ON THE ADMISSIBILITY OF ESTIMATORS IN EXPONENTIAL DISTRIBUTIONS

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Abstract: A sufficient condition for the admissibility of linear estimators of the mean of multivariate exponential distribution is obtained using exterior boundary value problem. This condition generalizes a result of Zidek (1970) to higher dimension. Also some results on characterization of admissible linear estimators as generalized Bayes are developed.

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Introduction

The problem of estimating the mean of general exponential family has drawn the attention of many authors. Karlin [ ] studied the problem and gave sufficiant condition for the admissibility of linear estimators in the 1-dimensional case. Zidek [ ] and Cheng Ping [ ] also have obtained results similar to Karlin in the 1-dimensional case.

We consider, in this paper, the problem of estimating the mean vector of m-dimensional general exponential family under squared error loss. In Section 2, we give a sufficient condition for the admissibility of linear estimators in m-dimensions using an exterior boundary value problem of elliptic differential equations. The result coincides with Karlin's (or Zidek's) when the dimension is 1. The sufficient condition can be used to obtain easily verifiable conditions on the normalizing function of parameter of density for the admissibility of the linear estimators.

In Section 3, we deal with the question of whether admissible estimators of the mean are generalized Bayes. We have been able to answer this question positively in the 1-dimensional case under mild conditions.

1. Preliminaries

Let $x = (x_1,\dots,x_n)$ be an m-dimensional random variable with density $f_\theta(x) = \beta(\theta)e^{\theta\cdot x}(\theta \in R^m)$ with respect to a σ-finite measure μ with support E^m. Let

$\Theta = \{\theta: \int e^{\theta x}\mu(dx) < \infty\}$. Assume Θ is an unbounded open convex set. It is desired to estimate the mean of x ie $-\frac{\nabla\beta(\theta)}{\beta(\theta)} = \eta(\theta)$ under the squared error loss $L(\theta,t) = ||\eta-t||^2$-($||\cdot||$ is the usual Euclidean norm). Let Π be an absolutely continuous (with respect to Lebesque measure) measure on the Borel subsets of Θ with density $\pi(\theta)$. We assume $\pi(\theta) > 0$ in Θ and non-negative continuously differentiable in R^m. Let δ_π denote the generalized Bayes estimator of Π.

Also, let E^θ and E^x_π denote the expectations with respect to $f_\theta(x)$ and the formal posterion distribution with respect to Π. The formal posteriori distribution has density with respect to the Lebesgue measure and it will be denoted by $p(\theta/x)$. Let E denote the expectation with respect to the marginal of x.

We will need the following notion of almost admissibility. Let $R(\eta,\delta)$ be the risk of δ (note that η is a function of θ ie $\eta(\theta)$, but we will drop θ for notational convenience). We say δ is almost admissible with respect to η if δ_1 is an estimator of η satisfying $R(\eta,\delta_1) \leq R(\eta,\delta)$ then $R(\eta,\delta_1) = R(\eta,\delta)$ a.e. Π. The results of this paper will be consequence of the following result due to Stein [8].

Let S_r be the sphere of radius r with origin as centre. Let J_r denote the class of all non-negative real valued functions j defined on R^m satisfying $j(\theta) \geq 1$ on S_r and $\int j(\theta)R(\theta,\delta\pi)\Pi(\theta)d\theta < \infty$.

Theorem 1.1 (Stein): The estimator δ_π is almost admissible with respect to Π if for ever $\epsilon > 0$ and $r > 0$, there exists $j \in J_r$ such that

$$E\left[\frac{||E^x_\pi\, j(\theta)(\eta-\delta_\pi)||^2}{E^x_\pi(j)}\right] < \epsilon.$$

Proof: See Stein [8].

2. Admissibility of linear estimators of the mean.

In this section we consider the admissibility of estimators of the form $\delta(x) = \frac{x}{\lambda+1}$ for $\lambda > 0$. This estimator was considered by Karlin [7] (also see Zidek) in 1-dimensional case and he gave a sufficient condition for its admissibility. We generalize his result to higher dimensions in this section.

We need the following exterior boundary value problem. Let L_π be the differential operator $L_\pi u = \Delta u + \frac{\nabla\pi}{\pi}\cdot\nabla u$ where Δ is the Laplacian. The only non-negative solution of $L_\pi u = 0$ in the exterior domain $R^m - S_1$ is the constant functions $u \equiv 1$ then we say Boundary Problem (BP) is solvable for L_π. The following result about BP is known (See Brown [1], Srinivasan [10]).

