POISSON MIXTURES AND QUASI-INFINITE DIVISIBILITY OF DISTRIBUTIONS

by

Prem S. Puri* and Charles M. Goldie
Purdue University, Lafayette, Indiana, and
University of Sussex, U. K.

Department of Statistics
Division of Mathematical Sciences
Mimeograph Series #509

September 1977

*These investigations got started while working on the project at the Statistics Laboratory at the University of California, Berkeley, supported by the auspices of the U.S. Energy Research and Development Agency and were completed under the support of U.S. National Science Foundation Grant No: MCS77-04075, at Purdue University.

POISSON MIXTURES AND QUASI-INFINITE DIVISIBILITY OF DISTRIBUTIONS

by

Prem S. Puri* and Charles M. Goldie
Purdue University, Lafayette, Indiana, and
University of Sussex, U. K.

1. INTRODUCTION.

Let Z be a proper nonnegative integer-valued random variable (r.v.) with probability generating function (p.g.f.) G(s) defined for $|s| \leq 1$, as

(1) $$G(s) = E(s^Z) = \sum_{k=0}^{\infty} s^k P(Z=k).$$

For every such r.v. dealt herewith we assume that $P(Z=0) > 0$. This is without loss of generality, for otherwise we can always subtract from Z an appropriate positive integer in order to satisfy this. The distribution of Z is said to be a mixture of Poisson or simply a Poisson mixture, with a mixing distribution function (d.f.) F concentrated on $[0,\infty)$, if for $k = 0,1,2,\ldots,$

(2) $$P(Z=k) = \int_0^{\infty} \frac{x^k}{k!} e^{-x}\, dF(x),$$

so that

(3) $$G(s) = \int_0^{\infty} \exp[-x(1-s)]dF(x), \qquad |s| \leq 1.$$

*These investigations got started while working on the project at the Statistics Laboratory at the University of California, Berkeley, supported by the auspices of the U.S. Energy Research and Development Agency and were completed under the support of U.S. National Science Foundation Grant No: MCS77-04075, at Purdue University.

Let the Laplace Stieltjes transform (L.T.) of F (or Laplace transform (L.T.) of the probability density function (p.d.f.) of F, whenever F is absolutely continuous) be given by

$$F^*(\theta) = \int_0^\infty \exp(-\theta x)dF(x), \quad \Re e(\theta) \geq 0. \tag{4}$$

By an abuse of language, if the d.f. F of a r.v. X has a certain property, then we shall say that the r.v. X or the corresponding F or F^* or G, as the case may be, has such a property.

The Poisson mixture defined by (2) is also known in literature as compound Poisson with perhaps the only exception being Feller (1968, 1971), although in his 1943 paper (Feller, 1943), he too called these as compound Poisson. But what he called than as Generalised Poisson, in (Feller, 1968, 1971) he calls them as compound Poisson (see also Haight (1967), p. 45). Anyway without worrying about the existing terminology, we shall call these as Poisson mixtures.

The Poisson mixtures have arisen many times in the past in several live situations. In the well known studies on accident proneness due to Bates and Neyman (1952a,b), these mixtures, among others, were used as possible models of the underlying mechanisms there. The reader may also refer to Neyman (1939) and Feller (1943), for their relevance in the context of Neyman's contagious distributions. Feller (1943) also gives several examples as special cases of the Poisson mixtures. Basically in any situation, where the events are rare and can be reasonably considered as Poisson distributed for each individual member of the population and furthermore if it is believed that the population is inhomogeneous with respect to the Poisson parameter, then the Poisson mixture may be considered reasonable as an appropriate model for the situation.

Again, it is well known (see Teicher (1961)) that a Poisson mixture as given by (2) identifies F, i.e. if

$$\int_0^\infty \frac{x^k}{k!} e^{-x} dF_1(x) = \int_0^\infty \frac{x^k}{k!} e^{-x} dF_2(x), \tag{5}$$

for $k = 0,1,2,\ldots$, then $F_1 = F_2$. Thus it is natural to study certain properties of F through those of G and vice versa. In particular, we shall be interested here in the infinite divisibility properties of F and G. For this, it is not too difficult to see through examples (see example (b), Section 5) that not **all** Poisson mixtures are infinitely divisible (i.d.). On the other hand, it is known (see Maceda (1948)) that if F in (3) is i.d., so will be the corresponding p.g.f. G. And since from (3) we have $G(s) = F^*(1-s)$, $|s| \leq 1$, one might expect the converse also to hold true; namely if G is i.d., so will be F^*. Unfortunately this is **false**. In fact in Section 5, we shall discuss an example (e) in some detail, where G is i.d. while the corresponding F is not. Among others, the questions that are raised and answered in this paper are as follows:

(A) Given a p.g.f. G, obtain necessary and sufficient conditions in order that it be a Poisson mixture (Section 2).

(B) Given a Poisson mixture G, obtain necessary and sufficient conditions in order that its mixing distribution F be i.d. (Section 2).

(C) Given a mixing distribution F, obtain necessary and sufficient conditions in order that the corresponding Poisson mixture be i.d.

In order to study questions such as (C), we call a d.f. F concentrated on $[0,\infty)$ as quasi-infinitely divisible (q.i.d.), if when used as a mixing distribution, it renders the Poisson mixture i.d. In section 3, we generalize this concept of q.i.d. to what we call λ-q.i.d. defined for $\lambda > 0$, so that the q.i.d. class becomes a λ-q.i.d. class with $\lambda = 1$. The classes of λ-q.i.d. distributions for various λ have the interesting property, namely that they provide a nested family of classes of distributions which partially to various extents, fulfil the conditions of infinite divisibility. In fact, as it turns out, the intersection of all λ-q.i.d. classes is precisely the class of i.d. distributions over $[0,\infty)$. We label the union of λ-q.i.d. classes over all $\lambda > 0$, as *-q.i.d. class. In section 3, we study some necessary and sufficient conditions for an F to be λ-q.i.d. It is known that the cumulants of any i.d. F over $[0,\infty)$, as far as they exist, are nonnegative (see Steutel (1970), p. 90). We show that this property holds more generally for all F's that are *-q.i.d. In section 4, we study some closure properties of Poisson mixtures, λ-q.i.d. and *-q.i.d. classes with respect to operations such as translation, change of scale, convolution, mixing, convergence in law, etc. In section 5, we give examples of distributions of various types.

