On The Exact Non-null Distribution Of Wilks' L_{VC} Criterion.

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1. Introduction and Summary. Let $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_N$ be a random sample of size N from a p-variate normal population with unknown mean vector $\underset{\sim}{\mu}$ and covariance matrix $\underset{\sim}{\Sigma}$, i.e., $\underset{\sim}{x}_i \sim N(\underset{\sim}{\mu},\underset{\sim}{\Sigma})$, $\underset{\sim}{\Sigma}$ is symmetric and positive definite. Let

$$(1.1)\qquad \bar{\underset{\sim}{x}} = N^{-1} \sum_{i=1}^{N} \underset{\sim}{x}_i \quad \text{and} \quad \underset{\sim}{S} = \sum_{i=1}^{N} (\underset{\sim}{x}_i-\bar{\underset{\sim}{x}})(\underset{\sim}{x}_i-\bar{\underset{\sim}{x}})'$$

Then $\bar{\underset{\sim}{x}}$ has a normal distribution $N(\underset{\sim}{\mu},N^{-\frac{1}{2}}\underset{\sim}{\Sigma})$ and $\underset{\sim}{S}$ has an independent Wishart distribution $W(\underset{\sim}{\Sigma},p,n)$ with $n = N-1$. Wilks [17] likelihood ratio criterion L_{VC} for testing H: $\underset{\sim}{\Sigma} = \sigma^2[(1-\rho)\underset{\sim}{I} + \rho\, \underset{\sim}{e}\, \underset{\sim}{e}']$, ρ,σ unknown, against the alternative $A \neq H$ can be expressed as

$$(1.2)\qquad L_{VC} = |\underset{\sim}{S}|\ [tr(\underset{\sim}{E}.\underset{\sim}{S})/p]^{-1}/[tr(p\underset{\sim}{I}-\underset{\sim}{E})\underset{\sim}{S}/p^{p-1}]^{p-1},$$

where $\underset{\sim}{E} = \underset{\sim}{e}\,\underset{\sim}{e}'$ and $\underset{\sim}{e}' = (1,1,\ldots,1)$ and $\underset{\sim}{\mu}$ is unknown. Wilks [17] derived the exact null distribution of L_{VC} for the special cases p=2 and p=3. Varma [16] obtained the exact null distribution in a series form using Mellin integral transform (see [14]) and factorial series expansion [11], and computed some percentage points for specific values of ρ. Nagarsenker [12] derived the null distribution employing Box's chisquare series approximation and tabulated percentage points for p=4(1)10. Khatri and Srivastava [8] obtained the exact non-null distribution of L_{VC} in a series form involving Meijer's G-functions [9] and certain $a_\delta(J)$ coefficients which are not easy to compute. In this paper, we derive the distribution of

L_{VC} in three series forms and compute powers for p=2 and 3 for five percent critical points for various values of N and the parameters. In section 2 we present some definitions and lemmas which are needed in the sequel. We derive in section 3, the non-null density of L_{VC} as a series involving Meijer's G-functions using Mellin integral transform. Some special cases have also been discussed which are used to compute powers for the case p=2. In section 4, we obtain the non-null density in an alternate series form through the method of contour integration and in section 5, the non-null moments of the criterion are used to obtain the distribution as a chisquare series employing methods similar to those of Box[2]. Section 6 is devoted to power computations. The densities derived in sections 4 and 5 have been used for power computation for various alternatives for the case p=3 and various values of N.

2. Some definitions and results. In this section we give a few definitions and some lemmas which will be used in the sequel.

Definitions. Let k be a non negative integer and let $\kappa = (k_1,k_2,\dots,k_p)$ be a partion of k such that $k_1 \geq k_2 \geq \dots \geq k_p \geq 0$, $\sum_{i=1}^{p} k_i = k$ and let

(2.1) $$(a)_\kappa = \prod_{i=1}^{p} (a-(i-1)/2)_{k_i} = \Gamma_p(a,\kappa)/\Gamma_p(a), \text{ where}$$

(2.2) $$(a)_k = (a)(a+1)\dots(a+k-1) \quad \text{and}$$

(2.3) $$\Gamma_p(a) = \pi^{p(p-1)/4} \prod_{i=1}^{p} \Gamma(a-(i-1)/2)$$

Now Meijer's G-function [9] may be defined by

(2.4) $$G^{m\ n}_{p\ q}\left[x \,\middle|\, \begin{matrix} a_1,a_2,\dots,a_p \\ b_1,b_2,\dots,b_p \end{matrix}\right] = (2\pi i)^{-1}\int_C \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}(a_j-s)}\, x^s\, ds$$

where an empty product is interpreted as unity and C is a curve separating the singularities of $\prod_{j=1}^{m} \Gamma(b_j - s)$ from those of $\prod_{j=1}^{n} \Gamma(1-a_j+s)$, $q \geq 1$, $0 \leq n \leq p \leq q$, $0 \leq m \leq q$; $x \neq 0$ and $|x| < 1$ if $q=p$; $x \neq 0$ if $q > p$. The definition above is on application of lemma 2.4 below.

Also we need the following special case

$$(2.5) \quad G^{2\ 0}_{2\ 2}\left[x \,\middle|\, \begin{matrix} a_1 & a_2 \\ b_1 & b_2 \end{matrix}\right] = x^{b_1} \frac{(1-x)^{a_1+a_2-b_1-b_2-1}}{\Gamma(a_1+a_2-b_1-b_2)} \cdot {}_2F_1(a_2-b_2, a_1-b_2, a_1+a_2-b_1-b_2; 1-x)$$

$$(2.6) \quad \text{where } {}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty} (a)_k \, (b)_k \, x^k/(c)_k \, k!$$

Futher, the hypergeometric function of a matrix variate (see James[6])

$$(2.7) \quad {}_pF_q\,(a_1,a_2,\ldots,a_p;b_1,b_2,\ldots,b_q;\underset{\sim}{S}) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_\kappa \cdots (a_p)_\kappa}{(b_1)_\kappa \cdots (b_q)_\kappa} \frac{C_\kappa(\underset{\sim}{S})}{k!}$$

where $C_\kappa(\underset{\sim}{A})$ denotes the zonal polynomial of the symmetric matrix $\underset{\sim}{A}$ of degree k corresponding to the partition κ. In particular, we have

$$(2.8) \quad {}_0F_0(\underset{\sim}{S}) = \exp(\mathrm{tr}\underset{\sim}{S}) \text{ and } {}_0F_1(a;\underset{\sim}{S}) = |\underset{\sim}{I} - \underset{\sim}{S}|^{-a}$$

<u>Lemmas</u>: We now state a few lemmas without proof which will be used in the following sections.

<u>Lemma 2.1</u>. Let $\underset{\sim}{\Sigma}$ be the matrix having the form $\underset{\sim}{\Sigma} = \sigma\underset{\sim}{I} + \rho\, \underset{\sim}{e}\, \underset{\sim}{e}'$ where $e' = (1,1,\ldots,1)$. $\underset{\sim}{\Sigma}$ can be represented in the form $\underset{\sim}{\Sigma} = \underset{\sim}{H}'\, \underset{\sim}{D}\, \underset{\sim}{H}$ where $\underset{\sim}{H}$ is any $p \times p$ orthoginal matrix having first row $p^{-\frac{1}{2}}\underset{\sim}{e}'$ and $\underset{\sim}{D} = \mathrm{diag}\,((\sigma+p\rho), \sigma, \sigma, \ldots \sigma)$.

Now using lemma 2.1, we note that the test of hypothesis H: $\underset{\sim}{\Sigma} = \sigma^2[(1-\rho)\underset{\sim}{I} + \rho\, \underset{\sim}{e}\, \underset{\sim}{e}']$ is equivalent to that of $\underset{\sim}{\Sigma} = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_2)$, $\sigma_1, \sigma_2 > 0$ and unknown. (see [5]).

Lemma 2.2. If $\underset{\sim}{R}$ is a positive definite $m\times m$ matrix then

$$\int_{\underset{\sim}{0}}^{\underset{\sim}{I}} (\det \underset{\sim}{S})^{t-(m+1)/2} (\det(\underset{\sim}{I}-\underset{\sim}{S}))^{u-(m+1)/2} C_\kappa(\underset{\sim}{R}\,\underset{\sim}{S})\, d\underset{\sim}{S} = \frac{\Gamma_m(t,\kappa)\,\Gamma_m(u)}{\Gamma_m(t+u,\kappa)} C_\kappa(\underset{\sim}{R}).$$

Proof. see Constantine [3].

Lemma 2.3. Let $\underset{\sim}{R}$ be a complex symmetric matrix whose real part is positive definite and let $\underset{\sim}{T}$ be an arbitrary complex symmetric matrix. Then

$$\int_{\underset{\sim}{S}>\underset{\sim}{0}} \exp(-\operatorname{tr} \underset{\sim}{R}\,\underset{\sim}{S})(\det \underset{\sim}{S})^{t-(m+1)/2} C_\kappa(\underset{\sim}{S}\,\underset{\sim}{T})d\underset{\sim}{S} = \Gamma_m(t,\kappa)(\det \underset{\sim}{R})^{-t} C_\kappa(\underset{\sim}{T}\,\underset{\sim}{R}^{-1}),$$

the integration being over the space of positive definite real $m\times m$ matrices, and valid for all complex numbers t satisfying $R(t) > (m-1)/2$.

Proof. See Constantine [3].

Finally we give a lemma defining the Mellin integral transform (see [14]).

Lemma 2.4. If s is any complex variate and $f(x)$ is a function of a real variate x, such that

$$F(s) = \int_0^\infty x^{s-1} f(x)dx$$

exists, then under certain regularity conditions

$$f(x) = (2\pi i)^{-1} \int_{C-i\infty}^{C+i\infty} x^{-s} F(s)ds$$

$F(s)$ is called the Mellin transform of $f(x)$ and $f(x)$ is the inverse Mellin transform of $F(s)$.

3. Exact Non-null distribution of L_{VC}. In this section, we derive the non-null density of L_{VC} as a series of Meijer's G-functions [9] using Mellin-integral transform (lemma 2.4). Using lemma 2.1, the test of $H: \underset{\sim}{\Sigma}=\sigma^2[(1-\rho)\underset{\sim}{I} + \rho\, \underset{\sim}{e}\, \underset{\sim}{e}']$ reduces to that of $H: \underset{\sim}{\Sigma} = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \underset{\sim}{I}_{p_2} \end{bmatrix}$

$\sigma_1, \sigma_2 > 0$ and unknown, and $p_2=p-1$. The L_{VC} can be expressed as

(3.1) $\quad L_{VC} = |\underset{\sim}{S}|/[s_{11}(\operatorname{tr} \underset{\sim}{S}_{22}/p_2)^{p_2}]$

where $\underset{\sim}{S} = \begin{bmatrix} \underset{\sim}{s}_{11} & \underset{\sim}{S}_{12} \\ \underset{\sim}{S}'_{12} & \underset{\sim}{S}_{22} \end{bmatrix} \begin{matrix} 1 \\ p_2 \end{matrix}$, $p_2 = p-1$, $n = N-1$,

N being the size of a sample from $N(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$, $\underset{\sim}{\Sigma} > 0$. Now we make a transformation $\begin{bmatrix} \underset{\sim}{x}_1 \\ \underset{\sim}{x}_2 \end{bmatrix} \to \begin{bmatrix} \underset{\sim}{x}_1/\sigma_1 \\ \underset{\sim}{x}_2/\sigma_2 \end{bmatrix} \begin{matrix} 1 \\ p_2 \end{matrix}$. Under this transformation the problem reduces to that of testing $H_1 : \underset{\sim}{\Sigma} = \begin{bmatrix} 1 & 0 \\ 0 & \underset{\sim}{I}_{p_2} \end{bmatrix}$ versus $A_1 \neq H_1$ where

$$\underset{\sim}{\Sigma} = \begin{bmatrix} 1 & \underset{\sim}{\Sigma}_{12}/\sigma_1\sigma_2 \\ \underset{\sim}{\Sigma}'_{12}/\sigma_1\sigma_2 & \underset{\sim}{\Sigma}_{22}/\sigma_2^2 \end{bmatrix} \begin{matrix} 1 \\ p_2 \end{matrix}, \quad \sigma_1 \text{ and } \sigma_2 \text{ being unknown.}$$

From now on we assume that this has been done and we are testing H_1 versus $A_1 \neq H_1$. Let us define

$$T = s_{11}^{-\frac{1}{2}} \underset{\sim}{S}_{12} \underset{\sim}{S}_{22}^{-1} \underset{\sim}{S}'_{12} s_{11}^{-\frac{1}{2}} \tag{3.2}$$

