SOME RESULTS ON SUBSET SELECTION PROCEDURES FOR DOUBLE EXPONENTIAL POPULATIONS*

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Shanti S. Gupta and Yoon-Kwai Leong
Purdue University

Department of Statistics
Division of Mathematical Sciences
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1. Introduction

In this paper we study the selection problems and some other related statistical inference problems for k double exponential (Laplace) populations. Before we do this, we give some discussion of the Laplace distribution, its characteristics (vs. normal, logistic and Cauchy) and its use as a model in statistics and probability.

The double exponential distribution arises as a model in some statistical problems as explained later. This distribution is also considered in robustness studies, which suggests that it provides a model with different characteristics than some of the other commonly used models such as the normal distribution. In particular, the tails of the double exponential distribution are thicker than the tails of the normal or logistic, but not as thick as the Cauchy (see p. 43, Hajék [14]). Yet the double exponential has not been used very extensively as a model. This could be due in part to the lack of available statistical techniques for this distribution, although it is likely that the experimenter has shied away from using the double exponential because it has a sharp peak in the center. However, many applications would be primarily concerned with tail probabilities, and it would seem

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that the double exponential would be a useful model if exponential tails are required.

The double exponential has some application as a model in the area of Actuarial Science, and it has been suggested as a model for the distribution of the strength of flaws in materials by Epstein [8]. Using the weakest link principle, the strength of the material should decrease as the number of flaws or volume increases. In particular, from extreme-value theory the double exponential assumption leads to the result that the mode or most probable strength decreases in proportion to log n, where n represents the size or number of flaws of the material. In comparison, the assumption of a normal model leads to a decrease in proportion to $(\log n)^{1/2}$. For most applications to material strength, only the minimum flaw strength would ordinarily be observable; however, Epstein [8] suggests that there may be many other types of problems, such as a system of components in series, which might be similar from a statistical point of view. Other possible applications of the double exponential are suggested by the fact that the difference of two independent (not necessary identical) two parameter exponential variables follows the double exponential distribution, and that the logarithm of the ratios of uniform or Pareto variables follows the double exponential distribution.

In classical theory, once having assumed the form of the parent distribution, we can derive a criterion which is appropriate to this assumption. For example, under the assumption of normality, for the comparison of two means we would derive the t-statistic. It is then customary to justify the use of such a normal theory criterion in the practical circumstance in which normality cannot be guaranteed by arguing that the distribution of the characteristic is but little affected by non-normality of the parent distribution - that is, it is robust under non-normality. However, this argument ignores the fact that if the parent distribution really differed from the normal, the appropriate criterion would no longer be the normal-theory statistic. Box and Tiao [4] reconsidered the analysis of Darwin's paired data on the heights of self and cross-fertilized plants quoted by Fisher in "The Design of Experiments (1935)". In this development the parent distribution is not assumed to be normal, but only a member of the following class of symmetric distributions

$$p(y|\theta,\sigma,\beta) = \frac{1}{\Gamma[1+\frac{1}{2}(1+\beta)]2^{1+\frac{1}{2}(1+\beta)}\sigma} \exp\{-\frac{1}{2}|\frac{y-\theta}{\sigma}|^{2/(1+\beta)}\} \qquad (1.1)$$

where $-\infty < y < \infty$, $0 < \sigma < \infty$, $-\infty < \theta < \infty$, $-1 < \beta \leq 1$. This class of distributions includes the normal (β=0) and the double exponential (β=1), and its kurtosis parameter is β.

If the probability density function of the double exponential is given by

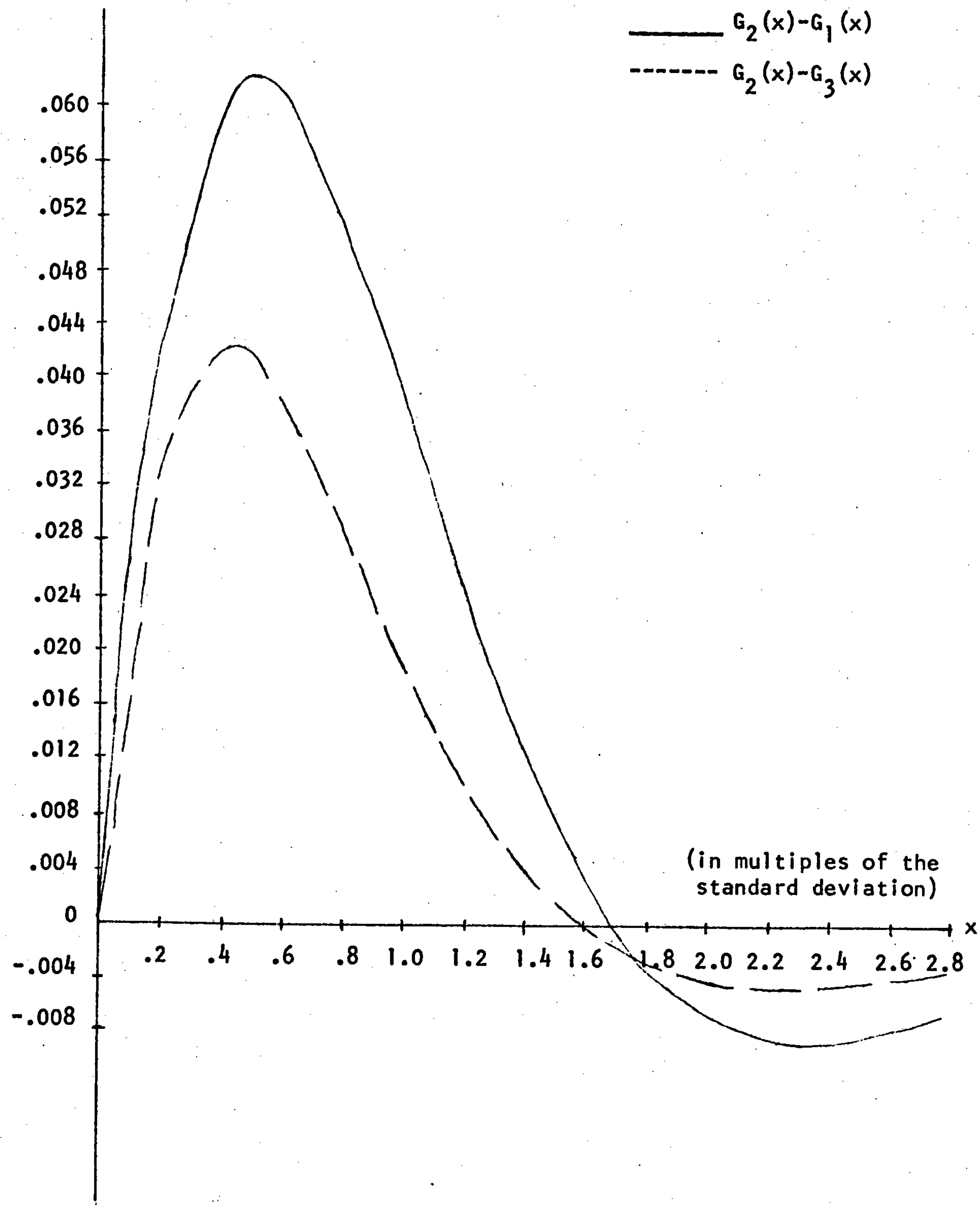
$$f(x,\theta,\sigma) = \frac{1}{2\sigma} e^{-|\frac{x-\theta}{\sigma}|}, \quad -\infty < x < \infty, -\infty < \theta < \infty, \quad \sigma > 0 \qquad (1.2)$$

