_{1\le j\le k} X_{(j)} + c_1)$$

$$= P_{\underline{\lambda}}(X_{(i)} \le c_1 X_{(j)} + c_1, \quad j = 1,2,\ldots,k,\ j \neq i)$$

$$= \sum_{x=0}^{\infty} e^{-\lambda_{[i]}} \frac{\lambda_{[i]}^x}{x!} \{ \prod_{\substack{j=1\\ j\neq i}}^{k} \sum_{\ell=<\frac{x}{c_1}-1>}^{\infty} e^{-\lambda_{[j]}} \frac{\lambda_{[j]}^{\ell}}{\ell!} \}$$

where $< \alpha >$ is the smallest integer $\ge \alpha$.

$$= \sum_{x=0}^{\infty} e^{-\lambda_{[i]}} \frac{\lambda_{[i]}}{x!} \prod_{\substack{j=1\\ j\neq i}}^{k} \int_0^{\lambda_{[j]}} \frac{1}{\Gamma(<\frac{x}{c_1} - 1>)} y^{<\frac{x}{c_1}-1>-1} e^{-y}\, dy$$

$$= \sum_{x=0}^{\infty} f(x;\ \lambda_{[1]},\ldots,\hat{\lambda}_{[i]},\ldots,\lambda_{[k]}) e^{-\lambda_{[i]}} \frac{\lambda_{[i]}^x}{x!}$$

$$= E_{\lambda_{[i]}} f(x;\ \lambda_{[1]},\ldots,\hat{\lambda}_{[i]},\ldots,\lambda_{[k]}) \qquad (1.2.1)$$

where $f(x;\lambda_{[1]},\ldots,\hat{\lambda}_{[i]},\ldots,\lambda_{[k]}) = \prod_{\substack{j=1\\ j\neq i}}^{k} \int_0^{\lambda_{[j]}} \frac{1}{\Gamma(<\frac{x}{c_1}->1)} y^{<\frac{x}{c_1}-1>-1} e^{-y}dy$

and $\hat{a}$ denote that a is delected. From (1.2.1), it is obvious that $p_{\underline{\lambda}}(i)$ is increasing in $\lambda_{[j]}$, $j \neq i$, when all other λ's are fixed. On the other hand, for fixed $\lambda_{[j]}$, $j \neq i$, f is a decreasing function in x and Poisson distribution belongs to the SI(Stochastically increasing) family, so by a lemma on P. 112 Lehmann [53],

$p_{\underline{\lambda}}(i) = E_{\lambda_{[i]}} f(x; \lambda_{[1]},\ldots,\hat{\lambda}_{[i]},\ldots,\lambda_{[k]})$ is a decreasing function in $\lambda_{[i]}$ when other λ's are fixed. Hence, the Theorem is proved.

Let $\Omega_0 = \{\underline{\lambda} = (\lambda_1,\lambda_2,\ldots,\lambda_k),\ \lambda_1 = \lambda_2 = \ldots = \lambda_k = \lambda,\ \lambda > 0\}$.

Corollary 1.2.1. $\inf\limits_{\underline{\lambda}\in\Omega} P_{\underline{\lambda}}(CS|R_1) = \inf\limits_{\underline{\lambda}\in\Omega_0} P_{\underline{\lambda}}(CS|R_1)$

$$= \inf_{\lambda > 0} \sum_{x=0}^{\infty} e^{-\lambda} \frac{\lambda^x}{x!} \{ \sum_{\ell=<\frac{x}{c_1}-1>}^{\infty} e^{-\lambda} \frac{\lambda^\ell}{\ell!} \}^{k-1} .$$

Proof. The proof follows directly from Theorem 1.2.1.

It should be pointed out that the infimum depends on the common unknown λ, $0 < \lambda$. In Section 1.7, we discuss numerical methods to determine this infimum and the constant for the selection rule.

Under the parameter space , the joint distribution of $X_1,X_2,\ldots,X_k$, given $\sum_1^k X_i = t$, is a multinomial distribution with parameters t; $\theta_1,\ldots,\theta_k$, where $\theta_j = \frac{\lambda_j}{\Sigma\lambda_j}$, $j = 1,\ldots,k$, i.e.

$$P(X_1 = x_1,\ldots,X_k = x_k | \sum_1^k X_i = t) = \frac{t!}{x_1!\ldots x_k!} \theta_1^{x_1} \ldots \theta_k^{x_k} \quad (1.2.2)$$

Lemma 1.2.1. For any t, $t \geq 0$, $c_1(t) \geq 1$ and for $\underline{\lambda} \in \Omega_0$

$$P_{\underline{\lambda}}(X_1 \leq c_1(t) \min_{2\leq j\leq k} X_j + c_1(t) \mid \sum_1^k X_i = t)$$

$$= \Sigma \frac{t!}{x_1! \ldots x_k!} (\frac{1}{k})^t$$

where the summation is over all x_i's such that $x_1 \leq c_1(t) \min\limits_{2\leq j\leq k} x_j + c_1(t)$, $x_i \geq 0$, for $i = 1,2,\ldots,k$, $\sum_1^k x_i = t$.

Let $$A(k,t,c_1(t)) = \sum_{\substack{x_1<c_1(t) \min\limits_{2\leq j\leq k} x_j+c_1(t) \\ x_i \geq 0,\ \Sigma x_i = t}} \frac{t!}{x_1! \ldots x_k!} (\frac{1}{k})^t \quad (1.2.3)$$

Theorem 1.2.2. For given P^*, any t, $t \geq 0$, let $c_1(t)$ be the smallest non-negative number such that $A(k,t,c_1(t)) \geq P^*$. If $c_1 = \sup\{c_1(t): t \geq 0\}$, then

$$\inf_{\underline{\lambda}\in\Omega} P_{\underline{\lambda}}(CS|R_1) \geq P^*.$$

Proof. For $\underline{\lambda} \in \Omega_0$,

$$\begin{aligned} P_{\underline{\lambda}}(CS|R_1) &= P_{\underline{\lambda}}(X_{(1)} \leq c_1 \min_{1\leq j\leq k} X_{(j)} + c_1) \\ &\geq P_{\underline{\lambda}}(X_{(1)} \leq c_1(t) \min_{1\leq j\leq k} X_{(j)} + c_1(t)) \\ &= \sum_{t=0}^{\infty} P_{\underline{\lambda}}(X_{(1)} \leq c_1(t) \min_{2\leq j\leq k} X_{(j)} + c_1(t) | \sum_1^k X_i = t) \\ &\qquad \cdot P_{\underline{\lambda}}(\sum_1^k X_i = t) \\ &= \sum_{t=0}^{\infty} A(k,t,c_1(t))\, P_{\underline{\lambda}}(\sum_1^k X_i = t) \\ &\geq P^*. \end{aligned}$$

Thus, the Theorem follows from Corollary 1.2.1.

(B) An Upper Bound on the Expected Subset Size Associated with R_1

Let S denote the size of the selected subset, then S is a random variable taking values $1,2,\ldots,k$. Let us consider the special case $\lambda_{[1]} = \delta\lambda$, $\delta < 1$, $\lambda_{[2]} = \cdots = \lambda_{[k]} = \lambda$, $\lambda > \lambda_0 > 0$, and denote the space of all slippage configurations of this type by Ω_1. We discuss the expected subset size as follows.

Theorem 1.2.3.
$$E_{\Omega_1}(S|R_1) \leq k - \left[\inf_{t\geq[c_1]+1} g(t) + (k-1) \inf_{t\geq[c_1]+1} h(t)\right] \int_0^{(1+\delta)\lambda_0} \frac{1}{([c_1])!} y^{[c_1]} e^{-y}\, dy$$

where

$$g(t) = \sum_{s=0}^{[\frac{t-c_1}{1+c_1}]} \binom{t}{s} \left(\frac{1}{1+\delta}\right)^s \left(\frac{\delta}{1+\delta}\right)^{t-s} \tag{1.2.4}$$

and

$$h(t) = \sum_{s=0}^{[\frac{t-c_1}{1+c_1}]} \binom{t}{s} \left(\frac{\delta}{1+\delta}\right)^s \left(\frac{1}{1+\delta}\right)^{t-s} \tag{1.2.5}$$

<u>Proof</u>. For $\underline{\lambda} \varepsilon \Omega_1$,

$$E_{\underline{\lambda}}(S|R_1) = P_{\underline{\lambda}}(X_{(1)} \le c_1 \min_{2\le j\le k} X_{(j)} + c_1) + (k-1) P_{\underline{\lambda}}(X_{(k)} \le c_1 \min_{1\le j\le k-1} X_{(j)} + c_1)$$

$$= k - P_{\underline{\lambda}}(X_{(1)} > c_1 \min_{2\le j\le k} X_{(j)} + c_1) - (k-1) P_{\underline{\lambda}}(X_{(k)} > c_1 \min_{1\le j\le k-1} X_{(j)} + c_1)$$

$$\le k - P_{\underline{\lambda}}(X_{(1)} > c_1 X_{(2)} + c_1) - (k-1) P_{\underline{\lambda}}(X_{(k)} > c_1 X_{(1)} + c_1)$$

$$= k - \sum_{t=0}^{\infty} P_{\underline{\lambda}}(X_{(1)} > c_1 X_{(2)} + c_1 \mid X_{(1)} + X_{(2)} = t) P_{\underline{\lambda}}(X_{(1)} + X_{(2)} = t)$$

$$- (k-1) \sum_{t=0}^{\infty} P_{\underline{\lambda}}(X_{(k)} > c_1 X_{(1)} + c_1 | X_{(1)} + X_{(k)} = t) P_{\underline{\lambda}}(X_{(1)} + X_{(k)} = t)$$

$$= k - \sum_{t=0}^{\infty} P_{\underline{\lambda}}(X_{(2)} < \frac{t-c_1}{1+c_1} \mid X_{(1)} + X_{(2)} = t) \; P_{\underline{\lambda}}(X_{(1)} + X_{(2)} = t)$$

$$- (k-1) \sum_{t=0}^{\infty} P_{\underline{\lambda}}(X_{(1)} < \frac{t-c_1}{1+c_1} \mid X_{(1)} + X_{(k)} = t) \; P_{\underline{\lambda}}(X_{(1)} + X_{(k)} = t)$$

$$= k - \sum_{t=[c_1]+1}^{\infty} \sum_{s=0}^{[\frac{t-c_1}{1+c_1}]} \binom{t}{s} \left(\frac{1}{1+\delta}\right)^s \left(\frac{\delta}{1+\delta}\right)^{t-s} e^{-(1+\delta)\lambda} \frac{[(1+\delta)\lambda]^t}{t!}$$

$$- (k-1) \sum_{t=[c_1]+1}^{\infty} \sum_{s=0}^{[\frac{t-c_1}{1+c_1}]} \binom{t}{s} \left(\frac{\delta}{1+\delta}\right)^s \left(\frac{1}{1+\delta}\right)^{t-s} e^{-(1+\delta)\lambda} \frac{[(1+\delta)\lambda]^t}{t!}$$

where $[a]$ is the greatest integer $\leq a$.

$$\leq k - \inf_{t \geq [c_1]+1} g(t) \sum_{t=[c_1]+1}^{\infty} e^{-(1+\delta)\lambda} \frac{[(1+\delta)\lambda]^t}{t!}$$

$$- (k-1) \inf_{t \geq [c_1]+1} h(t) \sum_{t=[c_1]+1}^{\infty} e^{-(1+\delta)\lambda} \frac{[(1+\delta)\lambda]^t}{t!}$$

$$= k - \left[\inf_{t \geq [c_1]+1} g(t) + (k-1) \inf_{t \geq [c_1]+1} h(t)\right] \int_0^{(1+\delta)\lambda} \frac{1}{([c_1])!} y^{[c_1]} e^{-y} dy$$

$$\leq k - \left[\inf_{t \geq [c_1]+1} g(t) + (k-1) \inf_{t \geq [c_1]+1} h(t)\right] \int_0^{(1+\delta)\lambda_0} \frac{1}{([c_1])!} y^{[c_1]} e^{-y} dy$$

This completes the proof.

1.3 The Conditional Procedure R_2

R_2: Select the population π_i in the subset if and only if

$$X_i \leq c_2(t) \min_{1 \leq j \leq k} X_j + c_2(t), \quad \text{given} \sum_1^k X_i = t \tag{1.3.1}$$

where $t \geq 0$ and $c_2(t) \geq 1$ is the smallest non-negative number chosen to satisfy the basic probability requirement (1.1.1).

For this rule R_2 we obtain an exact result for $k = 2$ in Theorem 1.3.1. For $k \geq 3$, we have a lower bound for the probability of a correct selection in Theorem 1.3.4.

(A) Property of the Rule R_2

A monotonicity property of the rule R_2 is discussed in the following theorem. As before, let $p_{\underline{\lambda}}(i)$ denote the probability of selecting population $\pi_{(i)}$ using rule R_2.

Theorem 1.3.1. For $i < j$, we have $p_{\underline{\lambda}}(i) \geq p_{\underline{\lambda}}(j)$.

Proof. Since

$$p_{\underline{\lambda}}(i) = P_{\underline{\lambda}}(\text{select population } \pi_{(i)})$$

$$= P_{\underline{\lambda}}(X_{(i)} \leq c_2(t) \min_{\ell \neq i} X_{(\ell)} + c_2(t) \mid \sum_{\ell=1}^{k} X_{(\ell)} = t)$$

$$= \sum_{\substack{x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_j,\ldots,x_k \\ x_i \leq c_2(t) \min x_\ell + c_2(t)}} \sum_{x_i \leq \frac{c_2(t)(x_i+x_j)+c_2(t)}{1+c_2(t)}} \binom{x_i+x_j}{x_i} \left(\frac{p_{[i]}}{p_{[i]}+p_{[j]}}\right)^{x_i} \left(\frac{p_{[j]}}{p_{[i]}+p_{[j]}}\right)^{x_j} \cdot$$

$$\binom{t}{x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_j,\ldots,x_k,\ x_i+x_j}$$

$$p_{[1]}^{x_1} \cdots \hat{p}_{[i]}^{x_i} \cdots \hat{p}_{[j]}^{x_j} \cdots p_{[k]}^{x_k} (p_{[i]}+p_{[j]})^{x_i+x_j} .$$

where $p_{[i]} = \dfrac{\lambda_{[i]}}{\sum_{\ell=1}^{k} \lambda_{[\ell]}}$ and $\hat{x}_i$ denote that x_i is deleted. Note that when x_i and x_j are interchanged, the second part in the above summand remains unchanged, and Binomial distribution belongs to the SI family, hence $p_{\underline{\lambda}}(i)$ is decreasing in $\dfrac{p_{[i]}}{p_{[i]}+p_{[j]}}$. So, if $i < j$, we have $p_{\underline{\lambda}}(i) \geq p_{\underline{\lambda}}(j)$.

(B) The Probability of a Correct Selection for R_2

Theorem 1.3.2. For a given P^*, $\frac{1}{k} < P^* < 1$, $k = 2$ and any $t \geq 0$, let $c_2(t)$ be the smallest value such that

$$P_{\Omega_0}\left(X_1 \leq \frac{c_2(t)\ (1+t)}{1 + c_2(t)} \;\middle|\; X_1 + X_2 = t\right) \geq P^*. \tag{1.3.2}$$

Then, $\inf\limits_{\underline{\lambda}\varepsilon\Omega} P_{\underline{\lambda}}(CS|R_2) = \inf\limits_{\underline{\lambda}\varepsilon\Omega_0} P_{\underline{\lambda}}(CS|R_2) \geq P^*$.

Proof. For $\underline{\lambda} \varepsilon \Omega$,

$$\begin{aligned} P_{\underline{\lambda}}(CS|R_2) &= P_{\underline{\lambda}}(X_{(1)} \leq c_2(t)X_{(2)}+c_2(t)|X_{(1)} + X_{(2)} = t) \\ &= P_{\underline{\lambda}}\left(X_{(1)} \leq \frac{c_2(t)(1+t)}{1 + c_2(t)} \;\middle|\; X_{(1)} + X_{(2)} = t\right) \\ &= \sum_{x=0}^{\left[\frac{c_2(t)(1+t)}{1 + c_2(t)}\right]} \binom{t}{x} \left(\frac{\lambda_{[1]}}{\lambda_{[1]} + \lambda_{[2]}}\right)^{x} \left(1 - \frac{\lambda_{[1]}}{\lambda_{[1]} + \lambda_{[2]}}\right)^{t-x} \end{aligned}$$

For fixed $\lambda_{[2]}$, $\dfrac{1}{1 + \frac{\lambda_{[2]}}{\lambda_{[1]}}}$ increases with $\lambda_{[1]}$ to $\frac{1}{2}$, and since Binomial distribution belongs to the SI family, so the right hand side of the above expression decreases as $\frac{\lambda_{[1]}}{\lambda_{[1]} + \lambda_{[2]}}$ increases to the value $\frac{1}{2}$.

Hence, $\inf\limits_{\underline{\lambda}\varepsilon\Omega} P_{\underline{\lambda}}(CS|R_2) = \inf\limits_{\underline{\lambda}\varepsilon\Omega_0} P_{\underline{\lambda}}(CS|R_2)$. Thus complete the proof.

For $k \geq 3$, we need the following definitions in order to discuss the least favorable configuration of the probability of a correct selection of the conditional rule.

Definition 1.3.1. If $a_1 \geq a_2 \geq \ldots \geq a_m$, $b_1 \geq b_2 \geq \ldots \geq b_m$, $\sum_{i=1}^{r} a_i \geq \sum_{i=1}^{r} b_i$ for $r = 1,2,\ldots,m-1$, and $\sum_{i=1}^{m} a_i = \sum_{i=1}^{m} b_i$, then

$\underline{a} = (a_1, a_2, \ldots, a_m)$ is said to majorize $\underline{b} = (b_1, b_2, \ldots, b_m)$, written $\underline{a} \succ \underline{b}$ or equivalently $\underline{b} \prec \underline{a}$.

Definition 1.3.2. If a function φ satisfies the property that $\varphi(\underline{x}) \leq \varphi(\underline{y})$ $(\varphi(\underline{x}) \geq \varphi(\underline{y}))$ whenever $\underline{x} \succ \underline{y}$, then φ is called a Schur-concave (Schur-convex) function.

We need the following lemma, due to Rinott [67], which is stated below without proof.

Lemma 1.3.1. Let $\underline{X} = (X_1, \ldots, X_k)$ have the multinomial distribution

$$P(\underline{X} = \underline{x}) = \binom{N}{x_1, \ldots, x_k} \prod_{i=1}^{k} \theta_i^{x_i}$$

where $\underline{x} = (x_1, \ldots, x_k)$, $\sum_{i=1}^{k} x_i = N$, and $\sum_{i=1}^{k} \theta_i = 1$.

Let $\phi(x_1, \ldots, x_k)$ be a Schur function. Then $E_{\underline{\theta}}\phi(\underline{X})$ is a Schur function.

Let $\Omega_2 = \{\underline{\lambda} = (\lambda_1, \ldots, \lambda_k)$: $\lambda_{[1]} = \ldots = \lambda_{[k-1]} = \lambda$, $\lambda_{[k]} = \lambda'$, $0 < \lambda < \lambda'\}$.

Theorem 1.3.3. $\inf_{\underline{\lambda}\varepsilon\Omega} P_{\underline{\lambda}}(CS|R_2) = \inf_{\underline{\lambda}\varepsilon\Omega_2} P_{\underline{\lambda}}(CS|R_2)$

Proof. $P_{\underline{\lambda}}(CS|R_2) = P_{\underline{\lambda}}(X_{(1)} \leq c_2(t) \min_{2\leq j\leq k} X_{(j)} + c_2(t) \mid \sum_{i=1}^{k} X_i = t)$

$$= \sum_{y_1=0}^{t} \binom{t}{y_1} p_1^{y_1} (1-p_1)^{t-y_1} \cdot \Sigma \binom{t-y_1}{y_2, \ldots, y_k} \prod_{j=2}^{k} \left(\frac{p_j}{1-p_1}\right)^{y_j}$$

where $p_i = \dfrac{\lambda_{[i]}}{\sum_{j=1}^{k} \lambda_{[j]}}$, $i = 1, \ldots, k$ and the summation is over all those $y_2, \ldots, y_k$ such that $y_j \geq \dfrac{y_1 - c_2(t)}{c_2(t)}$, $j = 2, \ldots, k$ and $\sum_{j=2}^{k} y_j = t - y_1$.

Let

$$\phi_{y_1}(y_2,\ldots,y_k) = \begin{cases} 1 & \text{if } \min\limits_{2\le i\le k} y_i \ge \dfrac{y_1 - c_2(t)}{c_2(t)} \\ 0 & \text{otherwise} \end{cases}$$

then $P_{\underline{\lambda}}(CS|R_2)$ can be written as $E_{Y_1}[E[\phi_{y_1}(Y_2,\ldots,Y_k)|\ Y_1 = y_1]]$ where the joint distribution of $Y_1,\ldots,Y_k$ is a multinomial distribution with parameters t; $p_1,\ldots,p_k$. It is easy to see that for a fixed y_1, $\phi_{y_1}(y_2,\ldots,y_k)$ is a Schur-concave function. By Lemma 1.3.1., $E[\phi_{y_1}(Y_2,\ldots,Y_k)|\ Y_1 = y_1]$ is a Schur-concave function in $\frac{p_2}{1-p_1},\ldots,\frac{p_k}{1-p_1}$. Now, since $p_j \ge p_1$, $j = 2,\ldots,k$ and $p_2 + \ldots + p_k = 1 - p_1$, we have $(\frac{p_2}{1-p_1},\ldots,\frac{p_k}{1-p_1}) \succ (\frac{p_1}{1-p_1},\ldots,\frac{p_1}{1-p_1}, \frac{1-(k-1)p_1}{1-p_1})$. Hence $P_{\underline{\lambda}}(CS|R_2)$ is minimized when $p_1 = p_2 = \ldots = p_{k-1}$ and $p_k = 1 - (k-1)\ p_1$, or when $\lambda_{[1]} = \ldots = \lambda_{[k-1]} = \lambda$, $\lambda_{[k]} = \lambda' > \lambda$. Thus the proof is completed.

Under the parameter space Ω_2, the joint distribution of $X_1,\ldots,X_k$ given $\sum_{i=1}^{k} X_i = t$, is a multinomial distribution with parameters t; $p_1,\ldots,p_k$; where $p_1 = \ldots = p_{k-1} = \frac{\lambda}{(k-1)\lambda+\lambda'} = p$, $p_k = \frac{\lambda'}{(k-1)\lambda+\lambda'} = q$, $p < q$, i.e.,

$$P_{\underline{\lambda}}(X_1=x_1,\ldots,X_k= x_k|\ \sum_{i=1}^{k} X_i = t) = \frac{t!}{x_1!\ \ldots\ x_k!}\, p^{\sum_1^{k-1} x_i}\, q^{x_k}\ .$$

Theorem 1.3.4.

$$\inf_{\underline{\lambda}\in\Omega} P_{\underline{\lambda}}(CS|R_2) = \inf_{0<\lambda<\lambda'} \sum_{\substack{x_1\le c_2(t)\min\limits_{j\ne 1} x_j + c_2(t) \\ x_i \ge 0,\ \Sigma x_i = t}} \frac{t!}{x_1!\ldots x_k!}\left[\frac{1}{k-1+\frac{\lambda'}{\lambda}}\right]^t \left(\frac{\lambda'}{\lambda}\right)^{x_k}$$

Proof. For $\underline{\lambda} \in \Omega_2$,

$$P_{\underline{\lambda}}(CS|R_2) = P_{\underline{\lambda}}\{X_{(1)} \leq c_2(t) \min_{j\neq 1} X_{(j)} + c_2(t) \mid \sum_{i=1}^{k} X_i = t\}$$

$$= \sum_{\substack{x_1 \leq c_2(t)\min_{j\neq 1} x_j + c_2(t) \\ x_i \geq 0,\ \Sigma x_i = t}} \frac{t!}{x_1!\cdots x_k!} \left[\frac{\lambda}{(k-1)\lambda+\lambda'}\right]^{\sum_1^{k-1} x_i} \left[\frac{\lambda'}{(k-1)\lambda+\lambda'}\right]^{x_k}$$

The Theorem follows from Theorem 1.3.3. after simplification.

Theorem 1.3.5. For $k \geq 3$, and for any P^*, let $P_2^* = 1 - \frac{1-p^*}{k-1}$, $0 \leq r \leq t$, let $c_2(r)$ be the smallest value such that

$$P_{\Omega_0}(X_1 \leq c_2(r)\, X_2 + c_2(r) \mid X_1 + X_2 = r) \geq P_2^* .$$

If $c_2(t) = \max\{c_2(r):\ 0 \leq r \leq t\}$, then

$$\inf_{\underline{\lambda}\in\Omega} P_{\underline{\lambda}}(CS|R_2) \geq P^*$$

Proof. For $\underline{\lambda} \in \Omega$,

$$P_{\underline{\lambda}}(CS|R_2) = P_{\underline{\lambda}}(X_{(1)} \leq c_2(t) \min_{2\leq j\leq k} X_{(j)} + c_2(t) \mid \sum_{i=1}^{k} X_i = t)$$

$$\geq 1 - \sum_{j=2}^{k} P_{\underline{\lambda}}(X_{(1)} > c_2(t)\, X_{(j)} + c_2(t) \mid \sum_{i=1}^{k} X_i = t)$$

$$= 1 - \sum_{j=2}^{k} [1 - P_{\underline{\lambda}}(X_{(1)} \leq c_2(t) X_{(j)} + c_2(t) \mid \sum_{i=1}^{k} X_i = t)$$

$$= 2 - k + \sum_{j=2}^{k} P_{\underline{\lambda}}(X_{(1)} \leq c_2(t) X_{(j)} + c_2(t) \mid \sum_{i=1}^{k} X_i = t)$$

now, we note that for fixed j $(j = 2,\ldots,k)$,

$$P_{\underline{\lambda}}(X_{(1)} \leq c_2(t)X_{(j)}+c_2(t) | \sum_{i=1}^{k} X_i=t) = \sum_{r=0}^{t} P_{\underline{\lambda}}(X_{(1)} \leq c_2(t)\, X_{(j)}$$

$$+ c_2(t) | X_{(1)}+X_{(j)}= r) \cdot P_{\underline{\lambda}}(X_{(1)}+X_{(j)}= r)$$

$$= \sum_{r=0}^{t} P_{\underline{\lambda}}(X_{(1)} \leq \frac{c_2(t)(1+r)}{1 + c_2(t)} \mid X_{(1)}+X_{(j)} = r)\, P_{\underline{\lambda}}(X_{(1)} + X_{(j)} = r)$$

$$= \sum_{r=0}^{t} \sum_{s=0}^{[\frac{c_2(t)(1+r)}{1 + c_2(t)}]} \binom{r}{s} \left(\frac{p_1}{p_1+p_j}\right)^s \left(\frac{p_j}{p_1+p_j}\right)^{r-s} P_{\underline{\lambda}}(X_{(1)} + X_{(j)} = r)$$

where $p_\ell = \frac{\lambda_{[\ell]}}{\sum_{i=1}^{k} \lambda_{[i]}}$, but Binomial distribution belongs to the SI family, so infimumn of the expression

$$\sum_{s=0}^{[\frac{c_2(t)(1+r)}{1 + c_2(t)}]} \binom{r}{s} \left(\frac{p_1}{p_1 + p_j}\right)^s \left(\frac{p_j}{p_1 + p_j}\right)^{r-s}$$

takes place when $p_1 = p_j$, i.e., when $\lambda_{[1]} = \lambda_{[j]}$. Hence,

$$\inf_{\underline{\lambda}\varepsilon\Omega} P_{\underline{\lambda}}(X_{(1)} \leq c_2(t)X_{(j)}+c_2(t) | \sum_{i=1}^{k} X_i = t) = \inf_{\underline{\lambda}\varepsilon\Omega_0} P_{\underline{\lambda}}(X_{(1)} \leq c_2(t)X_{(j)}$$

$$+ c_2(t) | \sum_{i=1}^{k} X_i = t) .$$

and for $\underline{\lambda} \varepsilon \Omega_0$,

$$P_{\underline{\lambda}}(X_{(1)} \leq c_2(t)\, X_{(j)} + c_2(t) \mid \sum_1^k X_i = t)$$

$$= \sum_{r=0}^{t} P_{\underline{\lambda}}(X_1 \leq c_2(t)\, X_2 + c_2(t) | X_1 + X_2 = r) \cdot P_{\underline{\lambda}}(X_1 + X_2 = r)$$

$$\geq \sum_{r=0}^{t} P_{\underline{\lambda}}(X_1 \leq c_2(r)\, X_2 + c_2(r) | \, X_1 + X_2 = r) \cdot P(X_1 + X_2 = r)$$

$$\geq P_2^*$$

Thus, we have the result.

Hence, for each k and P*, Theroem 1.3.5. guarantees the existence of $c_2(t)$ and gives a method to find $c_2(t)$ for given $\sum_1^k X_i = t$ such that $P_{\underline{\lambda}}(CS|R_2) \geq P^*$ for any $\underline{\lambda} \in \Omega$.

