A Limit Theorem for Point Processes with Applications*

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1. INTRODUCTION. Consider a population which reproduces according to a Bellman-Harris age-dependent branching process (B.H.P.) (see Harris [2]). Let $G(\cdot)$ be the common distribution function (D.F.) of the random length of life of an object, which produces j progeny with probability p_j $(j=0,1,2,\ldots)$ at the end of its life. The objects are assumed to develop independent of each other. Again, it is assumed that the population is being augmented by an independent immigration process defined below, where each immigrant, independent of others, generate a B.H.P. The immigration epochs occur in time according to a renewal process with D.F. of inter-immigration times given by $H(\cdot)$. Also, at each immigration epoch, j immigrants enter the population with probability h_j $(j=0,1,2,\ldots)$. These immigrant numbers are independent of each other and of everything else. Let for $|s| \leq 1$,

$$(1) \qquad f(s) = \sum_{j=0}^{\infty} p_j s^j \quad ; \quad h(s) = \sum_{j=0}^{\infty} h_j s^j,$$

and

$$(2) \qquad \alpha = f'(1-),\ \beta = h'(1-),\ \lambda_0 = \int_0^{\infty} t\,dH(t),\ \lambda_1 = \int_0^{\infty} t\,dG(t).$$

It is assumed that $p_0, h_0 < 1$ and $G(0+) = H(0+) = 0$. Let $X(t)$ denote the population size at time t. The process $\{X(t)\}$ as defined above will be called Bellman-Harris process with immigration (B.H.I.). We assume that $X(0) = 0$.

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Again let $Y(t)$ denote the population size at time t in a B.H.P. governed by $f(\cdot)$ and $G(\cdot)$, with $Y(0)=1$. Let $F(s,t)$ and $\Phi(s,t)$ denote the probability generating functions (p.g.f.) of $Y(t)$ and $X(t)$ respectively. Recently Pakes and Kaplan [14] (see also Kaplan and Pakes [10]) have considered the problem of existence of a limiting distribution of $X(t)$ as $t\to\infty$. In particular, among other results, they proved the following theorem.

THEOREM 1. (Pakes and Kaplan) *If $\lambda_0<\infty$ and $H(\cdot)$ is nonlattice, $X(t)$ has a limiting distribution iff*

$$(3) \qquad \int_0^\infty [1-h(F(0,t))]dt < \infty.$$

More specifically, if (3) holds then as $t\to\infty$, $\Phi(s,t)\to\Phi(s)$ $(0\le s<1)$, a p.g.f. given by

$$(4) \qquad \Phi(s) = 1 - \frac{1}{\lambda_0} \int_0^\infty \Phi(s,t)[1-h(F(s,t))]dt,$$

and if (3) fails to hold then $X(t) \overset{P}{\to}\infty$.

Jager [3], several years earlier, proved similar results under somewhat restrictive assumptions. Recently there has been a considerable interest shown in literature in processes with immigration. Kaplan [8] has considered a p-dimensional analog of the above process originally considered by Jager [3], and has generalized the above theorem. In the other direction Kaplan [9] has proved some results while specialising the above process to a GI/G/∞ queue. Again, Pakes [11] and Pakes and Parthasarthy [15] have considered Bellman-Harris processes with immigration, where the immigrations are assumed to occur not at the epoch of a renewal process, but instead at those of a nonhomogeneous Poisson process. For this case, in [11], Pakes also considers the problem of existence of a limit distribution of the process, somewhat analogous to the above theorem. Earlier, in a context relevant to a biological problem, Puri [16] considered and proved certain results in the case of a time homogeneous Markov branching processes with immigration also governed by a nonhomogeneous Poisson process. In the discrete time case, for Galton-Watson processes with immigration, the reader may refer to similar results and problems to Heathcote [4], [5], followed by Pakes [12], [13] and Kaplan [6], [7] (see these papers for other references). However, in the present paper, we shall, for convenience, restrict ourselves only to the continuous time case. The discrete time case can be dealt with in an analogous manner.

The present work was in part motivated by the results such as the one given in the above theorem, where, whenever the limit distribution exists, its p.g.f. $\Phi(s)$ unfortunately is given only by an implicit equation (4), which itself in turn needs $\Phi(s,t)$. As we shall see later, the solution $\phi(s)$ in fact has an interesting structure and can be explicitly written down in a rather curious but interesting form. (See section 6.0 and in particular equation (55)). The proof of Theorem 1 as given by Pakes and Kaplan [14] follows in part the lines of proof adopted earlier by Jager [3], and is based on setting up a renewal equation satisfied by $\Phi(s,t)$ and then using the key renewal theorem. By contrast, our proof is quite different. Also we shall prove in the next section a fundamental limit theorem 2, valid for a large class of point processes, which in particular includes, among others, renewal, nonhomogeneous Poisson, and stationary point processes. The results similar to the above theorem, and also related to many other characteristics of these processes (see for example section 6.1), follow then by a simple application of this theorem.

These applications are carried through in several contexts in sections 3, 5, 6 and 7. Also, section 4.1 deals with the question of positivity of the limit (11) of theorem 2. Section 4.2, gives an approximation to the limit (11), whenever it is positive, for a commonly occurring case. Also in Section 6.1, we study certain characteristics of a $GI/G/\infty$ queue, apparently not studied before.

2. A THEOREM ON POINT PROCESSES.

Let $N(t)$ be a point process representing the number of 'some' events occurring during $[0,t]$ and defined on some underlying probability space. It is assumed that almost every sample path of $N(t)$ is continuous from the right. Let $0 \leq T_1 \leq T_2 \leq \cdots$ be the epochs of occurrences of the events, so that

(5) $$N(t) = \sup \{k: \; T_k \leq t\}.$$

Let $U(t) = EN(t)$. We list below some of the conditions that occassionally we shall impose on the process $N(t)$.

(A_1) For every finite t, $U(t) < \infty$.

(A_2) $N(t) \xrightarrow{P} \infty$, as $t \to \infty$.

(A_3) There exist positive constants δ_1, δ_2 independent of t and a, such that

(6) $$U(t) - U(t-a) \leq \delta_1 a + \delta_2, \; \forall t \geq a \geq 0.$$

(A_4) For every $k \geq 1$, as $t \to \infty$,

(7) $$(t-T_{N(t)}, \; t-T_{N(t)-1}, \ldots, t-T_{N(t)-k+1}) \xrightarrow{\mathcal{L}} (\xi_1, \; \xi_2, \ldots, \xi_k),$$

where the distribution of $\xi_k = (\xi_1, \xi_2, \ldots, \xi_k)$ is necessarily such that $P(0 \leq \xi_1 \leq \xi_2 \leq \cdots \leq \xi_k) = 1$.

Note that the condition (A_3) easily implies the following condition needed later.

(8) $$U(t) - U(t-a-) \leq \delta_1 a + \delta_2, \ \forall\ t > a \geq 0.$$

REMARK 1 In condition (A_4), the random variables (r.v.) $\xi_1, \xi_2, \ldots,$ are not necessarily defined on the same probability space as N(t) and hence their introduction may appear an abuse of notation. Nevertheless we shall find them convenient even though we shall mostly be using them for their joint distribution, with the only exception, when $\xi_1, \xi_2 - \xi_1, \xi_3 - \xi_2, \ldots$ are mutually independent, the case where we see no problem (see section 4.1). Besides, introduction of such random variables even for convergence in law as in the above case, is not an uncommon practice in probability literature. Also, even though condition (A_1) is implied by (A_3) (assuming that $U(0) < \infty$), nevertheless it is stated here separately, for there will be occassions where only (A_1) suffices without the need of **the stronger** condition (A_3).

