ON STOPPING TIMES FOR n DIMENSIONAL BROWNIAN MOTION

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1. Introduction. Let $\overline{X}(t) = (X_1(t), X_2(t), \ldots, X_n(t)), t \geq 0$ be standard n dimensional Brownian motion, that is $X_1(t), X_2(t), \ldots, X_n(t)$ are independent Wiener processes. The following theorems and related results are proved.

Theorem 1.1 <u>For each positive number</u> p <u>there is a constant</u> C_p <u>not depending on</u> n <u>such that if</u> τ <u>is a stopping time for</u> $\overline{X}(t)$ <u>then</u>

$$(1.1) \quad E \sup_{0 \leq t \leq \tau} \left| |\overline{X}(t)| - ((n-1)t)^{\frac{1}{2}} \right| \leq C_p \, E\tau^{p/2}.$$

Here $|\overline{X}(t)| = (\sum_{i=1}^{n} X_i(t)^2)^{\frac{1}{2}}$ is the usual absolute value. As shown in Section 2, Theorem 1.1 can be viewed as a refinement of the recent result of D. L. Burkholder ([1], Theorem 2.1) that there are constants $c_{p,n}$ and $C_{p,n}$ both approaching 1 as $n \to \infty$ such that

$$(1.2) \quad c_{p,n} \, E(n\tau)^{p/2} \leq E \sup_{0 \leq t \leq \tau} |\overline{X}(t)|^p \leq C_{p,n} \, E(n\tau)^{p/2}.$$

Of course, for n=1, inequality (1.1) is the right hand side of the following well known inequality of Burkholder and P. W. Millar (for the exponents $p > 1$) and Burkholder and R. F. Gundy (for the exponents $0 < p \leq 1$). If $Z(t)$, $t \geq 0$, is a Wiener process there are constants K_p and k_p such that if T is a stopping time for $Z(t)$ then

$$(1.3) \quad k_p \, ET^{p/2} \leq E \sup_{0 \leq t \leq T} |Z(t)|^p \leq K_p \, ET^{p/2}.$$

The paper [2] is a good reference for these and related inequalities.

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The following theorem is also proved:

Theorem 1.2. Let k be a positive integer and $\varepsilon > 0$. There is an integer $j(k,\varepsilon) = j$ not depending on n such that if τ is a stopping time for $\overline{X}(t)$ then for all but j of the indices $i = 1,2,\ldots,n$

(1.4) $$(1-\varepsilon)E\tau^{k/2} \leq \frac{2^k k!}{(2k)!} EX_i(\tau)^k \leq (1+\varepsilon)E\tau^{k/2},$$

if k is even and $E\tau^{k/2} < \infty$,

and

(1.5) $$-\varepsilon\, E\tau^{k/2} \leq EX_i(\tau)^k \leq \varepsilon\, E\tau^{k/2},$$

if k is odd and $E\tau^{k/2} < \infty$.

Of course if $E\tau < \infty$ then $EX_i(\tau)^2 = E\tau$ for all i and if $E\tau^{\frac{1}{2}} < \infty$ then $EX_i(\tau) = 0$ for all i so there is nothing new here for the exponents $k = 1,2$. If Y is a nonnegative random variable which is independent of $\overline{X}(t)$ it is not hard to show that $EX_i(Y)^k = (2k)!\, EY^{k/2}/2^k k!$ if k is even and $EX_i(Y)^k = 0$ if k is odd and $EY^{k/2} < \infty$, so that Theorem 1.2 says that in some sense the stopping times τ don't pay much attention to most of the components $X_i(t)$ of $\overline{X}(t)$.