Theorem 2.1: The Boundary Problem for L_π is solvable if and only if

$$\inf_{\substack{u \geq 0 \\ u = 1 \text{ on } S_1 \\ u \to 0 \text{ as } ||y|| \to \infty}} \int ||\nabla u(y)||^2 \, \pi(y)\,dy = 0.$$

We will use this theorem in proving the main theorem of this section.

Let us make some observations before we state our assumptions. Note that the mapping $\theta \to \eta(\theta)$ is one-one and the range of $\eta(\theta)$ is R^m. Let $q(\eta) = q(\eta(\theta)) = \Pi(\theta)$. Then $q(\eta) > 0$ for all η. Moreover $q(\eta)$ is differentiable continuously.

Assumptions:

(I) The Jacobian of the transformation $\theta \to \eta(\theta)$ is bounded away from 0 and ∞ in Θ.

(II) $\int \beta(\theta) e^{\theta x} \beta^\lambda(\theta)\,d\theta < \infty$ a.e.$[\mu(dx)]$, $\lambda > 0$

(III) $\beta(\theta)\, e^{x\theta} \beta^\lambda(\theta) \to 0$ as $|\theta| \to \infty$, $\lambda > 0$.

The assumption (I) is easy to verify. It is equivalent to the condition that the determinant of the variance covariance matrix of $f_\theta(x)$ is bounded away from 0 and ∞. The assumptions (II) and (III) ensure that the estimator $\delta(x) = \frac{x}{\lambda+1}$ is generalized Bayes with respect to the prior $\Pi(\theta) = \beta^\lambda(\theta)$.

Theorem 2.2: Assume I, II & III. Then $\delta(x) - \frac{x}{\lambda+1}$ is generalized Bayes with respect to $\Pi(\theta) = \beta^\lambda(\theta)$. Moreover $\delta(x) = \frac{x}{\lambda+1}$ is admissible if BP is solvable for the equation $L_q u(\eta) = \Delta u(\eta) + \Delta u(\eta)\,\frac{\nabla q(\eta)}{q(\eta)} = 0$

Proof: Observe that it follows from assumptions II & III

$$\nabla(\beta(\theta)e^{x\cdot\theta}\beta^\lambda(\theta)) = \lambda+1[(\frac{x}{\lambda+1} - (-\frac{\nabla\beta(\theta)}{\beta(\theta)})]e^{\theta x}\beta^{\lambda+1}_{(\theta)} \tag{2.1}$$

for any x, where the derivatives are taken with respect to θ. Now, for any piecewise differentiable function $j \in J_r$

$$E^x_\Pi(j^2_{(\theta)}(\frac{x}{\lambda+1} - (-\frac{\nabla\beta(\theta)}{\beta(\theta)})) = E^x_\Pi[j^2_{(\theta)}\nabla(e^{\theta\cdot x}\beta^{\lambda+1}_{(\theta)})]$$

$$= \frac{\int j^2_{(\theta)}\nabla(e^{\theta\cdot x}\beta^{\lambda+1}_{(\theta)})d\theta}{\int e^{\theta x}\beta^{\lambda+1}_{(\theta)}d} \tag{2.2}$$

Integrating the numerator of (2.2) by parts and using assumption III and the fact $\beta(\theta)$ vanishes on the boundary of Θ we have

$$E^x_\pi(j^2(\theta)(\frac{x}{\lambda+1} - (-\frac{\nabla\beta(\theta)}{\beta(\theta)})) = \frac{\int \nabla j^2_{(\theta)}e^{\theta x}\beta^{\lambda+1}_{(\theta)}d\theta}{\int e^{\theta x}\beta^{\lambda+1}_{(\theta)}d\theta}.$$

Therefore, by Schwartz inequality,

$$||E^x_\pi(j^2(\theta)(\frac{x}{\lambda+1} + \frac{\nabla\beta(\theta)}{\beta(\theta)})||^2/E^x_\pi(j^2_{(\theta)} \leq 4\,\frac{\int |\nabla j(\theta)|^2 e^{\theta x}\beta^{\lambda+1}_{(\theta)}d\theta}{\int e^{\theta x}\beta^{\lambda+1}_{(\theta)}d\theta} \tag{2.3}$$