Again we shall talk of an arbitrary function R(t) defined for $t \geq 0$, for which we let

(9) $$R_*(t) = \inf_{u \geq t} R(u), \quad t \geq 0,$$

and for every finite $\tau \geq 0$,

(10) $$R^*_\tau(t) = \begin{cases} \sup\limits_{\tau \leq u \leq t} R(u), & t \geq \tau \geq 0 \\ R(t) & \tau \geq t \geq 0. \end{cases}$$

We shall write simply $R^*(t)$ for $R^*_\tau(t)$, whenever $\tau = 0$. Also we shall occasionally impose the following conditions on the function R(t).

(B_1) $0 \leq R(t) \leq 1, \quad \forall\ 0 \leq t < \infty.$

(B_2) $\lim_{t \to \infty} R(t) = 1.$

(B_3) For every $k \geq 1$, the set S_k of the discontinuity points of the function $\Psi_k(u_1,\ldots,u_k) = \prod_{i=1}^{k} R(u_i)$, defined for $u_i \geq 0$, $i=1,2,\ldots,k$, satisfies $P_k(S_k) = 0$, where $P_k(\cdot)$ is the probability measure associated with the random vector $\xi_k = (\xi_1,\ldots,\xi_k)$ of (A_4).

(B_4) The integral $\int_0^\infty u\,dR_*(u) = \int_0^\infty (1-R_*(u))du$, is finite.

It can be easily shown that for any R satisfying (B_1) and (B_2), both R_* and R^* also satisfy (B_1) and (B_2). Furthermore, both R_* and R^* are <u>nondecreasing</u> with R_* continuous from the left and R^* continuous from the right, while satisfying $R^*(t) \geq R(t)$ and $R_*(t) \leq R(t)$, $\forall\ t \geq 0$. In fact for an R satisfying (B_1), R_* is the maximal function of the class C_* of functions defined over $[0,\infty)$ and given by

$$C_* = \left\{ r(\cdot): \begin{array}{l} r \text{ nondecreasing, satisfies } (B_1) \text{ and} \\ r(t) \leq R(t),\ \forall\ t \geq 0. \end{array} \right\}.$$

Similarly R^* is the minimal function of the class C^* defined by

$$C^* = \left\{ r(\cdot): \begin{array}{l} r \text{ nondecreasing, satisfies } (B_1) \text{ and} \\ r(t) \geq R(t),\ \forall\ t \geq 0. \end{array} \right\}.$$

Similar remarks hold for the function $R_\tau^*(t)$, for every $\tau > 0$ and for $t \geq \tau$. Again, we shall consider the products of the form $\prod_{i=1}^{N(t)} R(t-T_i)$, which by convention will always be defined as unity, whenever $N(t) = 0$. We now state our fundamental result.

<u>THEOREM</u> 2. <u>For a point process</u> $N(t)$ <u>satisfying conditions</u> (A_1) <u>to</u> (A_4)

<u>and an arbitrary function</u> $R(t)$ <u>satisfying conditions</u> (B_1) <u>to</u> (B_4), <u>we have</u>

$$(11) \qquad \lim_{t\to\infty} E\{ \prod_{i=1}^{N(t)} R(t-T_i) \} = E\{ \prod_{i=1}^{\infty} R(\xi_i) \},$$

<u>where the right side is defined as</u> $\lim_{n\to\infty} E\{ \prod_{i=1}^{n} R(\xi_i) \}$.

The proof is accomplished by lemmas 2 and 3, given below. But first we need to prove the following lemma.

<u>LEMMA</u> 1. <u>Under the conditions</u> (A_1) <u>and</u> (A_3) <u>imposed on the process</u> $N(t)$, <u>and conditions</u> (B_1), (B_2) <u>and</u> (B_4) <u>on the function</u> R, <u>for every</u> $\epsilon > 0$, $\exists\ t_0 > 0$, <u>such that</u>

$$(12) \qquad E\{ \prod_{i=1}^{N(t-t_0)} R(t-T_i) \} \geq 1-\epsilon, \quad \forall\ t > t_0.$$

<u>Proof</u>. Since

$$(13) \qquad E\{ \prod_{i=1}^{N(t-t_0)} R(t-T_i) \} \geq E\{ \prod_{i=1}^{N(t-t_0)} R_*(t-T_i) \},$$

We shall show the existence of a $t_0 > 0$, such that the right side of (13) is no less than $1-\epsilon$ for all $t > t_0$. Using (A_3), it can be easily shown that $U(t)/t$ remains bounded as $t\to\infty$. And because of (B_4), since $t \log R_*(t)$ tends to zero as $t\to\infty$, it follows that $U(t) \log R_*(t)$ tends to zero as $t\to\infty$. Thus, using (B_4) and the fact that $R_*(t)\nearrow 1$ as $t\to\infty$, it is possible to choose a $t_0 > 0$, such that

(i) $2\delta_1 \int_{t_0}^{\infty} u\,dR_*(u) \leq \epsilon/4$ (ii) $R_*(t_0) \geq \max\,(1/2,\ 1-\frac{\epsilon}{8\delta_2})$,

and (iii) $|U(t) \log R_*(t)| \leq \epsilon/2$, $\forall\ t > t_0$. We shall show that such a t_0

works. For $t > t_0$, the following representations can be easily justified.

$$(14)\qquad \prod_{i=1}^{N(t-t_0)} R_*(t-T_i) = \exp[\sum_{i=1}^{N(t-t_0)} \log R_*(t-T_i)]$$

$$= \exp[-\sum_{i=1}^{N(t-t_0)} \int_{[t-T_i,\infty)} (R_*(u))^{-1} dR_*(u)]$$

$$= \exp[-\int_{[0,\infty)} \{\sum_{i=1}^{N(t-t_0)} I_{[u \geq t-T_i]}\}(R_*(u))^{-1} dR_*(u)]$$

$$= \exp[-\int_{[t_0,t)} \{N(t-t_0) - N(t-u-)\}(R_*(u))^{-1} dR_*(u)$$

$$+ N(t-t_0) \log R_*(t)].$$

Here $I_{[A]}$ denotes the indicator function of the set A. Thus using Jenson inequality, it follows that $\forall t > t_0$,

$$(15)\qquad E\left\{\prod_{i=1}^{N(t-t_0)} R_*(t-T_i)\right\} \geq \exp[-\int_{[t_0,t)} \{U(t-t_0)-U(t-u-)\}(R_*(u))^{-1} dR_*(u)$$

$$+ U(t-t_0) \log R_*(t)]$$

$$\geq \exp[-2\delta_1 \int_{[t_0,t)} (u-t_0) dR_*(u) - 2\delta_2(1-R_*(t_0)) - \epsilon/2]$$

$$\geq \exp[-2\delta_1 \int_{[t_0,t)} u\, dR_*(u) - \epsilon/4 - \epsilon/2]$$

$$\geq \exp(-\epsilon) \geq 1 - \epsilon,$$

where the second inequality follows from condition (A_3) (in particular from (8)), (ii), and (iii); third inequality from (ii) and fourth from (i). □

LEMMA 2. Under the conditions (A_1), (A_3), (A_4) and (B_1) to (B_4), we have

(16) $$\liminf_{t\to\infty} E\left\{\prod_{i=1}^{N(t)} R(t-T_i)\right\} \geq E\left\{\prod_{i=1}^{\infty} R(\xi_i)\right\}.$$

PROOF. Let $\epsilon_1^2 = \epsilon$ be an arbitrary positive number. Then choose $t_0 > 0$ as in lemma 1, such that (12) holds. Let $t > t_0$, and

(17) $$\eta(t) = \prod_{i=1}^{N(t-t_0)} R(t-T_i),$$

then we may write

(18) $$\prod_{i=1}^{N(t)} R(t-T_i) = \eta(t)\left\{\prod_{i=1}^{N(t)-N(t-t_0)} R(t-T_{N(t)-i+1})\right\}.$$