(C) An Upper Bound on the Expected Subset Size for R_2

For the procedure R_2, the subset size S of the selected subset is a random variable which can take on only integer values from 1 to k, inclusively. For any fixed values of k and P*, the expected size of the selected subset is a function of the true configuration $\underline{\lambda} = (\lambda_1,\ldots,\lambda_k)$. Now, consider the special case, $\lambda_{[1]} = \delta\lambda$, $\delta < 1$, $\lambda_{[2]} = \ldots = \lambda_{[k]} = \lambda$, $\lambda > 0$. Let us denote the space of all slippage configuration of the type discussed here by Ω_3. We investigate the expected subset size as follows:

Theorem 1.3.6. $$E_{\Omega_3}(S|R_2) \leq k - \sum_{r=0}^{t} \sum_{s=0}^{\left[\frac{r-c_2(t)}{1+c_2(t)}\right]-1} \binom{r}{s} \{\delta^{r-s} + (k-1)\delta^s\} \cdot \binom{t}{r} \cdot \frac{(k-2)^{t-r}}{(k-1+\delta)^t}$$

Proof.

$$E_{\Omega_3}(S|R_2) = P_{\underline{\lambda}}(\pi_{(1)} \text{ is selected}) + \sum_{j=2}^{k} P_{\underline{\lambda}}(\pi_{(j)} \text{ is selected})$$

$$= P_{\underline{\lambda}}(X_{(1)} \leq c_2(t) \min_{2\leq j\leq k} X_{(j)} + c_2(t) \mid \sum_{i=1}^{k} X_i = t)$$

$$+ (k-1)\, P_{\underline{\lambda}}(X_{(k)} \leq c_2(t) \min_{1\leq j\leq k-1} X_{(j)} + c_2(t) \mid \sum_{i=1}^{k} X_i = t)$$

$$= k - P_{\underline{\lambda}}(X_{(1)} > c_2(t) \min_{2\leq j\leq k} X_{(j)} + c_2(t) \mid \sum_{i=1}^{k} X_i = t)$$

$$- (k-1)\, P_{\underline{\lambda}}(X_{(k)} > c_2(t) \min_{1\leq j\leq k-1} X_{(j)}+c_2(t) \mid \sum_{i=1}^{k} X_i=t)$$

$$\leq k - P_{\underline{\lambda}}(X_{(1)} > c_2(t)X_{(2)} + c_2(t) \mid \sum_{1}^{k} X_i = t)$$

$$- (k-1)\, P_{\underline{\lambda}}(X_{(k)} > c_2(t)\, X_{(1)} + c_2(t) \mid \sum_{1}^{k} X_i = t)$$

$$= k - \frac{1}{P_{\underline{\lambda}}(\sum_{1}^{k} X_i = t)} P_{\underline{\lambda}}(X_{(1)} > c_2(t)\, X_{(2)} + c_2(t), \sum_{1}^{k} X_i = t)$$

$$- \frac{k-1}{P_{\underline{\lambda}}(\sum_{1}^{k} X_i = t)} P_{\underline{\lambda}}(X_{(k)} > c_2(t)\, X_{(1)} + c_2(t), \sum_{1}^{k} X_i = t).$$

Now, we note that

$$P_{\underline{\lambda}}(X_{(1)} > c_2(t)\, X_2(t) + c_2(t), \sum_{1}^{k} X_i = t)$$

$$= \sum_{r=0}^{t} P_{\underline{\lambda}}(X_{(1)} > c_2(t)X_{(2)}+c_2(t),\ X_{(1)}+X_{(2)} = r,\ \sum_{i\neq 1,2} X_{(i)} = t-r)$$

$$= \sum_{r=0}^{t} P_{\underline{\lambda}}(X_{(1)} > c_2(t)X_{(2)}+c_2(t),$$

$$X_{(1)}+X_{(2)}=r \mid \sum_{i\neq 1,2} X_{(i)}=t-r)\cdot P_{\underline{\lambda}}(\sum_{i\neq 1,2} X_{(i)} = t-r)$$

$$= \sum_{r=0}^{t} P_{\underline{\lambda}}(X_{(1)} > c_2(t)X_{(2)}+c_2(t),\ X_{(1)}+X_{(2)}=r)\cdot P_{\underline{\lambda}}(\sum_{i\neq 1,2} X_{(i)}=t-r)$$

since the event $\{\sum_{i\neq 1,2} X_{(i)} = t-r\}$ is independent of the event $\{X_{(1)} > c_2(t)\, X_{(2)} + c_2(t),\ X_{(1)} + X_{(2)} = r\}$.

$$= \sum_{r=0}^{t} P_{\underline{\lambda}}(X_{(1)} > c_2(t)X_{(2)}+c_2(t)\,|\,X_{(1)}+X_{(2)}=r)\cdot P_{\underline{\lambda}}(X_{(1)}+X_{(2)}=r) \cdot P_{\lambda}(\sum_{i\neq 1,2} X_{(i)}=t-r)$$

and a similar expression for $P_{\underline{\lambda}}(X_{(k)} > c_2(t)X_{(1)}+c_2(t),\ \sum_{i=1}^{k} X_i = t)$. So,

$$E_{\Omega_3}(S|R_2) \le k - \frac{1}{P_{\underline{\lambda}}(\sum_{i=1}^{k} X_i=t)} \sum_{r=0}^{t} P_{\underline{\lambda}}(X_{(1)} > c_2(t)X_{(2)}+c_2(t)\,|\,X_{(1)}+X_{(2)}=r) \cdot P_{\underline{\lambda}}(X_{(1)}+X_{(2)}=r)\cdot P_{\underline{\lambda}}(\sum_{i\neq 1,2} X_{(i)}=t-r)$$

$$- \frac{k-1}{P_{\underline{\lambda}}(\sum_{i=1}^{k} X_i=t)} \sum_{r=0}^{t} P_{\underline{\lambda}}(X_{(k)} > c_2(t)X_{(1)}+c_2(t)\,|\,X_{(1)}+X_{(k)}=r) \cdot P_{\underline{\lambda}}(X_{(1)}+X_{(k)}=r)\cdot P_{\underline{\lambda}}(\sum_{i\neq 1,k} X_{(i)}=t-r)$$

$$= k - \frac{1}{P_{\underline{\lambda}}(\sum_{i=1}^{k} X_i=t)} \sum_{r=0}^{t} P_{\underline{\lambda}}\left(X_{(2)} < \frac{r - c_2(t)}{1 + c_2(t)}\,\middle|\, X_{(1)}+X_{(2)}=r\right) \cdot P_{\underline{\lambda}}(X_{(1)}+X_{(2)}=r)\cdot P_{\underline{\lambda}}(\sum_{i\neq 1,2} X_{(i)}=t-r)$$

$$- \frac{k-1}{P_{\underline{\lambda}}(\sum_{i=1}^{k} X_i=t)} \sum_{r=0}^{t} P_{\underline{\lambda}}\left(X_{(1)} < \frac{r-c_2(t)}{1+c_2(t)}\,\middle|\, X_{(1)}+ X_{(k)}=r\right). \cdot P_{\underline{\lambda}}(X_{(1)}+X_{(k)}=r)\cdot P_{\underline{\lambda}}(\sum_{i\neq 1,k} X_{(i)}=t-r)$$

$$= k - \frac{t!}{[(k-1+\delta)\lambda]^t e^{-(k-1+\delta)\lambda}} \sum_{r=0}^{t} \sum_{s=0}^{[\frac{r-c_2(t)}{1+c_2(t)}]-1} \binom{r}{s} \left(\frac{1}{1+\delta}\right)^s \left(\frac{\delta}{1+\delta}\right)^{r-s}$$

$$\frac{[(1+\delta)\lambda]^r e^{-(1+\delta)\lambda}}{r!} \quad \frac{[(k-2)\lambda]^{t-r} e^{-(k-2)\lambda}}{(t-r)!}$$

$$- \frac{(k-1)t!}{[(k-1+\delta)\lambda]^t e^{-(k-1+\delta)\lambda}} \sum_{r=0}^{t} \sum_{s=0}^{[\frac{r-c_2(t)}{1+c_2(t)}]-1} \binom{r}{s} \left(\frac{\delta}{1+\delta}\right)^s \left(\frac{1}{1+\delta}\right)^{r-s}$$

$$\frac{[(1+\delta)\lambda]^r e^{-(1+\delta)\lambda}}{r!} \cdot \frac{[(k-2)\lambda]^{t-r} e^{-(k-2)\lambda}}{(t-r)!}$$

After simplifying, we have the result.

1.4 A Different Selection Procedure

(A) The Selection Procedure R_3 and Its Expected Subset Size

In this section we consider a selection procedure of the type suggested by Seal [70].

R_3: Select population π_i if and only if

$$X_i \le c_3 + \frac{c_3}{k-1} \sum_{j \ne i} X_i$$

where $c_3 \ge 1$ is the smallest constant determined from $P(CS|R_3) \ge P^*$.

Theorem 1.4.1. $\inf_{\underline{\lambda} \varepsilon \Omega} P_{\underline{\lambda}}(CS|R_3) = \inf_{\underline{\lambda} \varepsilon \Omega_0} P_{\underline{\lambda}}(CS|R_3)$

Proof. For $\underline{\lambda} \varepsilon \Omega$,

$$P_{\underline{\lambda}}(CS|R_3) = P_{\underline{\lambda}}\{X_{(1)} \le c_3 + \frac{c_3}{k-1} \sum_{j=2}^{k} X_{(j)}\}$$

$$= \sum_{i=0}^{\infty} e^{-\lambda_{[1]}} \frac{\lambda_{[1]}^i}{i!} \sum_{\ell=<(k-1)(\frac{i}{c_3}-1)>}^{\infty} e^{-\sum_{j=2}^{k} \lambda_{[j]}} \frac{(\sum_{j=2}^{k} \lambda_{[j]})^\ell}{\ell!}$$

$$= \sum_{i=0}^{\infty} e^{-\lambda_{[1]}} \frac{\lambda_{[1]}^{i}}{i!} \left[\int_0^{\sum_{j=2}^{k} \lambda_{[j]}} \frac{1}{\Gamma(<(k-1)(\frac{i}{c_3}-1)>)} y^{<(k-1)(\frac{i}{c_3}-1)>-1} e^{-y} dy \right] .$$

The proof follows by observing that the expression in the square brackets is a monotone increasing function of $\sum_{j=2}^{k} \lambda_{[j]}$.

By using an analogous argument as in the proof of Theorem 1.2.2., we have the following theorem.

Theorem 1.4.2. For any P*, any t, $t \geq 0$, let $c_3(t)$ be the smallest value such that

$$\sum_{i=0}^{\left[\frac{(k-1)c_3(t)+tc_3(t)}{k-1+c_3(t)}\right]} \binom{t}{i} \left(\frac{1}{k}\right)^{i} \left(\frac{k-1}{k}\right)^{t-i} \geq P^*.$$

If $c_3 = \text{Sup}\,\{c_3(t):\ t \geq 0\}$, then $\inf_{\underline{\lambda}\varepsilon\Omega} P_{\underline{\lambda}}(CS|R_3) \geq P^*$.

Consider the special configuration $\lambda_{[1]} = \delta\lambda$, $\delta < 1$; $\lambda_{[2]} = \ldots = \lambda_{[k]} = \lambda$, $\lambda > \lambda_o > 0$. Using the same notation as in Section 1.2, the space of all such slippage configuration is denoted by Ω_1. In the following theorem, we give an upper bound for the expected subset size S.

Theorem 1.4.3. $E_{\Omega_1}(S|R_3) \leq \sup_{r \geq 2} g(r) + (k - \sup_{r \geq 2} g(r))(1+(k-1+\delta)\lambda_o)e^{-(k-1+\delta)\lambda_o}$

where

$$g(r) = \sum_{s=0}^{\left[\frac{c_3(r+k-1)}{c_3+k-1}\right]} \left\{ \binom{r}{s} \left(\frac{\delta}{k-1+\delta}\right)^s \left(1 - \frac{\delta}{k-1+\delta}\right)^{r-s} + (k-1)\binom{r}{s}\left(\frac{1}{k-1+\delta}\right)^s \left(1 - \frac{1}{k-1+\delta}\right)^{r-s} \right\}$$

Proof.

$$E_{\Omega_1}(S|R_3) = P\{X_{(1)} \le c_3 + \frac{c_3}{k-1}\sum_{j=2}^{k} X_{(j)}\} + (k-1)P\{X_{(k)} \le c_3 + \frac{c_3}{k-1}\sum_{j=1}^{k-1} X_{(j)}\}$$

$$= \sum_{r=0}^{\infty} P\{X_{(1)} \le c_3 + \frac{c_3}{k-1}\sum_{j=2}^{k} X_{(j)} \mid \sum_{i=1}^{k} X_i = r\} P\{\sum_{i=1}^{k} X_i = r\}$$

$$+ (k-1)\sum_{r=0}^{\infty} P\{X_{(k)} \le c_3 + \frac{c_3}{k-1}\sum_{j=1}^{k-1} X_{(j)} \mid \sum_{i=1}^{k} X_i = r\} P\{\sum_{i=1}^{k} X_i = r\}$$

$$= \sum_{r=0}^{\infty} P\{X_{(1)} \le \frac{c_3(r+k-1)}{c_3+k-1} \mid \sum_{i=1}^{k} X_i = r\} P\{\sum_{i=1}^{k} X_i = r\}$$

$$+ (k-1)\sum_{r=0}^{\infty} P\{X_{(k)} \le \frac{c_3(r+k-1)}{c_3+k-1} \mid \sum_{i=1}^{k} X_i = r\} P\{\sum_{i=1}^{k} X_i = r\}$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\left[\frac{c_3(r+k-1)}{c_3+k-1}\right]} \left\{ \binom{r}{s}\left(\frac{\delta}{k-1+\delta}\right)^s \left(1 - \frac{\delta}{k-1+\delta}\right)^{r-s} + (k-1)\binom{r}{s}\left(\frac{1}{k-1+\delta}\right)^s \left(1 - \frac{1}{k-1+\delta}\right)^{r-s} \right\} \cdot e^{-(k-1+\delta)\lambda} \frac{[(k-1+\delta)\lambda]^r}{r!}$$

$$\le k\{e^{-(k-1+\delta)\lambda} + e^{-(k-1+\delta)\lambda}(k-1+\delta)\lambda\} + \sup_{r \ge 2} g(r) \cdot \sum_{r=2}^{\infty} e^{-(k-1+\delta)\lambda} \frac{[(k-1+\delta)\lambda]^r}{r!}$$

$$= \sup_{r\geq 2} g(r) + (k-\sup_{r\geq 2} g(r)) \{e^{-(k-1+\delta)\lambda} + e^{-(k-1+\delta)\lambda}(k-1+\delta)\lambda\}$$

$$\leq \sup_{r\geq 2} g(r) + (k-\sup_{r\geq 2} g(r)) [1 + (k-1+\delta)\lambda_o] e^{-(k-1+\delta)\lambda_o}$$

The proof is completed.

(B) A Conditional Procedure R_4

We also consider a conditional rules as follows.

R_4: Select the population π_i if and only if

$$X_i \leq c_4(t) + \frac{c_4(t)}{k-1} \sum_{j\neq i} X_j \quad \text{given} \quad \sum_{i=1}^{k} X_i = t .$$

We know that $X_1,\ldots,X_k$ given $\sum_{i=1}^{k} X_i = t$ is distributed as multinomial with parameters $t; \frac{\lambda_1}{\sum_{j=1}^{k}\lambda_j},\ldots,\frac{\lambda_k}{\sum_{j=1}^{k}\lambda_j}$, and the marginal distribution of X_i given $\sum_{i=1}^{k} X_i = t$ is binomial with parameters t and $\frac{\lambda_i}{\sum_{j=1}^{k}\lambda_j}$.

Theorem 1.4.4. $\inf_{\underline{\lambda}\epsilon\Omega} P_{\underline{\lambda}}(CS|R_4) = \inf_{\underline{\lambda}\epsilon\Omega_o} P_{\underline{\lambda}}(CS|R_4)$

Proof. For $\underline{\lambda} \in \Omega$,

$$P_{\underline{\lambda}}(CS|R_4) = P_{\underline{\lambda}}\{X_{(1)} \leq c_4(t) + \frac{c_4(t)}{k-1} \sum_{j=2}^{k} X_{(j)} \mid \sum_{i=1}^{k} X_i = t\}$$

$$= P_{\underline{\lambda}}\{X_{(1)} \leq D(t) \mid \sum_{i=1}^{k} X_i = t\}$$

where $D(t) = \frac{c_4(t)(t+k-1)}{c_4(t)+k-1}$. Since $X_{(1)}$ given $\sum_{i=1}^{k} X_i = t$ is distributed as $B(t, \frac{\lambda_{[1]}}{\sum_{j=1}^{k}\lambda_{[j]}})$ which belongs to the SI family, hence

$$\inf_{\underline{\lambda}\in\Omega} P_{\underline{\lambda}}(CS|R_4) = \inf_{\underline{\lambda}\in\Omega_o} P_{\underline{\lambda}}(CS|R_4) = \sum_{i=0}^{[D(t)]} \binom{t}{i}\left(\frac{1}{k}\right)^i\left(\frac{k-1}{k}\right)^{t-i}.$$

Note that the infimum of the probability of a correct selection is independent of the common value λ and $c_4(t)$ is the smallest constant determined from the following inequality

$$\sum_{i=0}^{[D(t)]} \binom{t}{i}\left(\frac{1}{k}\right)^i\left(\frac{k-1}{k}\right)^{t-i} \geq P^*.$$

__Theorem 1.4.5.__ $E_{\Omega_1}(S|R_4) = \sum_{s=0}^{[D(t)]} \left\{ \binom{t}{s}\left(\frac{\delta}{k-1+\delta}\right)^s\left(\frac{k-1}{k-1+\delta}\right)^{t-s} + (k-1)\binom{t}{s}\left(\frac{1}{k-1+\delta}\right)^s\left(\frac{k-2+\delta}{k-1+\delta}\right)^{t-s}\right\}$

__Proof.__ $E_{\Omega_1}(S|R_4) = P_{\underline{\lambda}}\{X_{(1)} \leq c_4 + \frac{c_4}{k-1}\sum_{j=2}^{k} X_{(j)} | \sum_{i=1}^{k} X_i = t\}$

$$+ (k-1)\, P\{X_{(k)} \leq c_4(t) + \frac{c_4(t)}{k-1}\sum_{j=1}^{k-1} X_{(j)} | \sum_{i=1}^{k} X_i = t\}$$

$$= P_{\underline{\lambda}}\{X_{(1)} \leq D(t) | \sum_{i=1}^{k} X_i = t\} + (k-1)P\{X_{(k)} \leq D(t) | \sum_{i=1}^{k} X_i = t\}$$

The theorem follows easily.

1.5 Selecting a Subset which Contains All Populations Better Than a Standard

In this section, we discuss a related problem.

Let $\pi_o, \pi_1, \ldots, \pi_k$ be k+1 independent Poisson populations with parameters $\lambda_o, \lambda_1, \ldots, \lambda_k$ respectively. We use the same notation and definitions as in Section 1.2. Population π_i is said to be better than the standard if $\lambda_i \leq \lambda_o$. The procedure described in this section control the probability that the selected subset contains all those populations better than the standard and guarantees this probability of such a correct decision to be at least P^*. Let X_i denote the observation from π_i $(i = 1,2,\ldots,k)$. Let r_1 and r_2 denote the number of populations with $\lambda \leq \lambda_o$ and $\lambda > \lambda_o$ respectively. We discuss the following cases:

Case (i): Known Standard

We assume λ_o is known, and propose a procedure as follows:

R_{d_1}: Retain in the selected subset those and only those populations π_i for which

$$X_i \leq d_1(\lambda_o + 1) \tag{1.5.1}$$

where $d_1 \geq 1$ is the smallest number to be determined below.

The probability P_1 of a correct decision is given by

$$\begin{aligned} P_1 &= \prod_{i=1}^{r_1} P\{X_{(i)} \leq d_1(\lambda_o + 1)\} \\ &\geq \prod_{i=1}^{r_1} P_{\lambda_o}\{X_{(i)} \leq d_1(\lambda_o + 1)\} \\ &\geq \left[\sum_{j=0}^{[d_1(\lambda_o+1)]} e^{-\lambda_o} \frac{\lambda_o^j}{j!}\right]^k \end{aligned} \tag{1.5.2}$$

Remark 1.5.1. By solving for the smallest value d_1 satisfying

$$\left[\sum_{j=0}^{[d_1(\lambda_o+1)]} e^{-\lambda_o} \frac{\lambda_o^j}{j!}\right]^k \geq P^*$$

we obtain the procedure.

Case (ii): λ_o unknown.

Let X_o be an observation from π_o. Then we propose the following procedure.

R_{d_2}: Retain in the selected subset those and only those populations π_i for which

$$X_i \leq d_2(X_o + 1) \tag{1.5.3}$$

where $d_2 \geq 1$ is the smallest value to be determined below.

Then the probability P_2 of a correct decision is given by

$$\begin{aligned} P_2 &= P\{X_{(i)} \leq d_2(X_o + 1), \quad i=1,\ldots,r_1\} \\ &= \sum_{x=0}^{\infty} \{ \prod_{j=1}^{r_1} \sum_{x_j=0}^{[d_2(x+1)]} e^{-\lambda_{[j]}} \frac{\lambda_{[j]}^{x_j}}{x_j!} \} e^{-\lambda_o} \frac{\lambda_o^x}{x!} \\ &\geq \sum_{x=0}^{\infty} \{ \sum_{y=0}^{[d_2(x+1)]} e^{-\lambda_o} \frac{\lambda_o^y}{y!} \}^{r_1} e^{-\lambda_o} \frac{\lambda_o^x}{x!} \\ &\geq \{ \sum_{x=0}^{\infty} (\sum_{y=0}^{[d_2(x+1)]} e^{-\lambda_o} \frac{\lambda_o^y}{y!}) e^{-\lambda_o} \frac{\lambda_o^x}{x!} \}^{r_1} \\ &\geq \{ \sum_{x=0}^{\infty} (\sum_{y=0}^{[d_2(x+1)]} e^{-\lambda_o} \frac{\lambda_o^y}{y!}) e^{-\lambda_o} \frac{\lambda_o^x}{x!} \}^{k} \end{aligned} \tag{1.5.4}$$

$$= \{P(X_1 \leq d_2(X_o + 1))\}^k$$

where X_o, X_1 are iid Poisson with parameter λ_o.

$$= \{ \sum_{r=0}^{\infty} P(X_1 \leq d_2(X_o + 1) \mid X_o + X_1 = r) \cdot P(X_o + X_1 = r)\}^k$$

For any fixed $r \geq 0$, let $d_2(r)$ be the smallest number such that

$$A(2,\ r,\ d_2(r)) \geq \sqrt[k]{P^*}$$

where $A(2, r, d_2(r))$ is defined in (1.2.3). Let $d_2 = \sup\{d_2(r): r \geq 0\}$, we have $P_2 \geq P^*$.

1.6 Applications to a Test of Homogeneity for $\lambda_1 = \lambda_2 = \ldots = \lambda_k$

In some practical situations one wishes to know whether λ_i are significantly different or not. This is the problem of the test of homogeneity of the Poisson populations. In order to test the homogeneity of k experiments, i.e. to test H: $\lambda_1 = \lambda_2 = \ldots = \lambda_k = \lambda_o$ against the H_A: not H, we proposed the following rules ϕ_1 and $\phi_2(T)$,

(1) The procedure ϕ_1: H is accepted if, and only if, $X_{max} - c\, X_{min} \leq c$, where c is some constant depending on k, λ_o, and the level of significance α.

(2) The procedure $\phi_2(T)$: H is accepted if and only if $X_{max} - c(t)\, X_{min} \leq c(t)$, given that $T = \sum_{i=1}^{k} X_i = t$.

For the procedure ϕ_1, if we choose $c = \sup\{c(t): t \geq 0\}$, where for any t, $t \geq 0$, c(t) is the smallest constant such that

$$A(k, t, c(t)) \geq 1 - \frac{\alpha}{k} ,$$

then, for $\underline{\lambda} \in \Omega_o$, i.e., when H is true

$$P_{\underline{\lambda}}\{ \max_{1 \leq j \leq k} X_j - c \min_{1 \leq j \leq k} X_j \leq c\}$$

$$= 1 - P_{\underline{\lambda}}\{ \max_{1 \leq j \leq k} X_j > c \min_{1 \leq j \leq k} X_j + c\}$$

$$\geq 1 - \sum_{i=1}^{k} P_{\underline{\lambda}}\{X_i > c \min_{1 \leq j \leq k} X_j + c\}$$

$$= 1 - k + \sum_{i=1}^{k} P_{\underline{\lambda}}\{X_i \leq c \min_{\substack{1 \leq j \leq k \\ j \neq i}} X_j + c\}$$

$$= 1-k + \sum_{i=1}^{k} \sum_{t=0}^{\infty} P_{\underline{\lambda}}\{X_i \leq c \min_{\substack{1 \leq j \leq k \\ j \neq i}} X_j + c \mid \sum_{i=1}^{k} X_i = t\} P(\sum_{i=1}^{k} X_i = t)$$

$$\geq 1-k + \sum_{i=1}^{k} \sum_{t=0}^{\infty} P_{\underline{\lambda}}\{X_i \leq c(t) \min_{\substack{1 \leq j \leq k \\ j \neq i}} X_j + c(t) \mid \sum_{i=1}^{k} X_i = t\} P(\sum_{i=1}^{k} X_i = t)$$

$$\geq 1 - \alpha$$

by Lemma 1.2.1. Hence $P_{\underline{\lambda} \varepsilon \Omega_o}$ [Reject H] $\leq \alpha$.

The probability of the error of the first kind for $\phi_2(T)$ is then given by

$$E(\phi_2(T) \mid H,t) = P(\max_{1 \leq j \leq k} X_j - c(t) \min_{1 \leq j \leq k} X_j > c(t) \mid \sum_{i=1}^{k} X_i = t)$$

$$= P(X_i - c(t) \min_{1 \leq j \leq k} X_j > c(t) \text{ for some } i \mid \sum_{i=1}^{k} X_i = t)$$

$$\leq \sum_{i=1}^{k} P(X_i > c(t) \min_{1 \leq j \leq k} X_j + c(t) \mid \sum_{i=1}^{k} X_i = t)$$

$$= k\, P(X_1 > c(t) \min_{1 \leq j \leq k} X_j + c(t) \mid \sum_{i=1}^{k} X_i = t)$$

$$= k[1 - A(k, t, c(t)]$$

by Lemma 1.2.1. Hence, for given significance level α, we can find $c(t)$ such that $E(\phi_2(T) \mid H,\alpha) \leq \alpha$.

1.7 Explanations of the Tables

(1) Tables I, II and III list the infimum of the probability of a correct selection (approximate value) for the rules R_1, R_3 and

R_1'. R_1 and R_3 are proposed and studied in this paper and R_1' is the selection procedure for selecting a subset to include the population associated with $\lambda_{[k]}$ discussed in Gupta and Huang [43]. It should be pointed out that the probability of a correct selection for all these rules is decreasing with λ when λ is small and then it is increasing again with λ. Hence, the approximate infimum can be determined numerically by computing the probability as a function of λ, for fixed values of c. For given k and P*, the selection constants (approximately) can be found from these tables. For example, for $P^* = .8504$, and $k = 4$, the approximate value of c associated with R_1 is 2.4.

(2) In Tables IVA, IVB, IVC and IVD, the first entry denotes the probability of selecting the best population, the second entry denotes the probability of selecting a non-best population and the third entry is the expected proportion, all under the slippage configuration $\lambda_{[1]} = \delta\lambda$, $\delta < 1$; $\lambda_{[2]} = \cdots = \lambda_{[k]} = \lambda$, when the rule R_1 is used. The three entries in Table VA, VB, VC, VD define the same quantities for the rule R_3. For example, from Table IVC, we find that λ for the rule R_1 if $\lambda = 2.50$ and $c = 1.50$, (k=5 and δ=.3, the probability of a correct selection is .9461, the probability of selecting a non-best population is .4879 and the expected proportion of populations in the selected subset is .5796.

1.8 Some Remarks on the Comparison of R_1 and R_3

We define a rule R to be better than another rule R′ if the expected proportion for R is smaller than the expected proportion for R′. We compare the performance of the rules R_1 and R_3 in the aspect. For example, when $k = 5$, $P^* = 0.92$, we obtain the approximate values of selection constants for R_1 and R_3 as $c_1 = 3.0$, $c_3 = 1.55$ from Table I and Table II respectively. For these constants Tables IV, V show that if δ is kept fixed R_3 seems to be better than R_1 when λ is small, while R_1 performs better than R_3 for large values of λ.

Table I

Table of inf $P(CS|R_1)$ (Approximate) Using the Rule R_1

k \ c	1.50	1.60	1.70	1.80	1.90	2.00	2.20	2.40	2.60	2.80	3.00	3.50	4.00	4.50	5.00
2	0.8543	0.8577	0.8756	0.8762	0.8762	0.9353	0.9353	0.9391	0.9515	0.9517	0.9771	0.9792	0.9902	0.9906	0.9956
3	0.7588	0.7627	0.7892	0.7895	0.7895	0.8845	0.8845	0.8904	0.9116	0.9118	0.9566	0.9604	0.9811	0.9817	0.9913
4	0.6897	0.6936	0.7245	0.7246	0.7246	0.8431	0.8431	0.8504	0.8782	0.8784	0.9380	0.9433	0.9724	0.9733	0.9872
5	0.6369	0.6394	0.6740	0.6740	0.6740	0.8076	0.8076	0.8151	0.8478	0.8484	0.9209	0.9277	0.9643	0.9654	0.9832
6	0.5937	0.5963	0.6311	0.6313	0.6313	0.7769	0.7769	0.7845	0.8207	0.8212	0.9053	0.9135	0.9566	0.9578	0.9793
8	0.5266	0.5322	0.5643	0.5644	0.5644	0.7263	0.7263	0.7341	0.7747	0.7750	0.8774	0.8881	0.9425	0.9439	0.9720
10	0.4769	0.4807	0.5144	0.5144	0.5144	0.6858	0.6858	0.6943	0.7372	0.7374	0.8532	0.8641	0.9289	0.9314	0.9651

For given k and c, this table represents the minimum value (approximately) of

$$P_\lambda\left[X_k \le c \min_{1\le j\le k-1} X_j + c\right] = \sum_{i=0}^{\infty} e^{-\lambda}\frac{\lambda^i}{i!}\left\{\sum_{j=<\frac{i}{c}-1>}^{\infty} e^{-\lambda}\frac{\lambda^j}{j!}\right\}^{k-1}$$

where $X_1,\ldots,X_k$ are i.i.d. Poisson variables with parameter λ.

Table II

Table of inf $P(CS|R_3)$ (Approximate) Using the Rule R_3

$k \backslash c_3$	1.50	1.60	1.70	1.80	1.90	2.00	2.20	2.40	2.60	2.80	3.00	3.50	4.00	4.50	5.00
2	0.8542	0.8577	0.8756	0.8762	0.8762	0.9353	0.9353	0.9391	0.9515	0.9517	0.9771	0.9792	0.9902	0.9906	0.9956
3	0.8924	0.8996	0.9244	0.9407	0.9407	0.9575	0.9726	0.9772	0.9887	0.9887	0.9950	0.9965	0.9989	0.9990	0.9996
4	0.8949	0.9201	0.9331	0.9452	0.9624	0.9730	0.9802	0.9826	0.9924	0.9937	0.9953	0.9985	0.9995	0.9997	0.9999
5	0.9160	0.9260	0.9445	0.9573	0.9657	0.9733	0.9824	0.9889	0.9925	0.9955	0.9979	0.9993	0.9995	0.9998	0.9999
6	0.9212	0.9389	0.9550	0.9611	0.9703	0.9796	0.9852	0.9911	0.9951	0.9964	0.9982	0.9993	0.9998	0.9999	0.9999
8	0.9272	0.9453	0.9579	0.9676	0.9752	0.9828	0.9896	0.9938	0.9961	0.9973	0.9987	0.9995	0.9999	0.9999	0.9999
10	0.9277	0.9465	0.9598	0.9678	0.9789	0.9845	0.9902	0.9940	0.9969	0.9981	0.9987	0.9997	0.9999	0.9999	0.9999

For given k and c_3, this table represents the minimum value (approximately) of

$$P_\lambda\left[X_k \le \frac{c_3}{k-1}\sum_{j=1}^{k-1} X_j + c_3\right] = \sum_{i=0}^{\infty} e^{-\lambda}\frac{\lambda^i}{i!}\left\{\sum_{j=\langle (k-1)(\frac{i}{c_3}-1)\rangle}^{\infty} e^{-(k-1)\lambda}\frac{((k-1)\lambda)^j}{j!}\right\}$$

$$= \sum_{i=0}^{\infty} e^{-\lambda}\frac{\lambda^i}{i!}\left\{\int_0^{(k-1)\lambda}\frac{1}{\Gamma(\langle (k-1)(\frac{i}{c_3}-1)\rangle)}\, y^{\langle (k-1)(\frac{i}{c_3}-1)\rangle - 1} e^{-y}\,dy\right\}$$

where $X_1,\ldots,X_k$ are i.i.d. Poisson variables with parameter λ.