then the mode of the distribution is $x = \theta$ where it has a sharp peak. The expected value and standard deviation of (1.2) are θ and $\sqrt{2}\,\sigma$ respectively. Moments of the standardized double exponential order statistics can be obtained by using the closed-form expressions for the moments of the standardized negative exponential order statistics derived by Epstein and Sobel [9]. Govindarajulu [10] has given the expressions for these moments.

Chew [6] gives the graphs of the standardized density functions of normal, logistic and double exponential distributions, from which it is clear that the tails of the double exponential distribution are thicker than that of the normal or logistic, in the sense that the curve of double exponential is above that of the others to the left and right of some points. In the case of the normal distribution this point is 2.64.

If the cumulative distribution functions $G_1(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}u^2} du$ and $G_2(x) = \begin{cases} \frac{1}{2} e^{\sqrt{2}x} & , x < 0 \\ 1 - \frac{1}{2} e^{-\sqrt{2}x} & , x \geq 0 \end{cases}$ of the standardized normal and double exponential distributions are compared, (also similar comparison between standardized logistic $G_3(x) = 1/(1 + e^{-\frac{\pi}{\sqrt{3}}x})$ and the double exponential distribution) the differences $G_2(x) - G_1(x)$ (as well as $G_2(x) - G_3(x)$) vary in the way shown in the graph below. Since $G_1(x)$, $G_2(x)$ and $G_3(x)$ are symmetric about $x = 0$ only the values for $x \geq 0$ are shown.

With regard to point estimation, it is well known that the maximum likelihood estimates based on the complete sample of size n are given by $\hat{\Theta} = \tilde{X}$ and $\hat{\sigma} = \frac{1}{n} \sum_{i=1}^{n} |X_i - \tilde{X}|$, where $\tilde{X}$ denotes the sample median. Also best linear estimators (based on order statistics) under symmetric censoring are given by Govindarajulu [11] for sample sizes up to 20, and some alternate estimates are suggested by Raghunandanan and Srinivasan [16]. Interval estimation for the parameters of the


G2(x)-G1(x)
G2(x)-G3(x)
.060
.056
.052
.048
.044
.040
.036
.032
.028
.024
.020
.016
.012
.008
.004
0
-.004
-.008
.2 .4 .6 .8 1.0 1.2 1.4 1.6 1.8 2.0 2.2 2.4 2.6 2.8
x
(in multiples of the
standard deviation)

two-parameter double exponential distribution is considered by Bain and Engelhardt [1].

Now we discuss the problem of comparison of $k(\geq 2)$ double exponential distributions. First we study the selection problem for the largest mean (location).

2 Selecting a Subset Containing the Best of Several Double Exponential Populations with Respect to the Location Parameter

(A) Formulation of the Problem

Let X_i, $i = 1,2,\ldots,k$ be k independent random variables from double exponential population π_i, $i = 1,2,\ldots,k$ respectively, with probability density function

$$f(X;\ \theta_i,\sigma) = \frac{1}{2\sigma}\exp\left[-|X-\theta_i|/\sigma\right],\quad -\infty < x < \infty,\quad -\infty < \theta_i < \infty,\ \sigma > 0$$

where σ is a common, known constant for each of the k populations. We may, without loss of generality, assume σ to be one. The ranked parameters are denoted by $\theta_{[1]} \leq \theta_{[2]} \leq \cdots \leq \theta_{[k]}$. As before, it is assumed that there is no a priori information available about the correct pairing of the ordered $\theta_{[i]}$ and the k given populations from which observations are taken. Any population whose parameter value equals $\theta_{[k]}$ will be defined as a best population. A correct selection (CS) is defined as the selection of any subset of the k given populations which contains at least one best population.

Suppose we take (2n+1) independent observations from π_i, $i = 1,2,\ldots,k$; the sample size (2n+1) is assumed to be given in the primary problem below. Let $P^*(\frac{1}{k} < P^* < 1)$ be a preassigned constant.

Let $P(CS; k, n, \underline{\theta}, R)$ denote the probability of a correct selection when the procedure R is used with the given k, n and when the true configuration of parameter values is $\underline{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)$; let the space of all possible values of $\underline{\theta}$ be denoted by Ω.

The problem of primary interest is to define a procedure R which selects a subset of the k given populations that is small, never empty, and large enough so that it contains the best population with probability at best P*, regardless of the true configurations $\underline{\theta}$ in Ω, i.e., so that

$$\inf_{\Omega} P(CS; k, n, \underline{\theta}, R) \geq P^* . \tag{2.1}$$

After having defined a particular procedure $R = R(k, n, P^*)$ for each possible set of values of k, n and P*, we discuss the expected size $E\{S; k, n, \underline{\theta}, P^*, R\}$ of the selected subset when the procedure R is used with the given k, n, P* and where $\underline{\theta}$ is the true parameter configuration in Ω.

Let Y_i denote the sample median of the (2n+1) observations $X_{i1}, \ldots, X_{i,2n+1}$, from the ith population, and let $Y_{(i)}$ denote that unknown variable which is associated with $\theta_{[i]}$. The probability density $g_n(\cdot)$ and the cumulative distribution $G_n(\cdot)$ of Y_i are given by

$$g_n(y; \theta_i) = \frac{(2n+1)!}{n!n!} \left(\frac{1}{2} e^{-|y-\theta_i|}\right)^{n+1} \left(1 - \frac{1}{2} e^{-|y-\theta_i|}\right)^{n} \tag{2.2}$$

$$G_n(y; \theta_i) = \begin{cases} 1 - \sum_{j=0}^{n} \binom{2n+1}{j} \left(\frac{1}{2} e^{y-\theta_i}\right)^{j} \left(1 - \frac{1}{2} e^{y-\theta_i}\right)^{2n+1-j} , & y<\theta_i \\ \sum_{j=0}^{n} \binom{2n+1}{j} \left(\frac{1}{2} e^{-(y-\theta_i)}\right)^{j} \left(1-\frac{1}{2}e^{-(y-\theta_i)}\right)^{2n+1-j} , & y\geq\theta_i \end{cases} \tag{2.3}$$

Now, we propose the selection procedure R_1 as follows:

R_1: Retain in the selected subset only those populations π_i for which

$$Y_i \geq \max_{1 \leq j \leq k} Y_j - d \tag{2.4}$$

where $d = d(k, n, P^*)$ is the smallest non-negative constant to be determined that will satisfy the basic probability requirement (2.1) for all configurations $\underline{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)$.