Again from Markov inequality we have

(19) $$P(1- \eta(t) \leq \epsilon_1) \geq 1 - \frac{1-E\eta(t)}{\epsilon_1}.$$

From this and (12), it follows that

(20) $$P(\eta(t) < 1 -\epsilon_1) \leq \epsilon_1.$$

Thus from (18),we have

(21) $$\begin{aligned} E\left\{\prod_{i=1}^{N(t)} R(t-T_i)\right\} &\geq (1-\epsilon_1)E\left\{I_{[\eta(t) \geq 1 - \epsilon_1]} \prod_{i=1}^{N(t)-N(t-t_0)} R(t-T_{N(t)-i+1})\right\} \\ &\geq (1-\epsilon_1)E\left\{\prod_{i=1}^{N(t)-N(t-t_0)} R(t-T_{N(t)-i+1})\right\} -P(\eta(t) < 1 - \epsilon_1) \\ &\geq (1-\epsilon_1)E\left\{I_{[N(t)-N(t-t_0) < k]} \prod_{i=1}^{N(t)-N(t-t_0)} R(t-T_{N(t)-i+1})\right\}-\epsilon_1 \\ &\geq (1-\epsilon_1)E\left\{\prod_{i=1}^{k} R(t-T_{N(t)-i+1})\right\}-(1-\epsilon_1)P(N(t)-N(t-t_0)\geq k)-\epsilon_1, \end{aligned}$$

where the above is valid $\forall k \geq 1$. Letting $t\to\infty$, in view of conditions (A_4) and (B_3) and on using a weak convergence theorem, we have

$$(22) \qquad \liminf_{t\to\infty} E\left\{\prod_{i=1}^{N(t)} R(t-T_i)\right\} \geq (1-\epsilon_1)E\left\{\prod_{i=1}^{k} R(\xi_i)\right\} - \frac{1}{k}(1-\epsilon_1)(\delta_1 t_0+\delta_2) - \epsilon_1.$$

Here we have also used the fact that by Markov inequality and (A_3),

$$(23) \qquad P(N(t) - N(t-t_0) \geq k) \leq \frac{U(t) - U(t-t_0)}{k} \leq \frac{\delta_1 t_0+\delta_2}{k}.$$

Finally, letting $k\to\infty$ in (22) and since ϵ_1 is arbitrary, (16) follows. □

LEMMA 3. Under the conditions (A_2), (A_4), (B_1) and (B_3), we have

$$(24) \qquad \limsup_{t\to\infty} E\left\{\prod_{i=1}^{N(t)} R(t-T_i)\right\} \leq E\left\{\prod_{i=1}^{\infty} R(\xi_i)\right\}.$$

PROOF. It is easily seen using (B_1) that

$$(25) \qquad E\left\{\prod_{i=1}^{N(t)} R(t-T_i)\right\} = E\left\{(I_{[N(t)>k]} + I_{[N(t)\leq k]})\cdot \prod_{i=1}^{N(t)} R(t-T_{N(t)-i+1})\right\}$$
$$\leq E\left\{\prod_{i=1}^{k} R(t-T_{N(t)-i+1})\right\} + P(N(t)\leq k),$$

valid $\forall k \geq 1$. Letting first $t\to\infty$ and then $k\to\infty$, (24) now easily follows in view of (A_2), (A_4) and (B_3). □

REMARK 2. The limit result (11) of theorem 2 lends itself to the following interesting interpretation. Note that starting from t (t being infinitely large) and moving backward in time (so that taking t as the origin), ξ_1 is the point in time when the immediately preceding event occurs, ξ_2 is the point when next to the preceding event occurs, and so on. This is essentially looking at the process in the reverse direction beginning from t but after an infinitely large t. Now in an infinite server queue (see Kaplan [9]), if R(u) represents the probability that an arriving customer completes his service within a length u of time after arrival, and if the customers are independently being served, then $E\left\{\prod_{i=1}^{N(t)} R(t-T_i)\right\}$ is just the probability that at time t, no customer is getting served. On the other hand its limit $E\left\{\prod_{i=1}^{\infty} R(\xi_i)\right\}$ is also the probability that no customer is getting served at an infinitely large time t, while using however the backward "reverse process" based on the sequence $\{\xi_i\}$.

3. AN APPLICATION TO STATIONARY PROCESSES.

We consider briefly in this section, the case where N(t) is a stationary process with N(0) = 0, so that for every $k \geq 1$, and $0 \leq t_1 < t_2 < \ldots < t_k < \infty$, the joint distribution of $N(t_2) - N(t_1), \ldots, N(t_k) - N(t_{k-1})$ is the same as that of $N(t_2 - t_1) - N(0), \ldots, N(t_k - t_1) - N(t_{k-1} - t_1)$. We assume that $N(t) \xrightarrow{P} \infty$, and that $U(1) = EN(1) < \infty$. Evidently $U(t) = tU(1)$, so that (A_1) is satisfied. Furthermore, because of stationarity $U(t) - U(t-a) = U(a) = aU(1)$, $t \geq a \geq 0$, so that condition (A_3) is also satisfied. Finally, the event $[t - T_{N(t)} > u]$ is the same as $[N(t) - N(t-u) = 0]$, so that due to stationarity we have

$$P(t - T_{N(t)} > u) = P(N(u) - N(0) = 0),$$

which is independent of t. Similarly for $0 \leq u_1 < u_2$,

$$[t-T_{N(t)} > u_1,\ t-T_{N(t)-1} > u_2]$$

$$= [N(t)-N(t-u_2)=0] \cup [N(t)-N(t-u_1) = 0,\ N(t-u_1)-N(t-u_2)=1],$$

so that from stationarity it follows that

$$P(t-T_{N(t)} > u_1,\ t-T_{N(t)-1} > u_2)$$
$$= P(N(t)-N(t-u_2) = 0) + P(N(t)-N(t-u_1) = 0,\ N(t-u_1)-N(t-u_2) = 1)$$

$$= P(N(u_2)-N(0) = 0) + P(N(u_2)-N(u_2-u_1) = 0,\ N(u_2-u_1)-N(0) = 1),$$

which is again independent of t. Similar but somewhat more involved argument shows that for every $k \geq 1$, the joint distribution of $(t-T_{N(t)},\ldots, t-T_{N(t)-k+1})$ is independent of t, so that the condition (A_4) is trivially satisfied. Thus, since conditions (A_1) to (A_4) are all satisfied, the results of section 2 hold for all stationary point processes with $U(1)<\infty$ and satisfying the condition (A_2).

4.1 ON THE QUESTION OF POSITIVITY OF THE LIMIT (11):

Having proved theorem 2, it is natural to ask about the conditions under which the limit (11) will be positive. We answer this question in detail first for the commonly occuring case, where in condition (A_4), the process N(t) is such that the nonnegative random variables $\eta_1 = \xi_1$, $\eta_k = \xi_k - \xi_{k-1}$, $k=2,3,\ldots,$ are mutually independent. We prove the following theorem 3 for this case. Later in theorem 4 we treat the more general case briefly.

THEOREM 3. Let $\eta_1 = \xi_1$, $\eta_k = \xi_k - \xi_{k-1}$, $k \geq 2$, be mutually independent random variables with $\xi_k/k \to \mu$, a.s.$(\underset{\sim}{\xi})$, as $k\to\infty$, for some positive μ, possibly a r.v. Then for an R satisfying (B_1), the following hold.

(i) Let $\int_0^\infty (1-R_*(u))du < \infty$. Then $\prod_{i=1}^{\infty} R(\xi_i) > 0$, a.s $(\underset{\sim}{\xi})$, so that $E\left\{\prod_{i=1}^{\infty} R(\xi_i)\right\}$ is positive.

(ii) Let $\int_0^\infty (1-R_\tau^*(u))du = \infty$, for some finite $\tau \geq 0$. Then $\prod_{i=1}^{\infty} R(\xi_i) = 0$, a.s ($\underset{\sim}{\xi}$), so that $E\left\{\prod_{i=1}^{\infty} R(\xi_i)\right\} = 0$.