Then, the L_{VC} can be written as

$$L_{VC} = |\underset{\sim}{S}_{22}| \, (1-T)/(\mathrm{tr}\, \underset{\sim}{S}_{22}/p_2)^{p_2} \tag{3.3}$$

we now need the following lemma in order to compute the non-null moments of L_{VC}

Lemma 3.1. The joint p.d.f. of $\underset{\sim}{T}$, $\underset{\sim}{S}_{11}$ and $\underset{\sim}{S}_{22}$ is given by

$$\begin{aligned} f(\underset{\sim}{T}, \underset{\sim}{S}_{11}, \underset{\sim}{S}_{22}) = {} & k(p_1, p_2, n; \underset{\sim}{\Sigma}) |\underset{\sim}{S}_{11}|^{(n-p_1-1)/2} |\underset{\sim}{S}_{22}|^{(n-p_2-1)/2} \exp(-\tfrac{1}{2}\mathrm{tr}\underset{\sim}{\Sigma}_{1\cdot 2}^{-1}\underset{\sim}{S}_{11}) \\ & \exp(-\tfrac{1}{2}\mathrm{tr}\underset{\sim}{\Sigma}_{2\cdot 1}^{-1}\underset{\sim}{S}_{22}) |\underset{\sim}{I}-\underset{\sim}{T}|^{(n-p_1-p_2-1)/2} |\underset{\sim}{T}|^{(p_2-p_1-1)/2} \\ & \sum_{k=0}^{\infty} \sum_{\kappa} C_\kappa(\underset{\sim}{S}_{11}^{-\frac{1}{2}} \underset{\sim}{\Sigma}_{1\cdot 2}^{-1} \underset{\sim}{\beta} \underset{\sim}{S}_{12} \underset{\sim}{\beta}' \underset{\sim}{\Sigma}_{1\cdot 2}^{-1} \underset{\sim}{S}_{11}^{\frac{1}{2}} \underset{\sim}{T}/4)/[k!(p_2/2)_\kappa], \\ & \underset{\sim}{0} < \underset{\sim}{T} < \underset{\sim}{I}, \quad \underset{\sim}{S}_{11}, \underset{\sim}{S}_{22} > 0, \end{aligned} \tag{3.4}$$

where $\underset{\sim}{\Sigma}_{1\cdot 2} = \underset{\sim}{\Sigma}_{11} - \beta\, \underset{\sim}{\Sigma}'_{12}$, $\underset{\sim}{\Sigma}_{2\cdot 1} = \underset{\sim}{\Sigma}_{22} - \underset{\sim}{\Sigma}'_{12}\underset{\sim}{\Sigma}_{11}^{-1}\underset{\sim}{\Sigma}_{12}$, $\underset{\sim}{\beta} = \underset{\sim}{\Sigma}_{12}\underset{\sim}{\Sigma}_{22}^{-1}$,

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12} & \Sigma_{22} \end{bmatrix}\begin{matrix} p_1 \\ p_2 \end{matrix} \quad \text{and} \quad S = \begin{bmatrix} S_{11} & S_{12} \\ S'_{12} & S_{22} \end{bmatrix}\begin{matrix} p_1 \\ p_2 \end{matrix}$$

$p_1 + p_2 = p$, $p_2 \geq p_1 \geq 1$ (without loss of generality), $k^{-1}(p_1, p_2, n, \Sigma) = 2^{n(p_1+p_2)/2}\Gamma_{p_1}(p_2/2)\Gamma_{p_1}((n-p_2)/2)\Gamma_{p_2}(n/2)|\Sigma_{1\cdot 2}|^{n/2}|\Sigma_{22}|^{n/2}$ and

$T = S_{11}^{-\frac{1}{2}} S_{12} S_{22}^{-1} S_{12}' (S_{11}^{-\frac{1}{2}})'$. (see Khatri and Srivastava [8])

Now before finding $E[L_{VC}^h]$, we will prove the following theorem.

Theorem 3.1.

(3.6) $$E[\exp(-t \operatorname{tr} S_{22}/2)|S_{22}|^h (1-T)^h] = K_3(p_2, n, \Sigma, h) \cdot \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} (t+1)^{-(p_2(h+n/2)+k+j)} \cdot (h)_\kappa (n/2)_J C_\kappa(I - \Sigma_{2\cdot 1}^{-1}) C_J(I - \Sigma_{2\cdot 1}^{-1} + \Sigma_{1\cdot 2}^{-1} \beta' \beta)/k!j!$$

(3.7) where $K_3(p_2, n, \Sigma, h) = 2^{p_2 h}\Gamma_{p_2}((n+1)/2+h)/[\Gamma_{p_2}((n-1)/2)|\Sigma_{22}|^{n/2}]$

Proof. Let $W = \exp(-t \operatorname{tr} S_{22}/2) \; |S_{22}|^h \; (1-T)^h$

Using lemma (3.1) with $p_1 = 1$ we obtain

(3.8) $$E[W] = K(1, p_2, n, \Sigma) \int_{s_{11}>0} \int_{S_{22}>0} \int_{T:0}^{I} (s_{11})^{n/2-1} |S_{22}|^{n/2+h-(p_2+1)/2} \cdot \exp(-\operatorname{tr}\Sigma_{1\cdot 2}^{-1} s_{11}) \cdot \exp(-\operatorname{tr}(\Sigma_{2\cdot 1}^{-1} + t I) S_{22}/2) |T|^{(p_2-p_1-1)/2} \cdot |I-T|^{(n-p_1-p_2-1)/2+h} \sum_{k=0}^{\infty} \sum_{\kappa} C_\kappa(s_{11}^{\frac{1}{2}} \Sigma_{1\cdot 2}^{-1} \beta S_{22} \beta' \Sigma_{1\cdot 2}^{-1} s_{11}^{\frac{1}{2}} T)/[(p_2/2)_\kappa k!] \quad ds_{11} dS_{22} dT$$

Now using monotone convergence theorem, the interchange of the integral and summation signs is valid and using lemma (2.3) in order to integrate with respect to T, one obtains

$$(3.9)\quad E[W] = k_2 \sum_{k=0}^{\infty} \sum_{\kappa} \int_{s_{11}>0} s_{11}^{n/2-1} \exp(-\mathrm{tr}\Sigma_{1\cdot 2}^{-1} s_{11}/2) \int_{\underset{\sim}{S}_{22}>\underset{\sim}{0}} |\underset{\sim}{S}_{22}|^{n/2+h-(p_2+1)/2}$$

$$\exp(-\mathrm{tr}(t\underset{\sim}{I}+\underset{\sim}{\Sigma}_{2\cdot 1}^{-1})\underset{\sim}{S}_{22}/2) C_\kappa(s_{11}\underset{\sim}{\beta}'\underset{\sim}{\beta}\Sigma_{1\cdot 2}^{-2}\underset{\sim}{S}_{22}/4)/(k!(\tfrac{n}{2}+h)_\kappa)\, ds_{11} d\underset{\sim}{S}$$

where

$$(3.10)\quad k_2 = k(1,p_2,n,\underset{\sim}{\Sigma})\Gamma(p_2/2)\Gamma((n-p_2)/2+h)/\Gamma(n/2+h).$$

Now using lemma (2.3) to integrate with respect to $\underset{\sim}{S}_{22}$ and then in turn using monotone convergence theorem and the relation ${}_0F_0(\underset{\sim}{S}) = \exp(\mathrm{tr}\,\underset{\sim}{S})$, we get

$$(3.11)\quad E[W] = k_4 |(t\underset{\sim}{I}+\underset{\sim}{\Sigma}_{2\cdot 1}^{-1})/2|^{-(n/2+h)} \int_{s_{11}>0} s_{11}^{n/2-1} \exp(-(\Sigma_{1\cdot 2}^{-1} - \underset{\sim}{\beta}\Sigma_{1\cdot 2}^{-2}$$

$$(t\underset{\sim}{I}+\underset{\sim}{\Sigma}_{2\cdot 1}^{-1})^{-1}\beta')s_{11}/2)\, ds_{11},$$

where $k_4 = k_2\, \Gamma_{p_2}(n/2+h)$. Now integrating with respect to s_{11}, we get

$$(3.12)\quad E[W] = k_4 |(t\underset{\sim}{I}+\underset{\sim}{\Sigma}_{2\cdot 1}^{-1})/2|^{-(n/2+h)} \Gamma(n/2)\, ((\Sigma_{1\cdot 2}^{-1} - \underset{\sim}{\beta}\Sigma_{1\cdot 2}^{-2}(t\underset{\sim}{I}+\underset{\sim}{\Sigma}_{2\cdot 1}^{-1})^{-1}\underset{\sim}{\beta}')/2)^{-n/2}.$$

Rewriting (3.12), one obtains

$$(3.13)\quad E[W] = k_3(p_2,n,\underset{\sim}{\Sigma},h_1) |t\underset{\sim}{I}+\underset{\sim}{\Sigma}_{2\cdot 1}^{-1}|^{-h} |t\underset{\sim}{I}+\underset{\sim}{\Sigma}_{2\cdot 1}^{-1} - \underset{\sim}{\beta}'\underset{\sim}{\beta}\Sigma_{1\cdot 2}^{-1}|^{-n/2}.$$

Now adding and subtracting $\underset{\sim}{I}$ inside each of the two determinants and using (2.8) we have

$$(3.14)\quad E[W] = k_3(p_2,n,\underset{\sim}{\Sigma},h)\, (t+1)^{-p_2(h+n/2)}\, {}_1F_0(h;(t+1)^{-1}(\underset{\sim}{I}-\underset{\sim}{\Sigma}_{2\cdot 1}^{-1})).$$

$${}_1F_0(n/2;(t+1)^{-1}(\underset{\sim}{I}-\underset{\sim}{\Sigma}_{2\cdot 1}^{-1}+\Sigma_{1\cdot 2}^{-1}\underset{\sim}{\beta}'\underset{\sim}{\beta})),$$

which can be expressed as (3.6) after using (2.7)

<u>Theorem 3.2</u>. For any finite p, the p.d.f. of L_{VC} is given by

$$(3.15)\quad p(L_{VC}) = C_1(p_2,n;\underset{\sim}{\Sigma})(L_{VC})^{-(p_2/2+1)} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{j=0}^{\infty} \sum_{J} p_2^{-(k+j)}$$

$$A(J,\kappa,p_2,n,\Sigma)\ G^{2p_2\ \ 0}_{2p_2\ \ 2p_2}\left[L_{VC}\left|\begin{matrix} c_1,c_2,\ldots,c_{p_2};\ d_1,d_2,\ldots,d_{p_2} \\ a_1,a_2,\ldots,a_{p_2};\ b_1,b_2,\ldots,b_{p_2}\end{matrix}\right.\right]$$

where

(3.16) $C_1(p_2,n,\Sigma)=(2\pi)^{(p_2-1)/2}p_2^{(1-np_2)/2}/(\prod_{i=1}^{p_2}\Gamma((n-i)/2)|\Sigma_{22}|^{n/2})$,

$$A(J,\kappa,p_2,n,\Sigma)=(n/2)_J\ \Gamma(np_2/2+k+j)C_\kappa(I-\Sigma_{2\cdot 1}^{-1})C_J(I-\Sigma_{2\cdot 1}^{-1}+\Sigma_{1\cdot 2}\beta'\beta)\ j!k!$$

(3.17) $a_i = (p_2-i+1)/2 + k_i$, $b_i = (n+p_2-i)/2$; $i = 1,2,\ldots,p_2$

$c_i = (p_2-i+1)/2$ and $d_i = (p_2+n)/2 + (k+j+i-1)/p_2$; $i = 1,2,\ldots,p_2$

Proof. First we evaluate the h-th moment of L_{VC} as the method of derivation of the density of L_{VC} depends on lemma (2.4) concerning the Mellin-transform. Integrating both sides of (3.6) with respect to t, p_2h times under the integral sign and putting t = 0 in the final result we get

(3.18) $$E[L_{VC}]^h = k_3(p_2,n,\Sigma,h)(p_2/2)^{p_2h}\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{j=0}^{\infty}\sum_{J}(n/2)_J(h)_\kappa C_\kappa(I-\Sigma_{2\cdot 1}^{-1}) C_J(I-\Sigma_{2\cdot 1}^{-1}+\Sigma_{1\cdot 2}^{-1}\beta'\beta)/(j!k!(np_2/2+k+j)_{hp_2}).$$

(3.19) Now let $C(p_2,n,\Sigma) = 1/(\prod_{i=1}^{p_2}\Gamma((n-i)/2)\ |\Sigma_{22}|^{n/2})$. then