It is an observation of Poincare that if $(z_1,\ldots,z_n)$ is a point uniformly distributed on the surface S_n of the ball of radius $\sqrt{n}$ about 0 in R^n then, if n is large, z_1 is about normally distributed. This has a trivial proof: Let $X_1, X_2, \ldots$ be independent standard normal random variables. Then $(X_1,X_2,\ldots,X_n)/(X_1^2+\ldots+X_n^2)^{\frac{1}{2}}$ is uniformly distributed on the surface of the unit ball, and $(X_1^2+\ldots+X_n^2)^{\frac{1}{2}}/\sqrt{n} \to 1$ a.s. as $n \to \infty$. Both Theorems 1.1 and 1.2 have some of the character of this result and since it is not hard to give an alternative proof of it using Theorem 1.1 we do this now. Let $Z_1(t), Z_2(t), \ldots$ be a sequence of independent Wiener processes, and let $\tau_n = \inf\{t > 0: \sum_{i=1}^{n} Z_1(t)^2 = n\}$. Then $E\tau_n = 1$ and $(Z_1(\tau_n), Z_2(\tau_n), \ldots, Z_n(\tau_n))$ is uniformly distributed on S_n. Using (1.1) with p=2,

$$\begin{aligned} E(n^{\frac{1}{2}}-(n-1)\tau_n^{\frac{1}{2}})^2 &= E((\sum_{i=1}^{n} Z_i(\tau_n)^2)^{\frac{1}{2}}-((n-1)\tau_n)^{\frac{1}{2}})^2 \\ &\leq C_2\, E\tau_n \\ &= C_2 \end{aligned}$$

Thus $\tau_n \to 1$ in probability and $Z_1(\tau_n) \to Z_1(1)$ in distribution, as $n \to \infty$.

The following generalization of this is proved at the end of Section 3. If 2k is any fixed positive even integer, if T_n is the surface $\{(z_1, z_{21}, \ldots, z_n): \Sigma z_i^{2k} = n(2k)!/2^k k!\}$ and if μ_n is harmonic measure relative to $(0,0,\ldots,0)$ (i.e. μ_n is the first hitting of T_n distribution of $\overline{X}(t)$) then the distribution of z_1 under μ_n is about standard normal, if n is large.

In the final section a simple proof of the inequalities (1.3) for the exponents $0 < p \leq 1$ is given. There are a number of proofs of (1.3), including the original one, for the exponents $p > 1$ which do not extend to the exponents $0 < p \leq 1$.

The proofs of Theorems 1.1 and 1.2 are based on Ito's stochastic calculus and the methods and results of Burkholder and Gundy which deal with one dimensional Brownian motion.

2. Inequalities for $|\overline{X}(t)|$. The integer n will always stand for the dimension of the Brownian motion $\overline{X}(t)$. If $Z(t)$, $t \geq 0$, is a stochastic process (possibly multi-dimensional) we let $Z(t)^* = \sup_{0 \leq s \leq t} |Z(s)|$. The L^p norm of a random variable Y will be defined by $||Y||_p = (E|Y|^p)^{1/p}$ for all $p > 0$. If $0 < p < 1$ the triangle inequality doesn't hold but the weaker $||X + Y||_p \leq 2^p(||X||_p + ||Y||_p)$ is always a sufficient substitute for our purposes. Ef^p will always mean $E(f^p)$, not $(Ef)^p$.

A technical point that comes up several times can be illustrated by the following example. Inequality (1.3) is stated for stopping times for $Z(t)$ and does not apply directly to prove $E \sup_{0 \leq t \leq \tau} |X_1(t)|^p \leq K_p E\tau^{p/2}$ if τ is a stopping time with respect to the σ-fields $\sigma(\overline{X}(s), s \leq t) = Q(t)$. Nonetheless this inequality is still true since the only thing used about τ in the original proof of (1.3) (and all other proofs) is that $X_1(\tau+s)-X_1(\tau)$, $s \geq 0$, is standard Brownian motion which is independent of $Q(\tau)$.

For future reference we state now a generalization of (1.3) also due to Burkholder and Gundy (see [2]). Let Φ be any positive increasing function on $[0,\infty)$ satisfying $\Phi(0) = 0$ and $\Phi(2\lambda) \le \alpha\Phi(\lambda)$ for some constant $\alpha_\Phi = \alpha$ and all positive numbers λ. Then there are constants k and K depending only on α_Φ such that if $Z(t)$ is a Wiener process and τ is a stopping time for $Z(t)$ then

(2.1) $$kE\Phi(\tau^{\frac{1}{2}}) \le E\Phi(Z(\tau)^*) \le KE\Phi(\tau^{\frac{1}{2}}).$$

The following theorem extends Theorem 1.1.