(B) Probability of a Correct Selection and Its Infimum

The following result concerning the rule R_1 can be proved.

Theorem 2.1. $\inf_{\underline{\theta} \varepsilon \Omega} P_{\underline{\theta}}(CS|R_1) = \inf_{\underline{\theta} \varepsilon \Omega_o} P_{\underline{\theta}}(CS|R_1) = \int_{-\infty}^{\infty} G_n^{k-1}(y+d) g_n(y) dy$

where $\Omega_o = \{\underline{\theta} = (\theta_1, \ldots, \theta_k): \theta_1 = \theta_2 = \ldots = \theta_k = \theta\}$, $G_n(y)$, $g_n(y)$ are the cdf and pdf of the sample median of $(2n+1)$ observations from the standard double exponential distribution.

Proof. For $\underline{\theta} \varepsilon \Omega$,

$$P_{\underline{\theta}}(CS|R_1) = P_{\underline{\theta}}\{Y_{(k)} \geq \max_{1 \leq j \leq k} Y_{(j)} - d\}$$

$$= P_{\underline{\theta}}\{Y_{(k)} - \theta_{[k]} \geq Y_{(j)} - \theta_{[j]} + \theta_{[j]} - \theta_{[k]} - d,\ j = 1, 2, \ldots, k-1\}$$

$$= \int_{-\infty}^{\infty} \left[\prod_{j=1}^{k-1} \int_{-\infty}^{y+\theta_{[k]}-\theta_{[j]}+d} g_n(z)\, dz \right] g_n(y)\, dy. \tag{2.5}$$

Note that $\theta_{[k]} - \theta_{[j]} \geq 0$ for $j = 1, \ldots, k-1$; thus the result follows. Hence, if we choose d to be the smallest constant to satisfy

$$\int_{-\infty}^{\infty} G_n^{k-1}(y+d)\, g_n(y)\, dy = P^*, \tag{2.6}$$

then we have determined the constant d for which

$$\inf_{\underline{\theta}\in\Omega} P_{\underline{\theta}}(CS|R_1) = P^* . \tag{2.7}$$

(c) Some Properties of R_1

For $\underline{\theta} \in \Omega$ and $\underline{\theta} = (\theta_{[1]},\ldots,\theta_{[k]})$ define $P_{\underline{\theta}}(i) = P_{\underline{\theta}}$ {R select population $\pi_{(i)}$}, and recall the following definitions (see Santner [17]).

Definition 2.1. The rule R is strongly monotone in $\pi_{(i)}$ means

$$P_{\underline{\theta}}(i) \text{ is } \begin{cases} \uparrow \text{ in } \theta_{[i]} \text{ when all other components of } \underline{\theta} \text{ are fixed} \\ \downarrow \text{ in } \theta_{[j]}\ (j\neq i) \text{ when all other components of } \underline{\theta} \text{ are fixed} \end{cases}$$

Definition 2.2. R is a monotone procedure means for every $\underline{\theta} \in \Omega$ and $1 \leq i < j \leq k$, $P_{\underline{\theta}}(i) \leq P_{\underline{\theta}}(j)$.

Definition 2.3. R is an unbiased procedure means for every $\underline{\theta} \in \Omega$ and $1 \leq j < k$,

$$P_{\underline{\theta}}\{\text{R does not select } \pi_{(j)}\} \geq P_{\underline{\theta}}\{\text{R does not select } \pi_{(k)}\}$$

Of course, if R is monotone it is also unbiased.

Theorem 2.2. For any $i = 1,2,\ldots,k$, the procedure R_1 is strongly monotone in $\pi_{(i)}$.

Proof. The proof follows easily from the expression

$$P_{\underline{\theta}}(i) = \int_{-\infty}^{\infty} \{\prod_{\substack{j=1 \\ j\neq i}}^{k} G_n(y + \theta_{[i]} - \theta_{[j]} + d)\}\ g_n(y)\, dy .$$

Corollary 2.1. The rule R_1 is monotone and unbiased.

Proof. It is known and easy to see that if R is strongly monotone in

$\pi_{(i)}$, for all $i = 1,2,\ldots,k$, then it is monotone.

Now we consider some special configurations of $\underline{\theta} \in \Omega$.

$$\begin{cases} \theta_{[i]} = \theta \quad , \quad i = 1,2,\ldots,k-1 \\ \theta_{[k]} = \theta + \Delta \ , \quad \Delta > 0 \end{cases} \tag{2.8}$$

$$\theta_{[i]} = \theta + (i-1)\Delta \ , \ \Delta > 0, \quad i = 1,2,\ldots,k. \tag{2.9}$$

Under (2.8),

$$P_{\underline{\theta}}(i) = \int_{-\infty}^{\infty} [G_n(y+d)]^{k-2} \, G_n(y+d-\Delta) \, g_n(y) \, dy \quad \text{for } i=1,2,\ldots,k-1 \tag{2.10}$$

$$P_{\underline{\theta}}(k) = \int_{-\infty}^{\infty} [G_n(y+d+\Delta)]^{k-1} \, g_n(y) \, dy \, . \tag{2.11}$$

While under (2.9),

$$P_{\underline{\theta}}(i) = \int_{-\infty}^{\infty} \{ \prod_{\substack{j=1 \\ j \neq i}}^{k} G_n(y+d+(i-j)\Delta)\} \, g_n(y) \, dy, \quad i=1,2,\ldots,k.$$

From the above equations we can make the following remarks:

Remark 2.1. For fixed P^*, k, n, i ($i = 1,2,\ldots,k-1$), the probability of selecting population $\pi_{(i)}$ decreases from P^* to zero as Δ increases from zero to infinity.

Remark 2.2. For fixed P^*, k and n, the probability of selecting $\pi_{(k)}$ increases from P^* to one as Δ increases from zero to infinity.

Remark 2.3. For fixed P^*, k, i ($i=1,\ldots,k-1$) and Δ, the probability of selecting population $\pi_{(i)}$ tends to zero as $n \to \infty$. While the probability of selecting $\pi_{(k)}$ tends to one as $n \to \infty$.