Now taking expectation with respect to the marginal of x on both sides of (2.3) we have

$$E\{\frac{||E^x_\pi(j^2(\theta)(\frac{x}{\lambda+1} + \frac{\nabla\beta(\theta)}{\beta(\theta)})||^2}{E^x_\pi(j^2(\theta))}\} \leq 4\int ||\nabla j(\theta)||^2\beta^\lambda_{(\theta)}d\theta \tag{2.4}$$

Thus $\delta(x) = \frac{x}{\lambda+1}$ is almost admissible if for every $r > 0$,

$$\inf_{j \in J_r} \int ||\nabla j(\theta)||^2 \beta^\lambda_{(\theta)} d\theta = 0 \tag{2.5}$$

Switching from the variable θ to η and using the fact the assumption I it is easy to see that (2.5) holds if

$$\inf_{u \in U} \int ||\nabla u(\eta)||^2 q(\eta) d\eta = 0 \tag{2.6}$$

where $U = \{u: \; u \geq 0, \; u \geq 1 \text{ on } C \; \; Su(\eta)q(\eta)d\eta < \infty\}$ and C is a fixed compact set in R^m. Now by Theorem 2.1 we have (2.6) if BP II is solvable for $L_q u = 0$. Therefore $\delta(x) = \frac{x}{\lambda+1}$ is almost admissible if BP is solvable for $L_q u = 0$. Finally, since the underlying family of distributions is exponential and the risk function of $\delta(x) = \frac{x}{\lambda+1}$ is a Laplace transform of a function, it is continuous and hence almost admissibility of $\frac{x}{\lambda+1}$ is equivalent to admissibility. Hence the proof.

The above theorem can be used to obtain verifiable sufficient conditions on $q(\eta)$ and hence $\Pi(\theta)$ for the admissibility of $\frac{x}{1+\lambda}$ sufficient conditions for the solvability of $L_q u=0$ are available (See Brown [1], Srinivasin [10]). We list a couple of them below without pay.

Corollary 2.3: Suppose $q(\eta) \leq b||\eta||^{2-m}$ for all $||\eta|| > b_0$ for some b_0. Then BP for $L_q u = 0$ is solvable and hence $\frac{x}{\lambda+1}$ is admissible.

The next result is for spherically symmetric μ. Observe that if μ is spherically symmetric then so is $\beta(\theta)$ie $\beta(\theta) = \beta(||\theta||)$ and $\beta(||\theta||)$ exists and possitive for all $||\theta||$.

Corollary 2.4: Suppose μ is spherically symmetric. Then $\frac{x}{\lambda+1}$ is admissible if

$$\int_1^\infty \frac{1}{\beta^\lambda_{(||\theta||)}} \frac{1}{||\theta||^{m-1}} d||\theta|| = \infty. \tag{2.7}$$

Proof: It is well known that (2.7) implies that BP is solvable for $L_q u = 0$ and hence by Theorem 2.2 $\frac{x}{\lambda+1}$ is admissible.

The Corollary 2.4 is a generalization of a result of Karlin [7] to m-dimensions. He obtained a similar condition for the admissibility of $\frac{x}{\lambda+1}$ for one dimensional case. Cheng Ping [3] has studied admissibility of estimators of the form a + bx, b > 0, in the one dimensional case and obtained sufficient conditions similar to Karlins. We can generalize his results to m-dimensions along the same lines as in Theorem 2.2.

One of the major assumptions in proving Theorem 2.2 is that δ_π is generalized Bayes. This assumption does not hold always. Indeed, it is easy to construct examples of exponential distribution where x is an admissible estimator of $\eta(\theta)$, but it is not generalized Bayes. However, for dimension m = 1 it is possible to show that admissible estimators of $\eta(\theta)$ are generalized Bayes under some conditions. We prove this in the next section.

3. Generalized Bayes Estimators.

3.1 In this subsection we prove, for dimension m = 1 that every admissible estimator of $\eta(\theta)$ is generalized Bayes with respect to some σ-finite measure under mild conditions. The proof is similar to the one given by Farrell [5] so we only sketch the proof often referrring to Farrell's paper.

Since the dimension m = 1, it is clear that Θ is an interval. We assume throughout this section that Θ is an open interval ie the upper and lower end points, say $\bar{\theta}$ and $\underline{\theta}$, do not belong to Θ. Moreover we assume that the support of μ is $(-\infty,\infty)$.

