Power Comparisons of Two-Sided Tests of Equality of Two Covariance Matrices Based on Six Criteria

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1. Introduction. Let $\underset{\sim}{X}_1(p\mathrm{x}n_1)$ and $\underset{\sim}{X}_2(p\mathrm{x}n_2)$, $p \leq n_i$, $i = 1,2$, be independent matrix variates, columns of $\underset{\sim}{X}_1$ being independently distributed as $N(\underset{\sim}{0},\underset{\sim}{\Sigma}_1)$ and those of $\underset{\sim}{X}_2$ independently distributed as $N(\underset{\sim}{0},\underset{\sim}{\Sigma}_2)$. Let $0 < c_1 \leq \ldots \leq c_p < \infty$ be the characteristic roots of $|\underset{\sim}{X}_1\underset{\sim}{X}_1' - c\underset{\sim}{X}_2\underset{\sim}{X}_2'| = 0$ and $\lambda_1,\ldots,\lambda_p$, the characteristic roots of $|\Sigma_1 - \lambda\Sigma_2| = 0$. Power studies of tests of $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$ or equivalently $\lambda_1=\ldots=\lambda_p=1$ against the alternative of a one-sided nature:

$$\lambda_i \geq 1, \quad \sum_{i=1}^{p} \lambda_i > p, \qquad i = 1,\ldots,p,$$

were carried out by Pillai and Jayachandran [12] based on the following four criteria:

1) Roy's largest root, c_p, [Roy, 15] or $L_p^{(p)} = c_p/(1+c_p)$,

2) Hotelling's trace, $U^{(p)} = \sum_{i=1}^{p} c_i$, [Pillai, 7],

3) Pillai's trace, $V^{(p)} = \sum_{i=1}^{p} [c_i/(1+c_i)]$, [Pillai, 7],

4) Wilks' criterion, $W^{(p)} = \prod_{i=1}^{p} (1+c_i)^{-1}$, [Wilks, 21].

Exact power tabulations were made in the two-roots case for various (λ_1,λ_2) and different degrees of freedom n_1 and n_2 (actually in terms of $m = (n_1-p-1)/2$ and $n = (n_2-p-1)/2$). Power comparisons were also made of tests of the above

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hypothesis against the one-sided alternative, based on each c_i (equivalently $c_i/(1+c_i)$), $i = 1,\ldots,p$, in the two- and three-roots case [Pillai and Al-Ani, 10]. In this paper, a power comparison study has been attempted of tests of the hypothesis $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$ against $\underset{\sim}{\Sigma}_1 \neq \underset{\sim}{\Sigma}_2$ based on the above four criteria as well as

5) Roy's largest-smallest roots, c_1 $(L_1^{(p)} = c_1/(1+c_1))$ and c_p $(L_p^{(p)} = c_p/(1+c_p))$, to be denoted by $LS^{(p)}$ defined in terms of $L_p^{(p)}$ and $L_1^{(p)}$, [Roy, 16],

6) Modified likelihood ratio (mℓr), $Z^{(p)} = \prod_{i=1}^{p} [c_i^{\frac{1}{2}n_1}/(1+c_i)^{\frac{1}{2}(n_1+n_2)}]$, [Anderson, 1; Bartlett, 2].

Sugiura and Nagao [19] have shown the unbiasedness of the mℓr test $Z^{(p)}$. The distribution of $Z^{(p)}$ for $p = 1$ and 2 and $n_1 = n_2$ in the null case has been available in the literature for sometime [Anderson, 1]. Powers of $Z^{(2)}$ for $n_1 = n_2$ have been tabulated by Pillai and Young [14], after deriving the appropriate non-central distribution. Conditions for the unbiasedness of the largest-smallest roots test $LS^{(p)}$ could be obtained from the treatment of Roy [17]. Since the critical values a and b obtained through the "condition of local unbiasedness" are difficult to compute. Thompson has suggested an approximate approach [20] given by

$$(1.1) \qquad P(a \leq L_1^{(p)}) = 1 - \tfrac{1}{2}\alpha \text{ and } P(a \leq L_1^{(p)} \leq L_p^{(p)} \leq b) = 1 - \alpha.$$

(actually Thompson has suggested the approach for the one sample i.e. Wishart case). This will be called $LS_1^{(p)}$. A variation of $LS_1^{(p)}$, called $LS_2^{(p)}$, is also considered in this paper, which is given by

$$(1.2) \qquad P(L_p^{(p)} \leq b) = 1 - \tfrac{1}{2}\alpha \text{ and } P(a \leq L_1^{(p)} \leq L_p^{(p)} \leq b) = 1-\alpha.$$

Further, Krishnaiah [18] has suggested the following alternate approach:

(1.3) $$P(1-b \leq L_1^{(p)} \leq \ldots \leq L_p^{(p)} \leq b) = 1-\alpha.$$

This will be called $LS_3^{(p)}$. Obviously, $LS_1^{(p)}$, $LS_2^{(p)}$ and $LS_3^{(p)}$ provide biased tests. Similarly the tests 1) to 4) are biased.

In this paper, a theorem is proved first obtaining the condition of local unbiasedness for a class of tests of which 1) to 5) are special cases. Using the theorem, relations between the two critical values for each of the five tests are obtained as special cases for tests 1) to 5) for the two-roots case. Further, critical values for level α = .05 (five percent points) for the five tests are computed for p = 2 and values of m = 0,1,2,5 and n = 5, 10, 15, 20, 25, 30, 40, 60, 80, 100, and are given in Table 1. Also, powers of the criteria 1) to 5) have been tabulated for various values of (λ_1,λ_2), m = 0, 1, 2, 5 and n = 5, 15, 30, 60, and these are presented in Table 4. In addition, power tabulations have also been carried out from the equal tail areas point of view, of tests 1) to 4) which are observed to be biased although the bias is not serious. These tabulations are also available in Table 4 for the same values of (λ_1,λ_2), m and n as before facilitating comparisons with powers in the unbiased case. The critical values in this case are also given in Table 1.

For studying the mℓr test $Z^{(p)}$ and comparing its powers with those of others, the non-central distribution of $Z^{(2)}$ is obtained using zonal polynomials up to the sixth degree for $n_2 = 2n_1$. Tabulations of powers of $Z^{(2)}$ are carried out obtaining the lower five percent points for n_1 = 3, 5, 7, 13 and for comparison those of test 1) to 5) in the unbiased case and 1) to 4) in the equal tail areas case. These are given in Table 5. The critical values are given in Table 2.

In order to compare the largest-smallest roots test with the three approximations as well as the largest root, a separate study has been made and the results on the powers of these five tests are given in Table 6 for selected (λ_1,λ_2) and m = 0, 1, 2, 5 and n = 5, 15, 30 and the critical values in Table 3. The approximations $LS_1^{(2)}$, $LS_2^{(2)}$ and $LS_3^{(2)}$ are all biased and the largest root seems to fare better than all except for the largest-smallest roots for two-sided larger deviations.

A few findings seem to emerge from the numerical results of powers tabulated and in general it is observed that the largest root has some power advantage over the other criteria studied since it is less (very slightly) biased, and except for two-sided larger deviations has better power generally than $Z^{(2)}$ and the largest-smallest roots. These findings are presented in Section 7. In view of this, condition of local unbiasedness for p = 3 has been studied for the largest root and critical values obtained in the unbiased as well as equal tail areas cases which are presented in Table 1. The condition of local unbiasedness has been explicitly obtained in Section 4 for $U^{(3)}$ as well.

2. The condition of local unbiasedness. The acceptance regions based on criteria 1) to 5) with local unbiasedness property and α level of significance can be written in one form:

$$R: \quad a(p,n_1,n_2) \leq w(c_1,\ldots,c_p) \leq b(p,n_1,n_2)$$

where a and b are so chosen as to satisfy

$$(2.1) \quad (i) \quad P(a \leq w(c_1,\ldots,c_p) \leq b \,|\, \lambda_1=\ldots=\lambda_p = 1) = 1-\alpha,$$

$$(2.2)\quad (ii)\quad \left.\frac{\partial P(a \le w(c_1,\ldots,c_p) \le b|\lambda_1,\ldots,\lambda_p)}{\partial\lambda_i}\right|_{\lambda_1=\ldots=\lambda_p=1} = 0,\ i = 1,\ldots,p,$$

where $w(c_1,\ldots,c_p) = c_p/(1+c_p)$ for test 1), $\sum_{i=1}^{p} c_i$ for 2), $\sum_{i=1}^{p} [c_i/(1+c_i)]$ for 3), $\prod_{i=1}^{p} (1+c_i)^{-1}$ for 4) and $c_1/(1+c_1)$, $c_p/(1+c_p)$ for 5).

In this section, we will show that the p equations given in (ii) are really equivalent to one equation and are in turn equivalent to

$$(2.3)\quad (ii')\quad \left.\frac{\partial P(a \le w(c_1,\ldots,c_p) \le b|\lambda_1=\ldots=\lambda_p=\lambda)}{\partial\lambda}\right|_{\lambda=1} = 0.$$

This enables us to compute a and b in a much simpler way for each of the five criteria. We call (ii) or equivalently (ii') "the condition of local unbiasedness".

Theorem 1. The p equations $\left.\frac{\partial P(a \le w(c_1,\ldots,c_p) \le b|\lambda_1,\ldots,\lambda_p)}{\partial\lambda_i}\right|_{\lambda_1=\ldots=\lambda_p=1} = 0,$

$i = 1,\ldots,p$, are equivalent to one equation and are in turn equivalent to $\left.\frac{\partial P(a \le W(c_1,\ldots,c_p) \le b|\lambda_1=\ldots=\lambda_p = \lambda)}{\partial\lambda}\right|_{\lambda=1} = 0.$

Proof. The joint density of $\underset{\sim}{X}_1$, $\underset{\sim}{X}_2$ defined in the Introduction is given by

$$(2\pi)^{-\frac{1}{2}p(n_1+n_2)} |\underset{\sim}{\Sigma}_1|^{-\frac{1}{2}n_1} |\underset{\sim}{\Sigma}_2|^{-\frac{1}{2}n_2} \exp[-\tfrac{1}{2}\mathrm{tr}(\underset{\sim}{\Sigma}_1^{-1}\underset{\sim}{X}_1\underset{\sim}{X}_1' + \underset{\sim}{\Sigma}_2^{-1}\underset{\sim}{X}_2\underset{\sim}{X}_2')].$$

Without loss of generality, we may start directly from the following canonical form:

$$(2\pi)^{-\frac{1}{2}p(n_1+n_2)} \prod_{i=1}^{p} \lambda_i^{-\frac{1}{2}n_1} \exp[-\tfrac{1}{2}\mathrm{tr}(\underset{\sim}{D}_{\lambda_k^{-1}}\underset{\sim}{X}_1\underset{\sim}{X}_1'+\underset{\sim}{X}_2\underset{\sim}{X}_2')]$$

where $\underset{\sim}{D}_{\lambda_k^{-1}} = \mathrm{diag}(1/\lambda_k)$. Then

$$P(a \le w(c_1,\dots,c_p) \le b \mid \lambda_1,\dots,\lambda_p)$$

$$= (2\pi)^{-\frac{1}{2}p(n_1+n_2)} \int_{a\le w(c_1,\dots,c_p)\le b} \prod_{i=1}^{p} \lambda_i^{-\frac{1}{2}n_1} \exp[-\tfrac{1}{2}\mathrm{tr}(D_{\lambda_k^{-1}} X_1X_1' + X_2X_2')]\,dX_1\,dX_2$$

Hence

$$\frac{\partial P(a\le w(c_1,\dots,c_p) \le b \mid \lambda_1,\dots,\lambda_p)}{\partial \lambda_i^{-1}} \tag{2.4}$$

$$= (2\pi)^{-\frac{1}{2}p(n_1+n_2)} \int_{a\le w(c_1,\dots,c_p)\le b} \prod_{j=1}^{p} \lambda_j^{-\frac{1}{2}n_1} [\tfrac{1}{2}n_1\lambda_i - \tfrac{1}{2}(X_1X_1')_{ii}] \exp[-\tfrac{1}{2}\mathrm{tr}(D_{\lambda^{-1}} X_1X_1' + X_2X_2')]$$

$$dX_1\,dX_2.$$

Transform $X_1 = UD_{\sqrt{c_k}}L_1$, $\quad X_2 = UL_2$

where U is non-singular and $L_1L_1' = L_2L_2' = I$ and integrate out over L_{1I} and L_{2I}, the independent elements of L_1 and L_2 respectively [Roy, 17]. Then (2.4) becomes

$$c\int_{R^*} \prod_{j=1}^{p} \lambda_j^{-\frac{1}{2}n_1} [\tfrac{1}{2}n_1\lambda_i - \tfrac{1}{2}(UD_{c_k}U')_{ii}] \exp[-\tfrac{1}{2}\mathrm{tr}(D_{\lambda_k^{-1}} UD_{c_k}U' + UU')] \tag{2.5}$$

$$\times |U|^{n_1+n_2-p}\, dU \prod_{j=1}^{p} c_j^{\frac{1}{2}(n_1-p-1)} \prod_{j>j'} (c_j - c_{j'}) \prod_{j=1}^{p} dc_j$$

where R^* is $a \le w(c_1,\dots,c_p) \le b$, $-\infty <$ all $u_{ij} < \infty$ and c is a positive and constant factor of proportionality. Thus we have

$$\left.\frac{\partial P(a\le w(c_1,\dots,c_p) \le b \mid \lambda_1,\dots,\lambda_p)}{\partial \lambda_i^{-1}}\right|_{\lambda_1=\dots=\lambda_p=1} \tag{2.6}$$

$$= c\int_{R^*} [\tfrac{1}{2}n_1 - \tfrac{1}{2}(UD_{c_k}U')_{ii}] \exp[-\tfrac{1}{2}\mathrm{tr}(UD_{c_k}U' + UU')]\, |U|^{n_1+n_2-p}\, dU$$

$$\times \prod_{j=1}^{p} c_j^{\frac{1}{2}(n_1-p-1)} \prod_{j>j'} (c_j - c_{j'}) \prod_{j=1}^{p} dc_j$$

where R^* is $a \leq w(c_1,\ldots,c_p) \leq b$, $-\infty <$ all $u_{ij} < \infty$. The only term in the above integrand which depends on i is $(\underset{\sim}{U}\underset{\sim}{D}_{c_k}\underset{\sim}{U}')_{ii} = (u_{i1}^2 c_1 + u_{i2}^2 c_2 + \ldots + u_{ip}^2 c_p)$ and the integral is taken over the domain R^*. Therefore, $u_{i1},\ldots,u_{ip}$ are just dummy variables. Thus the integral is invariant under a change of the subscript i. So

$$\left.\frac{\partial P(a \leq w(c_1,\ldots,c_p) \leq b \mid \lambda_1,\ldots,\lambda_p)}{\partial \lambda_i^{-1}}\right|_{\lambda_1=\ldots=\lambda_p=1} \quad \text{is the same for } i = 1,\ldots,p.$$

Hence the p equations are really equivalent to one equation. Now adding up the p formally different looking integrals like (2.6) over i = 1,...,p, we have

$$(2.7) \quad c\int_{R^*} [\tfrac{1}{2}n_1 p - \tfrac{1}{2}\mathrm{tr}(\underset{\sim}{U}\underset{\sim}{D}_{c_k}\underset{\sim}{U}')] \exp[-\tfrac{1}{2}\mathrm{tr}(\underset{\sim}{U}\underset{\sim}{D}_{c_k}\underset{\sim}{U}' + \underset{\sim}{U}\underset{\sim}{U}')] |\underset{\sim}{U}|^{n_1+n_2-p} d\underset{\sim}{U}$$

$$\times \prod_{j=1}^{p} c_j^{\frac{1}{2}(n_1-p-1)} \prod_{j>j'} (c_j - c_{j'}) \prod_{j=1}^{p} dc_j,$$

where R^* is $a \leq w(c_1,\ldots,c_p) \leq b$, $-\infty <$ all $u_{ij} < \infty$. We claim that this is the same as $\left.\dfrac{\partial P(a \leq W(c_1,\ldots,c_p) \leq b \mid \lambda_1=\ldots=\lambda_p=\lambda)}{\partial \lambda^{-1}}\right|_{\lambda=1}$.

In order to see this, consider

$$P(a \leq w(c_1,\ldots,c_p) \leq b \mid \lambda_1=\ldots=\lambda_p=\lambda)$$

$$= (2\pi)^{-\frac{1}{2}p(n_1+n_2)} \int_{a \leq w(c_1,\ldots,c_p) \leq b} \lambda^{-\frac{1}{2}n_1 p} \exp[-\tfrac{1}{2}\mathrm{tr}(\lambda^{-1}\underset{\sim}{X}_1\underset{\sim}{X}_1' + \underset{\sim}{X}_2\underset{\sim}{X}_2')] d\underset{\sim}{X}_1 d\underset{\sim}{X}_2.$$

Then we have

$$\left.\frac{\partial P(a \le w(c_1,\ldots,c_p) \le b \mid \lambda_1=\ldots=\lambda_p=\lambda)}{\partial \lambda^{-1}}\right|_{\lambda=1} \tag{2.8}$$

$$= (2\pi)^{-\frac{1}{2}p(n_1+n_2)} \int_{a \le w(c_1,\ldots,c_p) \le b} [\tfrac{1}{2}n_1 p - \tfrac{1}{2}\mathrm{tr}(\underset{\sim}{X}_1\underset{\sim}{X}_1')] \exp[-\tfrac{1}{2}\mathrm{tr}(\underset{\sim}{X}_1\underset{\sim}{X}_1'+\underset{\sim}{X}_2\underset{\sim}{X}_2')] d\underset{\sim}{X}_1 d\underset{\sim}{X}_2 .$$

Now make the same transformation as before and integrate out over L_{1I} and L_{2I}. Then (2.8) becomes

$$c \int_{R^*} [\tfrac{1}{2}n_1 p - \tfrac{1}{2}\mathrm{tr}(\underset{\sim}{U}\underset{\sim}{D}_{c_k}\underset{\sim}{U}')] \exp[-\tfrac{1}{2}\mathrm{tr}(\underset{\sim}{U}\underset{\sim}{D}_{c_k}\underset{\sim}{U}'+\underset{\sim}{U}\underset{\sim}{U}')] |\underset{\sim}{U}|^{n_1+n_2-p} d\underset{\sim}{U}$$

$$\times \prod_{j=1}^{p} c_j^{\frac{1}{2}(n_1-p-1)} \prod_{j>j'} (c_j - c_{j'}) \prod_{j=1}^{p} dc_j ,$$

where R^* is $a \le w(c_1,\ldots,c_p) \le b$, $-\infty <$ all $u_{ij} < \infty$, which is the same as (2.7). Hence (ii) is equivalent to (ii').

<u>Theorem 2.</u> Condition (ii') can be written as

$$c(p,m,n)\Big[\int_{a \le w(c_1,\ldots,c_p) \le b} \tfrac{1}{2}p(2m+p+1) \prod_{j=1}^{p} c_j^m (1+c_j)^{-(m+n+p+1)} \prod_{j>j'} (c_j - c_{j'}) \prod_{j=1}^{p} dc_j -$$

$$\int_{a \le w(c_1,\ldots,c_p) \le b} (m+n+p+1) [\sum_{j=1}^{p} c_j/(1+c_j)] \prod_{j=1}^{p} c_j^m (1+c_j)^{-(m+n+p+1)} \prod_{j>j'} (c_j - c_{j'}) \prod_{j=1}^{p} dc_j\Big] = 0, \tag{2.9}$$

where $m = (n_1-p-1)/2$, $n = (n_2-p-1)/2$ and

$$c(p,m,n) = \pi^{\frac{1}{2}p} \prod_{i=1}^{p} \Gamma[\tfrac{1}{2}(2m+2n+p+i+2)]/\{\Gamma[\tfrac{1}{2}(2m+i+1)]\Gamma[\tfrac{1}{2}(2n+i+1)]\Gamma(\tfrac{1}{2}i)\}. \tag{2.10}$$

<u>Proof</u>. If $\underset{\sim}{U}$ in Theorem 1 is taken with a positive first row,

$$\left.\frac{\partial P(a \le w(c_1,\ldots,c_p) \le b \mid \lambda_1=\ldots=\lambda_p=\lambda)}{\partial \lambda^{-1}}\right|_{\lambda=1} \tag{2.11}$$

$$= c\int_{R^*} [\tfrac{1}{2}n_1 p - \tfrac{1}{2}\mathrm{tr}(\underset{\sim}{U}\underset{\sim}{D}_{c_k}\underset{\sim}{U}')] \exp[-\tfrac{1}{2}\mathrm{tr}(\underset{\sim}{U}\underset{\sim}{D}_{c_k}\underset{\sim}{U}'+\underset{\sim}{U}\underset{\sim}{U}')] |\underset{\sim}{U}|^{n_1+n_2-p} d\underset{\sim}{U}$$

$$\cdot \prod_{j=1}^{p} c_j^{\frac{1}{2}(n_1-p-1)} \prod_{j>j'} (c_j - c_{j'}) \prod_{j=1}^{p} dc_j .$$

where $c = \dfrac{2^p}{(2\pi)^{\frac{1}{2}p(n_1+n_2)}} \cdot \dfrac{\pi^{\frac{1}{2}pn_1 - \frac{1}{4}p(p-1)}}{\prod_{i=1}^{p} \Gamma[\frac{1}{2}(n_1-i+1)]} \cdot \dfrac{\pi^{\frac{1}{2}pn_2 - \frac{1}{4}p(p-1)}}{\prod_{i=1}^{p} \Gamma[\frac{1}{2}(n_2-i+1)]}$

and R* is $a \le w(c_1,\ldots,c_p) \le b$. $\underset{\sim}{U}$ nonsingular with a positive first row. Now transform $\underset{\sim}{U} = \underset{\sim}{B}\underset{\sim}{D}_{(1+c_k)^{-\frac{1}{2}}}$. Then (2.11) becomes

(2.12) $$c\int_{R^{**}} [\tfrac{1}{2}n_1 p - \tfrac{1}{2}\mathrm{tr}(\underset{\sim}{B}\underset{\sim}{D}_{c_k(1+c_k)^{-1}}\underset{\sim}{B}')] \exp[-\tfrac{1}{2}\mathrm{tr}\underset{\sim}{B}\underset{\sim}{B}'] |\underset{\sim}{B}|^{n_1+n_2-p} d\underset{\sim}{B}$$

$$\times \prod_{j=1}^{p} c_j^{\frac{1}{2}(n_1-p-1)} (1+c_j)^{-\frac{1}{2}(n_1+n_2)} \prod_{j>j'} (c_j - c_{j'}) \prod_{j=1}^{p} dc_j ,$$

where R** is $a \le w(c_1,\ldots,c_p) \le b$. $\underset{\sim}{B}$ nonsingular with a positive first row. For integration with respect to $\underset{\sim}{B}$, use the result [see Roy, 17 Appendix]

$$\int_{\underset{\sim}{B}} \exp[-\tfrac{1}{2}\mathrm{tr}\underset{\sim}{B}\underset{\sim}{B}'] |\underset{\sim}{B}|^q d\underset{\sim}{B} = 2^{\frac{1}{2}p(p+q)-p} \pi^{\frac{1}{2}p^2} \prod_{i=1}^{p} \Gamma[\tfrac{1}{2}(q+p-i+1)]/\Gamma[\tfrac{1}{2}(p-i+1)] .$$