Table III

Table of inf $P(CS|R_1')$ (Approximate) Using the Rule R_1': $X_i + 1 \geq c' \max_{1 \leq j \leq k} X_j$

k \ c′	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90
2	0.9902	0.9771	0.9517	0.9391	0.9353	0.9352	0.8762	0.8577	0.8542	0.8103	0.8031	0.7396	0.7184	0.6761
3	0.9821	0.9591	0.9176	0.8976	0.8922	0.8922	0.8015	0.7740	0.7692	0.7063	0.6961	0.6090	0.5795	0.5254
4	0.9753	0.9448	0.8904	0.8651	0.8596	0.8596	0.7505	0.7150	0.7103	0.6372	0.6255	0.5261	0.4951	0.4376
5	0.9687	0.9328	0.8664	0.8391	0.8341	0.8340	0.7123	0.6705	0.6651	0.5866	0.5737	0.4689	0.4357	0.3800
6	0.9629	0.9211	0.8467	0.8182	0.8130	0.8130	0.6772	0.6356	0.6289	0.5448	0.5315	0.4257	0.3929	0.3358
8	0.9529	0.9021	0.8170	0.7859	0.7793	0.7792	0.6281	0.5785	0.5756	0.4869	0.4698	0.3625	0.3307	0.2757
10	0.9449	0.8875	0.7967	0.7575	0.7526	0.7525	0.5960	0.5366	0.5330	0.4489	0.4269	0.3212	0.2890	0.2368

For given k and c′, this table represents the minimum value (approximately) of

$$P_\lambda\left[X_k + 1 \geq c' \max_{1 \leq j \leq k-1} X_j\right] = \sum_{i=0}^{\infty} e^{-\lambda} \frac{\lambda^i}{i!} \left\{ \sum_{j=0}^{\left[\frac{i+1}{c'}\right]} e^{-\lambda} \frac{\lambda^j}{j!} \right\}$$

where $X_1, \ldots, X_k$ are i.i.d. Poisson variables with parameter λ.

Table IVA

Using the rule R_1 and under the configuration $(\delta\lambda,\lambda,\ldots,\lambda)$, this table gives in order the triple (a) the probability of selecting a best population, (b) the probability of selecting any non-best population and (c) the expected proportion of the selected populations ([(a)+(k-1)(b)]/k).

$k = 3,\ \delta = 0.3$

λ \ c_1	1.5	1.75	2.0	2.5	3.0	3.5	4.0	4.5	5.0
0.50	0.9913	0.9913	0.9995	0.9995	0.9999	0.9999	0.9999	0.9999	0.9999
	0.9146	0.9146	0.9863	0.9864	0.9983	0.9983	0.9998	0.9998	0.9999
	0.9402	0.9402	0.9907	0.9907	0.9988	0.9988	0.9988	0.9998	0.9999
0.75	0.9842	0.9842	0.9988	0.9988	0.9999	0.9999	0.9999	0.9999	0.9999
	0.8443	0.8443	0.9636	0.9637	0.9934	0.9934	0.9990	0.9990	0.9998
	0.8909	0.8909	0.9754	0.9754	0.9956	0.9956	0.9993	0.9993	0.9999
1.00	0.9777	0.9777	0.9978	0.9978	0.9998	0.9998	0.9999	0.9999	0.9999
	0.7761	0.7761	0.9322	0.9327	0.9841	0.9841	0.9969	0.9969	0.9995
	0.8433	0.8433	0.9541	0.9544	0.9893	0.9893	0.9979	0.9979	0.9996
1.50	0.9695	0.9696	0.9956	0.9956	0.9995	0.9995	0.9999	0.9999	0.9999
	0.6654	0.6658	0.8580	0.8615	0.9526	0.9527	0.9866	0.9866	0.9967
	0.7668	0.7671	0.9038	0.9062	0.9682	0.9683	0.9910	0.9910	0.9978
2.00	0.9678	0.9679	0.9940	0.9941	0.9991	0.9991	0.9999	0.9999	0.9999
	0.5889	0.5912	0.7857	0.7974	0.9114	0.9125	0.9678	0.9678	0.9898
	0.7152	0.7168	0.8551	0.8630	0.9406	0.9413	0.9785	0.9785	0.9932
2.50	0.9699	0.9702	0.9932	0.9935	0.9988	0.9988	0.9998	0.9998	0.9999
	0.5333	0.5405	0.7239	0.7491	0.8698	0.8734	0.9434	0.9437	0.9783
	0.6788	0.6837	0.8137	0.8306	0.9128	0.9152	0.9622	0.9624	0.9855

Table IVA (continued)

k = 3, δ = 0.3

λ \ c_1	1.5	1.75	2.0	2.5	3.0	3.5	4.0	4.5	5.0
3.00	0.9736	0.9742	0.9932	0.9938	0.9986	0.9986	0.9997	0.9997	0.9999
	0.4880	0.5034	0.6729	0.7146	0.8332	0.8415	0.9179	0.9190	0.9632
	0.6499	0.6604	0.7797	0.8077	0.8883	0.8939	0.9452	0.9459	0.9755
3.50	0.9775	0.9786	0.9937	0.9945	0.9986	0.9986	0.9997	0.9997	0.9999
	0.4478	0.4741	0.6304	0.6890	0.8029	0.8182	0.8947	0.8972	0.9469
	0.6244	0.6423	0.7515	0.7908	0.8681	0.8783	0.9297	0.9314	0.9646
4.50	0.9811	0.9827	0.9944	0.9954	0.9987	0.9987	0.9997	0.9997	0.9999
	0.4111	0.4495	0.5945	0.6680	0.7783	0.8020	0.8752	0.8803	0.9314
	0.6011	0.6272	0.7278	0.7771	0.8518	0.8676	0.9167	0.9201	0.9542
5.00	0.9866	0.9890	0.9960	0.9971	0.9990	0.9991	0.9997	0.9997	0.9999
	0.3481	0.4083	0.5360	0.6307	0.7411	0.7822	0.8480	0.8609	0.9075
	0.5609	0.6018	0.6893	0.7528	0.8271	0.8545	0.8986	0.9072	0.9383
6.00	0.9904	0.9931	0.9973	0.9983	0.9993	0.9994	0.9998	0.9998	0.9999
	0.2980	0.3722	0.4892	0.5961	0.7134	0.7679	0.8313	0.8537	0.8940
	0.5288	0.5791	0.6586	0.7302	0.8087	0.8451	0.8874	0.9024	0.9293
8.00	0.9952	0.9973	0.9988	0.9994	0.9997	0.9998	0.9999	0.9999	0.9999
	0.2252	0.3052	0.4168	0.5400	0.6723	0.7400	0.8108	0.8466	0.8837
	0.4818	0.5359	0.6108	0.6931	0.7815	0.8266	0.8739	0.8977	0.9225
10.00	0.9977	0.9988	0.9995	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999
	0.1738	0.2479	0.3616	0.4996	0.6421	0.7186	0.7984	0.8382	0.8797
	0.4485	0.4982	0.5742	0.6663	0.7614	0.8124	0.8655	0.8921	0.9198
15.00	0.9996	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
	0.0955	0.1614	0.2647	0.4257	0.5901	0.8920	0.7838	0.8337	0.8810
	0.3969	0.4409	0.5098	0.6171	0.7267	0.7947	0.8559	0.8891	0.9206

Table IVB

Using the rule R_1 and under the configuration $(\delta\lambda, \lambda, \ldots, \lambda)$, this table gives in order the triple (a) the probability of selecting a best population, (b) the probability of selecting any non-best population and (c) the expected proportion of the selected populations $([(a)+(k-1)(b)]/k)$.

$k = 3,\ \delta = 0.5$

λ \ c_1	1.5	1.75	2.0	2.5	3.0	3.5	4.0	4.5	5.0
0.50	0.9775	0.9775	0.9981	0.9981	0.9998	0.9998	0.9999	0.9999	0.9999
	0.9174	0.9174	0.9868	0.9868	0.9984	0.9984	0.9998	0.9998	0.9999
	0.9375	0.9375	0.9906	0.9906	0.9988	0.9988	0.9998	0.9998	0.9999
0.75	0.9601	0.9601	0.9951	0.9952	0.9995	0.9995	0.9999	0.9999	0.9999
	0.8541	0.8541	0.9660	0.9661	0.9939	0.9939	0.9991	0.9991	0.9998
	0.8894	0.8894	0.9757	0.9758	0.9957	0.9957	0.9993	0.9993	0.9999
1.00	0.9452	0.9452	0.9913	0.9913	0.9989	0.9989	0.9998	0.9998	0.9999
	0.7871	0.7971	0.9388	0.9395	0.9857	0.9857	0.9972	0.9972	0.9995
	0.8465	0.8465	0.9563	0.9568	0.9901	0.9901	0.9981	0.9981	0.9996
1.50	0.9281	0.9282	0.9835	0.9838	0.9971	0.9971	0.9995	0.9995	0.9999
	0.7159	0.7168	0.8809	0.8857	0.9609	0.9612	0.9890	0.9890	0.9973
	0.7866	0.7873	0.9151	0.9184	0.9730	0.9731	0.9925	0.9925	0.9982
2.00	0.9249	0.9256	0.9782	0.9794	0.9951	0.9952	0.9990	0.9990	0.9998
	0.6679	0.6728	0.8322	0.8467	0.9334	0.9347	0.9760	0.9761	0.9924
	0.7536	0.7570	0.8809	0.8910	0.9540	0.9549	0.9837	0.9837	0.9949
2.50	0.9284	0.9306	0.9761	0.9786	0.9938	0.9939	0.9985	0.9985	0.9997
	0.6349	0.6483	0.7960	0.8248	0.9105	0.9146	0.9619	0.9623	0.9854
	0.7327	0.7424	0.8560	0.8761	0.9383	0.9410	0.9741	0.9744	0.9902

Table IVB (continued)

$k = 3,\ \delta = 0.5$

λ \ c_1	1.5	1.75	2.0	2.5	3.0	3.5	4.0	4.5	5.0
3.00	0.9339	0.9384	0.9762	0.9802	0.9933	0.9935	0.9981	0.9981	0.9995
	0.6066	0.6332	0.7695	0.8138	0.8945	0.9029	0.9501	0.9512	0.9779
	0.7157	0.7350	0.8384	0.8692	0.9274	0.9331	0.9661	0.9668	0.9851
3.50	0.9394	0.9466	0.9777	0.9828	0.9935	0.9939	0.9980	0.9980	0.9994
	0.5805	0.6227	0.7495	0.8076	0.8841	0.8982	0.9419	0.9442	0.9713
	0.7001	0.7307	0.8255	0.8660	0.9206	0.9301	0.9606	0.9621	0.9807
4.00	0.9443	0.9542	0.9798	0.9857	0.9941	0.9946	0.9980	0.9980	0.9993
	0.5570	0.6146	0.7338	0.8028	0.8777	0.8977	0.9372	0.9413	0.9666
	0.6861	0.7278	0.8158	0.8638	0.9165	0.9300	0.9574	0.9602	0.9775
5.00	0.9534	0.9667	0.9842	0.9904	0.9957	0.9964	0.9984	0.9985	0.9994
	0.5187	0.6004	0.7108	0.7940	0.8714	0.9007	0.9345	0.9431	0.9629
	0.6636	0.7225	0.8019	0.8594	0.9128	0.9326	0.9558	0.9616	0.9751
6.00	0.9617	0.9756	0.9881	0.9936	0.9971	0.9978	0.9989	0.9990	0.9995
	0.4897	0.5845	0.6943	0.7871	0.8695	0.9037	0.9359	0.9483	0.9641
	0.6470	0.7149	0.7923	0.8559	0.9120	0.9351	0.9569	0.9652	0.9759
8.00	0.9750	0.9861	0.9936	0.9970	0.9988	0.9992	0.9996	0.9997	0.9998
	0.4454	0.5482	0.6719	0.7827	0.8713	0.9084	0.9417	0.9563	0.9696
	0.6219	0.6942	0.7791	0.8542	0.9138	0.9386	0.9610	0.9708	0.9797
10.00	0.9839	0.9917	0.9966	0.9986	0.9995	0.9997	0.9998	0.9999	0.9999
	0.4110	0.5221	0.6568	0.7837	0.8761	0.9158	0.9484	0.9621	0.9750
	0.6020	0.6786	0.7700	0.8554	0.9172	0.9438	0.9655	0.9747	0.9833
15.00	0.9947	0.9979	0.9993	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999
	0.3474	0.4849	0.6325	0.7909	0.8911	0.9347	0.9634	0.9759	0.9853
	0.5631	0.6559	0.8548	0.8605	0.9274	0.9564	0.9756	0.9839	0.9902

Table IVC

Using the rule R_1 and under the configuration $(\delta\lambda,\lambda,\ldots,\lambda)$, this table gives in order the triple (a) the probability of selecting a best population, (b) the probability of selecting any non-best population and (c) the expected proportion of the selected populations ($[(a)+(k-1)(b)]/k$).

<u>$k = 5$, $\delta = 0.3$</u>

λ \ c_1	1.50	1.75	2.00	2.50	3.00	3.50	4.00	4.50	5.00
0.50	0.9900	0.9900	0.9995	0.9995	0.9999	0.9999	0.9999	0.9999	0.9999
	0.9105	0.9105	0.9857	0.9857	0.9982	0.9982	0.9998	0.9998	0.9999
	0.9264	0.9264	0.9884	0.9884	0.9986	0.9986	0.9998	0.9998	0.9999
0.75	0.9798	0.9798	0.9985	0.9985	0.9999	0.9999	0.9999	0.9999	0.9999
	0.8315	0.8315	0.9606	0.9606	0.9929	0.9929	0.9989	0.9989	0.9998
	0.8612	0.8612	0.9682	0.9682	0.9943	0.9943	0.9991	0.9991	0.9998
1.00	0.9689	0.9689	0.9969	0.9969	0.9997	0.9997	0.9999	0.9999	0.9999
	0.7518	0.7518	0.9247	0.9249	0.9822	0.9822	0.9965	0.9965	0.9994
	0.7952	0.7952	0.9391	0.9393	0.9857	0.9857	0.9972	0.9972	0.9995
1.50	0.9516	0.9516	0.9930	0.9930	0.9992	0.9992	0.9999	0.9999	0.9999
	0.6221	0.6221	0.8382	0.8406	0.9453	0.9454	0.9845	0.9845	0.9963
	0.6880	0.6880	0.8692	0.8711	0.9561	0.9562	0.9876	0.9876	0.9970
2.00	0.9447	0.9447	0.9896	0.9897	0.9985	0.9985	0.9998	0.9998	0.9999
	0.5399	0.5408	0.7568	0.7665	0.8975	0.8985	0.9626	0.9627	0.9882
	0.6208	0.6216	0.8034	0.8112	0.9177	0.9185	0.9700	0.9701	0.9906
2.50	0.9461	0.9463	0.9877	0.9882	0.9978	0.9978	0.9996	0.9996	0.9999
	0.4879	0.4919	0.6915	0.7152	0.8515	0.8550	0.9352	0.9354	0.9751
	0.5796	0.5828	0.7508	0.7698	0.8808	0.8836	0.9481	0.9483	0.9800

Table IVC (continued)

k = 5, δ = 0.3

λ \ c_1	1.50	1.75	2.00	2.50	3.00	3.50	4.00	4.50	5.00
3.00	0.9518	0.9524	0.9874	0.9882	0.9975	0.9975	0.9995	0.9995	0.9999
	0.4487	0.4595	0.6408	0.6827	0.8135	0.8221	0.9078	0.9088	0.9587
	0.5493	0.5581	0.7101	0.7438	0.8503	0.8572	0.9261	0.9270	0.9669
3.50	0.9585	0.9598	0.9881	0.9893	0.9974	0.9974	0.9994	0.9994	0.9999
	0.4139	0.4353	0.6004	0.6609	0.7840	0.8002	0.8841	0.8868	0.9416
	0.5228	0.5402	0.6780	0.7266	0.8267	0.8396	0.9072	0.9094	0.9532
4.00	0.9648	0.9670	0.9894	0.9910	0.9975	0.9975	0.9994	0.9994	0.9998
	0.3812	0.4154	0.5671	0.6437	0.7612	0.7865	0.8654	0.8708	0.9260
	0.4980	0.5257	0.6516	0.7132	0.8084	0.8287	0.8922	0.8966	0.9408
5.00	0.9748	0.9787	0.9923	0.9943	0.9981	0.9982	0.9995	0.9998	0.9998
	0.3242	0.3825	0.5139	0.6122	0.7278	0.7715	0.8406	0.8643	0.9032
	0.4542	0.5017	0.6096	0.6886	0.7819	0.8168	0.8724	0.9321	0.9225
6.00	0.9818	0.9866	0.9948	0.9966	0.9987	0.9988	0.9996	0.9996	0.9998
	0.2794	0.3530	0.4718	0.5817	0.7034	0.7603	0.8261	0.8496	0.8912
	0.4199	0.4797	0.5764	0.6647	0.7624	0.8080	0.8608	0.8796	0.9129
8.00	0.9908	0.9946	0.9978	0.9988	0.9995	0.9996	0.9998	0.9998	0.9999
	0.2141	0.2940	0.4062	0.5315	0.6668	0.7360	0.8084	0.8448	0.8825
	0.3694	0.4341	0.5245	0.6250	0.7334	0.7888	0.8467	0.8758	0.9060
10.00	0.9955	0.9977	0.9991	0.9996	0.9998	0.9999	0.9999	0.9999	0.9999
	0.1673	0.2408	0.3552	0.4948	0.6392	0.7166	0.7972	0.8374	0.8792
	0.3329	0.3922	0.4840	0.5957	0.7113	0.7733	0.8377	0.8699	0.9034
15.00	0.9992	0.9997	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
	0.0937	0.1594	0.2628	0.4244	0.5859	0.6917	0.7837	0.8336	0.8809
	0.2748	0.3274	0.4102	0.5395	0.6716	0.7533	0.8269	0.8669	0.9047

Table IVD

Using the rule R_1 and under the configuration $(\delta\lambda,\lambda,\ldots,\lambda)$, this table gives in order the triple (a) the probability of selecting a best population, (b) the probability of selecting any non-best population and (c) the expected proportion of the selected populations $([(a)+(k-1)(b)]/k)$.

$k = 5,\ \delta = 0.5$

λ \ c_1	1.5	1.75	2.0	2.5	3.0	3.5	4.0	4.5	5.0
0.50	0.9741	0.9741	0.9978	0.9978	0.9998	0.9998	0.9999	0.9999	0.9999
	0.9109	0.9109	0.9858	0.9858	0.9982	0.9982	0.9998	0.9998	0.9999
	0.9236	0.9236	0.9882	0.9982	0.9985	0.9985	0.9998	0.9998	0.9999
0.75	0.9492	0.9492	0.9938	0.9938	0.9994	0.9994	0.9999	0.9999	0.9999
	0.8342	0.8342	0.9613	0.9613	0.9930	0.9930	0.9989	0.9989	0.9998
	0.8572	0.8572	0.9678	0.9678	0.9943	0.9943	0.9991	0.9991	0.9998
1.00	0.9239	0.9239	0.9878	0.9878	0.9985	0.9985	0.9998	0.9998	0.9999
	0.7601	0.7601	0.9273	0.9276	0.9829	0.9828	0.9967	0.9967	0.9994
	0.7929	0.7929	0.9394	0.9396	0.9860	0.9860	0.9973	0.9973	0.9995
1.50	0.8873	0.8874	0.9739	0.9742	0.9953	0.9953	0.9993	0.9993	0.9999
	0.6516	0.6519	0.8518	0.8551	0.9503	0.9505	0.9860	0.9860	0.9966
	0.6988	0.6989	0.8762	0.8789	0.9593	0.9595	0.9886	0.9886	0.9973
2.00	0.8748	0.8751	0.9629	0.9643	0.9915	0.9916	0.9983	0.9983	0.9997
	0.5954	0.5973	0.7899	0.8029	0.9139	0.9151	0.9687	0.9688	0.9902
	0.6513	0.6529	0.8245	0.8351	0.9294	0.9304	0.9747	0.9747	0.9921
2.50	0.8776	0.8792	0.9576	0.9610	0.9887	0.9888	0.9973	0.9973	0.9994
	0.5651	0.5730	0.7478	0.7772	0.8853	0.8895	0.9507	0.9511	0.9811
	0.6276	0.6342	0.7898	0.8140	0.9060	0.9094	0.9600	0.9603	0.9848

Table IVD (continued)

k = 5, δ = 0.5

λ \ c_1	1.5	1.75	2.0	2.5	3.0	3.5	4.0	4.5	5.0
3.00	0.8861	0.8904	0.9569	0.9628	0.9875	0.9878	0.9965	0.9965	0.9991
	0.5415	0.5615	0.7199	0.7684	0.8674	0.8770	0.9367	0.9379	0.9718
	0.6104	0.6273	0.7673	0.8073	0.8914	0.8992	0.9486	0.9496	0.9773
3.50	0.8950	0.9035	0.9589	0.9673	0.9876	0.9882	0.9961	0.9962	0.9989
	0.5186	0.5554	0.7008	0.7664	0.8577	0.8744	0.9282	0.9309	0.9645
	0.5939	0.6251	0.7524	0.8065	0.8837	0.8971	0.9418	0.9440	0.9713
4.00	0.9030	0.9163	0.9623	0.9724	0.9886	0.9895	0.9961	0.9962	0.9988
	0.4971	0.5520	0.6869	0.7654	0.8531	0.8769	0.9244	0.9294	0.9598
	0.5783	0.6249	0.7420	0.8068	0.8802	0.8995	0.9387	0.9427	0.9676
5.00	0.9171	0.9381	0.9701	0.9814	0.9916	0.9929	0.9969	0.9970	0.9988
	0.4634	0.5474	0.6687	0.7622	0.8512	0.8856	0.9247	0.9349	0.9576
	0.5541	0.6256	0.7289	0.8060	0.8793	0.9071	0.9391	0.9473	0.9659
6.00	0.9306	0.9543	0.9772	0.9875	0.9944	0.9957	0.9979	0.9981	0.9991
	0.4402	0.5395	0.6573	0.7599	0.8534	0.8923	0.9288	0.9429	0.9605
	0.5383	0.6225	0.8213	0.8055	0.8816	0.9130	0.9426	0.9539	0.9682
8.00	0.9536	0.9737	0.9875	0.9942	0.9976	0.9985	0.9992	0.9994	0.9996
	0.4061	0.5123	0.6439	0.7644	0.8614	0.9019	0.9381	0.9538	0.9681
	0.5156	0.6045	0.7126	0.8103	0.8886	0.9212	0.9503	0.9629	0.9744
10.00	0.9696	0.9839	0.9933	0.9973	0.9990	0.9994	0.9997	0.9998	0.9999
	0.3799	0.4934	0.6358	0.7714	0.8701	0.9123	0.9466	0.9609	0.9743
	0.4979	0.5915	0.7073	0.8166	0.8959	0.9298	0.9573	0.9687	0.9794
15.00	0.9897	0.9959	0.9986	0.9996	0.9999	0.9999	0.9999	0.9999	0.9999
	0.3298	0.4700	0.6224	0.7864	0.8895	0.9340	0.9631	0.9758	0.9852
	0.4618	0.5752	0.6977	0.8291	0.9116	0.9472	0.9705	0.9806	0.9882

Table VA

Using the rule R_3 and under the configuration $(\delta\lambda,\lambda,\ldots,\lambda)$, this table gives in order the triple (a) the probability of selecting a best population, (b) the probability of selecting any non-best population and (c) the expected proportion of the selected populations $([(a)+(k-1)(b)]/k)$.

$k = 3,\ \delta = 0.3$

λ \ c_3	1.50	1.75	2.00	2.50	3.00	3.50	4.00	4.50	5.00
.50	.9960	.9962	.9998	.9998	.9999	.9999	.9999	.9999	.9999
	.9477	.9521	.9918	.9924	.9990	.9990	.9999	.9999	.9999
	.9638	.9668	.9945	.9948	.9993	.9993	.9999	.9999	.9999
.75	.9945	.9950	.9996	.9996	.9999	.9999	.9999	.9999	.9999
	.9179	.9305	.9817	.9843	.9971	.9972	.9995	.9995	.9999
	.9434	.9520	.9877	.9894	.9981	.9981	.9997	.9997	.9999
1.00	.9939	.9948	.9994	.9995	.9999	.9999	.9999	.9999	.9999
	.8931	.9165	.9701	.9766	.9945	.9947	.9989	.9990	.9998
	.9267	.9426	.9799	.9842	.9963	.9965	.9993	.9993	.9998
1.50	.9940	.9957	.9992	.9994	.9999	.9999	.9999	.9999	.9999
	.8530	.8974	.9461	.9662	.9891	.9903	.9973	.9973	.9993
	.9000	.9302	.9638	.9773	.9927	.9935	.9981	.9982	.9995
2.00	.9948	.9969	.9991	.9995	.9999	.9999	.9999	.9999	.9999
	.8176	.8782	.9227	.9605	.9848	.9881	.9958	.9960	.9987
	.8766	.9178	.9482	.9735	.9898	.9921	.9972	.9973	.9991
2.50	.9956	.9977	.9992	.9996	.9999	.9999	.9999	.9999	.9999
	.7855	.8583	.9010	.9559	.9814	.9876	.9949	.9955	.9983
	.8555	.9048	.9337	.9705	.9875	.9917	.9966	.9970	.9988

Table VA (continued)

k = 3, δ = 0.3

λ \ c_3	1.50	1.75	2.00	2.50	3.00	3.50	4.00	4.50	5.00
3.00	.9963	.9983	.9993	.9997	.9999	.9999	.9999	.9999	.9999
	.7590	.8412	.8828	.9511	.9784	.9875	.9945	.9957	.9981
	.8381	.8936	.9216	.9673	.9856	.9916	.9963	.9971	.9987
3.50	.9969	.9987	.9994	.9998	.9999	.9999	.9999	.9999	.9999
	.7387	.8279	.8692	.9463	.9758	.9871	.9941	.9962	.9982
	.8248	.8849	.9126	.9641	.9839	.9914	.9961	.9974	.9988
4.00	.9975	.9990	.9995	.9999	.9999	.9999	.9999	.9999	.9999
	.7235	.8175	.8599	.9419	.9739	.9865	.9937	.9966	.9983
	.8149	.8780	.9064	.9612	.9826	.9910	.9958	.9977	.9988
5.00	.9985	.9995	.9997	.9999	.9999	.9999	.9999	.9999	.9999
	.7006	.7996	.8503	.9365	.9724	.9857	.9931	.9970	.9985
	.7999	.8663	.9001	.9577	.9816	.9905	.9954	.9980	.9990
6.00	.9991	.9997	.9998	.9999	.9999	.9999	.9999	.9999	.9999
	.6818	.7826	.8471	.9363	.9734	.9864	.9933	.9972	.9986
	.7876	.8550	.8980	.9575	.9822	.9909	.9955	.9981	.9990
8.00	.9997	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999
	.6537	.7637	.8475	.9426	.9780	.9900	.9951	.9979	.9990
	.7690	.8425	.8983	.9617	.9853	.9933	.9967	.9986	.9993
10.00	.9999	.9999	.9999	.9999	.9999	.9999	.9999	1.0000	1.0000
	.6336	.7641	.8514	.9494	.9827	.9930	.9969	.9987	.9994
	.7557	.8427	.9009	.9662	.9884	.9953	.9979	.9991	.9996
15.00	.9999	.9999	.9999	.9999	1.0000	1.0000	1.0000	1.0000	1.0000
	.6003	.7607	.8658	.9647	.9909	.9973	.9991	.9997	.9999
	.7335	.8404	.9105	.9764	.9939	.9982	.9994	.9998	.9999

Table VB

Using the rule R_3 and under the configuration $(\delta\lambda,\lambda,\ldots,\lambda)$, this table gives in order the triple (a) the probability of selecting a best population, (b) the probability of selecting any non-best population and (c) the expected proportion of the selected populations $([(a)+(k-1)(b)]/k)$.