We will need the following result of Farrell [4]. We state in our set up.

Theorem 3.1: A decision procedure δ is admissible if and only if there exists a sequence of procedures δ_n Bayes with respect to finite measures λ_n having compact supports $D_n \uparrow \infty$ satisfying

(i) for every compact set E, $\sup_{n \geq 1} \lambda_n(E) < \infty$

(ii) there exists a compact set E_0 such that $\inf_n \lambda_n(E_0) \geq 1$

(iii) $\int (R(\eta(\theta),\delta) - R(\eta(\theta),\delta_n))\lambda_n(d\theta) \to 0$ as $n \to \infty$.

Proof: See Farrell [4].

We now state the main theorem of this section.

Theorem 3.2: An estimator $\delta(x)$ is admissible only if it is generalized Bayes ie. there exists a σ-finite measure F on $\Theta = (\underline{\theta},\overline{\theta})$ such that

$$\delta(x) = \frac{\int \eta(\theta)\beta(\theta)e^{\theta x}F(d\theta)}{\int \beta(\theta)e^{\theta x}F(d\theta)}$$

we present the proof of the theorem after a series of lemmas.

Let $V(\theta) = ||\eta(\theta)||+1$ and $V_1(\theta,t) = (\eta(\theta)-t)$. It is easy to see that if $E \subset (-\infty,\infty)$ is a compact set then $V_1(\theta,t)/V(\theta)$ is uniformly continuous in θ and t on ΘxE. This follows from the fact that $\eta(\theta)$ is monotone in θ and $\lim_{\theta\to\overline{\theta}} \eta(\theta) = \infty$ and $\lim_{\theta\to\underline{\theta}} \eta(\theta) = -\infty$ and hence $\lim_{\theta\to\overline{\theta}} \frac{V_1(\theta,t)}{V(\theta)} = 1$ and $\lim_{\theta\to\underline{\theta}} \frac{V_1(\theta,t)}{V(\theta)} = -1$ for every t in E.

Suppose now $\delta(x)$ is admissible. Then by Theorem 3.1 there exists a sequence of finite measures $\{\lambda_n\}$ satisfying (i), (ii) & (iii) of the theorem. Define a sequence of probability measures on Θ by setting

$$V_n(x,E) = \frac{1}{K_n(x)} \int_E V(\theta)f_\theta(x)\lambda_n(d\theta)$$

where E is a Borel subset of Θ. Note that the normalizing function $K_n(x)$ is defined and finite for every x since λ_n has compact support.

Let Θ^* be a metrizable compactification of Θ such that Θ is borel subset of Θ^* and extend the probability measures $V_n(x,E)$ to Θ^*. We will denote the extended measures also by $V_n(x,E)$. From now on we will be using these extended measures.

Let $F_n(E) = \int V(\theta)\lambda_n(d\theta)$. Since $\sup_{n\geq 1} \lambda_n(E) < \infty$ for every compact set $E \subset \Theta$, there exists a subsequence $\{F_{n_i}\}$ converging weakly to a σ-finite measure F' on, the weak convergence being with respect to the class of continuous functions on Θ vanishing outside compact sets. Assume, without loss of generality that $F_n \to F'$ weakly in the above sense.

<u>Lemma 3.3</u>: There exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\nu_{n_i}(x,\cdot) \to \nu(x,\cdot) \text{ weekly for almost all } x[\mu]$$

<u>Proof</u>: Observe that $\nu_n(x,\cdot)$ are continuous bilinear functionals on $L,(R,B,\mu) x C(\Theta^*)$ where $C(\theta^*)$ is the Banach Space of continuous functions with supremum norm and $L_1(R,B,\mu)$ is the Banach Space of absolutely integreble functions with respect to μ. Since $L_1(R,B,\mu)$ is separable, the unit ball of bilinear functionals is sequentially compact. Now a standard diagonalization argument along with the fact that μ is σ-finite gives the result.

<u>Lemma 3.4</u>: For almost all $x,y[\mu]$

$$\liminf_{n \geq 1} \frac{K_n(x)}{K_n(y)} > 0.$$

<u>Proof</u>: Observe that

$$\frac{K_n(x)}{K_n(y)} = \int_{\Theta} f_\theta(x)/f_\theta(y) \; \nu_n(y,d\theta)$$

Therefore if Θ is a finite interval the result is trivial and if Θ is an infinite interval it follows from Lemma 4.2 of Farrell [5].