Theorem 2.1 Let Φ be a function of the type described just above. There is a constant A depending on α_Φ but not on n such that if τ is a stopping time for $\overline{X}(t)$ then

(2.2) $$E\Phi(\sup_{0 \le t \le \tau}||\overline{X}(t)|-((n-1)t)^{\frac{1}{2}}|) \le AE\Phi(\tau^{\frac{1}{2}}).$$

Proof. For n=1, (2.2) is the right side of (2.1). Thus we assume $n \ge 2$. We use the notation of McKean's book [3]. See especially page 47. Thus let $r(t) = |\overline{X}(t)|$. The stochastic differential of r is

$$dr = da + (n-1)(2r)^{-1}\,dt,$$

where a(t) is the one dimensional Brownian motion given by

(2.3) $$a(t) = \int_0^t \frac{\Sigma X_i(s)dX_i(s)}{r(s)}$$

This is problem 6 on page 47 of [3]. Of course the stochastic differential of the non-random function $((n-1)t)^{\frac{1}{2}}$ is

$$d((n-1)t)^{\frac{1}{2}} = ((n-1)^{\frac{1}{2}}/2t^{\frac{1}{2}})dt$$

Thus $\gamma(t) = r(t)-((n-1)t)^{\frac{1}{2}}$ satisfies

$$d\gamma = da + [(n-1)/2r- (n-1)^{\frac{1}{2}}/2t^{\frac{1}{2}}]dt$$
$$= da-(\gamma/2rt^{\frac{1}{2}})dt.$$

Using Ito's lemma

$$d\gamma^2 = 2\gamma da-[2(\gamma^2/2rt^{\frac{1}{2}})+ 1]dt,$$

so that

$$d(\gamma^2-t) = 2\gamma da-2(\gamma^2/2rt^{\frac{1}{2}})dt,$$

which implies that $\gamma^2 - t$ is a submartingale since $\gamma^2/2rt$ is nonegative. We note that $\gamma^2 - t$ is a submartingale with respect to the σ-fields $(\overline{X}(s),\ s \leq t)$ so that the results which follow are applicable to stopping times for $\overline{X}(t)$. In the paper [2], p.27, D.L. Burkholder outlines a proof of (2.1) in which the only property of Z(t) that is used is that $Z(t)^2 - t$ is a martingale. This proof is readily adapted to the present case, and the adaption will not be given here. It yields the result that there is a constant A depending only on α_Φ such that if τ is a stopping time for $\overline{X}(t)$ then $E\Phi(\gamma(\tau)^*) \leq AE\Phi(\tau^{\frac{1}{2}})$, which is Theorem 2.1.

We remark that, in the statement of Theorem 2.1, $((n-1)t)^{\frac{1}{2}}$ may be replaced by $(nt)^{\frac{1}{2}}$, although the constant A must be increased a little. We have

$$
\begin{aligned}
(2.4)\quad & E\Phi(\sup_{0 \leq t \leq \tau} ||\overline{X}(t)| - (nt)^{\frac{1}{2}}|) \\
& \leq E\Phi(\gamma(\tau)^* + (n^{\frac{1}{2}} - (n-1)^{\frac{1}{2}})\tau^{\frac{1}{2}}) \\
& \leq E\Phi(\gamma(\tau)^* + \tau^{\frac{1}{2}}) \\
& \leq E\Phi(2\gamma(\tau)^*) + E\Phi(2\tau^{\frac{1}{2}}) \\
& \leq \alpha_\Phi E\Phi(\gamma(\tau)^*) + \alpha_\Phi E\Phi(\tau^{\frac{1}{2}}) \\
& \leq \alpha_\Phi (A+1) E\Phi(\tau^{\frac{1}{2}}).
\end{aligned}
$$