$k = 3,\ \delta = 0.5$

λ \ c_3	1.50	1.75	2.00	2.50	3.00	3.50	4.00	4.50	5.00
0.50	.9894	.9901	.9991	.9992	.9999	.9999	.9999	.9999	.9999
	.9520	.9565	.9925	.9931	.9991	.9991	.9999	.9999	.9999
	.9644	.9677	.9947	.9951	.9994	.9994	.9999	.9999	.9999
0.75	.9852	.9873	.9982	.9984	.9998	.9998	.9999	.9999	.9999
	.9268	.9394	.9838	.9864	.9975	.9976	.9996	.9996	.9999
	.9463	.9554	.9886	.9904	.9983	.9983	.9997	.9997	.9999
1.00	.9830	.9868	.9975	.9980	.9997	.9997	.9999	.9999	.9999
	.9068	.9294	.9743	.9806	.9955	.9957	.9991	.9991	.9998
	.9322	.9485	.9820	.9864	.9969	.9970	.9994	.9994	.9999
1.50	.9818	.9883	.9964	.9977	.9996	.9996	.9999	.9999	.9999
	.8750	.9160	.9554	.9740	.9917	.9928	.9979	.9980	.9995
	.9106	.9401	.9691	.9819	.9943	.9950	.9986	.9986	.9996
2.00	.9819	.9901	.9958	.9981	.9995	.9996	.9999	.9999	.9999
	.8472	.9019	.9377	.9711	.9891	.9919	.9971	.9973	.9991
	.8921	.9313	.9571	.9801	.9926	.9944	.9980	.9981	.9994
2.50	.9825	.9914	.9956	.9985	.9996	.9997	.9999	.9999	.9999
	.8235	.8884	.9224	.9687	.9873	.9920	.9968	.9972	.9989
	.8765	.9227	.9468	.9787	.9914	.9946	.9978	.9981	.9993

Table VB (continued)

<u>$k = 3, \delta = 0.5$</u>

λ \ c_3	1.50	1.75	2.00	2.50	3.00	3.50	4.00	4.50	5.00
3.00	.9837	.9925	.9957	.9989	.9997	.9998	.9999	.9999	.9999
	.8059	.8782	.9111	.9663	.9858	.9923	.9967	.9976	.9989
	.8652	.9163	.9393	.9771	.9904	.9948	.9978	.9983	.9993
3.50	.9854	.9936	.9961	.9991	.9997	.9998	.9999	.9999	.9999
	.7941	.8713	.9040	.9639	.9847	.9924	.9966	.9980	.9990
	.8479	.9121	.9347	.9756	.9897	.9948	.9977	.9986	.9993
4.00	.9873	.9946	.9966	.9993	.9998	.9999	.9999	.9999	.9999
	.7860	.8662	.9005	.9622	.9942	.9923	.9965	.9983	.9991
	.8531	.9090	.9325	.9746	.9894	.9948	.9977	.9988	.9994
5.00	.9908	.9963	.9978	.9995	.9998	.9999	.9999	.9999	.9999
	.7743	.8568	.8996	.9617	.9849	.9926	.9966	.9986	.9993
	.8464	.9033	.9323	.9743	.9899	.9950	.9977	.9990	.9995
6.00	.9934	.9974	.9986	.9997	.9999	.9999	.9999	.9999	.9999
	.7655	.8489	.9023	.9645	.9867	.9937	.9970	.9988	.9994
	.8414	.8984	.9344	.9762	.9911	.9958	.9980	.9992	.9996
8.00	.9965	.9987	.9995	.9999	.9999	.9999	.9999	.9999	.9999
	.7554	.8491	.9107	.9719	.9907	.9962	.9983	.9993	.9997
	.8358	.8989	.9403	.9812	.9938	.9975	.9988	.9995	.9998
10.00	.9982	.9994	.9998	.9999	.9999	.9999	.9999	.9999	.9999
	.7507	.8588	.9198	.9781	.9937	.9978	.9991	.9996	.9998
	.8332	.9057	.9465	.9853	.9958	.9985	.9994	.9997	.9999
15.00	.9996	.9999	.9999	.9999	.9999	.9999	1.0000	1.0000	1.0000
	.8488	.8742	.9402	.9886	.9977	.9994	.9998	.9999	.9999
	.8324	.9161	.9601	.9924	.9985	.9996	.9999	.9999	.9999

Table VC

Using the rule R_3 and under the configuration $(\delta\lambda,\lambda,\ldots.,\lambda)$, this table gives in order the triple (a) the probability of selecting a best population, (b) the probability of selecting any non-best population and (c) the expected proportion of the selected populations ([(a)+(k-1)(b)]/k).

$k = 5,\ \delta = 0.3$

λ \ c_3	1.50	1.75	2.00	2.50	3.00	3.50	4.00	4.50	5.00
0.50	.9982	.9983	.9996	.9999	.9999	.9999	.9999	.9999	1.0000
	.9721	.9739	.9886	.9961	.9995	.9996	.9999	.9999	.9999
	.9773	.9788	.9908	.9969	.9996	.9997	.9999	.9999	.9999
0.75	.9979	.9982	.9992	.9998	.9999	.9999	.9999	.9999	.9999
	.9562	.9635	.9759	.9929	.9988	.9993	.9998	.9999	.9999
	.9646	.9704	.9805	.9943	.9990	.9994	.9998	.9999	.9999
1.00	.9977	.9983	.9990	.9998	.9999	.9999	.9999	.9999	.9999
	.9394	.9540	.9655	.9899	.9979	.9990	.9996	.9998	.9999
	.9510	.9629	.9722	.9919	.9983	.9992	.9997	.9998	.9999
1.50	.9976	.9986	.9991	.9998	.9999	.9999	.9999	.9999	.9999
	.9097	.9359	.9561	.9861	.9962	.9987	.9991	.9997	.9999
	.9272	.9485	.9647	.9889	.9970	.9990	.9993	.9997	.9999
2.00	.9979	.9989	.9994	.9999	.9999	.9999	.9999	.9999	.9999
	.8871	.9218	.9526	.9849	.9953	.9985	.9989	.9996	.9999
	.9092	.9372	.9620	.9879	.9962	.9988	.9991	.9997	.9999
2.50	.9983	.9992	.9996	.9999	.9999	.9999	.9999	.9999	.9999
	.8704	.9150	.9505	.9845	.9950	.9983	.9989	.9996	.9999
	.8960	.9318	.9603	.9875	.9960	.9987	.9991	.9997	.9999

Table VC (continued)

k = 5, δ = 0.3

λ \ c_3	1.50	1.75	2.00	2.50	3.00	3.50	4.00	4.50	5.00
3.00	.9987	.9994	.9998	.9999	.9999	.9999	.9999	.9999	.9999
	.8597	.9140	.9499	.9845	.9953	.9983	.9991	.9997	.9999
	.8875	.9311	.9599	.9876	.9962	.9986	.9993	.9998	.9999
3.50	.9990	.9996	.9998	.9999	.9999	.9999	.9999	.9999	.0000
	.8539	.9154	.9506	.9853	.9957	.9984	.9993	.9998	.9999
	.8829	.9322	.9604	.9882	.9966	.9987	.9994	.9998	.9999
4.00	.9993	.9998	.9999	.9999	.9999	.9999	.9999	.9999	1.0000
	.8508	.9167	.9519	.9865	.9963	.9987	.9994	.9998	.9999
	.8805	.9333	.9615	.9892	.9970	.9989	.9995	.9999	.9999
5.00	.9997	.9999	.9999	.9999	.9999	.9999	.9999	1.0000	1.0000
	.8462	.9176	.9555	.9892	.9973	.9992	.9997	.9999	.9999
	.8769	.9341	.9644	.9914	.9979	.9993	.9997	.9999	.9999
6.00	.9998	.9999	.9999	.9999	.9999	1.0000	1.0000	1.0000	1.0000
	.8437	.9193	.9596	.9914	.9981	.9995	.9998	.9999	.9999
	.8750	.9355	.9677	.9931	.9985	.9996	.9998	.9999	.9999
8.00	.9999	.9999	.9999	.9999	1.0000	1.0000	1.0000	1.0000	1.0000
	.8446	.9265	.9676	.9946	.9991	.9998	.9999	.9999	.9999
	.8757	.9412	.9741	.9957	.9993	.9998	.9999	.9999	.9999
10.00	.9999	.9999	.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	.8484	.9357	.9744	.9967	.9996	.9999	.9999	.9999	.9999
	.8787	.9486	.9974	.9996	.9999	.9999	.9999	.9999	.9999
15.00	.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	.8624	.9534	.9861	.9990	.9999	.9999	.9999	.9999	1.0000
	.8899	.9627	.9889	.9992	.9999	.9999	.9999	.9999	1.0000

Table VD

Using the rule R_3 and under the configuration $(\delta\lambda,\lambda,\ldots,\lambda)$, this table gives in order the triple (a) the probability of selecting a best population, (b) the probability of selecting any non-best population and (c) the expected proportion of the selected population $([(a)+(k-1)(b)]/k)$.

$k = 5,\ \delta = 0.5$

λ \ c_3	1.50	1.75	2.00	2.50	3.00	3.50	4.00	4.50	5.00
0.50	.9948	.9952	.9985	.9996	.9999	.9999	.9999	.9999	.9999
	.9737	.9756	.9890	.9964	.9995	.9996	.9999	.9999	.9999
	.9779	.9795	.9909	.9971	.9996	.9997	.9999	.9999	.9999
0.75	.9930	.9944	.9970	.9994	.9999	.9999	.9999	.9999	.9999
	.9589	.9663	.9773	.9935	.9989	.9994	.9998	.9999	.9999
	.9657	.9720	.9813	.9947	.9991	.9995	.9998	.9999	.9999
1.00	.9915	.9941	.9961	.9993	.9999	.9999	.9999	.9999	.9999
	.9435	.9580	.9686	.9910	.9982	.9992	.9996	.9998	.9999
	.9531	.9652	.9741	.9927	.9985	.9993	.9997	.9998	.9999
1.50	.9898	.9940	.9963	.9993	.9999	.9999	.9999	.9999	.9999
	.9175	.9424	.9617	.9883	.9968	.9990	.9993	.9997	.9999
	.9320	.9527	.9687	.9905	.9974	.9992	.9994	.9998	.9999
2.00	.9898	.9943	.9972	.9995	.9999	.9999	.9999	.9999	.9999
	.8985	.9313	.9596	.9876	.9962	.9988	.9991	.9997	.9999
	.9168	.9439	.9672	.9900	.9970	.9990	.9993	.9998	.9999
2.50	.9905	.9952	.9979	.9996	.9999	.9999	.9999	.9999	.9999
	.8854	.9274	.9587	.9876	.9962	.9987	.9992	.9997	.9999
	.9064	.9410	.9665	.9900	.9969	.9990	.9994	.9998	.9999

Table VD (continued)

k = 5, δ = 0.5

λ \ c_3	1.50	1.75	2.00	2.50	3.00	3.50	4.00	4.50	5.00
3.00	.9917	.9963	.9985	.9997	.9999	.9999	.9999	.9999	.9999
	.8781	.9284	.9592	.9881	.9965	.9988	.9994	.9998	.9999
	.9009	.9420	.9671	.9904	.9972	.9990	.9995	.9995	.9999
3.50	.9931	.9973	.9989	.9998	.9999	.9999	.9999	.9999	.9999
	.8752	.9307	.9606	.9891	.9970	.9989	.9995	.9998	.9999
	.8988	.9440	.9683	.9912	.9976	.9991	.9996	.9999	.9999
4.00	.9945	.9980	.9992	.9999	.9999	.9999	.9999	.9999	.9999
	.8740	.9326	.9625	.9903	.9975	.9991	.9996	.9999	.9999
	.8981	.9457	.9698	.9922	.9980	.9993	.9997	.9999	.9999
5.00	.9964	.9989	.9996	.9999	.9999	.9999	.9999	.9999	.9999
	.8726	.9350	.9666	.9927	.9983	.9995	.9998	.9999	.9999
	.8974	.9478	.9732	.9941	.9986	.9996	.9998	.9999	.9999
6.00	.9976	.9993	.9998	.9999	9999	.9999	.9999	1.0000	1.0000
	.8733	.9382	.9708	.9944	.9989	.9997	.9999	.9999	.9999
	.8982	.9504	.9766	.9955	.9991	.9998	.9999	.9999	.9999
8.00	.9990	.9998	.9999	.9999	.9999	.9999	1.0000	1.0000	1.0000
	.8787	.9470	.9783	.9969	.9995	.9999	.9999	.9999	.9999
	.9028	.9576	.9826	.9975	.9996	.9999	.9999	.9999	.9999
10.00	.9996	.9999	.9999	.9999	.9999	1.0000	1.0000	1.0000	1.0000
	.8858	.9560	.9840	.9983	.9998	.9999	.9999	.9999	.9999
	.9086	.9648	.9872	.9986	.9998	.9999	.9999	.9999	.9999
15.00	.9999	.9999	.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	.9046	.9720	.9927	.9996	.9999	.9999	.9999	1.0000	1.0000
	.9237	.9776	.9941	.9997	.9999	.9999	.9999	1.0000	1.0000

CHAPTER II

SOME RESULTS ON SUBSET SELECTION PROCEDURES FOR DOUBLE EXPONENTIAL POPULATIONS

2.1 Introduction

In this chapter we study the selection problems and some other related statistical inference problems for the k double exponential (Laplace) populations. Before we do this, we give some discussion of the Laplace distribution, its characteristics (vs. normal, logistic and Cauchy) and its use as a model in statistics and probability.

The double exponential distribution arises as a model in some statistical problems as explained later. This distribution is also considered in robustness studies, which suggests that it provides a model with different characteristics than some of the other commonly used models such as the normal distribution. In particular, the tails of the double exponential distribution are thicker than the tails of the normal or logistic, but not as thick as the Cauchy (see p. 43, Hajék [47]). Yet the double exponential has not been used very extensively as a model. This could be due in part to the lack of available statistical techniques for this distribution, although it is likely that the experimentor has shied away from using the double exponential because it has a sharp peak in the center. However, many applications would be primarily concerned with tail probabilities, and it would seem that the double exponential would be a useful model if exponential tails are required.

The double exponential has some application as a model in the area of Actuarial Science, and it has been suggested as a model for the distribution of the strength of flaws in materials by Epstein [27]. Using the weakest link principle, the strength of the material should decrease as the number of flaws or volume increases. In particular, from extreme-value theory the double exponential assumption leads to the result that the mode or most probable strength decreases in proportion to log n, where n represents the size or number of flaws of the material. In comparison, the assumption of a normal model leads to a decrease in proportion to $(\log n)^{1/2}$. For most applications to material strength, only the minimum flaw strength would ordinarily be observable; however, Epstein [27] suggests that there may be many other types of problems, such as a system of components in series, which might be similar from a statistical point of view. Other possible applications of the double exponential are suggested by the fact that the difference of two independent (not necessary identical) two parameter exponential variables follows the double exponential distribution, and that the logarithm of the ratios of uniform or Pareto variables follows the double exponential distribution.

In classical theory, once having assumed the form of the parent distribution, we can derive a criterion which is appropriate to this assumption. For example, under the assumption of normality, for the comparison of two means we would derive the t-statistic. It is then customary to justify the use of such a normal theory criterion in the practical circumstance in which normality cannot be guaranteed by arguing that the distribution of the characteristic is but little affected by non-normality of the parent distribution - that is, it is robust under non-normality. However, this argument ignores the fact that if the parent distribution really differed from the normal, the appropriate criterion would no longer be the normal-theory statistic. Box and Tiao [19] reconsidered the analysis of Darwin's paired data on the heights of self and cross-fertilized plants quoted by Fisher in "The Design of Experiments (1935)". In this development the parent distribution is not assumed to be normal, but only a member of the following class of symmetric distributions

$$p(y|\theta,\sigma,\beta) = \frac{1}{\Gamma[1+\frac{1}{2}(1+\beta)]2^{1+\frac{1}{2}(1+\beta)}\sigma} \exp\{-\frac{1}{2}\left|\frac{y-\theta}{\sigma}\right|^{2/(1+\beta)}\} \quad (2.1.1)$$

where $-\infty < y < \infty$, $0 < \sigma < \infty$, $-\infty < \theta < \infty$, $-1 < \beta \leq 1$. This class of distributions includes the normal (β=0) and the double exponential (β=1), and its kurtosis parameter is β.

If the probability density function of the double exponential is given by

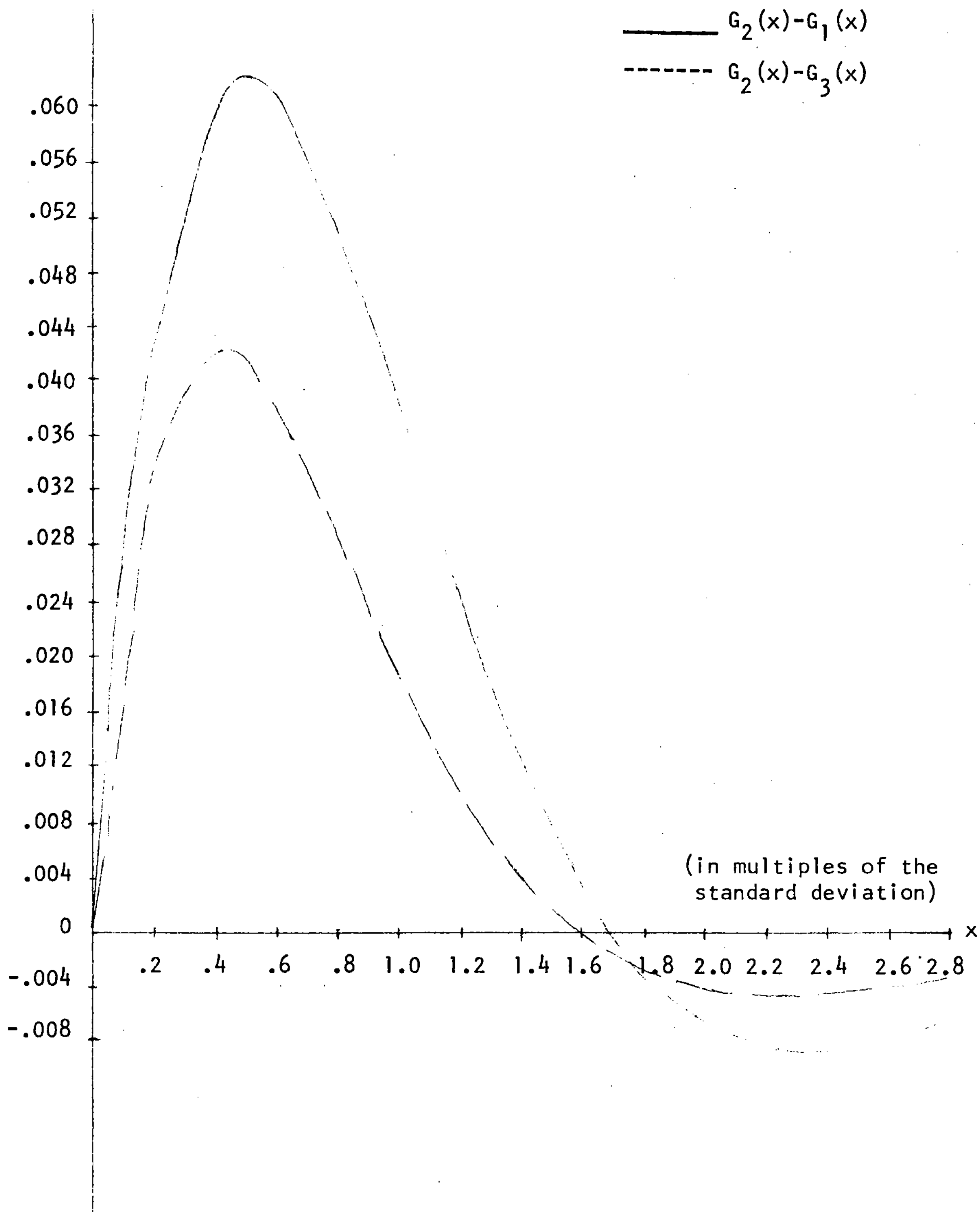
$$f(x,\theta,\sigma) = \frac{1}{2\sigma} e^{-\left|\frac{x-\theta}{\sigma}\right|}, \ -\infty < x < \infty, -\infty < \theta < \infty, \ \sigma > 0 \quad (2.1.2)$$

then the mode of the distribution is $x = \theta$ where it has a sharp peak. The expected value and standard deviation of (2.1.2) are θ and $\sqrt{2}\,\sigma$ respectively. Moments of the standardized double exponential order statistics can be obtained by using the closed-form expressions for the moments of the standardized negative exponential order statistics derived by Epstein and Sobel [28]. Govindarajulu [34] has given the expressions for these moments.

Chew [23] gives the graphs of the standardized density functions of normal, logistic and double exponential distributions, from which it is clear that the tails of the double exponential distribution are thicker than that of the normal or logistic, in the sense that the curve of double exponential is above that of the others to the left and right of some points. In the case of the normal distribution this point is 2.64.

If the cumulative distribution functions $G_1(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}u^2} du$ and $G_2(x) = \begin{cases} \frac{1}{2} e^{\sqrt{2}x} & , x < 0 \\ 1 - \frac{1}{2} e^{-\sqrt{2}x} & , x \geq 0 \end{cases}$ of the standardized normal and double exponential distributions are compared, (also similar comparison between standardized logistic $G_3(x) = 1/(1 + e^{-\frac{\pi}{\sqrt{3}}x})$ and the double exponential distribution) the differences $G_2(x) - G_1(x)$ (as well as $G_2(x) - G_3(x)$) vary in the way shown in the graph below. Since $G_1(x)$, $G_2(x)$ and $G_3(x)$ are symmetric about $x = 0$ only the values for $x \geq 0$ are shown.

With regard to point estimation, it is well known that the maximum likelihood estimates based on the complete sample of size n are given by $\hat{\Theta} = \tilde{X}$ and $\hat{\sigma} = \frac{1}{n} \sum_{i=1}^{n} |X_i - \tilde{X}|$, where $\tilde{X}$ denotes the sample median. Also best linear estimators (based on order statistics) under symmetric censoring are given by Govindarajulu [35] for sample sizes up to 20, and some alternate estimates are suggested by Raghunandanan and Srinivasan [66]. Interval estimation for the parameters of the



G2(x)-G1(x)
G2(x)-G3(x)
.060
.056
.052
.048
.044
.040
.036
.032
.028
.024
.020
.016
.012
.008
.004
0
-.004
-.008
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
2.8
x
(in multiples of the standard deviation)

two-parameter double exponential distribution is considered by Bain and Engelhardt [4].

Now we discuss the problem of comparison of $k(\geq 2)$ double exponential distributions. First we study the selection problem for the largest mean (location).

2.2 Selecting a Subset Containing the Best of Several Double Exponential Populations with Respect to the Location Parameter

(A) Formulation of the Problem

Let X_i, $i = 1,2,\ldots,k$ be k independent random variables from double exponential population π_i, $i = 1,2,\ldots,k$ respectively, with probability density function

$$f(X;\ \theta_i,\sigma) = \frac{1}{2\sigma}\exp\ [-|X-\theta_i|/\sigma],\quad -\infty < x < \infty,\quad -\infty < \theta_i < \infty,\ \sigma > 0$$

where σ is a common, known constant for each of the k populations. We may, without loss of generality, assume σ to be one. The ranked parameters are denoted by $\theta_{[1]} \leq \theta_{[2]} \leq \cdots \leq \theta_{[k]}$. As before, it is assumed that there is no a priori information available about the correct pairing of the ordered $\theta_{[i]}$ and the k given populations from which observations are taken. Any population whose parameter value equals $\theta_{[k]}$ will be defined as a best population. A correct selection (CS) is defined as the selection of any subset of the k given populations which contains at least one best population.

Suppose we take (2n+1) independent observations from π_i, $i = 1,2,\ldots,k$; the sample size (2n+1) is assumed to be given in the primary problem below. Let $P^*(\frac{1}{k} < P^* < 1)$ be a preassigned constant.

Let $P(CS; k, n, \underline{\theta}, R)$ denote the probability of a correct selection when the procedure R is used with the given k, n and when the true configuration of parameter values is $\underline{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)$; let the space of all possible values of $\underline{\theta}$ be denoted by Ω.

The problem of primary interest is to define a procedure R which selects a subset of the k given populations that is small, never empty, and large enough so that it contains the best population with probability at best P*, regardless of the true configurations $\underline{\theta}$ in Ω, i.e., so that

$$\inf_{\Omega} P(CS; k, n, \underline{\theta}, R) \geq P^* . \tag{2.2.1}$$

After having defined a particular procedure $R = R(k, n, P^*)$ for each possible set of values of k, n and P*, we discuss the expected size $E\{S; k, n, \underline{\theta}, P^*, R\}$ of the selected subset when the procedure R is used with the given k, n, P* and where $\underline{\theta}$ is the true parameter configuration in Ω.

Let Y_i denote the sample median of the (2n+1) observations $X_{i1}, \ldots, X_{i,2n+1}$; from the ith population, and let $Y_{(i)}$ denote that unknown variable which is associated with $\theta_{[i]}$. The probability density $g_n(\cdot)$ and the cumulative distribution $G_n(\cdot)$ of Y_i are given by

$$g_n(y; \theta_i) = \frac{(2n+1)!}{n!n!} \left(\frac{1}{2} e^{-|y-\theta_i|}\right)^{n+1} \left(1 - \frac{1}{2} e^{-|y-\theta_i|}\right)^{n} \tag{2.2.2}$$

$$G_n(y; \theta_i) = \begin{cases} 1 - \sum_{j=0}^{n} \binom{2n+1}{j} \left(\frac{1}{2} e^{y-\theta_i}\right)^{j} \left(1 - \frac{1}{2} e^{y-\theta_i}\right)^{2n+1-j} , & y<\theta_i \\ \sum_{j=0}^{n} \binom{2n+1}{j} \left(\frac{1}{2} e^{-(y-\theta_i)}\right)^{j} \left(1-\frac{1}{2}e^{-(y-\theta_i)}\right)^{2n+1-j} , & y\geq\theta_i \end{cases} \tag{2.2.3}$$

Now, we propose the selection procedure R_5 as follows:

R_5: Retain in the selected subset only those populations π_i for which

$$Y_i \geq \max_{1\leq j\leq k} Y_j - d \qquad (2.2.4)$$

where $d = d(k, n, P^*)$ is the smallest non-negative constant to be determined that will satisfy the basic probability requirement (2.2.1) for all configurations $\underline{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)$.

(B) Probability of a Correct Selection and Its Infimum

The following result concerning the rule R_5 can be proved.

Theorem 2.2.1. $\inf_{\underline{\theta}\in\Omega} P_{\underline{\theta}}(CS|R_5) = \inf_{\underline{\theta}\in\Omega_o} P_{\underline{\theta}}(CS|R_5) = \int_{-\infty}^{\infty} G_n^{k-1}(y+d) g_n(y) dy$

where $\Omega_o = \{\underline{\theta} = (\theta_1,\ldots,\theta_k): \theta_1 = \theta_2 = \ldots = \theta_k = \theta\}$, $G_n(y)$, $g_n(y)$ are the cdf and pdf of the sample median of $(2n+1)$ observations from the standard double exponential distribution.

Proof. For $\underline{\theta} \in \Omega$,

$$P_{\underline{\theta}}(CS|R_5) = P_{\underline{\theta}}\{Y_{(k)} \geq \max_{1\leq j\leq k} Y_{(j)} - d\}$$

$$= P_{\underline{\theta}}\{Y_{(k)}-\theta_{[k]} \geq Y_{(j)}-\theta_{[j]} + \theta_{[j]}-\theta_{[k]}-d,\ j=1,2,\ldots,k-1\}$$

$$= \int_{-\infty}^{\infty} \left[\prod_{j=1}^{k-1} \int_{-\infty}^{y+\theta_{[k]}-\theta_{[j]}+d} g_n(z)\, dz\right] g_n(y)\, dy \qquad (2.2.5)$$

Note that $\theta_{[k]} - \theta_{[j]} \geq 0$ for $j = 1,\ldots,k-1$; thus the result follows. Hence, if we choose d to be the smallest constant to satisfy

$$\int_{-\infty}^{\infty} G_n^{k-1}(y+d)\, g_n(y)\, dy = P^*, \qquad (2.2.6)$$

then we have determined the constant d for which

$$\inf_{\underline{\theta}\varepsilon\Omega} P_{\underline{\theta}}(CS|R_5) = P^* . \tag{2.2.7}$$

(c) Some Properties of R_5

For $\underline{\theta} \varepsilon \Omega$ and $\underline{\theta} = (\theta_{[1]},\ldots,\theta_{[k]})$ define $P_{\underline{\theta}}(i) = P_{\underline{\theta}}$ {R select population $\pi_{(i)}$}, and recall the following definitions (see Santner [69]).

Definition 2.2.1. The rule R is strongly monotone in $\pi_{(i)}$ means

$$P_{\underline{\theta}}(i) \text{ is } \begin{cases} \uparrow \text{ in } \theta_{[i]} \text{ when all other components of } \underline{\theta} \text{ are fixed} \\ \downarrow \text{ in } \theta_{[j]} \ (j\neq i) \text{ when all other components of } \underline{\theta} \text{ are fixed} \end{cases}$$

Definition 2.2.2. R is a monotone procedure means for every $\underline{\theta} \varepsilon \Omega$ and $1 \leq i < j \leq k$, $P_{\underline{\theta}}(i) \leq P_{\underline{\theta}}(j)$.

Definition 2.2.3. R is an unbiased procedure means for every $\underline{\theta} \varepsilon \Omega$ and $1 \leq j < k$,

$$P_{\underline{\theta}}\{R \text{ does not select } \pi_{(i)}\} \geq P_{\underline{\theta}}\{R \text{ does not select } \pi_{(k)}\}$$

Of course, if R is monotone it is also unbiased.

Theorem 2.2.2. For any $i = 1,2,\ldots,k$, the procedure R_5 is strongly monotone in $\pi_{(i)}$.

Proof. The proof follows easily from the expression

$$P_{\underline{\theta}}(i) = \int_{-\infty}^{\infty} \{\prod_{\substack{j=1 \\ j\neq i}}^{k} G_n(y + \theta_{[i]} - \theta_{[j]} + d)\} \ g_n(y)\, dy .$$

Corollary 2.2.1. The rule R_5 is monotone and unbiased.

Proof. It is known and easy to see that if R is strongly monotone in

$\pi_{(i)}$, for all $i = 1,2,\ldots,k$, then it is monotone.

Now we consider some special configurations of $\underline{\theta} \in \Omega$.

$$\begin{cases} \theta_{[i]} = \theta \quad , \quad i = 1,2,\ldots,k-1 \\ \theta_{[k]} = \theta + \Delta \ , \quad \Delta > 0 \end{cases} \tag{2.2.8}$$

$$\theta_{[i]} = \theta + (i-1)\Delta \ , \ \Delta > 0, \quad i = 1,2,\ldots,k. \tag{2.2.9}$$

Under (2.2.8),

$$P_{\underline{\theta}}(i) = \int_{-\infty}^{\infty} [G_n(y+d)]^{k-2} \, G_n(y+d-\Delta) \, g_n(y) \, dy \quad \text{for } i=1,2,\ldots,k-1 \tag{2.2.10}$$

$$P_{\underline{\theta}}(k) = \int_{-\infty}^{\infty} [G_n(y+d+\Delta)]^{k-1} \, g_n(y) \, dy \ . \tag{2.2.11}$$

While under (2.2.9),

$$P_{\underline{\theta}}(i) = \int_{-\infty}^{\infty} \{ \prod_{\substack{j=1 \\ j\neq i}}^{k} G_n(y+d+(i-j)\Delta)\} \, g_n(y) \, dy, \quad i=1,2,\ldots,k.$$

From the above equations we can make the following remarks:

<u>Remark 2.2.1.</u> For fixed P*, k, n, i ($i = 1,2,\ldots,k-1$), the probability of selecting population $\pi_{(i)}$ decreases from P* to zero as Δ increases from zero to infinity.

<u>Remark 2.2.2.</u> For fixed P*, k and n, the probability of selecting $\pi_{(k)}$ increases from P* to one as Δ increases from zero to infinity.