<u>Lemma 3.5</u>: $\nu(x,\Theta) > 0$ for almost all $x[\mu]$.

Proof: The proof is divided into two cases, Θ is finite and infinite interval.

Case (ii) Θ is a finite interval.

Suppose $\nu(x,\Theta) = 0$ on an x-set of positive μ measure. Then, for x in that set,

$$0 = \int_{\Theta^*} \frac{V_1(\theta,\delta(x))}{V(\theta)} \nu(x,d\theta) = \int_{\Theta} \frac{V_1(\theta,\delta(x))}{V(\theta)} (x,d\theta) +$$

$$+ \int_{\{\overline{\theta}\}} \frac{V_1(\theta,\delta(x))}{V(\theta)} \nu(x,d\theta) + \int_{\{\underline{\theta}\}} \frac{V_1(\theta,\delta(x))}{V(\theta)} \nu(x,d\theta)$$

Therefore, since $\frac{V_1(\overline{\theta},t)}{V(\theta)} = +1$ and $\frac{V_1(\underline{\theta},t)}{V(\theta)} = -1$,

We have $\nu(x,\overline{\theta}) = \nu(x,\underline{\theta})$ on a set of positive μ measure. It follows from Step (4.9) of Farrell [5] that this is not possible in our set up.

Case (ii) Θ is an infinite interval.

It follows from Lemma 4.3 of Farrell [5] that $\nu(x,\overline{\theta}) = 0$ if $\overline{\theta}$ is ∞ and $\nu(x,\underline{\theta}) = 0$ if $\underline{\theta} = -\infty$ for almost all x. Assume one of the end points, say $\overline{\theta}$, is infinity. Then

$$0 = \int_{\Theta^*} \frac{V_1(\theta,\delta(x))}{V(\theta)} \nu(x,d\theta) = -\nu(x,\underline{\theta}) + \nu(x,\infty) + \int_{\Theta} \frac{V_1(\theta,\delta(x))}{V(\theta)} \nu(x,d\theta)$$

$$= -\nu(x,\underline{\theta}).$$

But $\nu(x,\underline{\theta}) = 1$ and therefore we have a contradiction.

Lemma 3.6: For almost all $x[\mu]$, $\limsup_{n \geq 1} K_n(x) < \infty$.

Proof: It follows from Theorem 3.1 that there exists a subsequence $\{\delta_{n_i}\}$ of $\{\delta_n\}$ converging to δ: This fact follows from condition (iii) of Theorem 3.1. Assume without loss of generality δ_n converges to δ. Now, using Lemmas 3.3, 3.4, 3.5 and Lemma 3.2 of Farrell [5] it follows that there ixists an open set $U \subset \Theta$ having compact closure in Θ such that

$$0 < \liminf_{n \to \infty} \frac{\lambda_n(U)}{K_n(x)} < \limsup_{n \to \infty} \frac{\lambda_n(U)}{K_n(x)} < \infty.$$

for almost all $x[\mu]$. Since $\limsup_{n \to \infty} \lambda_n(\overline{U}) < \infty$, we have $\limsup_{n \geq 1} K_n(x) < \infty$ for almost all $x[\mu]$.

Lemma 3.7: For almost all $x[\mu]$ and all $t \in (-\infty,\infty)$,

(i) $\lim_{n\to\infty} \int_\Theta f_\theta(x) F_n(d\theta) = \int_\Theta f_\theta(x) F'(d\theta)$

(ii) $\lim_{n\to\infty} \int_\Theta \frac{V_1(\theta,t)}{V(\theta)} f_\theta(x) F_n(d\theta) = \int_\Theta \frac{V_1(\theta,t)}{V(\theta)} f_\theta(x) F'(d\theta)$

and the limits are finite.