Finally, it is appropriate to add that the characterization of Poisson mixtures (see section 2) and the concept of q.i.d. was first introduced and studied by Puri (see Puri (1976), for an abstract) in an earlier report. The generalization of q.i.d. concept to λ-q.i.d. and *-q.i.d. was suggested by the second author, who refereed that report. Both authors are grateful to the Editors of the Applied Probability Journals for their suggestion, which led to the present collaboration.

2. POISSON MIXTURES AND INFINITE DIVISIBILITY OF THE MIXING DISTRIBUTIONS.

For an arbitrary p.g.f. G as defined in (1), it is evident that

$$(6)\quad G(1) = 1,\ 0 \le G(s) \le 1,\ \text{and}\ 0 \le G^{(n)}(s) < \infty,\ n = 1,2,\ldots,$$

holds for all real $0 \le s < 1$, where $G^{(n)}(s)$ is the n^{th} derivative of G, which always exists for $n \ge 1$, and $|s| < 1$. The following theorem implies that if the condition (6) holds for all real $s < 1$, including the negative values, then G must be a Poisson mixture with some mixing distribution F.

THEOREM 1. *A function G is a p.g.f. corresponding to a Poisson mixture if and only if $G(\cdot)$ is defined, has continuous derivatives of all order and satisfies (6), for all $-\infty < s < 1$. Furthermore under these conditions, the mixing distribution F is characterized uniquely with its L.T. F^* given by*

(7) $$F^*(\theta) = G(1-\theta),\ \Re(\theta) \geq 0.$$

PROOF. If G is given to be a Poisson mixture for some F, as defined by (3), then the conditions of the theorem trivially hold. Conversely, let a G satisfy these conditions. Then for n = 0,1,2,..., the following equivalent inequalities hold.

$$G^{(n)}(s) \geq 0,\ \forall \text{ real } -\infty < s < 1 \Leftrightarrow G^{(n)}(1-u) \geq 0,\ \forall \text{ real } u > 0$$

$$\Leftrightarrow (-\frac{d}{du})^n R(u) \geq 0,\ \forall \text{ real } u > 0,$$

where $R(u) \equiv G(1-u)$. Thus the function R is completely monotone (c.m.) for real $u > 0$. As such by virtue of Bernstein's theorem (see Feller (1971), p. 439), since $R(0) = G(1) = 1$, there exists a probability distribution function F concentrated on $[0,\infty)$ such that for real $u \geq 0$.

$$R(u) = \int_0^\infty \exp(-ux)dF(x),$$

which is to say that G is a p.g.f. corresponding to a Poisson mixture with mixing distribution F. That the mixing distribution F is characterized uniquely through (7), follows from the analytic continuation of G from its domain of analyticity $\{s: |s| < 1\}$ to that of $F^*(1-s)$, namely $\{s: \Re(s) < 1\}$. □

It may be remarked the conditions of the above theorem in particular imply that G is strictly positive for all real $-\infty < s \leq 1$. Thus in order to answer the question (B) raised in Section 1, we consider such a p.g.f. G, so that log G is well defined for all real $-\infty < s \leq 1$. Again it is known (see Feller (1968), p. 290) that in order that a p.g.f. G be i.d., it is necessary and sufficient that for $|s| \leq 1$, it has the representation

(8) $$-\log G(s) = \lambda(1-H(s)),$$

for some positive λ and a p.g.f. H(s). Here H(s) can be expressed as $H(s) = 1 + \frac{1}{\lambda} \log G(s)$ and if G is defined and satisfies $0 < G(s) < 1$, $\forall$ real $-\infty < s \leq 1$, then using this relation, the definition of p.g.f. H(s) can be extended to all values of $-\infty < s \leq 1$. Based on such an H(s) corresponding to a given G, the next theorem gives a necessary and sufficient condition (9), in order that the mixing distribution F be i.d. Evidently, if one desires the mixing distribution F to be i.d. one must require that the corresponding mixture G be also i.d. Since our Theorem 2 is essentially an adaptation of a theorem in Feller ((1971), Theorem 1, p. 450), where we take $G(s) = F^*(1-s)$, $-\infty < s \leq 1$, and also since in part it follows from our Theorem 1, we shall omit its proof.

THEOREM 2. (a) Let a given p.g.f. G be defined and satisfy $0 < G(s) \leq 1$, $\forall$ real $-\infty < s \leq 1$, and let G be i.d., so that it satisfies (8) for $|s| \leq 1$, for some $\lambda > 0$ and a p.g.f. H. Let the extended definition of $H(s) \equiv 1 + \frac{1}{\lambda}\log G(s)$ for $-\infty < s < 1$, have the continuous derivatives $H^{(n)}(s)$ of all order, which satisfy the condition

$$(9) \qquad H^{(n)}(s) \geq 0, \ \forall n \geq 1, \text{ and real } -\infty < s < 1.$$

Then the G is a Poisson mixture with an i.d. mixing distribution F. Furthermore for every $n \geq 1$, $[G(s)]^{1/n}$ is also a Poisson mixture with an i.d. mixing distribution F_n satisfying $F_n^*(\theta) = [F^*(\theta)]^{1/n}$.