Note that

$$(2.5)\quad |\overline{X}(\tau)^* - n\tau| \leq \sup_{0 \leq t \leq \tau} ||\overline{X}(t)| - (nt)^{\frac{1}{2}}|,$$

so that (2.4) strengthens (1.2) by showing that not only are the L^p norms of $|\overline{X}(\tau)|$ and $(n\tau)^{\frac{1}{2}}$ close but the random variables $|\overline{X}(\tau)|$ and $(n\tau)^{\frac{1}{2}}$ are close in L^p. It is remarked that the constants in the analogue of (1.2) for functions Φ of the type discussed above need not approach 1 as $n \to \infty$, for some of these Φ. See [1]. The following inequality also holds.

$$(2.6)\quad ||\overline{X}(\tau)| - (n\tau)^{\frac{1}{2}}| \leq \sup_{0 \leq t \leq \tau} |\overline{X}(t) - (nt)^{\frac{1}{2}}|.$$

Burkholder proves (Theorem 2.2 of [1]) that if $p > 0$ there are constants $b_{p,n}$ which approach 1 as $n \to \infty$ such that

$$(2.7)\quad E(\overline{X}(\tau)^*)^p \leq b_{p,n}\, E|\overline{X}(\tau)|^p.$$

This can be derived, under the additional assumption that $E(\overline{X}(\tau)^*)^p < \infty$, from (2.4). The proof goes like this. If $E(\overline{X}(\tau)^*)^p < \infty$ then $E(n\tau)^{p/2} < \infty$, using (1.2). Using (2.4) and (2.6) it is shown that $E|\overline{X}(\tau)|^p/E(n\tau)^{p/2}$ is close to 1 if n is large, and using Theorem (2.4) and (2.5) it is shown that $E(X(\tau)^*)^p/E(n\tau)^{p/2}$ is close to 1 if n is large, so that $E(X(\tau)^*)^p/E|X(\tau)|^p$ is close to 1 if n is large.

2. Proof of Theorem 1.2. Unless otherwise indicated, sums will be taken over $i=1,2,\ldots,n$, so that $\sum_{i=1}^{n}$ will be shortened to Σ. As before, n is the dimension of the Brownian motion. For each nonnegative integer k we define $G_{k,n}(s) = G_k(s)$ and $H_{k,n}(s) = H_k(s)$ by

$$G_k(s) = \Sigma X_i(s)^k, \text{ and}$$

$$H_k(s) = \int_0^s \Sigma X_i(t)^k dX_i(t).$$

The following extension of (1.2), due to the same people responsible for (1.2), will be needed. Only the right hand side will be stated. Let $Z(t)$ be a Wiener process and $f(t,\omega)$ be a non-anticipating functional. Then for each number $p > 0$

$$\text{(3.1)} \quad E\sup_{0 \le t \le \tau}\left|\int_0^t f(s,\omega)dZ(s)\right|^p \le K_p E\left(\int_0^\infty f(t,\omega)^2 dt\right)^{p/2},$$

where K_p is the same constant as in (1.3).

By Ito's lemma, for each integer $k \geq 2$

$$\text{(3.2)} \quad G_k(t) = kH_{k-1}(t) + \frac{k(k-1)}{2}\int_0^t G_{k-2}(s)ds.$$

The next lemma estimates the first of these components. The constants K_q are those of (1.3).

Lemma 3.1. <u>Let</u> j <u>be a positive integer and</u> $p \geq 1$. <u>Then, if</u> τ <u>is a stopping time for</u> $\overline{X}(t)$, (3.3) $E\ \sup_{0 \le t \le \tau} |H_j(t)|^p \le g(p,n,j) E\tau^{p(j+1)/2}$,

<u>where</u> $g(p,n,j) = K_{p(j+1)}^{j(j+1)}\ n^{p/2}$ <u>if</u> $p \ge 2$,

$g(p,n,j) = K_{p(j+1)}^{j(j+1)} n$, if $1<p \le 2$, <u>and</u> $g(1,n,j)$

$= (1+K_{j+1})n^{(2j+1)(2j+2)}$.