Conclusion: Under either configuration (2.8), (2.9),

$E_{\underline{\theta}}(S|R_1) = \sum_{i=1}^{k} p_{\underline{\theta}}(i) \to 1$ as $\Delta \to \infty$ for fixed n and $E_{\underline{\theta}}(S|R_1) \to 1$ as $n \to \infty$ for fixed Δ.

(D) <u>Asymptotic Results for the Procedure R_1</u>

It suffices to consider the parameter space Ω_o. For n large, we discuss an asymptotic property of the procedure as follows. Let Y be the sample median from a sample of size (2n+1) with pdf $f(x;\theta) = \frac{1}{2} e^{-|x-\theta|}$, $-\infty < x < \infty$. Then it is known (see Chu [7]) that under Ω_o, $\frac{Y-\theta}{\sigma_n}$ is asymptotically normally distributed (here $\sigma_n^2 = \frac{1}{2n+1}$). Let Z denote a random variable which has a standard normal distribution, then $\frac{Y-\theta}{\sigma_n}$ is asymptotically distributed as Z. Hence, under Ω_o, the probability

$$Y_k \geq \max_{1 \leq j \leq k} Y_j - d$$

is asymptotically, the same as the probability

$$Z_k \geq \max_{1 \leq j \leq k} Z_j - \sqrt{2n+1}\, d \tag{2.13}$$

where Z_i, $i = 1,2,\ldots,k$, are iid standard normal variables. Hence,

$$\inf_{\underline{\theta}\in\Omega_o} P_{\underline{\theta}}(CS|R_1) \doteqdot P_{\underline{\theta}}\{Z_k \geq \max_{1 \leq j \leq k} Z_j - \sqrt{2n+1}\; d\}$$
$$= \int_{-\infty}^{\infty} \left[\Phi(z + \sqrt{2n+1}\; d)\right]^{k-1} d\Phi(z)$$

where $\Phi(.)$ is the cdf of the standard normal distribution.

(E) <u>The Monotone Likelihood Ratio Property of the Sample Median</u>

Suppose Y is the sample median of (2n+1) observations from the population with double exponential density function $f(x;\theta) = \frac{1}{2} e^{-|x-\theta|}$.

The pdf $g_n(y;\theta)$ and cdf $G_n(y;\theta)$ of Y are given by equations (2.2) and (2.3).

After some algebraic computations, we see that $G_n(\theta;\theta) = \frac{1}{2}$; also it is easy to show that $g_n(y;\theta)$ is differentiable at $y = \theta$.

Let $g_n(y;\theta) = \bar{g}_n(y-\theta)$. It is shown in Lehmann [15, p.330] that a necessary and sufficient condition for $\bar{g}_n(y-\theta)$ to have monotone likelihood ratio in y is that $-\log \bar{g}_n$ is convex. Our main goal in this section is to prove this assertion. Now

$$\bar{g}_n(y) = c_n(\tfrac{1}{2} e^{-|y|})^{n+1}(1 - \tfrac{1}{2} e^{-|y|})^n \text{ where } c_n = \frac{(2n+1)!}{n!n!}, \text{ so,}$$

$$-\log \bar{g}_n(y) = -\log c_n + (n+1)\log 2 + (n+1)|y| - n\log(1-\tfrac{1}{2} e^{-|y|}).$$

Let $h(y) = (n+1)|y| - n\log(1 - \frac{1}{2} e^{-|y|}) = \begin{cases} h_1(y), & y < 0 \\ h_2(y), & y \geq 0 \end{cases}$ which is a continuous function. For $y < 0$,

$h(y) = h_1(y) = -(n+1)y - n\log(1 - \frac{1}{2} e^y)$, we have

$$h_1'(y) = -(n+1) + \frac{\frac{n}{2} e^y}{1 - \frac{1}{2} e^y} < 0 \text{ since for } y < 0, \frac{\frac{1}{2} e^y}{1 - \frac{1}{2} e^y} < 1$$

and
$$h_1''(y) = \frac{\frac{n}{2} e^y}{(1 - \frac{1}{2} e^y)^2} > 0 .$$

Hence, for $y < 0$, $h_1(y)$ is a decreasing, convex function. Similarly, for $y \geq 0$,

$$h(y) = h_2(h) = (n+1)y - n\log(1 - \tfrac{1}{2} e^{-y})$$

$$h_2'(y) = n+1 - \frac{\frac{n}{2} e^{-y}}{1 - \frac{1}{2} e^{-y}} > 0 \text{ since for } y \geq 0, \frac{\frac{1}{2} e^{-y}}{1 - \frac{1}{2} e^{-y}} < 1$$

$$h_2''(y) = \frac{\frac{n}{2} e^{-y}}{(1 - \frac{1}{2} e^{-y})^2} > 0 .$$

Hence, for $y \geq 0$, $h_2(y)$ is an increasing, convex function. Note that $h(y)$ is continuous at $y = 0$, decreasing, convex for $y < 0$ and increasing, convex for $y \geq 0$. Hence, this concludes that $h(y)$ is a convex function, which implies $- \log \bar{g}_n(y)$ is also a convex function.

Theorem 2.3. $g_n(y;\theta)$ has monotone likelihood ratio in y.

(F) Expected Size of the Selected Subset

The procedure R_1 satisfies the basic probability requirement (2.1) and is defined by (2.4). Consistent with the basic probability requirement, we would like the size of the selected subset to be small. Now S, the size of the selected subset is a random variable which takes integer values 1,2,...,k. Hence, one criterion of the efficiency of the procedure R_1 is the expected value of the size of the subset. Now, we derive an expression for $E(S|R_1)$, the expected size of the selected subset using procedure R_1.

$$E(S|R_1) = \sum_{i=1}^{k} P\{\text{Selecting the population with parameter } \theta_{[i]}\}$$

$$= \sum_{i=1}^{k} P\{Y_{(i)} \geq \max_{1\leq j\leq k} Y_{(j)} - d\}$$

$$= \sum_{i=1}^{k} \int_{-\infty}^{\infty} \left[\prod_{\substack{j=1 \\ j\neq i}}^{k} G_n(y + d + \theta_{[i]} - \theta_{[j]}) \right] g_n(y)\, dy. \quad (2.16)$$

If we set the m smallest parameters θ_i $(1 \leq m < k)$ equal to a common value θ(say) and define

$$Q = E(S \mid \theta_{[1]} = \dots = \theta_{[m]} = \theta) \quad (2.17)$$

then by an analogous argument as in Gupta [13] one can prove the following theorem.

<u>Theorem 2.4.</u> For given k, $P^*(\frac{1}{k} < P^* < 1)$, the expected size of the selected subset $E(S \mid \theta_{[1]} = \theta_{[2]} = \ldots = \theta_{[m]} = \theta,\ m < k)$ in using the procedure R_1 is strictly increasing in θ.