<u>Remark 2.2.3</u>. For fixed P*, k, i($i=1,\ldots,k-1$) and Δ, the probability of selecting population $\pi_{(i)}$ tends to zero as $n \to \infty$. While the probability of selecting $\pi_{(k)}$ tends to one as $n \to \infty$.

<u>Conclusion</u>: Under either configuration (2.2.8), (2.2.9),

$E_{\underline{\theta}}(S|R_5) = \sum_{i=1}^{k} p_{\underline{\theta}}(i) \to 1$ as $\Delta \to \infty$ for fixed n and $E_{\underline{\theta}}(S|R_5) \to 1$ as $n \to \infty$ for fixed Δ.

(D) <u>Asymptotic Results for the Procedure R_5</u>

It suffices to consider the parameter space Ω_o. For n large, we discuss an asymptotic property of the procedure as follows. Let Y be the sample median from a sample of size (2n+1) with pdf $f(x;\theta) = \frac{1}{2} e^{-|x-\theta|}$, $-\infty < x < \infty$. Then it is known (see Chu [24]) that under Ω_o, $\frac{Y-\theta}{\sigma_n}$ is asymptotically normally distributed (here $\sigma_n^2 = \frac{1}{2n+1}$). Let Z denote a random variable which has a standard normal distribution, then $\frac{Y-\theta}{\sigma_n}$ is asymptotically distributed as Z. Hence, under Ω_o, the probability

$$Y_k \geq \max_{1 \leq j \leq k} Y_j - d$$

is asymptotically, the same as the probability that

$$Z_k \geq \max_{1 \leq j \leq k} Z_j - \sqrt{2n+1}\, d \tag{2.2.13}$$

where Z_i, $i = 1,2,\ldots,k$, are iid standard normal variables. Hence,

$$\inf_{\underline{\theta}\varepsilon\Omega_o} P_{\underline{\theta}}(CS|R_5) \doteqdot P_{\underline{\theta}}\{Z_k \geq \max_{1 \leq j \leq k} Z_j - \sqrt{2n+1})\, d$$

$$= \int_{-\infty}^{\infty} \left[\Phi(z + \sqrt{2n+1}\; d)\right]^{k-1} d\Phi(z)$$

where $\Phi(.)$ is the cdf of the standard normal distribution.

(E) <u>The Monotone Likelihood Ratio Property of the Sample Median</u>

Suppose Y is the sample median of (2n+1) observations from the population with double exponential density function $f(x;\theta) = \frac{1}{2} e^{-|x-\theta|}$.

The pdf $g_n(y;\theta)$ and cdf $G_n(y;\theta)$ of Y are given by equations (2.2.2) and (2.2.3).

After some algebraic computations, we see that $G_n(\theta;\theta) = \frac{1}{2}$; also it is easy to show that $g_n(y;\theta)$ is differentiable at $y = \theta$.

Let $g_n(y;\theta) = \bar{g}_n(y-\theta)$. It is shown in Lehmann [53, p.330] that a necessary and sufficient condition for $\bar{g}_n(y-\theta)$ to have monotone likelihood ratio in y is that $-\log \bar{g}_n$ is convex. Our main goal in this section is to prove this assertion. Now

$$\bar{g}_n(y) = c_n(\tfrac{1}{2} e^{-|y|})^{n+1}(1 - \tfrac{1}{2} e^{-|y|})^n \text{ where } c_n = \frac{(2n+1)!}{n!n!}, \text{ so,}$$

$$-\log \bar{g}_n(y) = -\log c_n + (n+1)\log 2 + (n+1)|y| - n\log(1-\tfrac{1}{2} e^{-|y|}).$$

Let $h(y) = (n+1)|y| - n\log(1 - \frac{1}{2} e^{-|y|}) = \begin{cases} h_1(y), & y < 0 \\ h_2(y), & y \geq 0 \end{cases}$ which is a continuous function. For $y < 0$,

$h(y) = h_1(y) = -(n+1)y - n\log(1 - \frac{1}{2} e^y)$, we have

$$h_1'(y) = -(n+1) + \frac{\frac{n}{2} e^y}{1 - \frac{1}{2} e^y} < 0 \text{ since for } y < 0, \frac{\frac{1}{2} e^y}{1 - \frac{1}{2} e^y} < 1$$

and $$h_1''(y) = \frac{\frac{n}{2} e^y}{(1 - \frac{1}{2} e^y)^2} > 0 .$$

Hence, for $y<0$, $h_1(y)$ is a decreasing, convex function. Similarly, for $y \geq 0$,

$$h(y) = h_2(h) = (n+1)y - n\log(1 - \tfrac{1}{2} e^{-y})$$

$$h_2'(y) = n+1 - \frac{\frac{n}{2} e^{-y}}{1 - \frac{1}{2} e^{-y}} > 0 \text{ since for } y \geq 0, \frac{\frac{1}{2} e^{-y}}{1 - \frac{1}{2} e^{-y}} < 1$$

$$h_2''(y) = \frac{\frac{n}{2} e^{-y}}{(1 - \frac{1}{2} e^{-y})^2} > 0 .$$

Hence, for $y \geq 0$, $h_2(y)$ is an increasing, convex function. Note that $h(y)$ is continuous at $y = 0$, decreasing, convex for $y < 0$ and increasing, convex for $y \geq 0$. Hence, this concludes that $h(y)$ is a convex function, which implies $-\log \bar{g}_n(y)$ is also a convex function.

Theorem 2.2.3. $g_n(y;\theta)$ has monotone likelihood ratio in y.

(F) Expected Size of the Selected Subset

The procedure R satisfies the basic probability requirement (2.2.1) and is defined by (2.2.4). Consistent with the basic probability requirement, we would like the size of the selected subset to be small. Now S, the size of the selected subset is a random variable which takes integer values 1,2,...,k. Hence, one criterion of the efficiency of the procedure R is the expected value of the size of the subset. Now, we derive an expression for $E(S|R_5)$, the expected size of the selected subset using procedure R_5.

$$E(S|R_5) = \sum_{i=1}^{k} P\{\text{Selecting the population with parameter } \theta_{[i]}\}$$

$$= \sum_{i=1}^{k} P\{Y_{(i)} \geq \max_{1\leq j\leq k} Y_{(j)} - d\}$$

$$= \sum_{i=1}^{k} \int_{-\infty}^{\infty} \left[\prod_{\substack{j=1 \\ j\neq i}}^{k} G_n(y + d + \theta_{[i]} - \theta_{[j]}) \right] g_n(y)\, dy \qquad (2.2.16)$$

If we set the m smallest parameters θ_i $(1 \leq m < k)$ equal to a common value θ(say) and define

$$Q = E(S \mid \theta_{[1]} = \dots = \theta_{[m]} = \theta) \qquad (2.2.17)$$

then by an analogous argument as in Gupta [41] one can prove the following theorem.

<u>Theorem 2.2.4</u>. For given k, $P^*(\frac{1}{k} < P^* < 1)$, the expected size of the selected subset $E(S \mid \theta_{[1]} = \theta_{[2]} = \cdots = \theta_{[m]} = \theta,\ m < k)$ in using the procedure R_5 is strictly increasing in θ.

<u>Corollary 2.2.2.</u> $\sup_{\underline{\theta}\varepsilon\Omega} E_{\underline{\theta}}(S|R_5) = k \int_{\infty}^{\infty} G_n^{k-1}(y+d)\ g_n(y)\ dy = k\ P^*.$

<u>Corollary 2.2.3.</u> In the subset $\Omega(\delta) = \{\underline{\theta}\colon\ \theta_{[i]} \leq \theta_{[k]} - \delta,$ $i = 1,2,\ldots,k-1\}$, the function $E_{\underline{\theta}}(S|R_5)$ takes on its maximum value when $\theta_{[i]} = \theta_{[k]} - \delta$, $i = 1,2,\ldots,k-1$, and so

$$\sup_{\underline{\theta}\varepsilon\Omega(\delta)} E_{\underline{\theta}}(S|R_5) = \int_{-\infty}^{\infty} G_n^{k-1}(y+d+\delta)\ g_n(y)\ dy + (k-1)\int_{-\infty}^{\infty} G_n^{k-2}(y+d)\ G_n(y+d-\delta)\ g_n(y)\ dy\ .$$

(G) <u>Minimax Property of the Rule R_5</u>

Suppose that $y_1,\ldots,y_k$ are the sample medians from the k populations $\pi_1,\ldots,\pi_k$, respectively, and with this set of observations, we select the ith population with probability $\phi_i(y_1,\ldots,y_k)$. Then the selection rule R is said to be invariant or symmetric if

$$\phi_i(y_1,\ldots,y_i,\ldots,y_j,\ldots,y_k) = \phi_j(y_1,\ldots,y_j,\ldots,y_i,\ldots,y_k)$$

for all i and j, i.e. if y_j is observed from π_i and y_i from π_j, then we select the jth population with the same probability $\phi_i(y_1,\ldots,y_k)$.

Notice that the rule R_5: $Y_i \geq \max_{1\leq j\leq k} Y_j - d$ satisfies the equations

$$\inf_{\underline{\theta}\varepsilon\Omega} P_{\underline{\theta}}(CS|R_5) = \inf_{\underline{\theta}\varepsilon\Omega_o} P_{\underline{\theta}}(CS|R_5) = P_{\underline{\theta}_o}(CS|R_5) = P^* \qquad (2.2.20)$$

and $$\sup_{\underline{\theta}\varepsilon\Omega} E_{\underline{\theta}}(S|R_5) = \sup_{\underline{\theta}\varepsilon\Omega_o} E_{\underline{\theta}}(S|R_5) = E_{\underline{\theta}_o}(S|R_5) = k\, P^* \qquad (2.2.21)$$

where $\underline{\theta}_o = (\theta_o,\ldots,\theta_o)$.

For any invariant rule R', $\underline{\theta}_o \varepsilon \Omega$

$$\begin{aligned} E_{\underline{\theta}_o}(S|R') &= \sum_{i=1}^{k} P_{\underline{\theta}_o}\{\text{select population } \pi_i|R'\} \\ &= \sum_{i=1}^{k} \int \phi_i(y_1,\ldots,y_k) \left[\prod_{j=1}^{k} g_n(y_j)\right] dy_1 \ldots dy_k \\ &= k\, P_{\underline{\theta}_o}(CS|R'). \end{aligned}$$

Hence for $\underline{\theta}_o \varepsilon \Omega_o$,

$$E_{\underline{\theta}_o}(S|R') - E_{\underline{\theta}_o}(S|R_5) = k\,[P_{\underline{\theta}_o}(CS|R') - P_{\underline{\theta}_o}(CS|R_5)] \qquad (2.2.22)$$

If the rule R' satisfies the basic P^* condition, it follows from (2.2.20) that the right hand side of (2.2.22) is non-negative. Thus

$$E_{\underline{\theta}_o}(S|R') \geq E_{\underline{\theta}_o}(S|R_5) = \sup_{\underline{\theta}\varepsilon\Omega} E_{\underline{\theta}}(S|R_5) .$$

So that $$\sup_{\underline{\theta}\varepsilon\Omega} E_{\theta}(S|R') \geq \sup_{\underline{\theta}\varepsilon\Omega} E_{\theta}(S|R_5)$$

i.e. the rule R_5 is minimax among all invariant rules satisfying the P^*-condition.

2.3. Selecting the Population with the Largest Location Parameter - Indifference Zone Approach

In this section, we would like to use the indifference zone approach of Bechhofer [11] to select one population which is guaranteed to be associated with the largest location parameter with a fixed probability P^* whenever the unknown parameters lie outside some subset, or zone of indifference, of the entire parameter space. The goal is to define a sequence of rules $\{R_6(n)\}$ each of which selects a single population $\pi_{(k)}$ and find the smallest n so that

$$P_{\underline{\theta}}(CS|R_6(n)) \geq P^*, \forall\ \underline{\theta}\varepsilon\Omega(\delta^*) = \{\underline{\theta}:\ \theta_{[k]} - \theta_{[k-1]} \geq \delta^*\} \qquad (2.3.1)$$

where P* and δ* are preassigned numbers.

For the sake of clarity, we will use the notation $Y_{[k]n}$ to denote the largest of the sample medians each based on (2n+1) observations.

$R_6(n)$: Select the population corresponding to $Y_{[k]n}$.

Let $\Omega_o(\delta^*) = \{\underline{\theta}: \theta_{[1]} = \dots = \theta_{[k-1]} = \theta_{[k]} - \delta^*\}$. Then we have the following theorem.

Theorem 2.3.1. $\inf\limits_{\underline{\theta}\varepsilon\Omega(\delta^*)} P_{\underline{\theta}}(CS|R_6(n)) = \inf\limits_{\underline{\theta}\varepsilon\Omega_o(\delta^*)} P_{\underline{\theta}}(CS|R_6(n))$

Proof. For $\underline{\theta}\ \varepsilon\ \Omega(\delta^*)$,

$$\begin{aligned} P_{\underline{\theta}}(CS|R_6(n)) &= P_{\underline{\theta}}\{\max_{1\leq j\leq k-1} Y_{(j)n} < Y_{(k)n}\} \\ &= P_{\underline{\theta}}\{Y_{(j)n} < Y_{(k)n},\ j = 1,2,\dots,k-1\} \\ &= P_{\underline{\theta}}\{Y_{(j)n}-\theta_{[j]} < Y_{(k)n}-\theta_{[k]}+\theta_{[k]}-\theta_{[j]},\ j = 1,2,\dots,k-1\} \\ &= \int_{-\infty}^{\infty} \left[\prod_{j=1}^{k-1} G_n(y+\delta_{kj})\right] d\,G_n(y) \qquad (2.3.2) \end{aligned}$$

where $G_n(y) = G_n(y;\ 0)$ is the cdf of the sample median of (2n+1) independent observation from the standard double exponential distribution with density function $\frac{1}{2}e^{-|x|}$, $-\infty < x < \infty$, and $\delta_{kj} = \theta_{[k]}-\theta_{[j]} \geq 0$. Hence the infimum of the probability of a correct selection occurs when $\theta_{[1]} = \theta_{[2]} = \dots = \theta_{[k-1]} = \theta_{[k]} - \delta^*$ provided $\theta_{[k]} - \theta_{[k-1]} \geq \delta^*$. This proves the theorem.

The minimum sample size required to achieve the P* condition (2.3.1) is the smallest integer n such that

$$\int_{-\infty}^{\infty} [G_n(y + \delta^*)]^{k-1} \, d\, G_n(y) \geq P^* . \qquad (2.3.3)$$

2.4 Selecting the t-Best Populations - Indifference Zone Approach

Now, we consider the problem of selecting the best t populations, i.e., the populations with location parameters $\theta_{[k-t+1]}, \theta_{[k-t+2]}, \ldots, \theta_{[k]}$, without regard to order. We are using the indifference zone approach based on the sample median Y_i of 2n+1 independent observations from population π_i, $i = 1, \ldots, k$. Define a sequence of procedures as follows:

$R_7(n)$: Select the t populations associated with t largest values of Y_i.

Let $\Omega'(\delta^*) = \{\underline{\theta}: \theta_{[k-t+1]} - \theta_{[k-t]} \geq \delta\}$ and let

$$\Omega'_o(\delta^*) = \{\underline{\theta}: \theta_{[1]} = \ldots = \theta_{[k-t]} = \theta, \; \theta_{[k-t+1]} = \ldots = \theta_{[k]} = \theta + \delta^*\}.$$

Theorem 2.4.1. $\inf\limits_{\underline{\theta}\in\Omega'(\delta^*)} P_{\underline{\theta}}\{CS|R_7(n)\} = \inf\limits_{\underline{\theta}\in\Omega'_o(\delta^*)} P_{\underline{\theta}}\{CS|R_7(n)\}$

Proof. It was shown in Theorem 2.2.3 that the pdf $g_n(y; 0)$ of the sample median has monotone likelihood ratio in y, which implies that it is stochastically increasing in θ. Using a theorem of Barr and Rizvi [8], it follows that, for $\underline{\theta} \in \Omega'(\delta^*)$

$$P_{\underline{\theta}}\{CS|R_7(n)\} = P_{\underline{\theta}}\{\max_{1\leq i\leq k-t} Y_{(i)} < \min_{k-t+1\leq j\leq k} Y_{(j)}\}$$

is a non-increasing function of $\theta_{[1]}, \ldots, \theta_{[k-t]}$ and a non-decreasing function of $\theta_{[k-t+1]}, \ldots, \theta_{[k]}$. Thus $P_{\underline{\theta}}\{CS|R_7(n)\}$ attains its infimum when $\theta_{[1]}, \ldots, \theta_{[k-t]}$ attain their maximum possible values, while

$\theta_{[k-t+1]},\ldots,\theta_{[k]}$ attain their minimum possible values subject to $\underline{\theta} \in \Omega'(\delta^*)$. The proof is thus completed.

Using the same notation as in Section 2.2, let $G_n(y;\theta_i)$ denotes the cdf of the sample median Y_i with parameter θ_i. Since θ_i is the location parameter, $G_n(y;\theta_i) = G_n(y-\theta_i;0)$ and G_n is stochastically increasing continuous in both y and θ_i. For $\underline{\theta} \in \Omega'(\delta^*)$

$$P_{\underline{\theta}}\{CS|R_7(n)\} = P_{\underline{\theta}}\{\max_{1\le i\le k-t} Y_{(i)} < \min_{k-t+1\le \ell\le k} Y_{(\ell)}\}$$

$$= P_{\underline{\theta}}\{\bigcup_{j=k-t+1}^{k} \{Y_{(j)} = \min_{k-t+1\le \ell\le k} Y_{(\ell)} \text{ and } \max_{1\le \ell\le k-t} Y_{(i)} < Y_{(j)}\}\}$$

$$= \sum_{j=k-t+1}^{k} \int_{-\infty}^{\infty} \prod_{\beta=1}^{k-t} G_n(y;\theta_{[\beta]}) \prod_{\substack{\alpha=k-t+1\\ \alpha\ne j}}^{k} \{1 - G_n(y;\theta_{[\alpha]})\}\, d\, G_n(y;\theta_{[j]})$$

In particular, for $\underline{\theta} \in \Omega_o'(\delta^*) \subset \Omega'(\delta^*)$,

$$P_{\underline{\theta}}\{CS|R_7(n)\} = t\int_{-\infty}^{\infty} G_n^{k-t}(y;\theta)\ \{1 - G_n(y;\theta+\delta^*)\}^{t-1}\ dG_n(y,\theta+\delta^*)$$

$$= t\int_{-\infty}^{\infty} G_n^{k-t}(y-\theta;0)\ \{1-G_n(y-\theta-\delta^*;0)\}^{t-1}\ dG_n(y-\theta-\delta^*;0)$$

$$= t\int_{-\infty}^{\infty} G_n^{k-t}(y+\delta^*;0)\ \{1 - G_n(y;0)\}^{t-1}\ dG_n(y;0)$$

which is independent of the parameter θ. Hence for specified values of δ^* and P^* $\left(\frac{1}{\binom{k}{t}} < P^* < 1\right)$, we can solve the equation

$$t\int_{-\infty}^{\infty} G_n^{k-t}(y+\delta^*;0)\ \{1 - G_n(y;0)\}^{t-1}\ dG_n(y;0) = P^*$$

for n.

2.5 Subset Selection with Respect to the Scale Parameter σ

Let X_i, $i = 1,2,\ldots,k$ be k independent random variables from double exponential population π_i, $i = 1,2,\ldots,k$, respectively, with π_i having the probability density function

$$f(x;\theta_i,\sigma_i) = \frac{1}{2\sigma_i} \exp\,[-|x-\theta_i|/\sigma_i], \quad -\infty < x < \infty, \; -\infty < \theta_i < \infty, \quad \sigma_i > 0.$$

Take n independent observations from π_i, $i = 1,2,\ldots,k$. From these data one wishes to select a subset contains the population with the largest σ_i. Let $\sigma_{[1]} \leq \cdots \leq \sigma_{[k]}$ be the ordered parameters. We consider two different cases.

Case (i): $\theta_1,\theta_2,\ldots,\theta_k$ known.

In this case, the maximum likelihood estimator of σ_i is $Y_i = \frac{1}{n}\sum_{j=1}^{n} |X_{ij} - \theta_i|$ which is distributed as a gamma variable with parameters n and $\frac{\sigma_i}{n}$, i.e. Y_i has density $\frac{n}{\sigma_i\Gamma(n)} \left(\frac{ny}{\sigma_i}\right)^{n-1} e^{-\frac{ny}{\sigma_i}}$, $y > 0$. Thus the problem reduces to the one considered by Gupta [40]. The selection procedure is

R: Select the population π_i in the subset if and only if

$$Y_i \geq c \max_{1\leq j\leq k} Y_j.$$

Case (ii): θ_i's are unknown.

When θ_i is unknown, it is well known that the maximum likelihood estimate of σ_i is given by $\hat{\sigma}_i = \frac{1}{n} \sum_{j=1}^{n} |X_{ij} - \tilde{X}_i|$, where $\tilde{X}_i$ denotes the sample median from population π_i. For this problem, we propose the following selection procedure.

R_8: Select the population π_i in the subset if and only if

$$\hat{\sigma}_i \geq c_8 \max_{1\leq j\leq k} \hat{\sigma}_j$$

where $0 < c_8 < 1$ is so determined as to satisfy the basic probability requirement regardless of what the unknown σ_i's may be.

Let $V_i = \frac{n\hat{\sigma}_i}{\sigma_i}$, $i = 1,2,\ldots,k$. Then

$$P(CS|R_8) = P\{\hat{\sigma}_{(k)} \geq c_8 \max_{1\leq j\leq k-1} \hat{\sigma}_{(j)}\}$$

$$= \int_0^\infty \left[\prod_{j=1}^{k-1} F_{V_{(j)}}\left(\frac{1}{c_8}\frac{\sigma_{[k]}}{\sigma_{[j]}} x\right)\right] dF_{V_{(k)}}(x).$$

So

$$\inf_{\underline{\sigma}\in\Omega'} P(CS|R_8) = \inf_{\underline{\sigma}\in\Omega'_0} P(CS|R_8) = \int_0^\infty F_V^{k-1}\left(\frac{x}{c_8}\right) dF_V(x),$$

where $\Omega' = \{\underline{\sigma} = (\sigma_1,\ldots,\sigma_k),\ \sigma_i > 0,\ i = 1,\ldots,k\}$, $\Omega'_0 = \{\underline{\sigma} = (\sigma,\ldots,\sigma),\ \sigma > 0\}$ and $F_V(\cdot)$, $F_{V_{(j)}}(\cdot)$, $j = 1,\ldots,k$ are the cdf's of $V = \frac{n\hat{\sigma}}{\sigma}$, $V_{(j)} = \frac{n\hat{\sigma}_{(j)}}{\sigma_{[j]}}$, $j = 1,\ldots,k$, respectively.

Hence if the distribution $F_V(\cdot)$ is known, then the constant c_8 can be determined by the equation

$$\int_0^\infty F_V^{k-1}\left(\frac{x}{c_8}\right) dF_V(x) = P^*.$$

The exact distribution F of V is worked out for $n = 3$ by Bain and Engelhardt [4], and a chi-square approximation is also given by them which is quite good even for small n. However, it follows from Chernoff, Gastwirth and Johns [22], that $\frac{1}{\sqrt{n}}(V-n) = \sqrt{n}\,[\frac{\hat{\sigma}}{\sigma} - 1]$ is asymptotically a standard normal variable. When all σ_i are identical

$$P(CS|R_8) = P\{\hat{\sigma}_k \geq c_8 \hat{\sigma}_j, \quad j = 1,\ldots,k-1\}$$
$$= P\{\sqrt{n}(\frac{\hat{\sigma}_k}{\sigma} - 1) \geq c_8 \sqrt{n}(\frac{\hat{\sigma}_j}{\sigma} - 1) + \sqrt{n}(c_8-1), \; j = 1,\ldots,k-1\}$$
$$\cong \int_{-\infty}^{\infty} \Phi^{k-1}(\frac{x-\sqrt{n}(c_8-1)}{c_8})\, d\Phi(x).$$

2.6 A Test of Homogeneity Based on the Sample Median Range

Let $\pi_1,\pi_2,\ldots,\pi_k$ be k independent double exponential populations such that the observations $X_{i1},\ldots,X_{i,2n+1}$ from π_i has density $\frac{1}{2} e^{-|X-\theta_i|}$, for $i = 1,2,\ldots,k$. As before, let the sample median of these (2n+1) observations be denoted as Y_i, $i = 1,\ldots,k$. In some practical situations one wishes to know whether θ_i are significantly different or not. This problem is to test the homogeneity of the double exponential populations. We are interested in using a test based on the sample range of Y's and hence we wish to derive the distribution of the sample median range $R = \max_{1\leq j\leq k} Y_j - \min_{1\leq j\leq k} Y_j$, considering all θ_i to be equal to a common unknown θ. When the value of R is large, the hypothesis of homogeneity is rejected. We wish to find a constant r, such that $P(R > r) \leq \alpha$ under the hypothesis H_o: $\theta_1 = \ldots = \theta_k = \theta$. This will provide an α-level test.

Theorem 2.5.1. For α, $0 < \alpha < 1$, let r be a constant such that

$$P_{\Omega_o}\{Y_k \geq \max_{1\leq j\leq k-1} Y_j - r\} \geq 1 - \frac{\alpha}{k}.$$

Then $P_{\Omega_o}(R > r) \leq \alpha$.

Proof. When H_o is true, i.e., under Ω_o,

$$P(R > r) = P\{\max_{1\le j\le k} Y_j - \min_{1\le j\le k} Y_j > r\}$$

$$\le k - \sum_{i=1}^{k} P\{Y_i \ge \max_{1\le j\le k} Y_j - r\}$$

$$= k - k\, P\{Y_k \ge \max_{1\le j\le k-1} Y_j - r\}$$

$$\le k - k\cdot(1 - \frac{\alpha}{k})$$

$$= \alpha.$$

The above theorem establishes a connection between the selection rule R_5 and the above test for equality of θ's.

2.7 On the Distribution of the Statistic Associated with R_5

Let X_i $(i = 0,1,\ldots,p)$ be $(p+1)$ independent and identically distributed random variables each representing the median in a random sample of size $(2n+1)$ from a population with standard double exponential density function $f(x) = \frac{1}{2} e^{-|x|}$. Consider the differences $Y_i = X_i - X_o$ $(i = 1,2,\ldots,p)$. The random variables Y_i $(i = 1,2,\ldots,p)$ are correlated and the distribution of the maximum of Y_i is of interest in problems of selection and ranking for double exponential distribution as explained earlier when discussing R_5. In this section, we give a closed form of the distribution of $Y = \max_{1\le i\le p} Y_i$ for some special cases. We have also computed tables of the upper percentage points of $Y = \max_{1\le i\le p} Y_i$ corresponding to the probability levels $\alpha = P^* = 0.75, 0.90, 0.95, 0.99$ for $p = 1(1)\,9$, $n = 1(1)10$.

For the special case $P = 1$ $(k=2)$, $n = 1$ (sample size = 3), straight forward integration gives the cdf of Y(see formulae (2.2.2), (2.2.3)) as

$$P(Y \leq y) = \int_{-\infty}^{\infty} G(x + y)\, g(x)\, dx$$

$$= 1 - \frac{9}{8} ye^{-2y} - \frac{3}{16} ye^{-3y} + \frac{9}{40} e^{-2y} - \frac{29}{40} e^{-3y}$$

Again, for p = 1 (k=2), n = 2 (sample size = 5),

$$P(Y \leq y) = 1 - \frac{75}{16} ye^{-3y} - \frac{225}{64} ye^{-4y} - \frac{45}{256} ye^{-5y} + \frac{10975}{1792} e^{-3y}$$

$$- \frac{5225}{896} e^{-4y} - \frac{203}{256} e^{-5y}$$

All computations related to and given at the end of this chapter were made on a CDC 6500 using Gauss Laguerre quadratue based on fifteen nodes to perform the numerical integration. Checks on the accuracy of the program for p = 1, n = 1 showed that these values seem to be correct to three decimal places.

Table VI

Upper $100(1-P^*)$ percentage points of $Y = \max_{1 \le i \le p} (X_i - X_o)$ where $X_o, X_1, \ldots, X_p$ are iid sample median random variables in samples of sizes $(2n+1)$ from the standard double exponential (Laplace) distribution.

p \ n	1	2	3	4	5	6	7	8	9	10
	.6781	.5145	.4274	.3721	.3334	.3049	.2828	.2651	.2504	.2378
1	1.3777	1.0311	.8496	.7357	.6568	.5987	.5541	.5184	.4888	.4634
	1.8508	1.3735	1.1261	.9718	.8658	.7885	.7294	.6825	.6436	.6102
	2.8631	2.0931	1.7002	1.4598	1.2994	1.1876	1.1070	1.0464	.9974	.9533
	1.0434	.7875	.6522	.5667	.5072	.4631	.4289	.4014	.3786	.3591
2	1.7380	1.2948	1.0640	.9195	.8195	.7459	.6892	.6437	.6060	.5738
	2.2092	1.6330	1.3354	1.1503	1.0231	.9302	.8590	.8024	.7555	.7513
	3.2186	2.3459	1.9015	1.6297	1.4479	1.3203	1.2274	1.1569	1.0998	1.0489
	1.2507	.9401	.7767	.6737	.6021	.5490	.5077	.4746	.4470	.4234
3	1.9451	1.4445	1.1847	1.0224	.9102	.8275	.7637	.7126	.6703	.6342
	2.4159	1.7811	1.4540	1.2509	1.1114	1.0094	.9313	.8690	.8176	.7735
	3.4247	2.4913	2.0166	1.7265	1.5322	1.3955	1.2956	1.2194	1.1576	1.1029
	1.3972	1.0468	.8632	.7476	.6673	.6078	.5616	.5244	.4935	.4672
4	2.0917	1.5496	1.2689	1.0938	.9729	.8838	.8151	.7601	.7146	.6758
	2.5624	1.8852	1.5369	1.3210	1.1728	1.0644	.9813	.9151	.7605	.8138
	3.5706	2.5936	2.0974	1.7942	1.5911	1.4479	1.3430	1.2629	1.1979	1.1405
	1.5109	1.1290	.9294	.8040	.7170	.6525	.6024	.5622	.5288	.5003
5	2.2053	1.6306	1.3336	1.1485	1.0208	.9268	.8543	.7962	.7482	.7074
	2.6759	1.9655	1.6007	1.3747	1.2197	1.1064	1.0195	.9503	.8932	.8444
	3.6838	2.6727	2.1596	1.8463	1.6363	1.4881	1.3794	1.2962	1.2287	1.1693
	1.6039	1.1958	.9830	.8496	.7570	.6885	.6353	.5925	.5571	.5269
6	2.2982	1.6965	1.3860	1.1928	1.0595	.9615	.8859	.8253	.7753	.7328
	2.7686	2.0308	1.6524	1.4183	1.2577	1.1403	1.0503	.9787	.9195	.8691
	3.7762	2.7371	2.2101	1.8855	1.6729	1.5207	1.4088	1.3232	1.2537	1.1925

Table VI (continued)

p \ n	1	2	3	4	5	6	7	8	9	10
7	1.6826	1.2521	1.0281	.8878	.7905	.7186	.6628	.6179	.5807	.5491
	2.3767	1.7520	1.4301	1.2299	1.0920	.9905	.9134	.8497	.7979	.7540
	2.8470	2.0859	1.6959	1.4548	1.2895	1.1687	1.0762	1.0024	.9416	.8898
	3.8543	2.7913	2.2526	1.9240	1.7037	1.5480	1.4336	1.3458	1.2745	1.2120
8	1.7507	1.3007	1.0669	.9207	.8193	.7444	.6863	.6396	.6009	.5681
	2.4447	1.7999	1.4681	1.2619	1.1199	1.0155	.9350	.8706	.8174	.7722
	2.9148	2.1334	1.7335	1.4863	1.3169	1.1932	1.0984	1.0228	.9606	.9075
	3.9219	2.8383	2.2894	1.9546	1.7302	1.5716	1.4548	1.3653	1.2925	1.2288
9	1.8109	1.3435	1.1010	.9495	.8446	.7670	.7069	.6586	.6186	.5847
	2.5047	1.8421	1.5014	1.2900	1.1444	1.0373	.9548	.8888	.8344	.7881
	2.9747	2.1753	1.7665	1.5139	1.3410	1.2146	1.1178	1.0407	.9772	.9231
	3.9816	2.8796	2.3217	1.9815	1.7535	1.5922	1.4735	1.3823	1.3083	1.2435

For given p,n and $P^* = .75$ (top), .90 (second), .95 (third), .99 (bottom), the entries in this table are the values of d for which $\int_{-\infty}^{\infty} G_n^P(x+d)\, dG_n(x) = P^*$ where $G_n(\cdot)$ is the cdf of the median in a sample of size (2n+1) from the standard double exponential distribution.