Proof. It can be shown, using the same reasoning suggested by McKean to prove that the process a(t) of (2.3) is a Brownian motion and that the equation preceeding (2.3) holds, that $W(s) = H_j(s)/(G_{2j}(s))^{\frac{1}{2}}$ is a standard Brownian motion and that

$$H_j(t) = \int_0^t (G_{2j}(s))^{\frac{j}{2}}\, dW(s).$$

Thus applying (3.1) to the functional $(G_{2j}(s))^{\frac{j}{2}} I(0 \le s \le \tau)$ we get

(3.4) $E\ \sup_{0 \le t \le \tau} |H_j(t)|^p \le K_p\ E(\int_0^\tau G_{2j}(s)ds)^{p/2}$.

Let $\gamma_i = \sup_{0 \le t \le \tau} |X_i(t)|$. Then

(3.5) $$E(\int_0^\tau G_{2j}(s)ds)^{p/2} = E(\Sigma \int_0^\tau X_i(s)^{2j} ds)^{p/2}$$

$$\le E(\Sigma \tau \gamma_i^{2j})^{p/2}.$$

If $p \ge 2$ we have, using (1.3),

$$E(\Sigma\tau\gamma_i^{2j})^{p/2} \le (\Sigma[E(\tau\gamma_i^{2j})^{p/2}]^{2/p})^{p/2}$$

$$\le n^{p/2} \max_{1 \le i \le n} E(\tau\gamma_i^{2j})^{p/2}$$

$$\le n^{p/2} \max(E\tau^{p(j+1)/2})^{1/(j+1)}\ E(\gamma_i^{p(j+1)})^{j/(j+1)}$$

$$\le n^{p/2} (E\tau^{p(j+1)/2})^{1/(j+1)} (K_{p(j+1)} E\tau^{p(j+1)/2})^{j/(j+1)}$$

$$= n^{p/2} K_{p(j+1)}^{j/(j+1)}\ E\tau^{p(j+1)/2}.$$

If $1 < p < 2$,

$$E(\Sigma\tau\gamma_i^{2j})^{p/2} \leq \Sigma E(\tau\gamma_i^{2j})^{p/2}$$

$$\leq \Sigma(E\tau^{p(j+1)/2})^{1/(j+1)}(K_{p(j+1)}E\tau^{p(j+1)/2})^{j/(j+1)}$$

$$= nK_{p(j+1)}^{j(j+1)} E\tau^{p(j+1)/2}.$$

If $p=1$, we let $M=M(n)$ be for the moment an arbitrary positive number. Define $\alpha_i = \gamma_i\ I(\gamma_i \leq M\tau^{\frac{1}{2}})$ and $\beta_i = \gamma_i\ I(\gamma_i > M\tau^{\frac{1}{2}})$.

Then

$$(3.6)\quad E(\Sigma\tau\gamma_i^{2j})^{\frac{1}{2}} = E(\Sigma\tau(\alpha_i^{2j} + \beta_i^{2j}))^{\frac{1}{2}}$$

$$\leq E(\Sigma\tau\alpha_i^{2j})^{\frac{1}{2}} + E(\Sigma\tau\beta_i^{2j})^{\frac{1}{2}}.$$

Now

$$E(\Sigma\tau\alpha_i^{2j})^{\frac{1}{2}} \leq E(\Sigma M^{2j}\tau^{j+1})^{\frac{1}{2}} = M^j n^{\frac{1}{2}} E\tau^{(j+1)/2}.$$

Also, for each $q > 0$,

$$M^q E\tau^{q/2} I(\gamma_i > M\tau^{\frac{1}{2}}) \leq E\gamma_i^q \leq K_q E\tau^{q/2},$$

so that

$$E(\Sigma\tau\beta_i^{2j})^{\frac{1}{2}} = E(\Sigma\tau I(\gamma_i > M\tau^{\frac{1}{2}})\gamma_i^{2j})^{\frac{1}{2}}$$

$$\leq \Sigma E(\tau I(\gamma_i > M\tau^{\frac{1}{2}})\gamma_i^{2j})^{\frac{1}{2}}$$

$$\leq \Sigma[E(\tau I(\gamma_i > M\tau^{\frac{1}{2}}))^{(j+1)/2}]^{1/(j+1)}(E\gamma_i^{j+1})^{j/(j+1)}$$

$$\leq \Sigma(K_{j+1}E\tau^{(j+1)/2}/M^{j+1})^{1/(j+1)}(K_{j+1}E\tau^{(j+1)/2})^{j/(j+1)}$$

$$= nK_{j+1}E\tau^{(j+1)/2}/M.$$

Taking $M=n^{1/2(j+1)}$ and using (3.6) we get the desired value for $g(1,n,j)$.