(3.20) $$E[L_{VC}]^h = C(p_2,n,\Sigma)\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{j=0}^{\infty}\sum_{J}A(J,\kappa,p_2,n,\Sigma)p_2^{p_2h}\prod_{i=1}^{p_2}\Gamma(n/2+h-i/2)\prod_{i=1}^{p_2}(h-(i-1)/2)_{k_i}/\Gamma(p_2(h+n/2)+k+j),$$

where $A(J,\kappa,p_2,n,\Sigma)$ is defined by (3.16). Now using lemma (2.4), take the Mellin integral transform on both sides of (3.20), we get the density of L_{VC} in the form

$$(3.21)\quad p(L_{VC}) = C(p_2,n,\Sigma)\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{j=0}^{\infty}\sum_{J} A(J,\kappa,p_2,n,\Sigma)(2\pi i)^{-1}$$

$$\int_{C-i\infty}^{C+i\infty}(L_{VC})^{-(h+1)}p_2^{p_2h}\prod_{i=1}^{p_2}(h-(i-1)/2)_{k_i}\frac{\prod_{i=1}^{p_2}\Gamma(n/2+h-i/2)}{\Gamma(p_2(h+n/2)+k+j)}\,dh.$$

we now need Gauss -Legendre's multiplication formula given by

$$(3.22)\quad \prod_{r=1}^{n}\Gamma(z+(r-1)/n) = 2\pi^{(n-1)/2}\,n^{\frac{1}{2}-nz}\,\Gamma(nz).$$

Applying the transformation $h \to h+p_2/2$ and using (3.22) on $\Gamma(p_2(h+n/2)+k+j)$, (3.21) can be written as

$$(3.23)\quad p(L_{VC}) = C_1(p_2,n,\Sigma)(L_{VC})^{-(p_2/2+1)}\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{j=0}^{\infty}\sum_{J}A(J,\kappa,p_2,n,\Sigma)$$

$$p_2^{-(k+j)}(2\pi i)^{-1}\int_{C_1-i\infty}^{C_1+i\infty}(L_{VC})^{-h}\frac{\prod_{i=1}^{p_2}\Gamma(h+(p_2-i+1)/2+k_i)\prod_{i=1}^{p_2}\Gamma(h+(n+p_2-i)/2)}{\prod_{i=1}^{p_2}\Gamma(h+(p-i+1)/2)\prod_{i=1}^{p_2}\Gamma(h+(p_2+n)/2+(k+j+i-1)/p_2)}\,dh$$

where $C_1 = C + p_2/2$ and $C_1(p_2,n,\Sigma)$ is given by (3.16). Now (3.23) can also be written as

$$(3.24)\quad p(L_{VC}) = C_1(p_2,n,\Sigma)(L_{VC})^{-(p_2/2+1)}\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{j=0}^{\infty}\sum_{J}p_2^{-(k+j)}$$

$$A(J,\kappa,p_2,n,\Sigma)(2\pi i)^{-1}\int_C(L_{VC})^{-h}\frac{\prod_{i=1}^{p_2}\Gamma(h+a_i)\prod_{i=1}^{p_2}\Gamma(h+b_i)}{\prod_{i=1}^{p_2}\Gamma(h+c_i)\prod_{i=1}^{p_2}\Gamma(h+d_i)}\,dh,$$

where a_i, b_i, c_i and d_i $i = 1, 2, \ldots, p_2$ are defined in (3.17) noticing that the integrals in (3.24) are in the form of Meijer's G-functions, we can write the density of L_{VC} in the form given in (3.15).

We now discuss special cases for $p_2 = 1$ and 2.

$p_2 = 1$. Putting $p_2 = 1$ in (3.15), we get

$$(3.25)\quad p(L_{VC}) = \frac{(L_{VC})^{-3/2}}{\Gamma((n-1)/2)} \sum_{k=0}^{\infty} \frac{\Gamma(n/2+k)}{k!} \quad (-\rho^2/(1-\rho^2))^k \; G^{2\;0}_{2\;2}\left[L_{VC}\middle|\begin{matrix}(n+1)/2+k, \frac{1}{2}\\ n/2 \quad , \quad k+\frac{1}{2}\end{matrix}\right],$$

where $\Sigma = \begin{bmatrix}1 & \rho\\ \rho & 1\end{bmatrix}$. Now using (2.5), (3.25) can be written as

$$(3.26)\quad p(L_{VC}) = \frac{(L_{VC})^{(n-1)/2-1}(1-L_{VC})^{\frac{1}{2}-1}}{\Gamma(n-1)/2\ \Gamma(\frac{1}{2})} \sum_{k=0}^{\infty} (n/2+k)(-\rho^2/(1-\rho^2))^k \cdot {}_2F_1(n/2,-k,$$
$$\tfrac{1}{2},1-L_{VC}),\ 0 < L_{VC} < 1.$$

In particular, under the null hypothesis H_1:$\rho = 0$, the null density is given by

$$(3.27)\quad p_1(L_{VC}) = \Gamma(n/2)/[\Gamma((n-1)/2)\Gamma(\tfrac{1}{2})]\cdot L_{VC}^{(n-1)/2-1}(1-L_{VC})^{-\frac{1}{2}},\ 0 < L_{VC} < 1.$$

$p_2 = 2$. In this case, $\Sigma = \begin{bmatrix}1 & \rho_{12} & c\rho_{13}\\ \rho_{12} & 1 & c\rho_{23}\\ c\rho_{13} & c\rho_{23} & c^2\end{bmatrix}$, $c = \sigma_3/\sigma_2$.

Putting $p_2 = 2$ in (3.15), we get

$$(3.28)\quad p(L_{VC}) = \frac{\pi\ 2^{1-n}\Gamma(n/2)}{\Gamma_2(n/2)\Gamma((n-2)/2)|\Sigma_{22}|^{n/2}} (L_{VC})^{-2} \sum_{k=0}^{\infty}\sum_{\kappa}\sum_{j=0}^{\infty}\sum_{J} 2^{-(k+j)}(n/2)_J$$

$$\Gamma(n+k+j)C_\kappa(I-\Sigma_{2\cdot1}^{-1})C_J(I-\Sigma_{2\cdot1}^{-1}+\Sigma_{1\cdot2}^{-1}\beta'\beta)/k!j!$$

$$G^{4\;0}_{4\;4}\left[L_{VC}\middle|\begin{matrix}1+(n+k+j)/2\ , & 1+(n+k+j+1)/2\ , & \frac{1}{2} & , & 1\\ n/2 & , \quad (n+1)/2 & , k_2+\frac{1}{2} & , & k_1+1\end{matrix}\right]$$

In particular, under the null hypothesis H_1: $\rho_{12} = \rho_{13} = \rho_{23} = 0$ and $c = 1$, the null density is given by

$$(3.29)\quad p_1(L_{VC}) = \pi\ 2^{1-n}\Gamma(n)\Gamma(n/2)[\Gamma_2(n/2)\Gamma((n-2)/2)](L_{VC})^{-2}\ G^{2\;0}_{2\;2}\left[L_{VC}\middle|\begin{matrix}1+n/2,(3+n)/2\\ n/2,\ (n+1)/2\end{matrix}\right]$$

Now using Legendre's duplication formula, namely

$\Gamma(2s) = \Gamma(s)\,(s+\frac{1}{2})\,2^{2s-1}/\pi^{\frac{1}{2}}$ and the well known result

$$(3.30)\quad (2\pi i)^{-1}\int_{C-i\infty}^{C+i\infty}[x^{-s}\Gamma(s)/\Gamma(s+\nu)]\,ds=(1-x)^{\nu-1}/\Gamma(\nu),\ 0\le x\le 1,\ C>0,$$

(see Titchmarsh [15]), we can write (3.29) in the form

$$(3.31)\quad p_1(L_{VC}) = \Gamma(n)/(2\Gamma(n-2))(L_{VC})^{\frac{1}{2}(n-2)}(1-(L_{VC})^{\frac{1}{2}}),\ 0<L_{VC}<1,$$

as was derived by Wilks[17].

4. The exact non-null distribution of L_{VC} through contour integration.

From (3.21) of section 3, we have the distribution of L_{VC} in the form

$$(4.1)\quad p(L_{VC})=C(p_2,n,\Sigma)\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{j=0}^{\infty}\sum_{J}A(J,\kappa,p_2,n,\Sigma)(2\pi i)^{-1}\int_{C-i\infty}^{C+i\infty}(L_{VC})^{-(h+1)}$$
$$p_2^{p_2h}\prod_{i=1}^{p_2}(h-(i-1)/2)_{k_i}\prod_{i=1}^{p_2}\Gamma(h+(n-i)/2)/\Gamma(p_2(h+n/2)+k+j)$$

For simplification, make use of the transformation $h+n/2\to h$. Then (4.1) can be written as

$$(4.2)\quad p(L_{VC})=C(p_2,n,\Sigma)\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{j=0}^{\infty}\sum_{J}A(J,\kappa,p_2,n,\Sigma)(L_{VC})^{n/2-1}\,p_2^{-np_2/2}$$
$$(2\pi i)^{-1}\int_{C+n/2-i\infty}^{C+n/2+i\infty}(L_{VC})^{-h}\,p_2^{p_2h}\prod_{i=1}^{p_2}(h-(n+i-1)/2)_{k_i}\prod_{i=1}^{p_2}\Gamma(h-i/2)/\Gamma(p_2h+k+j)\,dh$$

where

$$(4.3)\quad C^{-1}(p_2,n,\Sigma)=\prod_{i=1}^{p_2}\Gamma((n-i)/2)\,|\Sigma_{22}|^{n/2},\text{ and}$$

$$A(J,\kappa,p_2,n,\Sigma)=(n/2)_J\Gamma(np_2/2+k+j)C_\kappa(I-\Sigma_{2.1}^{-1})C_J(I-\Sigma_{2.1}^{-1}+\Sigma_{1.2}^{-1}\beta'\beta)/k!j!.$$

(4.4) Let $L_1 = L_{VC}/p_2^{p_2}$. Then (4.2) can be written as

$$(4.5)\quad p(L_{VC})=C(p_2,n,\Sigma)\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{j=0}^{\infty}\sum_{J}A(J,\kappa,p_2,n,\Sigma)(L_{VC})^{n/2-1}\,p_2^{-np_2/2}\,f_{J,\kappa}(L_{VC})$$

(4.6) $f_{J,\kappa}(L_{VC})=(2\pi i)^{-1}\int_{C_1-i\infty}^{C_1+i\infty} G_{J,\kappa}(h)dh$, $C_1 = C+n/2$,

(4.7) $G_{J,\kappa}(h)=(L_1)^{-h}\prod_{i=1}^{p_2}(h-(n+i-1)/2)_{k_i}\prod_{i=1}^{p_2}\Gamma(h-i/2)/\Gamma(p_2h+k+j)$.

Throughout the rest of this paper, functions $f_{J,\kappa}(\cdot)$ and $G_{J,\kappa}(\cdot)$ will be written as f and G respectively. We now start with a special case.

$p_2 = 2$. We have from (4.7)

(4.8) $G(h)=(L_1)^{-h}\prod_{i=1}^{2}(h-(n+i-1)/2)_{k_i}\,\Gamma(h-\frac{1}{2})\,\Gamma(h-1)/\Gamma(2h+k+j)$.

Using the duplication fomrula for gamma function in (4.8), we obtain

(4.9) $G(h)=(L_{VC})^{-h}\,D\prod_{i=1}^{2}(h-(n+i-1)/2)_{k_i}\,\Gamma(2h-2)/\Gamma(2h+k+j)$

where $D = 8(\pi)^{\frac{1}{2}}$. The integral in (4.6) will be evaluated by contour integration. The poles of the integrand (4.9) are at the points

(4.10) $h =-\ell/2$, $\ell = -2, -1, 0, 1, 2, 3, \ldots$

The residue at these poles can be found by putting $h = t-\ell/2$ in the integrand (4.9) and taking the residue of the integrand at $t = 0$. Substituting $h = t-\ell/2$ in (4.9), we obtain

(4.11) $G(t-\ell/2)=(L_{VC})^{-t+\ell/2}\,D\prod_{i=1}^{2}(t-(\ell+n+i-1)/2)_{k_i}\,\Gamma(2t-\ell-2)/\Gamma(2t-\ell+k+j)$.

To evaluate the integral (4.6), we need to consider separately the cases (A) $\ell \geq 0$ and (B) $\ell < 0$.

CASE A: Let $c = k+j-\ell$. We consider two subcases (A1) $c \leq 0$ and (A2) $c > 0$.

Subcase A1: $\ell \geq 0$ and $c \leq 0$. In this case, the integrand (4.11), after expanding the gamma functions can be written as

(4.12) $G(t-\ell/2)=(L_{VC})^{-t+\ell/2}\,D\prod_{i=1}^{2}(t-(\ell+n+i-1)/2)_{k_i}\prod_{\delta=1}^{-c}(2t-\delta)/\prod_{i=1}^{\ell+2}(2t-i)$,

$\ell \geq k + j$.