CHAPTER III

SOME CLASSIFICATION RULES FOR k UNIVARIATE NORMAL POPULATIONS

3.1 Introduction

In problems of classification, one usually assumes that an individual belongs to one of the k populations. Based on the observations from these populations one wishes to assign it to the correct population. Such problems of classification often arise in several branches of science. About forty years ago Fisher was consulted by Barnard [6] as to the best method of classifying skeletal remains unearthed by archaeological excavations. Fisher [29] suggested the use of the now well-known discriminant function. A general mathematical theory of statistical taxonomy was built by Welch [79] on foundations laid by Neyman and Pearson's theory of tests of hypotheses. The technique of discriminant functions which was devised by Fisher [29] has proved to be invaluable in tackling classification problems. But the construction of the discriminant function is possible only when we know the values of the parameters characterizing the populations to be discriminated between. This raises the question as to what is to be done when such knowledge is absent. In this chapter we describe some classification procedures suited to such situations.

The problem of classifying an individual into one of two categories, discriminant function analysis as some prefer to call it in the parametric case, has been considered by many authors in the statistical literature.

For an extensive bibliography, the reader is referred to Anderson, et.al. [1]. The probability of misclassification, inherent in such classification procedures is not necessarily known to the experimenter. Several authors have treated the problems dealing with the misclassification (e.g., see Smith [74], John [51], [52], Okamoto [59], Sedransk [73], Hills [48], and Sorum [76]). It should be stressed that the above papers deal with the case of two populations only.

In this chapter, we use the subset selection approach to the problem of classification where the probability of correct classification (P(CC)) is guaranteed to be at least a preassigned number P^* ($\frac{1}{k} < P^* < 1$) regardless of what the unknown state of nature might be. The classification rules proposed here are different from those considered by Gupta and Govindarajulu [37].

Let π_i denote a normal population with an unknown mean θ_i and variance σ_i^2 ($i = 0,1,\ldots,k$). From population π_i one observes a random sample X_{ij}, $j = 1,\ldots,n_i$, $i = 0,1,\ldots,k$. Based on the above data we allocate π_0 to one of the k populations with respect to the mean, variance, and the reciprocal of the coefficient of variation. In each case it is assumed that the parameter, for example, the mean θ_0 of π_0 is equal to one of the θ_i, $i = 1,\ldots,k$.

3.2 Classification Rules with Respect to the Mean

Case (i). Common known variance σ^2.

Without loss of generality σ^2 will be taken to be unity. Let

$$\overline{X}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} X_{ij}, \qquad i = 0,1,\ldots,k. \tag{3.2.1}$$

Then we propose the following classification rule,

R_9: Classify π_o as π_i if and only if

$$|\bar{X}_i - \bar{X}_o| \le c_9\sqrt{\frac{1}{n_i} + \frac{1}{n_o}} \tag{3.2.2}$$

where c_9 is chosen such that $P(CC|R_9)$ is at least P^* $(\frac{1}{k} < P^* < 1)$ which is a specified number. Then $P(CC|R_9)$ is given by

$$\begin{aligned} P(CC|R_9) &= \sum_{i=1}^{k} P(|\bar{X}_i-\bar{X}_o| \le c_9\sqrt{\frac{1}{n_i} + \frac{1}{n_o}} \mid \theta_i = \theta_o)\; P(\theta_i = \theta_o) \\ &= \sum_{i=1}^{k} [2\Phi(c_9) - 1]\, q_i \\ &= 2\Phi(c_9) - 1 \end{aligned} \tag{3.2.3}$$

where $\Phi(\cdot)$ denotes the distribution function of the standard normal variate and q_i is the a priori probability of $\theta_i = \theta_o$, $i = 1,\ldots,k$. Since $P(CC|R_9) \ge P^*$, we obtain c_9 as the smallest non-negative number which satisfies the equation

$$2\Phi(c_9) - 1 = P^*$$

or equivalently

$$c_9 = \Phi^{-1}\left(\frac{1+P^*}{2}\right) \tag{3.2.4}$$

<u>Theorem 3.2.1.</u> Let A_i be the event "classify π_o as π_i when $\theta_i \ne \theta_o$". Then $P(A_i|R_9) \to 0$ as each of n_o, $n_i \to \infty$.

<u>Proof.</u> $P(A_i|R_9) = P(|\bar{X}_i-\bar{X}_o| \le c_9\sqrt{\frac{1}{n_i} + \frac{1}{n_o}} \mid \theta_i \ne \theta_o)$

$$= P\left[-c_9 + \frac{\theta_o-\theta_i}{\sqrt{\frac{1}{n_i} + \frac{1}{n_o}}} \le \frac{\bar{X}_i-\bar{X}_o-(\theta_i-\theta_o)}{\sqrt{\frac{1}{n_i} + \frac{1}{n_o}}} \le c_9 + \frac{\theta_o-\theta_i}{\sqrt{\frac{1}{n_i} + \frac{1}{n_o}}} \,\middle|\, \theta_i \ne \theta_o\right]$$

$$\to 0 \quad \text{as } n_o, n_i \to \infty. \tag{3.2.5}$$

Case (ii). Common unknown variance σ^2.

$$\text{Let } \overline{X}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} X_{ij}, \quad s^2 = \frac{1}{\nu} \sum_{i=0}^{k} \sum_{j=1}^{n_i} (X_{ij} - \overline{X}_i)^2.$$

where $\nu = \sum_{i=0}^{k} (n_i - 1)$. In this case, we propose the classification rule as follows:

R_{10}: Classify π_o as π_i if and only if

$$|\overline{X}_i - \overline{X}_o| \leq c_{10}\, s\sqrt{\frac{1}{n_i} + \frac{1}{n_o}} \tag{3.2.7}$$

where, as before, c_{10} is determined as the smallest non-negative number which satisfies $P(CC|R_{10}) \geq P^*$. Here, we have

$$P(CC|R_{10}) = \sum_{i=1}^{k} P(|\overline{X}_i - \overline{X}_o| \leq c_{10}\, s\sqrt{\frac{1}{n_i} + \frac{1}{n_o}} \mid \theta_i = \theta_o)\, P(\theta_i = \theta_o)$$

$$= \sum_{i=1}^{k} P(|T_i| \leq c_{10})\, q_i \tag{3.2.8}$$

where T_i, $i = 1,\ldots,k$ are identically distributed (not independent) as Student's-t with ν degrees of freedom. Hence

$$P(CC|R_{10}) = P(|T| \leq c_{10})$$

where T is distributed as Student's-t with ν degrees of freedom. It should be pointed out that the joint distribution of T_i, $i = 1,\ldots,k$ is a multivariate t as studied, for example, in Gupta [39]. The covariance matrix is $\Sigma = (\sigma_{ij})$ where $\sigma_{ij} = \sigma^2$ for $i = j$ and

$$\sigma_{ij} = \sigma^2 \left[(1 + \frac{n_o}{n_i})(1 + \frac{n_o}{n_j})\right]^{-\frac{1}{2}} \quad \text{for } i \neq j.$$

$$P(CC|R_{10}) = F_T(c_{10}) - F_T(-c_{10}) = 2F_T(c_{10}) - 1 \tag{3.2.9}$$

where $F_T(\cdot)$ is the cumulative distribution function of T and c_{10} is determined to be the smallest non-negative number which satisfies the equation

$$c_{10} = F_T^{-1}\left(\frac{1+P^*}{2}\right) . \tag{3.2.10}$$

If A_i is the same event as defined in Theorem 3.2.1., then we have a similar result for the rule R_{10}.

<u>Theorem 3.2.2.</u> $P(A_i|R_{10}) \to 0$ as each of $n_o, n_1, \ldots, n_k \to \infty$.

<u>Proof</u>. The proof is straightforward and hence omitted.

3.3 <u>Classification Rule with Respect to the Variance</u>

It is assumed that the population π_i has unknown mean θ_i and variance σ_i^2, $i = 1,\ldots,k$. As before we assume that one of the variances σ_i^2, $i = 1,2,\ldots,k$ is equal to σ_o^2. We, then, wish to allocate π_o to one of the k populations $\pi_1,\ldots,\pi_k$ with respect to the variance. Assume we take $n_i \geq 4$ observations from π_i, $i = 0,1,\ldots,k$. We propose the classification rule:

R_{11}: Classify π_o as π_i if and only if

$$\left| \frac{n_o(n_i - 3)}{n_i(n_o - 1)} \frac{s_o^2}{s_i^2} - 1 \right| \leq c_{11} .$$

where $s_i^2 = \frac{1}{n_i} \sum_{j=1}^{n_i} (X_{ij} - \overline{X}_i)^2$, $i = 0,1,\ldots,k$ and c_{11} is the smallest non-negative number which satisfies the inequality $P(CC|R_{11}) \geq P^*$. We have

$$P(CC|R_{11}) = \sum_{i=1}^{k} P\{\left| \frac{n_o(n_i-3)}{n_i(n_o-1)} \frac{s_o^2}{s_i^2} - 1 \right| \le c_{11} \Big| \sigma_i=\sigma_o\} \; P\{\sigma_i=\sigma_o\}$$

$$= \sum_{i=1}^{k} P\{\left| \frac{n_o s_o^2/[\sigma_o^2(n_o-1)]}{n_i s_i^2/[\sigma_i^2(n_i-1)]} - \frac{n_i-1}{n_i-3} \right| \le \frac{n_i-1}{n_i-3} c_{11} \Big| \sigma_i=\sigma_o\} P\{\sigma_i=\sigma_o\}$$

$$= \sum_{i=1}^{k} q_i'\{F_{n_o-1,n_i-1}\left[\frac{n_i-1}{n_i-3}(1+c_{11})\right] - F_{n_o-1,n_i-1}\left[\frac{n_i-1}{n_i-3}(1-c_{11})\right]\} \tag{3.3.2}$$

where F_{ν_1,ν_2} is the cdf of an F random variable with ν_1 and ν_2 degrees of freedom, and q_i' is the a priori probability of $\sigma_i = \sigma_o$. In the special case when $n_1 = n_2 = \dots = n_k = n$, (3.3.2) becomes

$$P(CC|R_{11}) = F_{n_o-1,n-1}\left[\frac{n-1}{n-3}(1+c_{11})\right] - F_{n_o-1,n-1}\left[\frac{n-1}{n-3}(1-c_{11})\right] \tag{3.3.3}$$

and c_{11} is the smallest non-negative constant determined from the equation

$$F_{n_o-1,n-1}\left[\frac{n-1}{n-3}(1+c_{11})\right] - F_{n_o-1,n-1}\left[\frac{n-1}{n-3}(1-c_{11})\right] = P^*.$$

In using the rules R_9, R_{10}, R_{11}, we might classify π_o as none, one, two or k of the k populations. In the following sections, we use the subset (non-empty) selection approach to propose rules that will classify π_o to be at least one of the k populations. This overcomes the objection that π_o may not be classified as any one of the k populations.

3.4 A Subset Selection Approach to Classification Rule with Respect to the Mean

Without loss of generality, we may assume that $\sigma^2 = 1$. Again we assume that there is exactly one population with $\theta_i = \theta_o$. In this case, we propose the rollowing classification rule

R_{12}: Classify π_o as π_i if and only if

$$|\overline{X}_i - \overline{X}_o| \leq c_{12} \min_{1 \leq j \leq k} |\overline{X}_j - \overline{X}_o| \tag{3.4.1}$$

where c_{12} (≥ 1) is the smallest number which satisfies the inequality $P(CC|R_{12}) \geq P^*$.

The classification rule R_{12} has the following desirable asymptotic property, i.e., the probability of misclassification approaches zero as the sample sizes $n_o, n_1, \ldots, n_k$ become large. Before we prove this we need the following lemma.

<u>Lemma 3.4.1.</u> $P(MC|R_{12}) \leq \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j \neq i}}^{k} P(|\overline{X}_i - \overline{X}_o| > c_{12} |\overline{X}_j - \overline{X}_o| \,\big|\, \theta_i = \theta_o)$

where MC denotes misclassification.

<u>Proof.</u> Since

$$\begin{aligned}
P(MC|R_{12}) &= \sum_{i=1}^{k} P(\pi_o \text{ is not classified as } \pi_i | \theta_i = \theta_o) P(\theta_i = \theta_o) \\
&= \sum_{i=1}^{k} P\{|\overline{X}_i - \overline{X}_o| > c_{12} \min_{1 \leq j \leq k} |\overline{X}_j - \overline{X}_o| \,\big|\, \theta_i = \theta_o\} \cdot P\{\theta_i = \theta_o\} \\
&\leq \sum_{i=1}^{k} P\{|\overline{X}_i - \overline{X}_o| > c_{12} |\overline{X}_j - \overline{X}_o| \text{ for some } j \neq i \,\big|\, \theta_i = \theta_o\} \\
&\leq \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j \neq i}}^{k} P\{|\overline{X}_i - \overline{X}_o\} > c_{12} |\overline{X}_j - \overline{X}_o| \,\big|\, \theta_i = \theta_o\}
\end{aligned} \tag{3.4.2}$$

thus proved the lemma.

<u>Theorem 3.4.1</u>. With the above notation, if $\frac{n_j}{N} \to \lambda_j$, $j = 0,1,\ldots,k$, then

$$P(MC|R_{12}) \to 0 \qquad \text{as } N \to \infty$$

where $\quad N = \sum_{i=0}^{k} n_i.$

<u>Proof</u>. By Lemma 3.4.1., we have

$$P(MC|R_{12}) \le \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j\ne i}}^{k} P\{|\bar{X}_j-\bar{X}_o| \le \frac{1}{c_{12}} |\bar{X}_i - \bar{X}_o| \Big| \theta_i = \theta_o\}$$

$$= \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j\ne i}}^{k} \{P[-\frac{1}{c_{12}} (\bar{X}_i-\bar{X}_o) \le \bar{X}_j-\bar{X}_o \le \frac{1}{c_{12}} (\bar{X}_i-\bar{X}_o),\ \bar{X}_i>\bar{X}_o \Big| \theta_i=\theta_o]$$

$$+ P[-\frac{1}{c_{12}} (\bar{X}_o-\bar{X}_i) \le \bar{X}_j-\bar{X}_o \le \frac{1}{c_{12}} (\bar{X}_o-\bar{X}_i),\ \bar{X}_i \le \bar{X}_o \Big| \theta_i=\theta_o]\}$$

$$= \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j\ne i}}^{k} \{P[\bar{X}_o- \frac{1}{c_{12}}(\bar{X}_i-\bar{X}_o) \le \bar{X}_j \le \bar{X}_o+ \frac{1}{c_{12}} (\bar{X}_i-\bar{X}_o), \bar{X}_i>\bar{X}_o \Big| \theta_i=\theta_o]$$

$$+ P[\bar{X}_o- \frac{1}{c_{12}} (\bar{X}_o-\bar{X}_i) \le \bar{X}_j \le \bar{X}_o+ \frac{1}{c_{12}} (\bar{X}_o-\bar{X}_i),\ \bar{X}_i \le \bar{X}_o \Big| \theta_i=\theta_o]\}$$

$$= \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j\ne i}}^{k} \{P\Bigg[\sqrt{\frac{n_j}{n_o}}\, Z_o - \frac{1}{c_{12}} \left(\sqrt{\frac{n_j}{n_i}}\, Z_i - \sqrt{\frac{n_j}{n_o}}\, Z_o\right) + \sqrt{n_j}\,(\theta_o-\theta_j) \le Z_j$$

$$\le \sqrt{\frac{n_j}{n_o}}\, Z_o + \frac{1}{c_{12}} \left(\sqrt{\frac{n_j}{n_i}}\, Z_i - \sqrt{\frac{n_j}{n_o}}\, Z_o\right) + \sqrt{n_j}\,(\theta_o-\theta_j),$$

$$Z_i > \sqrt{\frac{n_i}{n_o}}\, Z_o \Big|\ \theta_i=\theta_o]$$

$$+ P[\sqrt{\frac{n_j}{n_o}}\, Z_o - \frac{1}{c_{12}}\left(\sqrt{\frac{n_j}{n_o}}\, Z_o - \sqrt{\frac{n_j}{n_i}}\, Z_i\right) + \sqrt{n_j}\,(\theta_o-\theta_j) \le Z_j$$

$$\le \sqrt{\frac{n_j}{n_o}}\, Z_o + \frac{1}{c_{12}}\left(\sqrt{\frac{n_j}{n_o}}\, Z_o - \sqrt{\frac{n_j}{n_i}}\, Z_i\right) + n_j\,(\theta_o-\theta_j), Z_i \le \sqrt{\frac{n_i}{n_o}}\, Z_o \Big| \theta_i=\theta_o]\}$$

where Z_i, $i = 0,1,\ldots,k$, are iid standard normal variates.

$$= \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j\neq i}}^{k} \Big\{ \int_{-\infty}^{\infty} \int_{\sqrt{\frac{n_i}{n_o}}x}^{\infty} \Big[\Phi\Big(\sqrt{\frac{n_j}{n_o}}\,x + \frac{1}{c_{12}}\Big(\sqrt{\frac{n_j}{n_i}}\,y - \sqrt{\frac{n_j}{n_o}}\,x\Big) + \sqrt{n_j}(\theta_o - \theta_j)\Big)$$

$$- \Phi\Big(\sqrt{\frac{n_j}{n_o}}\,x - \frac{1}{c_{12}}\Big(\sqrt{\frac{n_j}{n_i}}\,y - \sqrt{\frac{n_j}{n_o}}\,x\Big) + \sqrt{n_j}(\theta_o - \theta_j)\Big) \Big] d\Phi(y)\, d\Phi(x)$$

$$+ \int_{-\infty}^{\infty} \int_{-\infty}^{\sqrt{\frac{n_i}{n_o}}x} \Big[\Phi\Big(\sqrt{\frac{n_j}{n_o}}\,x + \frac{1}{c_{12}}\Big(\sqrt{\frac{n_j}{n_o}}\,x - \sqrt{\frac{n_j}{n_i}}\,y\Big) + \sqrt{n_j}(\theta_o - \theta_j)\Big)$$

$$- \Phi\Big(\sqrt{\frac{n_j}{n_o}}\,x - \frac{1}{c_{12}}\Big(\sqrt{\frac{n_j}{n_o}}\,x - \sqrt{\frac{n_j}{n_i}}\,y\Big) + \sqrt{n_j}(\theta_o - \theta_j)\Big) \Big] d\Phi(y)\, d\Phi(x) \Big\}$$

$$= \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j\neq i}}^{k} \{ I_1(i,j) + I_2(i,j) \}$$

where $I_1(i,j)$, $I_2(i,j)$ are the first and second double integrals inside the double summation signs.

Now, for every $\varepsilon > 0$, there exists $\delta(\varepsilon)$ such that

$$\int_{|x| > \delta(\varepsilon)} d\Phi(x) \leq \varepsilon \qquad (3.4.7)$$

Hence, we get

$$I_1(i,j) \le \int_{-\delta(\varepsilon)}^{\delta(\varepsilon)}\int_{-\delta(\varepsilon)}^{\delta(\varepsilon)} [\Phi((1-\frac{1}{c_{12}})\sqrt{\frac{n_j}{n_o}}\,x + \frac{1}{c_{12}}\sqrt{\frac{n_j}{n_i}}\,y + \sqrt{n_j}(\theta_o-\theta_j))$$

$$- \Phi((1+\frac{1}{c_{12}})\sqrt{\frac{n_j}{n_o}}\,x - \frac{1}{c_{12}}\sqrt{\frac{n_j}{n_i}}\,y + \sqrt{n_j}(\theta_o-\theta_j))]$$

$$d\Phi(y)d\Phi(x) + \varepsilon^2$$

$$\le \int_{-\delta(\varepsilon)}^{\delta(\varepsilon)} [\Phi((1-\frac{1}{c_{12}})\sqrt{\frac{n_j}{n_o}}\,x + \frac{1}{c_{12}}\sqrt{\frac{n_j}{n_i}}\,\delta(\varepsilon) + \sqrt{n_j}(\theta_o-\theta_j))$$

$$- \Phi((1+\frac{1}{c_{12}})\sqrt{\frac{n_j}{n_o}}\,x - \frac{1}{c_{12}}\sqrt{\frac{n_j}{n_i}}\,\delta(\varepsilon) + \sqrt{n_j}(\theta_o-\theta_j))]\ d\Phi(x) + \varepsilon^2$$

$$\le \Phi((1-\frac{1}{c_{12}})\sqrt{\frac{n_j}{n_o}}\,\delta(\varepsilon) + \frac{1}{c_{12}}\sqrt{\frac{n_j}{n_i}}\,\delta(\varepsilon) + \sqrt{n_j}\ (\theta_o-\theta_j))$$

$$- \Phi(-(1+\frac{1}{c_{12}})\sqrt{\frac{n_j}{n_o}}\,\delta(\varepsilon) - \frac{1}{c_{12}}\sqrt{\frac{n_j}{n_i}}\,\delta(\varepsilon) + \sqrt{n_j}\ (\theta_o-\theta_j)) + \varepsilon^2.$$

(3.4.7)

For every $\eta > 0$, there exists an $N(\varepsilon,\eta)$ such that for $N > N(\varepsilon,\eta)$,

$$\Phi((1-\frac{1}{c_{12}})\sqrt{\frac{\lambda_j}{\lambda_o}}\,\delta(\varepsilon) + \frac{1}{c_{12}}\sqrt{\frac{\lambda_j}{\lambda_i}}\,\delta(\varepsilon) + \sqrt{N\lambda_j}\ (\theta_o-\theta_j))$$

$$- \Phi(-(1+\frac{1}{c_{12}})\sqrt{\frac{\lambda_j}{\lambda_o}}\,\delta(\varepsilon) - \frac{1}{c_{12}}\sqrt{\frac{\lambda_j}{\lambda_i}}\,\delta(\varepsilon) + \sqrt{N\lambda_j}\ (\theta_o-\theta_j)) \le \eta$$

(3.4.8)

Now since ε and η are arbitrary, $I_1(i,j) \to 0$ as $N \to \infty$. Using an identical argument one can show that $I_2(i,j) \to 0$ as $N \to \infty$. Thus, $P(MC|R_{12}) \to 0$ as $N \to \infty$. This completes the proof of the theorem.

Suppose $n_o = n_1 = \ldots = n_k = n$, we have the following.

Theorem 3.4.2. If c_{12} is chosen to be the smallest constant such that

$$\int_0^\infty \Phi(\frac{\sqrt{2}x}{c_{12}})\ d\Phi(x) \le \frac{1}{4} + \frac{1-P^*}{4k(k-1)} \qquad (3.4.10)$$

then

$$P(CC|R_{12}) \geq P^*. \tag{3.4.11}$$

Proof. Since

$$P(CC|R_{12})=1-P(MC|R_{12})$$

$$\geq 1 - \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j\neq i}}^{k} \{P[\overline{X}_o - \frac{1}{c_{12}} (\overline{X}_i - \overline{X}_o) < \overline{X}_j < \overline{X}_o + \frac{1}{c_{12}} (\overline{X}_i - \overline{X}_o),\ \overline{X}_i > \overline{X}_o \Big| \theta_i = \theta_o]$$

$$+ P[\overline{X}_o - \frac{1}{c_{12}} (\overline{X}_o - \overline{X}_i) < \overline{X}_j < \overline{X}_o + \frac{1}{c_{12}} (\overline{X}_o - \overline{X}_i),\ \overline{X}_i \leq \overline{X}_o \Big| \theta_i = \theta_o]\}$$

$$= 1 - \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j\neq i}}^{k} \{P[Z_o - \frac{1}{c_{12}}(Z_i - Z_o) + \sqrt{n}(\theta_o - \theta_j) < Z_j < Z_o + \frac{1}{c_{12}} (Z_i - Z_o)$$

$$+ \sqrt{n}\ (\theta_o - \theta_j),\ Z_i > Z_o]$$

$$+ P[Z_o + \frac{1}{c_{12}} (Z_i - Z_o) + \sqrt{n}(\theta_o - \theta_j) < Z_j < Z_o - \frac{1}{c_{12}} (Z_i - Z_o)$$

$$+ \sqrt{n}\ (\theta_o - \theta_j),\ Z_i \leq Z_o]\} \tag{3.4.12}$$

where Z_i, $i = 0,1,\ldots,k$, are iid standard normal variables. But

$$P[Z_o + \sqrt{n}(\theta_o - \theta_j) - \frac{1}{c_{12}}(Z_i - Z_o) < Z_j < Z_o + \sqrt{n}(\theta_o - \theta_j) + \frac{1}{c_{12}} (Z_i - Z_o),\ Z_i - Z_o > 0]$$

$$\leq P\left[-\frac{\sqrt{2}}{c_{12}} \frac{(Z_i - Z_o)}{\sqrt{2}} < Z_j < \frac{\sqrt{2}}{c_{12}} \frac{(Z_i - Z_o)}{\sqrt{2}},\ \frac{Z_i - Z_o}{\sqrt{2}} > 0\right]$$

$$= \int_0^\infty \left[\Phi(\frac{\sqrt{2}}{c_{12}} x) - \Phi(-\frac{\sqrt{2}}{c_{12}} x)\right] d\Phi(x)$$

$$= \int_0^\infty \left[2\Phi(\frac{\sqrt{2}}{c_{12}} x) - 1\right] d\Phi(x)\ . \tag{3.4.13}$$

Similarly,

$$P\{Z_o + \frac{1}{c_{12}}(Z_i - Z_o) + \sqrt{n}(\theta_o - \theta_j) < Z_j < Z_o - \frac{1}{c_{12}}(Z_i - Z_o) + \sqrt{n}(\theta_o - \theta_j),\ Z_i - Z_o \le 0\}$$

$$\le \int_{-\infty}^{0} [2\Phi(-\frac{\sqrt{2}}{c_{12}}x) - 1]\, d\Phi(x)$$

$$= \int_{0}^{\infty} [2\Phi(\frac{\sqrt{2}}{c_{12}}x) - 1]\, d\Phi(x). \qquad (3.4.14)$$

So that

$$P(CC|R_{12}) \ge 1 - 2 \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j\ne i}}^{k} \int_{0}^{\infty} [2\Phi(\frac{\sqrt{2}}{c_{12}}x) - 1]\, d\Phi(x)$$

$$= 1 - 2k(k-1) \int_{0}^{\infty} [2\Phi(\frac{\sqrt{2}}{c_{12}}x) - 1]\, d\Phi(x) \qquad (3.4.15)$$

Hence, for any P^*, let c_{12} be the smallest non-negative number such that

$$\int_{0}^{\infty} \Phi(\frac{\sqrt{2}}{c_{12}}x)\, d\Phi(x) \le \frac{1}{4} + \frac{1 - P^*}{4k(k-1)},$$

then $P(CC|R_{12}) \ge P^*$.

3.5 A Subset Selection Approach to Classification Rule with Respect to the Reciprocal of the Coefficient of Variation

Let
$$\overline{X}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} X_{ij}, \quad i = 0,1,\ldots,k \qquad (3.5.1)$$

$$s_i^2 = \frac{1}{n_i} \sum_{j=1}^{n_i} (X_{ij} - \overline{X}_i)^2, \quad \frac{n_i s_i^2}{\sigma_i^2} \sim \chi^2_{n_i - 1} \qquad (3.5.2)$$

$$\alpha_i = \frac{\theta_i}{\sigma_i} = \frac{1}{\text{coefficient of variation}}, \quad i=0,1,\ldots,k \qquad (3.5.3)$$

We assume $\alpha_o = \frac{\theta_o}{\sigma_o}$ to be known and further that there exists only one population with $\alpha_1 = \alpha_o$. The classification rule proposed in this case is

R_{13}: Classify π_o as π_i if and only if

$$\left| \frac{\overline{X}_i}{s_i} - \alpha_o \right| \le c_{13} \min_{1\le j\le k} \left| \frac{\overline{X}_j}{s_j} - \alpha_o \right| \tag{3.5.4}$$

where $c_{13}(\geq 1)$ is chosen to be the smallest nonnegative number such that $P(CC|R_{13}) \geq P^*$.

The classification rule R_{13} has the asymptotic optimum property that $P(CC|R_{13})$ approaches unity as the sample sizes n_i's, $i = 1,\ldots,k$, become large. This result is proved in the following theorem where as before we write $N = \sum_{i=1}^{k} n_i$ and $\frac{n_i}{N} \to \lambda_i$, $i = 1,\ldots,k$.

<u>Theorem 3.5.1</u>. $P(MC|R_{13}) \to 0$ as $N \to \infty$, where $N = \sum_{i=1}^{k} n_i$.