<u>Corollary 2.2.</u> $\sup\limits_{\underline{\theta}\varepsilon\Omega} E_{\underline{\theta}}(S|R_1) = k \int_{\infty}^{\infty} G_n^{k-1}(y+d)\ g_n(y)\ dy = k\ P^*.$

<u>Corollary 2.3.</u> In the subset $\Omega(\delta) = \{\underline{\theta}:\ \theta_{[i]} \leq \theta_{[k]} - \delta,$ $i = 1,2,\ldots,k-1\}$, the function $E_{\underline{\theta}}(S|R_1)$ takes on its maximum value when $\theta_{[i]} = \theta_{[k]} - \delta$, $i = 1,2,\ldots,k-1$, and so

$$\sup_{\underline{\theta}\varepsilon\Omega(\delta)} E_{\underline{\theta}}(S|R_1) = \int_{-\infty}^{\infty} G_n^{k-1}(y+d+\delta)\ g_n(y)\ dy$$
$$+ (k-1) \int_{-\infty}^{\infty} G_n^{k-2}(y+d)\ G_n(y+d-\delta)\ g_n(y)\ dy\ .$$

(G) <u>Minimax Property of the Rule R_1</u>

Suppose that $y_1,\ldots,y_k$ are the sample medians from the k populations $\pi_1,\ldots,\pi_k$, respectively, and with this set of observations, we select the ith population with probability $\phi_i(y_1,\ldots,y_k)$. Then the selection rule R is said to be invariant or symmetric if

$$\phi_i(y_1,\ldots,y_i,\ldots,y_j,\ldots,y_k) = \phi_j(y_1,\ldots,y_j,\ldots,y_i,\ldots,y_k)$$

for all i and j, i.e. if y_j is observed from π_i and y_i from π_j, then we select the jth population with the same probability $\phi_i(y_1,\ldots,y_k)$.

Notice that the rule R_1: $Y_i \geq \max\limits_{1\leq j\leq k} Y_j - d$ satisfies the equations

$$\inf_{\underline{\theta}\varepsilon\Omega} P_{\underline{\theta}}(CS|R_1) = \inf_{\underline{\theta}\varepsilon\Omega_o} P_{\underline{\theta}}(CS|R_1) = P_{\underline{\theta}_o}(CS|R_1) = P^* \qquad (2.20)$$

and $$\sup_{\underline{\theta}\varepsilon\Omega} E_{\underline{\theta}}(S|R_1) = \sup_{\underline{\theta}\varepsilon\Omega_o} E_{\underline{\theta}}(S|R_1) = E_{\underline{\theta}_o}(S|R_1) = k\, P^* \tag{2.21}$$

where $\underline{\theta}_o = (\theta_o,\ldots,\theta_o)$.

For any invariant rule R', $\underline{\theta}_o \varepsilon \Omega$,

$$\begin{aligned} E_{\underline{\theta}_o}(S|R') &= \sum_{i=1}^{k} P_{\underline{\theta}_o}\{\text{select population } \pi_i | R'\} \\ &= \sum_{i=1}^{k} \int \phi_i(y_1,\ldots,y_k) \left[\prod_{j=1}^{k} g_n(y_j)\right] dy_1 \ldots dy_k \\ &= k\, P_{\underline{\theta}_o}(CS|R'). \end{aligned}$$

Hence for $\underline{\theta}_o \varepsilon \Omega_o$,

$$E_{\underline{\theta}_o}(S|R') - E_{\underline{\theta}_o}(S|R_1) = k\,[P_{\underline{\theta}_o}(CS|R') - P_{\underline{\theta}_o}(CS|R_1)] \tag{2.22}$$

If the rule R' satisfies the basic P^* condition, it follows from (2.20) that the right hand side of (2.22) is non-negative. Thus

$$E_{\underline{\theta}_o}(S|R') \geq E_{\underline{\theta}_o}(S|R_1) = \sup_{\underline{\theta}\varepsilon\Omega} E_{\underline{\theta}}(S|R_1)\ .$$

So that $$\sup_{\underline{\theta}\varepsilon\Omega} E_{\theta}(S|R') \geq \sup_{\underline{\theta}\varepsilon\Omega} E_{\theta}(S|R_1)$$

i.e. the rule R_1 is minimax among all invariant rules satisfying the P^*-condition.

3. Selecting the Population with the Largest Location Parameter - Indifference Zone Approach

In this section, we would like to use the indifference zone approach of Bechhofer [3] to select one population which is guaranteed to be associated with the largest location parameter with a fixed probability P^* whenever the unknown parameters lie outside some subset, or zone of indifference, of the entire parameter space. The goal is to define a sequence of rules $\{R_2(n)\}$ each of which selects a single population $\pi_{(k)}$ and find the smallest n so that

$$P_{\underline{\theta}}(CS|R_2(n)) \geq P^*, \forall\ \underline{\theta}\varepsilon\Omega(\delta^*) = \{\underline{\theta}:\ \theta_{[k]} - \theta_{[k-1]} \geq \delta^*\} \qquad (3.1)$$

where P^* and δ^* are preassigned numbers.

For the sake of clarity, we will use the notation $Y_{[k]n}$ to denote the largest of the sample medians each based on (2n+1) observations.

$R_2(n)$: Select the population corresponding to $Y_{[k]n}$.

Let $\Omega_o(\delta^*) = \{\underline{\theta}: \theta_{[1]} = \ldots = \theta_{[k-1]} = \theta_{[k]} - \delta^*\}$. Then we have the following theorem.

<u>Theorem 3.1</u>. $\inf\limits_{\underline{\theta}\varepsilon\Omega(\delta^*)} P_{\underline{\theta}}(CS|R_2(n)) = \inf\limits_{\underline{\theta}\varepsilon\Omega_o(\delta^*)} P_{\underline{\theta}}(CS|R_2(n))$

<u>Proof</u>. For $\underline{\theta} \varepsilon \Omega(\delta^*)$,

$$\begin{aligned} P_{\underline{\theta}}(CS|R_2(n)) &= P_{\underline{\theta}}\{\max_{1\leq j\leq k-1} Y_{(j)n} < Y_{(k)n}\} \\ &= P_{\underline{\theta}}\{Y_{(j)n} < Y_{(k)n},\ j = 1,2,\ldots,k-1\} \\ &= P_{\underline{\theta}}\{Y_{(j)n}-\theta_{[j]} < Y_{(k)n}-\theta_{[k]}+\theta_{[k]}-\theta_{[j]},\ j = 1,2,\ldots,k-1\} \\ &= \int_{-\infty}^{\infty} \left[\prod_{j=1}^{k-1} G_n(y + \delta_{kj})\right] d\, G_n(y) \qquad (3.2) \end{aligned}$$

where $G_n(y) = G_n(y;\ 0)$ is the cdf of the sample median of (2n+1) independent observation from the standard double exponential distribution with density function $\frac{1}{2} e^{-|x|}$, $-\infty < x < \infty$, and $\delta_{kj} = \theta_{[k]}-\theta_{[j]} \geq 0$. Hence the infimum of the probability of a correct selection occurs when $\theta_{[1]} = \theta_{[2]} = \ldots = \theta_{[k-1]} = \theta_{[k]} - \delta^*$ provided $\theta_{[k]} - \theta_{[k-1]} \geq \delta^*$. This proves the theorem.