(iii) Let R be such that either simultaneously the integrals $\int_0^\infty (1-R_*(u))du$ and $\int_0^\infty (1-R_\tau^*(u))du$, (for some $\tau \geq 0$), are both finite, or they are both infinite, then,

$$\int_0^\infty (1-R(u))du = \infty \Leftrightarrow E\left\{\prod_{i=1}^{\infty} R(\xi_i)\right\} = 0.$$

PROOF. Since $\xi_k/k \to \mu$, a.s($\underset{\sim}{\xi}$), as $k\to\infty$, almost for every sample path of $\{\xi_k\}$, $\exists\ k_0$ for an ϵ satisfying $0 < \epsilon < \mu$, such that

(26) $$(\mu-\epsilon)k \leq \xi_k \leq (\mu+\epsilon)k, \quad \forall\ k \geq k_0.$$

Also note that

(27) $$E\left\{\prod_{i=1}^{\infty} R(\xi_i)\right\} = 0 \Leftrightarrow \prod_{i=1}^{\infty} R(\xi_i) = 0, \text{ a.s}(\underset{\sim}{\xi})$$

$$\Leftrightarrow \sum_{i=1}^{\infty} (1-R(\xi_i)) = \infty, \text{ a.s}(\underset{\sim}{\xi}).$$

It is sufficient to prove (i) and (ii), since (iii) follows from these two.

In view of (27), using (26), we have for $k \geq k_0$, and a.s($\underset{\sim}{\xi}$),

(28) $$\sum_{i=k}^{\infty} (1-R(\xi_i)) \leq \sum_{i=k}^{\infty} (1-R_*(i(\mu-\epsilon)))$$

$$\leq \int_{k-1}^{\infty} [1-R_*(x(\mu-\epsilon))]dx \leq \frac{1}{\mu-\epsilon} \int_{(k-1)(\mu-\epsilon)}^{\infty} [1-R_*(u)]du,$$

from which (i) follows using (27). Again choose k_1 large enough such that $(\mu-\epsilon)k_1 \geq \tau$. Then using (26), we have a.s($\underset{\sim}{\xi}$), for $k \geq \max(k_0,k_1)$,

$$(29)\qquad \sum_{i=1}^{\infty} (1-R(\xi_i)) \geq \sum_{i=k}^{\infty} (1-R(\xi_i))$$

$$\geq \sum_{i=k}^{\infty} (1-R_\tau^*(\xi_i))$$

$$\geq \sum_{i=k}^{\infty} [1-R_\tau^*(i(\mu+\epsilon))]$$

$$\geq \int_{k+1}^{\infty} [1-R_\tau^*(x(\mu+\epsilon))]dx = \frac{1}{\mu+\epsilon} \int_{(k+1)(\mu+\epsilon)}^{\infty} (1-R_\tau^*(u))du,$$

from which (ii) follows using (27).

□

It may be remarked that the additional condition imposed on R in theorem 3 (iii), is not so uncommon. For instance, it will hold whenever R is non-decreasing (see also sections 6.0 and 6.1 for other examples). The following corollary immediately follows in part from Theorem 3 and lemma 3. In particular, part (ii) of the corollary follows from condition (B_4) and theorem 3(i).

COROLLARY 1. (i) Under the conditions of lemma 3 and theorem 3,

$$(30)\qquad \lim_{t\to\infty} E\left\{\prod_{i=1}^{N(t)} R(t-T_i)\right\} = 0,$$

whenever $\int_0^{\infty} [1-R_\tau^*(u)]du = \infty$, for some finite $\tau \geq 0$.

(ii) Under the conditions of theorems 2 and 3, we have

$$(31)\qquad \lim_{t\to\infty} E\left\{\prod_{i=1}^{N(t)} R(t-T_i)\right\} = E\left\{\prod_{n=1}^{\infty} R\left(\sum_{k=1}^{n} \eta_k\right)\right\},$$

which is always positive.

Finally the following theorem gives two sufficient conditions for the positivity of the limit (11), in the general case.

THEOREM 4. For an R satisfying condition (B_1), the expression $E\left\{\prod_{i=1}^{\infty} R(\xi_i)\right\}$ is positive provided either (a) $\sum_{i=1}^{\infty} E|\log R(\xi_i)|$ is finite, or (b) $\sum_{i=1}^{\infty} (1-ER(\xi_i))$

is less than one. Thus each of these conditions is sufficient for the limit (11) to be positive.

PROOF. Let (a) hold. Then for every $n \geq 1$, by Jenson inequality, we have

$$E\left\{\prod_{i=1}^{n} R(\xi_i)\right\} \geq \exp\left[\sum_{i=1}^{n} E \log R(\xi_i)\right],$$

so that the result follows by letting $n \to \infty$. For the case when (b) holds, the result follows from the inequality

$$\prod_{i=1}^{n} R(\xi_i) \geq 1 - \sum_{i=1}^{n} (1-R(\xi_i)), \ \forall\, n \geq 1,$$

after taking expectation and letting $n \to \infty$. □

4.2 AN APPROXIMATION TO THE LIMIT (11): For the case where the random variables $\eta_1 = \xi_1$, $\eta_k = \xi_k - \xi_{k-1}$, $k \geq 2$, are mutually independent with distribution functions $H_k(\cdot)$, $k = 1,2,\ldots$, respectively, we give in the following an approximate method of evaluating the expression $M = E\left\{\prod_{i=1}^{\infty} R(\xi_i)\right\}$ corresponding to an R satisfying conditions (B_1), (B_2) and (B_4). This expression in view of theorem 3(i), is of course positive. We define for $u \geq 0$, the functions

(32) $$M_{j,n}(u) = E\left\{\prod_{k=j}^{n} R(u+\sum_{i=j}^{k} \eta_i)\right\},$$

for $j = 1,2,\ldots,n$, and $n = 1,2,\ldots$, and let

(33) $$M_{j,\infty}(u) = \lim_{n\to\infty} M_{j,n}(u).$$

Since $M = M_{1,\infty}(0)$, we can approximate it by $M_{1,n}(0)$ by taking n large enough. On the other hand, it is easy to establish that for $j=1,2,\ldots,n$,

(34) $$M_{j,n}(u) = E\left\{R(u+\eta_j)\ M_{j+1,n}(u+\eta_j)\right\}$$

$$= \int_0^\infty R(u+v)\ M_{j+1,n}(u+v)dH_j(v), \qquad n \geq 2,$$

(35) $$M_{n,n}(u) = E\{R(u+\eta_n)\} = \int_0^\infty R(u+v)dH_n(v), \qquad n \geq 1.$$

Thus for any n, we can at least in principle, solve (34) and (35) recursively backward, to finally yield $M_{1,n}(u)$. For the special case, where η's are identically distributed with $H_k = H$, $k \geq 1$, let for $u \geq 0$,

(36) $$J_n(u) = M_{1,n}(u) \quad ; J(u) = M_{1,\infty}(u).$$

Then, since $M_{j,n}(u) = M_{1,n-j+1}(u)$, (34) and (35) simplify to

(37) $$J_n(u) = \int_0^\infty R(u+v)\ J_{n-1}(u+v)dH(v), \qquad n \geq 1,$$

with $J_0(u) \equiv 1$ These equations can now be solved in a forward recursive manner to yield $J_n(u)$ for any n, to approximate $M = J(0)$. Also, letting $n\to\infty$, it follows that J(u) itself satisfies the integral equation

(38) $$J(u) = \int_0^\infty R(u+v)\ J(u+v)dH(v), \qquad u \geq 0.$$

This equation, but with u = 0, is essentially an analog of equation (4). As before, for an appropriate function R, (38) in general does not lend itself to an easy <u>nonzero</u> solution, if there exists one, subject to $0 \leq J(u) \leq 1$.