Define $\lambda(k) = 2^k k!/(2k)!$ if $k \geq 1$, and define $\lambda(0) = 1$.

Lemma 3.3. _For each nonnegative integer_ j _and each_ $p \geq 1$ _there are constants_ $C(p,n,j)$ _which approach_ 0 _as_ n _approaches infinity such that if_ τ _is a stopping time for_ $\overline{X}(t)$ _then_

(3.7) $E \sup_{0 \le t \le \tau} |\lambda(k) G_{2k}(t)/n - t^k|^p \le C(p,n,2k) E\tau^{kp}$

and

(3.8) $E \sup_{0 \le t \le \tau} |G_{2k+1}(t)/n| \le C(p,n,2k+1) E\tau^{kp}$.

Proof. Let $\Gamma(p,n,j)$ denote the smallest possible value for $C(p,n,j)$ such that (3.7) and (3.8) hold for all stopping times τ. Clearly $C(p,n,0) = 0$ for all p and n. Using equation (3.2),

$$(E \sup_{0 \le t \le \tau} |\lambda(k) G_{2k}(t)/n - t^k|^p)^{1/p}$$

$$= (E \sup_{0 \le t \le \tau} |2k\lambda(k) H_{2k-1}(t)/n + k(2k-1)\lambda(k) \int_0^t G_{2k-2}(s)\,ds/n - t^k|^p)^{1/p}$$

$$\le (E \sup_{0 \le t \le \tau} |2k\lambda(k) H_{2k-1}(t)/n|^p)^{1/p}$$

$$+ (E \sup_{0 \le t \le \tau} |k(2k-1)\lambda(k) \int_0^t G_{2k-2}(s)\,ds/n - t^k|^p)^{1/p}$$

$$= I + II.$$

Using Lemma 3.1

$I \le \lambda(k) g(p,n,2k-1)^{1/p} (E\tau^{pk})^{1/p}/n$, while

$$II^p = E \sup_{0 \le t \le \tau} |k(2k-1)\lambda(k) \int_0^t G_{2k-2}(s)\,ds/n - \int_0^t ks^{k-1}\,ds|^p$$

$$\le E(\int_0^\tau |k(2k-1)\lambda(k) G_{2k-2}(s)/n - ks^{k-1}|\,ds)^p$$

$$\le E(\tau \sup_{0 \le s \le \tau} |k(2k-1)\lambda(k) G_{2k-2}(s)/n - ks^{k+1}|)^p$$

$$= k^p E(\tau \sup_{0 \le s \le \tau} |\lambda(k-1) G_{2k-2}(s)/n - s^{k-1}|)^p$$

$$\le k^p (E\tau^{kp})^{1/k} (E \sup_{0 \le s \le \tau} |\lambda(k-1)(G_{2k-2}(s)/n - s^{k-1}|^{kp/(k-1)})^{(k-1)/k}$$

$$\le k^p (E\tau^{kp})^{1/k} (\Gamma(\tfrac{kp}{k-1}, n, 2k-2) E\tau^{kp})^{(k-1)/k}$$

$$= k^p\, \Gamma(\tfrac{kp}{k-1}, n, 2k-2)^{(k-1)/k}\, E\tau^{kp}.$$

Thus

$$(I + II)^p \le 2^p(I^p + II^p)$$

$$\le (2^p\lambda(k)^p g(p,n,2k-1)/n^p + 2^p k^p \Gamma(\tfrac{kp}{k-1},n,2k-2)^{p(k-1)/k}\, E\tau^{kp}$$