We get

(2.13) $$c\int_{R^{**}} \tfrac{1}{2}n_1 p \exp[-\tfrac{1}{2}\mathrm{tr}\underset{\sim}{B}\underset{\sim}{B}'] |\underset{\sim}{B}|^{n_1+n_2-p} d\underset{\sim}{B} \prod_{j=1}^{p} c_j^{\frac{1}{2}(n_1-p-1)} (1+c_j)^{-\frac{1}{2}(n_1+n_2)} \prod_{j>j'} (c_j - c_{j'}) \prod_{j=1}^{p} dc_j$$

$$= c(p,m,n) \int_{a \le W(c_1,\ldots,c_p) \le b} \tfrac{1}{2}p(2m+p+1) \prod_{j=1}^{p} c_j^{m} (1+c_j)^{-(m+n+p+1)} \prod_{j>j'} (c_j - c_{j'}) \prod_{j=1}^{p} dc_j ,$$

where m, n and c(p,m,n) are given in (2.10). Now consider

(2.14) $$\int_{\underset{\sim}{B}} \tfrac{1}{2}\mathrm{tr}(\underset{\sim}{B}\underset{\sim}{D}_{c_k(1+c_k)^{-1}}\underset{\sim}{B}') \exp[\tfrac{1}{2}\mathrm{tr}\underset{\sim}{B}\underset{\sim}{B}'] |\underset{\sim}{B}|^{n_1+n_2-p} d\underset{\sim}{B}.$$

Transform $\underset{\sim}{B} = \underset{\sim}{T}\underset{\sim}{L}$ where $\underset{\sim}{T}$ is lower triangular, $\underset{\sim}{L}\underset{\sim}{L}' = \underset{\sim}{I}(p)$ and the first row of $\underset{\sim}{L}$ is to be positive. Then (2.14) becomes

$$(2.15)\quad \int_{\underset{\sim}{L}\underset{\sim}{L}'=\underset{\sim}{I}} \int_{\underset{\sim}{T}} \tfrac{1}{2}\mathrm{tr}(\underset{\sim}{T}\underset{\sim}{L}\underset{\sim}{D}_{c_k}(1+c_k)^{-1}\underset{\sim}{L}'\underset{\sim}{T}')\exp[-\tfrac{1}{2}\mathrm{tr}\underset{\sim}{T}\underset{\sim}{T}']\,|\underset{\sim}{T}\underset{\sim}{T}'|^{-\frac{1}{2}(n_1+n_2-p)} \prod_{i=1}^{p} t_{ii}^{p-i}\, d\underset{\sim}{T}\, d\underset{\sim}{L}_I \Big/ \left|\frac{\partial \underset{\sim}{L}\underset{\sim}{L}'}{\partial \underset{\sim}{L}_D}\right|_{\underset{\sim}{L}_I}.$$

Furthermore, transform $\underset{\sim}{S} = \underset{\sim}{T}'\underset{\sim}{T}$, $J(\underset{\sim}{T}:\underset{\sim}{S}) = 2^{-P} \prod_{i=1}^{p} t_{ii}^{-P+i-1}$. Then (2.15) becomes

$$(2.16)\quad 2^{-P}\int_{\underset{\sim}{L}\underset{\sim}{L}'=\underset{\sim}{I}} \int_{\underset{\sim}{S}>0} \tfrac{1}{2}\mathrm{tr}(\underset{\sim}{S}\underset{\sim}{L}\underset{\sim}{D}_{c_k}(1+c_k)^{-1}\underset{\sim}{L}')\exp[-\tfrac{1}{2}\underset{\sim}{S}]\,|\underset{\sim}{S}|^{\frac{1}{2}(n_1+n_2-P-1)} d\underset{\sim}{S}\, d\underset{\sim}{L}_I \Big/ \left|\frac{\partial \underset{\sim}{L}\underset{\sim}{L}'}{\partial \underset{\sim}{L}_D}\right|_{\underset{\sim}{L}_I}.$$

Apply equation (1) of Constantine [3], (2.16) becomes

$$2^{-P}\int_{\underset{\sim}{L}\underset{\sim}{L}'=\underset{\sim}{I}} \tfrac{1}{2}\Gamma_p(\tfrac{1}{2}(n_1+n_2),1)\,\mathrm{tr}(2\underset{\sim}{D}_{c_k}(1+c_k)^{-1})\,(\tfrac{1}{2})^{-\frac{1}{2}P(n_1+n_2)} d\underset{\sim}{L}_I \Big/ \left|\frac{\partial \underset{\sim}{L}\underset{\sim}{L}'}{\partial \underset{\sim}{L}_D}\right|_{\underset{\sim}{L}_I}$$

$$= [2^{-P}\pi^{\frac{1}{2}p^2-\frac{1}{4}p(p-1)} \Big/ \prod_{i=1}^{p}\Gamma[\tfrac{1}{2}(p-i+1)]]\,\tfrac{1}{2}\Gamma_p(\tfrac{1}{2}(n_1+n_2),1)\ \mathrm{tr}(2\underset{\sim}{D}_{c_k}(1+c_k)^{-1})\,(\tfrac{1}{2})^{-\frac{1}{2}p(n_1+n_2)}$$

$$= [2^{\frac{1}{2}p(n_1+n_2)-p}\,\pi^{\frac{1}{2}p^2} \prod_{i=1}^{p}\Gamma[\tfrac{1}{2}(n_1+n_2-i+1)]/\Gamma[\tfrac{1}{2}(p-i+1)]]\,\tfrac{1}{2}(n_1+n_2)\,[\sum_{j=1}^{p} c_j/(1+c_j)].$$

Thus we get

$$(2.17)\quad c\int_{R^{**}} \tfrac{1}{2}\mathrm{tr}(\underset{\sim}{B}\underset{\sim}{D}_{c_k}(1+c_k)^{-1}\underset{\sim}{B}')\exp[-\tfrac{1}{2}\mathrm{tr}\underset{\sim}{B}\underset{\sim}{B}']\,|\underset{\sim}{B}|^{n_1+n_2-p} d\underset{\sim}{B} \prod_{j=1}^{p} c_j^{\frac{1}{2}(n_1-p-1)} (1+c_j)^{-\frac{1}{2}(n_1+n_2)} \prod_{j>j'}(c_j-c_{j'}) \prod_{j=1}^{p} dc_j$$

$$= c(p,m,n)\int_{a\le w(c_1,\ldots,c_p)\le b} (m+n+p+1)[\sum_{j=1}^{p} c_j/(1+c_j)] \prod_{j=1}^{p} c_j^m(1+c_j)^{-(m+n+p+1)} \prod_{j>j'}(c_j-c_{j'}) \prod_{j=1}^{p} dc_j.$$

Therefore, by substracting (2.17) from (2.13), we get

$$\left.\frac{\partial P(a\le w(c_1,\ldots,c_p) \le b \mid \lambda_1=\ldots=\lambda_p=\lambda)}{\partial\lambda^{-1}}\right|_{\lambda=1}$$

$$= c(p,m,n)[\int_{a\le w(c_1,\ldots,c_p)\le b} \tfrac{1}{2}p(2m+p+1) \prod_{j=1}^{p} c_j^m(1+c_j)^{-(m+n+p+1)} \prod_{j>j'}(c_j-c_{j'}) \prod_{j=1}^{p} dc_j -$$

$$\int_{a\le w(c_1,\ldots,c_p)\le b} (m+n+p+1)[\sum_{j=1}^{p} c_j/(1+c_j)] \prod_{j=1}^{p} c_j^m(1+c_j)^{-(m+n+p+1)} \prod_{j>j'}(c_j-c_{j'}) \prod_{j=1}^{p} dc_j].$$

Now equating the above to zero, we get (2.9).

Thus, the acceptance region based on the criterion $w(c_1,\ldots,c_p)$ with local unbiasedness property (ℓup) and α level of significance can be written as

$$R: \quad a(p,m,n) \le w(c_1,\ldots,c_p) \le b(p,m,n)$$

where a and b are so chosen as to satisfy

$$\text{(i)} \quad c(p,m,n) \int_{a\le w(c_1,\ldots,c_p)\le b} \prod_{j=1}^{p} c_j^m (1+c_j)^{-(m+n+p+1)} \prod_{j>j'} (c_j - c_{j'}) \prod_{j=1}^{p} dc_j = 1-\alpha$$

and

$$\text{(ii')} \quad c(p,m,n)\Big[\int_{a\le w(c_1,\ldots,c_p)\le b} \tfrac{1}{2}p(2m+p+1) \prod_{j=1}^{p} c_j^m (1+c_j)^{-(m+n+p+1)} \prod_{j>j'} (c_j - c_{j'}) \prod_{j=1}^{p} dc_j - \int_{a\le w(c_1,\ldots,c_p)\le b} (m+n+p+1)\Big[\sum_{j=1}^{p} c_j/(1+c_j)\Big] \prod_{j=1}^{p} c_j^m (1+c_j)^{-(m+n+p+1)} \prod_{j>j'} (c_j - c_{j'}) \prod_{j=1}^{p} dc_j\Big] = 0.$$

3. <u>The acceptance regions based on the five criteria with ℓup for $p = 2$.</u>

In this section, we will consider the acceptance regions of tests 1) to 5) in that order.

1) <u>Roy's largest root, $L_2^{(2)} = c_2/(1+c_2)$.</u> By using Theorem 2 in the previous section, we know that

$$(3.1) \qquad \left.\frac{\partial P(a\le L_2^{(2)} \le b \mid \lambda_1 = \lambda_2 = \lambda)}{\partial \lambda^{-1}}\right|_{\lambda=1}$$

$$= c(2,m,n)\Big[\int_{a\le L_2^{(2)}\le b} g(c_1,c_2;m,n)\,dc_1 dc_2 - \int_{a\le L_2^{(2)}\le b} h(c_1,c_2;m,n)\,dc_1 dc_2\Big],$$

where $g(c_1,c_2;m,n) = (2m+3)(c_1c_2)^m[(1+c_1)(1+c_2)]^{-(m+n+3)}(c_2-c_1)$ and

$$h(c_1,c_2;m,n) = (m+n+3)[c_1/(1+c_1)+c_2/(1+c_2)](c_1c_2)^m[(1+c_1)(1+c_2)]^{-(m+n+3)}(c_2-c_1).$$

Now transform $\ell_1 = c_1/(1+c_1)$ and $\ell_2 = c_2/(1+c_2)$. Then (3.1) becomes

$$(3.2)\qquad c(2,m,n)\left[\int_a^b \int_0^{\ell_2} g_1(\ell_1,\ell_2;m,n)\,d\ell_1 d\ell_2 - \int_a^b \int_0^{\ell_2} h_1(\ell_1,\ell_2;m,n)\,d\ell_1 d\ell_2\right]$$

where $g_1(\ell_1,\ell_2;m,n) = (2m+3)(\ell_1\ell_2)^m[(1-\ell_1)(1-\ell_2)]^n(\ell_2-\ell_1)$ and

$$h_1(\ell_1,\ell_2;m,n) = (m+n+3)(\ell_1+\ell_2)(\ell_1\ell_2)^m[(1-\ell_1)(1-\ell_2)]^n(\ell_2-\ell_1).$$

Further, note that by making the same transformation,

$$(3.3)\qquad P(a \le L_2^{(2)} \le b|\lambda_1 = \lambda_2 = 1) = c(2,m,n)\int_a^b \int_0^{\ell_2} (\ell_1\ell_2)^m[(1-\ell_1)(1-\ell_2)]^n(\ell_2-\ell_1)\,d\ell_1 d\ell_2.$$

Now using Pillai's reduction formula, [Pillai, 6, 8],

$$(3.4)\qquad \int_0^b \int_0^{\ell_2} (\ell_1\ell_2)^m[(1-\ell_1)(1-\ell_2)]^n(\ell_2-\ell_1)\,d\ell_1 d\ell_2$$

$$= \left[2\int_0^b \ell_2^{2m+1}(1-\ell_2)^{2n+1}d\ell_2 - b^{m+1}(1-b)^{n+1}\int_0^b \ell_1^m(1-\ell_1)^n d\ell_1\right]/(m+n+2),$$

and

$$(3.5)\qquad \int_0^b \int_0^{\ell_2} (\ell_1+\ell_2)(\ell_1\ell_2)^m[(1-\ell_1)(1-\ell_2)]^n(\ell_2-\ell_1)\,d\ell_1 d\ell_2$$

$$= \left[2\int_0^b \ell_2^{2m+2}(1-\ell_2)^{2n+1}d\ell_2 - b^{m+2}(1-b)^{n+1}\int_0^b \ell_1^m(1-\ell_1)^n d\ell_1\right.$$

$$\left. + (m+2)\int_0^b \int_0^{\ell_2} (\ell_1\ell_2)^m[(1-\ell_1)(1-\ell_2)]^n(\ell_2-\ell_1)\,d\ell_1 d\ell_2\right]/(m+n+3),$$

we have proved the following:

Theorem 3. Let $T_1(x) = 2B_x(2m+2,2n+2)-x^{m+1}(1-x)^{n+1}B_x(m+1,n+1)$ and

$T_2(x) = 2B_x(2m+3,2n+2) - x^{m+2}(1-x)^{n+1}B_x(m+1,n+1)$ where $B_x(r,s) = \int_0^x t^{r-1}(1-t)^{s-1}dt$.
Then the acceptance region based on Roy's largest root, $L_2^{(2)} = c_2/(1+c_2)$ with ℓup and α level is given by $a \le L_2^{(2)} \le b$ where a and b are so chosen as to satisfy

(i) $c(2,m,n)[T_1(b)-T_1(a)]/(m+n+2) = 1-\alpha$ and

(ii) $c(2,m,n)\{[T_1(b)-T_1(a)][(m+1)/(m+n+2)] - [T_2(b)-T_2(a)]\} = 0.$

2) Hotelling's trace, $U^{(2)} = c_1+c_2$. From the previous section

(3.6) $$P(a \le U^{(2)} \le b|\lambda_1 = \lambda_2 = \lambda)$$

$$= (2\pi)^{-(n_1+n_2)} \int_{a\le c_1+c_2\le b} \lambda^{-n_1} \exp[-\tfrac{1}{2}tr(\lambda^{-1}X_1X_1'+X_2X_2')]dX_1dX_2.$$

Now transform $\lambda^{-1}X_1 = Y_1$ and $X_2 = Y_2$. $J(Y_1:X_1) = \lambda^{-n_1}$ and let $0 < d_1 \le d_2 < \infty$ be the characteristic roots of $|Y_1Y_1' - dY_2Y_2'| = 0$, Then (3.6) becomes

(3.7) $$c(2,m,n) \int_{a\le\lambda d_1+\lambda d_2\le b} (d_1d_2)^m[(1+d_1)(1+d_2)]^{-(m+n+3)}(d_2-d_1)dd_1dd_2.$$

Let $u = d_1+d_2$ and $g = d_1d_2$. We get

$$c(2,m,n)\int_{a\lambda^{-1}}^{b\lambda^{-1}} \int_0^{\frac{1}{4}u^2} g^m/(1+u+g)^{m+n+3}\, dgdu.$$

Thus $$\left.\frac{\partial P(a\le U^{(2)}\le b|\lambda_1=\lambda_2=\lambda)}{\partial\lambda^{-1}}\right|_{\lambda=1}$$

(3.8) $$= c(2,m,n)[b\int_0^{\frac{1}{4}b^2} g^m/(1+b+g)^{m+n+3}dg - a\int_0^{\frac{1}{4}a^2} g^m/(1+a+g)^{m+n+3}dg].$$

Furthermore, transform $t = g/(1+b+g)$, (3.8) becomes

$$c(2,m,n)[b(1+b)^{-(n+2)} \int_0^{b^2/(b+2)^2} t^m(1-t)^{n+1}dt - a(1+a)^{-(n+2)} \int_0^{a^2/(a+2)^2} t^m(1-t)^{n+1}dt].$$

And furthermore, $P(a \leq U^{(2)} \leq b|\lambda_1 = \lambda_2 = 1)$

(3.9) $$= c(2,m,n)\int_a^b \int_0^{\frac{1}{4}u^2} g^m/(1+u+g)^{m+n+3}dgdu.$$

Now using integration by parts, (3.9) becomes

$$c(2,m,n)\{[2\int_0^{b/(b+2)} t^{2m+1}(1-t)^{2n+2}dt-(1+b)^{-(n+1)} \int_0^{b^2/(b+2)^2} t^m(1-t)^{n+1}dt]$$

$$-[2\int_0^{a/(a+2)} t^{2m+1}(1-t)^{2n+2}dt - (1+a)^{-(n+1)} \int_0^{a^2/(a+2)^2} t^m(1-t)^{n+1}dt]\}/(n+1).$$

Therefore, we have proved the following:

Theorem 4. The acceptance region based on $U^{(2)} = c_1 + c_2$ with ℓup and α level is given by $a \leq U^{(2)} \leq b$ where a and b are so chosen as to satisfy

(i) $$c(2,m,n)\{[2B_{b/(b+2)}(2m+2, 2n+3) - (1+b)^{-(n+1)}B_{b^2/(b+2)^2}(m+1, n+2)]$$

$$- [2B_{a/(a+2)}(2m+2, 2n+3) - (1+a)^{-(n+1)}B_{a^2/(a+2)^2}(m+1,n+2)]\}/(n+1) = 1-\alpha$$ and

(ii) $$c(2,m,n)\{b(1+b)^{-(n+2)}B_{b^2/(b+2)^2}(m+1,n+2)-a(1+a)^{-(n+2)}B_{a^2/(a+2)^2}^{(m+1,n+2)}\} = 0.$$

3) Pillai's trace, $V^{(2)} = [c_1/(1+c_1)]+[c_2/(1+c_2)]$. From Theorem 2, we obtain $\left.\frac{\partial P(a\leq V^{(2)} \leq b|\lambda_1 = \lambda_2 = \lambda)}{\partial\lambda^{-1}}\right|_{\lambda=1}$ by replacing $L_2^{(2)}$ in the limits of the integrals in (3.1) by $V^{(2)}$. Now transform $\ell_1 = c_1/(1+c_1)$ and $\ell_2 = c_2/(1+c_2)$. Then

$$(3.10)\qquad \left.\frac{\partial P(a\le V^{(2)}\le b\,|\,\lambda_1 = \lambda_2 = \lambda)}{\partial\lambda^{-1}}\right|_{\lambda=1}$$

$$= c(2,m,n)\left[\int_{a\le \ell_1+\ell_2\le b} g_1(\ell_1,\ell_2;m,n)\,d\ell_1 d\ell_2 - \int_{a\le\ell_1+\ell_2\le b} h_1(\ell_1,\ell_2;m,n)\,d\ell_1 d\ell_2\right].$$

Let $v = \ell_1+\ell_2$ and $g = \ell_1\ell_2$. Then (3.10) becomes

$$c(2,m,n)\left[(2m+3)\int_a^b\int_0^{\frac{1}{4}v^2} g^m(1-v+g)^n dg\,dv - (m+n+3)\int_a^b\int_0^{\frac{1}{4}v^2} v g^m(1-v+g)^n dg\,dv\right].$$

and $P(a\le V^{(2)}\le b\,|\,\lambda_1 = \lambda_2 = 1) = c(2,m,n)\int_0^b\int_0^{\frac{1}{4}v^2} g^m(1-v+g)^n dg\,dv.$

Therefore, we have the following Theorem:

Theorem 5. Let

$$T_1(x) = \begin{cases} [2/(m+1)]\sum_{\gamma=0}^{n}(-1)^\gamma\left[\binom{n}{\gamma}/\binom{m+\gamma+1}{\gamma}\right]B_{\frac{1}{2}x}(2m+2\gamma+3,2n-2\gamma+1) & \text{if } 0\le x\le 1,\\ [2/(m+1)]\sum_{\gamma=0}^{n}(-1)^\gamma\left[\binom{n}{\gamma}/\binom{m+\gamma+1}{\gamma}\right]B_{\frac{1}{2}}(2m+2\gamma+3,2n-2\gamma+1) + \\ [2/(n+1)]\sum_{\gamma=0}^{m}(-1)^\gamma\left[\binom{m}{\gamma}/\binom{n+\gamma+1}{\gamma}\right]\left[B_{\frac{1}{2}x}(2m-2\gamma+1,2n+2\gamma+3)-B_{\frac{1}{2}}(2m-2\gamma+1,2n+2\gamma+3)\right] & \text{if } 1\le x\le 2,\end{cases}$$

and

$$T_2(x) = \begin{cases} [4/(m+1)]\sum_{\gamma=0}^{n}(-1)^\gamma\left[\binom{n}{\gamma}/\binom{m+\gamma+1}{\gamma}\right]B_{\frac{1}{2}x}(2m+2\gamma+4,2n-2\gamma+1) & \text{if } 0\le x\le 1,\\ [4/(m+1)]\sum_{\gamma=0}^{n}(-1)^\gamma\left[\binom{n}{\gamma}/\binom{m+\gamma+1}{\gamma}\right]B_{\frac{1}{2}}(2m+2\gamma+4,2n-2\gamma+1) + \\ [4/(m+1)]\sum_{\gamma=0}^{m}(-1)^\gamma\left[\binom{m}{\gamma}/\binom{n+\gamma+1}{\gamma}\right]\left[B_{\frac{1}{2}x}(2m-2\gamma+2,2n+2\gamma+3)-B_{\frac{1}{2}}(2m-2\gamma+2,2n+2\gamma+3)\right] & \text{if } 1\le x\le 2.\end{cases}$$

Then the acceptance region based on $V^{(2)} = [c_1/(1+c_1)] + [c_2/(1+c_2)]$ with ℓup and α level is given by $a \le V^{(2)} \le b$ where a and b are so chosen as to satisfy

(i) $c(2,m,n)[T_1(b)-T_1(a)] = 1-\alpha$ and

(ii) $c(2,m,n)\{(2m+3)[T_1(b)-T_1(a)] - (m+n+3)[T_2(b)-T_2(a)]\} = 0.$

4) Wilks' criterion, $W^{(2)} = [(1+c_1)(1+c_2)]^{-1}$. Again from Theorem 2, we obtain $\frac{\partial P(a \leq W^{(2)} \leq b \mid \lambda_1 = \lambda_2 = \lambda)}{\partial \lambda^{-1}}$ by replacing $L_2^{(2)}$ in the limits of the integrals in (3.1) by $W^{(2)}$. Now transform $\ell_1 = c_1/(1+c_1)$ and $\ell_2 = c_2/(1+c_2)$. Then

$$(3.11) \qquad \left.\frac{\partial P(a \leq W^{(2)} \leq b \mid \lambda_1 = \lambda_2 = \lambda)}{\partial \lambda^{-1}}\right|_{\lambda=1}$$

$$= c(2,m,n)\Big[\int_{a \leq (1-\ell_1)(1-\ell_2) \leq b} g_1(\ell_1,\ell_2;m,n)\,d\ell_1 d\ell_2 - \int_{a \leq (1-\ell_1)(1-\ell_2) \leq b} h_1(\ell_1,\ell_2;m,n)\,d\ell_1 d\ell_2\Big].$$

Let $w = (1-\ell_1)(1-\ell_2)$ and $g = \ell_1\ell_2$. Then (3.11) becomes

$$c(2,m,n)\Big[(2m+3)\int_a^b \int_0^{(1-w^{\frac{1}{2}})^2} g^m w^n\,dg\,dw - (m+n+3)\int_a^b \int_0^{(1-w^{\frac{1}{2}})^2} g^m w^n (1-w+g)\,dg\,dw\Big].$$

and

$$P(a \leq W^{(2)} \leq b \mid \lambda_1 = \lambda_2 = 1) = c(2,m,n) \int_a^b \int_0^{(1-w^{\frac{1}{2}})^2} g^m w^n\,dg\,dw.$$

Therefore, we have the following theorem:

Theorem 6. The acceptance region based on $W^{(2)} = [(1+c_1)(1+c_2)]^{-1}$ with ℓup and α level is given by $a \leq W^{(2)} \leq b$ where a and b are so chosen as to satisfy

(i) $c(2,m,n)2[B_{b^{\frac{1}{2}}}(2n+2,2m+3)-B_{a^{\frac{1}{2}}}(2n+2,2m+3)]/(m+1) = 1-\alpha$ and

(ii) $c(2,m,n)2\{(m+n+3)[B_{b^{\frac{1}{2}}}(2n+4,2m+3) - B_{a^{\frac{1}{2}}}(2n+4,2m+3)]/(m+1)$

$- (n-m)[B_{b^{\frac{1}{2}}}(2n+2,2m+3) - B_{a^{\frac{1}{2}}}(2n+2,2m+3)]/(m+1)$

$- (m+n+3)[B_{b^{\frac{1}{2}}}(2n+2,2m+5) - B_{a^{\frac{1}{2}}}(2n+2,2m+5)]/(m+2)\} = 0.$

5) <u>Roy's Largest-smallest roots, $LS^{(2)} = c_2/(1+c_2),\ c_1/(1+c_1)$.</u>

From Theorem 2, we have

$$(3.12)\qquad \left.\frac{\partial P(a \le LS^{(2)} \le b\,|\,\lambda_1 = \lambda_2 = \lambda)}{\partial \lambda^{-1}}\right|_{\lambda=1}$$

$$= c(2,m,n)\Big[\int\limits_{a\le c_1/(1+c_1)\le c_2/(1+c_2)\le b} g(c_1,c_2;m,n)dc_1dc_2 - \int\limits_{a\le c_1/(1+c_1)\le c_2/(1+c_2)\le b} h(c_1,c_2;m,n)dc_1dc_2\Big].$$

Now transform $\ell_1 = c_1/(1+c_1)$ and $\ell_2 = c_2/(1+c_2)$. Then (3.12) becomes

$$c(2,m,n)\Big[\int_a^b\int_a^{\ell_2} g_1(\ell_1,\ell_2;m,n)d\ell_1d\ell_2 - \int_a^b\int_a^{\ell_2} h_1(\ell_1,\ell_2;m,n)d\ell_1d\ell_2\Big].$$

Further, $P(a \le LS^{(2)} \le b\,|\,\lambda_1 = \lambda_2 = 1)$

$$= c(2,m,n)\int_a^b\int_a^{\ell_2} (\ell_1\ell_2)^m[(1-\ell_1)(1-\ell_2)]^n(\ell_2-\ell_1)d\ell_1d\ell_2.$$

Now using the same technique as in Pillai [6, 8, 9],

$$\int_a^b\int_a^{\ell_2} (\ell_1\ell_2)^m[(1-\ell_1)(1-\ell_2)]^n(\ell_2-\ell_1)d\ell_1d\ell_2$$

$$= \Big[2\int_a^b \ell_2^{2m+1}(1-\ell_2)^{2n+1}\,d\ell_2 - \{b^{m+1}(1-b)^{n+1}+a^{m+1}(1-a)^{n+1}\}\int_a^b \ell_1^m(1-\ell_1)^n d\ell_1\Big]/(m+n+2)$$

and $\int_a^b\int_a^{\ell_2}(\ell_1+\ell_2)(\ell_1\ell_2)^m[(1-\ell_1)(1-\ell_2)]^n(\ell_2-\ell_1)d\ell_1d\ell_2$

$$= \Big[2\int_a^b \ell_2^{2m+2}(1-\ell_2)^{2n+1}d\ell_2 - \{b^{m+2}(1-b)^{n+1}+a^{m+2}(1-a)^{n+1}\}\int_a^b \ell_1^m(1-\ell_1)^n d\ell_1$$

$$+ (m+2)\int_a^b\int_a^{\ell_2}(\ell_1\ell_2)^m[(1-\ell_1)(1-\ell_2)]^n(\ell_2-\ell_1)d\ell_1d\ell_2\Big]/(m+n+3).$$

Therefore, we have proved the following:

Theorem 7. The acceptance region based on Roy's largest-smallest roots $LS^{(2)} = c_2/(1+c_2),\ c_1/(1+c_1)$ with ℓup and α level is given by $a \le c_1/(1+c_1) \le c_2/(1+c_2) \le b$ where a and b are so chosen as to satisfy

(i) $c(2,m,n)[2B_{a,b}(2m+2,2n+2) - \{b^{m+1}(1-b)^{n+1}+a^{m+1}(1-a)^{n+1}\}B_{a,b}(m+1,n+1)]/(m+n+2) = 1-\alpha$ and

(ii) $c(2,m,n)\{[2B_{a,b}(2m+2,2n+2) - \{b^{m+1}(1-b)^{n+1}+a^{m+1}(1-a)^{n+1}\}B_{a,b}(m+1,n+1)][(m+1)/(m+n+2)] - [2B_{a,b}(2m+3,2n+2) - \{b^{m+2}(1-b)^{n+1}+a^{m+2}(1-a)^{n+1}\}B_{a,b}(m+1,n+1)]\} = 0$

where $B_{x,y}(r,s) = \int_x^y t^{r-1}(1-t)^{s-1}dt$.

4. The acceptance regions based on two criteria with ℓup for p = 3.

In this section, we consider the acceptance regions of Roy's largest root and Hotelling's trace for p = 3. Largest root is taken up first.

1) Roy's largest root, $L_3^{(3)} = c_3/(1+c_3)$. From Theorem 2, we have

$$\left.\frac{\partial P(a\le L_3^{(3)}\le b \mid \lambda_1 = \lambda_2 = \lambda_3 = \lambda)}{\partial \lambda^{-1}}\right|_{\lambda=1}$$

$$(4.1)\qquad = c(3,m,n)\Big[\int_{a\le L_3^{(3)}\le b} 3(m+2)\prod_{j=1}^{3} c_j^m(1+c_j)^{-(m+n+4)}\prod_{j'<j=2}^{3}(c_j-c_{j'})\prod_{j=1}^{3}dc_j - \int_{a\le L_3^{(3)}\le b}(m+n+4)\Big[\sum_{j=1}^{3}c_j/(1+c_j)\Big]\prod_{j=1}^{3}c_j^m(1+c_j)^{-(m+n+4)}\prod_{j'<j=2}^{3}(c_j-c_{j'})\prod_{j=1}^{3}dc_j\Big].$$

Now transform $\ell_j = c_j/(1+c_j)$, j = 1,2,3. Then (4.1) becomes

$$c(3,m,n)\Big[\int_a^b\int_0^{\ell_3}\int_0^{\ell_2} 3(m+2)\prod_{j=1}^{3}\ell_j^m(1-\ell_j)^n\prod_{j'<j=2}^{3}(\ell_j-\ell_{j'})\prod_{j=1}^{3}d\ell_j - \int_a^b\int_0^{\ell_3}\int_0^{\ell_2}(m+n+4)\Big(\sum_{j=1}^{3}\ell_j\Big)\prod_{j=1}^{3}\ell_j^m(1-\ell_j)^n\prod_{j'<j=2}^{3}(\ell_j-\ell_{j'})\prod_{j=1}^{3}d\ell_j\Big].$$

Following the notation of Pillai [8], denote

$$U(x;\ m+1,n,m,n;t) = \int_0^x \int_0^{\ell_2} (\ell_1 \ell_2)^m [(1-\ell_1)(1-\ell_2)]^n e^{t(\ell_1+\ell_2)} (\ell_2-\ell_1) d\ell_1 d\ell_2 .$$

Then from (3.4) we know that

$$(4.2)\quad U(x;\ m+1,n,m,n;\ 0) = \int_0^x \int_0^{\ell_2} (\ell_1 \ell_2)^m [(1-\ell_1)(1-\ell_2)]^n (\ell_2-\ell_1) d\ell_1 d\ell_2$$

$$= [2B_x(2m+2,2n+2) - x^{m+1}(1-x)^{n+1} B_x(m+1,n+1)]/(m+n+2).$$

And also from (3.5), we have

$$(4.3)\quad \left.\frac{\partial U(x;m+1,n,m,n;t)}{\partial t}\right|_{t=0} = \int_0^x \int_0^{\ell_2} (\ell_1+\ell_2)(\ell_1 \ell_2)^m [(1-\ell_1)(1-\ell_2)]^n (\ell_2-\ell_1) d\ell_1 d\ell_2$$

$$= [2B_x(2m+3,2n+2) - x^{m+2}(1-x)^{n+1} B_x(m+1,n+1) + (m+2)U(x;m+1,n,m,n;0)]/(m+n+3).$$

Now from Theorem 2 of Pillai [8]

$$U(x;m+2,n,m+1,n,m,n;t)$$

$$= \int_0^x \int_0^{\ell_3} \int_0^{\ell_2} \prod_{j=1}^{3} \ell_j^m (1-\ell_j)^n \exp(t\ell_j) \prod_{j'<j=2}^{3} (\ell_j-\ell_{j'}) \prod_{j=1}^{3} d\ell_j$$

$$= [-x^{m+2}(1-x)^{n+1} e^{tx} U(x;m+1,n,m,n;t) + 2B_x(2m+4,2n+2;2t) B_x(m+1,n+1;t)$$

$$- 2B_x(2m+3,2n+2;2t) B_x(m+2,n+1;t) + tU(x;m+2,n+1,m+1,n,m,n;t)]/(m+n+3).$$

where $B_x(r,s;t) = \int_0^x u^{r-1}(1-u)^{s-1} e^{tu} du$. Therefore we have

$$(4.4)\quad U(x;m+2,n,m+1,n,m,n;0)$$

$$= \int_0^x \int_0^{\ell_3} \int_0^{\ell_2} \prod_{j=1}^{3} \ell_j^m (1-\ell_j)^n \prod_{j'<j=2}^{3} (\ell_j-\ell_{j'}) \prod_{j=1}^{3} d\ell_j$$

$$[-x^{m+2}(1-x)^{n+1}U(x;m+1,n,m,n;0)+2B_x(2m+4,2n+2)\cdot B_x(m+1,n+1)$$

$$-2B_x(2m+3,2n+2)\cdot B_x(m+2,n+1)]/(m+n+3).$$

And also $\left.\frac{\partial U(x;m+2,n,m+1,n,m,n;t)}{\partial t}\right|_{t=0}$

$$(4.5)\quad = \int_0^x \int_0^{\ell_3} \int_0^{\ell_2} [\sum_{j=1}^{3} \ell_j] \prod_{j=1}^{3} \ell_j^m (1-\ell_j)^n \prod_{j'<j=2}^{3} (\ell_j - \ell_{j'}) \prod_{j=1}^{3} d\ell_j$$

$$= [-x^{m+3}(1-x)^{n+1}U(x;m+1,n,m,n;0)-x^{m+2}(1-x)^{n+1}\frac{\partial}{\partial t}U(x;m+1,n,m,n;t)\Big|_{t=0}$$

$$+4B_x(2m+5,2n+2)B_x(m+1,n+1)-2B_x(2m+4,2n+2)B_x(m+2,n+1)$$

$$-2B_x(2m+3,2n+2)B_x(m+3,n+1)+U(x;m+2,n,m+1,n,m,n;0)$$

$$-U(x;m+3,n,m+1,n,m,n;0)]/(m+n+3).$$

Note that $\frac{\partial}{\partial t}U(x;m+2,n,m+1,n,m,n;t)\Big|_{t=0} = U(x;m+3,n,m+1,n,m,n;0)$. So (4.5) gives $\frac{\partial U(x;m+2,n,m+1,n,m,n;t)}{\partial t}$

$$(4.6)\quad = [-x^{m+3}(1-x)^{n+1}U(x;m+1,n,m,n;0)-x^{m+2}(1-x)^{n+1}\frac{\partial}{\partial t}U(x;m+1,n,m,n;t)\Big|_{t=0}$$

$$+4B_x(2m+5,2n+2)B_x(m+1,n+1)-2B_x(2m+4,2n+2)B_x(m+2,n+1)$$

$$-2B_x(2m+3,2n+2)B_x(m+3,n+1)+U(x;m+2,n,m+1,n,m,n;0)]/(m+n+4).$$

Therefore, we have the following theorem:

Theorem 8. The acceptance region based on Roy's largest root, $L_3^{(3)} = c_3/(1+c_3)$ with ℓup and α level is given by $a \leq L_3^{(3)} \leq b$ where a and b are so chosen as to satisfy

(i) $c(3,m,n)[U(b;m+2,n,m+1,n,m,n;0)-U(a;m+2,n,m+1,n,m,n;0)] = 1-\alpha$ and

(ii) $c(3,m,n)\{3(m+2)[U(b;m+2,n,m+1,n,m,n;0)-U(a;m+2,n,m+1,n,m,n\ ;0)]$

$$- (m+n+4)\left[\frac{\partial}{\partial t}U(b;m+2,n,m+1,n,m,n;t)\Big|_{t=0} - \frac{\partial}{\partial t}U(a,m+2,n,m+1,n,m,m;t)\Big|_{t=0}\right]\}$$

$$= 0,$$

where U and $\frac{\partial}{\partial t}U$ are given in (4.4) and (4.6) respectively.

2) Hotellings' tace, $U^{(3)} = c_1 + c_2 + c_3$. From Section 2, we have

(4.7) $P(a \le U^{(3)} \le b \mid \lambda_1 = \lambda_2 = \lambda_3 = \lambda)$

$$= (2\pi)^{-\frac{3}{2}(n_1+n_2)} \int_{a\le c_1+c_2+c_3\le b} \lambda^{-\frac{3}{2}n_1} \exp[-\tfrac{1}{2}\mathrm{tr}(\lambda^{-1} X_1 X_1' + X_2 X_2')]\, dX_1\, dX_2.$$

Now transform $\lambda^{-1}X_1 = Y_1$ and $X_2 = Y_2$. $J(Y_1 : X_1) = \lambda^{-\frac{3}{2}n_1}$ and let $0 < d_1 \le d_2 \le d_3 < \infty$ be the characteristic roots of $|Y_1Y_1' - dY_2Y_2'| = 0$. Then (4.7) becomes

(4.8) $$c(3,m,n) \int_{a\le\lambda d_1+\lambda d_2+\lambda d_3\le b} \prod_{j=1}^{3} d_j^{m} (1+d_j)^{-(m+n+4)} \prod_{j'<j=2}^{3} (d_j - d_{j'}) \prod_{j=1}^{3} dd_j.$$

After a proper transformation, (4.8) is $\int_{a\lambda^{-1}}^{b\lambda^{-1}} T_2(u)du$ where $T_2(u)$ is the probability density function of $U^{(3)}$. Then

$$\frac{\partial P(a\le U^{(3)}\le b \mid \lambda_1 = \lambda_2 = \lambda_3 = \lambda)}{\partial \lambda^{-1}}\Bigg|_{\lambda=1} = bT_2(b) - aT_2(a).$$

Note that $T_2(u)$, the density of $U^{(3)}$, and $T_1(u)$, the distribution function of $U^{(3)}$, are given in equation (4.9) and (4.11) respectively by Pillai and Sudjana [13]. Therefore we have the following theorem

Theorem 9. The acceptance region based on $U^{(3)} = c_1+c_2+c_3$ with ℓ up and α level is given by $a \leq U^{(3)} \leq b$ where a and b are so chosen as to satisfy (i) $T_1(b) - T_1(a) = 1 - \alpha$ and (ii) $bT_2(b) - aT_2(a) = 0$.

5. $P(a\leq[c_1/(1+c_1)]\leq[c_2/(1+c_2)]\leq b)$ in the non-null case. The non-null distribution of $c_1,\ldots,c_p$ was obtained by Khatri [5] in the form

$$(5.1)\quad c(p,m,n)\,|\Lambda|^{-m-\frac{1}{2}p-\frac{1}{2}}|C|^m|I+C|^{-\frac{1}{2}\nu}\,{}_1F_0(\tfrac{1}{2}\nu;I-\Lambda^{-1},C(I+C)^{-1})\prod_{i>j}(c_i-c_j),$$

$$0 < c_1 \leq \ldots \leq c_p < \infty ,$$

where $\nu = n_1+n_2$ and the hypergeometric function of a matrix argument is defined by James [4]:

$${}_sF_t(a_1,\ldots,a_s;b_1,\ldots,b_t;S,T) = \sum_{k=0}^{\infty}\sum_{\kappa}\frac{(a_1)_\kappa\cdots(a_s)_\kappa}{(b_1)_\kappa\cdots(b_t)_\kappa}\frac{C_\kappa(S)C_\kappa(T)}{C_\kappa(I)k!}$$

where $a_1,\ldots,a_s$, $b_1,\ldots,b_t$ are real or complex constants and the coefficient $(a)_\kappa$ is defined by

$$(a)_\kappa = \prod_{i=1}^{p}(a-\tfrac{1}{2}(i-1))_{k_i}$$

where $(a)_k = a(a+1)\ldots(a+k-1)$ and κ of k is a partition of k, $\kappa = (k_1,\ldots,k_p)$, $k_1 \geq k_2 \geq \ldots \geq k_p \geq 0$ such that $k_1 + \ldots + k_p = k$ and the zonal polynomials $C_\kappa(S)$ are expressible in terms of elementary symmetric functions of the characteristic roots of S [James, 4].

Putting p = 2, the joint distribution of c_1, c_2 is

$$(5.2)\quad c(2,m,n)(\lambda_1\lambda_2)^{-m-\frac{3}{2}}(c_1c_2)^m[(1+c_1)(1+c_2)]^{-\frac{1}{2}\nu}\,{}_1F_0(\tfrac{1}{2}\nu;I-\Lambda^{-1},C(I+C)^{-1})(c_2-c_1).$$

Pillai and Jayachandran [11] have shown that by transforming $\ell_1 = c_1/(1+c_1)$ and $\ell_2 = c_2/(1+c_2)$ and using zonal polynomials up to the sixth degree, the joint distribution of ℓ_1,ℓ_2 can be written as

$$K''\left(\sum_{i+2j=k=0}^{6} c''_{ij}(\ell_1+\ell_2)^i(\ell_1\ell_2)^{j+m}+\dots\right)[(1-\ell_1)(1-\ell_2)]^n(\ell_2-\ell_1),\ell_1<\ell_2$$

where $K''=c(2,m,n)(\lambda_1\lambda_2)^{-m-\frac{3}{2}}$ and the c''_{ij}'s are functions of λ_1, λ_2, n_1, n_2 as given in terms of constants A''_{ij} by Pillai and Jayachandran. The A''_{ij}'s and c''_{ij}'s are given in Appendix A. Now using the technique given by Pillai [6, 8, 9], we have

$$h_0 = 0$$

$$h_j = \int_a^b \int_a^{\ell_2} (\ell_1\ell_2)^m[(1-\ell_1)(1-\ell_2)]^n(\ell_2^j-\ell_1^j)d\ell_1 d\ell_2$$

$$= [2B_{a,b}(2m+1+j,2n+2)-\{b^{m+j}(1-b)^{n+1}+a^{m+j}(1-a)^{n+1}\}B_{a,b}(m+1,n+1)+ (m+j)h_{j-1}]/(m+n+1+j)$$

$$j = 1,\dots,7.$$

Now defining $h_{ij} = \int_a^b \int_a^{\ell_2} (\ell_1+\ell_2)^i(\ell_1\ell_2)^{m+j}[(1-\ell_1)(1-\ell_2)]^n(\ell_2-\ell_1)d\ell_1 d\ell_2$, then

$$h_{0j}=[2B_{a,b}(2m+2+2j,2n+2)-\{b^{m+1+j}(1-b)^{n+1}+a^{m+1+j}(1-a)^{n+1}\}B_{a,b}(m+1+j,n+1)]/(m+n+2+j)$$

$$j = 1,2,3,$$

$$h_{1j}=[2B_{a,b}(2m+3+2j,2n+2)-\{b^{m+2+j}(1-b)^{n+1}+a^{m+2+j}(1-a)^{n+1}\}B_{a,b}(m+1+j,n+1) +(m+2+j)h_{0j}]/(m+n+3+j) \qquad j = 1,2,$$

$$h_{2j} = \{[2B_{a,b}(2m+4+2j,2n+2)-\{b^{m+3+j}(1-b)^{n+1}+a^{m+3+j}(1-a)^{n+1}\}B_{a,b}(m+1+j,n+1) + (m+3+j)h_{1j}]/(m+n+4+j)\}+h_{0j+1} \qquad j = 1,2,$$

$$h_{31}=\{[2B_{a,b}(2m+7,2n+2)-\{b^{m+5}(1-b)^{n+1}+a^{m+5}(1-a)^{n+1}\}B_{a,b}(m+2,n+1) +(m+5)(h_{21}-h_{02})]/(m+n+6)\}+2h_{12}$$

$$h_{41}=\{[2B_{a,b}(2m+8,2n+2)-\{b^{m+6}(1-b)^{n+1}+a^{m+6}(1-a)^{n+1}\}B_{a,b}(m+2,n+1) +(m+6)(h_{31}-h_{12})]/(m+n+7)\}+3h_{22}-h_{03},$$

where $B_{x,y}(\gamma,s) = \int_x^y t^{r-1}(1-t)^{s-1}dt.$

Then we have, in the non-null case,

$$P(a \leq c_1/(1+c_1) \leq c_2/(1+c_2) \leq b) = P(a \leq \ell_1 \leq \ell_2 \leq b)$$

$$= K'' \sum_{i+2j=k=0}^{6} c''_{ij} \int_a^b \int_0^{\ell_2} (\ell_1+\ell_2)^i \, (\ell_1\ell_2)^{j+m}[(1-\ell_1)(1-\ell_2)]^n(\ell_2-\ell_1)d\ell_1 d\ell_2 + \dots$$

$$= K''[c''_{00}h_1+c''_{10}h_2+c''_{20}(h_3+h_{01})+c''_{01}h_{01}+c''_{30}(h_4+2h_{11})+c''_{11}h_{11}$$
$$+c''_{40}(h_5+3h_{21}-h_{02})+c''_{21}h_{21}+c''_{02}h_{02}+c''_{50}(h_6+4h_{31}-3h_{12})+c''_{31}h_{31}$$
$$+c''_{12}h_{12}+c''_{60}(h_7+5h_{41}-6h_{22}+h_{03})+c''_{41}h_{41}+c''_{22}h_{22}+c''_{03}h_{03}]+\dots$$

6. <u>The likelihood ratio criterion, $Z^{(2)}$</u>. In order to study the density of $Z^{(2)}$, starting from (5.2), Pillai and Jayachandran [12] have shown that by transforming $a_2 = \ell_1\ell_2$ and $w = (1-\ell_1)(1-\ell_2)$ and using zonal polynomials up to the sixth degree, the joint distribution of a_2 and w is given by $K'' \sum_{i+2j=k=0}^{6} c''_{ij} a_2^{j+m} w^n (1-w+a_2)^i + \dots$, $0 \leq a_2^{\frac{1}{2}} + w^{\frac{1}{2}} \leq 1$. Now $Z^{(2)}$ (to be denoted by Z in the rest of this section) $= (\ell_1\ell_2)^{\frac{1}{2}n_1}[(1-\ell_1)(1-\ell_2)]^{\frac{1}{2}n_2} = a_2^{\frac{1}{2}n_1} w^{\frac{1}{2}n_2}$. Let $y = w^{\frac{1}{2}}$. Then the joint distribution of Z and y is

$$(4/n_1)K'' \sum_{i+2j=k=0}^{6} c''_{ij} z^{\frac{2}{n_1}j - \frac{1}{n_1}} y^{-\frac{2n_2}{n_1}j + \frac{n_2}{n_1} - 2}(1-y^2+z^{\frac{2}{n_1}} y^{-\frac{2n_2}{n_1}})^i + \dots$$

where $y^{\frac{n_2}{n_1}+1} - y^{\frac{n_2}{n_1}} + z^{\frac{1}{n_1}} \leq 0$.