(b) Conversely, suppose G is a Poisson mixture with an i.d. mixing distribution F. Then G is i.d. and satisfies (8) for some $\lambda > 0$ and a p.g.f. H. Furthermore G and hence $H(s) = 1 + \frac{1}{\lambda} \log G(s)$ is defined $\forall$ real $-\infty < s \leq 1$. Also H has the continuous derivatives of all order, satisfying (9).

Remark 1. Note that in the above theorem, the condition (9) is not required to hold for $n = 0$. On the other hand, if in addition to (9), we also have $0 \leq H(s) \leq 1$, $\forall$ $-\infty < s \leq 1$, then from Theorem 1, it would follow that the p.g.f. H itself is also a Poisson mixture. Consider, for example, the case where the mixing distribution F is a stable distribution with

(10) $F^*(\theta) = \exp(-\alpha\theta^\gamma)$, $0 < \gamma < 1$, $\alpha > 0$, $\theta \geq 0$.

The corresponding mixture is given by the p.g.f. $G(s) = \exp[-\alpha(1-s)^\gamma]$, which is i.d. as expected, and is expressible as $\exp[-\lambda(1-H(s))]$, for any $\lambda \geq \alpha$, with the corresponding $H(s)$ given by

(11) $$H(s) = 1 - \frac{\alpha}{\lambda}(1-s)^\gamma.$$

Clearly $H(s)$ satisfies (9), although not for $n = 0$, so that H is not a Poisson mixture. On the other hand, consider the case where the mixing distribution F is negative binomial with

(12) $F^*(\theta) = p^\alpha[1-q\exp(-\theta)]^{-\alpha}$, $\alpha > 0$, $0 < p = 1-q < 1$.

Then for any $\lambda > -\alpha \log p$, the $H(s)$ corresponding to the Poisson mixture $G(s) = F^*(1-s)$, is given by

(13) $$H(s) = 1 + \frac{\alpha}{\lambda}\log\{p[1-q\exp(s-1)]^{-1}\},$$

which satisfies (9) and also that $0 \leq H(s) \leq 1$, $\forall -\infty < s \leq 1$, so that here the p.g.f. $H(s)$ itself is a Poisson mixture and this can be easily verified.

Remark 2. If we wish to identify all Poisson mixtures with an i.d. mixing distribution, one way would be to start with an i.d. mixing distribution F concentrated on $[0,\infty)$ and then simply take the p.g.f. for the corresponding Poisson mixture as $G(s) = F^*(1-s)$. On the other hand being i.d., F^* must have the representation (see Feller (1971), p. 450)

(14) $$-\log F^*(\theta) = \int_0^\infty \frac{1-e^{-x\theta}}{x}\,dK(x),$$

for some nondecreasing K such that

(15) $$\int_1^\infty x^{-1}dK(x) < \infty.$$

Thus we have shown that <u>a p.g.f. G is a Poisson mixture with an i.d. mixing distribution, if and only if</u>

$$(16)\qquad -\log G(s) = \int_0^\infty \frac{1-e^{-x(1-s)}}{x}\,dK(x), \quad \text{for } -\infty < s \le 1,$$

for some nondecreasing K satisfying (15). This can also be easily obtained using the condition (9). In fact if H of Theorem 2 has the first moment $H^{(1)}(1)$ finite, so that $H^{(1)}(s)/H^{(1)}(1)$ itself is a p.g.f., one can easily show that G will have for some d.f. F_1 the representation

$$(17)\qquad -\log G(s) = \lambda H^{(1)}(1) \int_0^\infty \frac{1-e^{-x(1-s)}}{x}\,dF_1(x), \quad -\infty < s \le 1,$$

with $F_1^*(\theta) = H^{(1)}(1-\theta)/H^{(1)}(1)$, so that $H^{(1)}(s)/H^{(1)}(1)$ is a Poisson mixture with mixing distribution F_1.

We close this section with the following corollary, the proof of which, being rather straightforward, is omitted.

COROLLARY 1. *Let F be a distribution concentrated on $[0,\infty)$ and let $G(s) = F^*(1-s)$ be the corresponding Poisson mixture. Then F is i.d. if and only if for every positive integer n, $[G(s)]^{1/n}$ is a p.g.f. and is a Poisson mixture.*

3. QUASI-INFINITELY DIVISIBLE LAWS.

As mentioned earlier, a Poisson mixture is i.d. whenever the mixing distribution F is i.d. The converse however is not always true (see example (e), section 5). Thus in order to answer the questions such as (C) raised in section 1, we introduce the notions of quasi-infinite divisibility as follows.

DEFINITION 1. A distribution F concentrated on $[0,\infty)$ is said to be quasi-infinitely divisible (q.i.d.), if when used in (3) as a mixing distribution, it renders the Poisson mixture i.d. (see also Puri (1976))

DEFINITION 2. For a given $\lambda > 0$ and a d.f. F on $[0,\infty)$, the p.g.f. G defined by

$$(18)\qquad G(s) = \int_0^\infty \exp[-\lambda x(1-s)]\,dF(x), \quad |s| \le 1,$$

will be called as λ-Poisson mixture, mixed by F.

DEFINITION 3. A distribution F on $[0,\infty)$ is said to be λ-q.i.d. for a given $\lambda > 0$, if the corresponding λ-Poisson mixture is i.d. It will be called $*$-q.i.d. if it is λ-q.i.d. for some $\lambda > 0$.