The minimum sample size required to achieve the P* condition (3.1) is the smallest integer n such that

$$\int_{-\infty}^{\infty} [G_n(y + \delta^*)]^{k-1} \, d\, G_n(y) \geq P^* . \tag{3.3}$$

4 Selecting the t-Best Populations - Indifference Zone Approach

Now, we consider the problem of selecting the best t populations, i.e., the populations with location parameters $\theta_{[k-t+1]}, \theta_{[k-t+2]}, \ldots, \theta_{[k]}$, without regard to order. We are using the indifference zone approach based on the sample median Y_i of 2n+1 independent observations from population π_i, $i = 1, \ldots, k$. Define a sequence of procedures as follows:

$R_3(n)$: Select the t populations associated with t largest values of Y_i.

Let $\Omega'(\delta^*) = \{\underline{\theta}: \theta_{[k-t+1]} - \theta_{[k-t]} \geq \delta\}$ and let

$$\Omega'_o(\delta^*) = \{\underline{\theta}: \theta_{[1]} = \ldots = \theta_{[k-t]} = \theta, \theta_{[k-t+1]} = \ldots = \theta_{[k]} = \theta + \delta^*\}.$$

Theorem 4.1. $\inf_{\underline{\theta} \varepsilon \Omega'(\delta^*)} P_{\underline{\theta}}\{CS|R_3(n)\} = \inf_{\underline{\theta} \varepsilon \Omega'_o(\delta^*)} P_{\underline{\theta}}\{CS|R_3(n)\}$

Proof. It was shown in Theorem 2.3 that the pdf $g_n(y; 0)$ of the sample median has monotone likelihood ratio in y, which implies that it is stochastically increasing in θ. Using a theorem of Barr and Rizvi [2], it follows that, for $\underline{\theta} \varepsilon \Omega'(\delta^*)$

$$P_{\underline{\theta}}\{CS|R_3(n)\} = P_{\underline{\theta}}\{\max_{1 \leq i \leq k-t} Y_{(i)} < \min_{k-t+1 \leq j \leq k} Y_{(j)}\}$$

is a non-increasing function of $\theta_{[1]}, \ldots, \theta_{[k-t]}$ and a non-decreasing function of $\theta_{[k-t+1]}, \ldots, \theta_{[k]}$. Thus $P_{\underline{\theta}}\{CS|R_3(n)\}$ attains its infimum when $\theta_{[1]}, \ldots, \theta_{[k-t]}$ attain their maximum possible values, while

$\theta_{[k-t+1]},\ldots,\theta_{[k]}$ attain their minimum possible values subject to $\underline{\theta} \in \Omega'(\delta^*)$. The proof is thus completed.

Using the same notation as in Section 2, let $G_n(y;\theta_i)$ denote the cdf of the sample median Y_i with parameter θ_i. Since θ_i is the location parameter, $G_n(y;\theta_i) = G_n(y-\theta_i;0)$ and G_n is stochastically increasing, continuous in both y and θ_i. For $\underline{\theta} \in \Omega'(\delta^*)$,

$$P_{\underline{\theta}}\{CS|R_3(n)\} = P_{\underline{\theta}}\{\max_{1\le i\le k-t} Y_{(i)} < \min_{k-t+1\le \ell\le k} Y_{(\ell)}\}$$

$$= P_{\underline{\theta}}\{\bigcup_{j=k-t+1}^{k} \{Y_{(j)} = \min_{k-t+1\le \ell\le k} Y_{(\ell)} \text{ and } \max_{1\le \ell\le k-t} Y_{(i)} < Y_{(j)}\}\}$$

$$= \sum_{j=k-t+1}^{k} \int_{-\infty}^{\infty} \prod_{\beta=1}^{k-t} G_n(y;\theta_{[\beta]}) \prod_{\substack{\alpha=k-t+1\\ \alpha\ne j}}^{k} \{1-G_n(y;\theta_{[\alpha]})\}\, d\,G_n(y;\theta_{[j]}).$$

In particular, for $\underline{\theta} \in \Omega_o'(\delta^*) \subset \Omega'(\delta^*)$,

$$P_{\underline{\theta}}\{CS|R\ (n)\} = t\int_{-\infty}^{\infty} G_n^{k-t}(y;\theta)\ \{1-G_n(y;\theta+\delta^*)\}^{t-1}\ dG_n(y,\theta+\delta^*)$$

$$= t\int_{-\infty}^{\infty} G_n^{k-t}(y-\theta;0)\ \{1-G_n(y-\theta-\delta^*;0)\}^{t-1}\ dG_n(y-\theta-\delta^*;0)$$

$$= t\int_{-\infty}^{\infty} G_n^{k-t}(y+\delta^*;0)\ \{1-G_n(y;0)\}^{t-1}\ dG_n(y;0)$$

which is independent of the parameter θ. Hence for specified values of δ^* and P^* $\left(\frac{1}{\binom{k}{t}} < P^* < 1\right)$, we can solve the equation

$$t\int_{-\infty}^{\infty} G_n^{k-t}(y+\delta^*;\ 0)\ \{1-G_n(y;\ 0)\}^{t-1}\ dG_n(y;\ 0) = P^*$$

for n.

5. Subset Selection with Respect to the Scale Parameter σ

Let X_i, $i = 1,2,\ldots,k$ be k independent random variables from double exponential population π_i, $i = 1,2,\ldots,k$, respectively, with π_i having the probability density function

$$f(x;\theta_i,\sigma_i) = \frac{1}{2\sigma_i} \exp\,[-|x-\theta_i|/\sigma_i],\ -\infty < x < \infty,\ -\infty < \theta_i < \infty,\quad \sigma_i > 0.$$

Take n independent observations from π_i, $i = 1,2,\ldots,k$. From these data one wishes to select a subset contains the population with the largest σ_i. Let $\sigma_{[1]} \leq \ldots \leq \sigma_{[k]}$ be the ordered parameters. We consider two different cases.