Now putting $n_2 = 2n_1$, the joint distribution of Z and y can be written as

$$(4/n_1)K'' \sum_{i+2j=k=0}^{6} c''_{ij} z^{\frac{2}{n_1}j-\frac{1}{n_1}} y^{-4j}(1-y^2+z^{\frac{2}{n_1}}y^{-4})^i + \dots$$

where $y^3 - y^2 + z^{\frac{1}{n_1}} \leq 0$.

Note that the area bounded by $y^3 - y^2 + z^{\frac{1}{n_1}} \leq 0$ is the area bounded by $0 \leq z \leq (4/27)^{n_1}$ and

$$\tfrac{2}{3}\cos(\tfrac{\theta}{3} + \tfrac{4\pi}{3}) + \tfrac{1}{3} \leq y \leq \tfrac{2}{3}\cos(\tfrac{\theta}{3}) + \tfrac{1}{3} \quad \text{where } \cos\theta = 1 - \tfrac{27}{2} z^{\frac{1}{n_1}}.$$

Now integrate out y to obtained the density of the $m\ell r$ Z:

$$f(z) = (4/n_1)K'' \sum_{i+2j=k=0}^{6} c''_{ij} z^{\frac{2}{n_1}j-\frac{1}{n_1}} \int_a^b y^{-4j}(1-y^2+z^{\frac{2}{n_1}} y^{-4})^i \, dy + \dots$$

where $a = \frac{2}{3}\cos(\frac{\theta}{3} + \frac{4\pi}{3}) + \frac{1}{3}$, $b = \frac{2}{3}\cos(\frac{\theta}{3}) + \frac{1}{3}$ with $\cos\theta = 1 - \frac{27}{2} z^{\frac{1}{n_1}}$ and $0 \leq z \leq (4/27)^{n_1}$. Now let

$$g_{ij}(z) = \int_a^b y^{-4j}(1-y^2+ z^{\frac{2}{n_1}} y^{-4})^i dy.$$

Then the density of Z can be written as

$$f(z) = (4/n_1)K'' \sum_{i+2j=k=0}^{6} c''_{ij} z^{\frac{2}{n_1}j - \frac{1}{n_1}} g_{ij}(z) + \ldots \quad 0 \leq z \leq (4/27)^{n_1}.$$

The expressions for $g_{ij}(z)$ may be found in Appendix B. The method for determining the $g_{ij}(z)$'s will be illustrated by considering $g_{10}(z)$ which can be written as

$$\begin{aligned} g_{10}(z) &= \int_a^b (1-y^2+z^{\frac{2}{n_1}} y^{-4})dy \\ &= (b-a)-\tfrac{1}{3}(b^3-a^3)-\tfrac{1}{3}z^{\frac{2}{n_1}} (b^{-3}-a^{-3}) \\ &= (b-a)-\tfrac{1}{3}(b^3-a^3)-\tfrac{1}{3}z^{\frac{2}{n_1}}[z^{-\frac{2}{n_1}}(b-a)-z^{-\frac{2}{n_1}}(b^2-a^2)] \\ &= \tfrac{2}{3}(b-a), \end{aligned}$$

where we have made use of the relations

$$b^k-a^k = (b^{k-1}-a^{k-1})-z^{\frac{1}{n_1}} (b^{k-3}-a^{k-3}) \text{ for any integer } k.$$

Note that the $g_{ij}(z)$'s are expressed in terms of z, $b-a = \frac{2}{\sqrt{3}} \cos(\frac{\theta}{3} + \frac{\pi}{6})$ and $b^2-a^2 = \frac{4}{3\sqrt{3}} \cos(\frac{\theta}{3} + \frac{\pi}{6})[\cos(\frac{\pi}{3} - \frac{\theta}{3})+1]$ where $\cos\theta = 1-\frac{27}{2} z^{\frac{1}{n_1}}$,

7. <u>Numerical study of power</u>. The results in the previous sections were used to obtain five percent points for the tests of H_0: $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$ against $\underset{\sim}{\Sigma}_1 \neq \underset{\sim}{\Sigma}_2$ based on criteria 1) to 4) in the unbiased as well as equal tail

areas cases and criterion 5) in the unbiased case for $p = 2$, values of $m = 0, 1, 2, 5$ and $n = 5, 10, 15, 20, 25, 30, 40, 60, 80, 100$, and are given in Table 1. For mℓr $Z^{(2)}$, lower five percent points were obtained for $n_2 = 2n_1$ and for values of $n_1 = 3, 5, 7, 13$ and the five percentage points were also computed for tests 1) to 5) in the unbiased case and 1) to 4) in the equal tails case for the same values of n_1 and all these are presented in Table 2. In addition, five percent points were also computed for $LS_1^{(2)}$, $LS_2^{(2)}$ and $LS_3^{(2)}$ for $m = 0, 1, 2, 5$ and $n = 5, 15, 30, 60$ and are given in Table 3. Finally, for $p = 3$, five percentage points were computed for test 1), namely, Roy's largest root, in the unbiased as well as equal tail areas cases, which are also presented in Table 1.

The next step was to compute the powers of the various tests using the percentage points evaluated and the non-null distributions. For tests 1) to 4), non-null distributions were available in Pillai and Jayachandran [12] and for tests 5) and 6), they have been obtained in Sections 5 and 6. Before computing the power for a specific value of (λ_1, λ_2) using series involving zonal polynomials of degree 0 to 6, the total probability in that case over the whole range of the respective statistic for all the terms included in the formula was calculated and the number of decimal places included in the tables was determined depending on the number of places of accuracy obtained in the total probability as least as many decimal places as in the tables. Powers for tests 1) to 5) in the unbiased as well as equal tail areas cases for $p = 2$, for values of $m = 0, 1, 2, 5$, $n = 5, 15, 30, 60$ and various (λ_1, λ_2) are presented in Table 4. Further, powers for tests 1) and 6) under the condition $n_2 = 2n_1$ are given in Table 5 for $p = 2$, for values of n_1 =

3, 5, 7, 13 and various (λ_1,λ_2) in the unbiased case and also in the equal tail areas case for tests 1) to 4). Again, in order to compare the largest-smallest root test $(LS^{(P)})$ with the approximations, a tabulation of powers is presented in Table 6 for $LS^{(2)}$, $LS_1^{(2)}$, $LS_2^{(2)}$, $LS_3^{(2)}$ and Roy's largest root $L_2^{(2)}$ for m, n as in Table and various (λ_1,λ_2).

A few findings seem to emerge from tabulations of powers in Tables 4, 5, 6.

1. $\lambda_1 \geq 1$, $\lambda_2 \geq 1$. It may be seen from Table 4 that equal tail areas tests based on 1) to 4) generally seem to perform better than corresponding unbiased ones except when very close to H_0 in which case bias is observed in some instances, mostly when m is close to n.
2. $\lambda_1 < 1$, $\lambda_2 > 1$ or $\lambda_1 > 1$, $\lambda_2 < 1$. For tests 1) to 4), unbiased test is better than equal tails except when $\lambda_1 + \lambda_2 > 2$. When $\lambda_1 + \lambda_2 \leq 2$, bias is observed though small.
3. $\lambda_1 < 1$, $\lambda_2 < 1$. For tests 1) to 4), unbiased test seems to be better than equal tails. There exists some bias when close to H_0.
4. $L_2^{(2)}$ seems to be least biased, then $U^{(2)}$, then $W^{(2)}$ and lastly $V^{(2)}$.
5. $\lambda_1 \geq 1$, $\lambda_2 \geq 1$. In regard to comparative performance of the criteria, findings in the equal tail areas case are as in the one-sided case for 1) to 4) described by Pillai and Jayachandran [12], in the unbiased case when λ_1 and λ_2 are far apart but both greater than unity, in terms of power, $U^{(2)} > Z^{(2)} > W^{(2)} > V^{(2)} > L_2^{(2)} > LS^{(2)}$, but with only one large positive deviation, $L_2^{(2)} > U^{(2)} > W^{(2)} > Z^{(2)} > LS^{(2)} > V^{(2)}$. But if λ_1 and λ_2 are close, then $V^{(2)} > W^{(2)} > U^{(2)} > L_2^{(2)} > LS^{(2)} > Z^{(2)}$.

6. $\lambda_1 < 1$, $\lambda_2 > 1$ or $\lambda_1 > 1$, $\lambda_2 < 1$. In the unbiased case, $U^{(2)} > W^{(2)} > L_2^{(2)} > V^{(2)} > Z^{(2)} > LS^{(2)}$ when $\lambda_1 + \lambda_2 < 2$, $Z^{(2)} > LS^{(2)} > L_2^{(2)} > U^{(2)} > W^{(2)} > V^{(2)}$ when $\lambda_1 + \lambda_2 = 2$, $L_2^{(2)} > U^{(2)} > W^{(2)} > LS^{(2)} > Z^{(2)} > V^{(2)}$ when $\lambda_1 + \lambda_2 > 2$. In equal tail areas case, $L_2^{(2)} > U^{(2)} > W^{(2)} > V^{(2)}$ except for $\lambda_1 + \lambda_2 >> 2$, in which case $W^{(2)} > L_2^{(2)} > U^{(2)} > V^{(2)}$.

7. $\lambda_1 < 1$, $\lambda_2 < 1$. In the unbiased case, $V^{(2)} > W^{(2)} > U^{(2)} > L_2^{(2)} > Z^{(2)} > LS^{(2)}$, and in the equal tail areas case, $L_2^{(2)} > U^{(2)} > W^{(2)} > V^{(2)}$.

8. In comparing $LS^{(2)}$ with the approximate methods $LS_1^{(2)}$, $LS_2^{(2)}$ and $LS_3^{(2)}$ from Table 6, it is observed that $LS_1^{(2)}$ to $LS_3^{(3)}$ are all seriously biased and $LS_3^{(2)}$ especially so.

9. The bias in the tests 1) to 4), $LS_1^{(2)}$ and $LS_2^{(2)}$ disappears gradually with increasing m. Tests are practically unbiased when m = 5 i.e., $n_1 = 13$, $LS_3^{(2)}$ does not seem to become unbiased with large m.

10. If a single test has to be recommended on an overall basis over the whole parameter space, Roy's largest root seems to be the proper candidate. In the two-sided case as well as when both λ_1 and λ_2 are less than unity, among tests 1) to 4), largest root performs best in the equal tail areas case. Since the largest root is the least biased, even equal tail areas could be adequate. However, for the two-sided case, the unbiased largest root test compare favorably with $Z^{(2)}$ and $LS^{(2)}$ when $\lambda_1 + \lambda_2 = 2$ and is even the best when $\lambda_1 + \lambda_2 > 2$.

Table 1

Percentage points of $L_2^{(2)}$, $U^{(2)}$, $V^{(2)}$, $(W^{(2)})^{1/2}$, $LS^{(2)}$ and $L_3^{(3)}$

m \ n		5	10	15	20	25	30	40	60	80	100
						5% points of $L_2^{(2)}$ with ℓup					
0	a	.078233	.046349	.032920	.025523	.020839	.017608	.013440	.0091213	.0069031	.0055527
	b	.631304	.436723	.332050	.267488	.223832	.192383	.150139	.104288	.079879	.064727
1	a	.158169	.097653	.070633	.055325	.045470	.038596	.029635	.020237	.015365	.012384
	b	.697949	.505015	.392957	.321009	.271141	.234607	.184730	.129553	.099737	.081071
2	a	.229499	.146966	.108125	.085530	.070748	.060323	.046593	.032019	.024390	.019697
	b	.743316	.556725	.441531	.365025	.310846	.270561	.214756	.151938	.117522	.095809
5	a	.391002	.272387	.209192	.169814	.142937	.123500	.096624	.068000	.051235	.041585
	b	.821942	.659518	.545565	.463759	.402798	.355941	.287492	.209072	.161090	.132237
						Lower 2.5% points of $L_2^{(2)}$					
0		.073548	.042195	.029589	.022783	.018523	.015606	.011867	.0080232	.0060602	.0048690
1		.156683	.094226	.067399	.052468	.042954	.036362	.027822	.018931	.014346	.011550
2		.231254	.144878	.105550	.083029	.068433	.058203	.044808	.030686	.023333	.018823
5		.398551	.273826	.208810	.168804	.141682	.122077	.095624	.066719	.051235	.041585
						Upper 2.5% points of $L_2^{(2)}$					
0		.617908	.417886	.313858	.250942	.208930	.178926	.138963	.096023	.073347	.059332
1		.695660	.496309	.382840	.310983	.261643	.225739	.177051	.123631	.094955	.077072
2		.745265	.552837	.435546	.358447	.304254	.264180	.208985	.147294	.113692	.092565
5		.827152	.661201	.544999	.462038	.400460	.353132	.285373	.205996	.161090	.132237

Table 1 (continued)

m \ n		5	10	15	20	25	30	40	60	80	100
		5% points of $U^{(2)}$ with ℓup									
0	a	.106792	.061166	.042822	.032934	.026752	.022523	.017110	.011550	.0087301	.0070119
	b	1.89953	.877542	.567392	.418687	.331591	.274431	.204017	.134728	.100717	.080357
1	a	.258332	.149172	.104797	.080757	.065683	.055349	.042100	.028468	.021505	.017278
	b	2.67383	1.21937	.784457	.577371	.456546	.377454	.280257	.184932	.137980	.110038
2	a	.429501	.249669	.175920	.135792	.110564	.093230	.070988	.048050	.036316	.029188
	b	3.43163	1.55037	.993530	.729694	.576218	.475913	.352973	.232625	.173459	.138279
5	a	.988636	.582690	.413327	.320295	.261460	.220928	.167803	.114412	.084408	.067858
	b	5.66433	2.51423	1.59841	1.16857	.920040	.758492	.558353	.368596	.268521	.213779
		Lower 2.5% points of $U^{(2)}$									
0		.098891	.054938	.038041	.029095	.023555	.019788	.014993	.010098	.0076128	.0061093
1		.252579	.141904	.098716	.075691	.061378	.051618	.039164	.026417	.019931	.016002
2		.428758	.242889	.169574	.130283	.105782	.089040	.067636	.045678	.034483	.027696
5		1.00831	.580524	.408370	.315129	.256603	.216431	.164842	.111643	.084408	.067858
		Upper 2.5% points of $U^{(2)}$									
0		1.78212	.808820	.520238	.382990	.302930	.250509	.186070	.122836	.091673	.073120
1		2.61830	1.16901	.747279	.548380	.432890	.357515	.265110	.174722	.130290	.103870
2		3.42571	1.51260	.962659	.704750	.555510	.458321	.339420	.223403	.166470	.132660
5		5.78781	2.50510	1.58046	1.15140	.904690	.744770	.549980	.360907	.268520	.213780
		5% points of $V^{(2)}$ with ℓup									
0	a	.100390	.058783	.041597	.032193	.026258	.022172	.016909	.011483	.0086742	.0069757
	b	.797466	.532218	.399122	.319178	.265879	.227819	.177096	.122689	.093654	.075794
1	a	.224051	.136509	.098263	.076778	.063008	.053429	.040973	.027945	.021204	.017083
	b	.982657	.677643	.516977	.417800	.350511	.301871	.236273	.164678	.126375	.102527
2	a	.343947	.217315	.159019	.125424	.103560	.088192	.068012	.046663	.035516	.028667
	b	1.11565	.793692	.615388	.502339	.424316	.367244	.289365	.203160	.156519	.127291
5	a	.639300	.439525	.335369	.271287	.227762	.196333	.150650	.107400	.080524	.065324
	b	1.36073	1.03938	.838627	.702397	.603950	.529710	.418538	.304509	.232806	.190511

Table 1 (continued)

m \ n		5	10	15	20	25	30	40	60	80	100
					Lower 2.5% points of $V^{(2)}$						
0		.092062	.052764	.036986	.028473	.023146	.019498	.014826	.010022	.0075696	.0060814
1		.216162	.129637	.092620	.072054	.058965	.049901	.038167	.025960	.019669	.015833
2		.338074	.210867	.153321	.120472	.099221	.084346	.064892	.044410	.033756	.027225
5		.639285	.435841	.331081	.267032	.223784	.192607	.150650	.104948	.080524	.065324
					Upper 2.5% points of $V^{(2)}$						
0		.769068	.504284	.374451	.297612	.246883	.210907	.163290	.112476	.085776	.069319
1		.966662	.659036	.499376	.401796	.336047	.288760	.225311	.156498	.119875	.097140
2		1.10667	.780503	.601943	.489614	.412520	.356360	.280053	.196040	.150788	.122506
5		1.36071	1.03441	.831923	.695107	.596708	.522614	.418538	.299220	.232806	.190511
					5% points of $(W^{(2)})^{\frac{1}{2}}$ with ℓup						
0	a	.556476	.714448	.789734	.833659	.862420	.882707	.909426	.937780	.952615	.961739
	b	.949462.	.970439	.979108	.983845	.986831	.988885	.991528	.994259	.995658	.996509
1	a	.469095	.641567	.730065	.783638	.819506	.845188	.879497	.916517	.936140	.948295
	b	.886413	.931071	.950501	.961381	.968339	.973172	.979446	.985996	.989380	.991447
2	a	.406686	.583893	.680522	.740932	.782193	.812142	.852687	.897124	.920972	.935846
	b	.824861	.889985	.919742	.936816	.947895	.955667	.965852	.976601	.982203	.985641
5	a	.292315	.462786	.568819	.640356	.691685	.730252	.783620	.846003	.882630	.904098
	b	.672940	.776766	.830277	.863039	.885176	.901140	.922357	.946053	.959621	.967261
					Lower 2.5% points of $(W^{(2)})^{\frac{1}{2}}$						
0		.571871	.730027	.803227	.845268	.872528	.891629	.916626	.942951	.956643	.965035
1		.476229	.651321	.739389	.792083	.827095	.852032	.885175	.920712	.939455	.951032
2		.409925	.590469	.687466	.747539	.788314	.817777	.857484	.900765	.923889	.938274
5		.291242	.464886	.572104	.644001	.695377	.733864	.787620	.848795	.882630	.904098

Table 1 (continued)

m \ n		5	10	15	20	25	30	40	60	80	100
					Upper 2.5% points of $(W^{(2)})^{\frac{1}{2}}$						
0		.953421	.973441	.981420	.985712	.988393	.990227	.992573	.994983	.996212	.996957
1		.889830	.934452	.953322	.963752	.970370	.974944	.980855	.986992	.990149	.992073
2		.827014	.893092	.922573	.939296	.950073	.957600	.967420	.977733	.983086	.986364
5		.671792	.778335	.832356	.865137	.887171	.903007	.924257	.947326	.959620	.967261
					5% points of $LS^{(2)}$ with ℓup						
0	a	.0031051	.0018394	.0013046	.0010104	.00082430	.00069606	.00053086	.00035996	.00027228	.00021895
	b	.673421	.477232	.367102	.297749	.250271	.215783	.169105	.117981	.090577	.073499
1	a	.024594	.015004	.010781	.0084102	.0068935	.0058395	.0044727	.0030460	.0023093	.0018596
	b	.719427	.528510	.414542	.340302	.288383	.250098	.197564	.139025	.107225	.087256
2	a	.058204	.036531	.026595	.020905	.017219	.014636	.011261	.0077048	.0058562	.0047227
	b	.756681	.573029	.457318	.379590	.324137	.282670	.224995	.159650	.123692	.100940
5	a	.173052	.116768	.088137	.070801	.059169	.050803	.039320	.027524	.010231	.0080733
	b	.826948	.667352	.554116	.472327	.411050	.363530	.293502	.214186	.149354	.122410
					5% points of $L_3^{(3)}$ with ℓup						
0	a	.185837	.115332	.083607	.065570	.053934	.045805	.035195	.024053	.018269	.014728
	b	.711361	.517835	.404052	.330596	.279524	.242034	.190756	.133911	.103144	.083867
1	a	.273346	.176655	.130502	.103471	.085716	.073163	.056587	.038938	.029795	.023242
	b	.761110	.575177	.458147	.379726	.323904	.282258	.224386	.158997	.123519	.097939
2	a	.344427	.230426	.173140	.138670	.115620	.098954	.076035	.052528	.040124	.032459
	b	.795605	.618927	.501474	.420252	.361181	.316042	.250795	.178756	.138810	.113441
5	a	.493845	.356089	.281400	.230543	.195261	.169347	.133828	.094281	.072776	.059260
	b	.856577	.705650	.596680	.512244	.447975	.397695	.324414	.236690	.186188	.153412
					Lower 2.5% points of $L_3^{(3)}$						
0		.186066	.112743	.080893	.063078	.051695	.043792	.033539	.022843	.017319	.013947
1		.277532	.176022	.128915	.101705	.083982	.071520	.055153	.037836	.028795	.023242
2		.351212	.231415	.172602	.137635	.114452	.097954	.076035	.052528	.040124	.032459
5		.503945	.361083	.281400	.230543	.195261	.169347	.133828	.094281	.072776	.059260

Table 1 (continued)

m \ n	5	10	15	20	25	30	40	60	80	100
					Upper 2.5% points of $L_3^{(3)}$					
0	.711674	.511979	.396465	.322770	.271953	.234871	.184458	.128982	.099137	.080502
1	.765208	.574130	.454863	.375584	.319495	.277842	.220239	.155561	.120210	.097940
2	.801042	.620274	.500551	.418225	.358696	.313809	.250795	.178756	.138810	.113441
5	.862473	.710482	.596680	.512244	.447976	.397695	.324414	.236690	.186188	.153412

Table 2

Percentage points of $L_2^{(2)}$, $U^{(2)}$, $V^{(2)}$, $(W^{(2)})^{1/2}$, $LS^{(2)}$ and $Z^{(2)}$

test \ n_1		3	5	7	13
		5% points of tests 1) to 5) with ℓup			
$L_2^{(2)}$	a	.150415	.194303	.217283	.249727
	b	.879749	.782155	.720098	.621018
$U^{(2)}$	a	.221589	.330719	.400672	.518881
	b	7.76039	4.07220	3.06799	2.14719
$V^{(2)}$	a	.202030	.277900	.324936	.402030
	b	1.23671	1.13620	1.07231	.969926
$(W^{(2)})^{1/2}$	a	.287479	.381477	.430628	.499561
	b	.899426	.858903	.834674	.796084
$LS^{(2)}$	a	.0058353	.030358	.054955	.106398
	b	.903133	.800316	.734105	.629220
		Lower 2.5% points of tests 1) to 4)			
$L_2^{(2)}$		.153770	.195723	.218213	.250412
$U^{(2)}$		.225627	.330052	.398204	.515278
$V^{(2)}$		.192957	.270486	.318795	.398016
$(W^{(2)})^{1/2}$		.290421	.385738	.434498	.502147
		Upper 2.5% points of tests 1) to 4)			
$L_2^{(2)}$		.882622	.783731	.721217	.621892
$U^{(2)}$		7.90736	4.06403	3.04951	2.13282
$V^{(2)}$		1.21806	1.12340	1.06252	0.96421
$(W^{(2)})^{1/2}$		.901012	.861421	.837114	.797900
		Lower 5% points of $Z^{(2)}$			
$Z^{(2)}$		1.32301 (-5)	6.38366 (-7)	1.82925 (-8)	2.53203 (-12)

The numbers in parentheses indicate the power of 10 by which the tabulated values are to be multiplied.