Thus a q.i.d. law is λ-q.i.d. with $\lambda=1$. For a given $\lambda > 0$, let $\mathfrak{F}_\lambda$ denote the class of λ-q.i.d. distributions over $[0,\infty)$, so that $\mathfrak{F}_1$ is the class of all q.i.d. laws. The class $\mathfrak{F}_*$ of $*$-q.i.d. laws is then simply $\bigcup_{\lambda>0} \mathfrak{F}_\lambda$. Also let $\mathfrak{F}_\infty$ be the class of those distributions on $[0,\infty)$ which are λ-q.i.d. for every $\lambda > 0$, so that $\mathfrak{F}_\infty = \bigcap_{\lambda>0} \mathfrak{F}_\lambda$. Let $X \geq 0$ have d.f. F, with L.T. F^*. Consider a Poisson process with rate $\lambda>0$, which is independent of X. Let $N_\lambda(X)$ denote the number of events of the Poisson process occurring during the random time interval $[0,X]$. Then the distribution of $N_\lambda(X)$ is λ-Poisson mixture, mixed by F. Also F is λ-q.i.d. if and only if $N_\lambda(X)$ is i.d. Again, it is evident that every i.d. distribution F is λ-q.i.d. for all $\lambda > 0$. In fact as we shall see later, the set $\mathfrak{F}_\infty$ is precisely the set of all i.d. laws on $[0,\infty)$. However first we shall give a characterization of λ-q.i.d. distributions. It is known (see Feller (1971)) that F^* is i.d. if and only if $-F^{*(1)}(\theta)/F^*(\theta)$ is completely monotone (c.m.), i.e. iff

(19) $$\left(\frac{-d}{d\theta}\right)^n\left[-F^{*(1)}(\theta)/F^*(\theta)\right] \geq 0, \quad n=0,1,2,\ldots,$$

holds for all real $\theta > 0$. The following theorem shows that for any F^*, condition (19) holding <u>only</u> for $\theta = \lambda$, is necessary as well as sufficient, in order that F^* be λ-q.i.d. for a given $\lambda > 0$.

THEOREM 3. *For a given* $\lambda > 0$, $F \in \mathfrak{F}_\lambda$ *iff* (19) *holds for* $\theta = \lambda$.

PROOF. F^* is λ-q.i.d. if and only if $G(s) = \int_0^\infty \exp[-\lambda x(1-s)]dF(x)$, $|s| \leq 1$, be i.d. On the other hand, it is well known (see Feller (1968), p. 290) that p.g.f. G will be i.d. if and only if

(20) $$\log\left[\frac{G(s)}{G(0)}\right] = \sum_{k=1}^{\infty} a_k s^k, \quad 0 \leq s < 1,$$

with $a_k \geq 0$, $\forall\, k \geq 1$, or equivalently

(21) $$\left(\frac{d}{ds}\right)^n \left[\frac{G^{(1)}(s)}{G(s)}\right]\Bigg|_{s=0} \geq 0, \quad n=0,1,2,\ldots .$$

Replacing $\lambda(1-s)$ by θ and using the fact that $F^*(\theta) = G(1-\frac{\theta}{\lambda})$, (21) becomes equivalent to (19) holding only for $\theta=\lambda$. □

REMARK 3. Note that (20) and hence (21) are also equivalent to

(22) $$\left(\frac{d}{ds}\right)^n [G^{(1)}(s)/G(s)] \geq 0, \quad n=0,.,2,\ldots;\ \forall\, 0 \leq s < 1.$$

This in turn implies an equivalence between (19) holding for $\theta = \lambda$ and it holding $\forall\, 0 < \theta \leq \lambda$. In view of theorem 3 it therefore follows that for $0 < \lambda_1 < \lambda_2$, we have $\mathfrak{F}_{\lambda_2} \subset \mathfrak{F}_{\lambda_1}$. Thus the λ-q.i.d. classes $\mathfrak{F}_\lambda$ for various $\lambda > 0$, provide a nested family of collections of distributions which partially, to various extents, fulfil the conditions (such as (19)) of infinite divisibility. This is formally stated in the following theorem, part (b) of which follows, in part, from the fact that i.d. F^* satisfies (19), $\forall\, \theta > 0$.

THEOREM 4. (a) The family $\{\mathfrak{F}_\lambda;\ \lambda > 0\}$ is a nested family in the sense that for $0 < \lambda_1 < \lambda_2$, $\mathfrak{F}_{\lambda_2} \subset \mathfrak{F}_{\lambda_1}$.

(b) A nonnegative r.v. is i.d. iff it is λ-q.i.d. for an unbounded set of $\lambda > 0$.

(c) The set $\mathfrak{F}_\infty = \bigcap_{\lambda>0} \mathfrak{F}_\lambda$ is precisely the set of all i.d. laws on $[0,\infty)$.

The following corollary easily follows from the definition of λ-quasi-infinite divisibility and theorem 3.

COROLLARY 2. In order that an F be λ-q.i.d., it is necessary and sufficient that for every $n \geq 1$, $[F^*(\lambda(1-s))]^{1/n}$, $|s| \leq 1$, be a p.g.f. However, if a λ-q.i.d. F is not i.d. then there must exist an integer $n > 1$, such that the p.g.f. $[F^*(\lambda(1-s))]^{1/n}$ is not a λ-Poisson mixture.

Comparing corollaries 1 and 2, it is interesting to note that in order that F be i.d., one must require for every n not only that $[F^*(1-s)]^{1/n}$ be a p.g.f., but also that it be a Poisson mixture. On the other hand for it to be q.i.d., it is only sufficient that for every n, $[F^*(1-s)]^{1/n}$ be a p.g.f. Same comment also holds for λ-q.i.d. laws. Again it is now evident (see theorem 3 and remark 3) that any known condition for infinite divisibility of $F^*(\theta)$ required for all $\theta > 0$, when restricted only to $\theta = \lambda$, will typically become a condition for its λ-quasi-infinite divisibility. Of course, in view of remark 3, this in general will also be equivalent to holding $\forall\ 0 < \theta \leq \lambda$. Thus for instance, following Goldie (1967) (see also Steutel (1970), p. 81), in theorem 5 below, we give without proof another necessary and sufficient condition for an F^* to be λ-q.i.d. For this, we define for $\theta > 0$, the sequence

(23) $$b_k(\theta) = \frac{1}{k!}\int_0^\infty x^k \exp(-\theta x)dF(x),\ k=0,1,2,\ldots,$$

and the quantities $a_k(\theta)$ determined successively for $k \geq 1$, from the relations

(24) $$\sum_{k=0}^{n} b_k\, a_{n-k} = (n+1)b_{n+1},\ (n \geq 0).$$

THEOREM 5. *An F^* is λ-q.i.d. iff the quantities $a_k(\theta)$ defined by (23) and (24) are all nonnegative for $\theta = \lambda$ and hence $\forall\ 0 < \theta \leq \lambda$ in view of remark 3. A sufficient condition for this to hold is that the sequence $\{b_k(\theta)\}$ satisfy*

(25) $$b_{k+1}\, b_{k-1} \geq b_k^2,$$

for $k \geq 1$ and for $\theta = \lambda$.