<u>Proof</u>. Using a similar agrument as in Lemma 3.4.1, we have

$$P(MC|R_{13}) \le \sum_{i=1}^{k} \sum_{\substack{j=1\\j\ne i}}^{k} P\left\{ \left|\frac{\overline{X}_i}{s_i} - \alpha_o\right| > c_{13} \left|\frac{\overline{X}_j}{s_j} - \alpha_o\right| \,\middle|\, \alpha_i = \alpha_o \right\}$$

$$= \sum_{i=1}^{k} \sum_{\substack{j=1\\j\ne i}}^{k} [P\{\alpha_o - \frac{1}{c_{13}} (\frac{\overline{X}_i}{s_i} - \alpha_o) \le \frac{\overline{X}_j}{s_j} \le \alpha_o + \frac{1}{c_{13}} (\frac{\overline{X}_i}{s_i} - \alpha_o) ,$$

$$\frac{\overline{X}_i}{s_i} \ge \alpha_o \mid \alpha_i = \alpha_o\}$$

$$+ P\{\alpha_o + \frac{1}{c_{13}} (\frac{\overline{X}_i}{s_i} - \alpha_o) \le \frac{\overline{X}_j}{s_j} \le \alpha_o - \frac{1}{c_{13}} (\frac{\overline{X}_i}{s_i} - \alpha_o),$$

$$\frac{\overline{X}_i}{s_i} < \alpha_i \mid \alpha_i = \alpha_o\}]$$

$$= \sum_{i=1}^{k} \sum_{\substack{j=1\\j\ne i}}^{k} [P\{(1+ \frac{1}{c_{13}})\alpha_o - \frac{Z_i}{c_{13}U_i} - \frac{\sqrt{n_i}\alpha_o}{c_{13}U_i} - \frac{\sqrt{n_j}\alpha_j}{U_j} \le \frac{Z_j}{U_j} \le (1- \frac{1}{c_{13}})\alpha_o$$

$$+ \frac{Z_i}{c_{13}U_i} + \frac{\sqrt{n_i}\alpha_o}{c_{13}U_i} - \frac{\sqrt{n_j}\alpha_j}{U_j} ,$$

$$\frac{Z_i}{U_i} \ge \alpha_o - \frac{\sqrt{n_i}\alpha_o}{U_i} \,\middle|\, \alpha_i = \alpha_o\}$$

$$+ P\{(1-\frac{1}{c_{13}})\alpha_o + \frac{Z_i}{c_{13}U_i} + \frac{\sqrt{n_i}\alpha_o}{c_{13}U_i} - \frac{\sqrt{n_j}\alpha_j}{U_j} \leq \frac{Z_j}{U_j} \leq (1+\frac{1}{c_{13}})\alpha_o$$

$$- \frac{Z_i}{c_{13}U_i} - \frac{\sqrt{n_i}\alpha_o}{c_{13}U_i} - \frac{\sqrt{n_j}\alpha_j}{U_j}, \frac{Z_i}{U_i} < \alpha_o - \frac{\sqrt{n_i}\alpha_o}{U_i} \Bigg| \alpha_i = \alpha_o\}]$$

where

$$Z_i = \frac{\sqrt{n_i}(\bar{X}_i - \theta_i)}{\sigma_i}, \quad U_i = \frac{\sqrt{n_i}s_i}{\sigma_i}, \quad i = 1,\ldots,k.$$

$$= S_1 + S_2 \text{ (say)}$$

Now,

$$S_1 = \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j\neq i}}^{k} \int_0^\infty \int_0^\infty \int_{\alpha_o v - \sqrt{n_i}\alpha_o}^\infty [\Phi((1-\frac{1}{c_{13}})\alpha_o t + \frac{xt}{c_{13}v} + \frac{\sqrt{n_i}\alpha_o t}{c_{13}v} - \sqrt{n_j}\alpha_j)$$

$$- \Phi((1+\frac{1}{c_{13}})\alpha_o t - \frac{xt}{c_{13}v} - \frac{\sqrt{n_i}\alpha_o t}{c_{13}v} - \sqrt{n_j}\alpha_j)]$$

$$d\Phi(x)\, dF_i(v)\, dF_j(t)$$

where $\Phi(\cdot)$ is the cdf of unite normal variate and $F_i(\cdot)$ represents the cdf of the square root of a χ^2 random variable with (n_i-1) degrees of freedom.

For any $\varepsilon > 0$, there exists $\delta(\varepsilon)$ such that

$$\int_{|x|\geq\delta(\varepsilon)} d\Phi(x) \leq \varepsilon$$

and

$$\int_{|x|\geq\delta(\varepsilon)} dF_r(x) \leq \varepsilon \qquad \text{for } r = 1,2,\ldots,k.$$

Hence, we get

$$S_1 \le \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j\neq i}}^{k} \int_{|t|<\delta(\varepsilon)} \int_{|v|<\delta(\varepsilon)} \int_{|x|<\delta(\varepsilon)}$$

$$[\Phi((1-\frac{1}{c_{13}})\alpha_o t + \frac{xt}{c_{13}v} + \frac{\sqrt{n_i}\alpha_o t}{c_{13}v} - \sqrt{n_j}\alpha_j) - \Phi((1+\frac{1}{c_{13}})\alpha_o t - \frac{xt}{c_{13}v} - \frac{\sqrt{n_i}\alpha_o t}{c_{13}n}$$

$$- \sqrt{n_j}\alpha_j)]\ d\Phi(x)\ dF_i(v)\ dF_j(t) + \varepsilon^3.$$

$$= \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j\neq i}}^{k} \int_{|t|<\delta(\varepsilon)} \int_{|v|<\delta(\varepsilon)} \int_{|x|<\delta(\varepsilon)}$$

$$[\Phi((1-\frac{1}{c_{13}})\alpha_o t + \frac{xt}{c_{13}v} - (\alpha_j\sqrt{\lambda_j} - \frac{\sqrt{\lambda_i}\alpha_o t}{c_{13}v})\sqrt{N})$$

$$-\Phi((1+\frac{1}{c_{13}})\alpha_o t - \frac{xt}{c_{13}v} - (\alpha_j\sqrt{\lambda_j} + \frac{\sqrt{\lambda_i}\alpha_o t}{c_{13}v})\sqrt{N})]\ d\Phi(x) dF_i(x) dF_j(t) + \varepsilon^3.$$

If t, v and x are bounded, then for any $\varepsilon > 0$, there exists an $N(\varepsilon)$ such that whenever $N > N(\varepsilon)$,

$$\Phi((1-\frac{1}{c_{13}})\alpha_o t + \frac{xt}{c_{13}v} - (\alpha_j\sqrt{\lambda_j} - \frac{\sqrt{\lambda_i}\alpha_o t}{c_{13}v})\sqrt{N})$$

$$- \Phi((1+c_{13})\alpha_o t - \frac{xt}{c_{13}v} - (\alpha_j\sqrt{\lambda_j} + \frac{\sqrt{\lambda_i}\alpha_o t}{c_{13}v})\sqrt{N}) < \varepsilon .$$

Since $\varepsilon > 0$ is arbitrarily, hence $S_1 \to 0$ as $N \to \infty$. Similarly, one can show that $S_2 \to 0$ as $N \to \infty$. Then $P(MC|R_{13}) \to 0$ as $N \to \infty$.

Theorem 3.5.2. Suppose $n_1 = \ldots = n_k = n$. If c_{13} is the smallest non-negative number such that

$$\int_0^\infty\int_0^\infty \{\int_{\alpha_o v-\sqrt{n}\alpha_o}^{\infty} 2\Phi(\frac{1}{c_{13}}(-\alpha_o+\frac{x}{v}+\frac{\sqrt{n}\alpha_o}{v})t)\,d\Phi(x)$$

$$-\int_{-\infty}^{\alpha_o v-\sqrt{n}\alpha_o} 2\Phi(\frac{1}{c_{13}}(-\alpha_o+\frac{x}{v}+\frac{\sqrt{n}\alpha_o}{v})t)\,d\Phi(x) + 2\Phi(\alpha_o v-\sqrt{n}\alpha_o)\}$$

$$dF_n(v)\; dF_n(t) \le 1 + \frac{1-P^*}{k(k-1)}$$

Then $P(CC|R_{13}) \ge P^*$.

Proof.

$$P(CC|R_{13}) = 1 - P(MC|R_{13})$$

$$\ge 1 - \sum_{i=1}^{k}\sum_{\substack{j=1\\ j\ne i}}^{k} [P\{\frac{\alpha_o}{c_{13}} - \frac{Z_i}{c_{13}U_i} - \frac{\sqrt{n}\alpha_o}{c_{13}U_i} + \alpha_o - \frac{\sqrt{n}\alpha_j}{U_j} \le \frac{Z_j}{U_j} \le -\frac{\alpha_o}{c_{13}}$$

$$+\frac{Z_i}{c_{13}U_i} + \frac{\sqrt{n}\alpha_o}{c_{13}U_i} + \alpha_o - \frac{\sqrt{n}\alpha_j}{U_j},\ \frac{Z_i}{U_i} \ge \alpha_o - \frac{\sqrt{n}\alpha_o}{U_i}\}$$

$$+ P\{-\frac{\alpha_o}{c_{13}} + \frac{Z_i}{c_{13}U_i} + \frac{\sqrt{n}\alpha_o}{c_{13}U_i} + \alpha_o - \frac{\sqrt{n}\alpha_j}{U_j} \le \frac{Z_j}{U_j} \le \frac{\alpha_o}{c_{13}}$$

$$-\frac{Z_i}{c_{13}U_i} - \frac{\sqrt{n}\alpha_o}{c_{13}U_i} + \alpha_o - \frac{\sqrt{n}\alpha_j}{U_j},\ \frac{Z_i}{U_i} < \alpha_o - \frac{\sqrt{n}\alpha_o}{U_i}\}]$$

$$\ge 1 - \sum_{i=1}^{k}\sum_{\substack{j=1\\ j\ne i}}^{k} [P\{\frac{\alpha_o}{c_{13}} - \frac{Z_i}{c_{13}U_i} - \frac{\sqrt{n}\alpha_o}{c_{13}U_i} \le \frac{Z_j}{U_j} \le -\frac{\alpha_o}{c_{13}} + \frac{Z_i}{c_{13}U_i} + \frac{\sqrt{n}\alpha_o}{c_{13}U_i},$$

$$\frac{Z_i}{U_i} \ge \alpha_o - \frac{\sqrt{n}\alpha_o}{U_i}\}$$

$$+ P\{-\frac{\alpha_o}{c_{13}} + \frac{Z_i}{c_{13}U_i} + \frac{\sqrt{n}\alpha_o}{c_{13}U_i} \le \frac{Z_j}{U_j} \le \frac{\alpha_o}{c_{13}} - \frac{Z_i}{c_{13}U_i} - \frac{\sqrt{n}\,\alpha_o}{c_{13}U_i},$$

$$\frac{Z_i}{U_i} < \alpha_o - \frac{\sqrt{n}\,\alpha_o}{U_i}\}]$$

$$= 1 - \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j\neq i}}^{k} \Big[\int_0^\infty \int_0^\infty \int_{\alpha_o v-\sqrt{n}\alpha_o}^{\infty} \{\Phi(\frac{1}{c_{13}}(-\alpha_o + \frac{x}{v} + \frac{\sqrt{n}\,\alpha_o}{v})t)$$

$$- \Phi(\frac{1}{c_{13}}(\alpha_o - \frac{x}{v} - \frac{\sqrt{n}\alpha_o}{v})t)\}\ d\Phi(x)\ dF_n(v)\ dF_n(t)$$

$$+ \int_0^\infty \int_0^\infty \int_{-\infty}^{\alpha_o v-\sqrt{n}\alpha_o} \{\Phi(\frac{1}{c_{13}}(\alpha_o - \frac{x}{v} - \frac{\sqrt{n}\alpha_o}{v})t)$$

$$-\Phi(\frac{1}{c_{13}}(-\alpha_o + \frac{x}{v} + \frac{\sqrt{n}\alpha_o}{v})t)\}d\Phi(x)dF_n(v)dF_n(t)$$

where $F_n(\cdot)$ is the cdf of the square root of a x^2 random variable with (n-1) degrees of freedom.

$$= 1-k(k-1)\ \Big[\int_0^\infty \int_0^\infty \int_{\alpha_o v-\sqrt{n}\alpha_o}^{\infty} \{2\Phi(\frac{1}{c_{13}}(-\alpha_o + \frac{x}{v} + \frac{\sqrt{n}\alpha_o}{v})t)-1\}\ d\Phi(x)\ dF_n(v)\ dF_n(t)$$

$$+ \int_0^\infty \int_0^\infty \int_{-\infty}^{\alpha_o \nu-\sqrt{n}\alpha_o} \{1-2\Phi(\frac{1}{c_{13}}(-\alpha_o + \frac{x}{v} + \frac{\sqrt{n}\alpha_o}{v})t)\}\ d\Phi(x)\ dF_n(\nu)\ dF_n(t)$$

So if c_{13} is chosen to be the smallest nonnegative number such that

$$\int_0^\infty \int_0^\infty \int_{\alpha_o \nu-\sqrt{n}\alpha_o}^{\infty} 2\Phi(\frac{1}{c_{13}}(-\alpha_o + \frac{x}{v} + \frac{\sqrt{n}\alpha_o}{v})t)\ d\Phi(x) - \int_{-\infty}^{\alpha_o v-\sqrt{n}\alpha_o} 2\Phi(\frac{1}{c_{13}}(-\alpha_o + \frac{x}{v} + \frac{\sqrt{n}\alpha_o}{v})t)\ d\Phi(x)$$

$$+ 2\Phi(\alpha_o v-\sqrt{n}\alpha_o)\}\ dF_n(v)\ dF_n(t) \leq 1 + \frac{1 - P^*}{k(k-1)}$$

then

$$P(CC|R_{13}) \geq P^*.$$

CHAPTER IV

SELECTION PROCEDURES FOR NEGATIVE BINOMIAL POPULATIONS

4.1 Introduction

Let X_i, $i = 1,2,\ldots,k$ be k independent random observations from population π_i, $i = 1,2,\ldots,k$, respectively, which has a negative binomial distribution with parameters r_i, p_i. To select a subset of populations which contains the population associated with the largest p_i when $r_1 = \ldots = r_k = r$, Bartlett and Govindarajulu [10] proposed a rule which is based on the statistic $b \min_{1\leq j\leq k} X_j - X_i$, where b is a constant. However, in many situations one may be interested in small values of p_i. In this chapter, our aim is to select a subset of k negative binomial populations which contains the population with the smallest p_i based on a statistic of the type $c \max_{1\leq j\leq k} X_j - X_i$. It should be pointed out that although the two problems seem similar, they are not equivalent, i.e. the procedures proposed here cannot be obtained from the above paper [10]. In Section 4.2, we present a result which gives a conservative constant for the unconditional rule we propose, which is based on an exact computation of the conditional distribution of the statistic $c \max_{1\leq j\leq k} X_j - X_i$. In Section 4.3, we propose a similar rule as in Section 4.2, except that the rule is conditioned on the total number of observations $T = \sum_{i=1}^{k} X_i$. We obtain a lower bound for the infimum of the probability of a correct selection. It is shown that when $k = 2$, the infimum of this procedure is attained when $p_1 = p_2 = p$, and it is independent of p. A method leading to a conservative solution for the constant c(t) depending on T = t is also given. An upper bound for the expected subset size is derived which holds for all values of the parameters. The problem of selecting all populations better than a standard is also considered in Section 4.4.

4.2 An Unconditional Subset Selection Procedure

Let X be a random variable which has the negative binomial distribution with parameters r, p, i.e. X denotes the number of failures before the rth success is observed, p being the probability of a success in an independent trial. Then X is distributed with the probability mass function

$$P(X = x) = \binom{r+x-1}{x} p^r (1-p)^x \quad , \quad x = 0,1,2,\ldots \qquad (4.2.1)$$

It is known that the sum of n independent and identically distributed negative binomial random variables with parameters r, p is again a negative binomial random variable with parameters nr, p. We may, therefore, think of the selection problem as being one of picking a subset containing the negative binomial populations with the smallest p value, based on a single observation on X from each of the k populations.

Suppose then that X_i, $i = 1,2,\ldots,k$ are independent observations from populations π_i, $i = 1,2,\ldots,k$, respectively, which have negative binomial distributions with parameters r, p_i. Without loss of generality and in order to avoid a more complicated notation, we assume that $p_1 \le p_2 \le \ldots \le p_k$. Any population whose parameter value equals p_1 will be defined as a best population. A correct selection (CS) is defined as a selection of any subset of the k given populations which contains at least one best population. Let $P\{CS; k, \underline{p}, R\}$ denote the probability of a correct selection when the procedure R is used with the given k and when the true configuration of parameter values in $\underline{p} = (p_1,\ldots,p_k)$. Let Ω be the space of all configuration $\underline{p} = (p_1,\ldots,p_k)$ such that $0 < p_i < 1$, $i = 1,\ldots,k$.

(A) The Rule R_{14} and its Associated Probability of a Correct Selection

We propose the following selection rule:

R_{14}: Retain population π_i in the selected subset if and only if

$$\frac{r}{r + X_i} \leq d \min_{1 \leq j \leq k} \frac{r}{r + X_j} \tag{4.2.2}$$

where $d \geq 1$ is the smallest number such that the basic probability requirement

$$\inf_{\underline{p} \in \Omega} P_{\underline{p}}(CS|R_{14}) \geq P^* \tag{4.2.3}$$

is satisfied.

Letting $c_{14} = \frac{1}{d}$, the rule in (4.2.2) becomes

R_{14}: Retain population π_i in the selected subset if and only if

$$X_i \geq c_{14} \max_{1 \leq j \leq k} X_j - (1 - c_{14})r \tag{4.2.4}$$

where $0 \leq c_{14} \leq 1$ is the largest number for which the basic probability requirement (4.2.3) is satisfied for all parameter points $\underline{p} = (p_1,\ldots,p_k)$ in Ω.

Now for any $\underline{p} \varepsilon \Omega$,

$$P_{\underline{p}}(CS|R_{14}) = P_{\underline{p}}(X_{(1)} \geq c_{14} X_{(j)} - (1-c_{14})r, \quad j = 2,\ldots,k)$$

$$= \sum_{x=0}^{\infty} \binom{r+x-1}{x} p_1^r q_1^x \prod_{j=2}^{k} \left[\sum_{y=0}^{\left[\frac{x+(1-c_{14})r}{c_{14}}\right]} \binom{r+y-1}{y} p_j^r q_j^y \right] \tag{4.2.5}$$

where $q_i = 1 - p_i$, $i = 1,2,\ldots,k$ and $[a]$ denotes the greatest integer less than or equal to a.

Using the well-known identity for the incomplete beta function

$$q^r \sum_{j=0}^{s-1} \binom{r+j-1}{j} p^j = I_q(r,s) = p^s \sum_{j=r}^{\infty} \binom{s+j-1}{j} q^j \tag{4.2.6}$$

where $I_q(r,s) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \int_0^q t^{r-1}(1-t)^{s-1}\, dt$, it follows that

$$P_{\underline{p}}(CS|R_{14}) = \sum_{x=0}^{\infty} \binom{r+x-1}{x} p_1^r q_1^x \prod_{j=2}^{k}$$

$$\left[\frac{\Gamma\left(r + \left[\frac{x+(1-c_{14})r}{c_{14}}\right]+1\right)}{\Gamma(r)\, \Gamma\left(\left[\frac{x+(1-c_{14})r}{c_{15}}\right]+1\right)} \int_0^{p_j} t^{r-1}(1-t)^{\left[\frac{x+(1-c_{14})r}{c_{14}}\right]} dt \right].$$

It is immediately evident from the above that $P_{\underline{p}}(CS|R_{14})$ is an increasing function of p_j, for $j = 2,\ldots k$ and consequently $P_{\underline{p}}(CS|R_{14})$ is partially minimized as we let each p_j approach p_1 from above. Hence we have the following result.

<u>Theorem 4.2.1.</u> $\inf_{\underline{p}\in\Omega} P_{\underline{p}}(CS|R_{14}) = \inf_{\underline{p}\in\Omega_o} P_{\underline{p}}(CS|R_{14})$

$$= \inf_{0<p<1} \sum_{x=0}^{\infty} \binom{r+x-1}{x} p^r q^x \Big\{ \sum_{y=0}^{\left[\frac{x+(1-c_{14})r}{c_{14}}\right]} \binom{r+y-1}{y} p^r q^y \Big\}^{k-1} \qquad (4.2.7)$$

where $\Omega_o = \{\underline{p} = (p,\ldots,p),\ 0 < p < 1\}$.

It is difficult to determine analytically and also numerically where the infimum of the above expression with respect to the common p $(0 < p < 1)$ takes place. If we could find the infimum then we can solve for the constant c_{14} to satisfy (4.2.3). In order to overcome this difficulty, we give a method leading a conservative solution for the constant.

For any fixed non-negative integer t, let

$$N(t,c(t),r) = \sum_{x=0}^{\left[\frac{t+(1-c(t))r}{1+c(t)}\right]\wedge t} \binom{r+x-1}{x} \binom{r+t-x-1}{t-x} \qquad (4.2.8)$$

where [a] is defined as before and $a \wedge b = \min(a,b)$.

<u>Theorem 4.2.2.</u> For given P^*, let $P_1^* = (P^*)^{\frac{1}{k-1}}$, and $t \geq 0$, let $c_{14}(t)$ be the largest value such that

$$N(t,\ c_{14}(t),\ r) \geq \binom{2r+t-1}{t} P_1^* . \qquad (4.2.9)$$

If $c_{14} = \inf\{c_{14}(t):\ t \geq 0\}$, then

$$\inf_{\underline{p}\in\Omega} P_{\underline{p}}(CS|R_{14}) \geq P^*.$$

Proof. For $\underline{p} \in \Omega_o$

$$P_{\underline{p}}(CS|R_{14}) = P_{\underline{p}}(X_{(1)} \geq c_{14} \max_{2\leq j\leq k} X_{(j)} - (1-c_{14})r)$$

$$= \sum_{x=0}^{\infty} \binom{r+x-1}{x} p^r q^x \left[\sum_{y=0}^{\left[\frac{x+(1-c_{14})r}{c_{14}}\right]} \binom{r+y-1}{y} p^r q^y \right]^{k-1}$$

$$\geq \{ \sum_{x=0}^{\infty} \binom{r+x-1}{x} p^r q^x \left[\sum_{y=0}^{\left[\frac{x+(1-c_{14})r}{c_{14}}\right]} \binom{r+y-1}{y} p^r q^y \right] \}^{k-1}$$

$$= \{P_{\underline{p}}(X_1 \geq c_{14} X_2 - (1-c_{14})r)\}^{k-1}$$

$$= \{ \sum_{t=0}^{\infty} P_{\underline{p}}(X_1 \geq c_{14} X_2 - (1-c_{14})r \mid X_1+X_2=t) \cdot P_{\underline{p}}(X_1+X_2=t) \}^{k-1}$$

$$\geq \{ \sum_{t=0}^{\infty} P_{\underline{p}}(X_1 \geq c_{14}(t) X_2 - (1-c_{14}(t))r \mid X_1+X_2=t) \cdot P_{\underline{p}}(X_1+X_2=t) \}^{k-1}$$

$$= \{ \sum_{t=0}^{\infty} P_{\underline{p}}(X_2 \leq \frac{t+(1-c_{14}(t))r}{1+c_{14}(t)} \mid X_1+X_2=t) \cdot P_{\underline{p}}(X_1+X_2=t) \}^{k-1}$$

$$= \{ \sum_{t=0}^{\infty} \frac{N(t, c_{14}(t), r)}{\binom{2r+t-1}{t}} \cdot P_{\underline{p}}(X_1 + X_2 = t) \}^{k-1}$$

$$\geq (P_1^*)^{k-1} = P^*.$$

From Theorem 4.2.1., the proof is completed.

(B) Some Properties of the Procedure R_{14}

We now discuss some properties of the rule R_{14}. For $\underline{p} \in \Omega$, define

$$p_{\underline{p}}(i) = P_{\underline{p}}\{R_{14} \text{ selects } \pi_{(i)}\}$$

Theorem 4.2.3. $p_{\underline{p}}(i)$ is $\{$ $\downarrow$ in p_i when all other components of $\underline{p}$ are fixed.
$\uparrow$ in p_j $(j \neq i)$ when all other components of $\underline{p}$ are fixed.

Proof. $p_{\underline{p}}(i) = P_{\underline{p}}(R_{14} \text{ selects } \pi_{(i)})$

$$= \sum_{x=0}^{\infty} \binom{r+x-1}{x} p_i^r q_i^x \left[\prod_{\substack{j=1 \\ j\neq i}}^{k} \sum_{y=0}^{[\frac{x+(1-c_{14})r}{c_{14}}]} \binom{r+y-1}{y} p_j^r q_j^y \right]$$

$$= \sum_{x=0}^{\infty} \binom{r+x-1}{x} p_i^r q_i^x \left[\prod_{\substack{j=1 \\ j\neq i}}^{k} \frac{\Gamma(r + [\frac{x+(1-c_{14})r}{c_{14}}]+1)}{\Gamma(r)\Gamma([\frac{x+(1-c_{14})r}{c_{14}}] +1)} \cdot \int_0^{p_j} t^{r-1} (1-t)^{[\frac{x+(1-c_{14})r}{c_{14}}]} dt \right] .$$

It is obvious that $p_{\underline{p}}(i)$ is increasing in $p_j (j\neq i)$ when all other components of $\underline{p}$ are fixed. On the other hand, $p_{\underline{p}}(i)$ can be written as

$$p_{\underline{p}}(i) = E_{p_i} \ f(X; p_1,\ldots,\hat{p}_i,\ldots,p_k)$$

where $f(x; p_1,\ldots,\hat{p}_i,\ldots,p_k) = \prod_{\substack{j=1 \\ j\neq i}}^{k} \sum_{y=0}^{[\frac{x+(1-c_{14})r}{c_{14}}]} \binom{r+y-1}{y} p_j^r q_j^y$ and $\hat{p}_i$

denotes that p_i is deleted. Note that $f(x; p_1,\ldots,\hat{p}_i,\ldots,p_k)$ is a non-decreasing function in x and the cdf $F_p(x)$ of a negative binomial random variable is stochastically increasing in 1-p. Then for

$p_i \leq p_i'$,

$$E_{p_i} f(X; p_1,\ldots,\hat{p}_i,\ldots,p_k) = \sum_{x=0}^{\infty} [f(x; p_1,\ldots,\hat{p}_i,\ldots,p_k) - f(x+1; p_1,\ldots,\hat{p}_i,\ldots,p_k)] F_{p_i}(x)$$

$$\geq \sum_{x=0}^{\infty} [f(x; p_1,\ldots,\hat{p}_i,\ldots,p_k) - f(x+1; p_1,\ldots,\hat{p}_i,\ldots,p_k)] F_{p_i'}(x)$$

$$= E_{p_i'} f(X; p_1,\ldots,\hat{p}_i,\ldots,p_k).$$

Hence $p_{\underline{p}}(i)$ is non-increasing in p_i when all other components of $\underline{p}$ are fixed.

Corollary 4.2.1. For every $\underline{p} \in \Omega$ and $1 \leq i < j \leq k$, $p_{\underline{p}}(i) \geq p_{\underline{p}}(j)$.

Proof. The proof follows easily from the above theorem hence is omitted. We also have the following corollary.

Corollary 4.2.2. For every $\underline{p} \in \Omega$ and $1 < j \leq k$,

$$P_{\underline{p}}(R_{14} \text{ does not select } \pi_{(1)}) \leq P_{\underline{p}}(R_{14} \text{ does not select } \pi_{(j)}).$$

Remark 4.2.1. It also follows from Theorem 4.2.3. that $p_{\underline{p}}(1)$, the probability of a correct selection, attains its minimum when $p_2,\ldots,p_k$ tend to p_1 from above.

4.3 A Conditional Subset Selection Procedure

In this section, we use the same notation as in Section 4.2. We propose a rule R_{15}, similar to the rule R_{14} except that this rule is based on the total number of observations $T = \sum_{i=1}^{k} X_i$, as follows:

R_{15}: Selection π_i if and only if

$$X_i \geq c_{15}(t) \max_{1 \leq j \leq k} X_j - (1 - c_{15}(t))r, \quad \text{given} \quad \sum_{i=1}^{k} X_i = t,$$

where $t \geq 0$ and $0 \leq c_{15}(t) \leq 1$ is chosen to satisfy the basic probability requirement. For this rule R_{15}, we obtain an exact result for $k = 2$ in Theorem 4.3.1.. For $k \geq 3$, we have a lower bound for the probability of a correct selection in Theorem 4.3.2.

Theorem 4.3.1. For a given P^* $(\frac{1}{k} < P^* < 1)$, $k = 2$, and any $t \geq 0$, let $c_{15}(t)$ be the largest value such that

$$N(t, c_{15}(t), r) \geq P^*\binom{2r+t-1}{t}. \tag{4.3.1}$$

Then, $\inf_{\underline{p}\in\Omega} P_{\underline{p}}(CS|R_{15}) = \inf_{\underline{p}\in\Omega_0} P_{\underline{p}}(CS|R_{15}) \geq P^*$.