Table 3

Percentage points of $LS_1^{(2)}$, $LS_2^{(2)}$ and $LS_3^{(2)}$

m \ n		5	15	30	60
		5% points of $LS_1^{(2)}$			
0	a	.0019456	.00076691	.00040179	.00020581
	b	.616853	.313126	.178468	.095766
1	a	.020893	.0086368	.0045961	.0023746
	b	.694902	.382234	.225337	.123397
2	a	.054045	.023401	.012658	.0066008
	b	.744652	.435001	.263802	.147066
5	a	.172848	.084453	.047943	.025734
	b	.826744	.544551	.352784	.205771
		5% points of $LS_2^{(2)}$			
0	a	.0019751	.00077843	.00040782	.00020890
	b	.617908	.313858	.178926	.096023
1	a	.021036	.0086953	.0046272	.0023907
	b	.695660	.382804	.225739	.123631
2	a	.054288	.023506	.012715	.0066305
	b	.745265	.435546	.264180	.147294
5	a	.173256	.084655	.048060	.025797
	b	.827152	.544999	.353132	.205996
		5% points of $LS_3^{(2)}$			
0	1-b	.0039379	.0015531	.00081385	.00041693
	b	.9960621	.9984469	.99918615	.99958307
1	1-b	.030424	.012612	.0067177	.0034725
	b	.969576	.987388	.9932823	.9965275
2	1-b	.070509	.030681	.016625	.0086772
	b	.929491	.969319	.983375	.9913228
5	1-b	.173052	.098895	.056328	.030295
	b	.826948	.901105	.943672	.969705

Table 4

Powers of $L_2^{(2)}$, $U^{(2)}$, $V^{(2)}$, $W^{(2)}$ and $LS^{(2)}$ in the unbiased and equal tail areas cases for testing $\lambda_1=1$, $\lambda_2=1$ against different simple two-sided alternatives hypotheses, $\alpha=.05$

$m = 0,\ n = 5$

		With local unbiasedness property					With equal tail areas			
λ_1	λ_2	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$LS^{(2)}$	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
1	1.001	.050000	.050000	.050000	.050000	.050000	.050011	.050014	.050017	.050016
1	1.1	.050586	.050594	.050511	.050576	.050347	.051691	.051959	.052229	.052156
1.05	1.05	.050434	.050475	.050509	.050499	.050203	.051534	.051843	.052247	.052089
1	1.5	.0624	.0624	.0602	.0619	.0576	.0678	.0690	.0682	.0695
1.25	1.25	.0594	.0604	.0612	.0610	.0547	.0647	.0671	.0696	.0687
1	2	.0919	.0917	.0827	.0899	.0773	.1022	.1042	.0980	.1042
1.333	1.5	.0739	.0765	.0780	.0780	.0630	.0826	.0874	.0917	.0906
1	4	.267	.265	.224		.220	.288	.290	.256	
1	5	.353	.350	.301		.300	.375	.376	.336	
2	4	.342	.361	.357		.281	.367	.391	.393	
3	3	.339	.364	.371		.273	.364	.396	.409	
1	8	.546	.542	.491		.494	.566	.566	.523	
4.5	4.5	.579	.606	.610		.516	.601	.633	.642	
1	11	.665	.662	.616		.621	.681	.681	.664	
6	6	.731	.752	.754		.683	.747	.772	.777	
1.00001	0.9	.050649	.050663	.050590	.050648	.050379	.049520	.049260	.048793	.049015
1.00001	0.8	.052787	.052856	.052600	.052808	.051643	.050483	.049989	.048896	.049462
1.01	0.99	.050006	.050005	.050001	.050004	.050006	.050007	.050005	.050000	.050003
1.1	0.9	.050658	.050532	.050080	.050369	.050621	.050671	.050515	.049995	.050327
1.1	0.8	.052114	.051902	.050976	.051582	.051782	.051001	.050452	.048984	.049847
1.05	0.95	.050164	.050133	.050019	.050092	.050154	.050168	.050128	.049998	.050081
1.2	0.99	.0521	.0521	.0517	.0520	.0513	.0542	.0547	.0549	.0550
1.2	0.8	.0526	.0520	.0503	.0514	.0524	.0526	.0520	.0499	.0512
2	0.9	.091	.092	.077	.081	.077	.099	.103	.089	.094
2	0.7	.091	.087	.072	.081	.084	.098	.095	.080	.090
3	0.9	.15	.15	.12	.15	.13	.19	.19	.12	.17
5	0.9	.34	.33	.31	.32	.28	.36	.36	.30	.35
0.99999	0.9	.050649	.050663	.050590	.050648	.050379	.049520	.049260	.048793	.049015
0.99999	0.7	.0589	.0591	.0592	.0591	.0567	.0551	.0545	.0531	.0537
0.999	0.9	.050656	.050670	.050601	.050657	.050381	.049515	.049253	.048786	.049007
0.9	0.9	.051973	.052142	.052289	.052241	.050861	.049667	.049307	.048697	.048955
0.9	0.8	.054955	.055317	.055566	.055504	.052280	.051414	.050969	.050050	.050467
0.9	0.76	.0569	.0574	.0577	.0576	.0535	.0528	.0524	.0513	.0518
0.85	0.9	.053216	.053476	.053683	.053621	.051417	.050303	.049898	.049150	.049477
0.85	0.8	.0567	.0572	.0577	.0575	.0530	.0525	.0521	.0512	.0516
0.81	0.9	.054557	.054897	.055138	.055076	.052071	.051144	.050707	.049823	.050222
0.8	0.8	.0593	.0599	.0605	.0603	.0543	.0544	.0539	.0530	.0534

Table 4 (continued)

m = 1, n = 5

λ_1	λ_2	With local unbiasedness property $L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$LS^{(2)}$	With equal tail areas $L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
1	1.001	.050000	.050000	.050000	.050000	.050000	.050003	.050006	.050013	.050010
1	1.1	.050809	.050843	.050736	.050820	.050629	.051092	.051480	.052000	.051855
1.05	1.05	.050621	.050715	.050798	.050772	.050374	.050902	.051354	.052084	.051817
1	1.5	.0672	.0676	.0641	.0667	.0633	.0686	.0706	.0699	.0716
1.25	1.25	.0632	.0654	.0673	.0668	.0585	.0646	.0685	.0735	.0718
1	2	.1099	.1103	.0960	.1079	.0979	.1125	.1160	.1064	.1169
1.333	1.5	.0835	.0892	.0933	.0927	.0729	.0857	.0942	.1033	.1009
1	4	.384	.383	.342		.354	.388	.392	.357	
1	5	.508	.506	.454		.480	.512	.515	.469	
2	4	.489	.517	.520		.451	.493	.527	.539	
3	3	.486	.523	.541		.446	.491	.534	.561	
1	8	.740	.739	.701		.722	.743	.745	.711	
4.5	4.5	.779	.803	.812		.756	.782	.809	.822	
1	11	.849	.848	.822		.837	.851	.851	.829	
6	6	.903	.915	.919		.892	.905	.918	.924	
1.00001	0.9	.050895	.050948	.050867	.050934	.050713	.050603	.050283	.049517	.049846
1.00001	0.8	.054072	.054335	.054124	.054313	.053393	.053469	.052957	.051302	.052053
1.01	0.99	.050008	.050006	.050001	.050003	.050011	.050008	.050005	.049997	.050002
1.1	0.9	.050792	.050581	.050084	.050275	.051132	.050799	.050572	.049772	.050237
1.1	0.8	.052727	.052434	.051023	.051868	.053447	.052445	.051743	.049523	.050689
1.05	0.95	.050198	.050145	.050021	.050068	.050281	.050200	.050142	.049942	.050058
1.2	0.99	.0529	.0530	.0524	.0528	.0523	.0535	.0541	.0547	.0547
1.2	0.8	.0527	.0519	.0504	.0506	.0541	.0530	.0518	.0486	.0505
2	0.9	.103	.101	.080	.095	.095	.106	.106	.089	.103
2	0.7	.119	.112	.088	.102	.119	.120	.116	.093	.107
3	0.9	.22	.22	.17	.22	.20	.23	.23	.18	.23
5	0.9	.49	.48	.41	.49	.46	.49	.49	.43	.51
0.99999	0.9	.050895	.050948	.050867	.050934	.050715	.050603	.050283	.049518	.049846
0.99999	0.7	.0842	.0850	.0864	.0859	.0850	.0829	.0822	.0830	.0814
0.999	0.9	.050905	.050961	.050885	.050949	.050715	.050681	.050289	.049523	.049851
0.9	0.9	.052897	.053295	.053641	.053526	.051650	.052292	.051939	.050939	.051326
0.9	0.8	.058288	.059138	.059822	.059575	.055503	.057339	.057009	.055588	.056128
0.9	0.76	.0656	.0666	.0677	.0671	.0625	.0645	.0640	.0626	.629
0.85	0.9	.054819	.055437	.055943	.055778	.052833	.054050	.053714	.052515	.052986
0.85	0.8	.0630	.0640	.0649	.0646	.0589	.0618	.0614	.0599	.0604
0.81	0.9	.057340	.058146	.058785	.058566	.054707	.056430	.056105	.054724	.055260
0.8	0.8	.0747	.0755	.0763	.0758	.0697	.0732	.0723	.0701	.0707

Table 4 (continued)

$m = 2,\ n = 5$

		With local unbiasedness property					With equal tail areas			
λ_1	λ_2	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$LS^{(2)}$	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
1	1.001	.050000	.050000	.050000	.050000	.050000	.049997	.050001	.050009	.050006
1	1.1	.050968	.051036	.050932	.051015	.050857	.050652	.051100	.051805	.051613
1.05	1.05	.050760	.050905	.051035	.050994	.050516	.050447	.050969	.051926	.051598
1	1.5	.0708	.0716	.0677	.0707	.0678	.0693	.0719	.0716	.0734
1.25	1.25	.0660	.0693	.0724	.0715	.0615	.0646	.0696	.0766	.0744
1	2	.1259	.1271	.1106	.1250	.1164	.1231	.1276	.1173	.1301
1.333	1.5	.0907	.0991	.1060	.1049	.0808	.0884	.0996	.1128	.1096
1	4	.499	.499	.458		.481	.495	.499	.463	
1	5	.645	.644	.605		.630	.642	.645	.612	
2	4	.620	.648	.656		.598	.617	.649	.655	
3	3	.622	.657	.679		.599	.618	.658	.688	
1	8	.864	.864	.844		.858	.863	.864	.847	
4.5	4.5	.897	.910	.918		.889	.896	.911	.920	
1	11	.938	.938	.929		.935	.938	.938	.930	
6	6	.970	.974	.976		.967	.970	.974	.977	
1.00001	0.9	.051071	.051171	.051111	.051168	.050998	.051401	.051103	.050167	.050532
1.00001	0.8	.056131	.056628	.056651	.056697	.056080	.056824	.056485	.054644	.055351
1.01	0.99	.050008	.050006	.050001	.050002	.050015	.050008	.050006	.049996	.050001
1.1	0.9	.050859	.050597	.050097	.050185	.051551	.050848	.050596	.049665	.050159
1.1	0.8	.053395	.053112	.051630	.052403	.055185	.053709	.053041	.050521	.051703
1.05	0.95	.050215	.050149	.050022	.050046	.050385	.050213	.050149	.049915	.050039
1.2	0.99	.0535	.0536	.0530	.0534	.0531	.0529	.0537	.0546	.0545
1.2	0.8	.0509	.0510	.0508	.0510	.0537	.0529	.0499	.0462	.0489
2	0.9	.117	.115	.090	.107	.112	.115	.115	.096	.111
2	0.7	.155	.147	.121	.136	.161	.150	.148	.124	.135
3	0.9	.29	.30	.23	.28	.27	.28	.28	.24	.29
5	0.9	.62	.62	.57	.63	.61	.62	.62	.57	.64
0.99999	0.9	.051071	.051171	.051111	.051168	.050998	.051401	.051103	.050168	.050532
0.99999	0.7	.225	.226	.231	.230	.233	.227	.226	.235	.226
0.999	0.9	.051084	.051188	.051135	.051189	.051001	.051417	.051120	.050183	.050547
0.9	0.9	.053676	.054299	.054833	.054661	.052406	.054365	.054161	.052938	.053369
0.9	0.8	.068291	.069378	.070446	.069948	.065952	.069438	.069147	.067296	.067795
0.9	0.76	.1111	.1121	.1142	.1125	.1107	.1134	.1118	.1098	.1094
0.85	0.9	.056841	.057785	.058562	.058305	.054881	.057725	.057607	.056134	.056650
0.85	0.8	.0927	.0931	.0936	.0932	.0897	.0942	.0928	.0895	.0903
0.81	0.9	.064301	.065405	.066389	.065983	.061961	.065382	.065187	.063420	.063956
0.8	0.8	.1845	.1805	.1771	.1774	.1839	.1869	.1800	.1705	.1729

Table 4 (continued)

m = 5, n = 5

λ_1	λ_2	With local unbiasedness property $L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$LS^{(2)}$	With equal tail areas $L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
1	1.001	.050000	.050000	.050000	.050000	.050000	.049985	.049989	.050000	.049996
1	1.1	.051264	.051422	.051074	.051431	.051339	.049830	.050335	.051374	.051111
1.05	1.05	.051026	.051290	.051506	.051465	.050821	.049611	.050201	.051539	.051141
1	1.5	.0789	.0808	.0785	.0800	.0781	.0722	.0758	.0767	.0786
1.25	1.25	.0713	.0773	.0821	.0815	.0678	.0650	.0722	.0831	.0799
1	2	.1804	.1836	.1551	.1846	.1764	.1689	.1752	.1660	.1822
1.333	1.5	.1072	.1218	.1302	.1332	.0998	.0972	.1136	.1352	.1307
1	4	.780	.781	.740		.777	.774	.776	.770	
1	5	.898	.898	.887		.896	.895	.896	.889	
2	4	.876	.888	.893		.869	.872	.885	.895	
3	3	.883	.897	.906		.876	.879	.894	.909	
1	8	.986	.986	.984		.986	.986	.986	.985	
4.5	4.5	.993	.994	.995		.992	.993	.994	.995	
1	11	.997	.997	.997		.997	.997	.997	.997	
6	6	.999	.999	.999		.999	.999	.999	.999	
1.00001	0.9	.051453	.051690	.051563	.051743	.051702	.052962	.052859	.051743	.052090
1.00001	0.8	.132	.133	.136	.134	.135	.137	.137	.136	.135
1.01	0.99	.050009	.050006	.050002	.050000	.050024	.050008	.050006	.049995	.050000
1.1	0.9	.050917	.050593	.050132	.050004	.052444	.050839	.050605	.049586	.050021
1.1	0.8	.0645	.0645	.0638	.0639	.0689	.0660	.0658	.0638	.0644
1.05	0.95	.050234	.050152	.050034	.050005	.050607	.050214	.050155	.049898	.050008
1.2	0.99	.0546	.0549	.0544	.0548	.0548	.0519	.0529	.0544	.0542
1.2	0.8	.0782	.0792	.0818	.0806	.0812	.0786	.0795	.0818	.0807
2	0.9	.167	.164	.142	.155	.167	.156	.156	.136	.153
2	0.7	.318	.312	.292	.301	.331	.313	.309	.292	.300
3	0.9	.51	.51	.48	.51	.51	.50	.50	.47	.51
5	0.9	.89	.89	.88	.89	.89	.88	.88	.87	.89
0.99999	0.9	.051453	.051691	.051563	.051744	.051703	.052962	.052860	.051744	.052090
0.99999	0.7									
0.999	0.9	.051475	.051720	.051696	.051779	.051711	.053001	.052901	.051783	.052129
0.9	0.9	.062828	.063803	.064427	.064351	.061819	.066160	.066342	.064596	.065092
0.9	0.8									
0.9	0.76									
0.85	0.9	.150	.150	.149	.149	.150	.156	.154	.149	.150

Table 4 (continued)

m = 0, n = 15

λ_1	λ_2	With local unbiasedness property					With equal tail areas			
		$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$LS^{(2)}$	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
1	1.001	.050000	.050000	.050000	.050000	.050000	.050023	.050026	.050027	.050027
1	1.1	.050811	.050800	.050745	.050778	.050469	.053126	.053455	.053473	.053487
1.05	1.05	.050586	.050643	.050648	.050647	.050280	.052884	.053300	.053387	.053361
1	1.5	.0677	.0672	.0659	.0667	.0611	.0793	.0803	.0792	.0800
1.25	1.25	.0630	.0644	.0646	.0646	.0569	.0744	.0776	.0782	.0780
1	2	.1103	.1083	.1034	.1071	.0907	.1322	.1328	.1287	.1320
1.333	1.5	.0838	.0872	.0874	.0876	.0692	.1023	.1088	.1096	.1096
1	4	.334	.326	.287		.280	.373	.370	.357	
1	5	.428	.419	.400		.371	.467	.463	.448	
2	4	.418	.439	.436		.348	.463	.489	.488	
3	3	.418	.447	.448		.343	.465	.500	.502	
1	8	.620	.612	.593		.569	.653	.649	.634	
4.5	4.5	.658	.687	.687		.592	.696	.727	.728	
1	11	.728	.721	.705		.686	.754	.750	.738	
6	6	.792	.813	.813		.744	.818	.840	.841	
1.00001	0.9	.050873	.050866	.050820	.050850	.050489	.048573	.048200	.048064	.048123
1.00001	0.8	.053663	.053663	.053473	.053603	.052043	.049053	.048295	.047921	.048102
1.01	0.99	.050009	.050007	.050004	.050006	.050007	.050010	.050006	.050004	.050005
1.1	0.9	.050941	.050686	.050452	.050585	.050790	.051014	.050679	.050409	.050561
1.1	0.8	.052872	.052402	.051919	.052202	.052186	.050708	.049708	.049051	.049409
1.05	0.95	.050235	.050171	.050112	.050146	.050197	.050254	.050169	.050102	.050140
1.2	0.99	.0530	.0529	.0526	.0528	.0518	.0574	.0579	.0578	.0579
1.2	0.8	.0537	.0527	.0517	.0523	.0531	.0540	.0526	.0516	.0522
2	0.9	.106	.101	.095	.098	.091	.126	.123	.118	.122
2	0.7	.105	.097	.088	.093	.094	.121	.113	.105	.109
3	0.9	.21	.20	.19	.20	.17	.25	.24	.23	.25
5	0.9	.42	.40	.38	.40	.36	.46	.45	.43	.50
0.99999	0.9	.050873	.050866	.050820	.050851	.050489	.048573	.048200	.048063	.048123
0.99999	0.7	.0600	.0602	.0601	.0601	.0567	.0527	.0517	.0511	.0514
0.999	0.9	.050881	.050876	.050832	.050861	.050491	.048557	.048184	.048048	.048107
0.9	0.9	.052597	.052806	.052844	.052836	.051137	.047926	.047463	.047339	.047380
0.9	0.8	.056406	.056828	.056866	.056869	.052878	.049332	.048744	.048529	.048612
0.9	0.76	.0587	.0593	.0593	.0593	.0541	.0506	.0500	.0497	.0498
0.85	0.9	.054208	.054522	.054568	.054562	.051844	.048345	.047823	.047664	.047721
0.85	0.8	.0586	.0592	.0593	.0593	.0537	.0503	.0497	.0495	.0496
0.81	0.9	.055912	.056312	.056352	.056353	.052636	.049083	.048509	.048306	.048384
0.8	0.8	.0616	.0624	.0625	.0625	.0551	.0520	.0514	.0513	.0513

Table 4 (continued)

m = 1, n = 15

		With local unbiasedness property					With equal tail areas			
λ_1	λ_2	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$LS^{(2)}$	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
1	1.001	.050000	.050000	.050000	.050000	.050000	.050017	.050023	.050024	.050024
1	1.1	.051214	.051212	.051107	.051172	.050891	.052949	.053500	.053544	.053568
1.05	1.05	.050901	.051037	.051058	.051053	.050536	.052614	.053326	.053512	.053458
1	1.5	.0770	.0763	.0732	.0752	.0704	.0857	.0875	.0851	.0869
1.25	1.25	.0698	.0732	.0737	.0736	.0628	.0782	.0845	.0858	.0855
1	2	.1431	.1394	.1283	.1377	.1242	.1595	.1602	.1503	.1595
1.333	1.5	.1011	.1095	.1103	.1107	.0853	.1148	.1280	.1300	.1301
1	4	.475	.463	.431		.436	.499	.493	.464	
1	5	.597	.585	.553		.561	.618	.612	.584	
2	4	.581	.615	.612		.535	.606	.645	.644	
3	3	.581	.627	.630		.532	.607	.658	.662	
1	8	.802	.794	.771		.779	.814	.810	.790	
4.5	4.5	.837	.864	.865		.811	.850	.878	.880	
1	11	.888	.883	.869		.874	.896	.893	.880	
6	6	.932	.945	.945		.920	.938	.952	.953	
1.00001	0.9	.051296	.051320	.051230	.051289	.050955	.049569	.049005	.048740	.048852
1.00001	0.8	.055576	.055766	.055478	.055670	.054249	.052086	.051048	.050391	.050691
1.01	0.99	.050013	.050008	.050003	.050006	.050015	.050013	.050007	.050002	.050005
1.1	0.9	.051270	.050773	.050303	.050562	.051513	.051358	.050764	.050227	.050523
1.1	0.8	.053974	.053157	.052248	.052757	.054365	.052408	.050798	.049565	.050213
1.05	0.95	.050318	.050193	.050075	.050140	.050376	.050340	.050190	.050056	.050130
1.2	0.99	.0545	.0543	.0538	.0541	.0534	.0578	.0586	.0584	.0586
1.2	0.8	.0546	.0527	.0508	.0519	.0557	.0550	.0527	.0505	.0518
2	0.9	.136	.127	.112	.123	.121	.151	.146	.132	.143
2	0.7	.142	.126	.109	.118	.138	.153	.139	.121	.131
3	0.9	.30	.28	.25	.30	.27	.33	.31	.28	.33
5	0.9	.58	.56	.52	.63	.55	.60	.59	.56	.66
0.99999	0.9	.051296	.051320	.051230	.051290	.050955	.049569	.049005	.048740	.048852
0.99999	0.7	.0803	.0820	.0842	.0826	.0805	.0734	.0727	.0740	.0728
0.999	0.9	.051309	.051339	.051257	.051310	.050959	.049564	.049000	.048737	.048848
0.9	0.9	.054065	.054599	.054693	.054669	.052243	.050512	.049931	.049697	.049772
0.9	0.8	.060781	.061865	.062031	.061975	.056629	.055287	.054652	.054303	.054407
0.9	0.76	.0677	.0690	.0694	.0691	.0629	.0611	.0603	.0602	.0600
0.85	0.9	.056662	.057470	.057595	.057566	.053743	.052176	.051584	.051295	.051392
0.85	0.8	.0659	.0673	.0675	.0674	.0601	.0593	.0586	.0583	.0584
0.81	0.9	.059738	.060769	.060919	.060876	.055827	.054460	.053841	.053498	.053608
0.8	0.8	.0771	.0780	.0782	.0780	.0699	.0689	.0674	.0670	.0669