The condition in the above theorem of requiring $a_k(\theta)$ for $\theta = \lambda$ and hence for all $0 < \theta \leq \lambda$, to be nonnegative, can be shown (see Steutel (1970), pp. 90-91) to be equivalent to requiring the cumulants $K_n(\theta)$, n=1,2,..., of the distribution

(26) $$\exp(-\theta x)dF(x)/F^*(\theta),\ x \geq 0,$$

to be nonnegative for $\theta = \lambda$ and hence for all $0 < \theta \leq \lambda$. Note that as long as $\theta > 0$, the cumulants of (26) always exist. However letting $\theta \downarrow 0$, it follows that the cumulants of any *-q.i.d. distribution F, as far as they exist, are nonnegative. That they are nonnegative for i.d. F is well known (see Steutel (1970), p. 90). Thus we have

COROLLARY 3. (a) *A d.f. F is λ-q.i.d. iff the cumulants $K_n(\theta)$ of the distribution defined by (26) are all nonnegative for $n=1,2,\ldots$, and for $\theta = \lambda$ and hence, $\forall\ 0 < \theta \leq \lambda$.*

(b) *The cumulants of any $F \in \mathfrak{F}_*$, as far as they exist, are nonnegative.*

Part (b) of the above corollary raises the natural question about the existence of distributions concentrated on $[0,\infty)$, but not belonging to $\mathfrak{F}_*$ and yet having all their cumulants existing and nonnegative. The answer to this is that such distributions do exist; see example (f) of section 5. Finally, the following theorem gives another necessary and sufficient condition, which is easier to use in order to establish the λ-quasi-infinite divisibility of a given d.f. F over $[0,\infty)$.

THEOREM 6. *A d.f. F concentrated on $[0,\infty)$ is λ-q.i.d. iff $\log F^*(\theta)$, when restricted to $0 \leq \theta \leq \lambda$, has the representation*

$$\log F^*(\theta) = \mu[H(1-\tfrac{\theta}{\lambda})-1], \tag{27}$$

for some $\mu > 0$ and a p.g.f. H.

PROOF. It follows by noticing that the λ-Poisson mixture $G(s) = F^*(\lambda(1-s))$ is i.d. iff

$$\log F^*(\lambda(1-s)) = \mu(H(s)-1),\ 0 \leq s \leq 1, \tag{28}$$

for some $\mu > 0$ and a p.g.f. H. □

4. CLOSURE PROPERTIES OF $\mathfrak{F}_\lambda$ and $\mathfrak{F}_*$.

In the following subsections we study some of the closure properties of Poisson mixtures and the classes $\mathfrak{F}_\lambda$ and $\mathfrak{F}_*$, with respect to change of scale, translation, convolution, mixing and convergence in law.

4.1 CHANGE OF SCALE AND TRANSLATION.

The following theorem can be easily established by using the definitions of λ-q.i.d. and *-q.i.d. laws, theorem 3, remark 3 and the nested property of the family $\{\mathfrak{F}_\lambda, \lambda > 0\}$.

THEOREM 7. (a) If a nonnegative r.v. X belongs to $\mathfrak{F}_*$, then $c X \in \mathfrak{F}_*$, for every $c > 0$.

(b) If $X \in \mathfrak{F}_\lambda$, then $cX \in \mathfrak{F}_{\lambda/c}$ for $c > 0$ and hence $cX \in \mathfrak{F}_\lambda$, for all $0 < c \leq 1$.

(c) $X \in \mathfrak{F}_\lambda$ iff $\lambda X \in \mathfrak{F}_1$.

Again if a nonnegative r.v. X is λ-q.i.d., then so is X+c for every $c \geq 0$. This follows from the fact that if $N_\lambda(X)$ is i.d., then so will be $N_\lambda(X+c)$. Also let for some $c > 0$, $P(X \geq c) = 1$ and X be λ-q.i.d. Let F^* and F_c^* be the L.T. of the r.v.s. X and X-c respectively. Then since $F_c^*(\theta) = \exp(c\theta)F^*(\theta)$, we have

$$(29)\qquad \begin{aligned} &-F_c^{*(1)}(\theta)/F_c^*(\theta) = -c-F^{*(1)}(\theta)/F^*(\theta), \\ &(-d/d\theta)^n[-F_c^{*(1)}(\theta)/F_c^*(\theta)] = (-d/d\theta)^n[-F^{*(1)}(\theta)/F^*(\theta)],\ n \geq 1. \end{aligned}$$

Note since $P(X \geq c) = 1$,

$$(30)\qquad -F^{*(1)}(\theta)/F^*(\theta) = E_c(\xi),$$

where ξ has the d.f. F_θ given by

$$(31)\qquad dF_\theta(x) = \exp(-\theta x)dF(x)/F^*(\theta),\ x \geq c, \text{ and } \theta > 0.$$

Again since $E_c(\xi) \geq c$, it follows from the λ-quasi-infinite divisibility of X that

$$(32)\qquad (-d/d\theta)^n[-F_c^{*(1)}(\theta)/F_c^*(\ )]\Big|_{\theta=\lambda} \geq 0,\ n = 0,1,2,\ldots,$$

and hence in view of theorem 3, that X-c is also λ-q.i.d. Thus we have shown that all translations of mixing r.v.s. which keep them nonnegative preserve the λ-q.i.d. property. Consequently, by normalizing each such r.v. through appropriate translation, one may achieve the infima of their support as zero. Furthermore, in view of theorem 7(c), one could further standardize the *-q.i.d. random variables by multiplying them by suitable constants so that they belong to $\mathfrak{F}_1$. Thus we have established.