Proof: We shall give the proof of (ii). Proof of (i) is similar. If $\Theta = (\underline{\theta},\overline{\theta})$ is a finite interval then $f_\theta(x)$ is a bounded continuous function of θ for every x. Since $\frac{V_1(\theta,t)}{V(\theta)}$ is bounded continuous, the result follows from the fact that $F_n \to F'$ weakly. Suppose now Θ is an infinite interval. Let us assume $\Theta = (-\infty,\infty)$. Then for any $0 < A < \infty$,

$$\lim_{n\to\infty} \int_{-A}^{A} \frac{V_1(\theta,t)}{V(\theta)} f_\theta(x) F_n(d\theta) = \int_{-A}^{A} \frac{V_1(\theta,t)}{V(\theta)} f_\theta(x) F(d\theta)$$

we shall complete the proof by showing

$$\lim_{A\to\infty} \limsup_{n \to \infty} \int_A^\infty \left| \frac{V_1(\theta,t)}{V(\theta)} \right| f_\theta(x) F_n(d\theta) = 0 \qquad (3.1)$$

for almost all $x[\mu]$. Suppose (3.1) does not hold on a set of positive μ measure. Then there exists x' such that

$$\lim_{A\to\infty} \limsup_{n\to\infty} \int \left| \frac{V_1(\theta,t)}{V(\theta)} \right| f_\theta(x') F_n(d\theta) > 0 \tag{3.2}$$

and $\mu(x',\infty) > 0$ since the support of μ is the entire real line $(-\infty,\infty)$. Now the monotomicity of $f_\theta(x)$ as a function of x implies for $y > x'$

$$\int_A^\infty \left| \frac{V_1(\theta,t)}{V(\theta)} \right| f_\theta(x') F_n(d\theta) \le \sup_{\theta\geq A} \left[\left|\frac{V_1(\theta,t)}{V(\theta)}\right| e^{(x'-y)\theta}\right] \int_A^\infty f_\theta(y) F_n(d\theta). \tag{3.3}$$

Therefore taking lim sup and letting $A\to\infty$ in (3.3) we find, that

$$\lim_{A\to\infty} \sup_{\theta\geq A} \left[\left| \frac{V_1(\theta,t)}{V(\theta)} \right| e^{(x'-y)\theta}\right] = 0$$

and hence it follows from (3.2) that

$$\limsup_{n\to\infty} \int_{-\infty}^{\infty} f_\theta(y) F_n(d\theta) = \infty. \tag{3.4}$$

But this contradicts the fact $\limsup_{n\to\infty} K_n(x) < \infty$ for almost all $x[\mu]$ of Lemma 3.6. Therefore, (3.1) holds. The cases $\Theta = (-\infty,\overline{\theta})$ or $(\underline{\theta},\infty)$ are similar and we omit the proofs.

We are now in a position to prove Theorem 3.2.

Proof of Theorem 3.2:

We already observed that there exists a sequence $\{\delta_{n_i}\}$ Bayes with respect to λ_{n_i} given by Theorem 3.1 such that $\delta_{n_i}(x)$ converges to $\delta(x)$ for almost every x. Also for almost all n,

$$\frac{V_1(\theta,\delta_{n_i}(x))}{V(\theta)} \to \frac{V_1(\theta,\delta(x))}{V(\theta)} \text{ uniformly in } \theta.$$

Therefore by lemma 3.7

$$0 = \lim_{n_i \to \infty} \int \frac{V_1(\theta,\delta_{n_i}(x))}{V(\theta)} f_\theta(x) F_{n_i}(d\theta) = \int \frac{V_1(\theta,\delta(x))}{V(\theta)} f_\theta(x) F' \, d\theta \quad (3.5)$$

Now let $F(d\theta) = \frac{1}{V(\theta)} F'(d\ )$. Then it follows from (3.5)

$$\delta(x) = \frac{\int \eta(\theta) f_\theta(x) F(d\theta)}{\int f_\theta(x) F(d\theta)} .$$

3.2. The main result of the previous section does not generalize to higher dimensions easily. So we adopt a different approach using a recent result of Brown [2] for the care of general dimension m. In what follows, $\liminf_{\theta\to\infty} u(\theta) = \sup \{\inf \{u(\theta) = \theta \notin S\}: S \subset \Theta, S \text{ compact}\}$ for any real valued function $u(\theta)$. Recall that our loss function $L(\theta,t) = (\eta|\theta)-t|^2$ and $\P = \{\eta(\theta): \theta \in \Theta\}$ is an unbounded set.