Table 4 (continued)

m = 2, n = 15

		With local unbiasedness property					With equal tail areas			
1	2	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$LS^{(2)}$	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
1	1.001	.050000	.050000	.050000	.050000	.050000	.050013	.050019	.050022	.050021
1	1.1	.051552	.051574	.051430	.051519	.051271	.052801	.053516	.053590	.053620
1.05	1.05	.051168	.051390	.051429	.051420	.050768	.052397	.053333	.053610	.053532
1	1.5	.0850	.0842	.0796	.0826	.0788	.0914	.0937	.0900	.0928
1.25	1.25	.0755	.0809	.0819	.0817	.0681	.0815	.0905	.0926	.0922
1	2	.1729	.1680	.1509	.1667	.1552	.1846	.1853	.1698	.1855
1.333	1.5	.1158	.1293	.1310	.1319	.0997	.1255	.1449	.1484	.1488
1	4	.594	.581	.560		.567	.607	.600	.572	
1	5	.725	.714	.681		.703	.735	.729	.699	
2	4	.707	.742	.740		.678	.719	.759	.759	
3	3	.710	.756	.760		.679	.722	.773	.778	
1	8	.902	.897	.881		.893	.906	.903	.889	
4.5	4.5	.929	.945	.946		.919	.933	.949	.951	
1	11	.957	.954	.947		.952	.959	.957	.950	
6	6	.980	.985	.985		.977	.981	.986	.987	
1.00001	0.9	.051644	.051719	.051611	.051681	.051387	.050399	.049742	.049378	.049526
1.00001	0.8	.058015	.058529	.058315	.058460	.057274	.055449	.054414	.053666	.053962
1.01	0.99	.050015	.050008	.050001	.050005	.050021	.050015	.050008	.050000	.050004
1.1	0.9	.051510	.050813	.050149	.050502	.052170	.051593	.050805	.050058	.050457
1.1	0.8	.054999	.053989	.052892	.053444	.056646	.053890	.051954	.050348	.051155
1.05	0.95	.050378	.050203	.050037	.050125	.050539	.050399	.050201	.050014	.050114
1.2	0.99	.0557	.0556	.0549	.0553	.0548	.0582	.0593	.0589	.0592
1.3	0.8	.0538	.0511	.0502	.0507	.0563	.0542	.0511	.0481	.0506
2	0.9	.164	.151	.128	.145	.150	.175	.166	.145	.162
2	0.7	.184	.163	.140	.152	.186	.190	.173	.149	.160
3	0.9	.38	.36	.32	.38	.36	.40	.38	.34	.40
5	0.9	.71	.69	.65	.75	.69	.72	.71	.67	.77
0.99999	0.9	.051644	.051720	.051612	.051682	.051387	.050399	.049742	.049378	.049526
0.99999	0.7	.192	.198	.213	.203	.203	.182	.184	.197	.187
0.999	0.9	.051663	.051746	.051645	.051710	.051392	.050404	.049748	.049387	.049533
0.9	0.9	.055401	.056286	.056447	.056405	.053337	.052810	.052272	.051957	.052054
0.9	0.8	.070673	.072146	.072611	.072345	.066687	.066391	.065517	.065194	.065158
0.9	0.76	.1075	.1083	.1103	.1087	.1055	.1014	.0987	.0996	.0984
0.85	0.9	.059454	.060758	.060984	.060920	.056228	.056149	.055644	.055262	.055376
0.85	0.8	.0932	.0934	.0937	.0934	.0883	.0875	.0846	.0840	.0839
0.81	0.9	.067021	.068513	.068877	.068699	.063072	.062989	.062272	.061893	.061933
0.8	0.8	.1732	.1658	.1650	.1645	.1720	.1640	.1516	.1492	.1491

Table 4 (continued)

m = 5, n = 15

		With local unbiasedness property					With equal tail areas			
λ_1	λ_2	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$LS^{(2)}$	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
1	1.001	.050000	.050000	.050000	.050000	.050000	.050002	.050011	.050015	.050014
1	1.1	.052313	.052449	.052248	.052360	.052206	.052486	.053531	.053695	.053729
1.05	1.05	.051776	.052256	.052367	.052326	.051347	.051944	.053338	.053837	.053704
1	1.5	.1048	.1042	.0961	.1015	.0999	.1057	.1094	.1028	.1080
1.25	1.25	.0885	.0999	.1024	.1019	.0811	.0894	.1053	.1096	.1087
1	2	.2570	.2504	.2227	.2540	.2437	.2585	.2591	.2336	.2655
1.333	1.5	.1516	.1793	.1842	.1865	.1361	.1528	.1878	.1956	.1975
1	4	.836	.829	.820		.828	.836	.832	.822	
1	5	.927	.924	.912		.923	.927	.925	.914	
2	4	.915	.933	.934		.907	.916	.935	.936	
3	3	.920	.941	.944		.912	.921	.943	.947	
1	8	.991	.990	.988		.990	.991	.990	.989	
4.5	4.5	.995	.997	.997		.995	.995	.997	.997	
1	11	.998	.998	.997		.998	.998	.998	.997	
6	6	.999	.999	.999		.999	.999	.999	.999	
1.00001	0.9	.052484	.052753	.052639	.052725	.052555	.052312	.051634	.051104	.051294
1.00001	0.8	.126	.129	.132	.130	.129	.126	.125	.127	.126
1.01	0.09	.050020	.050008	.050000	.050003	.050037	.050020	.050008	.049997	.050002
1.1	0.9	.051966	.050846	.050175	.050306	.053813	.051982	.050841	.049707	.050262
1.1	0.8	.0653	.0647	.0641	.0644	.0709	.0651	.0634	.0621	.0626
1.05	0.95	.050496	.050215	.050156	.050080	.050946	.050501	.050214	.049929	.050068
1.2	0.99	.0586	.0586	.0575	.0581	.0583	.0590	.0607	.0601	.0607
1.2	0.8	.0750	.0766	.0798	.0803	.0805	.0750	.0762	.0792	.0799
2	0.9	.242	.221	.185	.213	.234	.243	.229	.194	.223
2	0.7	.349	.326	.302	.313	.362	.350	.329	.305	.307
3	0.9	.60	.58	.54	.61	.59	.61	.59	.55	.62
5	0.9	.92	.91	.90	.94	.91	.92	.91	.90	.93
0.99999	0.9	.052485	.052754	.052640	.052726	.052555	.052312	.051635	.051105	.051295
0.99999	0.7									
0.999	0.9	.052518	.052802	.052695	.052778	.052566	.052344	.051671	.051144	.051333
0.9	0.9	.065542	.067017	.067328	.067240	.063303	.065159	.064606	.064035	.064185
0.9	0.8									
0.9	0.76									
0.85	0.9	.147	.146	.138	.145	.146	.147	.141	.134	.140

Table 4 (continued)

m = 0, n = 30

		With local unbiasedness property					With equal tail areas			
λ_1	λ_2	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$LS^{(2)}$	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
1	1.001	.050000	.050000	.050000	.050000	.050000	.050028	.050031	.050031	.050031
1	1.1	.050905	.050886	.050844	.050862	.050522	.053698	.054012	.053999	.054013
1.05	1.05	.050649	.050715	.050707	.050707	.050313	.053418	.053840	.053866	.053859
1	1.5	.0700	.0692	.0683	.0688	.0626	.0840	.0846	.0839	.0844
1.25	1.25	.0646	.0660	.0660	.0660	.0578	.0783	.0817	.0818	.0818
1	2	.1179	.1148	.1120	.1143	.0967	.1440	.1435	.1410	.1432
1.333	1.5	.0878	.0915	.0914	.0916	.0720	.1102	.1169	.1172	.1172
1	4	.358	.348	.309		.304	.402	.396	.388	
1	5	.455	.444	.433		.399	.498	.492	.483	
2	4	.446	.466	.464		.376	.496	.521	.520	
3	3	.447	.476	.476		.372	.500	.533	.533	
1	8	.646	.635	.625		.596	.680	.674	.666	
4.5	4.5	.686	.713	.713		.622	.727	.754	.754	
1	11	.749	.740	.732		.709	.776	.770	.764	
6	6	.812	.833	.833		.767	.840	.860	.860	
1.00001	0.9	.050963	.050935	.050915	.050932	.050533	.048214	.047823	.047763	.047790
1.00001	0.8	.054006	.053943	.053858	.053917	.052197	.048535	.047709	.047528	.047615
1.01	0.99	.050010	.050007	.050006	.050007	.050008	.050011	.050007	.050005	.050006
1.1	0.9	.051058	.050741	.050607	.050679	.050859	.051164	.050744	.050589	.050672
1.1	0.8	.053175	.052570	.052302	.052457	.052343	.050634	.049450	.049094	.049284
1.05	0.95	.050264	.050185	.050151	.050169	.050214	.050291	.050186	.050147	.050167
1.2	0.99	.0533	.0532	.0530	.0531	.0520	.0587	.0591	.0590	.0591
1.2	0.8	.0541	.0529	.0524	.0527	.0534	.0546	.0529	.0523	.0526
2	0.9	.113	.107	.103	.114	.094	.143	.133	.130	.137
2	0.7	.111	.101	.096	.099	.099	.130	.120	.115	.118
3	0.9	.23	.22	.21	.22	.19	.27	.26	.25	.27
5	0.9	.44	.43	.42	.43	.39	.49	.48	.47	.49
0.99999	0.9	.050963	.050935	.050915	.050932	.050533	.048214	.047822	.047763	.047789
0.99999	0.7	.0604	.0605	.0604	.0605	.0566	.0519	.0508	.0505	.0506
0.999	0.9	.050972	.050946	.050927	.050944	.050535	.048194	.047803	.047744	.047770
0.9	0.9	.052844	.053043	.053072	.053070	.051248	.047274	.046822	.046784	.046796
0.9	0.8	.056972	.057366	.057393	.057398	.053110	.048581	.048000	.047921	.047949
0.9	0.76	.0594	.0599	.0599	.0599	.0544	.0498	.0492	.0491	.0491
0.85	0.9	.054599	.054896	.054929	.054929	.052014	.047624	.047114	.047062	.047080
0.85	0.8	.0593	.0599	.0600	.0600	.0540	.0495	.0490	.0489	.0489
0.81	0.9	.056441	.056816	.056845	.056849	.052857	.048336	.047771	.047697	.047724
0.8	0.8	.0625	.0633	.0634	.0633	.0554	.0511	.0507	.0506	.0506

Table 4 (continued)

m = 1, n = 30

λ_1	λ_2	With local unbiasedness property $L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$LS^{(2)}$	With equal tail areas $L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
1	1.001	.050000	.050000	.050000	.050000	.050000	.050024	.050029	.050030	.050029
1	1.1	.051400	.051364	.051294	.051333	.051014	.053778	.054312	.054285	.054316
1.05	1.05	.051027	.051170	.051177	.051176	.050613	.053370	.054119	.054179	.054163
1	1.5	.0816	.0799	.0779	.0792	.0739	.0937	.0945	.0927	.0939
1.25	1.25	.0729	.0765	.0766	.0766	.0649	.0845	.0912	.0916	.0915
1	2	.1582	.1518	.1440	.1518	.1368	.1805	.1784	.1715	.1788
1.333	1.5	.1092	.1181	.1182	.1188	.0914	.1281	.1420	.1425	.1430
1	4	.513	.495	.471		.472	.542	.531	.512	
1	5	.633	.616	.589		.596	.658	.647	.627	
2	4	.619	.651	.646		.571	.649	.685	.682	
3	3	.620	.665	.663		.570	.652	.700	.700	
1	8	.825	.814	.794		.803	.840	.832	.818	
4.5	4.5	.859	.884	.882		.834	.874	.900	.900	
1	11	.903	.896	.883		.889	.912	.907	.898	
6	6	.943	.955	.954		.931	.949	.961	.961	
1.00001	0.9	.051471	.051469	.051412	.051444	.051057	.049138	.048517	.048397	.048450
1.00001	0.8	.056216	.056323	.056150	.056242	.054589	.051544	.050348	.050019	.050168
1.01	0.99	.050015	.050008	.050005	.050007	.050016	.050016	.050008	.050005	.050006
1.1	0.9	.051491	.050841	.050555	.050706	.051679	.051636	.050839	.050509	.050684
1.1	0.8	.054523	.053417	.052865	.053163	.054736	.052463	.050431	.049719	.050090
1.05	0.95	.050373	.050210	.050138	.050176	.050418	.050410	.050209	.050127	.050170
1.2	0.99	.0552	.0549	.0546	.0547	.0539	.0598	.0605	.0602	.0604
1.2	0.8	.0556	.0530	.0518	.0526	.0564	.0562	.0530	.0517	.0525
2	0.9	.151	.138	.130	.102	.133	.172	.162	.153	.110
2	0.7	.153	.132	.121	.127	.147	.169	.148	.137	.143
3	0.9	.33	.31	.29	.34	.30	.37	.35	.33	.37
5	0.9	.62	.60	.56	.70	.58	.65	.63	.60	.73
0.99999	0.9	.051472	.051470	.051412	.051445	.051057	.049138	.048517	.048396	.048450
0.99999	0.7	.0785	.0806	.0823	.0809	.0782	.0695	.0691	.0702	.0691
0.999	0.9	.051487	.051490	.051434	.051466	.051061	.049129	.048509	.048389	.048442
0.9	0.9	.054569	.055126	.055157	.055149	.052493	.049780	.049200	.049127	.049149
0.9	0.8	.061829	.062938	.062992	.062958	.057070	.054476	.053844	.053726	.053744
0.9	0.76	.0686	.0699	.0701	.0699	.0630	.0598	.0591	.0591	.0589
0.85	0.9	.057449	.058283	.058318	.058311	.054121	.051421	.050833	.050737	.050767
0.85	0.8	.0672	.0685	.0687	.0686	.0605	.0583	.0577	.0576	.0576
0.81	0.9	.060750	.061805	.061849	.061828	.056275	.053680	.053062	.052944	.052971
0.8	0.8	.0780	.0789	.0791	.0788	.0698	.0672	.0657	.0656	.0654

Table 4 (continued)

m = 2, n = 30

λ_1	λ_2	With local unbiasedness property $L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$LS^{(2)}$	With equal tail areas $L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
1	1.001	.050000	.050000	.050000	.050000	.050000	.050020	.050027	.050028	.050028
1	1.1	.051842	.051818	.051706	.051766	.051477	.053855	.054562	.054523	.054568
1.05	1.05	.051367	.051614	.051622	.051619	.050895	.053340	.054358	.054452	.054428
1	1.5	.0923	.0900	.0865	.0888	.0845	.1026	.1036	.0997	.1026
1.25	1.25	.0803	.0863	.0866	.0866	.0716	.0900	.1001	.1007	.1007
1	2	.1963	.1868	.1706	.1882	.1753	.2149	.2110	.2037	.2131
1.333	1.5	.1285	.1433	.1433	.1448	.1096	.1442	.1654	.1653	.1674
1	4	.636	.617	.600		.607	.655	.642	.629	
1	5	.759	.743	.689		.737	.773	.762	.738	
2	4	.745	.778	.767		.715	.762	.799	.770	
3	3	.748	.792	.792		.716	.765	.812	.812	
1	8	.917	.910	.880		.908	.923	.918	.906	
4.5	4.5	.942	.956	.954		.932	.947	.961	.961	
1	11	.964	.961	.945		.960	.967	.964	.960	
6	6	.984	.988	.988		.981	.986	.990	.990	
1.00001	0.9	.051915	.051949	.051875	.051921	.051560	.049946	.049189	.049016	.049091
1.00001	0.8	.058864	.059297	.059276	.059230	.057746	.054849	.053598	.053378	.053372
1.01	0.99	.050018	.050009	.050004	.050007	.050024	.050020	.050008	.050003	.050006
1.1	0.9	.051840	.050890	.050460	.050684	.052455	.052000	.050886	.050397	.050652
1.1	0.8	.055763	.054309	.053626	.053956	.057264	.054078	.051488	.050607	.051006
1.05	0.95	.050461	.050222	.050114	.050171	.050610	.050502	.050221	.050099	.050163
1.2	0.99	.0569	.0565	.0560	.0562	.0556	.0608	.0617	.0613	.0615
1.2	0.8	.0552	.0515	.0502	.0517	.0570	.0575	.0516	.0498	.0517
2	0.9	.187	.167	.141	.165	.170	.204	.189	.220	.188
2	0.7	.199	.170	.155	.162	.199	.211	.184	.167	.176
3	0.9	.43	.40	.38	.43	.40	.45	.43	.42	.46
5	0.9	.75	.72	.70	.82	.72	.76	.74	.71	.84
0.99999	0.9	.051916	.051949	.051876	.051921	.051560	.049946	.049189	.049016	.049091
0.99999	0.7	.175	.185	.203	.188	.187	.160	.166	.187	.168
0.999	0.9	.051937	.051979	.051908	.051953	.051566	.049946	.049192	.049021	.049094
0.9	0.9	.056200	.057130	.057200	.057184	.053753	.052112	.051562	.051463	.051490
0.9	0.8	.071670	.073227	.073722	.073307	.066852	.064965	.064108	.064407	.063973
0.9	0.76	.1051	.1062	.1089	.1065	.1023	.0955	.0932	.0961	.0931
0.85	0.9	.060647	.062006	.062119	.062072	.056808	.055448	.054934	.054835	.054839
0.85	0.8	.0932	.0933	.0940	.0933	.0873	.0843	.0813	.0818	.0810
0.81	0.9	.068206	.069767	.070102	.069839	.063459	.061887	.061170	.061288	.061042
0.8	0.8	.1671	.1588	.1599	.1581	.1654	.1527	.1395	.1410	.1384

Table 4 (continued)

m = 5, n = 30

λ_1	λ_2	With local unbiasedness property $L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$LS^{(2)}$	With equal tail areas $L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
1	1.001	.050000	.050000	.050000	.050000	.050000	.050011	.050022	.050025	.050023
1	1.1	.052886	.052978	.052873	.052907	.052720	.054093	.055191	.055147	.055204
1.05	1.05	.052154	.052760	.052835	.052808	.051662	.053325	.054973	.055199	.055114
1	1.5	.1209	.1171	.1137	.1149	.1138	.1271	.1279	.1214	.1261
1.25	1.25	.0987	.1126	.1145	.1136	.0893	.1044	.1236	.1252	.1251
1	2	.3032	.2869	.2539	.2974	.2858	.3134	.3044	.2835	.3165
1.333	1.5	.1781	.2109	.2148	.2165	.1589	.1873	.2281	.2355	.2347
1	4	.866	.859	.848		.857	.870	.860	.847	
1	5	.942	.936	.922		.938	.944	.939	.923	
2	4	.935	.950	.948		.927	.937	.954	.955	
3	3	.938	.958	.961		.931	.941	.961	.964	
1	8	.993	.992	.989		.992	.993	.992	.991	
4.5	4.5	.997	.998	.998		.996	.997	.998	.998	
1	11	.998	.998	.997		.998	.998	.998	.998	
6	6	.999	.999	.999		.999	.999	.999	.999	
1.00001	0.9	.053123	.053309	.052990	.053246	.052984	.051947	.051062	.050887	.050895
1.00001	0.8	.122	.126	.127	.127	.125	.118	.119	.121	.120
1.01	0.99	.050026	.050009	.050002	.050006	.050045	.050027	.050010	.050002	.050005
1.1	0.9	.052613	.050937	.050260	.050546	.054552	.052758	.050934	.050300	.050502
1.1	0.8	.0656	.0647	.0641	.0645	.0716	.0645	.0621	.0614	.0617
1.05	0.95	.050658	.050237	.050048	.050140	.051130	.050695	.050237	.050033	.050128
1.2	0.99	.0611	.0606	.0607	.0602	.0604	.0635	.0648	.0627	.0645
1.2	0.8	.0735	.0749	.0783	.0799	.0802	.0735	.0741	.0718	.0792
2	0.9	.288	.252	.226	.252	.276	.298	.269	.238	.269
2	0.7	.369	.333	.309	.323	.381	.375	.340	.317	.320
3	0.9	.66	.62	.60	.67	.64	.67	.64	.60	.69
5	0.9	.93	.92	.92	.97	.93	.94	.93	.92	.97
0.99999	0.9	.053123	.053310	.052992	.053247	.052984	.051947	.051062	.050887	.050896
0.99999	0.7									
0.999	0.9	.053164	.053368	.053105	.053307	.052997	.051975	.051097	.050805	.050932
0.9	0.9	.067117	.068634	.068723	.068679	.063985	.064502	.063819	.063578	.063667
0.9	0.8									
0.9	0.76									
0.85	0.9	.145	.143	.132	.143	.143	.141	.134	.126	.134

Table 4 (continued)

m = 0, n = 60

		With local unbiasedness property					With equal tail areas			
λ_1	λ_2	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$LS^{(2)}$	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
1	1.001	.050000	.050000	.050000	.050000	.050000	.050030	.050034	.050034	.050034
1	1.1	.050961	.050959	.050857	.050914	.050554	.054037	.054328	.054314	.054323
1.05	1.05	.050687	.050780	.050787	.050744	.050333	.053733	.054147	.054154	.054152
1	1.5	.0714	.0705	.0696	.0701	.0636	.0868	.0871	.0867	.0869
1.25	1.25	.0655	.0672	.0672	.0670	.0584	.0807	.0840	.0841	.0840
1	2	.1224	.1189	.1166	.1187	.1003	.1509	.1495	.1482	.1499
1.333	1.5	.0903	.0942	.0940	.0940	.0737	.1150	.1216	.1216	.1217
1	4	.372	.360	.355		.319	.418	.411	.407	
1	5	.470	.458	.451		.415	.515	.507	.503	
2	4	.462	.482	.479		.392	.515	.538	.537	
3	3	.464	.493	.493		.389	.520	.550	.551	
1	8	.659	.648	.643		.612	.694	.687	.684	
4.5	4.5	.702	.728	.728		.640	.743	.768	.769	
1	11	.760	.751	.747		.723	.787	.781	.778	
6	6	.824	.843	.843		.780	.851	.869	.870	
1.00001	0.9	.051016	.050952	.050927	.050981	.050561	.048007	.047611	.047585	.047597
1.00001	0.8	.054209	.054058	.054053	.054101	.052288	.048242	.047385	.047298	.047340
1.01	0.99	.050011	.050008	.050006	.050007	.050009	.050012	.050008	.050007	.050007
1.1	0.9	.051129	.050771	.050699	.050737	.050901	.051257	.050781	.050698	.050741
1.1	0.8	.053355	.052641	.052572	.052610	.052435	.050602	.049308	.049122	.049222
1.05	0.95	.050282	.050192	.050174	.050184	.050224	.050315	.050195	.050174	.050185
1.2	0.99	.0536	.0534	.0532	.0533	.0522	.0595	.0598	.0598	.0598
1.2	0.8	.0545	.0531	.0527	.0529	.0536	.0550	.0530	.0527	.0529
2	0.9	.116	.111	.109	.119	.099	.144	.139	.137	.105
2	0.7	.116	.103	.100	.102	.102	.136	.124	.121	.120
3	0.9	.24	.23	.22	.23	.20	.28	.27	.27	.28
5	0.9	.46	.44	.44	.44	.41	.50	.49	.49	.52
0.99999	0.9	.051017	.050952	.050927	.050981	.050561	.048006	.047611	.047585	.047597
0.99999	0.7	.0607	.0607	.0608	.0607	.0567	.0515	.0503	.0501	.0502
0.999	0.9	.051026	.050964	.050939	.050993	.050562	.047984	.047589	.047563	.047576
0.9	0.9	.052991	.053134	.053211	.053208	.051315	.046899	.046468	.046457	.046460
0.9	0.8	.057305	.057606	.057738	.057708	.053247	.048157	.047597	.047567	.047576
0.9	0.76	.0599	.0602	.0604	.0603	.0546	.0494	.0488	.0487	.0487
0.85	0.9	.054829	.055055	.055255	.055145	.052115	.047212	.046726	.046709	.046715
0.85	0.8	.0598	.0603	.0605	.0604	.0542	.0490	.0485	.0485	.0485
0.81	0.9	.056753	.057039	.057265	.057139	.052988	.047913	.047369	.047342	.047351
0.8	0.8	.0631	.0637	.0639	.0638	.0556	.0507	.0502	.0503	.0502