The integrand (4.12) does not have any pole at $t = 0$. Therefore integral (4.6) will be 0 for $\ell \geq k+j$.

Subcase A2: $\ell \geq 0$ and $c > 0$. In this case after expanding the gamma functions (4.11) can be written as

$$(4.13)\quad G(t-\ell/2)=(L_{VC})^{-t+\ell/2}(D/2)\prod_{i=1}^{2}(t-(\ell+n+i-1)/2)_{k_i}\;\Gamma(2t+1)(-1)^{\ell}/(t\,\Gamma(2t+c)\prod_{i=1}^{\ell+2}(i-2t)),\qquad \ell = 0, 1, 2, \ldots, k+j-1.$$

The integral in (4.13) has a sample pole of first order at $t = 0$ and the residue at this point is given by

$$R_{\ell}=\lim_{t\to 0} t\cdot G_{J,\kappa}(t-\ell/2)$$

$$(4.14)\quad R_{\ell}= (-1)^{\ell}(D/2)(L_{VC})^{\ell/2}\prod_{i=1}^{2}(-(\ell+n+i-1)/2)_{k_i}/(\Gamma(k+j-\ell)(\ell+2)!),\qquad \ell = 0, 1, 2, \ldots, k+j-1.$$

CASE B. $\ell < 0$. Here $\ell = -2, -1$ and the integrands are

$$(4.15)\quad G(t+1)=(L_{VC})^{-t-1}(1/t)(D/2)\prod_{i=1}^{2}(t+1-(n+i-1)/2)_{k_i}\Gamma(2t+1)/\Gamma(2t+k+j+2),$$

and

$$(4.16)\quad G(t+\tfrac{1}{2})=(L_{VC})^{-t+\frac{1}{2}}(-1/t)(D/2)\prod_{i=1}^{2}(t+1-(n+i)/2)_{k_i}\Gamma(2t+1)/((1-2t)\Gamma(2t+k+j+1)).$$

Thus for $\ell = -1$ and $\ell = -2$, we have a simple pole of first order at $t = 0$ and the residue at these poles are given by

$$(4.17)\quad R_{-2}=(L_{VC})^{-1}(D/2)\prod_{i=1}^{2}((3-n-i)/2)_{k_i}/\Gamma(2+k+j)\text{ and}$$

$$(4.18)\quad R_{-1}=(L_{VC})^{-\frac{1}{2}}(D/2)(-1)\prod_{i=1}^{2}(1-(n+i)/2)_{k_i}/\Gamma(1+k+j).$$

Hence finally from (4.14), (4.17) and (4.18) and using Cauchy's residue theorem, the integral (4.6) for this case is given by

$$f_{J,\kappa}(L_{VC}) = R_{-1} + R_{-2} + \sum_{\ell=0}^{k+j-1} R_{\ell}$$

(4.19) $f_{J,\kappa}(L_{VC})=4\pi^{\frac{1}{2}}[(L_{VC})^{-1}\prod_{i=1}^{2}((3-n-i)/2)_{k_i}/\Gamma(2+k+j)+(L_{VC})^{-\frac{1}{2}}(-1)\cdot$

$$\prod_{i=1}^{2}(1-(n+i)/2)_{k_i}/\Gamma(1+k+j)+\sum_{\ell=0}^{k+j-1}(-1)^{\ell}(L_{VC})^{\ell/2}\prod_{i=1}^{2}(-(\ell+n+i-1)/2)_{k_i}/$$

$$((\ell+2)!\ \Gamma(k+j-\ell)).$$

Hence from (4.5) and using (4.19) the non-null density of L_{VC} for $p_2 = 2$ is given by

(4.20) $p(L_{VC})=C(p_2,n,\underset{\sim}{\Sigma})(L_{VC})^{n/2-1}\ p_2^{-np_2/2}\sum_{j=0}^{\infty}\sum_{J}\sum_{k=0}^{\infty}\sum_{\kappa}A(J,\kappa,p_2,\underset{\sim}{\Sigma})f_{J,\kappa}(L_{VC})$

where $f_{J,\kappa}(L_{VC})$ is as in (4.19).

This form of the density is useful for power computations and the power thus computed from (4.20) are given in table (6.2). The null density of L_{VC} from (4.20) reduces to that given in (3.31).

Now for finding the density of L_{VC} for $p_2 \geq 3$, we still use the method of contour integration but the density will involve psi functions and their derivatives. We will make use of the following lemma due to Nair[10] in this connection.

<u>Lemma 4.1</u>. Let (a_i) be a sequence of numbers, finite or infinite and let

(4.21) $F(x;t,a_2,a_3,\ldots.) = \exp(xt+a_2\ t^2/2!+a_3\ t^3/3!+\cdots.)$.

Then the n-th derivative of $F(x;t,a_2,a_3,\ldots.)$ at $t = 0$ is

(4.22) $$D_n(x;a)=\begin{vmatrix} x & -1 & 0 & 0 & 0 & \cdots & 0 \\ a_2 & x & -1 & 0 & 0 & \cdots & 0 \\ a_3 & 2a_2 & x & -1 & 0 & \cdots & 0 \\ a_4 & 3a_3 & 3a_2 & x & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n & \binom{n-1}{1}a_{n-1} & \binom{n-1}{2}a_{n-2} & \cdots & \cdots & \cdots & x \end{vmatrix}$$

Now we proceed to derive the densities of L_{VC} for the following two cases separately, namely (i) p_2 = even and (ii) p_2 = odd. We specify here that all the empty products in the following derivation will be interpreted as unity and all empty sums will be regarded as 0.

CASE (i): $p_2 = 2r,\ r \geq 1$. Now starting with (4.7) with $p_2 = 2r$ we have the integrand given by

(4.23) $G(h)=(L_1)^{-h} \prod_{i=1}^{2r} (h-(n+i-1)/2)_{k_i} \prod_{i=1}^{2r} \Gamma(h-i/2)/\Gamma(2rh+k+j).$

Using duplication formula of gamma function, (4.23) can be written as

(4.24) $G(h)=(L_1)^{-h}\ 2^{-2rh}\ D \prod_{i=1}^{2r} (h-(n+i-1)/2)_{k_i} \prod_{i=1}^{r} \Gamma(2h-2i)/\Gamma(p_2h+k+j),$

where $D = \pi^{r/2}\ 2^{r(r+2)}$, Let $L = L_1 2^{2r} = L_{VC}\ 2^{2r}\ p_2^{-p_2}$

(4.25) $G(h) = L^{-h}\ D \prod_{i=1}^{2r} (h-(n+i-1)/2)_{k_i} \prod_{i=1}^{r} \Gamma(2h-2i)/\Gamma(p_2h+k+j).$

The poles of the integrand G(h) are at the points

(4.26) $h = -\ell/2,\ \ell = -2r, -2r+1, \ldots, -2, -1, 0, 1, 2. \ldots\ , r \geq 1$ and the residue at these points is equal to the residue of $G(t-\ell/2)$ at $t = 0$. Now (4.25) can be written as

(4.27) $G(t-\ell/2) = L^{\ell/2}\ D \prod_{i=1}^{p_2} (t-(\ell+n+i-1)/2)_{k_i}\ L^{-k}\ GP(t)$, where

(4.28) $GP(t) = \prod_{i=1}^{r} (2t-\ell-2i)/\Gamma(p_2t+c),\quad \ell = -2r, -2r+1, \ldots, 0, 1, 2, \ldots$

and $c = k+j-r\ell$. Three cases arise: (A) $\ell \geq 0$, (B) $\ell < 0$, ℓ = even and (C) $\ell < 0$, ℓ = odd.

CASE A: $\ell \geq 0$. Two subcases: (A1) $c \leq 0$ and (A2) $c > 0$.

SUBCASE A1: $\ell \geq 0$ and $c \leq 0$ i.e., $k+j \leq r\ell$.

Expanding the gamma functions in (4.28), we have

(4.29) $GP(t) = (\Gamma(2t+1))^r \prod_{i=1}^{-c} (p_2t-i)\ t^{-(r-1)}\ p_2/(\Gamma(p_2t+1) \prod_{i=1}^{r} \prod_{\delta=1}^{\ell+2i} (2t-\delta)\ 2^r)$

Thus for $\ell \geq 0$, and $k+j \leq r\ell$, the pole of $G(t-\ell/2)$ is of order r-1, rewriting (4.29), we have

(4.30) $GP(t) = (-1)^{k+j} p_2\ t^{-(r-1)} (\Gamma(2t+1))^r (-c)! \prod_{i=1}^{-c} (1-p_2t/i)/(\Gamma(p_2t+1) \prod_{i=1}^{r} (\ell+2i)!$

$\prod_{i=1}^{r} \prod_{\delta=1}^{\ell+2i} (1-2t/\delta) 2^r).$

Hence from (4.27), we have

(4.31) $G(t-\ell/2)=L^{\ell/2}\ D\ p_2(-1)^{k+j}(-c)! \prod_{i=1}^{p_2} (-(\ell+n+i-1)/2)_{k_i} / (2^r \prod_{i=1}^{r} (\ell+2i))$

$t^{-(r-1)}(L)^{-t} \prod_{i=1}^{p_2} \prod_{\alpha=0}^{k_i-1} (1+t/(\alpha-(\ell+n+i-1)/2))(\Gamma(2t+1))^r \prod_{i=1}^{-c} (1-p_2t/i)/(\Gamma(p_2t+1)$

$\prod_{i=1}^{r} \prod_{\delta=1}^{\ell+2i} (1-2t/\delta)).$

This can be written as

(4.32) $G(t-\ell/2) = L^{\ell/2}\ D\ p_2\ a_0\ 2^{-r}\ t^{-(r-1)}\ \exp(\log A(t))$ where

(4.33) $a_0 = (-1)^{k+j} \prod_{i=1}^{p_2} (-(\ell+n+i-1)/2)_{k_i}\ (-c)!/ \prod_{i=1}^{r} (\ell+2i)$

(4.34) $A(t) = L^{-t} \prod_{i=1}^{p_2} \prod_{\delta=0}^{k_i-1} (1+t/(\alpha-(\ell+n+i-1)/2))(\Gamma(2t+1))^r \prod_{i=1}^{-c} (1-p_2t/i)/$

$(\Gamma(p_2t+1) \prod_{i=1}^{r} \prod_{\delta=1}^{\ell+2i} (1-2t/\delta)).$

Now the residue at the pole $t = 0$ of order $r - 1$ is given by

(4.35) $R_\ell=L^{\ell/2}\ D\ p_2\ a_0/(2^r\ (r\ \ell)\ \left(\frac{d}{dt}\right)^{r-2}_{t=0} \exp(\log A(t)).$

Using the formulae (see Erdelyi; [4])

(4.36) $\log \Gamma(x+a) = \log \Gamma(a) + x\psi(a) + x^2\ \psi(a)/2! + x^3\psi_2(a)/3! + \ldots$

(4.37) where $\psi(a)=\frac{d}{dx} \log \Gamma(x)\ _{x=a}$ and $\psi_j(a) = \left(\frac{d}{dx}\right)^j \psi(x)\ _{x=a}$ and

(4.38) $\log(1+z) = \sum_{n=0}^{\infty} (-1)^n z^{n+1}/(n+1)$ for $|z| < 1$,

Now log A(t) can be written as

(4.39) $\log A(t) = a_1 t + a_2 t^2/2! + a_3 t^3/3! + \ldots.$

where

(4.40) $$a_1 = -\log L + (2r-p_2)\psi(1) + \sum_{i=1}^{p_2} \sum_{\alpha=0}^{k_i-1} 1/(\alpha-(i+n+\ell-i)/2)$$

$$- \sum_{i=1}^{-c} (p_2/i) + \sum_{i=1}^{r} \sum_{\delta=1}^{\ell+2i} (2/\delta)$$ and for $s \geq 2$, we have

$$a_s = (r2^s-p_2^s)\psi_{s-1}(1)+(s-1)!\left[\sum_{i=1}^{p_2} \sum_{\alpha=0}^{k_i-1} (-1)^{s+1}/(\alpha-(i+n+\ell-1)/2)^s - \sum_{i=1}^{-c} (p_2/i)^s\right.$$

$$\left.+ \sum_{i=1}^{r} \sum_{\delta=1}^{\ell+2i} (2/\delta)^s\right].$$

using (4.39) in (4.35) and lemma (4.1), we get

(4.41) $R_\ell=L^{\ell/2}\ D\ p_2\ a_0/(2^r\Gamma(r-1))\ D_{r-2}(L;a)$, where

(4.42) $$D_{r-2}(L;a) = \begin{vmatrix} a_1 & -1 & 0 & \ldots & 0 \\ a_2 & a_1 & -1 & \ldots & 0 \\ a_3 & 2a_2 & a_1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & 0 \\ a_{r-2} & \binom{r-3}{1} a_{r-3} & \binom{r-4}{2} a_{r-4} & \ldots & a_1 \end{vmatrix}$$

where a's are defined in (4.40).