Proof. Let $D(t) = \langle \frac{tc_{15}(t)-(1-c_{15}(t))r}{1+c_{15}(t)} \rangle$, where $\langle \alpha \rangle$ denote the smallest integer $\geq \alpha$, then for any $\underline{p} \in \Omega$,

$$P_{\underline{p}}(CS|R_{15}) = P_{\underline{p}}(X_{(1)} \geq c_{15}(t)\, X_{(2)} - (1-c_{15}(t))r \mid X_{(1)}+X_{(2)}=t)$$

$$= P_{\underline{p}}\left(X_{(1)} \geq \frac{tc_{15}(t)-(1-c_{15}(t))r}{1+c_{15}(t)} \mid X_{(1)} + X_{(2)} = t\right)$$

$$= \frac{\sum_{x=D(t)}^{t} P_{\underline{p}}(X_{(1)} = x, X_{(2)} = t-x)}{\sum_{x=0}^{t} P_{\underline{p}}(X_{(1)} = x, X_{(2)} = t-x)}$$

$$= \frac{\sum_{x=D(t)}^{t} \binom{r+x-1}{x}\binom{r+t-x-1}{t-x} p_1^r q_1^x p_2^r q_2^{t-x}}{\sum_{x=0}^{t} \binom{r+x-1}{x}\binom{r+t-x-1}{t-x} p_1^r q_1^x p_2^r q_2^{t-x}}$$

$$= \frac{\sum_{x=D(t)}^{t} \binom{r+x-1}{x}\binom{r+t-x-1}{t-x} \lambda^x}{\sum_{x=0}^{t} \binom{r+x-1}{x}\binom{r+t-x-1}{t-x} \lambda^x} \qquad \text{where } \lambda = \frac{q_1}{q_2}$$

$$= \{1 + \frac{\sum_{x=0}^{D(t)-1} \binom{r+x-1}{x}\binom{r+t-x-1}{t-x} \lambda^x}{\sum_{x=D(t)}^{t} \binom{r+x-1}{x}\binom{r+t-x-1}{t-x} \lambda^x}\}^{-1}$$

$$= \{1 + \phi(\lambda)\}^{-1} \qquad (4.3.2)$$

By differentiating $\phi(\lambda)$ with respect to λ, we get

$$\phi'(\lambda) = \frac{1}{\{\sum_{x=D(t)}^{t} \binom{r+x-1}{x}\binom{r+t-x-1}{t-x}\lambda^x\}^2} \Bigg[\{\sum_{x=0}^{D(t)-1} x\binom{r+x-1}{x}\binom{r+t-x-1}{t-x}\lambda^{x-1}\} \{\sum_{x=D(t)}^{t} \binom{r+x-1}{x}\binom{r+t-x-1}{t-x} \lambda^x\}$$

$$- \{\sum_{x=D(t)}^{t} x\binom{r+x-1}{x}\binom{r+t-x-1}{t-x}\lambda^{x-1}\} \{\sum_{x=0}^{D(t)-1} \binom{r+x-1}{x}\binom{r+t-x-1}{t-x}\lambda^x\} \Bigg]$$

$$\leq 0.$$

Hence $\phi(\lambda)$ is non-increasing in λ and the right hand side of (4.3.2) is non-decreasing in λ. Since $\lambda \geq 1$, the infimum of $P_{\underline{p}}(CS|R_{15})$ occurs at $\lambda = 1$ i.e. when $p_1 = p_2$. Thus $\inf_{\underline{p}\varepsilon\Omega} P_{\underline{p}}(CS|R_{15}) = \inf_{\underline{p}\varepsilon\Omega_o} P_{\underline{p}}(CS|R_{15})$. Note that this infimum probability does not depend on the common value of $p_1 = p_2 = p$.

For $k = 2$ and selected values of r and t, tables of the constants $c_{15}(t)$ satisfying (4.3.1) are given at the end of the chapter.

Theorem 4.3.2. For $\underline{p} \varepsilon \Omega$,

$$P_{\underline{p}}(CS|R_{15}) \geq 2-k + \sum_{j=2}^{k} \{ \sum_{\ell=0}^{t} P_{\underline{p}}(X_{(1)} \geq c_{15}(t)X_{(j)} - (1-c_{15}(t))r | X_{(1)}+X_{(j)}=\ell) \cdot P_{\underline{p}}(X_{(1)}+X_{(j)} = \ell, \sum_{i=1}^{k} X_i = t)/P_{\underline{p}}(\sum_{i=1}^{k} X_i = t)\}$$

Proof. The proof is straightforward and hence omitted.

Theorem 4.3.3. For given P^*, $\frac{1}{k} < P^* < 1$, let $P_2^* = 1 - \frac{1-P^*}{k-1}$, $0 \leq \ell \leq t$, let $c_{15}(\ell)$ be the largest value such that

$$N(\ell, c_{15}(\ell), r) \geq P_2^* \binom{2r+\ell-1}{\ell}$$

If, $c_{15}(t) = \min \{c_{15}(\ell): \ 0 \leq \ell \leq t\}$, then

$$\inf_{\underline{p}\varepsilon\Omega} P_{\underline{p}}(CS|R_{15}) \geq P^*.$$

Proof. It follows from Theorem 4.3.2. that for $\underline{p} \varepsilon \Omega$,

$$P_{\underline{p}}(CS|R_{15}) \geq 2-k + \sum_{j=2}^{k} \{ \sum_{\ell=0}^{t} P_{\underline{p}}(X_{(1)} \geq c_{15}(t)X_{(j)} - (1-c_{15}(t))r | X_{(1)}+X_{(j)}=\ell) \cdot P_{\underline{p}}(X_{(1)}+X_{(j)}=\ell, \sum_{i=1}^{k} X_i=t)/P_{\underline{p}}(\sum_{i=1}^{k} X_i=t)\}$$

$$\geq 2-k + \sum_{j=2}^{k} \{ \sum_{\ell=0}^{t} P_{\underline{p}}(X_{(1)} \geq c_{15}(\ell)X_{(j)} - (1-c_{15}(\ell))r | X_{(1)}+X_{(j)}=\ell) \cdot P_{\underline{p}}(X_{(1)}+X_{(j)}=\ell, \sum_{i=1}^{k} X_i=t)/P_{\underline{p}}(\sum_{i=1}^{k} X_i=t)\}$$

and from Theorem 4.3.1., for any j, j = 2,...,k,

$$\inf_{\underline{p}\varepsilon\Omega} P_{\underline{p}}(X_{(1)} \geq c_{15}(\ell)\, X_{(j)} - (1-c_{15}(\ell))r \mid X_{(1)} + X_{(j)} = \ell)$$

$$= \inf_{\underline{p}\varepsilon\Omega_o} P_{\underline{p}}(X_1 \geq c_{15}(\ell)\, X_j - (1-c_{15}(\ell))r \mid X_1 + X_j = \ell)$$

$$= \frac{N(\ell, c_{15}(\ell), r)}{\binom{2r+\ell-1}{\ell}} \geq P_2^* ,$$

we have

$$P_{\underline{p}}(CS|R_{15}) \geq 2-k+\sum_{j=2}^{k}\sum_{\ell=0}^{t} P_2^* \cdot P_{\underline{p}}(X_{(1)}+X_{(j)}=\ell, \sum_{i=1}^{k} X_i=t)/P_{\underline{p}}(\sum_{i=1}^{k} X_i=t)$$

$$= 2-k+(k-1)\, P_2^* = P^*.$$

Thus the proof is completed.

Hence, for each k, P^*, Theorem 4.3.3. guarantees the existence of $c_{15}(t)$ for the rule R_{15} and gives a method to find $c_{15}(t)$ for given $\sum_{i=1}^{k} X_i = t$ such that $P_{\underline{p}}(CS|R_{15}) \geq P^*$ for any $\underline{p} \in \Omega$.

<u>An Upper Bound on the Expected Subset Size for R_{15}</u>

For the procedure R_{15}, the subset size S of the selected subset is a random variable which takes on only integer values from 1 to k, inclusively. For any fixed values of k, and P^*, then expected size of the selected subset is a function of the true configuration $\underline{p} = (p_1,\ldots,p_k)$.

<u>Lemma 3.3.1.</u> For k = 2, $a = \left< \frac{t\, c_{15}(t)-(1-c_{15}(t))r}{1+c_{15}(t)} \right>$, $b = \left[\frac{t+(1-c_{15}(t))r}{1+c_{15}(t)}\right]$.

$$\sup_{\underline{p}\in\Omega} E_{\underline{p}}(S|R_{15}) \leq 1 + \cfrac{1}{1+\cfrac{\sum_{x=b+1}^{t}\binom{r+x-1}{x}\binom{r+t-x-1}{t-x}}{\sum_{x=a}^{b}\binom{r+x-1}{x}\binom{r+t-x-1}{t-x}}}$$

<u>Proof.</u> For $\underline{p} \in \Omega$, let $\lambda = \frac{q_1}{q_2}$, then $\lambda \geq 1$

$$E_{\underline{p}}(S|R_{15}) = P_{\underline{p}}(X_{(1)} \geq c_{15}(t)X_{(2)}-(1-c_{15}(t))r|X_{(1)}+X_{(2)}=t)$$

$$+ P_{\underline{p}}(X_{(2)} \geq c_{15}(t)\ X_{(1)}-(1-c_{15}(t))r|X_{(1)}+X_{(2)} = t)$$

$$= P_{\underline{p}}\left(X_{(1)} \geq \frac{tc_{15}(t)-(1-c_{15}(t))r}{1 + c_{15}(t)} \,\middle|\, X_{(1)}+X_{(2)}=t\right)$$

$$+ P_{\underline{p}}\left(X_{(1)} \leq \frac{t+(1-c_{15}(t))r}{1 + c_{15}(t)} \,\middle|\, X_{(1)}+X_{(2)} = t\right)$$

$$= 1 + P_{\underline{p}}\left(\frac{tc_{15}(t)-(1-c_{15}(t))r}{1 + c_{15}(t)} \leq X_{(1)} \leq \frac{t+(1-c_{15}(t))r}{1 + c_{15}(t)} \,\middle|\, X_{(1)}+X_{(2)}=t\right)$$

$$= 1 + P_{\underline{p}}(a \leq X_{(1)} \leq b \mid X_{(1)} + X_{(2)} = t)$$

$$= 1 + \frac{\sum_{x=a}^{b} \binom{r+x-1}{x}\binom{r+t-x-1}{t-x} p_1^r q_1^x p_2^r q_2^{t-x}}{\sum_{x=0}^{t} \binom{r+x-1}{x}\binom{r+t-x-1}{t-x} p_1^r q_1^x p_2^r q_2^{t-x}}$$

$$= 1 + \frac{\sum_{x=a}^{b} \binom{r+x-1}{x}\binom{r+t-x-1}{t-x} \lambda^x}{\sum_{x=0}^{t} \binom{r+x-1}{x}\binom{r+t-x-1}{t-x} \lambda^x}$$

$$= 1 + \frac{1}{1 + \dfrac{\sum_{x=0}^{a-1} \binom{r+x-1}{x}\binom{r+t-x-1}{t-x} \lambda^x}{\sum_{x=a}^{b} \binom{r+x-1}{x}\binom{r+t-x-1}{t-x} \lambda^x} + \dfrac{\sum_{x=b+1}^{t} \binom{r+x-1}{x}\binom{r+t-x-1}{t-x} \lambda^x}{\sum_{x=a}^{b} \binom{r+x-1}{x}\binom{r+t-x-1}{t-x} \lambda^x}} \tag{4.3.2}$$

Let $g(\lambda) = \dfrac{\sum_{x=0}^{a-1} \binom{r+x-1}{x}\binom{r+t-x-1}{t-x} \lambda^x}{\sum_{x=a}^{b} \binom{r+x-1}{x}\binom{r+t-x-1}{t-x} \lambda^x}$ and $h(\lambda) = \dfrac{\sum_{x=b+1}^{t} \binom{r+x-1}{x}\binom{r+t-x-1}{t-x}\lambda^x}{\sum_{x=a}^{b} \binom{r+x-1}{x}\binom{r+t-x-1}{t-x}\lambda^x}$

Then, by differentiating $g(\lambda)$ and $h(\lambda)$ with respect to λ, it is easily seen that $g(\lambda)$ is a non-increasing function of λ, $h(\lambda)$ is a non-decreasing function of λ. Hence, from (4.3.2) the Lemma follows.

<u>Theorem 4.3.4.</u> For $k \geq 2$, $0 \leq \ell \leq t$, let $b(\ell) = \left[\dfrac{\ell+(1-c_{15}(\ell))r}{1 + c_{15}(\ell)}\right]$

$$a(\ell) = \left< \frac{\ell c_{15}(\ell)-(1-c_{15}(\ell))r}{1 + c_{15}(\ell)} \right>$$

$$\sup_{\underline{p}\in\Omega} E_{\underline{p}}(S|R_{15}) \leq \frac{k}{2} \max_{0\leq\ell\leq t} \left\{1 + \frac{1}{1 + \dfrac{\sum_{x=b(\ell)+1}^{\ell} \binom{r+x-1}{x}\binom{r+\ell-x-1}{\ell-x}}{\sum_{x=a(\ell)}^{b(\ell)} \binom{r+x-1}{x}\binom{r+\ell-x-1}{\ell-x}}}\right\}$$

<u>Proof.</u> For $\underline{p} \in \Omega$,

$$E_{\underline{p}}(S|R_{15}) = \sum_{i=1}^{k} P_{\underline{p}}(X_{(i)} \geq c_{15}(t) \max_{j\neq i} X_{(j)}-(1-c_{15}(t))r \mid \sum_{i=1}^{k} X_i = t)$$

$$\leq \frac{1}{k-1} \sum_{i=1}^{k} \sum_{j\neq i} P_{\underline{p}}(X_{(i)} \geq c_{15}(t)\ X_{(j)}-(1-c_{15}(t))r \mid \sum_{i=1}^{k} X_i = t)$$

$$= \frac{1}{(k-1)P_{\underline{p}}(\sum_{i=1}^{k} X_i=t)} \sum_{i=1}^{k} \sum_{j\neq i} \sum_{\ell=0}^{t} P_{\underline{p}}(X_{(i)}\geq c_{15}(t)X_{(j)}-(1-c_{15}(t))r,$$

$$X_{(i)}+X_{(j)}=\ell,\ \sum_{s\neq i,j} X_{(s)}=t-\ell)$$

$$= \frac{1}{(k-1)\ P_{\underline{p}}(\sum_{i=1}^{k} X_i=t)} \sum_{i=1}^{k} \sum_{j\neq i} \sum_{\ell=0}^{t} P_{\underline{p}}(X_{(i)}\geq c_{15}(t)X_{(j)}-(1-c_{15}(t))r \mid$$

$$X_{(i)}+X_{(j)}=\ell)\cdot P_{\underline{p}}(X_{(i)}+X_{(j)}=\ell,\ \sum_{s\neq i,j} X_{(s)}=t-\ell)$$

$$= \frac{1}{(k-1)P_{\underline{p}}(\sum_{i=1}^{k} X_i = t)} \sum_{i<j} \sum_{\ell=0} \{P_{\underline{p}}(X_{(i)} \geq c_{15}(t) X_{(j)} - (1-c_{15}(t))r \mid X_{(i)} + X_{(j)} = \ell)$$

$$+ P_{\underline{p}}(X_{(j)} \geq c_{15}(t) X_{(i)} - (1-c_{15}(t))r \mid X_{(i)} + X_{(j)} = \ell)\}$$

$$\cdot P_{\underline{p}}(X_{(i)} + X_{(j)} = \ell, \sum_{i=1}^{k} X_{(i)} = t)$$

$$\leq \frac{1}{(k-1)P_{\underline{p}}(\sum_{i=1}^{k} X_i = t)} \sum_{i<j} \sum_{\ell=0}^{t} \{\sup_{\Omega} [P_{\underline{p}}(X_{(i)} \geq c_{15}(t) X_{(j)} - (1-c_{15}(t))r \mid X_{(i)} + X_{(j)} = \ell)$$

$$+ P_{\underline{p}}(X_{(j)} \geq c_{15}(t) X_{(i)} - (1-c_{15}(t))r \mid X_{(i)} + X_{(j)} = \ell)]\}$$

$$\cdot P_{\underline{p}}(X_{(i)} + X_{(j)} = \ell, \sum_{i=1}^{k} X_i = t)$$

$$\leq \frac{1}{(k-1)P_{\underline{p}}(\sum_{i=1}^{k} X_i = t)} \sum_{i<j} \sum_{\ell=0}^{t} \{1 + \frac{1}{1 + \frac{\sum_{x=b(\ell)+1}^{\ell} \binom{r+x-1}{x}\binom{r+\ell-x-1}{\ell-x}}{\sum_{x=a(\ell)}^{b(\ell)} \binom{r+x-1}{x}\binom{r+\ell-x-1}{\ell-x}}}\}$$

$$\cdot P_{\underline{p}}(X_{(i)} + X_{(j)} = \ell, \sum_{i=1}^{k} X_i = t)$$

by Lemma 4.3.1.,

$$\leq \frac{1}{k-1} \sum_{i<j} \max_{0 \leq \ell \leq t} \{1 + \frac{1}{1 + \frac{\sum_{x=b(\ell)+1}^{\ell} \binom{r+x-1}{x}\binom{r+\ell-x-1}{\ell-x}}{\sum_{x=a(\ell)}^{b(\ell)} \binom{r+x-1}{x}\binom{r+\ell-x-1}{\ell-x}}}\}$$

The proof is thus completed.

4.4 Selection of all Populations Better Than a Standard

Case 1. Known Standard.

We consider the problem where k negative binomial populations π_i with parameters r_i, p_i $(i = 1,\ldots,k)$ are to be compared with a standard π_o with parameters r_o, p_o where p_o is the known value of the probability of a success in the standard population. Since p_i is the probability of a success, we define π_i to be better than π_o when $p_i > p_o$. Let X_i denote the number of failures before the r_ith success is observed in population π_i. Then for selecting all populations better than the standard we define the following procedure:

R_{D_1} : Retain in the selected subset those and only those populations π_i $(i = 1,\ldots,k)$ for which

$$X_i \leq p_o + D_1. \tag{4.4.1}$$

Let ℓ_1 and ℓ_2 denote the number of populations with $p_i \geq p_o$ and $p_i < p_o$, respectively, so that $\ell_1 + \ell_2 = k$. In general, ℓ_1 and ℓ_2 are unknown. Then the probability P_1 of a correct decision is given by

$$P_1 = \prod_{i=1}^{\ell_1} P\{X_i' \leq p_o + D_1\}$$

$$= \prod_{i=1}^{\ell_1} \left\{ \sum_{x=0}^{[p_o+D_1]} \binom{r_i+x-1}{x} p_i'^{\,r_i} q_i'^{\,x} \right\}$$

where primes refer to the ℓ_1 populations with $p_i \geq p_o$. Let $m = [p_o+D_1]$. Then

$$P_1 = \prod_{i=1}^{\ell_1} \frac{\Gamma(r_i+m+1)}{\Gamma(r_i)\Gamma(m+1)} \int_0^{p_i'} t^{r_i-1} (1-t)^m \, dt$$

A lower bound on the above probability is obtained by setting $p_i = p_o$ $(i = 1,\ldots,k)$ and $\ell_1 = k$. Hence the inequality determining D_1 becomes

$$\prod_{i=1}^{k} \left\{ \sum_{x=0}^{[p_o+D_1]} \binom{r_i+x-1}{x} p_o^{r} q_o^{x} \geq P^* \right. , \tag{4.4.2}$$

and the solution is the smallest value of D_1 satisfying (4.4.2). If $r_i = r$ $(i = 1,\ldots,k)$, then (4.4.2) reduces to

$$\sum_{x=0}^{[p_o+D_1]} \binom{r+x-1}{x} p_o^{r} q_o^{x} \geq (P^*)^{\frac{1}{k}} .$$

This is easily solved by consulting a table of cumulative negative binomial probabilities.

Case 2. Unknown Standard.

The assumptions are the same as in case 1 except that p_o is not known. Let X_o be the number of failures before the r_oth success is observed in population π_o. Consider the following procedure:

R_{D_2}: Retain in the selected subset those and only those populations π_i $(i = 1,\ldots,k)$ for which

$$X_i \leq \frac{1}{D_2} X_o + \left(\frac{1}{D_2} - 1\right) r_i . \tag{4.4.3}$$

The probability P_2 of retaining all the ℓ_1 populations with $p_i \geq p_o$ attains a minimum where $p_i = p_o$ $(i = 1,\ldots,k)$ and $\ell_1 = k$ and is given by

$$P_2(p_o,D_2) = \sum_{x=0}^{\infty} \prod_{i=1}^{k} \left\{ \sum_{y=0}^{[\frac{1}{D_2}x+(\frac{1}{D_2}-1)r_i]} \binom{r_i+y-1}{y} p_o^{r_i} q_o^{y} \right\} \binom{r_o+x-1}{x} p_o^{r_o} q_o^{x} .$$

Then the desired value of D_2 for the rule defined in (4.4.3) is the largest number for which

$$\min_{0<p_o<1} P_2(p_o, D_2) \geq P^* . \qquad (4.4.4)$$

For $r_i = r$ $(i = 0,1,\ldots,k)$,

$$P_2 \geq \{ \sum_{x=0}^{\infty} \binom{r+x-1}{x} p_o^r q_o^x \left[\sum_{y=0}^{[\frac{1}{D_2} x + (\frac{1}{D_2} - 1) r]} \binom{r+y-1}{y} p_o^r q_o^y \right] \}^k$$

$$= \{P(X_1 \leq \frac{1}{D_2} X_o + (\frac{1}{D_2} - 1) r)\}^k$$

For a given P^*, let $P_1^* = (P^*)^{\frac{1}{k}}$ and for any $t \geq 0$, let $D_2(t)$ be the largest value such that

$$N(t, D_2(t), r) \geq \binom{2r+t-1}{t} P_1^* ,$$

where $N(t, D_2(t), r)$ is defined in (4.2.8). Let $D_2 = \inf \{D_2(t): \ t \geq 0\}$. Then a conservative value of D_2 is obtained such that $P_2 \geq P^*$. The arguments for the above statement are essentially the same as those in Section 4.2.

4.5 Application

If a structure consists of n components it will be called a structure of order n. The state of all components of such a system will be described by a vector $\underline{x} = (x_1,\ldots,x_n)$ where $x_i = 1$ means "ith component performs" and $x_i = 0$ means "ith component fails". In reliability theory, structures satisfying the following conditions

(i) $\phi(\underline{1}) = 1$ where $\underline{1} = (1,\ldots,1)$

(ii) $\phi(\underline{0}) = 0$ where $\underline{0} = (0,\ldots,0)$

(iii) $\phi(\underline{x}) \geq \phi(y)$ whenever $x_i \geq y_i$, $i = 1,\ldots,n$

are called coherent (see Birnbaum, Esary and Saunders [17]) or monotonic (see Barlow and Proschan [5]).

Now consider k independent structures each of which consists of n components in parallel. We put each of the components of the structure on test. Suppose a component might not perform at the first trial. Let Y_{ij} denote the number of failures of the jth component of the structure π_i before it performs, $j = 1,2,\ldots,n$, $i = 1,2,\ldots,k$. It is easy to see that $Z_i = \min(Y_{i1},\ldots,Y_{in})$ is a geometric random variable with parameter $p_i = 1 - \prod_{j=1}^{n} (1-p_{ij})$ where p_{ij} is the probability that the jth component of the ith structure will perform. Hence, the problem of selecting the structure with the smallest reliability is the same as that of selecting the geometric population with the smallest parameter as discussed in this chapter when r=1.

Table VII

For k=2, r=5, this table gives the value of $c_{15}(t)$ for various t and P^* value.

t	$P^* = 0.75$	$P^* = 0.90$	$P^* = 0.95$	$P^* = 0.99$	t	$P^* = 0.75$	$P^* = 0.90$	$P^* = 0.95$	$P^* = 0.99$
1	.8333	.8333	.8333	.8333	26	.7142	.5000	.3846	.2857
2	.7142	.7142	.7142	.7142	27	.6817	.4800	.4230	.2758
3	.8571	.6250	.6250	.6250	28	.6521	.5199	.4073	.2666
4	.7500	.7500	.5555	.5555	29	.6956	.5000	.3928	.2580
5	.8750	.6666	.5000	.5000	30	.6666	.4814	.3793	.2903
6	.7778	.6000	.6000	.4545	31	.7083	.4642	.4137	.2812
7	.6999	.5454	.5454	.4166	32	.6800	.5000	.3999	.2727
8	.8000	.6363	.5000	.3846	33	.6538	.4827	.3871	.2646
9	.7272	.5833	.4615	.3571	34	.6922	.4666	.3750	.2571
10	.6666	.5384	.5384	.3333	35	.6666	.5000	.4062	.2500
11	.7500	.5000	.5000	.3999	36	.7037	.4838	.3939	.2778
12	.6922	.5714	.4666	.3750	37	.6785	.4687	.3823	.2702
13	.7692	.5333	.4375	.3529	38	.6551	.5000	.3714	.2631
14	.7142	.5000	.4117	.3333	39	.6896	.4848	.3999	.2564
15	.6666	.5625	.4706	.3157	40	.6666	.4706	.3888	.2500
16	.7333	.5293	.4444	.3000	41	.6999	.4571	.3783	.2439
17	.6875	.5000	.4210	.3500	42	.6773	.4856	.3684	.2682
18	.7500	.5555	.3999	.3333	43	.6562	.4721	.3947	.2619
19	.7059	.5263	.4499	.3182	44	.6875	.4594	.3846	.2558
20	.6666	.5000	.4285	.3043	45	.6666	.4864	.3750	.2500
21	.7221	.4761	.4090	.2916	46	.6969	.4736	.3658	.2444
22	.6842	.5238	.3913	.2799	47	.6764	.4615	.3902	.2666
23	.7368	.5000	.4347	.2692	48	.6571	.4499	.3809	.2608
24	.6999	.4782	.4166	.3077	49	.6856	.4750	.3721	.2553
25	.6666	.5217	.3999	.2962	50	.6666	.4634	.3636	.2500

Table VIII

For k=2, r=6, this table gives the value of $c_{15}(t)$ for various t and P^* value.

t	$P^* = 0.75$	$P^* = 0.90$	$P^* = 0.95$	$P^* = 0.99$	t	$P^* = 0.75$	$P^* = 0.90$	$P^* = 0.95$	$P^* = 0.99$
1	.8571	.8571	.8571	.8571	26	.7272	.5199	.4615	.3103
2	.7500	.7500	.7500	.7500	27	.6956	.5600	.4444	.3448
3	.8750	.6666	.6666	.6666	28	.7391	.5384	.4285	.3333
4	.7778	.7778	.6000	.6000	29	.7083	.5185	.4642	.3226
5	.8888	.6999	.5454	.5454	30	.7500	.5555	.4482	.3125
6	.8000	.6363	.6363	.5000	31	.7200	.5357	.4333	.3030
7	.7272	.5833	.5833	.4615	32	.6922	.5172	.4193	.3333
8	.8182	.6666	.5384	.4285	33	.7307	.5000	.4516	.3235
9	.7500	.6153	.5000	.3999	34	.7037	.5333	.4375	.3143
10	.8333	.5714	.5714	.4666	35	.7407	.5161	.4242	.3055
11	.7692	.6428	.5333	.4375	36	.7142	.5000	.4545	.2972
12	.7142	.6000	.5000	.4117	37	.6896	.5312	.4411	.3243
13	.7857	.5625	.4706	.3888	38	.7241	.5151	.4285	.3157
14	.7333	.650	.5293	.3684	39	.6999	.5000	.4166	.3077
15	.8000	.5882	.5000	.3500	40	.7333	.5293	.4444	.3000
16	.7500	.5555	.4736	.3999	41	.7096	.5143	.4324	.2926
17	.7059	.5263	.4499	.3809	42	.6875	.5000	.4210	.2857
18	.7646	.5789	.5000	.3636	43	.7187	.5278	.4102	.3095
19	.7221	.5500	.4761	.3478	44	.6969	.5135	.5358	.3023
20	.7778	.5238	.4545	.3333	45	.7272	.5000	.4250	.2954
21	.7368	.5714	.4347	.3750	46	.7059	.5263	.4146	.2889
22	.6999	.5454	.4782	.3600	47	.6856	.5128	.4047	.2826
23	.7500	.5217	.4583	.3461	48	.7142	.5000	.4285	.3043
24	.7142	.5652	.4399	.3333	49	.6944	.5249	.4186	.2978
25	.7619	.5416	.4800	.3214	50	.7221	.5121	.4090	.2916

Table IX

For k=2, r=10, this table gives the value of $c_{15}(t)$ for various t and P^* value.

t	$P^* = 0.75$	$P^* = 0.90$	$P^* = 0.95$	$P^* = 0.99$	t	$P^* = 0.75$	$P^* = 0.90$	$P^* = 0.95$	$P^* = 0.99$
1	.9090	.9090	.9090	.9090	26	.7692	.6428	.5862	.4838
2	.8333	.8333	.8333	.8333	27	.8077	.6206	.5666	.4687
3	.9166	.7692	.7692	.7692	28	.7778	.6551	.5483	.4545
4	.8461	.8461	.7142	.7142	29	.8147	.6333	.5806	.4411
5	.9230	.7857	.7857	.6666	30	.7857	.6666	.5625	.4706
6	.8571	.7333	.7333	.6250	31	.7586	.6451	.5454	.4571
7	.8000	.8000	.6875	.5882	32	.7931	.6250	.5757	.4444
8	.8666	.7500	.6470	.5555	33	.7666	.6562	.5588	.4324
9	.8125	.7059	.7059	.6111	34	.8000	.6363	.5428	.4594
10	.8750	.7646	.6666	.5789	35	.7742	.6176	.5714	.4473
11	.8235	.7221	.6315	.5500	36	.8064	.6470	.5555	.4358
12	.8823	.6842	.6842	.5238	37	.7812	.6285	.5405	.4250
13	.8333	.7368	.6499	.5714	38	.7575	.6111	.5675	.4146
14	.7894	.6999	.6190	.5454	39	.7878	.6388	.5526	.4389
15	.8421	.6666	.5909	.5217	40	.7646	.6216	.5384	.4285
16	.8000	.7142	.6363	.5000	41	.7940	.6052	.5641	.4186
17	.8500	.6817	.6086	.4800	42	.7714	.6315	.5500	.4090
18	.8095	.6521	.5833	.5199	43	.8000	.6151	.5365	.4317
19	.7727	.6956	.6250	.5000	44	.7778	.6000	.5238	.4222
20	.8182	.6666	.6000	.4814	45	.7567	.6250	.5476	.4130
21	.7826	.6399	.5769	.4642	46	.7837	.6097	.5348	.4042
22	.8260	.6800	.6153	.5000	47	.7631	.6341	.5227	.4255
23	.7916	.6538	.5926	.4827	48	.7894	.6190	.5454	.4166
24	.8333	.6296	.5714	.4666	49	.7692	.6046	.4186	.4081
25	.8000	.6666	.5517	.4516	50	.7948	.6278	.4090	.3999

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The main purpose of this paper is to propose and study the subset selection approach for some new problems. Chapter I deals with some subset selection procedures for Poisson populations. A procedure of the type discussed by Seal is proposed and compared with the main proposed procedure which is an unconditional rule. Some selection procedure for populations better than a standard are also investigated. In Chapter II, a subset selection procedure based on the sample median for selecting the largest of the k location parameters of double exponential (Laplace) distributions is studied. For this distribution the problem of

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selection for the scale parameters is also investigated. A test of homogeneity is proposed which is based on the range of sample medians. An indifference zone approach to the problem of selecting the populations with the t-largest unknown means is also studied. Chapter III discusses some classification rules for k univariate normal populations using the subset selection approach. The classification problem is studied in terms of (i) the mean (ii) the variance and (iii) the reciprocal of the coefficient of variation. Chapter IV deals with a conditional and an unconditional procedure for selecting a subset which contains the negative binomial population with the smallest unknown probability of a success. Selection of populations better than a standard is also investigated. An application of the procedure to reliability theory is described.