Table 4 (continued)

m = 1, n = 60

λ_1	λ_2	With local unbiasedness property $L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$LS^{(2)}$	With equal tail areas $L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
1	1.001	.050000	.050000	.050000	.050000	.050000	.050027	.050033	.050033	.050033
1	1.1	.051518	.051455	.051413	.051435	.051089	.054294	.054788	.054757	.054778
1.05	1.05	.051106	.051251	.051253	.051253	.050658	.053838	.054583	.054601	.054596
1	1.5	.0845	.0822	.0811	.0818	.0761	.0986	.0987	.0976	.0983
1.25	1.25	.0749	.0785	.0786	.0786	.0663	.0884	.0952	.0953	.0953
1	2	.1677	.1592	.1554	.1607	.1448	.1933	.1891	.1855	.1907
1.333	1.5	.1144	.1234	.1234	.1239	.0953	.1363	.1503	.1504	.1510
1	4	.534	.513	.504		.494	.566	.552	.544	
1	5	.653	.634	.625		.617	.680	.667	.659	
2	4	.641	.671	.669		.594	.673	.707	.706	
3	3	.643	.686	.686		.593	.677	.722	.722	
1	8	.838	.826	.820		.817	.853	.844	.840	
4.5	4.5	.872	.895	.895		.848	.887	.910	.911	
1	11	.911	.904	.900		.898	.920	.914	.912	
6	6	.949	.960	.960		.938	.956	.966	.967	
1.00001	0.9	.051580	.051556	.051522	.051540	.051122	.048880	.048239	.048184	.048210
1.00001	0.8	.056609	.056644	.056538	.056591	.054798	.051229	.049956	.049791	.049866
1.01	0.99	.050016	.050009	.050007	.050008	.050017	.050018	.050008	.050006	.050007
1.1	0.9	.051631	.050878	.050719	.050801	.051781	.051816	.050882	.050698	.050793
1.1	0.8	.054864	.053561	.053251	.053418	.054962	.052518	.050220	.049833	.050037
1.05	0.95	.050408	.050219	.050179	.050200	.050443	.050455	.050220	.050174	.050198
1.2	0.99	.0557	.0553	.0551	.0552	.0542	.0610	.0616	.0614	.0615
1.2	0.8	.0562	.0532	.0525	.0530	.0569	.0571	.0532	.0525	.0530
2	0.9	.160	.145	.140	.138	.141	.184	.172	.167	.131
2	0.7	.160	.135	.130	.132	.152	.179	.154	.148	.150
3	0.9	.35	.33	.32	.36	.32	.39	.37	.36	.41
5	0.9	.64	.61	.61	.70	.61	.67	.65	.64	.71
0.99999	0.9	.051581	.051557	.051523	.051541	.051122	.048879	.048239	.048184	.048209
0.99999	0.7	.0775	.0798	.0804	.0798	.0767	.0674	.0671	.0675	.0670
0.999	0.9	.051596	.051579	.051546	.051563	.051127	.048867	.048228	.048174	.048199
0.9	0.9	.054878	.055436	.055445	.055443	.052653	.049343	.048794	.048774	.048780
0.9	0.8	.062468	.063562	.063569	.063552	.057341	.054007	.053411	.053365	.053363
0.9	0.76	.0691	.0705	.0705	.0704	.0631	.0590	.0584	.0584	.0582
0.85	0.9	.057930	.058758	.058765	.058764	.054358	.050976	.050423	.050392	.050401
0.85	0.8	.0680	.0693	.0693	.0693	.0608	.0578	.0572	.0572	.0571
0.81	0.8	.061367	.062408	.062414	.062404	.056552	.053226	.052644	.052600	.052604
0.8	0.8	.0786	.0795	.0795	.0794	.0697	.0661	.0648	.0648	.0646

Table 4 (continued)

m = 2, n = 60

λ_1	λ_2	With local unbiasedness property: $L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$LS^{(2)}$	With equal tail areas: $L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
1	1.001	.050000	.050000	.050000	.050000	.050000	.050025	.050033	.050032	.050032
1	1.1	.052034	.051961	.051894	.051929	.051608	.054541	.055196	.055146	.055179
1.05	1.05	.051496	.051745	.051748	.051747	.050975	.053950	.054980	.055007	.055000
1	1.5	.0972	.0937	.0917	.0930	.0884	.1100	.1097	.1078	.1091
1.25	1.25	.0835	.0897	.0898	.0899	.0738	.0956	.1059	.1061	.1061
1	2	.2114	.1984	.1918	.2023	.1885	.2342	.2266	.2203	.2309
1.333	1.5	.1368	.1520	.1520	.1532	.1161	.1564	.1779	.1781	.1793
1	4	.661	.637	.625		.632	.683	.665	.654	
1	5	.779	.761	.750		.757	.795	.781	.772	
2	4	.768	.798	.797		.737	.786	.819	.818	
3	3	.771	.813	.813		.739	.790	.833	.834	
1	8	.927	.918	.914		.918	.932	.926	.922	
4.5	4.5	.949	.963	.963		.940	.954	.967	.968	
1	11	.969	.965	.963		.965	.971	.968	.966	
6	6	.987	.991	.991		.984	.988	.992	.992	
1.00001	0.9	.052090	.052094	.052047	.052074	.051672	.049663	.048868	.048786	.048823
1.00001	0.8	.059400	.059767	.059670	.059709	.058034	.054489	.053135	.052946	.053017
1.01	0.99	.050021	.050009	.050006	.050008	.050026	.050022	.050010	.050006	.050007
1.1	0.9	.052058	.050931	.050684	.050811	.052638	.052277	.050933	.050650	.050795
1.1	0.8	.056255	.054496	.054058	.054291	.057646	.054227	.051210	.050660	.050943
1.05	0.95	.050516	.050233	.050171	.050203	.050657	.050571	.050234	.050162	.050198
1.2	0.99	.0577	.0571	.0567	.0569	.0562	.0625	.0632	.0629	.0630
1.2	0.8	.0561	.0518	.0508	.0524	.0581	.0587	.0519	.0508	.0525
2	0.9	.202	.177	.169	.178	.183	.223	.203	.195	.205
2	0.7	.210	.175	.167	.170	.208	.226	.191	.182	.200
3	0.9	.46	.42	.41	.46	.43	.48	.45	.44	.50
5	0.9	.77	.74	.73	.86	.75	.78	.76	.75	.80
0.99999	0.9	.052090	.052095	.052047	.052074	.051672	.049663	.048868	.048786	.048823
0.99999	0.7	.164	.177	.183	.179	.176	.147	.155	.159	.156
0.999	0.9	.052113	.052128	.052081	.052107	.051679	.049660	.048869	.048788	.048824
0.9	0.9	.056709	.057657	.057677	.057673	.054022	.051680	.051168	.051139	.051146
0.9	0.8	.072283	.073887	.073993	.073885	.066924	.064088	.063323	.063336	.063247
0.9	0.76	.1036	.1050	.1057	.1050	.1002	.0918	.0900	.0905	.0898
0.85	0.9	.061401	.062776	.062800	.062790	.057176	.055020	.054553	.054512	.054518
0.85	0.8	.0931	.0933	.0933	.0932	.0866	.0822	.0794	.0794	.0792
0.81	0.9	.068944	.070535	.070602	.070537	.063683	.061212	.060565	.060548	.060501
0.8	0.8	.1631	.1545	.1545	.1540	.1609	.1456	.1325	.1323	.1317

Table 4 (continued)

m = 5, n = 60

		With local unbiasedness property					With equal tail areas			
λ_1	λ_2	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$LS^{(2)}$	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
1	1.001	.050000	.050000	.050000	.050000	.050000	.050018	.050030	.050030	.050024
1	1.1	.053156	.053365	.053259	.053224	.053060	.055268	.056293	.056188	.056256
1.05	1.05	.052280	.053133	.053176	.053067	.051856	.054323	.056061	.056121	.056104
1	1.5	.1320	.1262	.1215	.1245	.1241	.1430	.1405	.1359	.1395
1.25	1.25	.1054	.1215	.1219	.1215	.0952	.1155	.1361	.1367	.1367
1	2	.3347	.3119	.2966	.3280	.3159	.3520	.3346	.3202	.3526
1.333	1.5	.1963	.2326	.2330	.2368	.1755	.2122	.2551	.2556	.2605
1	4	.885	.869	.858		.877	.891	.877	.868	
1	5	.952	.944	.938		.948	.954	.948	.943	
2	4	.946	.961	.960		.939	.949	.964	.963	
3	3	.950	.967	.967		.943	.953	.970	.970	
1	8	.995	.993	.992		.994	.994	.994	.993	
4.5	4.5	.998	.999	.999		.997	.997	.998	.998	
1	11	.999	.999	.998		.999	.999	.998	.998	
6	6	.999	.999	.999		.999	.999	.999	.999	
1.00001	0.9	.053716	.053644	.053538	.053688	.053306	.051696	.050709	.050568	.050630
1.00001	0.8	.119	.124	.125	.125	.122	.113	.115	.116	.115
1.01	0.99	.050031	.050010	.050005	.050008	.050050	.050033	.050011	.050005	.050007
1.1	0.9	.053070	.050986	.050497	.050744	.055067	.053355	.050987	.050434	.050713
1.1	0.8	.0660	.0646	.0645	.0646	.0721	.0641	.0612	.0607	.0609
1.05	0.95	.050773	.050250	.050126	.050189	.051257	.050844	.050251	.050112	.050181
1.2	0.99	.0626	.0620	.0614	.0615	.0619	.0667	.0676	.0669	.0673
1.2	0.8	.0728	.0738	.0790	.0794	.0802	.0731	.0727	.0743	.0785
2	0.9	.319	.274	.253	.280	.306	.336	.295	.277	.303
2	0.7	.384	.338	.323	.331	.395	.395	.348	.332	.350
3	0.9	.69	.65	.63	.72	.68	.71	.67	.65	.74
5	0.9	.94	.93	.93	.99	.95	.95	.94	.93	.98
0.99999	0.9	.053716	.053651	.053539	.053689	.053306	.051697	.050710	.050568	.050631
0.99999	0.7									
0.999	0.9	.053763	.053715	.053605	.053755	.053321	.051721	.050745	.050606	.050667
0.9	0.9	.068517	.069627	.069596	.069810	.064487	.064025	.063356	.063300	.063306
0.9	0.8									
0.9	0.76									
0.85	0.9	.144	.141	.141	.141	.142	.136	.129	.129	.129

Table 5

Powers of $L_2^{(2)}$, $U^{(2)}$, $V^{(2)}$, $W^{(2)}$, $LS^{(2)}$ (in the unbiased and equal tail areas cases) and $Z^{(2)}$ for testing $\lambda_1 = 1$, $\lambda_2 = 1$ against different simple two-sided alternative hypotheses, $\alpha = .05$

$n_1 = 3$

			With local unbiasedness property					With equal tail areas			
λ_1	λ_2	$Z^{(2)}$	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$LS^{(2)}$	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
1	1.001	.0500	.050000	.050000	.050000	.050000	.050000	.049997	.049997	.050007	.050002
1	1.1	.0502	.050340	.050350	.050317	.050354	.050216	.050053	.050139	.051065	.050586
1.05	1.05	.0501	.050262	.050281	.050368	.050327	.050122	.049975	.050069	.051133	.050561
1	1.5	.0552	.0568	.0569	.0558	.0569	.0543	.0555	.0560	.0590	.0580
1.25	1.25	.0520	.0554	.0559	.0578	.0569	.0527	.0541	.0549	.0614	.0580
1	2	.0691	.0724	.0729	.0675	.0725	.0648	.0700	.0711	.0731	.0744
1.333	1.5	.0562	.0635	.0646	.0692	.0673	.0571	.0614	.0631	.0749	.0691
1.5	1.5	.0584	.0683	.0699	.0766		.0597	.0659	.0681	.0833	
2	2	.0724	.1096	.1148	.1338		.0872	.1053	.1115	.1455	
1.00001	0.9	.0502	.050391	.050405	.050384	.050414	.050251	.050696	.050631	.049567	.050166
1.00001	0.8	.0517	.051739	.051806	.051758	.051850	.051141	.052375	.052277	.050038	.051333
1.01	0.99	.0500	.050003	.050003	.050001	.050001	.050004	.050003	.050003	.049997	.050001
1.1	0.9	.0505	.050353	.050323	.050018	.050160	.050426	.050357	.050329	.049782	.050148
1.1	0.8	.0520	.051244	.051211	.050303	.050908	.051273	.051563	.051452	.049325	.050634
1.05	0.95	.0501	.050088	.050080	.050032	.050039	.050105	.050089	.050082	.049945	.050036
1.2	0.99	.0508	.0512	.0512	.0510	.0512	.0508	.0506	.0508	.0523	.0516
1.2	0.8	.0527	.0513	.0512	.0510	.0505	.0516	.051[illegible]	.0512	.0490	.0505
2	0.9	.071	.076	.075	.066	.069	.073	.074	.074	.070	.069
2	0.7		.076	.076	.062	.071	.075	.075	.074	.063	.072
3	0.9	.09	.10	.10	.07	.09	.09	.10	.10	.08	.10
5	0.9		.23	.23	.18	.22	.20	.23	.23	.19	.22
0.99999	0.9	.0502	.050391	.050405	.050384	.050414	.050251	.050696	.050631	.049567	.050166
0.99999	0.7		.0574	.0576	.0579	.0577	.0562	.0584	.0583	.0551	.0569
0.999	0.9	.0502	.050395	.050410	.050393	.050420	.050251	.050704	.050638	.049568	.050170
0.9	0.9	.0504	.051234	.051318	.051696	.051514	.050544	.051861	.051780	.050063	.051011
0.9	0.8	.0521	.053190	.053385	.054157	.053793	.051548	.054171	.054106	.051617	.053012
0.9	0.76	.0556	.0547	.0549	.0559	.0554	.0526	.0558	.0558	.0530	.0545
0.85	0.9	.0508	.052027	.052161	.052732	.052459	.050914	.052826	.052758	.050659	.051821
0.85	0.8	.0530	.0544	.0546	.0558	.0552	.0520	.0555	.0555	.0528	.0543
0.81	0.9	.0517	.052914	.053096	.053826	.053482	.051386	.053857	.053789	.051382	.052730
0.8	0.8	.0556	.0563	.0567	.0582	.0574	.0531	.0577	.0577	.0547	.0563

Table 5 (continued)

$n_1 = 5$

			With local unbiasedness property					With equal tail areas			
λ_1	λ_2	$Z^{(2)}$	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$LS^{(2)}$	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
1	1.001	.0500	.050000	.050000	.050000	.050000	.050000	.049998	.050000	.050009	.050005
1	1.1	.0505	.050676	.050713	.050635	.050703	.050542	.050477	.050763	.051536	.051267
1.05	1.05	.0502	.050527	.050604	.050713	.050675	.050321	.050329	.050653	.051633	.051244
1	1.5	.0659	.0641	.0646	.0619	.0641	.0611	.0632	.0649	.0659	.0667
1.25	1.25	.0562	.0611	.0628	.0653	.0645	.0571	.0601	.0631	.0697	.0672
1	2	.1001	.0993	.1003	.0885	.0987	.0899	.0976	.1008	.0955	.1035
1.333	1.5	.0679	.0779	.0824	.0881	.0867	.0690	.0764	.0827	.0951	.0911
1.5	1.5	.0727	.0880	.0945	.1031		.0763	.0863	.0949	.1114	
2	2	.0851	.1799	.1993	.2215		.1530	.1769	.2001	.2355	
1.00001	0.9	.0506	.050759	.050811	.050759	.050811	.050627	.050967	.050758	.049784	.050212
1.00001	0.8	.0534	.053544	.053797	.053693	.053829	.053071	.053978	.053687	.051645	.052578
1.01	0.99	.0500	.050006	.050004	.050001	.050001	.050009	.050006	.050004	.049997	.050001
1.1	0.9	.0513	.050639	.050502	.050068	.050193	.050998	.050636	.050501	.049715	.050169
1.1	0.8	.0533	.052306	.052143	.050821	.051580	.053105	.052512	.052088	.049674	.050923
1.05	0.95	.0503	.050159	.050125	.050012	.050048	.050248	.050159	.050125	.049928	.050042
1.2	0.99	.0522	.0524	.0525	.0520	.0524	.0520	.0520	.0526	.0537	.0534
1.2	0.8	.0543	.0520	.0515	.0502	.0503	.0535	.0520	.0515	.0483	.0502
2	0.9	.090	.093	.092	.074	.086	.086	.091	.092	.080	.090
2	0.7		.111	.107	.086	.098	.114	.110	.108	.088	.101
3	0.9		.20	.20	.16	.19	.18	.19	.20	.17	.20
5	0.9		.45	.45	.40	.46	.43	.45	.45	.41	.46
0.99999	0.9	.0506	.050759	.050811	.050759	.050811	.050627	.050968	.050758	.049785	.050212
0.99999	0.7		.0852	.0859	.0890	.0867	.0861	.0861	.0857	.0851	.0842
0.999	0.9	.0506	.050768	.050822	.050776	.050825	.050629	.050979	.050769	.049792	.050220
0.9	0.9	.0512	.052491	.052823	.053283	.053115	.051442	.052924	.052715	.051331	.051900
0.9	0.8	.0567	.057398	.058121	.059044	.058689	.055077	.058080	.057951	.055977	.056779
0.9	0.76		.0648	.0657	.0669	.0663	.0622	.0657	.0654	.0633	.0640
0.85	0.9	.0533	.054174	.054692	.055377	.055130	.052507	.054725	.054555	.052897	.053586
0.85	0.8	.0607	.0619	.0628	.0640	.0635	.0585	.0627	.0626	.0604	.0612
0.81	0.9	.0558	.056489	.057173	.058042	.057718	.054292	.057143	.057010	.055101	.055887
0.8	0.8	.0727	.0737	.0744	.0755	.0750	.0695	.0747	.0742	.0710	.0721

Table 5 (continued)

$n_1 = 7$

λ_1	λ_2	$Z^{(2)}$	With local unbiasedness property $L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$LS^{(2)}$	With equal tail areas $L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
1	1.001	.0500	.050000	.050000	.050000	.050000	.050000	.049998	.050002	.050010	.050007
1	1.1	.0508	.051018	.051084	.050970	.051058	.050893	.050837	.051322	.051952	.051792
1.05	1.05	.0504	.050796	.050948	.051069	.051033	.050538	.050616	.051186	.052071	.051773
1	1.5	.0722	.0720	.0727	.0685	.0717	.0687	.0711	.0739	.0730	.0751
1.25	1.25	.0613	.0668	.0704	.0732	.0724	.0621	.0660	.0715	.0780	.0760
1	2	.1269	.1298	.1307	.1133	.1284	.1196	.1282	.1328	.1210	.1347
1.333	1.5	.0792	.0928	.1018	.1081	.1073	.0824	.0915	.1037	.1158	.1131
1.5	1.5	.0835	.1087	.1216	.1312		.0951	.1071	.1238	.1404	
2	2	.2001	.2568	.2918	.3138		.2291	.2543	.2953	.3278	
1.00001	0.9	.0511	.051121	.051222	.051152	.051214	.051034	.051310	.050973	.050095	.050437
1.00001	0.8	.0563	.056304	.056813	.056801	.056863	.056199	.056699	.056286	.054555	.055220
1.01	0.99	.0500	.050009	.050006	.050002	.050002	.050015	.050009	.050006	.049996	.050001
1.1	0.9	.0521	.050914	.050619	.050159	.050209	.051607	.050907	.050616	.049678	.050179
1.1	0.8	.0558	.053536	.053202	.051708	.052493	.055325	.053714	.052941	.050471	.051640
1.05	0.95	.0504	.050229	.050154	.050039	.050052	.050399	.050227	.050154	.049918	.050044
1.2	0.99	.0532	.0537	.0538	.0531	.0536	.0533	.0533	.0542	.0549	.0549
1.2	0.8	.0530	.0512	.0501	.0500	.0501	.0539	.0511	.0501	.0463	.0490
2	0.9		.121	.118	.093	.110	.115	.120	.120	.100	.115
2	0.7		.157	.149	.123	.137	.163	.156	.150	.126	.140
3	0.9		.30	.29	.24	.29	.28	.29	.29	.25	.30
5	0.9		.63	.63	.58	.64	.62	.63	.63	.59	.65
0.99999	0.9	.0511	.051122	.051223	.051153	.051214	.051034	.051310	.050973	.050096	.050438
0.99999	0.7		.2227	.2242	.2400	.2286	.2308	.2241	.2224	.2330	.2231
0.999	0.9	.0511	.051135	.051241	.051177	.051236	.051037	.051325	.050989	.050110	.050452
0.9	0.9	.0528	.053832	.054486	.054980	.054823	.052492	.054225	.053974	.052856	.053245
0.9	0.8	.0676	.068522	.069654	.070665	.070187	.066041	.069177	.068799	.067137	.067558
0.9	0.76		.1115	.1118	.1140	.1122	.1104	.1124	.1106	.1090	.1084
0.85	0.9	.0548	.057079	.058067	.058784	.058550	.055007	.057583	.057410	.056065	.056529
0.85	0.8		.0927	.0931	.0937	.0932	.0896	.0936	.0920	.0891	.0897
0.81	0.9	.0631	.064557	.065708	.066629	.066246	.062078	.065174	.064903	.063304	.063771
0.8	0.8		.1836	.1792	.1762	.1764	.1830	.1850	.1774	.1687	.1707

Table 5 (continued)

$n_1 = 13$

			With local unbiasedness property					With equal tail areas			
λ_1	λ_2	$Z^{(2)}$	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$LS^{(2)}$	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
1	1.001	.0500	.050000	.050000	.050000	.050000	.050000	.049997	.050006	.050011	.050010
1	1.1	.0519	.052054	.052203	.052011	.052134	.051988	.051807	.052771	.053102	.053088
1.05	1.05	.0509	.051596	.052022	.052157	.052122	.051214	.051355	.052591	.053268	.053085
1	1.5	.0971	.0981	.0984	.0905	.0960	.0942	.0968	.1012	.0954	.1005
1.25	1.25	.0757	.0842	.0943	.0976	.0969	.0777	.0830	.0971	.1030	.1016
1	2	.2315	.2372	.2340	.2044	.2359	.2261	.2350	.2386	.2125	.2438
1.333	1.5	.1245	.1402	.1652	.1717	.1734	.1267	.1384	.1696	.1803	.1811
1.5	1.5	.1517	.1754	.2107	.2210		.1583	.1733	.2159	.2309	
2	2		.4916	.5559	.5717		.4657	.4891	.5616	.5822	
1.00001	0.9	.0523	.052223	.052505	.052422	.052484	.052358	.052473	.051912	.051253	.051479
1.00001	0.8		.128	.130	.133	.131	.131	.129	.128	.130	.128
1.01	0.99	.0500	.050017	.050008	.050002	.050002	.050034	.050016	.050008	.049996	.050002
1.1	0.9	.0541	.051694	.050795	.050206	.050215	.053485	.051672	.050792	.049614	.050180
1.1	0.8		.0651	.0647	.0641	.0643	.0705	.0653	.0640	.0626	.0631
1.05	0.95	.0511	.050428	.050202	.050042	.050057	.050865	.050422	.050201	.049905	.050048
1.2	0.99	.0572	.0576	.0577	.0566	.0573	.0574	.0571	.0588	.0586	.0591
1.2	0.8		.0758	.0773	.0816	.0804	.0807	.0758	.0770	.0811	.0801
2	0.9		.222	.207	.168	.197	.216	.220	.211	.175	.204
2	0.7		.341	.322	.299	.310	.354	.340	.324	.301	.312
3	0.9		.58	.56	.52	.58	.57	.[illegible]8	.57	.52	.59
5	0.9		.91	.91	.90	.93	.91	.91	.91	.90	.93
0.99999	0.9	.0523	.052224	.052505	.052423	.052485	.052358	.052473	.051912	.051253	.051479
0.99999	0.7										
0.999	0.9	.0522	.052254	.052549	.052474	.052533	.052369	.052507	.051950	.051293	.051517
0.9	0.9	.0626	.064878	.066278	.066678	.066562	.062975	.065432	.064998	.064203	.064411
0.9	0.8										
0.9	0.76										
0.85	0.9		.148	.147	.147	.146	.147	.149	.144	.143	.142