SUBCASE A2: $\ell \geq 0$ and $c > 0$ i.e., $k+j>r\ell$ Expanding the gamma function in (4.28), we get

$$(4.43)\quad GP(t) = (\Gamma(2t+1))^r(2t)^{-r}/(\Gamma(p_2t+c)\prod_{i=1}^{r}\prod_{\delta=1}^{\ell+2i}(2t-\delta)).$$

Thus for $c > 0$, we have a pole of order r at $t = 0$ and from (4.27) and (4.43), we have

$$(4.44)\quad G(t-\ell/2) = L^{\ell/2}\, D\, b_0'\, 2^{-r}\, t^{-r}\exp(\log F(t))$$

where

$$(4.45)\quad b_0' = (-1)^{r\ell}\prod_{i=1}^{p_2}(-(\ell+n+i-1)/2)_{k_i} / \prod_{i=1}^{r}(\ell+2i)! \text{ and}$$

$$(4.46)\quad F(t)=L^{-t}\prod_{i=1}^{p_2}\prod_{\alpha=0}^{k_i-1}(1+t/(\alpha-(\ell+n+i-1)/2))(\Gamma(2t+1))^r/[\Gamma(p_2t+c)\prod_{i=1}^{r}\prod_{\delta=1}^{\ell+2i}(1-2t/\delta)]$$

The residue at the pole $t = 0$ is given by

$$(4.47)\quad R_\ell=[L^{\ell/2}\, D\, b_0'\, 2^{-r}/\Gamma(r)]\left(\frac{d}{dt}\right)^{r-1}_{t=0}\exp(\log F(t)).$$

Now using (4.36), (4.37) and (4.38), $\log F(t)$ can be written as

$$(4.48)\quad \log F(t) = b_0'' + b_1t + b_2t^2/2! + b_3t^3/3! + \ldots.$$

Further, using (4.48) in (4.47) and lemma (4.1), we obtain

$$(4.49)\quad R_\ell= L^{\ell/2}\, D\, 2^{-r}\, b_0/\Gamma(r)\, D_{r-1}(L;b) \quad \text{for } \ell \geq 0 \quad \text{s.t. } r\ell<k+j,$$

where

$$(4.50)\quad b_0=b_0'b_0'', \quad b_0''= -\log(c) \text{ and } b_0' \text{ is given in (4.45)}$$

$$b_1=-\log L+\sum_{i=1}^{p_2}\sum_{\alpha=1}^{k_i-1}1/(\alpha-(i+n+\ell-1)/2)+2r\psi(1)-p_2\psi(c)+\sum_{i=1}^{r}\prod_{\delta=1}^{\ell+2i}(2/\delta)$$

and for $s \geq 2$, we have

$$b_s=r\,2^s\psi_{s-1}(1)-p_2^s\psi_{s-1}(c)+(s-1)!\,[\sum_{i=1}^{r}\sum_{\delta=1}^{\ell+2i}(2/\delta)^s+\sum_{i=1}^{p_2}\sum_{\alpha=0}^{k_i-1}(-1)^{s+1}/$$

$$(\alpha-(i-1+n+\ell)/2)^s]$$

and the determinant $D_{r-1}(L;b)$ is equal to the determinant on the right hand side of (4.22) with x replaced by b_1', n by r-1 and a_s's by $b_s'^{s}$; s = 1, 2, ..., r-1

CASE B: $\ell < 0$ and $\ell = -2u$, u = 1, 2, ... r, with $p_2 = 2r$. For this case we can write (4.28) as

$$(4.51)\quad GP(t) = \Gamma(2t+2u-2i)/\Gamma(p_2t+c), \quad u = 1, 2, \ldots, r \text{ and } c = k+j-r\ell > 0$$

expanding the gamma function in (4.51), we obtain

$$(4.52)\quad GP(t)=(\Gamma(2t+1))^{r-u+1} \prod_{i=1}^{u-1} \Gamma(2t+2u-2i)\ (2t)^{-(r-u+1)}/[\Gamma(p_2t+c) \prod_{i=u+1}^{r} \prod_{\delta=1}^{2i-2u} (2t-\delta)]$$

(All empty products are treated as 1 and empty sums as 0) It is clear from (4.52) that we have a pole of order r-u+1 at t = 0, u = 1, 2, ..., r. It is easy to check that $G(t-\ell/2)$ can be written as

$$(4.53)\quad G(t+u) = L^{-u} D C_0' (2t)^{-(r-u+1)} \exp(\log H(t))$$

where after using (4.36), (4.37) and (4.38) log H(t) can be written as

$$(4.54)\quad \log H(t) = C_0'' + C_1 t + C_2 t^2/2! + C_3 t^3/3! + \ldots \quad .$$

Now using (4.54) in (4.53) and appealing to the lemma (4.1), we get the residue as

$$(4.55)\quad R_u = L^{-u} D C_0 2^{-(r-u+1)}/\Gamma(r-u+1) D_{r-u}(L;C);\ u=1,2,\ldots,r \qquad r \geq 1,$$

where the determinant $D_{r-u}(L;C)$ can be obtained from the right hand side of (4.22) with x replaced by C_1, n by r-u and a_s's by C_s's , s = 1, 2, ... r-u. The coefficients C_s's are given by

$$(4.56)\quad C_0' = \prod_{i=1}^{p_2} (u-(n+i-1)/2)_{k_i} / \prod_{i=u+1}^{r} (2i-2u)!,$$

$$C_0'' = \prod_{i=1}^{u-1} (2u-2i)/\Gamma(c), \qquad C_0 = C_0' C_0''$$

$$C_1=-\log L+2(r-u+1)\psi(1)-p_2\psi(c)+2\sum_{i=1}^{u-1}\psi(2u-2i)+\sum_{i=u+1}^{r}\sum_{\delta=1}^{2i-2u}(2/\delta)$$

$$+\sum_{i=1}^{p_2}\sum_{\alpha=0}^{k_i-1}1/(\alpha-(i-1+n)/2+u),$$

and for $s \geq 2$

$$C_s=(r-u+1)2^s\psi_{s-1}(1)-p_2^s\psi_{s-1}(c)+\sum_{i=1}^{u-1}2^s\psi_{s-1}(2u-2i)+(s-1)!\,[$$

$$\sum_{i=u+1}^{r}\sum_{\delta=1}^{2i-2u}(2/\delta)^s+\sum_{i=1}^{p_2}\sum_{\alpha=0}^{k_i-1}(-1)^{s+1}/(\alpha-(i-1+n)/2+u)^s].$$

CASE C: $\ell < 0$ and $\ell = -2\nu+1$, $\nu = 1, 2, \ldots, r.$ Now (4.28) can be written as

$$(4.57)\quad GP(t)=\prod_{i=1}^{r}\Gamma(2t+2\nu-1-2i)/\Gamma(p_2t+c),\qquad \nu = 1, 2, \ldots, r.$$

After the expansion of gamma functions, one obtains

$$(4.58)\quad GP(t)=(\Gamma(2t+1))^{r-\nu+1}(2t)^{-(r-\nu+1)}\prod_{i=1}^{\nu-1}\Gamma(2t+2u-2i-1)/(\Gamma(p_2t+c)\prod_{i=\nu}^{r}\sum_{\delta=1}^{1+2i-2\nu}(2t-\delta)).$$

Thus, here we have a pole of order $r-\nu+1$ at $t = 0$, $\nu = 1, 2, \ldots, r.$ Proceeding as before, we have $G(t-\ell/2)$ in the form

$$(4.59)\quad G(t+\nu-\tfrac{1}{2})=(L)^{-\nu+\frac{1}{2}}\,D\,d_0'\,(2t)^{-(r-\nu+1)}\exp(\log I(t)),\text{ where}$$

$$(4.60)\quad \log I(t)=d_0'' + d_1t + d_2t^2/2! + d_3t^3/3! + \ldots \quad .$$

Now using (4.60) in (4.59) and applying lemma (4.1), we have the residue R_ν given by

$$(4.61)\quad R_\nu=(L)^{-\nu+\frac{1}{2}}\,D\,d_0\,2^{-(r-\nu+1)}D_{r-\nu}(L;d)/\Gamma(r-\nu+1),\qquad \nu = 1, 2, \ldots, r,$$

where $D_{r-\nu}(L;d)$ is equal to the determinant on the right hand side of (4.22) with x replaced by d_1, n by $r-\nu$ and a_q's by d_q's, $q = 1, 2, \ldots, r-\nu$. The coefficients d_q's are given by

$$d_0' = (-1)^{r-\nu+1} \prod_{i=1}^{p_2} (\nu-(n+i)/2)_{k_i} \Big/ \prod_{i=\nu}^{r} (1+2i-2\nu)!,$$

$$d_0'' = \prod_{i=1}^{\nu-1} \Gamma(2\nu-2i-1)/\Gamma(c) \quad \text{and } d_0 = d_0' d_0''$$

$$d_1 = -\log L + \sum_{i=1}^{p_2} \sum_{\alpha=0}^{k_i-1} 1/(\alpha-(\ell+n+i-1)/2) + 2 \sum_{i=1}^{\nu-1} \psi(2\nu-2i-1) - p_2\psi(c)$$

$$+2(r-\nu+1)\psi(1) + \sum_{i=\nu}^{r} \sum_{\delta=1}^{1+2i-2\nu} (2/\delta)$$

and for $s \geq 2$, we have

$$d_s = \sum_{i=1}^{\nu-1} 2^s \psi_{s-1}(2\nu-2i-1) - p_2^s\, \psi_{s-1}(c) + (r-\nu+1)2^s\psi_{s-1}(1) +$$

$$(s-1)!\ \Big[\sum_{i=\nu}^{r} \sum_{\delta=1}^{1+2i-2\nu} (2/\delta)^s + \sum_{i=1}^{p_2} \sum_{\alpha=0}^{k_i-1} (-1)^{s+1}/(\alpha+\nu-(n+i)/2)^s\Big].$$

Hence, for the case p_2=even, we have from (4.5) and Cauchy's residue theorem, the non-null density of L_{VC} in the form

$$(4.62)\quad p(L_{VC}) = C(p_2, n, \Sigma) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} A(J,\kappa,p_2,n,\Sigma)(L_{VC})^{n/2-1} p_2^{-np_2/2}$$

$$\left[\sum_{\substack{\ell\geq 0 \\ r\ell\geq k+j}} R_\ell + \sum_{\substack{\ell\geq 0 \\ r\ell< k+j}} R_\ell + \sum_{u=1}^{r} R_u + \sum_{\nu=1}^{r} R_\nu \right]$$

where R_ℓ, R_u, R_ν are given in (4.41), (4.49), (4.55) and (4.61) respectively. In particular, if we put p_2=2 in (4.62), we get (4.20).

CASE(ii): $p_2=2s+1$, $s \geq 0$ (s=0, covers the case p_2=1). Once again in the following discussion, all empty products will be interpreted as unity and empty sums as 0. The functions $f_{J,\kappa}$, $G_{J,\kappa}$, $GP_{J,\kappa}$, $R_{J,\kappa}$ will be written as f, G, GP and R respectively. Now starting with (4.7) and using the duplication formula for gamma functions, we have

(4.63) $G(h)=L^{-h}\, D \prod_{i=1}^{p_2} (h-(n+i-1)/2)_{k_i} \prod_{i=1}^{s} \Gamma(2h-2i)\Gamma(h-s-\tfrac{1}{2})/\Gamma(p_2h+k+j)$,

(4.64) where $D = \pi^{s/2} \quad 2^{s(s+2)}$. and $L = L_{VC}\ 2^{2s}/p_2^{p_2h}$.

The poles of the integrand G(h) are at the points

(4.65) $h=-\ell/2,\ \ell = -2s-1, -2s, \ldots, 0, 1, 2, \ldots$ and the residue of

G(h) at these points can be obtained by finding the residue of $G(t-\ell/2)$ at $t = 0$. Now

(4.66) $G(t-\ell/2)=L^{\ell/2} D \prod_{i=1}^{p_2} (t-(n+\ell+i-1)/2)_{k_i} L^{-t} GP(t)$, where

(4.67) $GP(t) = \prod_{i=1}^{s} \Gamma(2t-(\ell+2i))\Gamma(t-s-(\ell+1)/2)/\Gamma(p_2+k+j-p_2\ell/2)$.