Theorem 3.2.1 (Brown): Let $\delta(x)$ be any admissible estimator of η. Suppose there exists another procedure $\delta'(x)$ and a positive real valued function $g(\eta)$ such that

$$\liminf_{\eta\to\infty} g(\eta)(R(\eta,\delta)-R(\eta,\delta')) = c > 0.$$

Then $\delta(x)$ is generalized Bayes for almost all $x[\mu]$ on the set $S = \{x: \lim_{\eta\to\infty} g(\eta)L(\eta,t)f_\theta(x) = 0 \ \forall t \text{ and } \lim_{\eta\to\infty} g(\eta)R(\eta,\delta)f_\theta(x) = 0\}$.

Proof: See Brown [2].

As a consequence of the above result we have the following. Assume in what follows

(A) $\lim_{\eta\to\infty} f_\theta(x) = 0$ for almost all $x[\mu]$.

Theorem 3.2.2: Suppose $-\Delta\log\beta(\theta) < K, \forall\theta$. Then any estimator $\delta_\lambda(x) = \frac{x}{1+\lambda}$ $\lambda > 0$, is generalized Bayes if it is admissible.

Proof: Observe that $-\Delta\log\beta(\theta) < K$ is equivalent to the fact that $\delta_0(x) = x$ has bounded risk. Now, for any estimator $\delta_\lambda(x) = \frac{x}{1+\lambda}$, $\lambda > 0$, we have

$$R(\eta,\delta_\lambda) = \frac{1}{(1+\lambda)^2} R(\eta,\delta_0) + \frac{\lambda^2}{(1+\lambda)^2} |\eta|^2.$$

Therefore

$$\liminf_{\eta\to\infty} \frac{1}{1+|\eta|^2} (R(\eta,\delta_\lambda) - R(\eta,\delta_0)) = \frac{\lambda^2}{1+\lambda^2} > 0.$$

Now setting $g(\eta) = \frac{1}{1+|\eta|^2}$ it follows from Theorem 3.2.1 and assumption (A) that $\delta_\lambda(x)$ is generalized Bayes.

As an example to illustrate the above theorem consider the following distribution. Let $\mu(dx) = e^{-\frac{1}{2}|\eta|^2}|\eta|dx$ $\alpha > 1$. Then $\beta(\theta)$ is approximately $|\theta|^\alpha e^{-\frac{1}{2}|\theta|^2}$ for large $|\theta|$ and $(|\theta|^\alpha+c)\, e^{-\frac{1}{2}|\theta|^2}$ in a neighborhood of the origin. Using this fact it is not difficult to show that $-\Delta\log\beta(\theta) < K, \forall\theta$. Moreover assumption (A) holds because $\eta(\theta) \simeq (\frac{\alpha}{|\theta|^2} -1)\theta$ for large $|\theta|$ and $\eta(\theta) \to 0$ as $\theta \to 0$. Moreover by Theorem 2.2 $\delta_\lambda(x) = \frac{x}{1+\lambda}$ is admissible. Therefore Theorem 3.2.2 implies $\frac{x}{1+\lambda}$ is a generalized Bayes estimator.

References

[1] Brown, L. (1971) Admissible estimators, recurrent diffusions, insoluble boundary value problems. Ann. Math. Statist. 42, 855-903.

[2] Brown, L. A necessary condition for admissibility.

[3] Cheng Ping (1964) Minimax estimates of parameters of distributions belonging to Exponential family. Chinese Math-Acta 5, 277-299.

[4] Farrell, R. (1968) A necessary and sufficient condition for admissibility. Ann. Math. Statist. 39, 24-29.

[5] Farrell, R. (1965) Weak limits of sequences of Bayes procedures in estimation theory Proc. Fifth. Berkeley Symp. Vol. 1, 83-111.

[6] Ghosh, M. and Meeden G. Admissibility of Linear Estimators in the one parameter exponential family. Technical Report, Iowa State Univ.

[7] Karlin, S. (1958) Admissibility for estimation with quadratic loss. Ann. Math. Statist. 29, 406-436.

[8] Stein, C. Notes on Decision Theory.

[9] Stein, C. (1965) Approximation of improper prior measures by proper prior measures. Bernoulli, Bayes, Laplace, Anniversary Volume. Springer-Verlag, New York.

[10] Srinivasan, C. (1976) A study of admissibility through exterior boundary value problem. Ph.D. Thesis. Indian Statistical Institute.

[11] Zidek, J. V. (1970) Sufficient conditions for admissibility under squared error loss of formal Bayes estimators. Ann. Math. Statist. 41, 446-456.