Table 6

Powers of $LS_1^{(2)}$, $LS_2^{(2)}$, $LS_3^{(2)}$, $L_2^{(2)}$ (unbiased) and $LS^{(2)}$ for testing $\lambda_1 = 1$, $\lambda_2 = 1$

against different simple two-sided alternative hypotheses, $\alpha = .05$

λ_1	λ_2	$LS_1^{(2)}$	$LS_2^{(2)}$	$LS_3^{(2)}$	$L_2^{(2)}$	$LS^{(2)}$	$LS_1^{(2)}$	$LS_2^{(2)}$	$LS_3^{(2)}$	$L_2^{(2)}$	$LS^{(2)}$	$LS_1^{(2)}$	$LS_2^{(2)}$	$LS_3^{(2)}$	$L_2^{(2)}$	$LS^{(2)}$
		m = 0, n = 5					m = 0, n = 15					m = 0, n = 30				
1	1.001	.050029	.050028	.049975	.050000	.050000	.050041	.050041	.049975	.050000	.050000	.050046	.050045	.049975	.050000	.050000
1.05	1.05	.053180	.053108	.047681	.050434	.050203	.054582	.054498	.047678	.050586	.050286	.055132	.055043	.047677	.050649	.050313
1	1.5	.0736	.0733	.0418	.0624	.0576	.0852	.0848	.0418	.0677	.0611	.0899	.0895	.0418	.0700	.0626
1.25	1.25	.0704	.0701	.0402	.0594	.0547	.0802	.0798	.0402	.0630	.0569	.0842	.0837	.0402	.0646	.0578
1.333	1.5	.0900	.0893	.0358	.0739	.0630	.1097	.1090	.0358	.0838	.0692	.1177	.1170	.0357	.0878	.0720
1	4	.298	.296	.090	.267	.220	.381	.380	.090	.334	.280	.410	.409	.090	.358	.304
3	3	.371	.369	.115	.339	.273	.472	.470	.115	.418	.343	.506	.505	.115	.447	.372
1	8	.573	.571	.335	.546	.494	.658	.657	.335	.620	.569	.685	.684	.335	.646	.596
1.00001	0.9	.047498	.047569	.052697	.050649	.050379	.046469	.046550	.052700	.050873	.050489	.046085	.046170	.052702	.050963	.050533
1.1	0.8	.048935	.049006	.053909	.052114	.051782	.048521	.048598	.053915	.052872	.052186	.048409	.048488	.053918	.053175	.052343
1.05	0.95	.05165	.050165	.050121	.050164	.050154	.050247	.050247	.050121	.050235	.050197	.050283	.050283	.050122	.050264	.050214
2	0.7	.1064	.1059	.0600	.0914	.0848	.1291	.1285	.0600	.1057	.0947	.1384	.1378	.0600	.1118	.0992
3	0.9	.195	.193	.035	.151	.133	.259	.258	.035	.216	.176	.282	.281	.035	.235	.193
5	0.9	.373	.372	.116	.340	.286	.468	.467	.116	.421	.364	.499	.498	.116	.449	.393
0.99999	0.8	.045947	.046088	.056110	.052787	.051643	.044305	.044464	.056119	.053663	.052043	.043721	.043886	.056122	.054006	.052197
0.99999	0.7	.0474	.0476	.0646	.0589	.0567	.0446	.0448	.0646	.0600	.0567	.0436	.0439	.0646	.0604	.0566
0.999	0.9	.047471	.047542	.052721	.050656	.050381	.046430	.046512	.052725	.050881	.050491	.046042	.046128	.052726	.050972	.050535
0.9	0.8	.043844	.044053	.058877	.054955	.052280	.041412	.041647	.058889	.056406	.052878	.040551	.040796	.058893	.056972	.053110
0.85	0.9	.044358	.044532	.056978	.053216	.051417	.042140	.042338	.056988	.054208	.051844	.041337	.041544	.056991	.054599	.052014
0.8	0.8	.0430	.0433	.0630	.0593	.0543	.0401	.0404	.0630	.0616	.0551	.0391	.0394	.0630	.0625	.0554
		m = 1, n = 5					m = 1, n = 15					m = 1, n =30				
1	1.001	.050021	.050021	.049956	.050000	.050000	.050038	.050037	.049955	.050000	.050000	.050045	.050044	.049955	.050000	.050000
1.05	1.05	.052570	.052492	.045878	.050621	.050374	.054439	.054347	.045842	.050901	.050536	.055247	.055148	.045830	.051027	.050613
1	1.5	.0747	.0743	.0368	.0672	.0633	.0921	.0916	.0367	.0770	.0704	.1001	.0995	.0367	.0816	.0739
1.25	1.25	.0697	.0693	.0336	.0632	.0585	.0837	.0833	.0335	.0698	.0628	.0901	.0896	.0334	.0729	.0649
1.333	1.5	.0918	.0912	.0274	.0835	.0729	.1212	.1204	.0273	.1011	.0853	.1346	.1337	.0273	.1092	.0914
1	4	.396	.395	.207	.384	.354	.506	.504	.207	.475	.436	.548	.547	.207	.513	.472
3	3	.494	.493	.278	.486	.446	.610	.608	.278	.581	.532	.654	.653	.278	.620	.570
1	8	.747	.746	.621	.740	.722	.817	.817	.621	.802	.779	.842	.842	.621	.825	.803
1.00001	0.9	.048512	.048590	.055019	.050895	.050713	.047228	.047318	.055073	.051296	.050955	.046715	.046808	.055090	.051471	.051057
1.1	0.8	.051058	.051144	.057874	.052727	.053447	.050706	.050797	.057983	.053974	.054365	.050651	.050744	.058018	.054523	.054736
1.05	0.95	.050270	.050270	.050266	.050198	.050281	.050404	.050404	.050270	.050318	.050376	.050471	.050471	.050272	.050373	.050418
2	0.7	.1328	.1324	.0846	.1190	.1197	.1659	.1653	.0848	.1423	.1381	.1814	.1808	.0849	.1534	.1470
3	0.9	.243	.242	.081	.228	.206	.340	.338	.081	.307	.273	.379	.377	.0813	.339	.304
5	0.9	.505	.504	.311	.492	.465	.616	.615	.311	.588	.552	.656	.655	.311	.625	.589
0.99999	0.8	.048872	.049034	.062079	.054072	.053393	.046846	.047027	.062222	.055576	.054249	.046083	.046272	.062269	.056216	.054589
0.99999	0.7	.0751	.0754	.1084	.0842	.0850	.0643	.0647	.1087	.0803	.0805	.0601	.0605	.1088	.0785	.0782
0.999	0.9	.048492	.048572	.055064	.050905	.050715	.047194	.047285	.055118	.051309	.050958	.046673	.046769	.055135	.051487	.051061
0.9	0.8	.048769	.049010	.068489	.058288	.055503	.045575	.045845	.068688	.060781	.056629	.044371	.044653	.068753	.061829	.057070
0.05	0.9	.047353	.047549	.063372	.054819	.052833	.044635	.044858	.063519	.056662	.053743	.043591	.043824	.063567	.057449	.054121
0.8	0.8	.0597	.0601	.0901	.0747	.0697	.0535	.0539	.0904	.0771	.0699	.0511	.0515	.0905	.0780	.0698

Table 6 (continued)

λ_1	λ_2	$LS_1^{(2)}$	$LS_2^{(2)}$	$LS_3^{(2)}$	$L_2^{(2)}$	$LS^{(2)}$	$LS_1^{(2)}$	$LS_2^{(2)}$	$LS_3^{(2)}$	$L_2^{(2)}$	$LS^{(2)}$	$LS_1^{(2)}$	$LS_2^{(2)}$	$LS_3^{(2)}$	$L_2^{(2)}$	$LS^{(2)}$
		m = 2, n = 5					m = 2, n = 15					m = 2, n = 30				
1	1.001	.050015	.050014	.049940	.050000	.050000	.050034	.050033	.049939	.050000	.050000	.050043	.050042	.049939	.050000	.050000
1.05	1.05	.052031	.051948	.044525	.050760	.050516	.054246	.054145	.044405	.051168	.050768	.055281	.055173	.044356	.051367	.050895
1	1.5	.0754	.0750	.0338	.0708	.0678	.0979	.0974	.0336	.0850	.0788	.1093	.1087	.0335	.0923	.0845
1.25	1.25	.0691	.0686	.0292	.0660	.0615	.0865	.0860	.0228	.0755	.0681	.0952	.0947	.0287	.0803	.0716
1.333	1.5	.0934	.0928	.0229	.0907	.0808	.1310	.1302	.0224	.1158	.0997	.1498	.1489	.0223	.1285	.1096
1	4	.502	.500	.351	.499	.481	.612	.611	.350	.594	.567	.660	.659	.350	.636	.607
3	3	.620	.618	.460	.622	.599	.723	.722	.459	.710	.679	.767	.766	.459	.748	.716
1	8	.865	.865	.806	.864	.858	.908	.907	.806	.902	.893	.924	.924	.806	.917	.908
1.00001	0.9	.049437	.049523	.056903	.051071	.050998	.048006	.048105	.057087	.051644	.051387	.047396	.047501	.057147	.051915	.051560
1.1	0.8	.053363	.053465	.061767	.053395	.055185	.053096	.053203	.062158	.054999	.056646	.053106	.053214	.062292	.055763	.057264
1.05	0.95	.050367	.050368	.050406	.050215	.050385	.050550	.050550	.050423	.050378	.050539	.050650	.050650	.050430	.050461	.050610
2	0.7	.1688	.1684	.1238	.1553	.1614	.2079	.2073	.1243	.1844	.1862	.2284	.2278	.1246	.1998	.1992
3	0.9	.298	.297	.145	.292	.277	.413	.412	.144	.389	.362	.464	.463	.144	.433	.404
5	0.9	.632	.631	.498	.628	.614	.730	.729	.498	.714	.693	.770	.769	.498	.751	.729
0.99999	0.8	.052736	.052922	.068600	.056131	.056080	.050265	.050473	.069111	.058015	.057274	.049257	.049473	.069283	.058864	.057746
0.99999	0.7	.219	.220	.299	.225	.233	.172	.173	.301	.192	.203	.150	.151	.301	.175	.187
0.999	0.9	.049425	.049512	.056964	.051084	.051001	.047977	.048078	.057149	.051663	.051392	.047360	.047466	.057210	.051937	.051566
0.9	0.8	.060728	.061017	.085981	.068291	.065952	.055596	.055922	.086697	.070673	.066687	.053390	.053729	.086935	.071670	.066852
0.85	0.9	.050915	.051135	.069676	.056841	.054881	.047801	.048051	.070193	.059454	.056228	.046528	.046790	.070365	.060647	.056808
0.8	0.8	.1722	.1729	.2354	.1845	.1839	.1458	.1465	.2366	.1732	.1720	.1332	.1339	.2370	.1671	.1654

λ_1	λ_2	$LS_1^{(2)}$	$LS_2^{(2)}$	$LS_3^{(2)}$	$L_2^{(2)}$	$LS^{(2)}$	$LS_1^{(2)}$	$LS_2^{(2)}$	$LS_3^{(2)}$	$L_2^{(2)}$	$LS^{(2)}$	$LS_1^{(2)}$	$LS_2^{(2)}$	$LS_3^{(2)}$	$L_2^{(2)}$	$LS^{(2)}$
		m = 5, n = 5					m = 5, n = 15					m = 5, n = 30				
1	1.001	.050000	.049999	.050000	.050000	.050000	.050023	.050022	.049904	.050000	.050000	.050036	.050035	.049902	.050000	.050000
1.05	1.05	.050869	.050773	.050821	.051026	.050821	.053705	.053585	.041388	.051776	.051347	.055291	.055158	.041203	.052154	.051662
1	1.5	.0783	.0778	.0781	.0789	.0781	.1126	.1120	.0301	.1048	.0999	.1343	.1336	.0299	.1209	.1138
1.25	1.25	.0680	.0676	.0678	.0713	.0678	.0932	.0926	.0207	.0885	.0811	.1086	.1079	.0203	.0987	.0893
1.333	1.5	.1001	.0994	.0998	.1072	.0998	.1564	.1554	.0188	.1516	.1361	.1910	.1899	.0184	.1781	.1589
1	4	.777	.776	.777	.780	.777	.839	.838	.720	.836	.828	.872	.871	.720	.866	.857
3	3	.877	.876	.876	.883	.876	.919	.919	.825	.920	.912	.940	.940	.825	.938	.931
1	8	.986	.986	.986	.986	.986	.991	.991	.982	.991	.990	.993	.993	.982	.993	.992
1.00001	0.9	.051650	.051755	.051702	.051453	.051702	.050159	.050282	.061873	.052484	.052555	.049422	.049555	.062219	.053123	.052984
1.1	0.8	.0689	.0690	.0689	.0645	.0689	.0677	.0679	.0842	.0653	.0709	.0670	.0672	.0851	.0656	.0716
1.05	0.95	.050606	.050609	.050607	.050234	.050607	.050931	.050932	.050857	.050496	.050946	.051139	.051139	.050899	.050658	.051130
2	0.7	.3320	.3317	.3318	.3189	.3318	.3708	.3704	.3060	.3498	.3620	.3963	.3958	.3072	.3699	.3810
3	0.9	.510	.509	.510	.513	.510	.617	.616	.414	.609	.597	.677	.676	.414	.663	.649
5	0.9	.891	.891	.891	.892	.891	.924	.924	.861	.922	.919	.942	.942	.861	.939	.935
0.99999	0.8	.1350	.1354	.1352	.1329	.1352	.1217	.1221	.1695	.1266	.1298	.1133	.1137	.1706	.1223	.1256
0.99999	0.7															
0.999	0.9	.051658	.051764	.051711	.051475	.051711	.050147	.050271	.061977	.052518	.052566	.049401	.049534	.062325	.053164	.052997
0.9	0.8															
0.85	0.9	.1501	.1505	.1503	.1507	.1503	.1373	.1378	.1890	.1477	.1466	.1297	.1302	.1900	.1456	.1438
0.8	0.8															

Appendix A

The coefficients c''_{ij} for the non-central distribution of the mℓr and Roy's largest and smallest roots for p = 2 are given below in terms of the constants A''_{ij} which are also provided here.

$c''_{00} = 1$, $c''_{10} = A''_{11}$, $c''_{20} = 3A''_{21}$, $c''_{01} = A''_{22}-4A''_{21}$, $c''_{30} = 5A''_{31}$, $c''_{11} = A''_{32}-12A''_{31}$,

$c''_{40} = 35A''_{41}$, $c''_{21} = 3A''_{42}-120A''_{41}$, $c''_{02} = A''_{43}-4A''_{42}+48A''_{41}$, $c''_{50} = 63A''_{51}$,

$c''_{31} = 5A''_{52}-280A''_{51}$, $c''_{12} = A''_{53}-12A''_{52}+240A''_{51}$, $A''_{60} = 231A''_{61}$,

$c''_{41} = 35A''_{62}-1260A''_{61}$, $c''_{22} = 3A''_{63}-120A''_{62}+1680A''_{61}$,

$c''_{03} = A''_{64} - 4A''_{63} + 48A''_{62}-320A''_{61}$,

where

$A''_{11} = \nu b_1/4$, $A''_{21} = \nu_{(1)}b_{21}/(8.4!)$, $A''_{22} = \nu(\nu-1)b_2/6$,

$A''_{31} = \nu_{(2)}b_{31}/(2^5.5!)$, $A''_{32} = \nu_{(1)}(\nu-1)b_1b_2/40$, $A''_{41} = 3\nu_{(3)}b_{41}/(2^7.8!)$,

$A''_{42} = \nu_{(2)}(\nu-1)b_{42}/(7.2^4.4!)$, $A''_{43} = \nu_{(1)}(\nu^2-1)b_2^2/120$, $A''_{51} = \nu_{(4)}b_{51}/(3.2^9.8!)$,

$A''_{52} = \nu_{(3)}(\nu-1)b_{52}/(3.2^5.6!)$, $A''_{53} = \nu_{(2)}(\nu^2-1)b_1b_2^2/(5.7.2^5)$,

$A''_{61} = \nu_{(5)}b_{61}/(33.2^{13}.8!)$, $A''_{62} = 3\nu_{(4)}(\nu-1)b_{62}/(11.2^8.8!)$,

$A''_{63} = 5\nu_{(3)}(\nu^2-1)b_{63}/(3.2^5.7!)$, $A''_{64} = \nu_{(2)}(\nu^2-1)(\nu+3)b_2^3/(5.7.9.2^4)$

where

$\nu = n_1 + n_2$, $\nu_{(i)} = \nu(\nu+2)\ldots(\nu+2i)$

$b_1 = 2-[(1/\lambda_1) + (1/\lambda_2)]$, $b_2 = [1-(1/\lambda_1)][1-(1/\lambda_2)]$,

$b_{21}=3b_1^2 - 4b_2$, $b_{22} = b_2$, $b_{31} = 5b_1^3 - 12b_1b_2$, $b_{32} = b_1b_2$,

$b_{41}=35b_1^4 - 120b_1^2b_2 + 48b_2^2$, $b_{42} = b_2b_{21}$, $b_{43} = b_2^2$,

$b_{51}=63b_1^5 - 280b_1^3b_2 + 240b_1b_2^2$, $b_{52} = b_2b_{31}$, $b_{53} = b_1b_2^2$,

$b_{61}=231b_1^6 - 1260b_1^4b_2 + 1680b_1^2b_2^2 - 320b_2^3$, $b_{62} = b_2b_{41}$, $b_{63} = b_2^2b_{21}$,

$b_{64}=b_2^3$.

Appendix B

Tabulated below are the functions $g_{ij}(z)$ appearing in the non-central density function of the mℓr criterion $Z^{(2)}$ for testing the hypothesis $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$.

The following notations are used:

$$b-a = \frac{2}{\sqrt{3}} \cos\left(\frac{\theta}{3} + \frac{\pi}{6}\right)$$

$$b^2-a^2 = \frac{4}{3\sqrt{3}} \cos\left(\frac{\theta}{3} + \frac{\pi}{6}\right)\left[\cos\left(\frac{\theta}{3} - \frac{\pi}{3}\right)+1\right] \text{ where } \cos \ = 1-\frac{27}{2} z^{\frac{1}{n_1}}$$

$$g_{00}(z) = b - a, \quad g_{10}(z) = 0.66666667(b - a)$$

$$g_{20}(z) = (0.1904762 + 1.37142858z^{\frac{1}{n_1}})(b - a)+(0.34285714-2.05714286z^{\frac{1}{n_1}})(b^2-a^2)$$

$$g_{01}(z) = (-0.33333333\ z^{\frac{-2}{n_1}})(b - a)+(0.33333333z^{\frac{-2}{n_1}})(b^2-a^2)$$

$$g_{30}(z) = (-0.51948052 + 3.94805195z^{\frac{1}{n_1}} + 3.36623377z^{\frac{2}{n_1}})(b - a)$$
$$+ (0.97662338 - 5.79740261z^{\frac{1}{n_1}})(b^2-a^2)$$

$$g_{11}(z) = (-0.47619047\ z^{\frac{-2}{n_1}} + 0.57142858z^{\frac{-1}{n_1}})(b - a)$$
$$+ (0.47619047\ z^{\frac{-2}{n_1}} - 0.85714286z^{\frac{-1}{n_1}})(b^2-a^2)$$

$$g_{40}(z) = (-1.6207792 + 6.49696986z^{\frac{1}{n_1}} + 13.46493514z^{\frac{2}{n_1}})(b-a)$$
$$+ (2.02712842 - 10.84675336z^{\frac{1}{n_1}})(b^2-a^2)$$

$$g_{21}(z) = (-0.7099567z^{\frac{-2}{n_1}} + 0.63376626z^{\frac{-1}{n_1}} + 1.3090909)(b-a)$$
$$+ (0.7099567z^{\frac{-2}{n_1}} - 1.56883118z^{\frac{-1}{n_1}})(b^2-a^2)$$

$$g_{02}(z) = (-0.14285714z^{\frac{-4}{n_1}} - 0.42857142z^{\frac{-3}{n_1}})(b-a)$$
$$+ (0.14285714z^{\frac{-4}{n_1}} + 0.14285714z^{\frac{-3}{n_1}})(b^2-a^2)$$

$$g_{50}(z) = (-3.3902939 + 5.29940273z^{\frac{1}{n_1}} + 33.8642416z^{\frac{2}{n_1}} + 6.54166889z^{\frac{3}{n_1}})(b-a)$$
$$+ (3.75970227 - 15.10573663z^{\frac{1}{n_1}} - 0.48456834z^{\frac{2}{n_1}} - 9.81250329z^{\frac{3}{n_1}}(b^2-a^2)$$

$$g_{31}(z) = (-1.10129869z^{\frac{-2}{n_1}} - 0.87965352z^{\frac{-1}{n_1}} + 3.92727281)(b-a)$$
$$+ (1.10129869z^{\frac{-2}{n_1}} - 1.46839838z^{\frac{-1}{n_1}})(b^2-a^2)$$

$$g_{12}(z) = (-0.23376623z^{\frac{-4}{n_1}} - 1.13766232z^{\frac{-3}{n_1}} + 0.10909091z^{\frac{-2}{n_1}})(b-a)$$
$$+ (0.23376623z^{\frac{-4}{n_1}} + 0.48831168z^{\frac{-3}{n_1}})(b^2-a^2)$$

$$g_{60}(z) = (-6.32030669 - 10.47471613z^{\frac{1}{n_1}} + 73.91738429z^{\frac{2}{n_1}} + 35.12960155z^{\frac{3}{n_1}}$$
$$+ 17.16560576z^{\frac{4}{n_1}})(b-a)$$
$$+ (7.75220811 - 19.74911937z^{\frac{1}{n_1}} + 8.19005658z^{\frac{2}{n_1}} - 58.05863903z^{\frac{3}{n_1}})(b^2-a^2)$$

$$g_{41}(z) = (-1.76951467z^{\frac{-2}{n_1}} - 6.62254186z^{\frac{-1}{n_1}} + 9.642863 + 7.54146909z^{\frac{1}{n_1}})(b-a)$$
$$+ (1.76951467z^{\frac{-2}{n_1}} + 1.27681404z^{\frac{-1}{n_1}} - 1.89196093 - 5.31220359z^{\frac{1}{n_1}})(b^2-a^2)$$

$$g_{22}(z) = (-0.39134198z^{\frac{-4}{n_1}} - 3.02453086z^{\frac{-3}{n_1}} + 0.21818191z^{\frac{-2}{n_1}})(b-a)$$
$$+ (0.39134198z^{\frac{-4}{n_1}} + 1.6115439z^{\frac{-3}{n_1}})(b^2-a^2)$$

$$g_{03}(z) = (-0.09090909z^{\frac{-6}{n_1}} - 0.9090909z^{\frac{-5}{n_1}} - 0.09090909z^{\frac{-4}{n_1}})(b-a)$$
$$+(0.09090909z^{\frac{-6}{n_1}} + 0.54545454z^{\frac{-5}{n_1}})(b^2-a^2).$$

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