$$\le (2^p\lambda(k)^p g(p,n,2k-1)n^p + 2^p k^p \Gamma(\tfrac{kp}{k-1},n,2k-2)^{p(k-1)/k})E\tau^{kp}$$

$$= M(p,k,n)E\tau^{kp}.$$

The rest of the proof of (3.7) is by induction. Since $\Gamma(\frac{kp}{k-1},n,\ 2k-2) = 0$ if $k=1$ and since as $n \to \infty$ $g(p,n,1)/n^p \to 0$ for each $p \ge 1$, we have $M(p,2,n) \to 0$ as $n \to \infty$ so that $\Gamma(n,p,2) \to 0$ for each $p \ge 1$, as $n \to \infty$. This fact and the fact that $g(p,n,3) \to 0$ as $n \to \infty$ give that $\Gamma(p,n,4) \to 0$ as $n \to \infty$ for each $p \ge 1$, and so on.

The proof of (3.8) is similar. Since $\frac{G_1(s)}{n^{\frac{1}{2}}}$ is standard Brownian motion,

$$E\ \sup_{0 \le s \le \tau}\left|\frac{G_1(s)}{n}\right|^p \le n^{-p/2}K_p E\tau^{p/2},$$

so that $\Gamma(n,p,1) \to 0$ as $n \to \infty$. If $k \ge 1$,

$$(E\ \sup_{0 \le t \le \tau}\left|\frac{G_{2k+1}(t)}{n}\right|^p)^{1/p}$$

$$\le \frac{(2k+1)}{n}(E(H_{2k}(\tau)^*)^p)^{1/p} + \frac{(2k+1)k}{n}(E\ \sup_{0 \le t \le \tau}\left|\int_0^t G_{2k-1}(s)ds\right|^p)^{1/p}$$

$$= I + II.$$

The first of these terms goes to 0 as $n \to \infty$ for each fixed k and p by Lemma 3.1, and

$$E\ \sup_{0 \le t \le \tau}\left|\int_0^t G_{2k-1}(s)ds/n\right|^p$$

$$\le E(\tau\, G_{2k-1}(s)^*/n)^p$$

$$\le (E\tau^{kp})^{1/k}(E(G_{2k-1}(s)^*/n)^{pk/(k-1)})^{(k-1)/k}$$

$$\le (E\tau^{kp})^{1/k}(\Gamma(\tfrac{pk}{k-1},n,2k-1)E\tau^{pk})^{(k-1)/k}$$

$$= \Gamma(\tfrac{pk}{k-1},n,2k-1)^{(k-1)/k}E\tau^{kp},$$

and (3.8) follows by induction.

Proof of Theorem 1.2. Inequality (3.7) with p=1 implies

$$\left|E\frac{\lambda(k)G_{2k}(\tau)}{n} - E\tau^k\right| \leq C(1,n,2k)E\tau^k.$$

If j(1),j(2),...,j(m) is any subset of 1,2,...,n then $\overline{Y}(t) = (X_{j(1)}(t),\ldots, X_{j(m)}(t))$ is standard m dimensional Brownian motion so that

$$(3.9)\qquad \left|\frac{\lambda(k)}{m}\sum_{i=1}^{m} EX_{j(i)}(\tau)^{2k} - E\tau^k\right| \leq C(1,m,2k)E\tau^k.$$

This holds even though τ is a stopping time for $\overline{X}(t)$ and not $\overline{Y}(t)$, by reasoning similar to that used in the second paragraph of Section 2. Thus if $N(\varepsilon)$ is the number of i such that $\lambda(k)EX_i(\lambda)^{2k} > (1+\varepsilon)E\tau^k$, we have, by picking these i for j(1),...,j(m) in (3.9),

$$|(1+\varepsilon)E\tau^k - E\tau^k| \leq C(1,N(\varepsilon),2k)E\tau^k$$

Thus $N(\varepsilon)$ can at most be the largest integer ℓ such that $C(1,\ell,2k) \geq \varepsilon$. Since $C(1,n,2k) \to 0$ as $n \to \infty$ this is a finite integer, and gives an estimate which does not depend on n. The proof of the rest of Theorem 2.1 is similar.