Case (i): $\theta_1,\theta_2,\ldots,\theta_k$ known.

In this case, the maximum likelihood estimator of σ_i is $Y_i = \frac{1}{n}\sum_{j=1}^{n} |X_{ij} - \theta_i|$ which is distributed as a gamma variable with parameters n and $\frac{\sigma_i}{n}$, i.e. Y_i has density $\frac{n}{\sigma_i\Gamma(n)}\left(\frac{ny}{\sigma_i}\right)^{n-1} e^{-\frac{ny}{\sigma_i}}$, $y > 0$. Thus the problem reduces to the one considered by Gupta [12]. The selection procedure is

R: Select the population π_i in the subset if and only if

$$Y_i \geq c \max_{1\leq j\leq k} Y_j.$$

Case (ii): θ_i's are unknown.

When θ_i is unknown, it is well known that the maximum likelihood estimate of σ_i is given by $\hat{\sigma}_i = \frac{1}{n}\sum_{j=1}^{n} |X_{ij} - \tilde{X}_i|$, where $\tilde{X}_i$ denotes the sample median from population π_i. For this problem, we propose the following selection procedure.

R_4: Select the population π_i in the subset if and only if

$$\hat{\sigma}_i \geq c_4 \max_{1\leq j\leq k} \hat{\sigma}_j$$

where $0 < c_4 < 1$ is so determined as to satisfy the basic probability requirement regardless of what the unknown σ_i's may be.

Let $V_i = \frac{n\hat{\sigma}_i}{\sigma_i}$, $i = 1,2,\ldots,k$. Then

$$P(CS|R_4) = P\{\hat{\sigma}_{(k)} \geq c_4 \max_{1\leq j\leq k-1} \hat{\sigma}_{(j)}\}$$

$$= \int_0^\infty \left[\prod_{j=1}^{k-1} F_{V_{(j)}}\left(\frac{1}{c_4}\,\frac{\sigma_{[k]}}{\sigma_{[j]}}\,x\right)\right] dF_{V_{(k)}}(x) \cdot$$

So

$$\inf_{\underline{\sigma}\in\Omega'} P(CS|R_4) = \inf_{\underline{\sigma}\in\Omega'_0} P(CS|R_4) = \int_0^\infty F_V^{k-1}\left(\frac{x}{c_4}\right) dF_V(x),$$

where $\Omega' = \{\underline{\sigma} = (\sigma_1,\ldots,\sigma_k),\ \sigma_i > 0,\ i = 1,\ldots,k\}$, $\Omega'_0 = \{\underline{\sigma} = (\sigma,\ldots,\sigma),\ \sigma > 0\}$ and $F_V(\cdot)$, $F_{V_{(j)}}(\cdot)$, $j = 1,\ldots,k$ are the cdf's of $V = \frac{n\hat{\sigma}}{\sigma}$, $V_{(j)} = \frac{n\hat{\sigma}_{(j)}}{\sigma_{[j]}}$, $j = 1,\ldots,k$, respectively.

Hence if the distribution $F_V(\cdot)$ is known, then the constant c_4 can be determined by the equation

$$\int_0^\infty F_V^{k-1}\left(\frac{x}{c_4}\right) dF_V(x) = P^*.$$

The exact distribution F of V is worked out for n = 3 by Bain and Engelhardt [1], and a chi-square approximation is also given by them which is quite good even for small n. However, it follows from Chernoff, Gastwirth and Johns [5], that $\frac{1}{\sqrt{n}}(V-n) = \sqrt{n}\,[\frac{\hat{\sigma}}{\sigma} - 1]$ is asymptotically a standard normal variable. When all σ_i are identical

$$P(CS|R_4) = P\{\hat{\sigma}_k \geq c_4\, \hat{\sigma}_j,\quad j = 1,\ldots,k-1\}$$

$$= P\{\sqrt{n}(\frac{\hat{\sigma}_k}{\sigma} - 1) \geq c_4\, \sqrt{n}(\frac{\hat{\sigma}_j}{\sigma} - 1) + \sqrt{n}(c_4-1),\ j = 1,\ldots,k-1\}$$

$$\cong \int_{-\infty}^{\infty} \Phi^{k-1}\left(\frac{x-\sqrt{n}(c_4-1)}{c_4}\right) d\Phi(x).$$

6. A Test of Homogeneity Based on the Sample Median Range

Let $\pi_1,\pi_2,\ldots,\pi_k$ be k independent double exponential populations such that the observations $X_{i1},\ldots,X_{i,2n+1}$ from π_i has density $\frac{1}{2} e^{-|X-\theta_i|}$, for $i = 1,2,\ldots,k$. As before, let the sample median of these (2n+1) observations be denoted as Y_i, $i = 1,\ldots,k$. In some practical situations one wishes to know whether θ_i are significantly different or not. This problem is to test the homogeneity of the double exponential populations. We are interested in using a test based on the sample range of Y's and hence we wish to derive the distribution of the sample median range $R = \max_{1\leq j\leq k} Y_j - \min_{1\leq j\leq k} Y_j$, considering all θ_i to be equal to a common unknown θ. When the value of R is large, the hypothesis of homogeneity is rejected. We wish to find a constant r, such that $P(R > r) \leq \alpha$ under the hypothesis H_o: $\theta_1 = \ldots = \theta_k = \theta$. This will provide an α-level test.

Theorem 6.1. For α, $0 < \alpha < 1$, let r be a constant such that

$$P_{\Omega_o}\{Y_k \geq \max_{1\leq j\leq k-1} Y_j - r\} \geq 1 - \frac{\alpha}{k}.$$

Then $P_{\Omega_o}(R > r) \leq \alpha$.

Proof. When H_o is true, i.e., under Ω_o,

$$P(R > r) = P\{\max_{1\le j\le k} Y_j - \min_{1\le j\le k} Y_j > r\}$$

$$\le k - \sum_{i=1}^{k} P\{Y_i \ge \max_{1\le j\le k} Y_j - r\}$$

$$= k - k\,P\{Y_k \ge \max_{1\le j\le k-1} Y_j - r\}$$

$$\le k - k\cdot(1 - \frac{\alpha}{k})$$

$$= \alpha.$$

The above theorem establishes a connection between the selection rule R_1 and the above test for equality of θ's.