<u>REMARK</u> 3. Not unexpectedly, under suitable conditions on R, (38) does lend itself to a nonzero solution, when the common $H(\cdot)$ is an exponential distribution given by

(39) $$H(x) = 1 - \exp(-\gamma x), \qquad x \geq 0,\ \gamma > 0.$$

For this from (38) we have

$$(40) \qquad J(u) = \gamma \exp(\gamma u) \int_u^\infty R(v)J(v)\exp(-\gamma v)dv; \quad u \geq 0.$$

Let R be continuous with $\int_0^\infty (1-R(u))du < \infty$. Then (40) easily yields the differential equation

$$(41) \qquad \frac{dJ(u)}{du} = \gamma J(u)(1-R(u)), \qquad u \geq 0.$$

The right side of (41), being nonnegative, implies that J(u) is nondecreasing. Also, since in our case $J(0) = M_{1,\infty}(0) = M$, is positive, we have $J(u) > 0$, $\forall u \geq 0$. Thus dividing both sides of (41) by J(u), we have

$$(42) \qquad \frac{d\log J(u)}{du} = \gamma(1-R(u)).$$

Let $\lim_{u\to\infty} J(u) = c$, where $0 < c \leq 1$. Then subject to this (42) yields the solution,

$$(43) \qquad J(u) = c \exp[-\gamma \int_u^\infty (1-R(v))dv].$$

In particular, when c = 1, we have

$$(44) \qquad J(0) = E\left\{\prod_{i=1}^\infty R(\xi_i)\right\} = \exp[-\gamma \int_0^\infty (1-R(u))du],$$

which is a familiar formula, known to arise when the immigration process is governed by a Poisson process with parameter γ. Of course the above derivation is intended to be only heuristic in spirit.

5. A COMMONLY ARISING FUNCTION R IN APPLICATIONS:

In applications, as we shall see later, it often happens that the function R is a probability generating function (p.g.f.), so that besides depending on $t \geq 0$, it also depends on a dummy variable $\underset{\sim}{s} = (s_1,\ldots,s_k)$ with $0 \leq s_i \leq 1$, $i=1,2,\ldots,k$. Thus in what we have considered so far, the function R may be regarded as a function $R(\underset{\sim}{s},t)$ of two arguments $\underset{\sim}{s}$ and t, with $\underset{\sim}{s}$ considered fixed and hence suppressed. We shall now invoke the variation in $\underset{\sim}{s}$, subject to $\underset{\sim}{0} \leq \underset{\sim}{s} \leq \underset{\sim}{1}$. The conditions (B_1) to (B_4) could now be visualised for every fixed $\underset{\sim}{0} \leq \underset{\sim}{s} \leq \underset{\sim}{1}$, with $R_*(\underset{\sim}{s},u) = \inf_{t \geq u} R(\underset{\sim}{s},t)$. Thus under the conditions of theorems 2 and 3, the result (31) now becomes

$$\lim_{t\to\infty} E\left\{\prod_{i=1}^{N(t)} R(\underset{\sim}{s},t-T_i)\right\} = E\left\{\prod_{n=1}^{\infty} R(\underset{\sim}{s}, \sum_{i=1}^{n} \eta_i)\right\}, \tag{45}$$

for $\underset{\sim}{0} \leq \underset{\sim}{s} < \underset{\sim}{1}$, where $\underset{\sim}{s} < \underset{\sim}{1}$ means $s_i < 1$, $\forall\ i=1,2,\ldots,k$. When $R(\underset{\sim}{s},t)$ is a p.g.f., the question that we now pose and answer is whether or not the right side of (45) is also a p.g.f. This is done in the next theorem, lemma 4 and corollaries 2 and 3.

THEOREM 5. Let $R(\underset{\sim}{s},t)$ (not necessarily a p.g.f.) be a function defined for $\underset{\sim}{0} \leq \underset{\sim}{s} \leq \underset{\sim}{1}$, $t \geq 0$, nondecreasing in each s_i, $i=1,2,\ldots,k$, and satisfying $0 \leq R(\underset{\sim}{s},t) \leq 1$. Then under the conditions of theorem 3,

$$\lim_{\underset{\sim}{s}\uparrow\underset{\sim}{1}} E\left\{\prod_{n=1}^{\infty} R(\underset{\sim}{s}, \sum_{k=1}^{n} \eta_k)\right\} = E\left\{\prod_{n=1}^{\infty} R(\underset{\sim}{1}, \sum_{k=1}^{n} \eta_k)\right\}, \tag{46}$$

whenever there exists an $\underset{\sim}{0} \leq \underset{\sim}{s}_0 < \underset{\sim}{1}$, such that

$$\int_0^\infty (1-R_*(\underset{\sim}{s}_0,u))du < \infty. \tag{47}$$

PROOF. By monotone convergence theorem it follows that

$$\lim_{\underset{\sim}{s}\to\underset{\sim}{1}-} E\left\{\prod_{n=1}^{\infty} R(\underset{\sim}{s}, \sum_{k=1}^{n} \eta_k)\right\} = E\left\{\lim_{\underset{\sim}{s}\to\underset{\sim}{1}-} \prod_{n=1}^{\infty} R(\underset{\sim}{s}, \sum_{k=1}^{n} \eta_k)\right\}. \tag{48}$$

Note that since (47) implies that $\int_0^\infty (1-R_*(\underset{\sim}{s},u))du < \infty$, $\forall\ \underset{\sim}{s}_0 \leq \underset{\sim}{s} \leq \underset{\sim}{1}$, in view of theorem 3(i), the product $\prod_{i=1}^{n} R(\underset{\sim}{s},\xi_i)$ converges a.s.$(\underset{\sim}{\xi})$, for every fixed $\underset{\sim}{s}$

satisfying $s_0 \leq s \leq 1$, to a positive limit as $n\to\infty$. That

(49) $$\lim_{s\to 1-} \prod_{i=1}^{\infty} R(s,\xi_i) = \prod_{i=1}^{\infty} R(1,\xi_i) \quad , \text{ a.s.}(\xi),$$

and hence the theorem will follow, once we show that for every sequence of values of $\{\xi_i\}$, with the exception of a set of probability zero, the convergence of the product $\prod_{i=1}^{n} R(s,\xi_i)$ as $n\to\infty$, is uniform in s, for $s_0 \leq s \leq 1$. For every such sequence $\{\xi_i\}$, this follows from the uniform convergence as $n\to\infty$ of the series $\sum_{i=1}^{n} (1-R(s,\xi_i))$, shown below. From an argument similar to the one used in (28), it follows that for the given sequence $\{\xi_i\}$ and $0 < \epsilon_1 < \mu$, $\exists\ k_0 \ni$ for $k \geq k_0$, and $s_0 \leq s \leq 1$,

(50) $$\sum_{i=k}^{\infty} (1-R(s,\xi_i)) \leq \frac{1}{\mu-\epsilon_1} \int_{(k-1)(\mu-\epsilon_1)}^{\infty} (1-R_*(s,u))du$$

$$\leq \frac{1}{\mu-\epsilon_1} \int_{(k-1)(\mu-\epsilon_1)}^{\infty} (1-R_*(s_0,u))du.$$

Since in view of (47), by taking k large enough, the right side can be made arbitrarily small, this establishes the desired uniform convergence and hence the theorem. □

<u>REMARK</u> 4. Note that theorem 5 is applicable also to the case where $R(s,t)$ instead is a Laplace Stieltjes transform (L.S.T.) of nonnegative random variables. For this all one needs to do is to replace s_i in $R(s,t)$ by $\exp(-\theta_i)$ with $\theta_i \geq 0$, $i=1,2,\ldots,k$ (see section 6.1, for an example).

We shall need the following lemma, the proof of which is based on an argument due to Pakes (see for instance [14]).

<u>LEMMA</u> 4. <u>Let</u> $R(s,t)$ <u>be a p.g.f. for each</u> t. <u>Then</u> (i) <u>the integral</u> $\int_0^{\infty} (1-R_*(s,t))dt$ <u>is finite for some</u> $0 \leq s < 1$, <u>if and only if it is finite for all</u> $0 \leq s < 1$.

(ii) Furthermore, (i) also holds with R_* replaced by R^*_τ, for every finite $\tau \geq 0$.