Now let k be a fixed integer ≥ 1 and let $T_n(k) = T_n = \{(z_1,z_2,\ldots,z_n): \Sigma z_i^{2k} = n(2k)!/2^k k!\}$. Let $Z_1(t), Z_2(t),\ldots$ be an infinite sequence of independent Wiener processes, and let

$$s_n = \inf\{t > 0: \quad (Z_1(t),\ldots,Z_n(t)) \in T_n\}.$$

Then, using (3.7) with p=1,

$$E|1-s_n^k| = E\left|\lambda(k)\sum_{i=1}^{n} Z_i(s_n)^{2k}/n - s_n^k\right|$$

$$\leq C(p,n,2k)Es_n^k,$$

implying

$$E|1-s_n^k| \leq \frac{C(p,n,2k)}{1-C(p,n,2k)} \to 0 \text{ as } n \to \infty.$$

Thus $s_n^k \to 1$ in probability implying $s_n \to 1$ in probability so that $X_1(s_n) \to X_1(1)$ in distribution.

4. A proof of (1.3) for $0 < p \le 1$. We first prove the right hand side. Let $\beta_p = E \sup_{0 \le t \le 1} |Z(t)|^p$. Note β_p is a lower bound for K_p. Let $\nu = \inf\{t \ge T: \; t=2^k$ for some integer $k\}$. Then $\nu \le 2T$, and $\sum_{k=-\infty}^{\infty} P(\nu=2^k) = P(\nu > 0)$. We have $E(Z(T)^*)^p \le E(Z(\nu)^*)^p$

$$\le \sum_{k=-\infty}^{\infty} E \sup_{2^k \le s \le 2^{k+1}} |Z(s)-Z(2^k)|^p I(\nu > 2^k)$$

$$= \sum_{k=-\infty}^{\infty} E \sup_{0 \le s \le 2^k} |Z(s)|^p \; P(\nu > 2^k)$$

$$= \sum_{k=-\infty}^{\infty} \beta_p 2^{kp/2} \; P(\nu > 2^k)$$

$$\le \beta_p \sum_{k=-\infty}^{\infty} \Big(\sum_{j=-\infty}^{k-1} 2^{jp/2}\Big) P(\nu=2^k)$$

$$= \beta_p \sum_{k=-\infty}^{\infty} \Big(\frac{2^{kp/2}}{2^{p/2}-1}\Big) P(\nu=2^k)$$

$$= \beta_p/(2^{p/2}-1) E\nu^{p/2}$$

$$\le \frac{2^p \beta_p}{2^{p/2}-1} ET^{p/2}.$$

The left hand side is similar. Let $\mu(\lambda) = \inf\{t > 0: \; |Z(t)| = \lambda\}$ and let $\alpha_p = E\mu(1)^{p/2}$. Note α_p is an upper bound for k_p. Let $\tau_i = \inf\{t > 0: \; |Z(t)| = 2^i\}$, and let $\eta = \inf\{t \ge T: \; t = \tau_i$ for some integer $i\}$. Then $Z(\eta)^* \le 2Z(T)^*$ and $\sum_{i=-\infty}^{\infty} P(Z(\eta)^* = 2^i) = P(Z(\eta)^* > 0)$. Also, $\tau_{i+1} - \tau_i$ is smaller in distribution than $\mu(2^i 3)$. We have

$$ET^{p/2} \le E\eta^{p/2}$$

$$\leq \sum_{i=-\infty}^{\infty} E(\tau_{i+1}-\tau_i)^p \, I(\eta > \tau_i)$$

$$\leq \sum_{i=-\infty}^{\infty} E\, \mu(2^i 3\ )^p \, P(\eta > \tau_i)$$

$$= \sum_{i=-\infty}^{\infty} \alpha_p (2^i 3\ )^p \, P(Z(\eta)^* > 2^i)$$

$$= \frac{\alpha p 3^p}{1-2^p} E(Z(\eta)^*)^p$$

$$\leq \frac{\alpha_p 3^p 2^p}{1-2^p} E(Z(T)^*)^p.$$

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