7. On the Distribution of the Statistic Associated with R_1

Let Y_i $(i = 0,1,\ldots,p)$ be $(p+1)$ independent and identically distributed random variables each representing the median in a random sample of size $(2n+1)$ from a population with standard double exponential density function $f(x) = \frac{1}{2} e^{-|x|}$. Consider the differences $Z_i = Y_i - Y_o$ $(i = 1,2,\ldots,p)$. The random variables Z_i $(i = 1,2,\ldots,p)$ are correlated and the distribution of the maximum of Z_i is of interest in problems of selection and ranking for double exponential distribution as explained earlier when discussing R_1. In this section, we give a closed form of the distribution of $Z = \max_{1\le i\le p} Z_i$ for some special cases. We have also computed tables of the upper percentage points of $Z = \max_{1\le i\le p} Z_i$ corresponding to the probability levels $\alpha = P^* = 0.75, 0.90, 0.95, 0.99$ for $p = 1(1)\,9$, $n = 1(1)10$.

For the special case $P = 1$ $(k=2)$, $n = 1$ (sample size $= 3$), straight forward integration gives the cdf of Z (see formulae (2.2), (2.3)) as

$$P(Z \leq z) = \int_{-\infty}^{\infty} G(x + z)\, g(x)\, dx$$

$$= 1 - \frac{9}{8} ze^{-2z} - \frac{3}{16} ze^{-3z} + \frac{9}{40} e^{-2z} - \frac{29}{40} e^{-3z}.$$

Again, for $p = 1$ ($k=2$), $n = 2$ (sample size = 5),

$$P(Z \leq z) = 1 - \frac{75}{16} ze^{-3z} - \frac{225}{64} ze^{-4z} - \frac{45}{256} ze^{-5z} + \frac{10975}{1792} e^{-3z}$$

$$- \frac{5225}{896} e^{-4z} - \frac{203}{256} e^{-5z}.$$

All computations related to and given at the end of this chapter were made on a CDC 6500 using Gauss Laguerre quadratue based on fifteen nodes to perform the numerical integration. Checks on the accuracy of the program for $p = 1$, $n = 1$ showed that these values seem to be correct to three decimal places.

Upper $100(1-P^*)$ percentage points of $Z = \max_{1 \leq i \leq p} (Y_i - Y_o)$ where $Y_o, Y_1, \ldots, Y_p$ are iid sample median random variables in samples of sizes (2n+1) from the standard double exponential (Laplace) distribution.

p \ n	1	2	3	4	5	6	7	8	9	10
	.6781	.5145	.4274	.3721	.3334	.3049	.2828	.2651	.2504	.2378
1	1.3777	1.0311	.8496	.7357	.6568	.5987	.5541	.5184	.4888	.4634
	1.8508	1.3735	1.1261	.9718	.8658	.7885	.7294	.6825	.6436	.6102
	2.8631	2.0931	1.7002	1.4598	1.2994	1.1876	1.1070	1.0464	.9974	.9533
	1.0434	.7875	.6522	.5667	.5072	.4631	.4289	.4014	.3786	.3591
2	1.7380	1.2948	1.0640	.9195	.8195	.7459	.6892	.6437	.6060	.5738
	2.2092	1.6330	1.3354	1.1503	1.0231	.9302	.8590	.8024	.7555	.7513
	3.2186	2.3459	1.9015	1.6297	1.4479	1.3203	1.2274	1.1569	1.0998	1.0489
	1.2507	.9401	.7767	.6737	.6021	.5490	.5077	.4746	.4470	.4234
3	1.9451	1.4445	1.1847	1.0224	.9102	.8275	.7637	.7126	.6703	.6342
	2.4159	1.7811	1.4540	1.2509	1.1114	1.0094	.9313	.8690	.8176	.7735
	3.4247	2.4913	2.0166	1.7265	1.5322	1.3955	1.2956	1.2194	1.1576	1.1029
	1.3972	1.0468	.8632	.7476	.6673	.6078	.5616	.5244	.4935	.4672
4	2.0917	1.5496	1.2689	1.0938	.9729	.8838	.8151	.7601	.7146	.6758
	2.5624	1.8852	1.5369	1.3210	1.1728	1.0644	.9813	.9151	.7605	.8138
	3.5706	2.5936	2.0974	1.7942	1.5911	1.4479	1.3430	1.2629	1.1979	1.1405
	1.5109	1.1290	.9294	.8040	.7170	.6525	.6024	.5622	.5288	.5003
5	2.2053	1.6306	1.3336	1.1485	1.0208	.9268	.8543	.7962	.7482	.7074
	2.6759	1.9655	1.6007	1.3747	1.2197	1.1064	1.0195	.9503	.8932	.8444
	3.6838	2.6727	2.1596	1.8463	1.6363	1.4881	1.3794	1.2962	1.2287	1.1693
	1.6039	1.1958	.9830	.8496	.7570	.6885	.6353	.5925	.5571	.5269
6	2.2982	1.6965	1.3860	1.1928	1.0595	.9615	.8859	.8253	.7753	.7328
	2.7686	2.0308	1.6524	1.4183	1.2577	1.1403	1.0503	.9787	.9195	.8691
	3.7762	2.7371	2.2101	1.8855	1.6729	1.5207	1.4088	1.3232	1.2537	1.1925

(continued)

p \ n	1	2	3	4	5	6	7	8	9	10
7	1.6826	1.2521	1.0281	.8878	.7905	.7186	.6628	.6179	.5807	.5491
	2.3767	1.7520	1.4301	1.2299	1.0920	.9905	.9134	.8497	.7979	.7540
	2.8470	2.0859	1.6959	1.4548	1.2895	1.1687	1.0762	1.0024	.9416	.8898
	3.8543	2.7913	2.2526	1.9240	1.7037	1.5480	1.4336	1.3458	1.2745	1.2120
8	1.7507	1.3007	1.0669	.9207	.8193	.7444	.6863	.6396	.6009	.5681
	2.4447	1.7999	1.4681	1.2619	1.1199	1.0155	.9350	.8706	.8174	.7722
	2.9148	2.1334	1.7335	1.4863	1.3169	1.1932	1.0984	1.0228	.9606	.9075
	3.9219	2.8383	2.2894	1.9546	1.7302	1.5716	1.4548	1.3653	1.2925	1.2288
9	1.8109	1.3435	1.1010	.9495	.8446	.7670	.7069	.6586	.6186	.5847
	2.5047	1.8421	1.5014	1.2900	1.1444	1.0373	.9548	.8888	.8344	.7881
	2.9747	2.1753	1.7665	1.5139	1.3410	1.2146	1.1178	1.0407	.9772	.9231
	3.9816	2.8796	2.3217	1.9815	1.7535	1.5922	1.4735	1.3823	1.3083	1.2435

For given p,n and $P^* = .75$ (top), .90 (second), .95 (third), .99 (bottom), the entries in this table are the values of d for which $\int_{-\infty}^{\infty} G_n^P(x+d)\, dG_n(x) = P^*$ where $G_n(\cdot)$ is the cdf of the median in a sample of size (2n+1) from the standard double exponential distribution.

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