(iii) If $R(\underset{\sim}{0},t)$ is a nondecreasing function of t, then the integral $\int_0^\infty (1-R_*(\underset{\sim}{0},t))dt$ is finite (infinite) if and only if the integrals $\int_0^\infty (1-R_*(\underset{\sim}{s},t))dt$ and $\int_0^\infty (1-R^*_\tau(\underset{\sim}{s},t))dt$ for finite $\tau \geq 0$, are all finite (infinite) and hence if and only if the integral $\int_0^\infty (1-R(\underset{\sim}{s},t))dt$, is finite (infinite), $\forall\ \underset{\sim}{0} \leq \underset{\sim}{s} < \underset{\sim}{1}$.

PROOF. Following [14], the proof of (i) follows from the fact that being a p.g.f., $R(\underset{\sim}{s},t)$ satisfies $\forall\ \underset{\sim}{0} \leq \underset{\sim}{s} \leq \underset{\sim}{1}$, and $\forall\ t \geq 0$,

$$R(\underset{\sim}{0},t) \leq R(\underset{\sim}{s},t) \leq s^* + (1-s^*)R(\underset{\sim}{0},t),$$

so that

$$(1-s^*)(1-R_*(\underset{\sim}{0},t)) \leq 1 - R_*(\underset{\sim}{s},t) \leq 1 - R_*(\underset{\sim}{0},t),$$

where for each $\underset{\sim}{s}$, $s^* = \max(s_1,\ldots,s_k)$. A similar argument holds for R^*_τ and hence for (ii). Also (iii) follows from (i) and (ii) and the fact that $R_*(\underset{\sim}{0},t) = R^*_\tau(\underset{\sim}{0},t) = R(\underset{\sim}{0},t)$. ◻

The following corollary now immediately follows.

COROLLARY 2. Let $R(\underset{\sim}{s},t)$ be a p.g.f. for each $t \geq 0$. Then under the conditions of theorems 2 and 3 with (B_4) replaced by (47), the limit (45) exists and is a valid p.g.f., so that there exists a limit distribution.

Following corollary follows from lemmas 3 and 4 and theorem 3(ii).

COROLLARY 3. Let $R(\underset{\sim}{s},t)$ be a p.g.f. for each $t \geq 0$. Then under the conditions of lemma 3 and theorem 3,

$$\lim_{t\to\infty} E\left\{\prod_{i=1}^{N(t)} R(\underset{\sim}{s},t-T_i)\right\} = 0, \tag{51}$$

holds whenever for some finite $\tau \geq 0$, $\int_0^\infty (1-R_\tau^*(\underset{\sim}{s},u))du = \infty$, for some $\underset{\sim}{0} \leq \underset{\sim}{s} < \underset{\sim}{1}$ and hence $\forall\ \underset{\sim}{0} \leq \underset{\sim}{s} < \underset{\sim}{1}$, so that the random variable $\underset{\sim}{X}(t)$ corresponding to the p.g.f. $E\left\{\prod_{i=1}^{N(t)} R(\underset{\sim}{s},t-T_i)\right\}$ tends in probability to infinity, as $t\to\infty$.

REMARK 5. Let us assume that the limit (45) exists and is a valid p.g.f. Also we introduce a random vector $\underset{\sim}{Z} = (Z_1,\ldots,Z_k)$, where each Z_i is nonnegative integer valued, such that the p.g.f. of $\underset{\sim}{Z}$ is same as the limit (45). Then the form of (45) lends itself to the following representation for $\underset{\sim}{Z}$, which is quite helpful in studying the behavior, computing moments etc., of this limit distribution (see section 6.1). Let for each $t \geq 0$, $\underset{\sim}{Y}(t) = (Y_1(t),\ldots,Y_k(t))$ be a nonnegative integer valued vector r.v., such that its p.g.f. is $R(\underset{\sim}{s},t)$. Then from (45) it follows that $\underset{\sim}{Z}$ has the same distribution as that of the random vector $\underset{\sim}{Q} = (Q_1,\ldots,Q_k)$, where

$$Q_i = \sum_{n=1}^{\infty} Y_i^{(n)}\left(\sum_{\ell=1}^{n} \eta_\ell\right), \quad i=1,2,\ldots,k, \tag{52}$$

and conditionally given the random sequence $\{\eta_i\}$, the vectors

$$\underset{\sim}{Y}^{(n)}\left(\sum_{\ell=1}^{n}\eta_\ell\right) = \left(Y_1^{(n)}\left(\sum_{\ell=1}^{n}\eta_\ell\right),\ldots,Y_k^{(n)}\left(\sum_{\ell=1}^{n}\eta_\ell\right)\right), \tag{53}$$

for $n = 1,2,\ldots$, are mutually independent and are distributed as $\underset{\sim}{Y}(t)$, with argument t replaced by $\sum_{\ell=1}^{n} \eta_\ell$, for each n. Thus to construct $\underset{\sim}{Q}$, we first observe the sequence of independent random variables $\{\eta_i\}$. Given these we then observe the sequence $\underset{\sim}{Y}^{(n)}\left(\sum_{\ell=1}^{n}\eta_\ell\right)$, $n=1,2,\ldots$.

2.2. SPECIALIZATION TO RENEWAL PROCESSES.

We now consider the case discussed in the introduction, where $N(t)$ is a renewal process with the common distribution of (T_i-T_{i-1}), $i=2,3,\ldots$, given by

$H(\cdot)$ and that of T_1 possibly different. The following theorem is more or less known.

THEOREM 6. Let H be a nonlattice distribution with $H(0) < 1$ and $\lambda_0 = \int_0^\infty t\, dH(t) < \infty$. Then the renewal process $N(t)$ satisfies the conditions (A_1) to (A_4). Furthermore in (A_4), the random variables $\eta_1 = \xi_1$, $\eta_i = \xi_i - \xi_{i-1}$, $i \geq 2$, are all mutually independent. The distribution of η_1 is given by the probability density $\frac{1}{\lambda_0}(1-H(x))$, for $x \geq 0$, whereas η_i, $i \geq 2$, all have the same distribution function H.

OUTLINE OF PROOF. That $N(t)$ satisfies (A_1) and (A_2), is well known. Also it satisfies (A_3) with $\delta_1 = (cd)^{-1}$ and $\delta_2 = d^{-1}$, where c and d are two positive constants chosen so that $1-H(c) > d$, which is always possible since $H(0) < 1$ (see Feller [1]). That it satisfies (A_4) in the manner stated in the theorem should also be a known result. It is well known for $k = 1$, however for any $k > 1$, it must be burried somewhere in literature, although the author failed to find it. Nevertheless, it can be easily proven in a rather standard manner, using the key renewal theorem. □

Thus under the conditions on H as stated in theorem 6, all the results of Sections 4 and 5 hold for the renewal processes. In particular, since $\xi_1 = \eta_1$ and hence $\xi_k = \sum_{i=1}^{k} \eta_i$, $k \geq 2$, all admit probability densities, the condition (B_3) for the commonly arising R is automatically satisfied. Also, theorem 1 of Pakes and Kaplan [14] as stated in the introuduction and a generalised version due to Kaplan [8] for the case of multidimensional age-dependent branching processes, now easily follow from our corollaries 2 and 3 and lemma 4(iii), while taking

$$R(\underset{\sim}{s},t) = h(F(\underset{\sim}{s},t)). \tag{54}$$

However, our approach goes further in giving an explicit solution for the limiting p.g.f. $\Phi(\underset{\sim}{s})$ which satisfies equation of the type (4). For instance, in the one dimensional case we have the solution of (4) given by

$$(55) \qquad \Phi(s) = E\left\{\prod_{k=1}^{\infty} h(F(s, \sum_{i=1}^{k} \eta_i))\right\},$$

where $\eta_1, \eta_2, \ldots$, are as defined in theorem 6. Also, for practical purposes, this could be approximated by

$$(56) \qquad E\left\{\prod_{k=1}^{n} h(F(s, \sum_{i=1}^{k} \eta_i))\right\}$$

for large enough n, for the cases where (56) may be more convenient than (55) for computation (see also section 4.2).