THEOREM 8. It is possible to normalize through appropriate change of scale and translation any r.v. belonging to $\mathfrak{F}_*$, so that the resulting r.v. belongs to $\mathfrak{F}_1$ and has zero as the infima of its support.

4.2. CONVOLUTION. Consider the class of λ-Poisson mixtures defined by (18) (see definition 2, section 3) and generated by varying the mixing distribution F over $[0,\infty)$. Since convolution of mixing distributions corresponds to the convolution of their Poisson mixtures, it follows that the class of λ-Poisson mixtures is closed under convolution. In fact, since Poisson mixtures are 'power mixture' in the sense of Keilson and Steutel (1974), it also follows from their results. A similar argument leads to the conclusion that for every $\lambda > 0$ the class $\mathfrak{F}_\lambda$ of λ-q.i.d. distributions over $[0,\infty)$ is also closed under convolution. Same holds also for the larger class $\mathfrak{F}_*$. Thus we have

THEOREM 9. (a) For each $\lambda > 0$, the class of λ-Poisson mixtures and the class $\mathfrak{F}_\lambda$ are both closed under convolution.

(b) If $F_1 \in \mathfrak{F}_{\lambda_1}$ and $F_2 \in \mathfrak{F}_{\lambda_2}$, then $F_1 * F_2 \in \mathfrak{F}_{\lambda_1 \wedge \lambda_2}$, where $\lambda_1 \wedge \lambda_2 = \min(\lambda_1, \lambda_2)$ and $\lambda_i > 0, i=1,2$. It follows that the class $\mathfrak{F}_*$ is closed under convolution.

4.3. MIXING. It is immediate from the definition that the Poisson mixtures are closed under mixing. However neither $\mathfrak{F}_\lambda$ for any $\lambda > 0$ nor the class $\mathfrak{F}_*$ of q.i.d. laws is closed under finite mixing. This follows from the fact that a simple mixture of two different Poisson distributions can never be i.d. (see theorem II of Tortrat (1969), Keilson and Steutel (1972) and also example (b), section 5).

4.4 CONVERGENCE IN DISTRIBUTION.

Let a sequence of d.f. F_n concentrated over $[0,\infty)$ converge weakly to a proper d.f. F, as $n\to\infty$. It follows that

$$(33)\qquad \lim_{n\to\infty} \int_0^\infty \exp[-x(1-s)]dF_n(x) = \int_0^\infty \exp[-x(1-s)]dF(x), \quad |s| \le 1.$$

On the other hand, for each n let Z_n be a r.v. having a Poisson mixture distribution with mixing d.f. F_n over $[0,\infty)$ and suppose Z_n converges in law to some proper r.v. Z. Let F_{n_k} be a subsequence of F_n converging to a possibly defective d.f. F, then for $|s| \le 1$,

$$(34)\qquad \int_0^\infty \exp[-x(1-s)dF(x) = \lim \int_0^\infty \exp[-x(1-s)]dF_{n_k}(x)$$
$$= \lim E(s^{Z_{n_k}})$$
$$= E(s^Z),$$

so that F has to be a proper d.f. and $F^*(1-s)$ is uniquely determined for $|s| \le 1$. By analytic continuation $F^*(\theta)$ is uniquely determined for all $\mathrm{Re}(\theta) > 0$. As such, F_n converges weakly to F, and the distribution of Z is a Poisson mixture with mixing d.f. F. Thus we have shown that Poisson mixtures can converge to a Poisson mixture only and this happens if and only if the corresponding mixing distributions converge weakly to the mixing distribution of the limit. The same closure property with respect to convergence in law holds for the class of distributions F_λ for fixed $\lambda > 0$. This in part follows from a similar well known closure property of i.d. laws. Finally the class F_* of q.i.d. laws is <u>not</u> closed under convergence in distribution, as is exhibited through a counter example (see example (g)) given in the next section. Thus we have

THEOREM 10. (a) *Poisson mixtures can converge to a Poisson mixture only and this happens iff the mixing distributions converge weakly to the mixing distribution of the limit.*

(b) For every fixed $\lambda > 0$, the class $\mathfrak{F}_\lambda$ is closed under convergence in distribution. However the class $\mathfrak{F}_*$ is not closed under the same operation.

5. EXAMPLES. In this section, we consider examples of p.g.f.'s of the types (a)-(e) listed below, some briefly others in detail. Some of these are well known, but are touched here for completeness sake only. The types (a)-(e) are mutually exclusive as well as exhaustive over all p.g.f.'s. Again examples (f) and (g) serve as counterexamples (see sections 3 and 4.4). In particular, example (g) shows the existence of a sequence F_n converging weakly to an F, with $F_n \in \mathfrak{F}_* \ \forall n \geq 1$, but with $F \notin \mathfrak{F}_*$.

(a) A p.g.f. G, which is neither i.d. nor a Poisson mixture.

(b) A p.g.f. G, which is a Poisson mixture, but is not i.d.

(c) A p.g.f. G, which is i.d. but is not a Poisson mixture.

(d) A Poisson mixture G generated by an i.d. mixing distribution.

(e) A Poisson mixture G generated by a mixing distribution F, which is *-q.i.d. but is not i.d.

(f) A d.f. F concentrated on $[0,\infty)$, which does not belong to F_* and yet has all its cumulants existing and positive.

(g) A sequence of d.f. $F_n \in \mathfrak{F}_*$ converging weakly to a d.f. F not belonging to F_*.