We have to consider separately the cases (A) $\ell \geq 0$, ℓ = even, (B) $\ell \geq 0$, ℓ = odd, (C) $\ell < 0$, ℓ = even and (D) $\ell < 0$, ℓ = odd. Let $d=k+j-p_2\ell/2$. Now (4.66) can be written as

(4.68) $G(t-\ell/2)=L^{\ell/2} D \prod_{i=1}^{p_2} (-(n+\ell+i-1)/2) \prod_{i=1}^{p_2} \prod_{\alpha=0}^{k_i-1} (1+t/(\alpha-(n+\ell+i-1)/2))L^{-t}GP(t)$.

<u>CASE A</u>: Two subcases arise (A1) $d \leq 0$ and (A2) $d > 0$.

<u>SUBCASE A1: $\ell \geq 0$, $\ell = 2u_2$, $u_2 = 0, 1, 2, \ldots$, $d \leq 0$.</u> After expanding the gamma functions in (4.67), we have

(4.69) $GP(t)=(\Gamma(2t+1))^{s}\Gamma(t+\tfrac{1}{2})p_2\ 2^{-s}\ t^{-(s-1)} \prod_{\delta=1}^{-d} (p_2t-\delta)/(\Gamma(p_2t+1) \prod_{\delta=0}^{u_2+s} (t-s-\tfrac{1}{2}))$.

So we have a pole of order (s-1) at t = 0. Proceeding as before, we have

(4.70) $G(t-u_2)=L^{u_2} D\ p_2\ 2^{-s}\ t^{-(s-1)}\ f_0' \exp(\log P(t))$ and the residue R_{u_2} is given by

(4.71) $R_{u_2}=L^{u_2} D\ p_2\ 2^{-s}\ f_0 D_{s-2}(L;f)/\Gamma(s-1)$, $\quad p_2u_2 \geq k+j, u_2 = 0,1,2,\ldots,$

where the determinant $D_{s-2}(L;f)$ is the same as the one in (4.22) with n replaced by s-2, x by f and a_q's by f_q's, q = 1, 2, ..., s-2 and the coefficients f_q's are given by

$$(4.72)\quad f_0'=(-1)^{k+j+s+1}(-d)!\prod_{i=1}^{p_2}(-(n+\ell+i-1)/2)_{k_i}/\Big(\prod_{i=1}^{s}(\ell+2i)!\prod_{\delta=1}^{s+u_2}(\delta+\tfrac{1}{2})\Big)$$

and $f_0 = f_0'\,\Gamma(\tfrac{1}{2})$

$$f_1=-\log L+\psi(\tfrac{1}{2})+\sum_{i=1}^{p_2}\sum_{\alpha=0}^{k_i-1}(1/(\alpha-(i+n+\ell-1)/2))-\psi(1)-\sum_{\delta=1}^{-d}(p_2/\delta)$$

$$+\sum_{i=1}^{s}\sum_{\delta=1}^{\ell+2i}(2/\delta)+\sum_{\delta=0}^{u_2+s}1/(\delta+\tfrac{1}{2})$$

and for $q \geq 2$, we have

$$f_q=\psi_{q-1}(\tfrac{1}{2})+\psi_{q-1}(1)[s2^q-p_2^q]+(q-1)!\Bigg[\sum_{i=1}^{p_2}\sum_{\alpha=0}^{k_i-1}(-1)^{q+1}/(\alpha-(i+n+\ell-1)/2)^q$$

$$-\sum_{\delta=1}^{-d}(p_2/\delta)^q+\sum_{i=1}^{s}\sum_{\delta=1}^{\ell+2i}(2/\delta)^q+\sum_{\delta=0}^{u_2+s}(1/(\delta+\tfrac{1}{2}))^q\Bigg].$$

SUBCASE A2: $\ell > 0$, $\ell=2u_2$, $d > 0$, $u_2 = 0, 1, 2, \ldots$

Expanding the gamma product in (4.67), we have

$$(4.73)\quad GP(t)=(\Gamma(2t+1))^s\Gamma(t+\tfrac{1}{2})(2t)^{-s}/\Big(\Gamma(p_2t+d)\prod_{i=1}^{s}\prod_{\delta=1}^{\ell+2i}(2t-\delta)\prod_{\delta=0}^{s+u_2}(t-\delta-\tfrac{1}{2})\Big).$$

In this case, we have a pole of order s at t = 0. Following the same procedures as earlier, we have

$$(4.74)\quad G(t-u_2)=L^{u_2}D\,g_0'\exp(\log Q(t))/(2t)^s,\quad \text{where}$$

$$\log Q(t)=g_0''+g_1t+g_2t^2/2!+\ldots.$$ and the residue

R_{u_2} is given by

$$(4.75)\quad R_{u_2} = L^{u_2}\,D2^{-s}g_0D_{s-1}(L;g),\quad u_2 = 0, 1, 2, \ldots,\quad p_2u_2<k+j \text{ where}$$

the determinant $D_{s-1}(L;g)$ is similar to the determinant on the right hand side of

(4.22) having (s-1) rows and the elements a_q's replaced by g_q's and x by g_1, where g_q's are

$$(4.76)\quad g_0' = (-1)^{u_2+s+1} \prod_{i=1}^{p_2} (-(n+\ell+i-1)/2)_{k_i} / [\prod_{\delta=1}^{s} (\ell+2i)! \prod_{\delta=0}^{u_2+s} (\delta+\tfrac{1}{2})],$$

$$g_0'' = \Gamma(\tfrac{1}{2})/\Gamma(d) \quad \text{and } g_0 = g_0' \, g_0'',$$

$$g_1 = -\log L + 2s\psi(1) + \psi(\tfrac{1}{2}) - p_2\psi(d) + \sum_{i=1}^{p_2} \sum_{\alpha=0}^{k_i-1} 1/(\alpha-(i+n+\ell-1)/2) + \sum_{i=1}^{s} \sum_{\delta=1}^{\ell+2i} (2/\delta) + \sum_{\delta=1}^{u_2+s} (\delta+\tfrac{1}{2})^{-1}$$

and for $q \geq 2$, we have

$$g_q = (q-1)! \left[\sum_{i=1}^{p_2} \sum_{\alpha=0}^{k_i-1} (-1)^{q+1}/(\alpha-(i+n+\ell-1)/2)^q + \sum_{i=1}^{s} \sum_{\delta=1}^{\ell+2i} (2/\delta)^q + \sum_{\delta=0}^{u_2+s} (\delta+\tfrac{1}{2})^{-q} \right]$$

$$+\psi_{q-1}(\tfrac{1}{2}) - p_2^q \psi_{q-1}(d) + s\, 2^q \, \psi_{q-1}(1).$$

<u>CASE B</u>: <u>$\ell \geq 0$, $\ell = 2\nu_2+1$, $\nu_2 \geq 0$.</u> The gamma product in (4.67) can be written as

$$(4.77)\quad GP(t) = \prod_{i=1}^{s} \Gamma(2t-(\ell+2i))\, \Gamma(t-\nu_2-s-1)/\Gamma(p_2t+d-\tfrac{1}{2}), \quad \text{where}$$

$$(4.78)\qquad d = k+j-s-p_2\nu_2.$$

Two subcases arise (B1) $d \leq 0$, (B2) $d > 0$.

<u>SUBCASE B1</u>: <u>$\ell \geq 0$, $= 2\nu_2+1$, $\nu_2 \geq 0$ s.t. $p_2\nu_2 \geq k+j-s$.</u> Now (4.77) can be written as

$$(4.79)\quad GP(t) = (\Gamma(2t+1))^s \Gamma(t+1) 2^{-s} t^{-(s+1)} \prod_{\delta=0}^{-d} (p_2t-\delta-\tfrac{1}{2})/(\Gamma(p_2t+\tfrac{1}{2}) \prod_{\delta=1}^{\nu_2+s+1} (t-\delta) \prod_{i=1}^{s} \prod_{\delta=1}^{\ell+2i} (2t-\delta)).$$

So we have a pole of order s+1 at t = 0. As before, we have

(4.80) $G(t-\nu_2-\frac{1}{2})=L^{\nu_2+\frac{1}{2}}\ D\ m_0'\ 2^{-s}\ \exp(\log R(t))/t^{s+1}$ and using lemma (4.1), the residue R_{ν_2} is given by

(4.81) $R_{\nu_2}=D(L)^{\nu_2+\frac{1}{2}}\ m_0\ 2^{-s}\ D_s(L;m)/\Gamma(s+1),\ \nu_2\geq 0,\ \text{s.t. } p_2\nu_2\geq k+j-s$

with $D_s(L;m)$ being the determinant of order s and can be obtained from (4.22) by replacing x by m_1 and a_q's by m_q's, where m_q's are given by

(4.82) $m_0'=(-1)^{k+j-s}\prod_{i=1}^{p_2}((-n-\ell-i+1)/2)_{k_i}\ \prod_{\alpha=0}^{d}(\alpha+\frac{1}{2})/((\nu_2+s+1)!\prod_{i=1}^{s}(\ell+2i)!)$

$m_0=m_0'/\Gamma(\frac{1}{2})$ and

$$m_1=-\log L+p_2\psi(1)-p_2\psi(\tfrac{1}{2})+\sum_{i=1}^{p_2}\sum_{\alpha=0}^{k_i-1}1/(\alpha-(i-1+n+\ell)/2)-\sum_{\alpha=0}^{-d}p_2/(\alpha+\tfrac{1}{2})$$
$$+\sum_{\delta=1}^{\nu_2+s+1}(1/\delta)+\sum_{i=1}^{s}\sum_{\delta=1}^{\ell+2i}(2/\delta)$$

and for $q\geq 2$, we have

$$m_q=\psi_{q-1}(1)[1+s2^q]-p_2^q\psi_{q-1}(\tfrac{1}{2})+(q-1)!\left[\sum_{i=1}^{p_2}\sum_{\alpha=0}^{k_i-1}(-1)^{q+1}/(\alpha-(i+n+\ell-1)/2)^q\right.$$
$$\left.-\sum_{\alpha=0}^{-d}(p_2/(\alpha+\tfrac{1}{2}))^q+\sum_{i=1}^{s}\sum_{\delta=1}^{\ell+2i}(2/\delta)^q+\sum_{\delta=1}^{\nu_2+s+1}(1/\delta)^q\right].$$

<u>SUBCASE B2</u>: <u>$\ell\geq 0,\ \ell=2\nu_2+1,\ \nu_2\geq 0,\ p_2\nu_2<k+j-s$.</u> Now (4.77) can be written as

(4.83) $GP(t)=\Gamma(t+1)(\Gamma(2t+1))^s2^{-s}/[\Gamma(p_2t+d-\frac{1}{2})\prod_{i=1}^{s}\prod_{\delta=1}^{\ell+2i}(2t-\delta)\prod_{\delta=1}^{\nu_2+s+1}(t-\delta)t^{s+1}]$.

Here also we have a pole of order s+1 at t = 0, and an earlier using lemma (4.1), we have

(4.84) $G(t-\nu_2-\frac{1}{2})=L^{\nu_2+\frac{1}{2}}\ D\ n_0'\ 2^{-s}\ t^{-(s+1)}\ \exp(\log S(t))$ and

(4.85) $R_{\nu_2}=D(L)^{\nu_2+\frac{1}{2}}\ n_0\ D_s(L;n)/(2^s\Gamma(s+1)),\ \nu_2\geq 0$ s.t. $p_2\nu_2<k+j-s$

where the determinant $D_s(\mathbb{L};n)$ is defined similarly as in (4.81) with m's replaced by n's and the coefficients n's are defined as

$$(4.86)\quad n_0' = (-1)^{-\nu_2+1} \prod_{i=1}^{p_2} (-(n+\ell+i-1)/2)_{k_i} / [\prod_{i=1}^{s} (\ell+2i)!(\nu_2+s+1)!],$$

$$n_0 = n_0'/\Gamma(d-\tfrac{1}{2}) \quad \text{and}$$

$$n_1 = -\log L + p_2\psi(1) - p_2\psi(d-\tfrac{1}{2}) + \sum_{i=1}^{p_2} \sum_{\alpha=0}^{k_i-1} 1/(\alpha-(i-1+n+\ell)/2)$$

$$+ \sum_{i=1}^{s} \sum_{\delta=1}^{\ell+2i} (2/\delta) + \sum_{\delta=1}^{\nu_2+s+1} (1/\delta)$$

and for $q \geq 2$

$$n_q = \psi_{q-1}(1)[1+s2^q] - p_2^q\psi_{q-1}(d-\tfrac{1}{2}) + (q-1)!\left[\sum_{i=1}^{r} \sum_{\delta=1}^{\ell+2i} (2/\delta)^q + \sum_{\delta=1}^{\nu_2+s+1} (1/\delta)^q + \sum_{i=1}^{p_2} \sum_{\alpha=0}^{k_i-1} (-1)^{q+1}/(\alpha-(i-1+n+\ell)/2)^q\right].$$