6.1. AN EXAMPLE OF GI/G/∞ QUEUE.

Consider a GI/G/∞ queue (also sometimes called as Type II counter), where customers arrive at the epochs of a renewal process N(t), and are served immediately upon arrival by one of an infinite number of servers. Let the common distribution function of the interarrival times $T_i - T_{i-1}$, $i=2,3,\ldots$, be given by $H(\cdot)$ and that of T_1 may be different. The service times for various customers are I.I.D. with common distribution function G, and are also independent of the arrival process N(t). Let X(t) denote the number of customers in service at time t. Let $L_1(t)$ and $L_2(t)$ denote respectively the cumulative length of time of service received until time t, and the residual cumulative length of time of service yet to be received, by those in service at time t. The purpose of this section is to determine the limiting joint distribution of $L_1(t)$, $L_2(t)$ and X(t) as $t\to\infty$. For this we define the transform

$$(57) \qquad V(\theta_1, \theta_2, s; t) = E\left\{s^{X(t)} \exp[-\theta_1 L_1(t) - \theta_2 L_2(t)]\right\},$$

for $\operatorname{Re}(\theta_i) \geq 0$, i=1,2, and $|s| \leq 1$. Then it is easy to establish that

(58) $$V(\theta_1,\theta_2,s;t) = E\left\{\prod_{k=1}^{N(t)} R(\theta_1,\theta_2,s;t-T_{N(t)-k+1})\right\},$$

where

(59) $$R(\theta_1,\theta_2,s;t) = G(t) + s\exp(-\theta_1 t)\int_0^\infty \exp(-\theta_2 u)dG(u+t).$$

We now have the following theorem.

THEOREM 7. Let H be a nonlattice distribution with $H(0) < 1$ and $\lambda_0 = \int_0^\infty u\, dH(u) < \infty$. Then for $\operatorname{Re}(\theta_i) \geq 0$, i=1,2, $|s| \leq 1$, with $(\theta_1,\theta_2,s) \neq (0,0,1)$,

(60) $$\lim_{t\to\infty} V(\theta_1,\theta_2,s;t) = V(\theta_1,\theta_2,s)$$

exists. In particular

(61) $$V(\theta_1,\theta_2,s) = E\left\{\prod_{k=1}^{\infty} R(\theta_1,\theta_2,s;\sum_{k=1}^{k}\eta_i)\right\},$$

which is a bonafide transform (of a proper distribution) with $V(0,0,1) = 1$, if and only if

(62) $$\lambda_1 = \int_0^\infty (1-G(t)dt < \infty.$$

If (62) fails to hold, then limit (60) is zero, and $L_1(t)$, $L_2(t)$ and $X(t)$, each converges in probability to ∞, as $t\to\infty$. The η_i's of (61) are as defined in theorem 5.

PROOF. In view of theorem 6, the above theorem follows from corollary 1 and theorem 5, once we show that for $(\theta_1,\theta_2,s) \neq (0,0,1)$, and $\tau > 0$,

(63) $\lambda_1 < \infty \Leftrightarrow \int_0^\infty (1-R_*(t))dt < \infty \Leftrightarrow \int_0^\infty (1-R_\tau^*(t))dt < \infty,$

where we have suppressed the arguments θ_1, θ_2 and s in R_* and R_τ^*, which are defined as in (9) and (10). Here in order to apply theorem 5, we take k=3 and identify s_1, s_2 and s_3 of theorem 5 with $\exp(-\theta_1)$, $\exp(-\theta_2)$ and s respectively (see remark 4, following theorem 5). Consider first the case when $\theta_1 = 0$, but $(\theta_2, s) \neq (0,1)$. Then from (59) we easily have

(64) $R(0,\theta_2,s;u) = G(u)(1-s) + s\theta_2 \int_0^\infty \exp(-\theta_2 v)G(u+v)dv,$

which is a nondecreasing function of u, so that for $\tau \geq 0$,

(65) $R(0,\theta_2,s;t) = R_*(0,\theta_2,s;t) = R_\tau^*(0,\theta_2,s;t).$

On the other hand, using (64) and after some simplifications, it follows that

(66) $\int_0^\infty (1-R(0,\theta_2,s;t))dt = \int_0^\infty [(1-s) + s(1-\exp\{-\theta_2 t\})](1-G(t))dt.$

Since $(\theta_2, s) \neq (0,1)$, the last integral in (66) can easily be shown to be finite if and only if $\int_0^\infty (1-G(t))dt < \infty$. Thus in view of (65), the relations (63) follow. Consider now the case when $\theta_1 \neq 0$. Since from (59) we have

(67) $R(\theta_1,\theta_2,s;t) = G(t)(1-s\exp(-\theta_1 t))$

$$+ s\theta_2 \exp(-\theta_1 t) \int_0^\infty \exp(-\theta_2 v)G(t+v)dv,$$

it follows that for $t \geq 0$,

(68) $$(1-\exp(-\theta_1 t))(1-G(t)) \leq 1 - R(\theta_1,\theta_2,s;t) \leq 1-G(t).$$

Using this we have for $t \geq \tau > 0$,

$$(69) \qquad (1-\exp(-\theta_1\tau))(1-G(t)) \leq 1-R_\tau^*(t) \leq 1-R(t) \leq 1-R_*(t) \leq 1-G(t).$$

Finally, since $\tau > 0$, the relations (63) now easily follow from (69). □

REMARK 6. For the case, where $\lambda_1 = \int_0^\infty (1-G(u))\,du$ is finite and the interarrival time distribution H is exponential given by (39) with a parameter $\gamma > 0$, it can be shown either directly or by using the heuristic argument mentioned earlier (see remark 3, section 4.2) and in particular (44), that the limiting p.g.f. (61) takes the form

$$(70) \qquad V(\theta_1,\theta_2,s) = \exp[-\gamma \int_0^\infty (1-R(\theta_1,\theta_2,s;u))du],$$

which, using (59), further simplifies to

$$(71) \qquad V(\theta_1,\theta_2,s) = \exp[-\gamma \int_0^\infty (1-G(t))(1-g(\theta_1,\theta_2,s;t))dt],$$

where

$$(72) \qquad g(\theta_1,\theta_2,s;t) = \frac{s}{\theta_1-\theta_2}[\theta_1\exp(-\theta_1 t) - \theta_2\exp(-\theta_2 t)].$$

Following the approach suggested in remark 5 at the end of section 5, we shall now indicate briefly how one could find the moments of the limit distribution corresponding to the transform (61), assuming that (62) holds. Let (L_1^*, L_2^*, X^*) be a nonnegative random vector, where X^* is nonnegative integer-valued, such that the transform

$$(73) \qquad V^*(\theta_1,\theta_2,s) \equiv E\{s^{X^*} \exp[-\theta_1 L_1^* - \theta_2 L_2^*]\},$$

with $Re(\theta_i) \geq 0$, i=1,2, and $|s| \leq 1$, coincides with (61). Let for each $t \geq 0$, $(\tilde{L}_1(t), \tilde{L}_2(t), \tilde{X}(t))$ be a nonnegative random vector, with $\tilde{X}(t)$ nonnegative integer valued, such that its transform analogous to (73) is given by $R(\theta_1,\theta_2,s,t)$ of (59). Then it follows from (61) that the (L_1^*, L_2^*, X^*) has the <u>same joint distribution</u> as that of the random vector (S_1,S_2,S_3), where for i=1,2,