(a) Examples of this type are easy to construct. Any nonnegative integer valued nondegenerate r.v. which is bounded, has to be of this type, since being bounded, it cannot either be i.d. (see Lukacs (1960)) or a Poisson mixture. Thus binomial and hypergeometric distributions would fall under this type. Again an example of the unbounded case corresponds to the p.g.f. given by (11), which by virtue of Theorem 1 clearly is not a Poisson mixture. That it is also not i.d. follows from the fact that $H^{(1)}(s)/H(s)$ does not have the desired power series expansion.

(b) All Poisson mixtures with mixing distributions concentrated on two arbitrary points in $[0,\infty)$ are not i.d. This result is due to Tortrat (1969). Thus no distribution concentrated on two points in $[0,\infty)$ belongs to $\mathfrak{F}_*$ (see also section 4.3).

(c) Let $H(s)$ be a p.g.f. of r.v., which takes only nonnegative <u>even</u> integer values. Then for some $\lambda > 0$, the p.g.f. given by $G(s) = \exp[-\lambda(1-H(s))]$, is i.d., but cannot be a Poisson mixture, since the condition $0 \leq G(s) \leq 1$, for all $-\infty < s \leq 1$, is not satisfied (see Theorem 1).

(d) The examples of this type are plenty in the literature. The most common is the one where the mixing distribution F is a Gamma distribution, yielding negative binomial as the Poisson mixture. This arises in the theory of accident proneness (see Bates and Neyman (1952a,b)). Another less known example touched briefly in Remark 1, Section 2, is where the mixing distribution F corresponds to a stable law with F^* given by (10).

(e) Examples of Poisson mixtures, where the mixing distribution is *-q.i.d. but is not i.d., do not appear in literature so often. We discuss one in the following in some detail. It is well known that all the 'proper' mixtures (with positive coefficients) of exponential distributions are i.d. (see Goldie (1967)). However there are mixtures of exponential distributions with some coefficients negative and yet valid p.d.f.'s that are not i.d. (see Steutel (1970)). Often such mixtures turn out to be *-q.i.d. More specifically consider a mixing distribution, with

(35) $$F^*(\theta) = [abc + \theta^2][(\theta+a)(\theta+b)(\theta+c)]^{-1},$$

where without loss of generality $0 < a < b < c$, and are assumed to be such that the p.d.f. corresponding to F^*, given by

(36) $$f(x) = Ae^{-ax} - Be^{-bx} + Ce^{-cx},$$

is nonnegative for all $x \geq 0$. The reader may refer to Bartholomew (1969) for the desired conditions for this to hold. Since the characteristic function corresponding to F has real roots $\pm(abc)^{1/2}$, it cannot be i.d. (see Luckacs (1960)). However, we shall show that for every fixed $\lambda > 0$, it is possible to choose the constants a, b and c appropriately so that it is λ-q.i.d. From (35) we easily have

$$(37)\qquad F^{*(1)}(\theta)/F^{*}(\theta) = [\theta + i(abc)^{\frac{1}{2}}]^{-1} + [\theta - i(abc)^{\frac{1}{2}}]^{-1} - (\theta+a)^{-1} - (\theta+b)^{-1} - (\theta+c)^{-1},$$

so that for $n=0,1,2,\ldots,$

$$(38)\qquad \frac{1}{n!}(-d/d\theta)^{n}[-F^{*(1)}(\theta)/F^{*}(\theta)]\Big|_{\theta=\lambda} = (\lambda+a)^{-(n+1)} + (\lambda+b)^{-(n+1)} + (\lambda+c)^{-(n+1)} - 2r^{-(n+1)}\cos[(n+1)\beta],$$

where

$$(39)\qquad r = (\lambda^2+abc)^{\frac{1}{2}} \text{ and } \tan\beta = (abc)^{\frac{1}{2}}/\lambda.$$

Now in view of theorem 3, it is sufficient to choose a, b and c so that (36) is a p.d.f. and that (38) is nonnegative for all $n \geq 0$. This is always possible. For instance, if $b=a+\alpha, c=a+2\alpha$ with $\alpha > 0$, it turns out that $a \geq 1$ is sufficient to make (36) a p.d.f. and in addition taking $\alpha \geq 2\lambda$ suffices to make (38) nonnegative for all $n \geq 0$. Thus the family (35) of distributions so obtained is such that each member is not i.d. and yet it is λ-q.i.d. for the given λ. Here the p.d.f. is given by (36) with

$$(40)\qquad \begin{aligned} A &= \tfrac{1}{2}[2a + \frac{3a^2}{\alpha} + (\frac{a}{\alpha})^2(1+a)] \\ B &= (1 + \frac{a}{\alpha})[1 + 2a + \frac{a(1+a)}{\alpha}] \\ C &= \tfrac{1}{2}(B + 3 + \frac{2a}{\alpha}). \end{aligned}$$

(f) Consider the d.f. F with

$$(41)\qquad F^{*}(\theta) = \frac{a^3+\theta^2}{(\theta+a)^3}, \quad a \geq 1.$$