CASE C. $\ell < 0,\ \ell = -2u,\ u = 1, 2, 3, \ldots, s$. For this case the gamma product in (4.67) can be expanded as

$$(4.87)\quad GP(t) = \frac{\prod_{i=1}^{u-1} \Gamma(2t+2u-2i)(\Gamma(2t+1))^{s-u+1}\Gamma(t+\frac{1}{2})}{(2t)^{s-u+1}\Gamma(p_2t+p_2u+k+j) \prod_{\alpha=0}^{s-u} (t-\alpha-\frac{1}{2}) \prod_{i=u+1}^{s} \prod_{\delta=1}^{2u-2i} (2t-\delta)}$$

We have a pole of order s-u+1, u = 1, 2, ..., s. Proceeding as before, we have

$$(4.88)\quad G(t+u) = L^{-u}Dy_0' \exp(\log V(t))/(2t)^{s-u+1}$$

and the residue R_u is given by

$$(4.89)\quad R_u = L^{-u}Dy_0D_{s-u}(L;y)/(2^{s-u+1}\Gamma(s-u+1)),\ u = 1, 2, \ldots, s$$

where the determinant $D_{s-u}(L;y)$ is equal to the R.H.S. of (4.22) with s-u rows and x replaced by y_1 and a_q's by y_q's, q = 1, 2, ..., s-u, and the coefficients y_q's are given by

$$(4.90)\quad y_0' = \prod_{i=1}^{p_2} (-(n+\ell+i-1)/2)_{k_i} (-1)^{s-u+1} / \left(\prod_{\alpha=0}^{s-u} (\alpha+\tfrac{1}{2}) \prod_{i=u+1}^{s} (2u-2i)! \right)$$

$$y_0 = y_0' \prod_{i=1}^{u-1} \Gamma(2u-2i)\Gamma(\tfrac{1}{2})/\Gamma(p_2u+k+j),$$

$$y_1 = -\log L + 2\sum_{i=1}^{u-1} \psi(2u-2i) + 2(s-u+1)\psi(1) + \psi(\tfrac{1}{2}) - p_2\psi(p_2u+k+j) +$$

$$\sum_{\alpha=0}^{s-u} (\alpha+\tfrac{1}{2})^{-1} + \sum_{i=u+1}^{s} \sum_{\delta=1}^{2u-2i} (2/\delta) + \sum_{i=1}^{p_2} \sum_{\alpha=0}^{k_i-1} 1/(\alpha-(i+n+\ell-1)/2)$$

and for q ≥ 2, we have

$$y_q = \sum_{i=1}^{u-1} 2^q\, \psi_{q-1}(2u-2i) + (s-u+1)\; 2^q\psi_{q-1}(1) + \psi_{q-1}(\tfrac{1}{2}) - p_2^q\psi_{q-1}(p_2u+k+j)$$

$$+(q-1)!\left[\sum_{\alpha=0}^{s-u} (\alpha+\tfrac{1}{2})^{-q} + \sum_{i=u+1}^{s} \sum_{\delta=1}^{2u-2i} (2/\delta)^q + \sum_{i=1}^{p_2} \sum_{\alpha=0}^{k_i-1} (-1)^{q+1}/(\alpha-(n+\ell+i-1)/2)^q\right].$$

CASE D: $\ell < 0,\ \ell = -2\nu+1,\ \nu = 1, 2, \ldots, s, s+1$. The gamma product in (4.67) can be written as

$$(4.91)\quad GP(t) = \frac{2\,\Gamma(t+1)(\Gamma(2t+1))^{s-\nu+1} \prod_{i=1}^{\nu-1} \Gamma(2t+2\nu-2i-1)}{(2t)^{s-\nu+2}\Gamma(p_2t+k+j+p_2(\nu-\tfrac{1}{2})) \prod_{\delta=1}^{s+1-\nu} (t-\delta) \prod_{i=\nu}^{s} \prod_{\delta=1}^{1+2i-2\nu} (2t-\delta)}$$

So here, we have a pole of order $s-\nu+2$ at $t = 0$, $\nu = 1, 2, \ldots, s+1$, and as earlier, we have

(4.92) $G(t+\nu-\tfrac{1}{2}) = 2L^{-\nu+\frac{1}{2}} D\, Z_0'\, \exp(W(t))/(2t)^{s-\nu+2}$ and using lemma (4.1), the residue R_ν is given by

(4.93) $R_\nu = D\, L^{-\nu+\frac{1}{2}}\, Z_0\, D_{s-\nu+1}(L;Z)\,(2^{s-\nu+1}\Gamma(s-\nu+2))$. $\nu = 1, 2, \ldots, s+1$

where the determinant $D_{s-\nu+1}(L;Z)$ can be obtained from (4.22) by replacing n by $s-\nu+1$, x by Z_1 and a_q's by Z_q's where the Z_q's are given by

$$(4.94)\quad Z_0' = \prod_{i=1}^{p_2} (-(\ell+n+i-1)/2)_{k_i} / ((s-\nu+1)! \prod_{i=\nu}^{s} (1+2i-2\nu)!),$$

$$Z_0 = Z_0' \prod_{i=1}^{\nu-1} \Gamma(2\nu-2i-1)/\Gamma(k+j-p_2\ell/2)$$

$$Z_1 = -\log L + (2(s-\nu+1)+1)\psi(1) + 2\sum_{i=1}^{\nu-1} \psi(2\nu-2i-1) - p_2\psi(k+j-p_2\ell/2)$$

$$+ \sum_{i=1}^{p_2} \sum_{\alpha=0}^{k_i-1} 1/(\alpha-(i-1+n+\ell/2)) + \sum_{\delta=1}^{s+\nu-1} 1/\delta + \sum_{i=\nu}^{s} \sum_{\delta=1}^{1+2i-2\nu} (2/\delta)$$

and for $q \geq 2$, we have

$$Z_q = [1+2^q(s-\nu+1)]\psi_{q-1}(1) + 2^q \sum_{i=1}^{\nu-1} \psi_{q-1}(2\nu-2i-1) - p_2^q \psi_{q-1}(k+j-p_2\ell/2)$$

$$+(q-1)! \left[\sum_{i=1}^{p_2} \sum_{\alpha=0}^{k_i-1} (-1)^{q+1}/(\alpha-(i+n+\ell-1)/2)^q + \sum_{\delta=1}^{s+\nu-1} (1/\delta)^q + \sum_{i=\nu}^{s} \sum_{\delta=1}^{1+2i-2\nu} (2/\delta)^q \right].$$

Hence, when p is odd, the density of L_{VC} is given by

$$(4.95)\quad p(L_{VC}) = C(p_2,n,\Sigma) \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{j=0}^{\infty} \sum_{J} A(J,\kappa, p_2,n,\Sigma)(L_{VC})^{n/2-1}\, p_2^{-np_2/2}$$

$$\left[\sum_{\substack{u_2=0 \\ p_2u_2 \geq k+j}}^{\infty} R_{u_2} + \sum_{\substack{u_2=0 \\ p_2u_2 < k+j}}^{\infty} R_{u_2} + \sum_{\substack{\nu_2=0 \\ p_2\nu_2 \geq k+j-s}}^{\infty} R_{\nu_2} + \sum_{\substack{\nu_2=0 \\ p_2\nu_2 < k+j-s}}^{\infty} R_{\nu_2} + \right.$$

$$\left. \sum_{u=1}^{s} R_u + \sum_{\nu=1}^{s+1} R_\nu \right]$$

where R_{u_2}, R_{ν_2}, R_u, R_ν are given in (4.71), (4.75), (4.81), (4.85) and (4.86) and (4.93) respectively.

Remark. Putting $\Sigma = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 I_{p_2} \end{bmatrix}$ in (4.62) and (4.95), we can deduce the results of Nagarsenker [12] and Wilks [17].

5. Distribution of L_{VC} as a chisquare series: In this section we express the density of L_{VC} as a chisquare series using methods similar to those of BOX [2]. Let $L=(L_{VC})^{n/2}$ and $\lambda^* = -2\rho \log L$ where ρ is chosen so that the rate of convergence of the resulting series can be controlled, $0 \le \rho \le 1$. Let $\phi(t)$ be the characteristic function of λ^*, then

(5.1) $\phi(t) = E(L_{VC})^{-it\rho n}$

In section 3, we obtained the non-null moments $E[L_{VC}]^h$ for integral values of h. But the result (3.20) can be extended to any complex number h by analytic continuation. So we have for any complex number h

(5.2) $$E[L_{VC}]^h = C(p_2,n,\Sigma) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} A(J,\kappa,p_2,n,\Sigma) p_2^{p_2 h} \prod_{i=1}^{p_2} \Gamma((n-i)/2+h) \prod_{i=1}^{p_2} (h-(i-1)/2)_{k_i} / \Gamma(p_2(h+n/2)+k+j)$$

Now using (5.2) in (5.1), we obtain

(5.3) $$\phi(t) = C(p_2,n,\Sigma) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} A(J,\kappa,p_2,n,\Sigma) p_2^{-it\rho n p_2} \prod_{\delta=1}^{p_2} \Gamma((n(1-2it\rho)-\delta)/2) \prod_{\delta=1}^{p_2} ((1-\delta-2it\rho n)/2)_{k_\delta} / \Gamma(np_2(1-2it\rho)/2+k+j).$$

For t = 0, we have $\phi(t)=1$ using $\Sigma_{22}^{-1} = \Sigma_{2\cdot 1}^{-1} - \Sigma_{1\cdot 2}^{-1} \beta'\beta$ and for $t \neq 0$ (5.3) can be written as

(5.4) $$\phi(t) = C(p_2,n,\Sigma) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} A(J,\kappa,p_2,n,\Sigma) \exp(\log G(t)),$$

where $G_{J,\kappa}(t)$ is denoted by G(t) and is given by

$$(5.5)\ G(t) = \frac{p_2^{-it\rho np_2} \prod_{\delta=1}^{p_2} \Gamma((n(1-2it\rho)-\delta)/2) \prod_{\delta=1}^{p_2} \Gamma((n(1-2it\rho)+1-\delta-n)/2+k_\delta)}{\Gamma(np_2(1-2it\rho)/2+k+j) \prod_{\delta=1}^{p_2} \Gamma((n(1-2it\rho)+1-\delta-n)/2)}.$$

In the following derivation, function G, W, w, R, all depend upon J and κ, for simplicity of notation the subscripts or the superscripts J,κ will not be explicitly given unless necessary. From (5.5) taking logarithm on both sides, we get

$$(5.6)\ \log G(t) = -it\rho np_2 \log p_2 + \sum_{\delta=1}^{p_2} \log \Gamma((n(1-2it\rho)-\delta)/2)$$

$$- \log \Gamma(np_2(1-2it\rho)/2+k+j) + \sum_{\delta=1}^{p_2} \log \Gamma((n(1-2it\rho)+1-\delta-n)/2+k_\delta)$$

$$- \sum_{\delta=1}^{p_2} \log \Gamma((n(1-2it\rho)+1-\delta-n)/2).$$

we now need the following expansion for gamma function (see Anderson [1]).

$$(5.7)\ \log \Gamma(x+h) = \log\sqrt{2\pi} + (x+h-\tfrac{1}{2})\log x - x - \sum_{r=1}^{m} (-1)^r B_{r+1}(h)/(r(r+1)x^r) + R_{m+1}(x).$$

where R_{m+1} is the remainder such that $|R_{m+1}(x)| \le A/|x^{m+1}|$. A is a constant independent of x and $B_r(h)$ is the Bernoulli polynomial of degree r and order unity defined by

$$\frac{\tau e^{h\tau}}{e^\tau - 1} = \sum_{r=0}^{\infty} \tau^r B_r(h)/r!$$

where the polynomials are given by

$$B_0(h) = 1,\ B_1(h) = h-\tfrac{1}{2},\ B_2(h) = h^2-h+1/6,\ B_3(h) = h^3-3h^2/2+h/2$$

and in general we have

$$B_r(h) = \sum_{\ell=0}^{r} \binom{r}{\ell} B_\ell\, h^{r-\ell},$$ where B_ℓ are the Bernoulli numbers and $\binom{r}{\ell} = r!/((r-\ell)!\,\ell!)$.