(74) $$S_i = \sum_{n=1}^{\infty} \tilde{L}_i^{(n)}(\sum_{\ell=1}^{n} \eta_\ell); \quad S_3 = \sum_{n=1}^{\infty} \tilde{X}^{(n)}(\sum_{\ell=1}^{n} \eta_\ell),$$

and conditionally given the sequence $\{\eta_\ell\}$, the vectors

(75) $$(\tilde{L}_1^{(n)}(\sum_{\ell=1}^{n} \eta_\ell), \tilde{L}_2^{(n)}(\sum_{\ell=1}^{n} \eta_\ell), \tilde{X}^{(n)}(\sum_{\ell=1}^{n} \eta_\ell))$$

for n=1,2,..., are mutually independent and are distributed as $(\tilde{L}_1(t),\tilde{L}_2(t),\tilde{X}(t))$, with the argument t replaced by $\sum_{\ell=1}^{n} \eta_\ell$, for each n. Note that, given the sequence $\{\eta_\ell\}$, we have for each $n \geq 1$, the conditional expectations

(76) $$\begin{aligned} E\{\tilde{L}_1^{(n)}(\sum_{\ell=1}^{n} \eta_\ell)|\{\eta_\ell\}\} &= [1-G(\sum_{\ell=1}^{n} \eta_\ell)](\sum_{\ell=1}^{n} \eta_\ell), \\ E\{\tilde{L}_2^{(n)}(\sum_{\ell=1}^{n} \eta_\ell)|\{\eta_\ell\}\} &= \int_0^{\infty} (t-\sum_{\ell=1}^{n} \eta_\ell)^+ dG(t), \\ E\{\tilde{X}^{(n)}(\sum_{\ell=1}^{n} \eta_\ell)|\{\eta_\ell\}\} &= [1-G(\sum_{\ell=1}^{n} \eta_\ell)], \end{aligned}$$

where $a^+ = \max(0,a)$. Thus from (74), (76) and the fact that the random variables are all nonnegative, we have

(77) $$\begin{aligned} E(L_1^*) &= \sum_{n=1}^{\infty} E\{(\sum_{\ell=1}^{n}\eta_\ell)[1-G(\sum_{\ell=1}^{n}\eta_\ell)]\}, \\ E(L_2^*) &= \sum_{n=1}^{\infty} \int_0^{\infty} E(t-\sum_{\ell=1}^{n}\eta_\ell)^+ \, dG(t), \\ E(X^*) &= \sum_{n=1}^{\infty} E\{1-G(\sum_{\ell=1}^{n}\eta_\ell)\}. \end{aligned}$$

Again, under the conditions of theorem 7 on H and from theorem 6, the random variables $\eta_1, \eta_2, \ldots,$ are mutually independent, with the distribution of η_1, given by the p.d.f. $\frac{1}{\lambda_0}(1-H(x))$ for $x \geq 0$, whereas η_i, $i \geq 2$, all have the same distribution function H. Let $H^{(n)}$ denote the n-fold convolution of H, and for $t \geq 0$,

(78) $$M(t) = \sum_{n=0}^{\infty} H^{(n)}(t),$$

the corresponding renewal function, with $H^{(0)}(t) \equiv 1$. Also, let $F(t) = P(\eta_1 \leq t) = \frac{1}{\lambda_0} \int_0^t (1-H(x))dx$, and

(79) $$M^*(t) = F * M(t),$$

the convolution between F and M. Then it easily follows from (77) that

(80) $$E(L_1^*) = \int_0^{\infty} u(1-G(u))\,dM^*(u),$$

(81) $$E(L_2^*) = \int_0^{\infty} \int_0^{\infty} (t-u)^+ \, dG(t)dM^*(u)$$

$$= \int_0^{\infty} (1-G(u))M^*(u)\, du,$$

and

(82) $$E(X^*) = \int_0^{\infty} (1-G(u))dM^*(u).$$

Of course, these expressions for the means may or may not be finite. Similar expressions can be obtained for higher moments using the above approach.

7. A FEW CONCLUDING REMARKS. One can think of many more interesting situations where the results of this paper are applicable. However, in order not to

overload this paper, we shall only discuss briefly the case of nonhomogeneous Poisson processes.

(a) NON-HOMOGENEOUS POISSON PROCESSES. Let us consider the case where $N(t)$ is governed by a nonhomogeneous Poisson process with intensity function $\gamma(t)$, about which we assume that (i) $\gamma(t)\to\gamma$, as $t\to\infty$, where $\gamma > 0$, and that (ii) the integral $\int_0^t \gamma(u)du$ exists and is finite for every $t>0$. Thus, since $N(t)$ has a Poisson distribution with $U(t) = E\,N(t) = \int_0^t \gamma(u)du$, which tends to infinity as $t\to\infty$, it follows that conditions (A_1) and (A_2) are satisfied. Again using the condition (i), one easily finds that the condition (A_3) is also satisfied. Finally using a standard analysis and the conditions (i) and (ii), one can also show that for every $k \geq 1$,

$$(83)\quad (t-T_{N(t)}, T_{N(t)}-T_{N(t)-1}, \ldots, T_{N(t)-k+2}-T_{N(t)-k+1}) \to (\eta_1, \eta_2, \ldots, \eta_k),$$

as $t\to\infty$, where η_i's are I.I.D. with a common exponential distribution given by (39), so that the condition (A_4) is also satisfied. Thus the results developed here are applicable. In particular, in the limit, this fits into the case discussed earlier in remarks 3 (section 4.2) and 6 (section 6.1), and consequently the results mentioned there also hold in the present case.

(b) Under the conditions of corollary 1(i), when the limit (30) is zero, one may like to study the rate of its convergence. For the same reason, when this limit is zero, one may also like to develop some local limit theorems (as done in special cases by several authors quoted in the introduction) in the present generality for the case, where R is a p.g.f. or a L.S.T.

(c) Again occassionally situations arise, where in

$$E\left\{\prod_{k=1}^{N(t)} R(t-T_{N(t)-k+1})\right\},$$

the function R itself may vary with k. It should be possible to extend the present results to cover such situations as well. These and other related investigations will be reported elsewhere.

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REFERENCES

[1] Feller, W. (1971) An Introduction to Probability Theory and its Applications 2, Second Edition, Wiley, New York.

[2] Harris, T. E. (1963) The Theory of Branching Processes. Springer-Verlag, Berlin.

[3] Jagers, P. (1968) Age-dependent Branching processes allowing Immigration. Theor. Probability Appl. 13, 225-236.

[4] Heathcote, C. R. (1965) A Branching Process allowing Immigration. J. Roy. Statist. Soc. B27, 138-143.

[5] Heathcote, C. R. (1966) Corrections and Comments on the paper,"A Branching Process allowing Immigration." J. Roy. Statist. Soc. B28, 213-217.

[6] Kaplan, N. (1973) The Multitype Galton-Watson process with Immigration, The Ann. Prob. 1, 947-953.

[7] Kaplan, N. (1974) The Supercritical p-dimensional Galton-Watson process with Immigration. Math. Biosci. 22, 1-18.

[8] Kaplan, N. (1974) Multidimensional Age-dependent Branching processes allowing Immigration: The Limiting distribution, J. Appl. Prob. 11, 225-236.

[9] Kaplan, N. (1975) Limit Theorems for a GI/G/∞ Queue, The Ann. Prob. 3, 780-789.

[10] Kaplan, N. and Pakes, A. G. (1974) Supercritical Age-dependent Branching processes with Immigration, Stochastic Processes Appl., 2, 371-389.

[11] Pakes, A. G. (1972). Limit theorems for an Age-dependent Branching process with immigration. Math. Biosci. 14, 221-234.

[12] Pakes, A. G. (1974) On supercritical Galton-Watson processes allowing immigration. J. Appl. Prob. 11, 814-817.

[13] Pakes, A. G. (1975) Some results for nonsupercritical Galton-Watson processes with immigration, Math. Biosci. 24, 71-92.

[14] Pakes, A. G. and Kaplan, N. (1974) On the subcritical Bellman-Harris process with immigration, J. Appl. Prob. 11, 652-668.

[15] Pakes, A. G. and Parthasarathy, P. R. (1975) Some convergence rate results for the Bellman-Harris process with immigration. Math. Biosci. 26, 207-216.

[16] Puri, P. S. (1967) Some limit theorems on Branching processes related to development of biological populations. Math. Biosci. 1, 77-94.