This is a special case of (35) with $a = b = c$. Its p.d.f. is given by

$$(42)\qquad f(x) = [1-2ax + \frac{a^2(1+a)}{2}x^2]\exp(-ax),$$

which can be easily shown to be positive for all $x \geq 0$ by using the fact that $a \geq 1$. As before, it is not i.d. Moreover, we shall show that there exists no positive λ for which it is λ-q.i.d. Suppose on the contrary, there exists one such λ. Then by virtue of theorem 3 and remark 3, (19) must hold for all $0 < \theta \leq \lambda$, and n=0,.,2,... Following the lines of example (e), this can be shown to be equivalent to the inequalities

$$(43) \qquad \left[\frac{\theta^2+a^{3/2}}{(\theta+a)^2}\right]^{\frac{n+1}{2}} \geq \frac{2}{3}\cos[(n+1)\beta],\ n=0,1,2,\ldots,$$

holding, $\forall\ 0 < \theta \leq \lambda$, where $\tan\beta = a^{3/2}/\theta$. Now there must exist $\theta\varepsilon(0,\lambda]$ for which

$$\nu = \limsup_{n\to\infty} \cos[(n+1)\beta]$$

is positive. On the other hand, since $a \geq 1$, the left side of (43) tends to zero as $n\to\infty$, contradicting (43). Thus there exists no interval $(0,\lambda]$ with $\lambda > 0$, in which $-\log F^*$ is completely monotone and hence (41) must <u>not</u> belong to $\mathfrak{F}_*$ and yet it can be easily shown that

$$(44) \qquad (-d/d\theta)^n[\log F^*(\theta)]\Big|_{\theta=0} > 0,\quad n=1,2,\ldots,$$

so that all its cumulants exist and are positive.

(g) Consider again for each n=1,2,..., the F^* of example (e) with $b = a+\alpha$, $c = a+2\alpha$, $a \geq 1$, and $\alpha = 2/n$, and call it F_n^*. Evidently for each n, $F_n^*\varepsilon\ \mathfrak{F}_*$ and yet as $n\to\infty$, F_n^* converges to the F^* of example (f), which was shown to be not belonging to $\mathfrak{F}_*$. Thus the class $\mathfrak{F}_*$ is <u>not</u> closed under convergence in distribution, as was claimed in theorem 10.

6. A FEW CONCLUDING REMARKS

(i) The notion of quasi-infinite divisibility as introduced in Section 3, strictly speaking, may be called as quasi-infinite divisibility with respect to Poisson mixtures. It can therefore be generalized to quasi-infinite divisibility with respect to mixtures of an arbitrary family of distributions suitably chosen.

(ii) Again, for the present results, we were concerned only with the univariate case, both with respect to the mixture itself as well as the mixing distribution. Some of these, including the notion of quasi-infinite divisibility, are easily generalizable to the multivariate case to cover, for instance, the multivariate Poisson mixture with p.g.f.,

$$G(s_1,s_2,\ldots,s_k) = \int_0^\infty \cdots \int_0^\infty \exp\Big[\sum_{i=1}^{k} x_i(s_i-1)\Big]dF(x_1,\ldots,x_k), \tag{45}$$

with $0 \leq s_i \leq 1$, $i = 1,2,\ldots,k$.

(iii) Finally one may also consider the more general question of infinite divisibility of mixtures of "Generalized Poisson" distributions. The c.f. ϕ of a generalized Poisson distribution is defined for an arbitrary c.f. ψ and $\lambda > 0$, as

$$\phi(u) = \exp[-\lambda(1-\psi(u))], \tag{46}$$

where u is a real (dummy) variable. The c.f. ϕ_F for the <u>generalized Poisson mixture</u> with mixing distribution F is then given by

$$\phi_F(u) = G(\psi(u)) = \int_0^\infty \exp[x(\psi(u)-1)]dF(x), \tag{47}$$

where G is the p.g.f. for the corresponding (simple) Poisson mixture, considered earlier. For a given ψ one may, as before, attempt to characterize all the mixing distributions, which render (47) i.d. As a partial answer to this, based on the results presented here, it is easy to show that <u>for any arbitrary ψ, every q.i.d. mixing distribution F renders the corresponding generalized Poisson mixture i.d.</u>

REFERENCES

Bartholomew, D. J. (1969). Conditions for mixtures of exponential densities to be probability densities, *Ann. Math. Statist.* **40**, 2183-2188.

Bates, G. E. and Neyman, J. (1952a). Contribution to the theory of accident proneness. I. An optimistic model of the correlation between light and severe accidents, *Univ. Calif. Pub. Statist.* **1**, 215-254.

Bates, G. E. and Neyman, J. (1952b). Contribution to the theory of accident proneness. II. True or false contagion, Univ. Calif. Pub. Statist. **1**, 255-276.

Feller, W. (1943). On a general class of "contagious" distributions, Ann. Math. Statist. **14**, 389-400.

Feller, W. (1968). An Introduction to Probability Theory and Its Applications. Vol. 1 (3rd ed.), J. Wiley, New York.

Feller, W. (1971). An Introduction to Probability Theory and Its Applications. Vol. 2 (2nd ed.), J. Wiley, New York.

Goldie, C. M. (1967). A class of infinitely divisible distributions, Proc. Cambridge Phil. Soc. **63**, 1141-1143.

Haight, F. A. (1967). Handbook of the Poisson Distribution. J. Wiley, New York.

Keilson, J. and Steutel, F. W. (1972). Families of infinitely divisible distributions closed under mixing and convolution, Ann. Math. Statist. **43**, 242-250.

Keilson, J. and Steutel, F. W. (1974). Mixtures of distributions, moment inequalities, and measures of exponentiality and normality, Ann. Prob. **2**, 112-130.

Lukacs, E. (1960). Characteristic Functions, Griffin, London.

Maceda, E. C. (1948). On the compound and generalized Poisson distributions, Ann. Math. Statist. **19**, 414-416.

Neyman, J. (1939). On a new class of contagious distributions, applicable in entomology and bacteriology, Ann. Math. Statist. **10**, 35-57.

Puri, P. S. (1976). Poisson mixtures and quasi-infinite divisibility of distributions, The I.M.S. Bulletin, **5**, (Abstract 76t-127) p.204.

Steutel, F. W. (1970). Preservation of Infinite Divisibility Under Mixing and Related Topics. Math. Centre Tracts 33, Amsterdam.

Teicher, H. (1961). Identifiability of mixtures, Ann. Math. Statist. **32**, 244-248.

Tortrat, A. (1969). Sur les mélanges de lois indéfinement divisibles, Comptes Rendus Acad. Sci. Paris, Sér. A, **269**, A784-A786.