Now using (5.7) in (5.6), we obtain

(5.8) $\log G(t) = ((p-1)/2) \log 2\pi - ((np_2-1/2+k+j) \log p_2$

$- ((p_2-1)/2+p_2(p_2+1)/4+j) \log(n(1-2it\rho)/2) +$

$+ \sum_{r=1}^{m} (n(1-2it\rho)/2)^{-r} w_r + R^0_{m+1}(n,t)$, where the coefficients w_r are

given by

$$w_r = \left[\sum_{\delta=1}^{p_2} [B_{r+1}((1-\delta-n)/2) - B_{r+1}((1-\delta-n)/2+k_\delta)] + B_{r+1}(k+j)/p_2^r - \sum_{\delta=1}^{p_2} B_{r+1}(-\delta/2)\right] (-1)^r/(r(r+1))$$

Therefore G(t) can be written as

(5.9) $G(t) = (2\pi)^{(p_2-1)/2} p_2^{(1-np_2)/2-(k+j)}] (n(1-2it\rho)/2)^{-[(p_2-1)/2+p_2(p_2+1)/4+j]}$

$\sum_{r=0}^{\infty} W_r((1-2it\rho)n/2)^{-r} + R'_{m+1}(n,t)$, where W_r is the coefficient of

$((1-2it\rho)n/2)^{-r}$ in the expansion of $\exp\left[\sum_{r=1}^{m} ((1-2it\rho)n/2^{-r} w_r\right]$.

Let $u = (p_2-1)/2+p_2(p_2+1)/4 + j$. Then (5.9) can be put in the form

(5.10) $G(t) = (2\pi)^{(p_2-1)/2} p_2^{(1-np_2)/2-(k+j)} \sum_{r=0}^{\infty} W_r((1-2it\rho)n/2)^{-(r+u)} + R'_{m+1}(n,t)$.

Hence the characteristic function of λ^* is given by

(5.11) $\phi(t) = C_1(p_2,n,\Sigma) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} A(J,\kappa,p_2,n,\Sigma) p_2^{-(k+j)}$

$\sum_{r=0}^{\infty} W_r((1-2it\rho)n/2)^{-(r+u)} + R''_{m+1}(n,t)$ where

$C_1(p_2,n,\Sigma) = C(p_2,n,\Sigma) (2\pi)^{(p_2-1)/2} p_2^{(1-np_2)/2}$.

Since $(1-i\beta t)^{-\alpha}$ is the characteristic function of the gamma density $g_\alpha(\beta,x)$ where

(5.12) $g_\alpha(\beta,x) = [\beta^\alpha \Gamma(\alpha)]^{-1} x^{\alpha-1} e^{-x/\beta}$, the density of λ^* can be derived from (5.11) in the form

$$(5.13)\quad p(\lambda^*) = C_1(p_2,n,\Sigma) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} A(J,\kappa,p_2,n,\Sigma) p_2^{-(k+j)}$$

$$\sum_{r=0}^{\infty} (2/n)^{r+u} W_r\, g_{r+u}(2\rho,\lambda^*) + R_{m+1}'''(n).$$

Hence the probability that λ^* is larger than any value, say λ_0 is

$$(5.14)\quad P[\lambda^* > \lambda_0] = C_1(p_2,n,\Sigma) \sum_{J=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} A(J,\kappa,p_2,n,\Sigma) p_2^{-(k+j)}$$

$$\sum_{r=0}^{\infty} (2/n)^{r+u} W_r\, G_{r+u}(2\rho,\lambda_0) + R_{m+1}(n) \quad \text{where}$$

$$(5.15)\quad G_{r+u}(2\rho,\lambda_0) = \int_{\lambda_0}^{\infty} g_{r+u}(2\rho,x)\, dx \quad \text{and}$$

$$(5.16)\quad R_{m+1}(n) = (2\pi)^{-1} C_1(p_2,n,\Sigma) \sum_{j=0}^{\infty} \sum_{J} \sum_{k=0}^{\infty} \sum_{\kappa} A(J,\kappa,p_2,n,\Sigma) p_2^{-(k+j)}$$

$$\int_{\lambda_0}^{\infty} \int_{-\infty}^{\infty} e^{-it\lambda^*} \sum_{r=0}^{\infty} W_r (2/n)^{r+u} (1-2it\rho)^{-(r+u)} [\exp(R_{m+1}'''(n))-1]\, dt\, d\lambda^*.$$

From (5.14), we obtain the distribution of λ^* as a series of gamma distributions. In particular, taking $\rho = 1$, we see that the distribution of λ^* may be expressed as a series of chisquare distributions. Now

$$(5.17)\quad P[\lambda^* > \lambda_0] = P[-2\rho \log L_{VC}^{n/2} > \lambda_0] = P[L_{VC} < \exp(-\lambda_0/n\rho)]$$

Therefore, once we know the distribution of λ^*, the distribution of L_{VC} can be obtained by using (5.17).

In particular, the null distribution of L_{VC} is given by

$$(5.18)\quad p_1(\lambda^*) = C_1(p_2,n,\Sigma)\Gamma(np_2/2) \sum_{r=0}^{\infty} (2/n)^{r+u_0} W_{0,r} g_{r+u}(2\rho,\lambda^*) + R_{0,m+1}(n),$$

where

$$(5.19)\quad C_1(p_2,n,\Sigma) = (2\pi)^{(p_2-1)/2}\, p_2^{(1-np_2)/2} / \prod_{i=1}^{p_2} \Gamma(n-i)/2,$$

which is the same as the one obtained by Nagarsenker [12]. $W_{0,r}$ being the coefficient of $((1-2it\rho)n/2)^{-r}$ in the expansion of $\exp[\sum_{r=1}^{m} ((1-2it\rho)n/2)^{-r} w_{0,r}]$, where

$$u_0 = (p_2-1)/2 + p_2(p_2+1)/4 \text{ and}$$

$$(5.20)\quad w_{0,r} = [B_{r+1}(0)/p_2^r - \sum_{\delta=1}^{p_2} B_{r+1}(-\delta/2)](-1)^r/(r(r+1)),$$ $R_{0,m+1}(n)$ is defined similarly as in (5.16) with $j = k = 0$.

6. Power Computations of L_{VC} Criterion. The distributions obtained in sections 3, 4, and 5 were used to study the power behavior of Wilks' L_{VC} criterion. Powers have been computed for $p = 2$ using (3.26) and for $p = 3$ using (4.20) and (5.14) which have been presented in tables (6.1) and (6.2) respectively. The computations involve zonal polynomials of degree 0 to 9 (see [7]). Lower five percent points of L_{VC} Criterion (see Wilks [17]) have been used for our computations. All the computations were carried out on CDC 6500 computer at the Purdue University Computing Center. Before computing the power for specific values of the parameter, the total probability for that case has been computed and the number of decimals included in the tables were determined depending upon the number of places of accuracy obtained in the total probabilities. The accuracy of the results have been checked by comparing the powers for specific values of the parameters based on (4.20) and (5.14).

From table (6.1), we observe that power increases with the sample size N as well as the only parameter involved, ρ. For the case $p = 3$, we observe from table (6.2) the power increases with N, each of the parameters c, ρ_{12} and ρ_{13}, but decreases with ρ_{23}.

Table 6.1

Power Computations For Wilks' L_{VC} Criterion

p=2

$N \backslash \rho^2$	$.0^41$	$.0^31$	$.0^21$	$.0^25$	.01	.05	.1
3	.0500005	.050005	.05005	.05025	.05050	.05258	.05534
4	.050001	.050014	.05014	.05070	.05140	.05727	.06523
5	.050002	.05002	.05024	.05123	.05247	.06288	.07721
6	.050004	.05004	.05036	.05179	.05359	.06883	.08999
7	.050005	.05005	.05047	.05235	.05474	.07493	.1031
8	.05006	.05006	.05058	.05293	.05590	.02111	.1165
9	.05007	.05007	.05070	.05350	.05706	.08735	.1299
10	.050008	.05008	.05081	.05408	.05822	.09362	.1434
15	.050014	.05014	.05138	.05698	.06409	.1234	.2115
20	.050020	.05020	.05196	.05989	.06999	.1576	.2789
25	.050025	.05025	.05253	.06281	.07593	.1899	.3443
30	.050031	.05031	.05311	.06573	.08190	.2223	.4070
40	.05004	.05042	.05426	.07161	.09392	.2864	.5217
60	.5007	.05065	.05657	.08347	.1182	.4086	.7030
70	.05008	.05077	.05773	.08944	.1305	.4654	.7704
80	.05009	.05088	.05889	.09544	.1429	.5188	.8243
110	.05012	.05123	.06237	.1136	.1802	.6566	.9254
120	.05013	.05134	.06353	.1197	.1926	.6951	.9448
140	.05016	.05157	.06586	.1319	.2176	.7617	.9703
200	.05023	.05226	.07289	.1689	.2917	.8926	.9959

Table 6.1 (Continued)

N \ ρ^2	.15	.2	.25	.3	.35	.4	.45
3	.05831	.06153	.06503	.06886	.07308	.07777	.0830
4	.07400	.08371	.09450	.1066	.1202	.1356	.1533
5	.09320	.1111	.1311	.1537	.1792	.2080	.2408
6	.1137	.1404	.1702	.2037	.2411	.2829	.3295
7	.1349	.1704	.2100	.2539	.3024	.3555	.413
8	.1562	.2006	.2496	.3034	.3616	.4240	.4899
9	.1777	.2307	.2887	.3513	.4179	.4875	.559
10	.1992	.2606	.3270	.3975	.4709	.5459	.621
15	.3052	.4029	.5012	.5962	.6846	.763	
20	.4051	.5283	.6417	.7403	.821	.9	
25	.4966	.6340	.7489	.8383	.903		
30	.5782	.7203	.827	.902			
40	.7114	.8427	.923	.993			
60	.8753	.9556	.99				
70	.9206	.9774					
80	.9502	.9889					
110	.9887	.99					
120	.993						

Table 6.2

Power Computations For Wilks' L_{VC} Criterion

p=3

c	1.0	1.0	1.0	1.0	1.025	1.025
ρ_{12}	.05	.05	.3	.4	.005	.05
ρ_{13}	.05	.1	.3	.3	.005	.05
ρ_{23}	.05	.2	0.	0.	.05	.05
n						
3	.0502	.052	.057	.061	.05006	.0502
4	.0505	.057	.067	.076	.0502	.0506
5	.0509	.062	.079	.093	.0504	.0510
6	.0511	.067	.087	.109	.0506	.0513
7	.0516	.080	.103	.134	.0508	.0518
8	.0519	.092	.117	.157	.0511	.052
10	.053	.125	.146	.207	.0516	.053
17	.055	.210	.207	.289	.053	.058
22	.057	.292	.261	.371	.056	.061

c	1.025	1.025	1.025	1.025	1.05	1.05
ρ_{12}	.05	.1	.3	.3	.005	.05
ρ_{13}	.1	.15	.3	.3	.005	.05
n ρ_{23}	.2	.2	.05	0.	.05	.05
3	.0516	.0522	.057	.057	.0501	.0502
4	.054	.0542	.067	.067	.0502	.0506
5	.057	.058	.078	.079	.0504	.0510
6	.059	.061	.086	.087	.0507	.05111
7	.065	.066	.102	.103	.0512	.0518
8	.069	.069	.115	.117	.0514	.052
10	.077	.079	.142	.145	.0517	.053
17	.14	.153	.22	.24	.055	.059
22	.17	.19	.29	.31	.059	.067

Table 6.2 (Continued)

n	c	1.05	1.05	1.05	1.05	1.05	1.05
	ρ_{12}	.05	.2	.4	.2	.25	.0
	ρ_{13}	.1	.15	.1	.2	.25	.3
	ρ_{23}	.2	.2	.1	.2	.25	.0
3		.051	.052	.051	.054	.056	.054
4		.052	.055	.052	.059	.065	.058
5		.056	.058	.053	.065	.075	.063
6		.058	.062	.055	.068	.082	.066
7		.061	.066	.057	.077	.092	.074
8		.064	.070	.059	.084	.109	.079
10		.070	.078	.063	.098	.12	.091
17		.14	.151	.077	.15	.20	.13
22		.17	.186	.097	.21	.25	.16

n	c	1.05	1.05	1.05	1.05	1.05
	ρ_{12}	.3	.3	.0	.4	.4
	ρ_{13}	.3	.3	.4	.3	.4
	ρ_{23}	0	.3	.0	.0	0.
3		.057	.058	.056	..0607	.064
4		.068	.069	.066	.076	.086
5		.079	.082	.076	.093	.111
6		.087	.090	.083	.109	.134
7		.103	.102	.097	.133	.170
8		.118	.113	.108	.155	.202
10		.15	.14	.13	.20	.27
17		.26	.22	.22	.37	.50
22		.34	.29	.28	.50	.60

n	c	1.2	1.2	1.2	1.2	1.2
	ρ_{12}	.005	.1	.05	.1	.2
	ρ_{13}	.005	.1	.1	.15	.2
	ρ_{23}	.05	.1	.2	.2	.2
3		.0507	.053	.053	.054	.058
4		.055	.057	.059	.060	.069
5		.059	.063	.065	.067	.081
6		.061	.068	.070	.073	.091
7		.066	.078	.081	.084	.108
8		.081	.087	.090	.094	.12
10		.098	.105	.108	.114	.15
17		.18	.20	.22	.23	.25
22		.21	.22	.25	.26	.28

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